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ROTATIONAL SYMMETRY OF RICCI SOLITONS IN HIGHER DIMENSIONS

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Abstract

Let (M, g) be a steady gradient Ricci soliton of dimension $n \geq 4$ which has positive sectional curvature and is asymptotically cylindrical. Under these assumptions, we show that (M, g) is rotationally symmetric. In particular, our results apply to steady gradient Ricci solitons in dimension 4 which are κ-noncollapsed and have positive isotropic curvature.

1. Introduction

This is a sequel to our earlier paper [**4**], in which we proved a uniqueness theorem for the three-dimensional Bryant soliton. Recall that the Bryant soliton is the unique steady gradient Ricci soliton in dimension 3, which is rotationally symmetric (cf. [**6**]). In [**4**], it was shown that the three-dimensional Bryant soliton is unique in the class of κ-noncollapsed steady gradient Ricci solitons:

Theorem 1.1 (Brendle [**4**])**.** *Let (M, g) be a three-dimensional complete steady gradient Ricci soliton which is non-flat and κ-noncollapsed. Then (M, g) is rotationally symmetric, and is therefore isometric to the Bryant soliton up to scaling.*

Theorem 1.1 resolves a problem mentioned in Perelman's first paper [**16**].

In this paper, we consider similar questions in higher dimensions. We will assume throughout that (M, g) is a steady gradient Ricci soliton of dimension $n \geq 4$ with positive sectional curvature. We may write $\text{Ric} = D^2 f$ for some real-valued function f. As usual, we put $X = \nabla f$, and denote by Φ_t the flow generated by the vector field $-X$.

Definition. We say that (M, g) is asymptotically cylindrical if the following holds:
(i) The scalar curvature satisfies $\frac{\Lambda_1}{d(p_0,p)} \leq R \leq \frac{\Lambda_2}{d(p_0,p)}$ at infinity, where Λ_1 and Λ_2 are positive constants.

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(ii) Let p_m be an arbitrary sequence of marked points going to infinity. Consider the rescaled metrics

$$\hat{g}^{(m)}(t) = r_m^{-1}\,\Phi_{r_m t}^*(g),$$

where $r_m\,R(p_m) = \frac{n-1}{2} + o(1)$. As $m \to \infty$, the flows $(M, \hat{g}^{(m)}(t), p_m)$ converge in the Cheeger-Gromov sense to a family of shrinking cylinders $(S^{n-1} \times \mathbb{R}, \overline{g}(t))$, $t \in (0,1)$. The metric $\overline{g}(t)$ is given by

$$\overline{g}(t) = (n-2)(2-2t)\,g_{S^{n-1}} + dz \otimes dz, \tag{1}$$

where $g_{S^{n-1}}$ denotes the standard metric on S^{n-1} with constant sectional curvature 1.

We now state the main result of this paper. This result is motivated in part by the work of Simon and Solomon [**17**], which deals with uniqueness questions for minimal surfaces with prescribed tangent cones at infinity.

Theorem 1.2. *Let (M, g) be a steady gradient Ricci soliton of dimension $n \geq 4$ which has positive sectional curvature and is asymptotically cylindrical. Then (M, g) is rotationally symmetric. In particular, (M, g) is isometric to the n-dimensional Bryant soliton up to scaling.*

In dimension 3, it follows from work of Perelman [**16**] that any complete steady gradient Ricci soliton which is non-flat and κ-noncollapsed is asymptotically cylindrical. Thus, Theorem 1.2 can be viewed as a higher dimensional version of Theorem 1.1.

Theorem 1.2 has an interesting implication in dimension 4. A four-dimensional manifold (M, g) has positive isotropic curvature if and only if $a_1 + a_2 > 0$ and $c_1 + c_2 > 0$, where a_1, a_2, c_1, c_2 are defined as in [**12**]. The notion of isotropic curvature was first introduced by Micallef and Moore [**15**] in their work on the index of minimal two-spheres. It also plays a central role in the convergence theory for the Ricci flow in higher dimensions (see e.g. [**2**], [**3**]).

Theorem 1.3. *Let (M, g) be a four-dimensional steady gradient Ricci soliton which is non-flat; is κ-noncollapsed; and satisfies the pointwise pinching condition*

$$0 \leq \max\{a_3, b_3, c_3\} \leq \Lambda\,\min\{a_1 + a_2, c_1 + c_2\},$$

where $a_1, a_2, a_3, c_1, c_2, c_3, b_3$ are defined as in Hamilton's paper [**12**] *and $\Lambda \geq 1$ is a constant. Then (M, g) is rotationally symmetric.*

We note that various authors have obtained uniqueness results for Ricci solitons in higher dimensions; see e.g. [**7**], [**8**], [**9**], and [**11**]. Moreover, Ivey [**14**] has constructed examples of Ricci solitons which are not rotationally symmetric.

In order to prove Theorem 1.2, we will adapt the arguments in [**4**]. While many arguments in [**4**] directly generalize to higher dimensions, there are several crucial differences. In particular, the proof of the roundness estimate in Section 2 is very different than in the three-dimensional case. Moreover, the proof in [**4**] uses an estimate of Anderson and Chow [**1**] for the linearized Ricci flow system. This estimate uses special properties of the curvature tensor in dimension 3, so we require a different argument to handle the higher dimensional case. This will be discussed in Section 4.

Finally, to deduce Theorem 1.3 from Theorem 1.2, we show that a steady gradient Ricci soliton (M, g) which satisfies the assumptions of Theorem 1.3 must have positive curvature operator (cf. Corollary 6.4 below). The proof of this fact uses the pinching estimates of Hamilton (see [**12**], [**13**]). Using results from [**10**], we conclude that (M, g) is asymptotically cylindrical. Theorem 1.2 then implies that (M, g) is rotationally symmetric.

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2. The roundness estimate

By scaling, we may assume that $R + |\nabla f|^2 = 1$. Since $R \to 0$ at infinity, we can find a point p_0 where the scalar curvature attains its maximum. Since (M, g) has positive sectional curvature, the Hessian of f is strictly positive definite at each point in M. The identity $\nabla R(p_0) = 0$ implies $\nabla f(p_0) = 0$. Since f is strictly convex, we conclude that $\liminf_{p\to\infty} \frac{f(p)}{d(p_0,p)} > 0$. On the other hand, since $|\nabla f|^2 \leq 1$, we have $\limsup_{p\to\infty} \frac{f(p)}{d(p_0,p)} < \infty$.

Using the fact that (M, g) is asymptotically cylindrical, we obtain the following result:

Proposition 2.1. *We have* $f\,R = \frac{n-1}{2} + o(1)$ *and* $f\,\mathrm{Ric} \leq \big(\frac{1}{2} + o(1)\big)\,g$. *Moreover, we have* $f^2\,\mathrm{Ric} \geq c\,g$ *for some positive constant* c.

Proof. Since (M, g) is asymptotically cylindrical, we have $\Delta R = o(r^{-2})$ and $|\mathrm{Ric}|^2 = \frac{1}{n-1}\,R^2 + o(r^{-2})$. This implies

$$-\langle X, \nabla R\rangle = \Delta R + 2\,|\mathrm{Ric}|^2 = \frac{2}{n-1}\,R^2 + o(r^{-2}),$$

hence

$$\Big\langle X, \nabla\Big(\frac{1}{R} - \frac{2}{n-1}\,f\Big)\Big\rangle = o(1).$$

Integrating this inequality along the integral curves of X gives

$$\frac{1}{R} - \frac{2}{n-1}\,f = o(r),$$

hence

$$f\,R = \frac{n-1}{2} + o(1).$$

Moreover, we have $\mathrm{Ric} \leq \big(\frac{1}{n-1} + o(1)\big)\, R\, g$ since (M, g) is asymptotically cylindrical. Therefore, $f\,\mathrm{Ric} \leq \big(\frac{1}{2} + o(1)\big)\, g$.

In order to verify the third statement, we choose an orthonormal frame $\{e_1, \ldots, e_n\}$ such that $e_n = \frac{X}{|X|}$. Since (M, g) is asymptotically cylindrical, we have

$$\mathrm{Ric}(e_i, e_j) = \frac{1}{n-1}\, R\, \delta_{ij} + o(r^{-1})$$

for $i, j \in \{1, \ldots, n-1\}$ and

$$2\,\mathrm{Ric}(e_i, X) = -\langle e_i, \nabla R\rangle = o(r^{-\frac{3}{2}}).$$

Moreover, we have

$$2\,\mathrm{Ric}(X, X) = -\langle X, \nabla R\rangle = \Delta R + 2\,|\mathrm{Ric}|^2 = \frac{2}{n-1}\, R^2 + o(r^{-2}).$$

Putting these facts together, we conclude that $\mathrm{Ric} \geq c\, R^2\, g$ for some positive constant c. From this, the assertion follows. q.e.d.

