ON THE EXTERIOR SPACE OF $(n, k, 2)$-LINK

By

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1. Introduction. Throughout this paper we shall only be concerned with the combinatorial category, consisting of simplicial complexes and piecewise linear maps (for the combinatorial category, see [5]). In this paper we study the exterior $Q = S^n - \mathit{Int}\, U_1 \cup \mathit{Int}\, U_2$ of $(n, k, 2)$-*links* $L = (S^n \supset K_1^k \cup K_2^k)$ where $U_i = U(K_i^k, S^n)$ is a regular neighborhood of K_i^k in S^n and classify Q by homeomorphisms. If $n = k+1$, the classification of $(n, k, 2)$-*links* is equivalent to PL annulus conjecture. So we restrict the case $n \geqq k+2$. First we will study the algebraic structure of the exterior Q and will show the following.

Theorem 1.

Case 1. *If* $2k+1 \neq n$, $2k+2 \neq n$ *and* $n \geqq k+3$, *for* $i=1$ *and* 2

$$H_p(Q, M_i) \cong \begin{cases} Z & p = n-k-1,\ k+0 \\ 0 & \text{otherwise} \end{cases}$$

where $M_i = \partial U_i$.

Case 2. *If* $2k+1 = n$ *and* $n \geqq k+3$, *according to the embedding* $f: S^k \to S^n - \mathit{Int}\, U_2$ *with* $f(S^k) = K_1^k$,

$$H_p(Q, M_1) \cong \begin{cases} Z & p = k,\ \ k+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\cong \begin{cases} Z_m & p = k \quad (m \neq 1) \\ 0 & \text{otherwise} \end{cases}$$

$$\cong \{\ 0\ \} \qquad \text{for all } p$$

Case 3. *If* $2k+2 = n$ *and* $n \geqq k+3$, *for* $i=1$ *and* 2

$$H_p(Q, M_i) \cong \begin{cases} Z+Z & p = k+1 \\ 0 & \text{otherwise} \end{cases}$$

Case 4. *If* $2k+1 > n$ *and* $n = k+2$ *and if* $(S^n \supset K_2^k)$ *is a trivial knot,*

$$H_p(Q, M_1) \cong \begin{cases} Z & p = 1,\ \ k+1 \\ 0 & \text{otherwise} \end{cases}$$

Case 5. *If* $2k+1=n$ *and* $n=k+2$ (*equally* $n=3$ *and* $k=1$) *and if* $(S^3 \supset K_2^1)$ *is a trivial knot, according to the embedding* $f: S^1 \to S^3 - Int\, U_2$ *with* $f(S^1)=K_1^1$,

$$H_p(Q, M_1) \cong \begin{cases} Z & p=1,\ 2 \\ 0 & otherwise \end{cases}$$

$$\cong \begin{cases} Z_m & p=1 \quad (m \neq 1) \\ 0 & otherwise \end{cases}$$

$$or \qquad \cong \{0\} \qquad for\ all\ 1.$$

Then we will obtain the following Main Theorem using the handle decomposition theorem [1], [2], [3] and the unknotting theorem [4], [5].

Theorem 3.

Case 1. *If* $n>2k+1$, Q *is homeomorphic to*

$$(S_1^{n-k-1} \times D_1^{k+1} - Int\, D_1^n) \underset{\partial D_1^n = \partial D_2^n}{\cup} (S_2^{n-k-1} \times D_2^{k+1} - Int\, D_2^n)$$

where $D_i^n \subset Int\,(S_i^{n-k-1} \times D_i^{k+1})$, $i=1, 2$.

Case 2. *If* $n=2k+1$ *and* $k \geqq 3$, Q *is homeomorphic to*

$$(S^k \times S^k \times I) \natural (S^k \times D^{k+1}) \underset{g}{\cup} (D^{k+1} \times D^k)$$

with $\partial Q \cong (S^k \times S^k) \cup (S^k \times S^k)$ *where* $\natural$ *is a boundary connected sum and* Q *is classified by the homology class of the embedding*

$$g \,|\, \partial D^{k+1} \times p : \partial D^{k+1} \times p \to (S^k \times S^k) \# (S^k \times S^k)$$

where $p \in D^k$ *and* $\#$ *is a connected sum.*

Case 3. *If* $2n \geqq 3k+5$ *and* $2k+1>n$, Q *is homeomorphic to*

$$(S^k \times S^{n-k-1} \times I) \natural (S^{n-k-1} \times D^{k+1}) \underset{g}{\cup} (D^{k+1} \times D^{n-k-1})$$

and it is classified by the homotopy class of

$$g \,|\, \partial D^{k+1} \times p : \partial D^{k+1} \times p \to (S^k \times S^{n-k-1}) \# (S^k \times S^{n-k-1}).$$

Case 4. *If* $n \geqq k+3$, $3k+4 \geqq 2n$ *and* $k \geqq 3$, Q *is same as case* 3 *and it is classified by an ambient isotopy class of locally fiat embedding*

$$g \,|\, \partial D^{k+1} \times p : \partial D^{k+1} \times p \to (S^k \times S^{n-k-1}) \# (S^k \times S^{n-k-1}).$$

I should like to express my sincere gratitude to the members of Kōbe Topology Seminar for many discussion of this problem.

