NOTE ON DECOMPOSITION SETS OF SEMI-PRIME RINGS

By

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Introduction. As has been observed by Jacobson the set $\mathfrak{P}=\mathfrak{P}(A)$ of all primitive ideals of a ring A may be made into a topological space endowed with Stone's topology, and recently, concerning topological properties of the structure space, Suliński [8] obtained some structure theorems of a semi-simple ring which is represented as a subdirect sum of simple rings with unity.

In this note, we shall extend his results to semi-prime rings and give necessary and sufficient conditions for a semi-prime ring to have a minimal decomposition set.

§ **1.** First of all, we shall prove the following extension of [1, Theorem 1].

Lemma 1. *Let T be an ideal of a ring A.*

(1) *If p is a prime ideal of A then $T\frown p$ is a prime ideal of the ring T and if moreover p does not contain T then $(p\frown T: T)$[1]$=p$.*

(2)[2] *If p_1 is a prime ideal of the ring T, then there exists a prime ideal p of A such that $p\frown T=p_1$ and, if $p_1\neq T$, then $(p_1: T)=p$.*

Proof. (1) By [6, Lemma 2], $T\frown p$ is a prime ideal of the ring T. Assume that p does not contain T. Then $T\cdot(p\frown T: T)\subseteqq p$ implies $(p\frown T: T)\subseteqq p$ and hence we have $(p\frown T: T)=p$.

(2) Let B be the ideal of A generated by p_1 and let x be an arbitrary element of $B\frown T$. Since $xTxTx\subseteqq TBT\subseteqq p_1$ and p_1 is a prime ideal in T, x belongs to p_1, and hence $T\frown B=p_1$. The complement C of p_1 in T is an *m-system* (in T whence) in A and does not meet B. By Zorn's lemma, there exists a prime ideal p of A containing B such that p does not meet C and satisfies $T\frown p=p_1$. Moreover, if $p_1\neq T$ then p can not contain T, and hence, by (1), we have $(p_1: T)=p$.

A ring A is called *a semi-prime ring* if it is isomorphic to a subdirect sum of prime rings, i.e., if there exist prime ideals p_α $(\alpha\in\Lambda)$ of

1) We shall denote by $(p\frown T: T)$ the set $\{a\in A;\ Ta\subseteqq p\frown T\}$.
2) Cf. [3] and [7].

A such that $\bigcap_{\alpha\in\Lambda} p_\alpha=0$.

As is easily seen, the annihilator[3] of a non-zero ideal in a semi-prime ring is always represented as the intersection of all prime ideals which contain the annihilator. However, we have

Corollary 1. *A non-zero ideal T of a semi-prime ring A is a prime ring if and only if the annihilator $(0\colon T)$ is a prime ideal in A.*

Let A be an arbitrary ring and let $\mathfrak{Q}=\mathfrak{Q}(A)$ be the set of all prime ideals of A other than A. For any non-empty subset $\mathfrak{N}$ of $\mathfrak{Q}$, we define the closure $\overline{\mathfrak{N}}$ of $\mathfrak{N}$ as the totality of those prime ideals p in $\mathfrak{Q}$ which contains $I(\mathfrak{N})$, where $I(\mathfrak{N})$ denotes the intersection of all prime ideals belonging to $\mathfrak{N}$. $\mathfrak{Q}$ becomes a topological space relative to this closure operation $\mathfrak{N}\to\overline{\mathfrak{N}}$, and is called the *structure space* of the ring A.

For the lower radical $R=I(\mathfrak{Q})$ of A, we set $T^*=(R\colon T)$ for any ideal T of A. If A is semi-prime, then the lower radical R of A is equal to 0 and hence T^* coincides with the right annihilator $r(T)$ of T as well as the left annihilator $l(T)$ of T.

Lemma 2. *Let A be a ring. Then, for any subset $\mathfrak{N}$ of $\mathfrak{Q}$, we have $I(\mathfrak{N})^*=I(\mathfrak{Q}-\overline{\mathfrak{N}})$*[4].