In the remainder of this section, we prove a roundness estimate. We begin with a lemma:

Lemma 2.2. *We have* $R_{ijkl}\, \partial^l f = O(r^{-\frac{3}{2}})$.

Proof. Using Shi's estimate, we obtain

$$R_{ijkl}\, \partial^l f = D_i \mathrm{Ric}_{jk} - D_j \mathrm{Ric}_{ik} = O(r^{-\frac{3}{2}}).$$

This proves the assertion. q.e.d.

We next define

$$T = (n-1)\,\mathrm{Ric} - R\, g + R\, df \otimes df.$$

Note that

$$\begin{aligned}
&\mathrm{tr}(T) = -R^2 = O(r^{-2}), \\
&T(\nabla f, \cdot) = (n-1)\,\mathrm{Ric}(\nabla f, \cdot) - R^2\, \nabla f = O(r^{-\frac{3}{2}}), \\
&T(\nabla f, \nabla f) = (n-1)\,\mathrm{Ric}(\nabla f, \nabla f) - R^2\, |\nabla f|^2 = O(r^{-2}).
\end{aligned}$$

Proposition 2.3. *We have* $|T| \leq O(r^{-\frac{3}{2}})$.

Proof. The Ricci tensor of (M, g) satisfies the equation

$$\Delta \mathrm{Ric}_{ik} + D_X \mathrm{Ric}_{ik} = -2 \sum_{j,l=1}^n R_{ijkl}\, \mathrm{Ric}^{jl}.$$

Moreover, using the identity $\Delta X + D_X X = 0$, we obtain

$$\begin{aligned}
&\Delta(R\, g_{ik} - R\, \partial_i f\, \partial_k f) + D_X(R\, g_{ik} - R\, \partial_i f\, \partial_k f) \\
&= (\Delta R + \langle X, \nabla R\rangle)\, (g_{ik} - \partial_i f\, \partial_k f) + O(r^{-\frac{5}{2}}) \\
&= -2\, |\mathrm{Ric}|^2\, (g_{ik} - \partial_i f\, \partial_k f) + O(r^{-\frac{5}{2}}).
\end{aligned}$$

Using Lemma 2.2, we conclude that

$$\Delta T_{ik} + D_X T_{ik} = -2 \sum_{j,l=1}^{n-1} R_{ijkl}\, T^{jl} - 2\, R\, \mathrm{Ric}_{ik} + 2\, |\mathrm{Ric}|^2\, (g_{ik} - \partial_i f\, \partial_k f) + O(r^{-\frac{5}{2}}),$$

hence

$$\begin{aligned} &\Delta(|T|^2) + \langle X, \nabla(|T|^2)\rangle \\ &= 2\, |DT|^2 - 4 \sum_{j,l=1}^{n-1} R_{ijkl}\, T^{ik}\, T^{jl} - 4\, R \sum_{i,k=1}^{n} \mathrm{Ric}_{ik}\, T^{ik} \\ &+ 4\, |\mathrm{Ric}|^2 \sum_{i,k=1}^{n} (g_{ik} - \partial_i f\, \partial_k f)\, T^{ik} + O(r^{-\frac{5}{2}})\, |T| \\ &= 2\, |DT|^2 - 4 \sum_{j,l=1}^{n-1} R_{ijkl}\, T^{ik}\, T^{jl} - \frac{4}{n-1}\, R\, |T|^2 \\ &+ 4\, \Big(|\mathrm{Ric}|^2 - \frac{1}{n-1}\, R^2\Big) \sum_{i,k=1}^{n} (g_{ik} - \partial_i f\, \partial_k f)\, T^{ik} + O(r^{-\frac{5}{2}})\, |T|. \end{aligned}$$

Since $\sum_{i,k=1}^{n} (g_{ik} - \partial_i f\, \partial_k f)\, T^{ik} = O(r^{-2})$, we obtain

$$\begin{aligned} &\Delta(|T|^2) + \langle X, \nabla(|T|^2)\rangle \\ &\geq -4 \sum_{j,l=1}^{n-1} R_{ijkl}\, T^{ik}\, T^{jl} - \frac{4}{n-1}\, R\, |T|^2 - O(r^{-\frac{5}{2}})\, |T| - O(r^{-4}). \end{aligned}$$

Moreover, since (M, g) is asymptotically cylindrical, we have

$$\begin{aligned} R_{ijkl} &= \frac{1}{(n-1)(n-2)}\, R\, (g_{ik} - \partial_i f\, \partial_k f)\, (g_{jl} - \partial_j f\, \partial_l f) \\ &- \frac{1}{(n-1)(n-2)}\, R\, (g_{il} - \partial_i f\, \partial_l f)\, (g_{jk} - \partial_j f\, \partial_k f) \\ &+ o(r^{-1}) \end{aligned}$$

near infinity. This implies

$$\sum_{j,l=1}^{n-1} R_{ijkl}\, T^{ik}\, T^{jl} = -\frac{1}{(n-1)(n-2)}\, R\, |T|^2 + O(r^{-\frac{5}{2}})\, |T| + o(r^{-1})\, |T|^2,$$

hence

$$\begin{aligned} &\Delta(|T|^2) + \langle X, \nabla(|T|^2)\rangle \\ &\geq -\frac{4(n-3)}{(n-1)(n-2)}\, R\, |T|^2 - o(r^{-1})\, |T|^2 - O(r^{-\frac{5}{2}})\, |T| - O(r^{-4}). \end{aligned}$$

We next observe that $|D_X \text{Ric}| \le O(r^{-2})$ and $|D^2_{X,X} \text{Ric}| \le O(r^{-\frac{5}{2}})$. This implies $|D_X T| \le O(r^{-2})$ and $|D^2_{X,X} T| \le O(r^{-\frac{5}{2}})$. From this, we deduce that

$$\begin{aligned} &\Delta_\Sigma(|T|^2) + \langle X, \nabla(|T|^2)\rangle \\ &\ge -\frac{2(n-3)}{n-2}\, f^{-1}\, |T|^2 - o(r^{-1})\, |T|^2 - O(r^{-\frac{5}{2}})\, |T| - O(r^{-4}), \end{aligned}$$

where Δ_Σ denotes the Laplacian on the level surfaces of f. Thus, we conclude that

$$\begin{aligned} &\Delta_\Sigma(f^2\, |T|^2) + \langle X, \nabla(f^2\, |T|^2)\rangle \\ &\ge \frac{2}{n-2}\, f\, |T|^2 - o(r)\, |T|^2 - O(r^{-\frac{1}{2}})\, |T| - O(r^{-2}) \ge -O(r^{-2}) \end{aligned}$$

outside some compact set. Since $f^2\, |T|^2 \to 0$ at infinity, the parabolic maximum principle implies that $f^2\, |T|^2 \le O(r^{-1})$. This completes the proof. q.e.d.

In the following, we fix ε sufficiently small; for example, $\varepsilon = \frac{1}{1000n}$ will work. By Proposition 2.3, we have $|T| \le O(r^{\frac{1}{2(n-2)} - \frac{3}{2} - 32\varepsilon})$. Moreover, it follows from Shi's estimates that $|D^m T| \le O(r^{-\frac{m+2}{2}})$ for each m. Using standard interpolation inequalities, we obtain $|DT| \le O(r^{\frac{1}{2(n-2)} - 2 - 16\varepsilon})$. Using the identity

$$\begin{aligned} D^k T_{ik} &= \frac{n-3}{2}\, \partial_i R + \langle \nabla f, \nabla R\rangle\, \partial_i f + R^2\, \partial_i f + R\, \text{Ric}_i^k\, \partial_k f \\ &= \frac{n-3}{2}\, \partial_i R + O(r^{-2}), \end{aligned}$$

we conclude that $|\nabla R| \le O(r^{\frac{1}{2(n-2)} - 2 - 16\varepsilon})$. This implies

$$|D\text{Ric}| \le C\, |DT| + C\, |\nabla R| + C\, R\, |D^2 f| \le O(r^{\frac{1}{2(n-2)} - 2 - 16\varepsilon}).$$

Using standard interpolation inequalities, we obtain

$$|D^2 \text{Ric}| \le O(r^{\frac{1}{2(n-2)} - \frac{5}{2} - 8\varepsilon}).$$

Proposition 2.4. *We have* $f\, R = \frac{n-1}{2} + O(r^{\frac{1}{2(n-2)} - \frac{1}{2} - 8\varepsilon})$.

