SOME STUDIES ON PROJECTIVE FROBENIUS EXTENSIONS[1)]

By

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Introduction. Let A be a Frobenius algebra over a field K, that is, an algebra such that its first and second regular representations are equivalent. Then there exist bases $\{l_i\}$, $\{r_i\}$ of A over K such that

$$l_i a = \sum_{j=1}^{n} \lambda_{ij}(a) l_j ,$$

$$a r_i = \sum_{j=1}^{n} r_j \lambda_{ji}(a) \qquad (\lambda_{ij}(a) \in K , \quad i = 1, 2, \cdots, n)$$

hold for every element a in A. Let h_i's and k_i's be the K-linear transformations of A into K defined by

$$h_i(a) = \lambda_i$$

$$k_i(a) = \lambda'_i \qquad (i = 1, 2, \cdots, n) ,$$

where $a = \sum_{j=1}^{n} \lambda_j l_j = \sum_{j=1}^{n} r_j \lambda'_j$, λ_j, $\lambda'_j \in K$. Then for every element a in A we have

$$(1) \qquad a = \sum_{j=1}^{n} h_j(a) l_j = \sum_{j=1}^{n} r_j k_j(a) ,$$

$$(2) \qquad h_i(l_j a) = k_j(a r_i) \qquad (1 \leqq i, j \leqq n) .$$

These properties do not require that K is a field, and suggest a generalization of the notion of a Frobenius algebra.

Let Γ be a ring with a unit element 1 and Λ be a subring of Γ containing 1. Then we consider the following condition (A) which corresponds to (1) and (2):

(A) There exist element $\{l_i\}$, $\{r_i\}$ in Γ and Λ-linear mappings $\{h_i\} \subseteqq \mathrm{Hom}({}_\Lambda\Gamma, {}_\Lambda\Gamma)$ and $\{k_i\} \subseteqq \mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda)$ such that both (1) and (2) hold for every element a in Γ.

On the other hand, in [6] Kasch defined that the ring extension Γ/Λ is

1) This is a doctoral dissertation at Hokkaido University.

a projective Frobenius extension if and only if the following conditions are fulfilled:

(B_r) $(r\,1)$ ${}_{\Lambda}\Gamma_{\Gamma} \cong \mathrm{Hom}(\Gamma_{\Lambda}, \Lambda_{\Lambda})$.

$(r\,2)$ Γ_{Λ} is finitely generated and projective.

One of the main purposes of the present paper is to prove the equivalence of (A) and (B_r) (Theorem 1). As corollaries of Theorem 1 some simple conditions for a projective Frobenius extension are derived. §1 will be mainly devoted to the proof of Theorem 1. In §2 we shall consider the endomorphism ring $\mathrm{Hom}_{\Lambda}(\Gamma, \Gamma)$ for a projective Frobenius extension and give a simplified proof of [6, Satz 1] based on Theorem 1. In §3 we shall generalize the Maschke–Ikeda–Kasch characterization of relatively projective and relative injective modules to the case of projective Frobenius extensions. In §4–§6 we shall generalize the results in [7] as a whole to the case of projective Frobenius extensions by the way quite analogously to that of [7] except some slight modifications and supplementary remarks. In §7, as applications, we shall give some remarks concerning Galois extensions of simple rings including [4, 3.c].

§1. Let Γ be a ring with a unit element 1[2] and Λ be a subring of Γ containing 1. By P we shall denote the centralizer of Λ in Γ. Let $A = {}_{\Gamma}A_{\Lambda}$ be a Γ–Λ-module and let $B = {}_{\Lambda}B_{\Lambda}$ be a Λ–Λ-module. Then $\mathrm{Hom}(A_{\Lambda}, B_{\Lambda})$, the module of Λ-homomorphisms of A into B, becomes a Λ–Γ-module by defining

$$(\lambda f \gamma)(a) = \lambda f(\gamma a) \qquad f \in \mathrm{Hom}(A_{\Lambda}, B_{\Lambda}) \qquad (\lambda \in \Lambda,\ \ \gamma \in \Gamma,\ \ a \in A).$$

Similarly, if ${}_{\Lambda}A_{\Gamma}$ is a Λ–Γ-module and ${}_{\Lambda}B_{\Lambda}$ is a Λ–Λ-module then $\mathrm{Hom}({}_{\Lambda}A, {}_{\Lambda}B)$ is a Γ–Λ-module by

$$(\gamma \circ f \circ \lambda)(a) = f(a\gamma)\lambda \qquad f \in \mathrm{Hom}({}_{\Lambda}A, {}_{\Lambda}B) \qquad (\lambda \in \Lambda,\ \ \gamma \in \Gamma,\ \ a \in A).$$

Moreover if modules ${}_{\Gamma}A_{\Lambda}$ $({}_{\Lambda}A_{\Gamma})$ and ${}_{\Gamma}B_{\Lambda}$ $({}_{\Lambda}B_{\Gamma})$ are given, then $\mathrm{Hom}(A_{\Lambda}, B_{\Lambda})$ $(\mathrm{Hom}({}_{\Lambda}A, {}_{\Lambda}B))$ is a Γ–Γ-module by

$$(\gamma_1 f \gamma_2)(a) = \gamma_1 f(\gamma_2 a) \qquad ((\gamma_1 \circ f \circ \gamma_2)(a) = f(a\gamma_1)\gamma_2).$$

Analogous remarks are used in other cases.

Lemma 1. *If Γ_{Λ} is finitely generated and projective, then for each Λ-module ${}_{\Lambda}C$ we have a Γ-isomorphism*

$$\Gamma \otimes_{\Lambda} C \cong \mathrm{Hom}({}_{\Lambda}\mathrm{Hom}(\Gamma_{\Lambda}, \Lambda_{\Lambda}), {}_{\Lambda}C) \tag{3}$$

2) In what follows, by a ring we shall always mean a ring with an identity element and by a module we shall always mean a unital one.

and similarly if ${}_{\Lambda}\Gamma$ is finitely generated and projective, then for each Λ-module ${}_{\Lambda}C$ we have a Γ-isomorphism

$$(4) \qquad C\underset{\Lambda}{\otimes}\Gamma \cong \mathrm{Hom}(\mathrm{Hom}({}_{\Lambda}\Gamma, {}_{\Lambda}\Lambda)_{\Lambda}, C_{\Lambda}).$$

Moreover if ${}_{\Lambda}C_{\Gamma'}({}_{\Gamma'}C_{\Lambda})$ is a Λ–Γ'-(Γ'–Λ-) module where Γ' is an arbitrary ring, then we have Γ–Γ'-(Γ'–Γ-) isomorphisms between the corresponding modules.

The proof of Lemma 1 proceeds just as in [8, §1], but for the sake of completeness we shall sketch here the proof.