§ 2. Definitions and Notations

From now on all spaces and maps assumed in the *PL* category. Thus each space is homeomorphic to a locally finite simplicial complex, and maps are simplicial with respect to some subdivision of the triangulations of domain and range. All manifolds with or without boundary will be assumed oriented and combinatorial in the sense of [5]. S^n is a standard n-sphere and D^n is a standard n-cell. An $(n, k, 2)$-*link* L is a pair $(S^n \supset K^k \cup K^k)$ of an n-sphere S^n and a disjoint union of locally flatly embeded k-spheres $K_i^k, i=1, 2$ in S^n. We assume $n \geqq k+2$ throughout this paper. ∂X and *Int* X mean the boundary and the interior of a manifold X. Let $U_i = U(K_i^k, S^n)$ be a regular neighborhood of K_i^k in S^n then U_i is homeomorphic to $S^k \times D^{n-k}$ and denoted by $U_i \cong S^k \times D^{n-k}$. Let $Q = S^n - Int\, U_1 \cup Int\, U_2$ and $M_i = \partial U_i$. We call Q the exterior of $K_1^k \cup K_2^k$ in S^n. We denote $H_*(K; Z)$ briefly $H_*(K)$ where Z is a ring of integers. $f \simeq g$ means map f homotopic to g and $X \simeq Y$ means that spaces X and Y are same homotopy type. $\sharp$ means a connected sum and $\natural$ means a boundary connected sum.

§3. Algebraic Part

Since M_i is homeomorphic to $S^k \times S^{n-k-1}$, we obtain the following.

Lemma 1.

$$H_p(M_i) \cong \begin{cases} Z & p=0,\ n-k-1,\ k,\ n-1 \\ 0 & otherwise \end{cases} \quad (if\ 2k+1 \neq n)$$

$$\cong \begin{cases} Z+Z & p=k \\ Z & p=0,\ n-1 \\ 0 & otherwise \end{cases} \quad (if\ 2k+1=n)$$

Corollary.

$$H_p(\partial Q) = H_p(M_1 \cup M_2)$$

$$\cong \begin{cases} Z+Z & p=0, n-k-1,\ n-1 \\ 0 & otherwise \end{cases} \quad (2k+1 \neq n)$$

$$\cong \begin{cases} Z+Z+Z+Z & p=k \\ Z+Z & p=0,\ n-1 \\ 0 & otherwise \end{cases} \quad (2k+1=n)$$

Lemma 2.

$$H_p(Q, \partial Q) \cong \begin{cases} Z+Z & p=k+1 \\ Z & p=1,\ n \\ 0 & otherwise \end{cases}$$

Proof. $H_p(Q, \partial Q) \cong H_p(S^n, U_1 \cup U_2)$ by excision and $U_i \cong S^k \times D^{n-k}$. Hence the lemma follows from the exact sequence of the pair $(S^n, U_1 \cup U_2)$.

Theorem 1.

Case 1. *If* $2k+1 \neq n$, $2k+2 \neq n$ *and* $n \geqq k+3$, *for* $i=1$ *and* 2

$$H_p(Q, M_i) \cong \begin{cases} Z & p=n-k-1, \ k+1 \\ 0 & \text{otherwise} \end{cases}$$

Case 2. *If* $2k+1=n$ *and* $n \geqq k+3$ *(equally* $2k+1=n$ *and* $k \geqq 2$*), according to the embedding* $f: S^k \to S^n - Int\, U_2$ *with* $f(S^k) = K_1^k$,

$$H_p(Q, M_1) \cong \begin{cases} Z & p=k, \ k+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\cong \begin{cases} Z_m & p=k \quad (m \neq 1) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{or} \qquad \cong \{ 0 \} \qquad \text{for all } p.$$

Case 3. *If* $2k+2=n$ *and* $n \geqq k+3$, *for* $i=1$ *and* 2,

$$H_p(Q, M_i) \cong \begin{cases} Z+Z & p=k+1 \\ 0 & \text{otherwise} \end{cases}$$

Case 4. *If* $2k+1>n$ *and* $n=k+2$ *(equally* $n=k+2$*) and* $k \geqq 2$ *and if* $(S^n \supset K_2^k)$ *is a trivial knot,*

$$H_p(Q, M_1) \cong \begin{cases} Z & p=1, \ k+1, \ \text{(equally } p=1, \ n-1) \\ 0 & \text{otherwise} \end{cases}$$