In particular, we have $I(\mathfrak{N})^*\frown I(\mathfrak{N})=R$.

Proof. $I(\mathfrak{N})\cdot I(\mathfrak{Q}-\overline{\mathfrak{N}})\subseteqq I(\mathfrak{N})\frown I(\mathfrak{Q}-\overline{\mathfrak{N}})=I(\overline{\mathfrak{N}})\frown I(\mathfrak{Q}-\overline{\mathfrak{N}})=I(\mathfrak{Q})=R$. Conversely, for any prime ideal $p\in\mathfrak{Q}-\overline{\mathfrak{N}}$, we have $I(\mathfrak{N})\cdot I(\mathfrak{N})^*\subseteqq R\subseteqq p$ and hence $I(\mathfrak{N})^*\subseteqq p$, thus $I(\mathfrak{N})^*\subseteqq I(\overline{\mathfrak{Q}}-\mathfrak{N})$.

Lemma 3. *Let A be a ring and let p be in $\mathfrak{Q}$. Then the following conditions are equivalent:*

(1) $p^*\neq R$.

(2) $p^{**}=p$.

(3) *$\overline{\{p\}}$ contains a non-empty open subset $\mathfrak{N}$ of $\mathfrak{Q}$.*

Moreover, if this is the case, p is a minimal prime ideal of A.

Proof. (1)$\rightleftarrows$(2). Assume that $p^*\neq R$. Then $p^*\nsubseteq p$ and, since $p^*p^{**}\subseteqq R\subseteqq p$, we have $p^{**}\subseteqq p$ and hence $p^{**}=p$. Conversely, assume that $p^{**}=p$ and $p^*=R$. Then $p=A$, a contradiction.

(1)$\rightleftarrows$(3). $p^*\neq R$ means $\overline{\mathfrak{Q}-\overline{\{p\}}}\neq\mathfrak{Q}$. Thus, $\mathfrak{N}=\mathfrak{Q}-\overline{\mathfrak{Q}-\overline{\{p\}}}$ $(\subseteqq\overline{\{p\}})$ is

3) In a semi-prime ring, the right annihilator $r(T)$ of any ideal T coincides with its left annihilator $l(T)$.

4) We shall denote by $\mathfrak{Q}-\overline{\mathfrak{N}}$ the set theoretical complement of $\overline{\mathfrak{N}}$ in $\mathfrak{Q}$.

open. Conversely, let $\mathfrak{R}$ be a non-empty open subset of $\overline{\{p\}}$. Then $p^* = I(\mathfrak{Q}-\overline{\{p\}}) \supseteqq I(\mathfrak{Q}-\mathfrak{R}) \supsetneqq R$ because $\mathfrak{Q}-\mathfrak{R}$ is closed. Thus $p^* \neq R$.

Now assume that $p^* \neq R$ and let p_1 be a prime ideal of A such that $p_1 \supsetneqq p$. Since $p_1 \notin \overline{\{p\}}$, $p^* = I(\mathfrak{Q}-\overline{\{p\}}) \subseteqq p_1$, and hence $p^* \subseteqq p$, which is a contradiction.

Corollary 2. *Let A be a semi-prime ring and let p be a prime ideal in A such that $p^* \neq 0$. Then p^* is a prime ring, and is maximal in the set of those ideals of A which are prime as ring.*

Proof. From Lemma 3 and Corollary 1, p^* is a prime ring. Let T be an ideal in A which is prime as a ring and $T \supsetneqq p^*$. Then $T \frown p$ and p^* are non zero ideals in the prime ring T and $(T \frown p) \cdot p^* = 0$. This is a contradiction.