Define

$$\tau: \ \Gamma\underset{\Lambda}{\otimes}C \longrightarrow \mathrm{Hom}({}_{\Lambda}\mathrm{Hom}(\Gamma_{\Lambda}, \Lambda_{\Lambda}), {}_{\Lambda}C)$$

by setting

$$\tau(\gamma\otimes c)f = f(\gamma)c \qquad\qquad (\gamma\in\Gamma, \quad c\in C, \quad f\in\mathrm{Hom}(\Gamma_{\Lambda}, \Lambda_{\Lambda})).$$

If $\Gamma=\Lambda$, then $\tau(\lambda\otimes c)=0$ implies $f(\lambda)c=0$ for all $f\in\mathrm{Hom}(\Lambda_{\Lambda}, \Lambda_{\Lambda})$. Setting here $f=1_{\Lambda}$, the identity mapping on Λ, we see that $\lambda c=0$ whence $\lambda\otimes c=1\otimes\lambda c=0$, and this implies that τ is a monomorphism. Clearly $\mathrm{Hom}(\Lambda_{\Lambda}, \Lambda_{\Lambda})=\Lambda\cdot 1_{\Lambda}$. Let $g\in\mathrm{Hom}({}_{\Lambda}\mathrm{Hom}(\Lambda_{\Lambda}, \Lambda_{\Lambda}), {}_{\Lambda}C)$. Then setting $c_0=g(1_{\Lambda})$ we have

$$\tau(1\otimes c_0)(\lambda 1_{\Lambda}) = \lambda c_0 = \lambda g(1_{\Lambda}) = g(\lambda 1_{\Lambda}),$$

that is, $\tau(1\otimes c_0)=g$, whence τ is an epimorphism. Since

$$\tau(\lambda_1(\lambda\otimes c))f = f(\lambda_1\lambda)c = (f\lambda_1(\lambda))c = (\lambda_1\circ\tau(\lambda\otimes c))f \qquad (\lambda_1\in\Lambda),$$

τ is a Λ-isomorphism. Therefore, by a direct sum argument it follows that τ is a Γ-isomorphism if Γ_{Λ} is finitely generated and projective. The rest of the assertions is easily verified.

When the following conditions $(r1)$, $(r2)$ ($(r1)$, $(r2)'$) are satisfied, then the ring extension Γ/Λ is called projective (free) Frobenius extention.

$(r\,1)$ ${}_{\Lambda}\Gamma_{\Gamma}\cong\mathrm{Hom}(\Gamma_{\Lambda}, \Lambda_{\Lambda})$.

$(r\,2)$ Γ_{Λ} is finitely generated and projective.

$(r\,2)'$ Γ_{Λ} is finitely generated and free.

From Lemma 1 it follows that $(r1)$, $(r2)$ ($(r1)$, $(r2)'$) are equivalent to the following conditions $(l1)$, $(l2)$ ($(l1)$, $(l2)'$):

$(l\,1)$ ${}_{\Gamma}\Gamma_{\Lambda}\cong\mathrm{Hom}({}_{\Lambda}\Gamma, {}_{\Lambda}\Lambda)$.

$(l\,2)$ ${}_{\Lambda}\Gamma$ is finitely generated and projective.

$(l\,2)'$ ${}_{\Lambda}\Gamma$ is finitely generated and free.

When Γ/Λ is a projective (free) Frobenius extension, denote by h the image of $1 \in \Gamma$ under a Λ–Γ-isomorphism of $(r1)$. Then $h \in \mathrm{Hom}({}_\Lambda\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda)$ and the isomorphism is denoted by

$$(5) \qquad \Gamma \ni \gamma \longleftrightarrow h\gamma \in \mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda) = h\Gamma .$$

As is easily seen, h is uniquely determined up to a right multiplication of a regular element of P, and is called a Frobenius homomorphism. From Lemma 1 it follows that using the same h we have a Γ–Λ-isomorphism of $(l1)$:

$$(6) \qquad \Gamma \ni \gamma \longleftrightarrow \gamma \circ h \in \mathrm{Hom}({}_\Lambda\Gamma, {}_\Lambda\Lambda) = \Gamma \circ h .$$

By (5) and (6) we have

$$hP = P \circ h = \mathrm{Hom}({}_\Lambda\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda) .$$

Thus for each $\rho \in P$, there exists $\rho' \in P$ such that $h\rho = \rho' \circ h$. Then the mapping

$$\sigma : \quad \rho \longrightarrow \rho' = \rho^\sigma$$

defines a automorphism of P by the formula

$$\begin{aligned} h\rho_1\rho_2(\gamma) &= h\rho_1(\rho_2\gamma) = \rho_1^\sigma \circ h(\rho_2\gamma) \\ &= h(\rho_2\gamma\rho_1^\sigma) = h\rho_2(\gamma\rho_1^\sigma) \\ &= \rho_2^\sigma \circ h(\gamma\rho_1^\sigma) = h(\gamma\rho_1^\sigma\rho_2^\sigma) \\ &= \rho_1^\sigma\rho_2^\sigma \circ h(\gamma) \qquad (\rho_1, \rho_2 \in P , \quad \gamma \in \Gamma) \end{aligned}$$

and is called Nakayama-automorphism of P. Since h is uniquely determined up to right multiplications of regular elements in P, the Nakayama-automorphism of P is uniquely determined up to inner automorphisms of P.

Now we shall give here an equivalent condition for a projective (free) Frobenius extension.

Theorem 1. *The ring extension Γ/Λ is a projective (free) Frobenius extension if and only if there exist elements $\{l_i\}$, $\{r_i\}$ in Γ, $\{h_i\}$ in* $\mathrm{Hom}({}_\Lambda\Gamma, {}_\Lambda\Lambda)$ *and $\{k_i\}$ in* $\mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda)$ *such that*

$$\gamma = \sum_{i=1}^n h_i(\gamma)l_i = \sum_{i=1}^n r_i k_i(\gamma)$$

$$h_i(l_j\gamma) = k_j(\gamma r_i) \qquad (\gamma \in \Gamma , \quad 1 \leqq i,j \leqq n) .$$

(For a free Frobenius extension moreover $h_i(l_j)$ $(=k_j(r_i)) = \delta_{ij}$, the Kronecker's delta.)

Before proving the theorem, we quote here the following lemma which is found in [1, Chap. VII, 3].

Lemma 2. *In order that a right Λ-module A be projective it is necessary and sufficient that there exist a family $\{a_\alpha\}$ of elements of A and a family $\{\varphi_\alpha\}$ of Λ-homomorphisms φ_α: $A \rightarrow \Lambda$ such that for all $a \in A$*

$$a = \sum_\alpha a_\alpha \varphi_\alpha(a),$$

where $\varphi_\alpha(a)$ is zero for all but a finite number of indices α.

Proof of Theorem 1. Sufficiency. By Lemma 2 Γ_Λ is finitely generated and projective. We define a scalar product of Γ and Γ in Λ by

$$(\gamma_1, \gamma_2) = \sum_{i=1}^{n} h_i(\gamma_1) k_i(\gamma_2) \qquad \gamma_1, \gamma_2 \in \Gamma.$$

The product is bilinear and associative, that is, it satisfies

$$\begin{aligned}
&\text{(i)} && (\lambda\gamma_1, \gamma_2) = \lambda(\gamma_1, \gamma_2), \qquad (\gamma_1, \gamma_2\lambda) = (\gamma_1, \gamma_2)\lambda \\
&\text{(ii)} && (\gamma_1 + \gamma_1', \gamma_2) = (\gamma_1, \gamma_2) + (\gamma_1', \gamma_2) \\
& && (\gamma_1, \gamma_2 + \gamma_2') = (\gamma_1, \gamma_2) + (\gamma_1, \gamma_2') \\
&\text{(iii)} && (\gamma_1\gamma, \gamma_2) = (\gamma_1, \gamma\gamma_2) \qquad (\lambda \in \Lambda, \quad \gamma_1, \gamma_1', \gamma_2, \gamma_2', \gamma \in \Gamma).
\end{aligned}$$