Proof. Using the inequality $|T| \le O(r^{-\frac{3}{2}})$, we obtain

$$|\text{Ric}| = \frac{1}{n-1}\, R\, |g - df \otimes df| + O(r^{-\frac{3}{2}}) = \frac{1}{\sqrt{n-1}}\, R + O(r^{-\frac{3}{2}}),$$

hence

$$|\text{Ric}|^2 = \frac{1}{n-1}\, R^2 + O(r^{-\frac{5}{2}}).$$

This implies

$$-\langle X, \nabla R\rangle = \Delta R + 2\, |\text{Ric}|^2 = \frac{2}{n-1}\, R^2 + O(r^{\frac{1}{2(n-2)} - \frac{5}{2} - 8\varepsilon}),$$

hence
$$\Big\langle X, \nabla\Big(\frac{1}{R} - \frac{2}{n-1}\,f\Big)\Big\rangle = O(r^{\frac{1}{2(n-2)}-\frac{1}{2}-8\varepsilon}).$$
Integrating this identity along the integral curves of X, we obtain
$$\frac{1}{R} - \frac{2}{n-1}\,f = O(r^{\frac{1}{2(n-2)}+\frac{1}{2}-8\varepsilon}).$$
From this, the assertion follows. q.e.d.

Proposition 2.5. *We have*
$$\begin{aligned} f\,R_{ijkl} &= \frac{1}{2(n-2)}\,(g_{ik} - \partial_i f\,\partial_k f)\,(g_{ik} - \partial_i f\,\partial_k f) \\ &\quad - \frac{1}{2(n-2)}\,(g_{il} - \partial_i f\,\partial_l f)\,(g_{jk} - \partial_j f\,\partial_k f) \\ &\quad + O(r^{\frac{1}{2(n-2)}-\frac{1}{2}-8\varepsilon}). \end{aligned}$$

Proof. It follows from Proposition 2.10 in [**3**] that
$$\begin{aligned} -D_X R_{ijkl} &= D^2_{i,k}\mathrm{Ric}_{jl} - D^2_{i,l}\mathrm{Ric}_{jk} - D^2_{j,k}\mathrm{Ric}_{il} + D^2_{j,l}\mathrm{Ric}_{ik} \\ &\quad + \sum_{m=1}^{n} \mathrm{Ric}_i^m\,R_{mjkl} + \sum_{m=1}^{n} \mathrm{Ric}_j^m\,R_{imkl}. \end{aligned}$$
Using Lemma 2.2 and Proposition 2.3, we obtain
$$\begin{aligned} \sum_{m=1}^{n} \mathrm{Ric}_i^m\,R_{mjkl} &= \frac{1}{n-1}\,R\sum_{m=1}^{n}(\delta_i^m - \partial_i f\,\partial^m f)\,R_{mjkl} + O(r^{-\frac{5}{2}}) \\ &= \frac{1}{n-1}\,R\,R_{ijkl} + O(r^{-\frac{5}{2}}). \end{aligned}$$
Thus, we conclude that
$$\begin{aligned} -D_X R_{ijkl} &= \frac{2}{n-1}\,R\,R_{ijkl} + O(r^{\frac{1}{2(n-2)}-\frac{5}{2}-8\varepsilon}) \\ &= f^{-1}\,R_{ijkl} + O(r^{\frac{1}{2(n-2)}-\frac{5}{2}-8\varepsilon}), \end{aligned}$$
hence
$$|D_X(f\,R_{ijkl})| \leq O(r^{\frac{1}{2(n-2)}-\frac{3}{2}-8\varepsilon}).$$
On the other hand, the tensor
$$\begin{aligned} S_{ijkl} &= \frac{1}{2(n-2)}\,(g_{ik} - \partial_i f\,\partial_k f)\,(g_{jl} - \partial_j f\,\partial_l f) \\ &\quad - \frac{1}{2(n-2)}\,(g_{il} - \partial_i f\,\partial_l f)\,(g_{jk} - \partial_j f\,\partial_k f) \end{aligned}$$
satisfies
$$|D_X S_{ijkl}| \leq O(r^{-\frac{3}{2}}).$$
Putting these facts together, we obtain
$$|D_X(f\,R_{ijkl} - S_{ijkl})| \leq O(r^{\frac{1}{2(n-2)}-\frac{3}{2}-8\varepsilon}).$$

Moreover, we have $|f\,R_{ijkl} - S_{ijkl}| \to 0$ at infinity. Integrating the preceding inequality along integral curves of X gives

$$|f\,R_{ijkl} - S_{ijkl}| \leq O(r^{\frac{1}{2(n-2)}-\frac{1}{2}-8\varepsilon}),$$

as claimed. q.e.d.

We next construct a collection of approximate Killing vector fields:

Proposition 2.6. *We can find a collection of vector fields U_a, $a \in \{1, \ldots, \frac{n(n-1)}{2}\}$, on (M, g) such that $|\mathscr{L}_{U_a}(g)| \leq O(r^{\frac{1}{2(n-2)}-\frac{1}{2}-2\varepsilon})$ and $|\Delta U_a + D_X U_a| \leq O(r^{\frac{1}{2(n-2)}-1-2\varepsilon})$. Moreover, we have*

$$\sum_{a=1}^{\frac{n(n-1)}{2}} U_a \otimes U_a = r\,\Big(\sum_{i=1}^{n-1} e_i \otimes e_i + O(r^{\frac{1}{2(n-2)}-\frac{1}{2}-2\varepsilon})\Big),$$

where $\{e_1, \ldots, e_{n-1}\}$ is a local orthonormal frame on the level set $\{f = r\}$.

The proof of Proposition 2.6 is analogous to the arguments in [**4**], Section 3. We omit the details.

3. An elliptic PDE for vector fields

Let us fix a smooth vector field Q on M with the property that $|Q| \leq O(r^{\frac{1}{2(n-2)}-1-2\varepsilon})$. We will show that there exists a vector field V on M such that $\Delta V + D_X V = Q$ and $|V| \leq O(r^{\frac{1}{2(n-2)}-\varepsilon})$.

Lemma 3.1. *Consider the shrinking cylinders $(S^{n-1} \times \mathbb{R}, \overline{g}(t))$, $t \in (0,1)$, where $\overline{g}(t)$ is given by (1). Let $\overline{V}(t)$, $t \in (0,1)$, be a one-parameter family of vector fields which satisfy the parabolic equation*

$$\frac{\partial}{\partial t}\overline{V}(t) = \Delta_{\overline{g}(t)}\overline{V}(t) + \mathrm{Ric}_{\overline{g}(t)}(\overline{V}(t)). \tag{2}$$

Moreover, suppose that $\overline{V}(t)$ is invariant under translations along the axis of the cylinder, and

$$|\overline{V}(t)|_{\overline{g}(t)} \leq 1 \tag{3}$$

for all $t \in (0, \frac{1}{2}]$. Then

$$\inf_{\lambda \in \mathbb{R}} \sup_{S^{n-1}\times\mathbb{R}} \Big|\overline{V}(t) - \lambda\,\frac{\partial}{\partial z}\Big|_{\overline{g}(t)} \leq L\,(1-t)^{\frac{1}{2(n-2)}}$$

for all $t \in [\frac{1}{2}, 1)$, where L is a positive constant.

Proof. Since $\overline{V}(t)$ is invariant under translations along the axis of the cylinder, we may write

$$\overline{V}(t) = \xi(t) + \eta(t)\,\frac{\partial}{\partial z}$$

for $t \in (0,1)$, where $\xi(t)$ is a vector field on S^{n-1} and $\eta(t)$ is a real-valued function on S^{n-1}. The parabolic equation (2) implies the following system of equations for $\xi(t)$ and $\eta(t)$:

$$\frac{\partial}{\partial t}\xi(t) = \frac{1}{(n-2)(2-2t)}\,(\Delta_{S^{n-1}}\xi(t) + (n-2)\,\xi(t)), \tag{4}$$

$$\frac{\partial}{\partial t}\eta(t) = \frac{1}{(n-2)(2-2t)}\,\Delta_{S^{n-1}}\eta(t). \tag{5}$$

Furthermore, the estimate (3) gives

$$\sup_{S^{n-1}} |\xi(t)|_{g_{S^{n-1}}} \leq L_1, \tag{6}$$

$$\sup_{S^{n-1}} |\eta(t)| \leq L_1 \tag{7}$$

for each $t \in (0, \frac{1}{2}]$, where L_1 is a positive constant.