Lemma 4. *Let A be a ring and let $\mathfrak{R} = \{p_\alpha\}_{\alpha \in \Lambda'}$ be a set of different minimal prime ideals in A. If $I(\mathfrak{R}) = 0$ then $r(p_\alpha) = l(p_\alpha) = I(\mathfrak{R}-\{p_\alpha\})$ for each $\alpha \in \Lambda'$.*

Proof. Let p_α be in $\mathfrak{R}$. Then for each p_β in $\mathfrak{R}$, we have either $p_\alpha \subseteqq p_\beta$ or $r(p_\alpha) \subseteqq p_\beta$. Since p_β is a minimal prime ideal in A, $r(p_\alpha) \subseteqq p_\beta$ for all p_β with $\beta \neq \alpha$. Therefore $r(p_\alpha)$ ($\subseteqq$whence)$= I(\mathfrak{R}-\{p_\alpha\})$. Similarly, we have $l(p_\alpha) = I(\mathfrak{R}-\{p_\alpha\})$.

§ 2. **Definition 1.** *Let A be a ring. We shall denote by $\mathfrak{D}$ the set of all prime ideals $p \in \mathfrak{Q}$ such that $p^* \neq R$, and call it the decomposition set for A.*

Definition 2. *Let A be a semi-prime ring. A subset $\mathfrak{R}$ of $\mathfrak{Q}$ will be called a minimal decomposition set for A if $I(\mathfrak{R}) = 0$ and $I(\mathfrak{R}-\{p\}) \neq 0$ for all p in $\mathfrak{R}$* (Goldie [1]).

In [4, Theorem 3], one of the present authors proved that a semi-prime ring has at most one minimal decomposition set for A, and, if it exists, it should coincide with $\mathfrak{D}$.

Now we shall give necessary and sufficient conditions for a semi-prime ring to have a minimal decomposition set.

Theorem 1. *If A is a semi-prime ring, then the following conditions are equivalent:*

(1) *There exists a minimal decomposition set $\mathfrak{M}$ for A.*

(2) *Every non-zero ideal T of A contains a non-zero ideal B of the ring T which is prime as a ring.*

(3) *The annihilator of the ideal generated by all those non-zero*

ideals of A which are prime as ring is zero.

(4) *There exists a subset $\mathfrak{N}$ of $\mathfrak{D}$ such that $I(\mathfrak{N})=0$.*

Proof. (1)$\to$(2). Let T be any non-zero ideal of A. There exists a prime ideal p in $\mathfrak{M}$ such that $T \nsubseteq p$. Then $T\cdot p^*$ is a non-zero ideal of the ring T. For otherwise, $p^* \neq R$ and so $p^{**}=p$ by Lemma 3, which would imply $T \subseteq p$. Besides, $T\cdot p^*$ is prime as a ring by Lemma 1 (1) because of $(T\cdot p^*)\frown p \subseteq (T\frown p^*)\frown p = T\frown(p^*\frown p)=0$.

(2)$\to$(3). It is easily seen, by Corollaries 1 and 2, that the ideal generated by all those non-zero ideals of A which are prime as ring coincides with the ideal $\sum p^*$ generated by all p^* with $p \in \mathfrak{D}$. Now $(\sum p^*)^* = \bigcap p^{**} = \bigcap p = I(\mathfrak{D})$ by Lemma 3.

Next, suppose that $I(\mathfrak{D}) \neq 0$. Then, by our assumption, there exists a non-zero ideal B of the ring $I(\mathfrak{D})$ which is prime as a ring. By Lemma 1 (2), there exists a prime ideal $p \notin \mathfrak{D}$ such that $0=(p\frown I(\mathfrak{D}))\frown B = p\frown B$. B contains a non-zero ideal B' of A by [2, Proposition IV. 3.2]. Since $B'\frown p=0$, we have $p^* \supseteq B' \neq 0$, which contradicts $p \notin \mathfrak{D}$.

(3)$\to$(4). This is clear by the proof of (2)$\to$(3).

(4)$\to$(1). Since every prime ideal p belonging to $\mathfrak{D}$ is a minimal prime ideal by Lemma 3, Lemma 4 yields our implication.