Case 5. *If* $2k+1=n$ *and* $n=k+2$ *(equally* $n=3$ *and* $k=1$*) and if* $(S^3 \supset K_2^1)$ *is a trivial knot, according to the embedding* $f: S \to S^3 - Int\, U_2$ *with* $f(S^1) = K_1^1$,

$$H_p(Q, M_1) \cong \begin{cases} Z & p=1, \ 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\cong \begin{cases} Z_m & p=1 \quad (m \neq 1) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{or} \qquad \cong \{ 0 \} \qquad \text{for all } i.$$

Proof. Case 1. Since the knot $(S^n \supset K_i^k)$ is trivial under $n \geqq k+3$ [4], $S^n - Int\, U_i = S^{n-k-1} \times D^{k+1}$. And by the excision theorem,

$$H_p(Q, M_1) \cong H_p(S^n - Int\, U_2, U_1).$$

It is obviously $H_0(Q, M_1)=\{0\}$. So by the exact sequence of $(S^n-Int\ U_2, U_1)$ $\cong(S^{n-k-1}\times D^{k+1}, S^k\times D^{k+1})$ and by the fact that any epimorphism $Z\to Z$ is an isomorphism, we obtain

$$H_p(Q, M_1)\cong\begin{cases} Z & p=n-k-1,\ k+1 \\ 0 & otherwise \end{cases}$$

Similarly for $H_p(Q, M_2)$.

Hence

$$H_p(Q, M_i)\cong\begin{cases} Z & p=n-k-1,\ k+1 \\ 0 & otherwise \end{cases}$$

for $i=1, 2$.

Case 2. Same as above we obtain $H_p(Q, M_1)=0$, $p\neq k$, $k+1$ using the exact sequence of $(S^n-Int\ U_2, U_1)\cong(S^k\times D^{k+1}, S^k\times D^{k+1})$.
When $p=k$ or $k+1$,

$$0\longrightarrow H_{k+1}(S^n-Int\ U_2, U_1)\xrightarrow{\partial_*}\underset{\underset{Z}{\wr\|}}{H_k(U_1)}\xrightarrow{i_*}\underset{\underset{Z}{\wr\|}}{H_k(S^n-Int\ U_2)}$$

$$\xrightarrow{j_*}H_k(S^n-Int\ U_2, U_1)\longrightarrow 0\,.$$

And i_* is induced by the inclusion $i: U_1\to S^n-Int\ U_2$ and hence induced by the imbedding $f: S^k\to S^n-Int\ U_2$ with $f(S^k)=K_1^k$. Since any subgroup of a free group is free, $Im\ i_*$ is a free group. So we many consider the case

(a) $Im\ i_*=0$ (b) $Im\ i_*\cong Z$ (c) $Im\ i_*\cong mZ$ (m: *integer*).

(a) $Im\ i_*=0\Rightarrow Im\ \partial_*=Ker\ i_*=Z$. Hence $H_{k+1}(S^n-Int\ U_1, U_1)\cong Z$.
On the other hand $Im\ i_*=0\Rightarrow Ker\ j_*=Im\ i_*=0$.

So j_* is an isomorphism because j_* is an epimorphism. Hence $H_k(S^n-Int\ U_2, U_1)\cong Z$.

(b) $Im\ i_*\cong Z\Rightarrow Ker\ i_*=Im\ \partial_*=0$.

Hence $H_{k+1}(S^n-Int\ U_2, U_1)=0$ becauce ∂_* is a monomorphism. And since $Ker\ j_*=Im\ i_*=Z$, $Im\ j_*=0$.

So $\quad H_k(S^n-Int\ U_2, U_1)=0$.

(c) $Im\ i_*=mZ\Rightarrow Ker\ i_*=Im\ \partial_*=0$.

Hence $\quad H_{k+1}(S^n-Int\ U_2, U_1)\cong 0$.

Furthermore since $Ker\ j_*=Im\ i_*=mZ$, so $H_k(S^n-Int\ U_2, U_1)\cong Z/Ker\ j_*=Z_m$

becauce j_* is an epimorphism. Hence we obtain the required result in Case 2. Geometrically the case (a), (b) and (c) corresponds to the following,

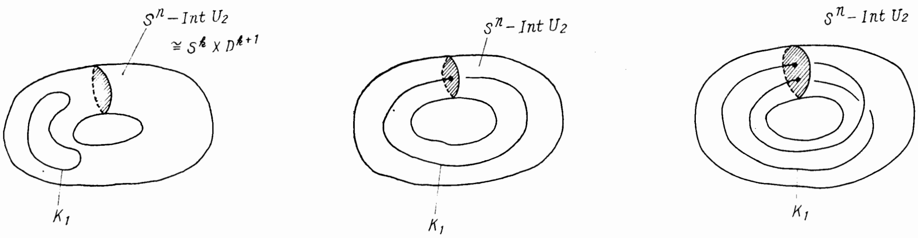

Fig. 1.