Let us consider the operator $\xi \mapsto -\Delta_{S^{n-1}}\xi - (n-2)\,\xi$, acting on vector fields on S^{n-1}. By Proposition A.1, the first eigenvalue of this operator is at least $-(n-3)$. Using (4) and (6), we obtain

$$\sup_{S^{n-1}} |\xi(t)|_{g_{S^{n-1}}} \leq L_2\,(1-t)^{-\frac{n-3}{2(n-2)}} \tag{8}$$

for all $t \in [\frac{1}{2}, 1)$, where L_2 is a positive constant. Similarly, it follows from (5) and (7) that

$$\inf_{\lambda \in \mathbb{R}} \sup_{S^{n-1}} |\eta(t) - \lambda| \leq L_3\,(1-t)^{\frac{n-1}{2(n-2)}} \tag{9}$$

for each $t \in [\frac{1}{2}, 1)$, where L_3 is a positive constant. Combining (8) and (9), the assertion follows. q.e.d.

Lemma 3.2 (cf. [**4**], Lemma 5.2)**.** *Let V be a smooth vector field satisfying $\Delta V + D_X V = Q$ in the region $\{f \leq \rho\}$. Then*

$$\sup_{\{f \leq \rho\}} |V| \leq \sup_{\{f = \rho\}} |V| + B\,\rho^{\frac{1}{2(n-2)} - 2\varepsilon}$$

for some uniform constant $B \geq 1$.

The proof of Lemma 3.2 is similar to the proof of Lemma 5.2 in [**4**]; we omit the details.

As in [**4**], we choose a sequence of real numbers $\rho_m \to \infty$. For each m, we can find a vector field $V^{(m)}$ such that $\Delta V^{(m)} + D_X V^{(m)} = Q$ in the region $\{f \leq \rho_m\}$ and $V^{(m)} = 0$ on the boundary $\{f = \rho_m\}$. We now define

$$A^{(m)}(r) = \inf_{\lambda \in \mathbb{R}} \sup_{\{f = r\}} |V^{(m)} - \lambda\, X|$$

for $r \leq \rho_m$.

Lemma 3.3. *Let us fix a real number $\tau \in (0, \frac{1}{2})$ so that $\tau^{-\varepsilon} > 2L$, where L is the constant in Lemma* 3.1*. Then we can find a real number ρ_0 and a positive integer m_0 such that*

$$2\,\tau^{-\frac{1}{2(n-2)}+\varepsilon}\,A^{(m)}(\tau r) \leq A^{(m)}(r) + r^{\frac{1}{2(n-2)}-\varepsilon}$$

for all $r \in [\rho_0, \rho_m]$ and all $m \geq m_0$.

Proof. We argue by contradiction. Suppose that the assertion is false. After passing to a subsequence, there exists a sequence of real numbers $r_m \leq \rho_m$ such that $r_m \to \infty$ and

$$A^{(m)}(r_m) + r_m^{\frac{1}{2(n-2)}-\varepsilon} \leq 2\,\tau^{-\frac{1}{2(n-2)}+\varepsilon}\,A^{(m)}(\tau r_m)$$

for all m. For each m, there exists a real number λ_m such that

$$\sup_{\{f=r_m\}} |V^{(m)} - \lambda_m\,X| = A^{(m)}(r_m).$$

Applying Lemma 3.2 to the vector field $V^{(m)} - \lambda_m\,X$ gives

$$\begin{aligned}\sup_{\{f\leq r_m\}} |V^{(m)} - \lambda_m\,X| &\leq \sup_{\{f=r_m\}} |V^{(m)} - \lambda_m\,X| + B\,r_m^{\frac{1}{2(n-2)}-2\varepsilon} \\ &\leq A^{(m)}(r_m) + r_m^{\frac{1}{2(n-2)}-\varepsilon}\end{aligned}$$

if m is sufficiently large. We next consider the vector field

$$\tilde{V}^{(m)} = \frac{1}{A^{(m)}(r_m) + r_m^{\frac{1}{2(n-2)}-\varepsilon}}\,(V^{(m)} - \lambda_m\,X).$$

The vector field $\tilde{V}^{(m)}$ satisfies

$$\sup_{\{f\leq r_m\}} |\tilde{V}^{(m)}| \leq 1. \tag{10}$$

Let

$$\hat{g}^{(m)}(t) = r_m^{-1}\,\Phi_{r_m t}^*(g)$$

and

$$\hat{V}^{(m)}(t) = r_m^{\frac{1}{2}}\,\Phi_{r_m t}^*(\tilde{V}^{(m)}).$$

Note that the metrics $\hat{g}^{(m)}(t)$ evolve by the Ricci flow. Moreover, the vector fields $\hat{V}^{(m)}(t)$ satisfy the parabolic equation

$$\frac{\partial}{\partial t}\hat{V}^{(m)}(t) = \Delta_{\hat{g}^{(m)}(t)}\hat{V}^{(m)}(t) + \mathrm{Ric}_{\hat{g}^{(m)}(t)}(\hat{V}^{(m)}(t)) - \hat{Q}^{(m)}(t),$$

where

$$\hat{Q}^{(m)}(t) = \frac{r_m^{\frac{3}{2}}}{A^{(m)}(r_m) + r_m^{\frac{1}{2(n-2)}-\varepsilon}}\,\Phi_{r_m t}^*(Q).$$

Using (10), we obtain

$$\limsup_{m\to\infty} \sup_{t\in[\delta,1-\delta]} \sup_{\{r_m-\delta^{-1}\sqrt{r_m}\leq f\leq r_m+\delta^{-1}\sqrt{r_m}\}} |\hat{V}^{(m)}(t)|_{\hat{g}^{(m)}(t)} < \infty$$

for any given $\delta \in (0, \frac{1}{2})$. Moreover, the estimate $|Q| \leq O(r^{\frac{1}{2(n-2)}-1-2\varepsilon})$ implies that

$$\limsup_{m \to \infty} \sup_{t \in [\delta, 1-\delta]} \sup_{\{r_m - \delta^{-1}\sqrt{r_m} \leq f \leq r_m + \delta^{-1}\sqrt{r_m}\}} |\hat{Q}^{(m)}(t)|_{\hat{g}^{(m)}(t)} = 0$$

for any given $\delta \in (0, \frac{1}{2})$.

We now pass to the limit as $m \to \infty$. To that end, we choose a sequence of marked points $p_m \in M$ such that $f(p_m) = r_m$. The manifolds $(M, \hat{g}^{(m)}(t), p_m)$ converge in the Cheeger-Gromov sense to a one-parameter family of shrinking cylinders $(S^{n-1} \times \mathbb{R}, \overline{g}(t))$, $t \in (0,1)$, where $\overline{g}(t)$ is given by (1). Furthermore, the rescaled vector fields $r_m^{\frac{1}{2}} X$ converge to the axial vector field $\frac{\partial}{\partial z}$ on $S^{n-1} \times \mathbb{R}$. Finally, the sequence $\hat{V}^{(m)}(t)$ converges in C^0_{loc} to a one-parameter family of vector fields $\overline{V}(t)$, $t \in (0,1)$, which satisfy the parabolic equation

$$\frac{\partial}{\partial t}\overline{V}(t) = \Delta_{\overline{g}(t)}\overline{V}(t) + \mathrm{Ric}_{\overline{g}(t)}(\overline{V}(t)).$$

As in [**4**], we can show that $\overline{V}(t)$ is invariant under translations along the axis of the cylinder. Moreover, the estimate (10) implies that

$$|\overline{V}(t)|_{\overline{g}(t)} \leq 1$$

for all $t \in (0, \frac{1}{2}]$. Hence, it follows from Lemma 3.1 that

$$\inf_{\lambda \in \mathbb{R}} \sup_{S^{n-1} \times \mathbb{R}} \Big|\overline{V}(t) - \lambda \frac{\partial}{\partial z}\Big|_{\overline{g}(t)} \leq L\,(1-t)^{\frac{1}{2(n-2)}} \tag{11}$$

for all $t \in (0, \frac{1}{2}]$. Finally, we have

$$\begin{aligned}
&\inf_{\lambda \in \mathbb{R}} \sup_{\Phi_{r_m(\tau-1)}(\{f = \tau r_m\})} \Big|\hat{V}^{(m)}(1-\tau) - \lambda\, r_m^{\frac{1}{2}}\, X\Big|_{\hat{g}^{(m)}(1-\tau)} \\
&= \inf_{\lambda \in \mathbb{R}} \sup_{\{f = \tau r_m\}} |\tilde{V}^{(m)} - \lambda\, X|_g \\
&= \frac{1}{A^{(m)}(r_m) + r_m^{\frac{1}{2(n-2)}-\varepsilon}} \inf_{\lambda \in \mathbb{R}} \sup_{\{f = \tau r_m\}} |V^{(m)} - \lambda\, X|_g \\
&= \frac{A^{(m)}(\tau r_m)}{A^{(m)}(r_m) + r_m^{\frac{1}{2(n-2)}-\varepsilon}} \\
&\geq \frac{1}{2}\, \tau^{\frac{1}{2(n-2)}-\varepsilon}.
\end{aligned}$$

If we send $m \to \infty$, we obtain

$$\inf_{\lambda \in \mathbb{R}} \sup_{S^{n-1} \times \mathbb{R}} \Big|\overline{V}(1-\tau) - \lambda \frac{\partial}{\partial z}\Big|_{\overline{g}(1-\tau)} \geq \frac{1}{2}\, \tau^{\frac{1}{2(n-2)}-\varepsilon}. \tag{12}$$

Since $\tau^{-\varepsilon} > 2L$, the inequality (12) is in contradiction with (11). This completes the proof of Lemma 3.3. q.e.d.