The third equality follows from

$$\begin{aligned}
(\gamma_1\gamma, \gamma_2) &= \sum_i h_i(\gamma_1\gamma) k_i(\gamma_2) = \sum_i h_i(\sum_j h_j(\gamma_1) l_j\gamma) k_i(\gamma_2) \\
&= \sum_{i,j} h_j(\gamma_1) h_i(l_j\gamma) k_i(\gamma_2) = \sum_{i,j} h_j(\gamma_1) k_j(\gamma r_i) k_i(\gamma_2) \\
&= \sum_j h_j(\gamma_1) k_j(\sum_i \gamma r_i k_i(\gamma)) \\
&= \sum_j h_j(\gamma_1) k_j(\gamma\gamma_2) \\
&= (\gamma_1, \gamma\gamma_2).
\end{aligned}$$

We consider the following mapping h of Γ into Λ:

$$h: \quad \Gamma \ni \gamma = \gamma_1\gamma_2 \longrightarrow h(\gamma) = (\gamma_1, \gamma_2).$$

Since by (iii) $h(\gamma) = (\gamma_1, \gamma_2) = (1, \gamma_1\gamma_2) = (1, \gamma)$, h is an well defined Λ–Λ-homomorphism of Γ into Λ. Then we assert that the correspondence

$$\varphi: \quad \Gamma \ni \gamma \longrightarrow h\gamma \in \mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda)$$

is a Λ–Γ-isomorphism of Γ onto $\mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda)$. Let $\varphi(\gamma) = h\gamma = 0$, then we have

$$\begin{aligned}
0 = h\gamma(r_i) = h(\gamma r_i) &= \sum_j h_j(\gamma) k_j(r_i) \\
&= \sum_j h_j(\gamma) h_i(l_j) = h_i(\sum_j h_j(\gamma) l_j) \\
&= h_i(\gamma) \qquad (i = 1, 2, \cdots, n),
\end{aligned}$$

whence $\gamma=\sum_i h_i(\gamma)l_i=0$, that is, φ is a monomorphism. Let $f\in\mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda)$ and set $\gamma'=\sum_i f(r_i)l_i$. Since

$$\gamma=\sum_i r_ik_i(\gamma)=\sum_i r_i(k_i(\sum_j r_jk_j(\gamma))$$
$$=\sum_{i,j} r_ik_i(r_j)k_j(\gamma)=\sum_{i,j} r_ih_j(l_i)k_j(\gamma),$$

we have

$$f(\gamma)=\sum_{i,j} f(r_i)h_j(l_i)k_j(\gamma)=\sum_j(\sum_i f(r_i)h_j(l_i))k_j(\gamma)$$
$$=\sum_j h_j(\sum_i f(r_i)l_i)k_j(\gamma)=\sum_j h_j(\gamma')k_j(\gamma)=h(\gamma'\gamma)$$
$$=h\gamma'(\gamma),$$

whence $f=h\gamma'=\varphi(\gamma')$, that is, φ is an epimorphism.

Necessity. Consider the following diagram of Γ-Γ-modules and Γ-Γ-isomorphisms where the Γ-Γ-isomorphisms in the upper row are those given in Lemma 1 (by setting $C=\Gamma$) and those in the columns are the induced ones from (5) and (6)

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathrm{Hom}({}_\Lambda\Gamma, {}_\Lambda\Lambda)_\Lambda, \Gamma_\Lambda) & \longleftrightarrow & \Gamma\otimes_\Lambda\Gamma & \longleftrightarrow & \mathrm{Hom}({}_\Lambda\mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda), {}_\Lambda\Gamma) \\
(\gamma\circ h\longrightarrow\gamma a) & & \sum_i r_i\otimes l_i & & (h\gamma\longrightarrow\gamma) \\
\updownarrow & & & & \updownarrow \\
(\gamma\longrightarrow\gamma a) & & & & (\gamma\longrightarrow\gamma) \\
\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda) & \longleftarrow\cdots\cdots & \cdots\cdots & \cdots\cdots\longrightarrow & \mathrm{Hom}({}_\Lambda\Gamma, {}_\Lambda\Gamma)
\end{array}$$

By connecting those isomorphisms we have a Γ-Γ-isomorphism of $\mathrm{Hom}({}_\Lambda\Gamma, {}_\Lambda\Gamma)$ onto $\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda)$. Let $f\in\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda)$ be the image of $1_\Gamma\in\mathrm{Hom}({}_\Lambda\Gamma, {}_\Lambda\Gamma)$ under the isomorphism. Then we have for all $\gamma\in\Gamma$, $\gamma f=f\gamma$, that is,

$$\gamma f(\gamma')=f(\gamma\gamma') \qquad \gamma, \gamma'\in\Gamma$$

whence f is a Γ-Λ-endomorphism of Γ. Thus f is given by a right multiplication of an element a in P. Let $\sum_i r_i\otimes l_i\in\Gamma\otimes_\Lambda\Gamma$ be the image of $(h\gamma\to\gamma)\in\mathrm{Hom}({}_\Lambda\mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda), {}_\Lambda\Gamma)$. Then by the construction of our isomorphism we have

$$\gamma=\sum_i h(\gamma r_i)l_i$$
$$\gamma a=\sum_i r_ih(l_i\gamma).$$

Set $\gamma=1$ in the first equality. Then we have $1=\sum_i h(r_i)l_i$ whence $\gamma=\sum_i h(r_i)l_i\gamma$. Then

$$h(\gamma)=\sum_i h(r_i)h(l_i\gamma)=h(\gamma a)=a\circ h(\gamma) \qquad (\gamma\in\Gamma),$$

that is, $h=a\circ h$ and this implies that $a=1$. Thus we have

$$\gamma=\sum_i h(\gamma r_i)l_i=\sum_i r_i h(l_i\gamma). \tag{7}$$

Setting here $h_i(\gamma)=h(\gamma r_i)$, $k_i(\gamma)=h(l_i\gamma)$ we see that $\{l_i\}$, $\{r_i\}$, $\{h_i\}$ and $\{k_i\}$ are the required ones in Theorem 1.

Let Γ/Λ be a free Frobenius extension and let $\{r_i\}$ be a right basis of Γ over Λ:

$$\Gamma=\sum\nolimits_{\oplus} r_i\Lambda.$$

Denote by d_i's the elements of $\mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda)$ defined by

$$d_i(r_j)=\delta_{ij} \qquad (1\leqq i,j\leqq n).$$

Then we have

$$\mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda)=\sum_{i}{}_{\oplus}\Lambda d_i.$$

Let φ be a Λ–Γ-isomorphism of Γ onto $\mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda)$ and let $l_i=\varphi^{-1}(d_i)$ $i=1,2,\cdots,n$. Then we see that $\{l_i\}$ is a left basis of Γ over Λ. Clearly $h=\varphi(1)\in\mathrm{Hom}({}_\Lambda\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda)$ and we have

$$\begin{aligned} h(l_ir_j)&=\varphi(1)(l_ir_j)=\varphi(l_i)(r_j)=d_i(r_j)\\ &=\delta_{ij} \qquad (1\leqq i,j\leqq n). \end{aligned}$$

Moreover for each element $\gamma=\sum_i \lambda_i l_i$ $(\lambda_i\in\Gamma)$ in Γ we have

$$\sum_i h(\gamma r_i)l_i=\sum_i \varphi(\gamma)(r_i)l_i=\sum_{i,j}\lambda_j d_j(r_i)l_i=\sum_j \lambda_j l_j$$

and similarly

$$\sum_i r_i h(l_i\gamma)=\gamma.$$

As is seen from the process of the proof of Theorem 1 together with the above remarks we have the following corollaries.

Corollary 1. *A ring extension Γ/Λ is a projective (free) Frobenius extension if and only if there exist element $\{l_i\}$, $\{r_i\}$ in Γ and h in $\mathrm{Hom}({}_\Lambda\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda)$ such that*

$$\gamma = \sum_i h(\gamma r_i) l_i = \sum_i r_i h(l_i \gamma).$$

(*For a free Frobenius extension, moreover* $h(l_i r_j) = \delta_{ij}$).