Case 3. Same as above we obtain $H_p(Q, M_1)=0$ for $p \neq k+1$ using the exact sequence of $(S^n - Int\ U_1, U_1)$ and using the facts $S^n - Int\ U_2 \cong S^{n-k-1} \times D^{k+1}$ and $U_1 \cong S^k \times D^{n-k}$. When $p=k+1$ $(=n-k-1$ because $n=2k+2)$,

$$0 \longrightarrow \underset{\underset{Z}{\wr\Vert}}{H_{k+1}}(S^n - Int\ U_2) \xrightarrow{j_*} H_{k+1}(S^n - Int\ U_2, U_1) \longrightarrow \underset{\underset{Z}{\wr\Vert}}{H_k(U_1)} \longrightarrow 0.$$

Since ∂ is an epimorphism,

$$Z \cong H_{k+1}(S^n - Int\ U_2, U_1)/_{ker\,\partial} \cong H_{k+1}/_{imj_*} \cong H_{k+1}/_{Z} \text{ or } H_{k+1}/_{mZ}.$$

Hence $H_{k+1}(S^n - Int\ U_2, U_1) \cong Z + Z$ in both cases.

Case 4. By excision theorem $H_p(Q, M_1) \cong H_p(S^n - Int\ U_2, U_1)$ and $(S^n - Int\ U_2 \supset U_1)$ is homeomorphic to $(S^1 \times D^{k+1} \supset S^k \times D^2)$ because the knot $(S^n \supset K_2^k)$ is trivial and $n=k+2$. Hence from the exact sequent of $(S^n - Int\ U_2, U_1)$,

$$\cdots \longrightarrow H_p(U_1) \longrightarrow H_p(S^n - Int\ U_2) \longrightarrow H_p(S^n - Int\ U_2, U_1)$$
$$\longrightarrow H_{p-1}(U_1) \longrightarrow \cdots$$

and using $H_0(S^n - Int\ U_2, U_1)=0$,

$$H_p(Q, M_1) \cong \begin{cases} Z & p=1,\ k+1 \\ 0 & otherwise \end{cases}$$

Case 5. Since the knot $(S^3 \supset K_2^1)$ is trivial, $H_p(Q, M_1) \cong H_p(S^3 - Int\ U_2, U_1)$ and $(S^3 - Int\ U_2 \supset U_1)$ is homeomorphic to $(S^1 \times D^2 \supset S^1 \times D^2)$ similary to the above. First it is obviously $H_0(Q, M_1) \cong H_3(Q, M_1)=0$.
Next from the exact sequence of $(S^3 - Int\ U_2, U_1)$

$$0 \longrightarrow H_2(S^3-Int\ U_2,\ U_1) \xrightarrow{\partial_*} H_1(U_1) \xrightarrow{i_*} H_1(S^3-Int\ U_2)$$
$$H_1(U_1)\cong Z,\quad H_1(S^3-Int\ U_2)\cong Z$$
$$\xrightarrow{j_*} H_1(S^3-Int\ U_2,\ U_1) \xrightarrow{\partial_*} H_0(U_1) \xrightarrow{i_*} H_0(S^3-Int\ U_2)$$
$$H_0(U_1)\cong Z,\quad H_0(S^3-Int\ U_2)\cong Z$$
$$\xrightarrow{j_*} H_0(S^3-Int\,U_2,\ U_1).\quad H_0(S^3-Int\,U_2,\ U_1)=0$$

Since $i_*: H_0(U_1)\to H_0(S^3-Int\ U_2)$ is epimorphic, its i_* is isomorphic. Hence $Im\ \partial_*=0$ where $\partial_*: H_1(S^3-Int\ U_2,\ U_1)\to H_0(U_1)$. So we may consider the following

$$0 \longrightarrow H_2(S^3-Int\ U_2,\ U_1) \xrightarrow{i_*} H_1(U_1) \xrightarrow{\partial_*} H_1(S^3-Int\ U_2)$$
$$H_1(U_1)\cong Z,\quad H_1(S^3-Int\ U_2)\cong Z$$
$$\xrightarrow{j_*} H_1(S^3-Int\ U_2,\ U_1) \longrightarrow 0\,.$$

But i_* is induced by the inclusion $i: U_1\to S^3-Int\ U_2$ and hence induced by the imbedding $f: S^1\to S^3-Int\ U_2$ with $f(S^1)=K_1^1$. Therefore it is completely the same as Case 2 and so we omit the rest of the proof. And according to the imbedding f we obtain

$$H_p(Q,\ M_1)\cong\begin{cases} Z & p=1,\ 2 \\ 0 & otherwise \end{cases}$$

$$H_p(Q,\ M_1)\cong\begin{cases} Z_m & p=1 \quad (m\neq 1) \\ 0 & otherwise \end{cases}$$
$$or \qquad \cong\{\ 0\ \} \qquad for\ all\ i.$$

Complete the proof.