Corresponding to [8, Theorem 5], we have

Theorem 2. *Let A be a semi-prime ring and let T be a non-zero ideal of A. Then we have $I(\mathfrak{D}_T)=I(\mathfrak{D})\frown T$, where $\mathfrak{D}_T$ denotes the decomposition set for the ring T.*

Proof. Let $\mathfrak{N}$ be the set of all p in $\mathfrak{Q}$ such that $p \supseteq T$. Then we have $I(\mathfrak{D})\frown T = I(\mathfrak{D}-(\mathfrak{D}\frown\mathfrak{N}))\frown I(\mathfrak{D}\frown\mathfrak{N})\frown T = I(\mathfrak{D}-(\mathfrak{D}\frown\mathfrak{N}))\frown T$. Now assume that $p' \in \mathfrak{D}-(\mathfrak{D}\frown\mathfrak{N})$ and $(T\frown p')^*\frown T=0$. Then $(T\frown p')^* \subseteq p'$ because $p' \nsupseteq T$ contradicting $p' \in \mathfrak{D}$. Hence, $(T\frown p')^*\frown T \neq 0$ and we have $T\frown p' \in \mathfrak{D}_T$. Thus $I(\mathfrak{D})\frown T = I(\mathfrak{D}-(\mathfrak{D}\frown\mathfrak{N}))\frown T \supseteq I(\mathfrak{D}_T)$.

Conversely, let p_1 be in $\mathfrak{D}_T$. Then there exists, by Lemma 1 (2), a prime ideal p in $\mathfrak{Q}$ such that $T\frown p = p_1$ and $(T\frown p)^*\frown T \neq 0$. Since $(((T\frown p)^*\frown T)\frown p)^2 = ((T\frown p)^*\frown(T\frown p))^2 \subseteq (T\frown p)^*\cdot(T\frown p)=0$, $((T\frown p)^*\frown T)\frown p=0$ and hence $(T\frown p)^*\frown T \subseteq p^*$. Thus $p^* \neq 0$, showing that $I(\mathfrak{D}_T)$ contains $T\frown I(\mathfrak{D})$. This completes our proof.

As a corollary of Theorem 2, we have the following second necessary and sufficient condition for a semi-prime ring to have a minimal decomposition set.

Corollary 3. *A semi-prime ring A has a minimal decomposition set* *if and only if A has an ideal T such that $T^*=0$ and T has a minimal decomposition set.*

Proof. Let $\mathfrak{M}$ be a minimal decomposition set for A and let $T=\sum p_\alpha^*$ with $p_\alpha\in\mathfrak{M}$. Then $T^*=0$ by Theorem 1 and for $\alpha\neq\beta$, $p_\alpha^*\frown p_\beta^*\subseteqq p_\alpha^*\frown p_\alpha=0$ since $p_\beta^*=I(\mathfrak{M}-\{p_\beta\})\subseteqq p_\alpha$ by Lemma 4. Thus for each α, $T=p_\alpha^*\oplus T_\alpha$ with $T_\alpha=\sum_{\beta\neq\alpha}p_\beta^*$. Moreover, $\bigcap T_\alpha\subseteqq\bigcap p_\alpha=I(\mathfrak{M})=0$. Hence, T is isomorphic to a special subdirect sum of p_α^* with $p_\alpha\in\mathfrak{M}$, by [5, Theorem 15]. Therefore, T has a minimal decomposition set for T by [4, Corollary to Theorem 4].

Conversely, let T be an ideal of A such that $T^*=0$ and $I(\mathfrak{D}_T)=0$. By Theorem 2, $I(\mathfrak{D})\frown T=I(\mathfrak{D}_T)=0$ and hence $I(\mathfrak{D})=0$ because $T^*=0$. By Theorem 1 this completes our proof.

Definition 3. *Let A be a ring. We shall denote by $\mathfrak{D}_0$ the intersection of all dense subsets of $\mathfrak{Q}$ and call it the minimal set for A.* (Suliński [8]).

Lemma 5. *Let A be a ring. Then $p\in\mathfrak{D}_0$ if and only if $\{p\}$ is open in $\mathfrak{Q}$.*

Proof. If we assume that $\{p\}$ is not open, then $\overline{\mathfrak{Q}-\{p\}}=\mathfrak{Q}$, and hence $\mathfrak{Q}-\{p\}\supseteqq\mathfrak{D}_0$. Thus $p\notin\mathfrak{D}_0$.