Corollary 2. *A ring extension* Γ/Λ *is a projective* (*free*) *Frobenius extension if and only if there exist elements* $\{l_i\}$, $\{r_i\}$ *in* Γ *and a* Λ*-bilinear form* (x, y) *defined in* Γ *which is* Γ*-associative and such that*

$$\gamma = \sum_i (\gamma, r_i) l_i = \sum_i r_i (l_i, \gamma).$$

(*Moreover* $(l_i, r_j) = \delta_{ij}$).

Theorem 2 (Pareigis). *Let* Γ, Δ *and* Λ *are rings such that* $\Gamma \supseteqq \Delta \supseteqq \Lambda$. *If* Γ/Δ *and* Δ/Λ *are both projective Frobenius extensions, then* Γ/Λ *is also a projective Frobenius extension.*

Proof. Let $\{l_i\}$, $\{r_i\}$ $(\subseteq \Gamma)$ and $h \in \mathrm{Hom}({}_\Delta\Gamma_\Delta, {}_\Delta\Delta_\Delta)$ satisfy

$$\gamma = \sum_i h(\gamma l_i) r_i = \sum_i r_i h(l_i \gamma) \qquad (\gamma \in \Gamma),$$

and let $\{l'_j\}$, $\{r'_j\}$ $(\subseteqq \Delta)$ and $h' \in \mathrm{Hom}({}_\Lambda\Delta_\Lambda, {}_\Lambda\Lambda_\Lambda)$ satisfy

$$\delta = \sum_j h'(\delta r'_j) l'_j = \sum_j r'_j h'(l'_j \delta) \qquad (\delta \in \Delta).$$

Set $l_{(i,j)} = l'_j l_i$, $r_{(i,j)} = r_i r'_j$ and $h^* = h'h$, then $h^* \in \mathrm{Hom}({}_\Lambda\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda)$ and we have

$$\begin{aligned} \gamma &= \sum_i h(\gamma r_i) l_i = \sum_{i,j} h'(h(\gamma r_i) r'_j) l'_j l_i \\ &= \sum_{i,j} h'(h(\gamma r_i r'_j)) l'_j l_i = \sum_{(i,j)} h^*(\gamma r_{(i,j)}) l_{(i,j)} \end{aligned}$$

and similarly

$$\gamma = \sum_{(i,j)} r_{(i,j)} h^*(l_{(i,j)} \gamma) \qquad (\gamma \in \Gamma).$$

By Corollary 1 this implies that Γ/Λ is a projective Frobenius extension.

Theorem 3. *Let* Γ/Λ *be a projective Frobenius extension. If* Δ/Λ *is a ring extension, then* $\Gamma \otimes_\Lambda \Delta/\Delta$ $(= 1 \otimes_\Lambda \Delta)$ *is a projective Frobenius extension.*

Proof. Let $\{l_i\}$, $\{r_i\}$ and h be as in Corollary 1. Define $\{l'_i\}$, $\{r'_i\}$ and h' by

$$l'_i = l_i \otimes 1, \qquad r'_i = r_i \otimes 1$$

$$h'(\gamma \otimes \delta) = h(\gamma) \otimes \delta \qquad (\gamma \otimes \delta \in \Gamma \otimes \Delta).$$

Then, as is easily seen, $h' \in \mathrm{Hom}({}_\Delta\Gamma \otimes \Delta_\Delta, {}_\Delta\Delta_\Delta)$ and we have for each $\gamma \otimes \delta \in \Gamma \otimes_\Lambda \Delta$

$$\sum_i h'((\gamma \otimes \delta) r'_i) l'_i = \sum_i h(\gamma r_i) l_i \otimes \delta = \gamma \otimes \delta$$

and

$$\sum_i r'_i h'(l'_i(\gamma\otimes\delta)) = \sum_i r_i h(l_i\gamma)\otimes\delta = \gamma\otimes\delta\,.$$

Our assertion then follows from Corollary 1 at once.

§ 2. Let Γ/Λ be a ring extension. The endomorphism ring $\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda)$ contains Γ_l, the ring of left multiplications of the ring Γ. Γ_l is ring isomorphic to Γ, whence we identify it with Γ. In [6] Kasch has proved the following theorem:

Theorem 4. *If Γ/Λ is a projective Frobenius extension then* $\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda)/\Gamma$ *is also a projective Frobenius extension.*

We shall give here a simplified proof of it based on Theorem 1 (or Corollary 1).

Let $\{l_i\}$, $\{r_i\}$ and h be as in Corollary 1. Let $\{L_i\}$, $\{R_i\}$ be the elements of $\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda)$ such that

$$L_i = hl_i\,, \qquad R_i = r_i h \qquad (i=1, 2, \cdots, n)\,,$$

and let H be the mapping of $\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda)$ into Γ defined by

$$H(f) = \sum_i f(r_i)l_i \qquad (f\in \mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda))\,.$$

Clearly H is a left Γ-homomorphism. Since

$$\begin{aligned} H(f\gamma) &= \sum_i f(\gamma r_i)l_i = \sum_i f(\sum_j r_j h(l_j\gamma r_i))l_i \\ &= \sum_j f(r_j)\sum_i h(l_j\gamma r_i)l_i) = \sum_j f(r_j)l_j\gamma \\ &= H(f)\gamma \end{aligned}$$

H is also a right Γ-homomorphism. Moreover for each $f\in\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda)$ we have

$$\begin{aligned} \sum_j H(fR_j)L_j(\gamma) &= \sum_{i,j} f(r_j h(r_i))l_i h(l_j\gamma) = \sum_j f(r_j)(\sum_i h(r_i)l_i)h(l_j\gamma) \\ &= \sum_j f(r_j)h(l_j\gamma) = f(\sum_j r_j h(l_j\gamma)) \\ &= f(\gamma) \end{aligned}$$

and

$$\begin{aligned} \sum_j R_j H(L_i f)(\gamma) &= \sum_{i,j} r_j h(h(l_j f(r_i))l_i\gamma) = \sum_{i,j} r_j h(l_j f(r_i))h(l_i\gamma) \\ &= \sum_i f(r_i)h(l_i\gamma) \\ &= f(\gamma)\,, \end{aligned}$$

whence we have

$$f=\sum_j H(fR_j)L_j=\sum_j R_jH(L_jf) \qquad (f\in \mathrm{Hom}(\Gamma_A, \Gamma_A))\,.$$

This implies by Corollary 1 that $\mathrm{Hom}(\Gamma_A, \Gamma_A)/\Gamma$ is a projective Frobenius extension.

As for the converse of Theorem 4 we have the following

Theorem 5. *Let Γ/Λ be a ring extension and* $\mathrm{Hom}(\Gamma_A, \Gamma_A)/\Gamma$ *satisfies the condition* $(l1)$. *Then there exists a Γ–Λ-monomorphism of* $\mathrm{Hom}(\Gamma_A, \Lambda_A)$ *into Γ. If, moreover, Γ_A have a Λ-submodule isomorphic to Λ_A as a Λ-direct summand, then there exists a Λ–Γ-isomorphism of* $\mathrm{Hom}(\Gamma_A, \Lambda_A)$ *onto Γ, that is, Γ/Λ satisfies* $(r1)$.

A concise proof of the theorem is found in [6, 2.4], but we shall repeat it here for completeness.