If we iterate the estimate in Lemma 3.3, we obtain

$$\sup_m \sup_{\rho_0 \leq r \leq \rho_m} r^{-\frac{1}{2(n-2)}+\varepsilon} A^{(m)}(r) < \infty.$$

From this, we deduce the following result:

Proposition 3.4. *There exists a sequence of real numbers λ_m such that*

$$\sup_m \sup_{\{f \leq \rho_m\}} f^{-\frac{1}{2(n-2)}+\varepsilon} |V^{(m)} - \lambda_m X| < \infty.$$

The proof of Proposition 3.4 is analogous to the proof of Proposition 5.4 in [**4**]. We omit the details. By taking the limit as $m \to \infty$ of the vector fields $V^{(m)} - \lambda_m X$, we obtain the following result:

Theorem 3.5. *There exists a smooth vector field V such that $\Delta V + D_X V = Q$ and $|V| \leq O(r^{\frac{1}{2(n-2)}-\varepsilon})$. Moreover, $|DV| \leq O(r^{\frac{1}{2(n-2)}-\frac{1}{2}-\varepsilon})$.*

4. Analysis of the Lichnerowicz equation

Throughout this section, we will denote by Δ_L the Lichnerowicz Laplacian; that is,

$$\Delta_L h_{ik} = \Delta h_{ik} + 2\, R_{ijkl}\, h^{jl} - \mathrm{Ric}_i^l\, h_{kl} - \mathrm{Ric}_k^l\, h_{il}.$$

Lemma 4.1. *Let us consider the shrinking cylinders $(S^{n-1} \times \mathbb{R}, \overline{g}(t))$, $t \in (0,1)$, where $\overline{g}(t)$ is given by (1). Let $\overline{h}(t)$, $t \in (0,1)$, be a one-parameter family of $(0,2)$-tensors which solve the parabolic equation*

$$\frac{\partial}{\partial t} \overline{h}(t) = \Delta_{L,\overline{g}(t)} \overline{h}(t). \tag{13}$$

Moreover, suppose that $\overline{h}(t)$ is invariant under translations along the axis of the cylinder, and

$$|\overline{h}(t)|_{\overline{g}(t)} \leq (1-t)^{-2} \tag{14}$$

for all $t \in (0, \frac{1}{2}]$. Then

$$\inf_{\lambda \in \mathbb{R}} \sup_{S^{n-1} \times \mathbb{R}} \big|\overline{h}(t) - \lambda\, \mathrm{Ric}_{\overline{g}(t)}\big|_{\overline{g}(t)} \leq N\,(1-t)^{\frac{1}{2(n-2)}-\frac{1}{2}}$$

for all $t \in [\frac{1}{2}, 1)$, where N is a positive constant.

Proof. Since $\overline{h}(t)$ is invariant under translations along the axis of the cylinder, we may write

$$\overline{h}(t) = \chi(t) + dz \otimes \sigma(t) + \sigma(t) \otimes dz + \beta(t)\, dz \otimes dz$$

for $t \in (0,1)$, where $\chi(t)$ is a symmetric $(0,2)$ tensor on S^{n-1}, $\sigma(t)$ is a one-form on S^{n-1}, and $\beta(t)$ is a real-valued function on S^{n-1}. The parabolic Lichnerowicz equation (13) implies the following system of equations for $\chi(t)$, $\sigma(t)$, and $\beta(t)$:

$$\frac{\partial}{\partial t}\chi(t) = \frac{1}{(n-2)(2-2t)}\,(\Delta_{S^{n-1}}\chi(t) - 2(n-1)\,\overset{\circ}{\chi}(t)), \tag{15}$$

$$\frac{\partial}{\partial t}\sigma(t) = \frac{1}{(n-2)(2-2t)}\,(\Delta_{S^{n-1}}\sigma(t) - (n-2)\,\sigma(t)), \tag{16}$$

$$\frac{\partial}{\partial t}\beta(t) = \frac{1}{(n-2)(2-2t)}\,\Delta_{S^{n-1}}\beta(t). \tag{17}$$

Here, $\overset{\circ}{\chi}(t)$ denotes the trace-free part of $\chi(t)$ with respect to the standard metric on S^{n-1}. Using the assumption (14), we obtain

$$\sup_{S^{n-1}} |\chi(t)|_{g_{S^{n-1}}} \leq N_1, \tag{18}$$

$$\sup_{S^{n-1}} |\sigma(t)|_{g_{S^{n-1}}} \leq N_1, \tag{19}$$

$$\sup_{S^{n-1}} |\beta(t)| \leq N_1 \tag{20}$$

for each $t \in (0, \frac{1}{2}]$, where N_1 is a positive constant.

We next analyze the operator $\chi \mapsto -\Delta_{S^{n-1}}\chi + 2(n-1)\,\overset{\circ}{\chi}$, acting on symmetric $(0,2)$-tensors on S^{n-1}. The first eigenvalue of this operator is equal to 0, and the associated eigenspace is spanned by $g_{S^{n-1}}$. Moreover, the other eigenvalues of this operator are at least $n-1$ (cf. Proposition A.2 below). Hence, it follows from (15) and (18) that

$$\inf_{\lambda\in\mathbb{R}} \sup_{S^{n-1}} |\chi(t) - \lambda\, g_{S^{n-1}}|_{g_{S^{n-1}}} \leq N_2\,(1-t)^{\frac{n-1}{2(n-2)}} \tag{21}$$

for all $t \in [\frac{1}{2}, 1)$, where N_2 is a positive constant. We now consider the operator $\sigma \mapsto -\Delta_{S^{n-1}}\sigma + (n-2)\sigma$, acting on one-forms on S^{n-1}. By Proposition A.1, the first eigenvalue of this operator is at least $n-1$. Using (16) and (19), we deduce that

$$\sup_{S^{n-1}} |\sigma(t)|_{g_{S^{n-1}}} \leq N_3\,(1-t)^{\frac{n-1}{2(n-2)}} \tag{22}$$

for all $t \in [\frac{1}{2}, 1)$, where N_3 is a positive constant. Finally, using (17) and (20), we obtain

$$\sup_{S^{n-1}} |\beta(t)| \leq N_4 \tag{23}$$

for all $t \in [\frac{1}{2}, 1)$, where N_4 is a positive constant. If we combine (21), (22), and (23), the assertion follows. q.e.d.

We now study the equation $\Delta_L h + \mathscr{L}_X(h) = 0$ on (M, g), where Δ_L denotes the Lichnerowicz Laplacian defined above.

Lemma 4.2. *Let h be a solution of the Lichnerowicz-type equation*

$$\Delta_L h + \mathscr{L}_X(h) = 0$$

on the region $\{f \leq \rho\}$. Then

$$\sup_{\{f \leq \rho\}} |h| \leq C\,\rho^2 \sup_{\{f = \rho\}} |h|$$

for some uniform constant C which is independent of ρ.

Proof. It suffices to show that

$$h \leq C\,\rho^2 \Big(\sup_{\{f=\rho\}} |h|\Big)\,g \tag{24}$$

for some uniform constant C. Indeed, if (24) holds, the assertion follows by applying (24) to h and $-h$.