Theorem 2. *If $n\geqq k+3$ or if $n=k+2$ and the knot $(S^3\supset K_i^k)$ $i=1$ or 2 is trivial,*

$$H_p(Q)\cong\begin{cases} Z+Z & p=n-k-1 \\ Z & p=0,\ n-1 \\ 0 & otherwise \end{cases}$$

Proof. Since Q is connected and with boundary, $H_0(Q)\cong Z$ and $H_n(Q)=0$ for all cases.

Using the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_{p+1}(S^n) \longrightarrow H_p(\partial Q) \longrightarrow H_p(Q)+H_p(U_1 \cup U_2) \longrightarrow H_p(S^n) \longrightarrow \cdots,$$

we obtain $H_p(Q) \cong 0$ under $n-1>p>0$ and $p \neq n-k-1$.
Futhermore if $2k+1 \neq n$, $H_{n-k-1}(Q) \cong H_{n-k-1}(\partial Q) \cong Z+Z$ by Cor. to Lemma 1. If $2k+1=n$, $H_{n-k-1}(\partial Q) \cong H_{n-k-1}(Q)+H_{n-k-1}(U_1 \cup U_2)$

and $\quad H_{n-k-1}(\partial Q) \cong Z+Z+Z+Z, \quad H_{n-k-1}(U_1 \cup U_2) \cong Z+Z.$

so $\quad H_{n-k-1}(Q) \cong Z+Z.$

Case 1. $n \geqq k+3$. From Theorem 1, $H_{n-1}(Q, M_i) \cong H_n(Q, M_i)=0$ if $n \geqq k+3$.
So using the relative Mayer-Vietoris sequeuce

$$\begin{aligned}\cdots &\longrightarrow H_n(Q, M_1)+H_n(Q, M_2) \longrightarrow H_n(Q, \partial Q)\\ &\longrightarrow H_{n-1}(Q) \longrightarrow H_{n-1}(Q, M_1)+H_{n-1}(Q, M_2)\\ &\longrightarrow \cdots,\end{aligned}$$

we obtain $H_{n-1}(Q) \cong H_n(Q, \partial Q) \cong Z$.

Case 2. $n=k+2$ and $k \geqq 2$. We may assume the knot $(S^n \supset K_2^{n-2})$ be trivial. Then $H_n(Q, M_1)=0$ and $H_{n-1}(Q, M_1) \cong Z$ by Theorem 1. So by the following sequence,

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_n(Q, M_1) & \xrightarrow{\partial_*} & H_{n-1}(M_1) & \xrightarrow{i_*} & H_{n-1}(Q) & \\ & \| & & \| & & & \\ & 0 & & Z & & & \\ \xrightarrow{j_*} & H_{n-1}(Q, M_1) & \xrightarrow{\partial_*} & H_{n-2}(M_1) & \xrightarrow{i_*} & H_{n-2}(Q) & \longrightarrow \cdots \\ & \| & & \| & & \| & \\ & Z & & Z & & 0 & \end{array}$$

$\partial_* : H_{n-1}(Q, M_1) \rightarrow H_{n-2}(M_1)$ is an isomorphism and hence $Im\, j_*=0$. Therefore $H_{n-1}(Q) \cong H_{n-1}(M_1)=Z$.

Case 3. $n=k+2$ and $k=1$. By Theorem 1,

$$(*)\quad \begin{array}{cccccc} \cdots \longrightarrow & H_3(Q, M_1) & \xrightarrow{\partial_*} & H_2(M_1) & \xrightarrow{i_*} & H_2(Q) \\ & \| & & \| & & \\ & 0 & & Z & & \\ \xrightarrow{j_*} & H_2(Q, M_1) & \longrightarrow \cdots & & & \\ & \| & & & & \\ & Z \textit{ or } 0 & & & & \end{array}$$

provided the knot $(S^3 \supset K_2^1)$ be trivial. Hence if $H_2(Q, M_1)=0$, $H_2(Q) \cong Z$.
So let $H_2(Q, M_1) \cong Z$. By the excision theorem

$$H_p(S^n, Q) \cong H_p\Big(\bigcup_{i=1}^{2} U_i, \bigcup_{i=1}^{2} M_i\Big)$$

and
$$\begin{array}{ccccccc} H_3(\bigcup_{i=1}^{2} U_i) & \longrightarrow & H_3(\bigcup_{i=1}^{2} U_i, \bigcup_{1=i}^{2} M_i) & \longrightarrow & H_2(\bigcup_{i=1}^{2} M_i) & \longrightarrow & H_2(\bigcup_{i=1}^{2} U_i), \\ \| & & & & \| & & \| \\ 0 & & & & Z+Z & & 0 \end{array}$$