Let $H\in\mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Gamma_A, \Gamma_A), {}_\Gamma\Gamma)$ be the image of $1_\Gamma\in\mathrm{Hom}(\Gamma_A, \Gamma_A)$ under the isomorphism

$${}_{\mathrm{Hom}(\Gamma_A\Gamma_A)}\mathrm{Hom}(\Gamma_A, \Gamma_A)_\Gamma \cong \mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Gamma_A, \Gamma_A), {}_\Gamma\Gamma)\,.$$

The H is a Γ–Γ-homomorphism and the isomorphism above is given by

$$\mathrm{Hom}(\Gamma_A, \Gamma_A)\ni f \longleftrightarrow f\circ H\in\mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Gamma_A, \Gamma_A), {}_\Gamma\Gamma\,.$$

Denoting by h the restriction of H on $\mathrm{Hom}(\Gamma_A, \Lambda_A)$ we see that h is a Λ–Γ-homomorphism of $\mathrm{Hom}(\Gamma_A, \Lambda_A)$ into Γ. If $h(f)=0$ for an element f in $\mathrm{Hom}(\Gamma_A, \Lambda_A)$ then we have

$$0=\Gamma h(f)=H(\Gamma f)=H(\mathrm{Hom}(\Gamma_A, \Gamma_A)f)=f\circ H(\mathrm{Hom}(\Gamma_A, \Gamma_A))$$

and this implies that $f\circ H=0$ whence $f=0$. Thus h is a monomorphism.

Let

$$\Gamma_A=x\Lambda_A\oplus A_A\,, \qquad x\Lambda_A\cong\Lambda_A\,.$$

Then we assert that h is also an epimorphism.

Define $d\in\mathrm{Hom}(\Gamma_A, \Lambda_A)$ by

$$d(x)=1\,, \qquad d(A)=0\,.$$

Then we have

$$\mathrm{Hom}(\Gamma_A, \Gamma_A)=\Gamma d\oplus\mathrm{Hom}(A_A, \Gamma_A)\,.$$

Let σ be an element of $\mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Gamma_A, \Gamma_A), {}_\Gamma\Gamma)$ defined by

$$\sigma(d)=1\,, \qquad \sigma(\mathrm{Hom}(A_A, \Gamma_A))=0\,,$$

and let g be the element of $\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda)$ such that $\sigma = g \circ H$. Then we have

$$\gamma = \sigma(d)\gamma = (g \circ H)(d)\gamma = H(dg)\gamma = H(dg\gamma) \qquad (\gamma \in \Gamma).$$

Since $dg\gamma \in \mathrm{Hom}(\Gamma_\Lambda, \Lambda_\Lambda)$ this implies that h is an epimorphism as asserted.

§ 3. Let Γ/Λ be a projective Frobenius extension and let $\{l_i\}$, $\{r_i\}$ and h be as in Corollary 1. For left Γ-modules ${}_\Gamma A$, ${}_\Gamma B$ and a Λ-homomorphism f of A into B we define trace f by the mapping

$$A \ni x \longrightarrow \sum_i r_i f(l_i x) \in B.$$

The followings are the fundamental properties of trace.

1°. If $f \in \mathrm{Hom}({}_\Lambda A, {}_\Lambda B)$, then trace $f \in \mathrm{Hom}({}_\Gamma A, {}_\Gamma B)$

2°. If $f, g \in \mathrm{Hom}({}_\Lambda A, {}_\Lambda B)$ then

$$\mathrm{trace}\,(f+g) = \mathrm{trace}\, f + \mathrm{trace}\, g$$

3°. If further ${}_\Gamma C$ and ${}_\Gamma D$ are left Γ-modules and $f \in \mathrm{Hom}({}_\Lambda A, {}_\Lambda B)$, $f_1 \in \mathrm{Hom}({}_\Gamma B, {}_\Gamma D)$, $f_2 \in \mathrm{Hom}({}_\Gamma C, {}_\Gamma A)$, then

$$\mathrm{trace}\,(f_1 f f_2) = f_1(\mathrm{trace}\, f) f_2.$$

The property 1° follow from

$$\begin{aligned} \mathrm{trace}\, f(\gamma x) &= \sum_i r_i f(l_i \gamma x) = \sum_i r_i f(\sum_j h(l_i \gamma r_j) l_j x) \\ &= \sum_{i,j} r_i h(l_i \gamma r_j) f(l_j x) = \gamma \sum_j r_j f(l_j x) \\ &= \gamma\, \mathrm{trace}\, f(x) \qquad (\gamma \in \Gamma, \quad x \in A). \end{aligned}$$

The properties 2° and 3° are easily verified.

Let Γ/Λ be a ring extension. A left Γ-module A is (Γ, Λ)-projective if for every (Γ, Λ)-exact sequence

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\alpha_{i+1}} A_i \xrightarrow{\alpha_i} A_{i-1} \longrightarrow \cdots$$

the sequence

$$\cdots \longrightarrow \mathrm{Hom}(A, A_{i+1}) \xrightarrow{\mathrm{Hom}(1,\, \alpha_{i+1})} \mathrm{Hom}(A, A_i) \xrightarrow{\mathrm{Hom}(1,\, \alpha_i)} \mathrm{Hom}(A, A_{i-1}) \longrightarrow \cdots$$

is also exact, or equivalently, $\Gamma \otimes_\Lambda A$ has a Γ-submodule isomorphic to ${}_\Gamma A$ as a Γ-direct summand. Similarly, a left Γ-module ${}_\Gamma B$ is (Γ, Λ)-injective if for every (Γ, Λ)-exact sequence

$$\cdots \longrightarrow B_{i+1} \xrightarrow{\beta_{i+1}} B_i \xrightarrow{\beta_i} B_{i-1} \longrightarrow \cdots$$

the sequence

$$\cdots \longrightarrow \mathrm{Hom}(B_{i-1}, B) \xrightarrow{\mathrm{Hom}(\beta_i, 1)} \mathrm{Hom}(B_i, B) \xrightarrow{\mathrm{Hom}(\beta_{i+1}, 1)} \mathrm{Hom}(B_{i+1}, B) \longrightarrow \cdots$$

is exact, or equivalently, $\mathrm{Hom}({}_A\Gamma, {}_AB)$ has a Γ-submodule isomorphic to ${}_\Gamma B$ as a Γ-direct summand.

In the same way as in [6, Satz 4], we have the following

Theorem 6. *Let Γ/Λ be a projective Frobenius extension. If ${}_\Gamma A$, ${}_\Gamma M$ and ${}_\Gamma N$ are arbitrary left Γ-modules, then we have for every* $g \in \mathrm{Hom}({}_\Lambda M, {}_\Lambda N)$

$$\mathrm{Ext}^i_{(\Gamma,\Lambda)}(1_A, \text{trace } g) = 0 \qquad i = 1, 2, \cdots .$$

Similarly if ${}_\Gamma A$, ${}_\Gamma B$ and ${}_\Gamma M$ are arbitrary left Γ-modules, then we have for every $g \in \mathrm{Hom}({}_\Lambda B, {}_\Lambda A)$

$$\mathrm{Ext}^i_{(\Gamma,\Lambda)}(\text{trace } g, 1_M) = 0 \qquad (i = 1, 2, \cdots) .$$

Theorem 7. *Let Γ/Λ be a projective Frobenius extension and let ${}_\Gamma A$ be a left Γ-module. Then the following conditions are equivalent.*