We now describe the proof of (24). By Proposition 2.1, we have $f^2\,\mathrm{Ric} \geq c\,g$ for some positive constant c. Therefore, the tensor $\mathrm{Ric} - \frac{c}{2}\,\rho^{-2}\,g$ is positive definite in the region $\{f \leq \rho\}$. Let θ be the smallest real number with the property that $\theta\,(\mathrm{Ric} - \frac{c}{2}\,\rho^{-2}\,g) - h$ is positive semi-definite at each point in the region $\{f \leq \rho\}$. There exists a point $p_0 \in \{f \leq \rho\}$ and an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_{p_0}M$ such that

$$\theta\,\mathrm{Ric}(e_1, e_1) - \frac{\theta\,c}{2}\,\rho^{-2} - h(e_1, e_1) = 0$$

at the point p_0. We now distinguish two cases:

Case 1: Suppose that $p_0 \in \{f < \rho\}$. In this case, we have

$$\theta\,(\Delta\mathrm{Ric})(e_1, e_1) - (\Delta h)(e_1, e_1) \geq 0$$

and

$$\theta\,(D_X\mathrm{Ric})(e_1, e_1) - (D_X h)(e_1, e_1) = 0$$

at the point p_0. Using the identity $\Delta_L h + \mathscr{L}_X(h) = 0$, we obtain

$$\begin{aligned}
0 &= (\Delta h)(e_1, e_1) + (D_X h)(e_1, e_1) + 2\sum_{i,k=1}^n R(e_1, e_i, e_1, e_k)\,h(e_i, e_k) \\
&\leq \theta\,(\Delta\mathrm{Ric})(e_1, e_1) + \theta\,(D_X\mathrm{Ric})(e_1, e_1) + 2\sum_{i,k=1}^n R(e_1, e_i, e_1, e_k)\,h(e_i, e_k) \\
&= -2\sum_{i,k=1}^n R(e_1, e_i, e_1, e_k)\,(\theta\,\mathrm{Ric}(e_i, e_k) - h(e_i, e_k)) \\
&= -\theta\,c\,\rho^{-2}\,\mathrm{Ric}(e_1, e_1) \\
&\quad - 2\sum_{i,k=1}^n R(e_1, e_i, e_1, e_k)\,\Big(\theta\,\mathrm{Ric}(e_i, e_k) - \frac{\theta\,c}{2}\,\rho^{-2}\,g(e_i, e_k) - h(e_i, e_k)\Big)
\end{aligned}$$

at the point p_0. Since (M, g) has positive sectional curvature, we have

$$\sum_{i,k=1}^{n} R(e_1, e_i, e_1, e_k) \Big(\theta \operatorname{Ric}(e_i, e_k) - \frac{\theta\, c}{2}\, \rho^{-2}\, g(e_i, e_k) - h(e_i, e_k)\Big) \geq 0.$$

Consequently, $\theta \leq 0$. This implies $h \leq 0$ at each point in the region $\{f \leq \rho\}$. Therefore, (24) is satisfied in this case.

Case 2: Suppose that $p_0 \in \{f = \rho\}$. Since $f^2 \operatorname{Ric} \geq c\, g$, we have

$$\frac{\theta\, c}{2} \leq \theta\, \rho^2 \operatorname{Ric}(e_1, e_1) - \frac{\theta\, c}{2} = \rho^2\, h(e_1, e_1) \leq \rho^2 \sup_{\{f=\rho\}} |h|.$$

Since $h \leq \theta\, (\operatorname{Ric} - \frac{c}{2}\, \rho^{-2}\, g)$, we conclude that

$$h \leq C\, \rho^2 \Big(\sup_{\{f=\rho\}} |h|\Big)\, g$$

at each point in the region $\{f \leq \rho\}$. This proves (24). q.e.d.

Lemma 4.3. *Let h be a solution of the Lichnerowicz-type equation*

$$\Delta_L h + \mathscr{L}_X(h) = 0$$

on the region $\{f \leq \rho\}$. Then

$$\sup_{\{f \leq \rho\}} f^2\, |h| \leq B\, \rho^2 \sup_{\{f=\rho\}} |h|,$$

where B is a positive constant that does not depend on ρ.

Proof. As above, it suffices to show that

$$f^2\, h \leq C\, \rho^2 \Big(\sup_{\{f=\rho\}} |h|\Big)\, g \tag{25}$$

for some uniform constant C. We now describe the proof of (25). By Proposition 2.1, we can find a compact set K such that $f \operatorname{Ric} < (1 - 3\, f^{-1}\, |\nabla f|^2)\, g$ on $M \backslash K$. Let us consider the smallest real number θ with the property that $\theta\, f^{-2}\, g - h$ is positive semi-definite at each point in the region $\{f \leq \rho\}$. By definition of θ, there exists a point $p_0 \in \{f \leq \rho\}$ and an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_{p_0} M$ such that

$$\theta\, f^{-2} - h(e_1, e_1) = 0$$

at the point p_0. Let us distinguish two cases:

Case 1: Suppose that $p_0 \in \{f < \rho\} \setminus K$. In this case, we have

$$\theta\, \Delta(f^{-2}) - (\Delta h)(e_1, e_1) \geq 0$$

and

$$\theta\, \langle X, \nabla(f^{-2}) \rangle - (D_X h)(e_1, e_1) = 0$$

at the point p_0. Using the identity $\Delta_L h + \mathscr{L}_X(h) = 0$, we obtain

$$\begin{aligned}
0 &= (\Delta h)(e_1, e_1) + (D_X h)(e_1, e_1) + 2 \sum_{i,k=1}^n R(e_1, e_i, e_1, e_k)\, h(e_i, e_k) \\
&\leq \theta\, \Delta(f^{-2}) + \theta\, \langle X, \nabla(f^{-2}) \rangle + 2 \sum_{i,k=1}^n R(e_1, e_i, e_1, e_k)\, h(e_i, e_k) \\
&= -2\theta\, f^{-3}\, (1 - 3\, f^{-1}\, |\nabla f|^2 - f\, \mathrm{Ric}(e_1, e_1)) \\
&- 2 \sum_{i,k=1}^n R(e_1, e_i, e_1, e_k)\, (\theta\, f^{-2}\, g(e_i, e_k) - h(e_i, e_k))
\end{aligned}$$

at the point p_0. Since (M, g) has positive sectional curvature, we have

$$\sum_{i,k=1}^n R(e_1, e_i, e_1, e_k)\, (\theta\, f^{-2}\, g(e_i, e_k) - h(e_i, e_k)) \geq 0,$$

hence

$$0 \leq -2\theta\, f^{-3}\, (1 - 3\, f^{-1}\, |\nabla f|^2 - f\, \mathrm{Ric}(e_1, e_1)).$$

On the other hand, we have $f\, \mathrm{Ric}(e_1, e_1) < 1 - 3\, f^{-1}\, |\nabla f|^2$ since $p_0 \in M \setminus K$. Consequently, we have $\theta \leq 0$. This implies that $h \leq 0$ at each point in the region $\{f \leq \rho\}$, and (25) is trivially satisfied.

Case 2: We next assume that $p_0 \in \{f = \rho\} \cup K$. Using Lemma 4.2, we obtain

$$\theta = f^2\, h(e_1, e_1) \leq \sup_{\{f=\rho\} \cup K} f^2\, |h| \leq C\, \rho^2 \sup_{\{f=\rho\}} |h|.$$

Since $f^2\, h \leq \theta\, g$, we conclude that

$$f^2\, h \leq C\, \rho^2 \Big(\sup_{\{f=\rho\}} |h| \Big)\, g$$

at each point in the region $\{f \leq \rho\}$. This proves (25). q.e.d.

Theorem 4.4. *Suppose that h is a solution of the Lichnerowicz-type equation*

$$\Delta_L h + \mathscr{L}_X(h) = 0$$

with the property that $|h| \leq O(r^{\frac{1}{2(n-2)} - \frac{1}{2} - \varepsilon})$. Then $h = \lambda\, \mathrm{Ric}$ for some constant $\lambda \in \mathbb{R}$.

Proof. Let us consider the function

$$A(r) = \inf_{\lambda \in \mathbb{R}} \sup_{\{f=r\}} |h - \lambda\, \mathrm{Ric}|.$$

Clearly, $A(r) \leq \sup_{\{f=r\}} |h| \leq O(r^{\frac{1}{2(n-2)} - \frac{1}{2} - \varepsilon})$. We consider two cases:

Case 1: Suppose that there exists a sequence of real numbers $r_m \to \infty$ such that $A(r_m) = 0$ for all m. In this case, we can find a sequence of real numbers λ_m such that $h - \lambda_m \operatorname{Ric} = 0$ on the level surface $\{f = r_m\}$. Using Lemma 4.3, we conclude that $h - \lambda_m \operatorname{Ric} = 0$ in the region $\{f \leq r_m\}$. Therefore, the sequence λ_m is constant. Moreover, h is a constant multiple of the Ricci tensor.