Conversely, assume that $\{p\}$ is open in $\mathfrak{Q}$ and $p\notin\mathfrak{D}_0$. Then there exists a dense subset $\mathfrak{N}$ of $\mathfrak{Q}$ such that $\mathfrak{N}\not\ni p$. Accordingly $\mathfrak{N}\subseteqq\mathfrak{Q}-\{p\}$ and $\mathfrak{Q}=\overline{\mathfrak{N}}\subseteqq\overline{\mathfrak{Q}-\{p\}}=\mathfrak{Q}-\{p\}$. This contradiction shows $p\in\mathfrak{D}_0$.

In general, the minimal set $\mathfrak{D}_0$ is contained in the decomposition set $\mathfrak{D}$ by Lemmas 3 and 5. However, in case the structure space of A is a T_1-space, $\mathfrak{D}$ coincides with $\mathfrak{D}_0$.

The following is an extension of [8, Theorem 7].

Theorem 3. *Let A be a semi-prime ring. Then the following conditions are equivalent:*

(1) *$\mathfrak{D}$ is empty.*

(2) *A has no non-zero ideal which is prime as a ring.*

Proof. Assume that $\mathfrak{D}$ is empty and there exists a non-zero ideal T of A which is prime as a ring. Then, by Lemma 1 (2), there is a prime ideal p of A such that $T\frown p=0$. Hence $p^*\supseteqq T\neq 0$, a contradiction.

The converse is easy from Corollary 2.

§ 3. Finally, we shall consider the case where $I(\mathfrak{D})\neq 0$ and $\mathfrak{D}\neq\phi$, that is, the case where A is neither *special* nor *completely non-special* in

Sulinski's sense [8].

Lemma 6. *Let A be a semi-prime ring and let T be a non-zero ideal of A such that $T^* \neq 0$.*

(1) *If the ring T has a minimal decomposition set, then the semi-prime ring A/T^**[5] *has a minimal decomposition set too.*

(2) *If both T and T^* have minimal decomposition sets, then the ring A has a minimal decomposition set too.*

Proof. Let $\mathfrak{N}$ and $\mathfrak{N}'$ be the sets of all prime ideals $p \in \mathfrak{Q}$ such that $p \supseteqq T$ and $p \supseteqq T^*$ respectively. Since $T \neq 0$ and $T^* \neq 0$, both $\mathfrak{N}$ and $\mathfrak{N}'$ are not empty, and $\mathfrak{N} \smile \mathfrak{N}' = \mathfrak{Q}$ and $(\mathfrak{D} \frown \mathfrak{N}) \frown (\mathfrak{D} \frown \mathfrak{N}') = \phi$.

Let $\tilde{p}$ be a prime ideal in the ring A/T^*. Then there exists a prime ideal $p \in \mathfrak{N}'$ such that $p/T^* = \tilde{p}$, and $\tilde{p}^*$[6] $= 0$ if and only if $(T^* : p) = T^*$.

(1) Suppose that T has a minimal decomposition set. Then, by Theorems 1 and 2, $0 = I(\mathfrak{D}_T) = I(\mathfrak{D}) \frown T = I(\mathfrak{D} - (\mathfrak{D} \frown \mathfrak{N})) \frown T$ as was seen in the proof of Theorem 2 and these are equal to $I(\mathfrak{D} \frown \mathfrak{N}') \frown T$. Hence $I(\mathfrak{D} \frown \mathfrak{N}')$ ($\subseteqq T^*$ whence) $= T^*$. Now, let p be in $\mathfrak{D} \frown \mathfrak{N}'$. Then $\tilde{p}^* \neq 0$. For otherwise, we would have $p^* \subseteqq (T^* : p) = T^* \subseteqq p$. Thus $\tilde{p}$ is contained in the decomposition set of the ring A/T^*. Therefore $I(\mathfrak{D}_{A/T^*}) \subseteqq I(\mathfrak{D} \frown \mathfrak{N}')/T^*$ and hence we have $I(\mathfrak{D}_{A/T^*}) = 0$.