(a) *A is (Γ, Λ)-projective,*

(b) *A is (Γ, Λ)-injective,*

(c) *There exists a Λ-endomorphism f of A such that*

$$\text{trace } f = 1_A \qquad (\textit{the identity mapping on } A) .$$

Proof. By (3) and $(r1)$, there holds for every left Λ-module ${}_\Lambda C$

$$\Gamma \underset{\Lambda}{\otimes} C \overset{(\Gamma\text{-})}{\cong} \mathrm{Hom}({}_\Lambda\Gamma, {}_\Lambda C) ,$$

whence the equivalence of (a) and (b) is clear. Next we shall show the equivalence of (a) and (c). Let $\{l_i\}$, $\{r_i\}$ and h be as in Corollary 1. Consider the Λ-endomorphism $h \underset{\Lambda}{\otimes} 1_A$ of $\Gamma \otimes A$:

$$h \otimes 1_A: \ \Gamma \underset{\Lambda}{\otimes} A \ni \gamma \otimes a \longrightarrow h(\gamma) \otimes a \in \Gamma \underset{\Lambda}{\otimes} A \qquad (\gamma \in \Gamma, \ a \in A) .$$

Then we have

$$\text{trace.}(h \otimes 1_A)(\gamma \otimes a) = \sum_i r_i h(l_i \gamma) \otimes a = \gamma \otimes a ,$$

that is, $\text{trace.}(h \otimes 1_A) = 1_{\Gamma \underset{\Lambda}{\otimes} A}$. Let

$$\Gamma \underset{\Lambda}{\otimes} A = {}_\Gamma A_1 \oplus {}_\Gamma A_2 , \qquad {}_\Gamma A_1 \cong {}_\Gamma A$$

and let p be the projection of $\Gamma\underset{\Lambda}{\otimes}A$ onto A_1. Denote by $(h\otimes 1_A)_{A_1}$ the restriction of $h\otimes 1_A$ on A_1. Then by 3° we have

$$\operatorname{trace}(p(h\otimes 1_A)_{A_1}) = p\cdot\operatorname{trace}(h\otimes 1_A)_{A_1} = 1_{A_1}.$$

Thus we see that (a) implies (c). The converse (and also (b) from (c)) follows from Theorem 6, but we shall add here an elementary proof of it.

Consider the subset A' of $\Gamma\underset{\Lambda}{\otimes}A$ such that

$$A' = \left\{\sum_i r_i\otimes f(l_i x)\,|\, x\in A\right\}.$$

Then A' is a Γ-submodule of A because we have

$$\begin{aligned}\gamma(\sum_i r_i\otimes f(l_i x)) &= \sum_i \gamma r_i\otimes f(l_i x) = \sum_{i,k} r_k h(l_k\gamma r_i)\otimes f(l_i x)\\ &= \sum_k r_k\otimes f(\sum_i h(l_k\gamma r_i)l_i x)\\ &= \sum_k r_k\otimes f(l_k\gamma x) \qquad\qquad (\gamma\in\Gamma,\quad x\in A).\end{aligned}$$

Consider the Γ-homomorphism ε of $\Gamma\otimes A$ onto A defined by

$$\varepsilon(\gamma\otimes x) = \gamma x.$$

Then A' is isomorphically mapped onto A by ε, whence we have

$$\Gamma\underset{\Lambda}{\otimes}A = A'\oplus\operatorname{Ker}(\varepsilon), \qquad {}_\Gamma A'\cong{}_\Gamma A.$$

This implies that A is (Γ, Λ)-projective.

§ **4.** Let again Γ/Λ be a projective Frobenius extension and let $\{l_i\}$, $\{r_i\}$ and h be as in Corollary 1. Denote by σ the Nakayama-automorphism of P (belonging to h). If ${}_\Gamma A_\Lambda$ is a Γ–Λ-module then we shall denote the right P-module $\operatorname{Hom}({}_\Lambda A_\Lambda, {}_\Lambda\Lambda_\Lambda)$ by A^*. For each $f\in A^*$ we have by 1°

$$(7)\qquad \operatorname{trace} f\in\operatorname{Hom}({}_\Gamma A_\Lambda, {}_\Gamma\Gamma_\Lambda)$$

and moreover

$$(8)\qquad \begin{aligned}\operatorname{trace} f\rho(x) &= \sum_i r_i f(\rho l_i x) = \sum_i r_i f(\sum_j h(\rho l_i r_j) l_j x)\\ &= \sum_{i,j} r_i h(\rho l_i r_j) f(l_j x) = \sum_{i,j} r_i h(l_i r_j \rho^\sigma) f(l_j x)\\ &= \sum_j r_j \rho^\sigma f(l_j x) = \sum_j r_j f(l_j x)\rho^\sigma\\ &= \operatorname{trace} f(x)\cdot\rho^\sigma \qquad\qquad (\rho\in P,\quad x\in A).\end{aligned}$$

For each left P-module ${}_PC$ we shall define a new P-module ${}_PC^0$ by the additive group C and the scalar multiplication $*$:

$$\rho * c = \rho^\sigma c \qquad \rho \in P\,,\quad c \in C\,.$$

Theorem 8. *If ${}_\Gamma A_\Lambda$ is a Γ–Λ-module such that ${}_\Gamma A_\Lambda$ is a Γ–Λ-direct summand of the direct sum of a finite number of copies of ${}_\Gamma\Gamma_\Lambda$, then the right P-module A^* is finitely generated and projective, and the mapping*

$$A^* \ni f \longrightarrow \operatorname{trace} f \in \operatorname{Hom}({}_\Gamma A_\Lambda, {}_\Gamma\Gamma_\Lambda)$$

is an isomorphism.

Proof. If $A=\Gamma$, then we have $A^*=\operatorname{Hom}({}_\Lambda\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda)=hP$ and $\operatorname{Hom}({}_\Gamma\Gamma_\Lambda, {}_\Gamma\Gamma_\Lambda=P_r$[3]. Since by (8)

$$\operatorname{trace} h\rho = \rho^\sigma_r$$

our assertion is obvious. In the general case, the assertion is also obtained by a simple direct sum argument.

Given modules ${}_\Gamma A_\Lambda$ and ${}_\Gamma C$, we define a mapping φ by

$$\varphi:\quad A^* \underset{P}{\otimes} C^0 \ni f\otimes c \longrightarrow (x \rightarrow \operatorname{trace} f(x)c) \in \operatorname{Hom}({}_\Gamma A_\Lambda, {}_\Gamma C_\Lambda)\,.$$

Then by (8) we see that φ is an well defined homomorphism. When $A=\Gamma$, we define the mapping ϕ by

$$\phi:\quad \operatorname{Hom}({}_\Gamma\Gamma_\Lambda, {}_\Gamma C_\Lambda) \ni g \longrightarrow h\otimes g(1) \in A^* \underset{P}{\otimes} C^0 = hP \underset{P}{\otimes} C^0\,.$$

Then we have

$$\begin{aligned}\varphi\phi(g)(\gamma) &= \varphi(h\otimes g(1))(\gamma) = \operatorname{trace} h(\gamma) g(1) = \gamma g(1)\\ &= g(\gamma) \qquad\qquad (g\in \operatorname{Hom}({}_\Gamma\Gamma_\Lambda, {}_\Gamma C_\Lambda), \quad \gamma\in\Gamma)\,,\end{aligned}$$

whence $\varphi\phi = 1_{\operatorname{Hom}(\Gamma_\Lambda, C_\Lambda)}$ and

$$\phi\varphi(h\rho\otimes c) = h\otimes \operatorname{trace} h\rho(1)c = h\otimes\rho^\sigma c = h\rho\otimes c \qquad (\rho\in P\,,\quad c\in C)\,,$$

whence $\phi\varphi = 1_{A^*\otimes_P C^0}$. This implies that φ is an isomorphism. Then again by a simple direct sum argument φ is also an isomorphism when ${}_\Gamma A_\Lambda$ is a Γ–Λ-direct summand of the direct sum of a finite number of copies of ${}_\Gamma\Gamma_\Lambda$.