Case 2: Suppose now that $A(r) > 0$ when r is sufficiently large. Let us fix a real number $\tau \in (0, \frac{1}{2})$ such that $\tau^{-\varepsilon} > 2N\,B$, where N and B are the constants in Lemma 4.1 and Lemma 4.3, respectively. Since $A(r) \leq O(r^{\frac{1}{2(n-2)} - \frac{1}{2} - \varepsilon})$, there exists a sequence of real numbers $r_m \to \infty$ such that

$$A(r_m) \leq 2\,\tau^{\frac{1}{2} - \frac{1}{2(n-2)} + \varepsilon}\,A(\tau r_m)$$

for all m. For each m, we can find a real number λ_m such that

$$\sup_{\{f = r_m\}} |h - \lambda_m \operatorname{Ric}| = A(r_m).$$

Applying Lemma 4.3 to the tensor

$$\tilde{h}^{(m)} = \frac{1}{A(r_m)}\,(h - \lambda_m \operatorname{Ric})$$

gives

(26)

$$\sup_{\{f=r\}} |\tilde{h}^{(m)}| \leq \frac{B\,r_m^2}{r^2} \sup_{\{f=r_m\}} |\tilde{h}^{(m)}| = \frac{B\,r_m^2}{r^2\,A(r_m)} \sup_{\{f=r_m\}} |h - \lambda_m \operatorname{Ric}| = \frac{B\,r_m^2}{r^2}$$

for $r \leq r_m$.

At this point, we define

$$\hat{g}^{(m)}(t) = r_m^{-1}\,\Phi_{r_m t}^*(g)$$

and

$$\hat{h}^{(m)}(t) = r_m^{-1}\,\Phi_{r_m t}^*(\tilde{h}^{(m)}).$$

The metrics $\hat{g}^{(m)}(t)$ evolve by the Ricci flow, and the tensors $\hat{h}^{(m)}(t)$ satisfy the parabolic Lichnerowicz equation

$$\frac{\partial}{\partial t}\hat{h}^{(m)}(t) = \Delta_{L,\hat{g}^{(m)}(t)}\hat{h}^{(m)}(t).$$

Using (26), we obtain

$$\limsup_{m\to\infty} \sup_{t \in [\delta, 1-\delta]} \sup_{\{r_m - \delta^{-1}\sqrt{r_m} \leq f \leq r_m + \delta^{-1}\sqrt{r_m}\}} |\hat{h}^{(m)}(t)|_{\hat{g}^{(m)}(t)} < \infty$$

for any given $\delta \in (0, \frac{1}{2})$.

We now pass to the limit as $m \to \infty$. Let us choose a sequence of marked points $p_m \in M$ satisfying $f(p_m) = r_m$. The manifolds $(M, \hat{g}^{(m)}(t), p_m)$ converge in the Cheeger-Gromov sense to a one-parameter family of shrinking cylinders $(S^{n-1} \times \mathbb{R}, \overline{g}(t))$, $t \in (0,1)$, where $\overline{g}(t)$ is given by (1). The vector fields $r_m^{\frac{1}{2}}\,X$ converge to the axial vector field

$\frac{\partial}{\partial z}$ on $S^{n-1} \times \mathbb{R}$. Furthermore, the sequence $\hat{h}^{(m)}(t)$ converges to a one-parameter family of tensors $\overline{h}(t)$, $t \in (0,1)$, which solve the parabolic Lichnerowicz equation

$$\frac{\partial}{\partial t}\overline{h}(t) = \Delta_{L,\overline{g}(t)}\overline{h}(t).$$

As in [**4**], we can show that $\overline{h}(t)$ is invariant under translations along the axis of the cylinder. Using (26), we obtain

$$|\overline{h}(t)|_{\overline{g}(t)} \leq B\,(1-t)^{-2}$$

for all $t \in (0, \frac{1}{2}]$. Hence, Lemma 4.1 implies that

$$\inf_{\lambda \in \mathbb{R}} \sup_{S^{n-1} \times \mathbb{R}} \big|\overline{h}(t) - \lambda\,\mathrm{Ric}_{\overline{g}(t)}\big|_{\overline{g}(t)} \leq N\,B\,(1-t)^{\frac{1}{2(n-2)} - \frac{1}{2}} \tag{27}$$

for all $t \in [\frac{1}{2}, 1)$. On the other hand, we have

$$\begin{aligned}
&\inf_{\lambda \in \mathbb{R}} \sup_{\Phi_{r_m(\tau-1)}(\{f = \tau r_m\})} \Big|\hat{h}^{(m)}(1-\tau) - \lambda\,\mathrm{Ric}_{\hat{g}^{(m)}(1-\tau)}\Big|_{\hat{g}^{(m)}(1-\tau)} \\
&= \inf_{\lambda \in \mathbb{R}} \sup_{\{f = \tau r_m\}} |\tilde{h}^{(m)} - \lambda\,\mathrm{Ric}_g|_g \\
&= \frac{1}{A(r_m)} \inf_{\lambda \in \mathbb{R}} \sup_{\{f = \tau r_m\}} |h - \lambda\,\mathrm{Ric}_g|_g \\
&= \frac{A(\tau r_m)}{A(r_m)} \\
&\geq \frac{1}{2}\,\tau^{\frac{1}{2(n-2)} - \frac{1}{2} - \varepsilon}.
\end{aligned}$$

If we send $m \to \infty$, we obtain

$$\inf_{\lambda \in \mathbb{R}} \sup_{S^{n-1} \times \mathbb{R}} \big|\overline{h}(1-\tau) - \lambda\,\mathrm{Ric}_{\overline{g}(1-\tau)}\big|_{\overline{g}(1-\tau)} \geq \frac{1}{2}\,\tau^{\frac{1}{2(n-2)} - \frac{1}{2} - \varepsilon}. \tag{28}$$

Since $\tau^{-\varepsilon} > 2N\,B$, the inequality (28) contradicts (27). This completes the proof of Theorem 4.4. q.e.d.

5. Proof of Theorem 1.2

Combining Theorems 3.5 and 4.4, we obtain the following symmetry principle:

Theorem 5.1. *Suppose that U is a vector field on (M,g) such that $|\mathscr{L}_U(g)| \leq O(r^{\frac{1}{2(n-2)} - \frac{1}{2} - 2\varepsilon})$ and $|\Delta U + D_X U| \leq O(r^{\frac{1}{2(n-2)} - 1 - 2\varepsilon})$ for some small constant $\varepsilon > 0$. Then there exists a vector field $\hat{U}$ on (M,g) such that $\mathscr{L}_{\hat{U}}(g) = 0$, $[\hat{U}, X] = 0$, $\langle \hat{U}, X\rangle = 0$, and $|\hat{U} - U| \leq O(r^{\frac{1}{2(n-2)} - \varepsilon})$.*

Proof. In view of Theorem 3.5, the equation

$$\Delta V + D_X V = \Delta U + D_X U$$

has a smooth solution which satisfies the bounds $|V| \leq O(r^{\frac{1}{2(n-2)}-\varepsilon})$ and $|DV| \leq O(r^{\frac{1}{2(n-2)}-\frac{1}{2}-\varepsilon})$. Hence, the vector field $W = U - V$ satisfies $\Delta W + D_X W = 0$. Using Theorem 4.1 in [**4**], we conclude that the Lie derivative $h = \mathscr{L}_W(g)$ satisfies the Lichnerowicz-type equation

$$\Delta_L h + \mathscr{L}_X(h) = 0.$$

Since $|h| \leq O(r^{\frac{1}{2(n-2)}-\frac{1}{2}-\varepsilon})$, Theorem 4.4 implies that $h = \lambda \,\mathrm{Ric}$ for some constant $\lambda \in \mathbb{R}$. Consequently, the vector field $\hat{U} := U - V - \frac{1}{2}\lambda X$ must be a Killing vector field. The identities $[\hat{U}, X] = 0$ and $\langle \hat{U}, X\rangle = 0$ follow as in [**4**]. q.e.d.

To complete the proof of Theorem 1.2, we apply Theorem 5.1 to the vector fields U_a constructed in Proposition 2.6. Consequently, there exist vector fields $\hat{U}_a$, $a \in \{1, \ldots, \frac{n(n-1)}{2}\}$, on (M, g) such that $\mathscr{L}_{\hat{U}_a}(g) = 0$, $[\hat{U}_a, X] = 0$, and $\langle \hat{U}_a, X\rangle = 0$. Moreover, we have

$$\sum_{a=1}^{\frac{n(n-1)}{2}} \hat{U}_a \otimes \hat{U}_a = r\,\Big(\sum_{i=1}^{n-1} e_i \otimes e_i + O(r^{\frac{1}{2(n-2)}-\frac{1}{2}-\varepsilon})\Big),$$

where $\{e_1, \ldots, e_{n-1}\}$ is a local orthonormal frame on the level set $\{f = r\}$. This shows that (M, g) is rotationally symmetric.