Hence
$$H_3(S^3, Q) \cong Z+Z.$$

Furthermore by the exact sequence of (S^3, Q),

$$(**) \quad \begin{array}{ccccccccc} H_3(Q) & \xrightarrow{\tilde{i}_*} & H_3(S^3) & \xrightarrow{\tilde{j}_*} & H_3(S^3, Q) & \xrightarrow{\tilde{\partial}} & H_2(Q) & \longrightarrow & H_2(S^3). \\ \| & & \| & & \| & & & & \| \\ 0 & & Z & & Z+Z & & & & 0 \end{array}$$

So $H_2(Q)$ has Z or mZ as the subgroup from (*) and there is an epimorphism of $Z+Z$ onto $H_2(Q)$ from (**). Hence $H_2(Q)$ is isomorphic to $Z+Z$, $Z+Z_m$ or Z. If we assume $H_2(Q)=Z+Z$, the epimorphism $\tilde{\partial}$ is an isomorphism and so $Im\ \tilde{j}_*=Ker\ \tilde{\partial}=0$. This contradicts the fact $Im\ \tilde{j}^*=Z$ or mZ. Next we assume $H_2(Q)\cong Z+Z_m$. Since $H_1(M_1)\cong H_1(Q)\cong Z+Z$ by Lemma 1 and the case 1 of Theorem 1 and $H_1(Q, M_1)=0$ by the following Theorem 2, the epimorphism $H_1(M_1)\to H_1(Q)$ is an isomorphism in (*). Hence $Im\ \partial=0$ and so the sequence (*) becomes as follows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(M_1) & \xrightarrow{i_*} & H_2(Q) & \xrightarrow{j_*} & H_2(Q, M_1) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & Z & & Z+Z_m & & Z & & \end{array}$$

Then Z or $mZ=Im\ i_*\cong Ker\ j_*\cong Z_m$. This is contradiction. Therefore $H_2(Q)\cong Z$.
Complete the proof.

§ 4. Geometrical Part

Notations and Definitions are same as §3. If $n\geqq k+3$, the pair (Q, M_i) is simply connected using the fact that any $(n, k, 1)$-*link* $(S^n\supset S^k\cup S^1)$ is trivial see ([5]). So by the handle decomposition theorem [1], [2], [3], we obtain the following.

Proposition. *If $n\geqq k+3$ and $k\geqq 3$, we have the handle decompositions.*

(1) *If* $n\leqq 2k+1$,
$$Q\cong M_1\times I\cup_f D^{n-k-1}\times D^{k+1}\cup_g D^{k+1}\times D^{n-k-1}$$
where
$$f: \partial D^{n-k-1}\times D^{k+1}\to M_1\times\{1\} \quad \text{and}$$
$$g: \partial D^{k+1}\times D^{n-k-1}\to \partial(M_1\times I\cup_f D^{n-k-1}\times D^{k+1})-M_1\times\{0\}$$
are embeddings.

(2) *If* $n>2k+1$,

$$Q\cong M_1\times I\cup_f D^{k+1}\times D^{n-k-1}\cup_g D^{n-k-1}\times D^{k+1}$$

where $f:\partial D^{k+1}\times D^{n-k-1}\to M_1\times\{1\}$ and

$$g:\partial D^{n-k-1}\times D^{k+1}\to\partial(M_1\times I\cup_f D^{k+1}\times D^{n-k-1})-M_1\times\{0\}$$

are embeddings.

Since $M_i\times I\cong S^k\times S^{n-k-1}\times I$, $M_i\times I$ is d-connected where $d=\min\{k, n-k-1\}-1$.