Let ${}_\Gamma B_\Lambda$ and ${}_\Gamma D$ be Γ–Λ- and left Γ-modules respectively and let $\alpha\in$

3) The right multiplications of the ring P.

$\mathrm{Hom}({}_\Gamma C, {}_\Gamma D)$, $\beta \in \mathrm{Hom}({}_\Gamma A_\Lambda, {}_\Gamma B_\Lambda)$. Then noting that for each $f \in B^* = \mathrm{Hom}({}_\Lambda B_\Lambda, {}_\Lambda \Lambda_\Lambda)$ there holds by 3°

$$(\mathrm{trace}\, f)\,\beta = \mathrm{trace}\,(f\beta)$$

we have the following commutative diagram:

$$\begin{array}{ccc} B^* \underset{P}{\otimes} C^0 & \longrightarrow & A^* \underset{P}{\otimes} D^0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}({}_\Gamma B, {}_\Gamma C) & \longrightarrow & \mathrm{Hom}({}_\Gamma A, {}_\Gamma D)\,. \end{array}$$

Thus we have the following

Theorem 9. *Let Γ/Λ be a projective Frobenius extension. For a Γ–Λ-module ${}_\Gamma A_\Lambda$ and a left Γ-module ${}_\Gamma C$ the functors*

$$T_1(A, C) = A^* \underset{P}{\otimes} C^0$$

$$T_2(A, C) = \mathrm{Hom}({}_\Gamma A, {}_\Gamma C)$$

define additive functors both contravariant in the first variable and covariant in the second variable, and the mapping φ is a natural transformation of T_1 into T_2. Moreover if we restrict ourselves to those ${}_\Gamma A_\Lambda$ which is a Γ–Λ-direct summand of the direct sum of finite number of copies of ${}_\Gamma \Gamma_\Lambda$, then φ is a natural isomorphism of T_1 and T_2.

§ 5. Let Γ be a ring. By a complete projective resolution of a left module ${}_\Gamma A$ we shall mean an exact sequence of Γ-projective modules

$$\mathfrak{A}: \quad \cdots \longrightarrow A_2 \xrightarrow{\alpha_2} A_1 \xrightarrow{\alpha_1} A_0 \xrightarrow{\alpha_0} A_{-1} \xrightarrow{\alpha_{-1}} A_{-2} \longrightarrow \cdots$$

such that

$$\mathrm{Im}\,(\alpha_0) = \mathrm{Ker}\,(\alpha_{-1}) \cong {}_\Gamma A\,.$$

For a left Γ-module ${}_\Gamma C$, we shall denote the k-th factor groups of the complex $\mathfrak{A} \otimes C$ and $\mathrm{Hom}(\mathfrak{A}, C)$ by $H_k^\Gamma(A, C)$ and $H_\Gamma^k(A, C)$ respectively.

Theorem 10. *Let Γ/Λ be a projective Frobenius extension. If a Γ–Λ-module ${}_\Gamma A_\Lambda$ considered as a left Γ-module have a complete projective resolution $\mathfrak{A}$ where each A_k is a Γ–Λ-module such that A_k is a Γ–Λ-direct summand of the direct sum of a finite number of copies of ${}_\Gamma \Gamma_\Lambda$, and each* $\mathrm{Ker}\,(\alpha_k)$ *is a Λ–Λ-direct summand of A_k, then for every left Γ-module ${}_\Gamma C$ we have*

$$H_k^P(A^*, C^0) \cong H_\Gamma^{-k-1}(A, C) \qquad (k = 0, \pm 1, \pm 2, \cdots)\,.$$

Proof. From $\mathfrak{A}$, we obtain the following exact sequence

$$\mathfrak{A}^*: \quad \cdots \longrightarrow A^*_{-2} \xrightarrow{\beta^1} A^*_{-1} \xrightarrow{\beta^0} A^*_0 \xrightarrow{\beta^{-1}} A^*_0 \xrightarrow{\beta^{-1}} A^*_2 \longrightarrow \cdots$$

such that

$$\mathrm{Im}(\beta^0) = \mathrm{Ker}(\beta^{-1}) \cong {}_P A^*,$$

where each A^*_k is P-projective by Theorem 8. This implies that $\mathfrak{A}^*$ is a complete projective resolution of ${}_P A^*$. Then our assertion follows from Theorem 9.

If a ring extension Γ/Λ is central, that is, the center of Γ contain Λ, then each left Γ-module A is also a right Λ-module by defining

$$x\lambda = \lambda x \qquad (\lambda \in \Lambda, \quad x \in A),$$

and we have $A^*_\Gamma (= \mathrm{Hom}({}_\Lambda A_\Lambda, {}_\Lambda \Lambda_\Lambda)) = \mathrm{Hom}({}_\Lambda A, {}_\Lambda \Lambda)$.

Theorem 11. *Let Γ/Λ be a central ring extension*[4] *such that ${}_\Lambda\Gamma$ is finitely generated and projective. If a left Γ-module ${}_\Gamma A$ considered as a Λ-module is finitely generated and projective, then ${}_\Gamma A$ has a (Γ, Λ)-exact complete projective resolution consisting finitely generated and free Γ-modules.*

Proof. There exists a finitely generated and free Γ-module A_0 such that

$$A_0 \xrightarrow{\alpha_0} A \longrightarrow 0$$

is exact. Since ${}_\Lambda A$ is projective, $\mathrm{Ker}(\alpha_0)$ is a Λ-direct summand of A_0. As a Λ-homomorphic image of the finitely generated Λ-module A_0 $\mathrm{Ker}(\alpha_0)$ is also finitely generated as a Λ-module. Then by induction we obtain a (Γ, Λ)-projective resolution of A:

$$\cdots \longrightarrow {}_\Gamma A_1 \xrightarrow{\alpha_1} {}_\Gamma A_0 \xrightarrow{\alpha_0} {}_\Gamma A \longrightarrow 0,$$

where each ${}_\Gamma A_k$ is finitely generated and free. Since A^*_Γ is also a finitely generated and projective Λ-module, we have similarly a (Γ, Λ)-projective resolution of A^*_Γ:

$$\cdots \longrightarrow B_{1\Gamma} \xrightarrow{\beta_1} B_{0\Gamma} \xrightarrow{\beta_0} A^*_\Gamma \longrightarrow 0,$$

where each $B_{k\Gamma}$ is finitely generated and free. This affords then the following (Γ, Λ)-exact sequence:

$$0 \longrightarrow {}_\Gamma A^{**} \xrightarrow{\beta^*_0} {}_\Gamma B^*_0 \xrightarrow{\beta^*_1} {}_\Gamma B^*_1 \longrightarrow \cdots$$

4) The condition that Γ is Noetherian in [6, Satz 3] is superfluous as Prof. Azumaya pointed out to the present author.

with ${}_\Gamma B_k^*$ finitely generated and free. Since ${}_\Gamma A^{**} = {}_\Gamma A$ we have consequently the following (Γ, Λ)-exact complete projective resolution of ${}_\Gamma A$ consisting of finitely generated and free Γ-modules:

$$\cdots \longrightarrow {}_\Gamma A_1 \xrightarrow{\alpha_1} {}_\Gamma A_0 \xrightarrow{\beta_0^* \alpha_0} {}_\Gamma B_0^* \xrightarrow{\beta_1^*} {}_\Gamma B_1^* \longrightarrow \cdots,$$

and this proves the theorem.