(2) Suppose that both T and T^* have minimal decomposition sets. Then, $0 = I(\mathfrak{D}_{T^*}) = I(\mathfrak{D}) \frown T^* = I(\mathfrak{D}) \frown I(\mathfrak{D} \frown \mathfrak{N}') = I(\mathfrak{D})$ since $T^* = I(\mathfrak{D} \frown \mathfrak{N}')$ as was seen above. Thus, A has a minimal decomposition set.

Combining Lemma 6 (1) with Theorem 2, we obtain a generalization of [1, Theorem 6].

Lemma 7. *Let A be a semi-prime ring and let T be a non-zero ideal of A such that $T^* \neq 0$.*

(1) *If the decomposition set of the ring T is empty, then that of the semi-prime ring A/T^* is also empty.*

(2) *If the decomposition sets of both T and T^* are empty, then that of the ring A is also empty.*

Proof. (1) Suppose that $\mathfrak{D}_T$ is empty. Then by Theorem 2 $I(\mathfrak{D}) \supseteqq I(\mathfrak{D}) \frown T = I(\mathfrak{D}_T) = T$ and hence $I(\mathfrak{D})^* \subseteqq T^*$. Let $\mathfrak{N}'$ be as in the proof of Lemma 6. Then $\mathfrak{D} \frown \mathfrak{N}' = \phi$. For otherwise, there would exist a prime ideal p such that $p \in \mathfrak{D}$ and $p \supseteqq T^*$. Then $p \supseteqq T^* \supseteqq I(\mathfrak{D})^*$, and hence p^*

5) As is remarked in § 1, $T^* = I(\mathfrak{N})$, $\mathfrak{N} = \{p \in \mathfrak{Q} : p \supseteqq T^*\}$, and hence the ring A/T^* is semi-prime.

6) Since no confusion can arise, we shall use this notation in the residue class ring A/T^*.

$\subseteqq I(\mathfrak{D})\subseteqq p$, because $I(\mathfrak{D})^{**}=I(\mathfrak{D})$, which is a contradiction. Let p be in $\mathfrak{N}'$. Then $p\cdot(T^*: p)\subseteqq T^*$, $p\cdot(T^*: p)\cdot T=0$, $(T^*: p)\cdot T\subseteqq p^*=0$ and hence $(T^*: p)$ ($\subseteqq$whence)$=T^*$. This completes our proof.

(2) Suppose that both $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$ are empty. Then we have, by Theorem 2, $T=I(\mathfrak{D}_T)=I(\mathfrak{D})\frown T\subseteqq I(\mathfrak{D})$ and $T^*=I(\mathfrak{D}_{T^*})=I(\mathfrak{D})\frown T^*\subseteqq I(\mathfrak{D})$. Therefore $I(\mathfrak{D})\supseteqq T^*\supseteqq I(\mathfrak{D})^*$, $(I(\mathfrak{D})^*)^2=0$, and hence $I(\mathfrak{D})^*=0$. Thus $I(\mathfrak{D})=I(\mathfrak{D})^{**}=A$. This completes our proof.

As an easy consequence of Lemmas 6 and 7, we have the following

Theorem 4. *Let A be a semi-prime ring and let $I(\mathfrak{D})\neq 0$ and $\neq A$.*

(1) *The ring $I(\mathfrak{D})^*$ has a minimal decomposition set.*

(2) *The decomposition set of the ring $I(\mathfrak{D})$ is empty.*

(3)[7] *The semi-prime ring $A/I(\mathfrak{D})$ has a minimal decomposition set.*

(4)[7] *The decomposition set of the semi-prime ring $A/I(\mathfrak{D})^*$ is empty.*

Proof. (1), (2) and (3), (4) follow from Theorem 2 and Lemmas 6 (1) and 7 (1) respectively.

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(Received April 30, 1962)

7) Cf. [9].