Case 1. If $n>2k+1$, any $(n, k, 2)$*-link* is trivial. Hence $Q=S^n-Int\,U(K_1^k\cup K_2^k, S^n)\simeq S^{n-k-1}\vee S^{n-k-1}$. So if $k\geqq 2$, $f:\partial D^{k+1}\times D^{n-k-1}\to M_1\times\{1\}=S^k\times S^{n-k-1}\times\{1\}$ homotopic to the embedding $\tilde{f}: S^k\to S^k\times p\times\{1\}$, $p\in S^{n-k-1}$. If $f, f':\partial D^{k+1}\times D^{n-k-1}\to M_1\times\{1\}$ are homotopic to $\tilde{f}$, then f and f' are ambient isotopic to $\tilde{f}$ by $k\geqq 2$ and by *Zeeman's Unknotting Theorem* [5]. Let $W=S^k\times S^{n-k-1}\times I\cup_f D^{k+1}\times D^{n-k-1}$. Then from the above argument $W\simeq S^{n-k-1}$ and $\partial W\cong S^k\times S^{n-k-1}\cup_f S^{n-1}$. Since $\partial(M_1\times I\cup D^{k+1}\times D^{n-k-1}-M_1\times\{0\}=S^{n-1}$, all embeddings $\partial D^{n-k-1}\times D^{k+1}\to\partial(M_1\times I\cup_f D^{k+1}\times D^{n-k-1})-M_1\times\{0\}$ are ambient isotopic. So the embedding $g:\partial D^{n-k-1}\times D^{k+1}\to\partial(M_1\times I\cup_f D^{k+1}\times D^{n-k-1})-M_1\times\{0\}$ is uniquely determined up to ambient isotopy. Hence Q is homeomorphic to

$$\{D^n-\phi(Int\,(S^k\times D^{n-k}))\}\cup_g(D^{n-k-1}\times D^{k+1})$$
$$\cong(S^{n-k-1}\times D^{k+1}-\psi(Int\,D^n))\cup_g(D^{n-k-1}\times D^{d+1})$$

where $\phi: S^k\times D^{n-k}\to Int\,D^n$, $\psi: D^n\to Int\,(S^{n-k-1}\times D^{k+1})$ and $g:\partial D^{n-k-1}\times D^{k+1}\to S^{n-1}$ are any embeddings.

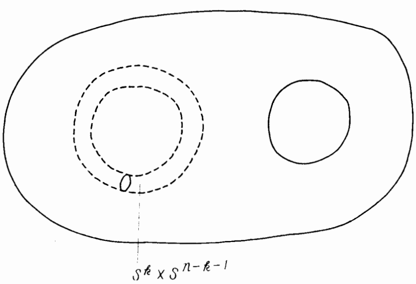

Fig. 2.

Next if $k=1$ and $n>2k+1$, the link $(S^n\supset K_1^1\cup K_2^1)$ is trivial. So we can assume $S^n=D_1^n\cup D_2^n$ so that $D_1^n\cap D_2^n=\partial D_i^n=S^{n-1}$ and $Int\,D_i^n\supset K_i^1$, $i=1, 2$. Then $W_i=D_i^n-Int\,U(K_i^1, D_i^n)\cong S^{n-2}\times D^2-Int\,D^n$ where D^n is contained in $Int\,(S^{n-2}\times D^2)$. Hence

$$Q=S^n-Int\,U(K_1^1\cup K_2^1, S^n)=W_1\cup W_2\cong(S_1^{n-2}\times D_1^2-Int\,D_1^n)$$
$$\bigcup_{\partial D_1^n=\partial D_2^n}(S_2^{n-2}\times D_2^2-Int\,D_2^n)\cong(S^n\times D^2-Int\,D^n)\cup_g(D^{n-2}\times D^2)$$

where D_i^n, $i=1, 2$ are contained in $Int\,(S_i^{n-2}\times D_i^2)$ and where $g:\partial D^{n-2}\times D^2\to S^n$ is embedding. Therefore in general

$$Q \cong (S^{n-k-1}\times D^{k-1}-\phi(Int\ D^n)) \underset{g}{\cup} (D^{n-k-1}\times D^{k+1})=$$
$$(S^{n-k-1}\times D_1^{k+1}-Int\ D_1^n) \underset{\partial D_1^n=\partial D_2^n}{\cup} (S_2^{n-k-1}\times D_2^{k+1}-Int\ D_2^n) \text{ if } n>2k+1.$$

Case 2. If $n=2k+1$ and $k\geqq 3$, any embedding $f:\partial D^{n-k-1}\times D^{k+1}\to S^n \times S^{n-k-1}$ is homotopic. And such f is isotopic by *Zeeman's Unknotting Theorem.* Let $W=M_1\times I\underset{f}{\cup} D^{n-k-1}\times D^{k+1}$, then

$$\partial W=S^k\times S^k\cup((S^k\times S^k)\#(S^k\times S^k)).$$

If we set $\widetilde{W}=W\cup U(K_1^k, S^n)$, then $\widetilde{W}\cong(S^k\times D^{k+1}) \natural (S^k\times D^{k+1})$. But it is not necessarily that one component K_1^k of the link is included in a component of the left side.