§ 6. Let Γ/Λ be a projective Frobenius extension and let $\{l_i\}$, $\{r_i\}$ be those in Corollary 1. For a left Γ-module ${}_\Gamma A$ we shall give here a complete (Γ, Λ)-projective resolution of ${}_\Gamma A$. Firstly construct the (Γ, Λ)-projective resolution of ${}_\Gamma A$:

$$\cdots \longrightarrow A_2 \xrightarrow{\alpha_2} A_1 \xrightarrow{\alpha_1} A_0 \xrightarrow{\alpha_0} A \longrightarrow 0$$

where

$$A_k = \underbrace{\Gamma \otimes \Gamma \otimes \cdots \Gamma}_{(k+1\text{-times})} \otimes A \qquad (k = 0, 1, 2, \cdots)$$

and

$$\alpha_k(\gamma_{k+1} \otimes \gamma_k \otimes \cdots \otimes \gamma_1 \otimes x) = \sum_{i=0}^{k} (-1)^{k-i} \gamma_{k+1} \otimes \cdots \otimes \gamma_{i+1}\gamma_i \otimes \cdots \otimes \gamma_1 \otimes x\,.$$

Next we consider the (Γ, Λ)-injective (whence also (Γ, Λ)-projective) modules

$$A^k = \underbrace{\mathrm{Hom}({}_\Lambda\Gamma, {}_\Lambda\mathrm{Hom}({}_\Lambda\Gamma, \cdots\cdots, {}_\Lambda\mathrm{Hom}}_{(k+1\text{-times})}({}_\Lambda\Gamma, {}_\Lambda A)\cdots)), \qquad (k = 0, 1, 2, \cdots)\,.$$

For each element f in A^k we shall use the following simplification:

$$f(\gamma_1)(\gamma_2) \cdots (\gamma_{k+1}) = f(\gamma_1, \gamma_2, \cdots, \gamma_{k+1}) \qquad (\gamma_1, \gamma_2, \cdots, \gamma_{k+1} \in \Gamma)\,.$$

Define the Γ-homomorphism α^k $(k = 1, 2, \cdots)$ of ${}_\Gamma A_k$ into ${}_\Gamma A_{k+1}$ by

$$\alpha^k f(\gamma_1, \gamma_2, \cdots, \gamma_{k+1}) = -\gamma_{k+2} f(\gamma_1, \gamma_2, \cdots, \gamma_{k+1}) + \sum_{i=1}^{k+1} (-1)^{k-i+1} f(\gamma_1, \gamma_2, \cdots, \gamma_{i+1}\gamma_i, \gamma_{i+2}, \cdots, \gamma_{k+2})$$

and $\alpha^0 \in \mathrm{Hom}({}_\Gamma A, {}_\Gamma A^0)$ by

$$\alpha^0 a(\gamma) = \gamma a \qquad (a \in A\,, \quad \gamma \in \Gamma)\,.$$

Then we have the (Γ, Λ)-injective resolution of the form:

$$0 \longrightarrow {}_\Gamma A \xrightarrow{\alpha^0} {}_\Gamma A^0 \xrightarrow{\alpha^1} {}_\Gamma A^1 \xrightarrow{\alpha^2} {}_\Gamma A^2 \longrightarrow \cdots\,.$$

Since by (3)

$$_{\Gamma}A_k \cong {_{\Gamma}A^k}\,, \qquad k=0, 1, 2, \cdots,$$

by connecting the both resolutions and replacing $_{\Gamma}A^k$ by $_{\Gamma}A_k$ and α^k by β^k we have the following complete (Γ, Λ)-projective resolution of $_{\Gamma}A$:

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow A_2 & \xrightarrow{\alpha_2} & A_1 & \xrightarrow{\alpha_1} & A_0 & & \xrightarrow{\beta_0\alpha_0} & & A_0 & \xrightarrow{\beta_1} & A_1 & \xrightarrow{\beta_2} & A_2 \longrightarrow \cdots, \\ & & & & & \alpha_0 \searrow & & \nearrow \beta_0 & & & & & \\ & & & & & & A & & & & & & \\ & & & & & \nearrow & & \searrow & & & & & \\ & & & & 0 & & & & 0 & & & & \end{array}$$

where

$$\beta^k(\gamma_k \otimes \cdots \otimes \gamma_1 \otimes a) = \sum_{i=0}^{k} \sum_{j=1}^{n} (-1)^{k-i} \gamma_k \otimes \cdots \otimes \gamma_{i+1} \otimes r_j \otimes l_j \gamma_i \otimes \cdots \gamma_1 \otimes a\,,$$
$$\gamma_k \otimes \cdots \otimes \gamma_1 \otimes a \in A_{k-1}\,.$$

§ 7. Let Γ be a simple ring with minimum condition with the center C and let $\mathfrak{G}$ be a semi-regular group of automorphisms of A in the sense of Nakayama [7]. Denote by Λ the fixed subring of $\mathfrak{G}$. Let $\mathfrak{J}$ be the invariant subgroup of $\mathfrak{G}$ consisting of the inner automorphisms of $\mathfrak{G}$, and let T be the (semi-simple) centralizer of Λ in Γ. If $\{\sigma_1=1, \sigma_2, \cdots, \sigma_r\}$ is a complete representative system of $\mathfrak{G}$ module $\mathfrak{J}$ then we have as is well known,

$$\mathrm{Hom}(\Gamma_\Lambda, \Gamma_\Lambda) = \mathfrak{G}\Gamma_l = \sum_i{}_{\oplus} \sigma_i \mathfrak{J}\Gamma_l = \sum_i \sigma_i T_r \Gamma_l = \sum_i \sigma_i (T_r \otimes_C \Gamma_l)\,.$$

Let h be the $\mathfrak{J}\Gamma_l - \mathfrak{J}\Gamma_l$-homomorphism of $\mathfrak{G}\Gamma_l$ into $\mathfrak{J}\Gamma_l$ defined by

$$h(\sum_i \sigma_i a_i) = a_1 \qquad\qquad (a_i \in \mathfrak{J}\Gamma_l)\,.$$

Then we have for each $x \in \mathfrak{G}\Gamma_l$

$$x = \sum_i^1 \sigma_i h(\sigma_i^{-1}x) = \sum_i h(x\sigma_i)\sigma_i^{-1}$$

and this implies by Corollary 1 that $\mathfrak{G}\Gamma_l/\mathfrak{J}\Gamma_l$ is a (free) Frobenius extension. Since T is a semi-simple ring with minimum condition T_r/C and whence $T_r \otimes_C \Gamma_l/\Gamma_l$ is a (free) Frobenius extension. Then by Theorem 2 we see that $\mathfrak{G}\Gamma_l/\Gamma_l$ is also a free Frobenius extension. Thus by Theorem 5 we have the following

Proposition 1 (Kasch). *Let Γ be a simple ring with minimum condition and let $\mathfrak{G}$ be a semi-regular group of automorphisms of Γ with Λ as its fixed subring. Then Γ/Λ is a free Frobenius extension.*

From Lemma 1 and the above proposition we can derive the following

Proposition 2. *Let Γ, Λ and $\mathfrak{G}$ be as in Proposition* 1. *Then we have the following $\mathfrak{G}\Gamma-\Gamma$-isomorphism* :

$$\Gamma\underset{\Lambda}{\otimes}\Gamma \overset{(\mathfrak{G}\Gamma-\Gamma)}{\cong} \mathfrak{G}\Gamma .$$

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