Next the embedding

$$g:\partial D^{k+1}\times D^k\to\partial(M_1\times I\underset{f}{\cup} D^k\times D^{n-k})-M_1\times\{0\}\subset(S^k\times D^{k+1})\natural(S^k\times D^{k+1})$$

represents the element of

$$\pi_k(S^k\times S^k\# S^k\times S^k)\cong H_k(S^k\times S^k\# S^k\times S^k)\cong Z+Z+Z+Z.$$

Since $\partial(M_1\times I\underset{f}{\cup} D^k\times D^{k+1})$ is $(k-1)$-connected, any embedding homotopic to g is ampient isotopic by *Zeeman's Unknotting Theorem.* Therefore Q is homeomorphic to

$$(S^k\times S^{n-k-1}\times I)\natural(S^{n-k-1}\times D^{k+1}) \underset{g}{\cup} (D^{k+1}\times D^{n-k-1})$$

with $\partial Q\cong(S^k\times S^{n-k-1})\cup(S^k\times S^{n-k-1})$ and Q is classified by the homology class of the embedding

$$g\,|\,\partial D^{k+1}\times p:\ D^{k+1}\times p\to(S^k\times S^{n-k-1})\#(S^k\times S^{n-k-1})$$

where $p\in D^{n-k-1}$.

Case 3. $2n\geqq 3k+5$ and $n<2k+1$, any two embeddings $f_1, f_2:\partial D^{n-k-1}\times D^{k+1}\to S^k\times S^{n-k-1}$ are homotopic and are ambient isotopic by *Zeeman's Unknotting Theorem.* Let $W=(M_1\times I)\underset{f}{\cup}(D^{n-k-1}\times D^{k+1})$, then W is same as *case* 2. And two homotopic embeddings

$$\partial D^{k+1}\times p\to(S^k\times S^{n-k-1})\#(S^k\times S^{n-k-1})$$

is ambient isotopic by *Zeeman's Unknotting Theorem.* So Q is homeomorphic to

$$(S^k\times S^{n-k-1}\times I)\natural(S^{n-k-1}\times D^{k+1}) \underset{g}{\cup} (D^{k+1}\times D^{n-k-1})$$

and is classified by the homotopy class of

$$g\,|\,\partial D^{k+1}\times p:\partial D^{k+1}\times p\to(S^k\times S^{n-k-1})\#(S^k\times S^{n-k-1}).$$

Case 4. If $n\geqq k+3, 2n\leqq 3k+4$ and $k\geqq 3$, Q is homeomorphic to

$$(S^k\times S^{n-k-1}\times I)\natural(S^{n-k-1}\times D^{k+1}) \underset{g}{\cup} (D^{k+1}\times D^{n-k-1})$$

and classified by an ambient isotopy class of locally flat embedding

$$g \mid \partial D^{k+1} \times p: D^{k+1} \times p \to (S^k \times S^{n-k-1}) \# (S^k \times S^{n-k-1}).$$

Hence we obtain the following Theorem.

Theorem 3. *Let* Q *be the exterior space*

$$S^n - Int\, U(K_1^k \cup K_2^k, S^n) \text{ of } (n, k, 2)\text{-link}.$$

Case 1. *If* $n > 2k+1$, Q *is homeomorphic to*

$$(S_1^{n-k-1} \times D_1^{k+1} - Int\, D_1^n) \underset{\partial D_1^n = \partial D_2^n}{\cup} (S_2^{n-k-1} \times D_2^{k+1} - Int\, D_2^n)$$

where $D_i^n \subset Int\, (S_i^{n-k-1} \times D_i^{k+1})$, $i = 1, 2$.

Case 2. *If* $n = 2k+1$ *and* $k \geqq 3$, Q *is homeomorphic to*

$$(S^k \times S^k \times I) \natural (S^k \times D^{k+1}) \underset{g}{\cup} (D^{k+1} \times D^k)$$

with $Q \cong (S^k \times S^k) \cup (S^k \times S^k)$ *and* Q *is classified by the homology class of the embedding*

$$g \mid \partial D^{k+1} \times p: D^{k+1} \times p \to (S^k \times S^k) \# (S^k \times S^k)$$

where $p \in D^k$.

Case 3. *If* $2n \geqq 3k+5$ *and* $n < 2k+1$, Q *is homeomorphic to*

$$(S^k \times S^{n-k-1} \times I) \natural (S^{n-k-1} \times D^{k+1}) \underset{g}{\cup} (D^{k+1} \times D^{n-k-1})$$

and it is classified by the homotopy class of

$$g \mid \partial D^{k+1} \times p: \partial D^{k+1} \times p \to (S^k \times S^{n-k-1}) \# (S^k \times S^{n-k-1}).$$

Case 4. *If* $n \geqq k+3$, $2n \leqq 3k+4$ *and* $k \geqq 3$, Q *is same as Case* 3 *and it is classified by an ambient isotopy class of locally fiat embedding*

$$g \mid \partial D^{k+1} \times p: \partial D^{k+1} \times p \to (S^k \times S^{n-k-1}) \# (S^k \times S^{n-k-1}).$$

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