6. Proof of Theorem 1.3

We now describe how Theorem 1.3 follows from Theorem 1.2. Let (M, g) be a four-dimensional steady gradient Ricci soliton which is non-flat; is κ-noncollapsed; and satisfies the pointwise pinching condition

$$0 \leq \max\{a_3, b_3, c_3\} \leq \Lambda \,\min\{a_1 + a_2, c_1 + c_2\}$$

for some constant $\Lambda \geq 1$. In particular, (M, g) has nonnegative isotropic curvature. Moreover, since the sum $R + |\nabla f|^2$ is constant, the scalar curvature of (M, g) is bounded from above; consequently, (M, g) has bounded curvature.

We next show that (M, g) has positive curvature operator. To that end, we adapt the arguments in [**12**] and [**13**]. We note that pinching estimates for ancient solutions to the Ricci flow were established in [**5**].

Lemma 6.1. *We have* $a_3 \leq (6\Lambda^2 + 1)\, a_1$ *and* $c_3 \leq (6\Lambda^2 + 1)\, a_1$.

Proof. Using the inequalities

$$\Delta a_1 + \langle X, \nabla a_1\rangle \leq -2a_2 a_3$$

and

$$\Delta a_3 + \langle X, \nabla a_3 \rangle \geq -a_3^2 - 2a_1 a_2 - b_3^2,$$

we obtain

$$\begin{aligned}
&\Delta((6\Lambda^2 + 1)\, a_1 - a_3) + \langle X, \nabla((6\Lambda^2 + 1)\, a_1 - a_3)\rangle \\
&\leq a_3^2 + 2a_1 a_2 + b_3^2 - (12\Lambda^2 + 2)\, a_2 a_3 \\
&\leq a_3^2 + b_3^2 - 12\Lambda^2\, a_2 a_3 \\
&\leq a_3^2 + b_3^2 - 3\Lambda^2\, (a_1 + a_2)^2 \\
&\leq -a_3^2.
\end{aligned}$$

Hence, the Omori-Yau maximum principle implies that $(6\Lambda^2 + 1)\, a_1 - a_3 \geq 0$. The inequality $(6\Lambda^2 + 1)\, c_1 - c_3 \geq 0$ follows similarly. q.e.d.

Lemma 6.2. *We have* $4b_3^2 \leq (a_1 + a_2)\,(c_1 + c_2)$.

Proof. Suppose that $\gamma = \sup_M \frac{2b_3}{\sqrt{(a_1+a_2)\,(c_1+c_2)}} > 1$. The function $u = \frac{1}{2}\sqrt{(a_1 + a_2)\,(c_1 + c_2)}$ satisfies

$$\begin{aligned}
&\Delta u + \langle X, \nabla u \rangle \\
&\leq -u \left[a_3 + c_3 + \frac{a_1^2 + a_2^2 + b_1^2 + b_2^2}{2(a_1 + a_2)} + \frac{c_1^2 + c_2^2 + b_1^2 + b_2^2}{2(c_1 + c_2)} \right].
\end{aligned}$$

On the other hand, we have

$$\Delta b_3 + \langle X, \nabla b_3 \rangle \geq -b_3(a_3 + c_3) - 2b_1 b_2.$$

Putting these facts together, we obtain

$$\begin{aligned}
&\Delta(\gamma u - b_3) + \langle X, \nabla(\gamma u - b_3)\rangle \\
&\leq -\gamma u \left[a_3 + c_3 + \frac{a_1^2 + a_2^2 + b_1^2 + b_2^2}{2(a_1 + a_2)} + \frac{c_1^2 + c_2^2 + b_1^2 + b_2^2}{2(c_1 + c_2)} \right] \\
&+ b_3(a_3 + c_3) + 2b_1 b_2 \\
&= -\gamma u\, \frac{(a_1 - b_1)^2 + (a_2 - b_2)^2 + 2a_2(b_2 - b_1)}{2(a_1 + a_2)} \\
&- \gamma u\, \frac{(c_1 - b_1)^2 + (c_2 - b_2)^2 + 2c_2(b_2 - b_1)}{2(c_1 + c_2)} \\
&- (\gamma u - b_3)\,(a_3 + c_3 + 2b_1) - 2b_1(b_3 - b_2).
\end{aligned}$$

Note that $\gamma u - b_3 \geq 0$ by definition of γ. Since $\gamma > 1$, we can find a positive constant δ such that

$$\begin{aligned}
3\delta\, |\mathrm{Ric}|^2 \leq \gamma u\, &\frac{(a_1 - b_1)^2 + (a_2 - b_2)^2 + 2a_2(b_2 - b_1)}{2(a_1 + a_2)} \\
+ \gamma u\, &\frac{(c_1 - b_1)^2 + (c_2 - b_2)^2 + 2c_2(b_2 - b_1)}{2(c_1 + c_2)} \\
+ (\gamma u &- b_3)\,(a_3 + c_3 + 2b_1) + 2b_1(b_3 - b_2).
\end{aligned}$$

This implies

$$\Delta(\gamma u - b_3) + \langle X, \nabla(\gamma u - b_3)\rangle \leq -3\delta\,|\mathrm{Ric}|^2,$$

hence

$$\Delta(\gamma u - b_3 - \delta\,R) + \langle X, \nabla(\gamma u - b_3 - \delta\,R)\rangle \leq -\delta\,|\mathrm{Ric}|^2.$$

Using the Omori-Yau maximum principle, we conclude that $\gamma u - b_3 - \delta\,R \geq 0$. This contradicts the definition of γ. Thus, $\gamma \leq 1$, as claimed. q.e.d.

Proposition 6.3. *We have $b_3^2 \leq a_1 c_1$.*

Proof. Suppose that $\gamma = \sup_M \frac{b_3}{\sqrt{a_1 c_1}} > 1$. The function $v = \sqrt{a_1 c_1}$ satisfies

$$\Delta v + \langle X, \nabla v\rangle \leq -v\left[\frac{a_1^2 + 2a_2 a_3 + b_1^2}{2a_1} + \frac{c_1^2 + 2c_2 c_3 + b_1^2}{2c_1}\right].$$

This implies

$$\begin{aligned}
&\Delta(\gamma v - b_3) + \langle X, \nabla(\gamma v - b_3)\rangle \\
&\leq -\gamma v\left[\frac{a_1^2 + 2a_2 a_3 + b_1^2}{2a_1} + \frac{c_1^2 + 2c_2 c_3 + b_1^2}{2c_1}\right] \\
&+ b_3(a_3 + c_3) + 2b_1 b_2 \\
&= -\gamma v\left[\frac{(a_1 - b_1)^2 + 2(a_2 - a_1)a_3}{2a_1} + \frac{(c_1 - b_1)^2 + 2(c_2 - c_1)c_3}{2c_1}\right] \\
&- (\gamma v - b_3)\,(a_3 + c_3 + 2b_1) - 2b_1(b_3 - b_2).
\end{aligned}$$

Note that $\gamma v - b_3 \geq 0$ by definition of γ. Using Lemma 6.2 and the inequality $\gamma > 1$, we obtain an estimate of the form

$$\begin{aligned}
3\delta\,|\mathrm{Ric}|^2 &\leq \gamma v\left[\frac{(a_1 - b_1)^2 + 2(a_2 - a_1)a_3}{2a_1} + \frac{(c_1 - b_1)^2 + 2(c_2 - c_1)c_3}{2c_1}\right] \\
&+ (\gamma v - b_3)\,(a_3 + c_3 + 2b_1) + 2b_1(b_3 - b_2)
\end{aligned}$$

for some positive constant δ. From this, we deduce that

$$\Delta(\gamma v - b_3) + \langle X, \nabla(\gamma v - b_3)\rangle \leq -3\delta\,|\mathrm{Ric}|^2,$$

hence

$$\Delta(\gamma v - b_3 - \delta\,R) + \langle X, \nabla(\gamma v - b_3 - \delta\,R)\rangle \leq -\delta\,|\mathrm{Ric}|^2.$$

As above, the Omori-Yau maximum principle implies that $\gamma v - b_3 - \delta\,R \geq 0$. This contradicts the definition of γ. Consequently, $\gamma \leq 1$, which proves the assertion. q.e.d.

Corollary 6.4. *The manifold (M, g) has positive curvature operator.*

Proof. The inequality $b_3^2 \leq a_1 c_1$ implies that (M, g) has nonnegative curvature operator. If (M, g) has generic holonomy group, then the strict maximum principle (cf. [**12**]) implies that (M, g) has positive curvature operator. On the other hand, if (M, g) has non-generic holonomy group, then (M, g) locally splits as a product. In this case, we can deduce from Proposition 6.3 that (M, g) is isometric to a cylinder. This contradicts the fact that (M, g) is a steady soliton. q.e.d.

Note that (M, g) satisfies restricted isotropic curvature pinching condition in [**10**]. Using the compactness theorem for ancient κ-solutions in [**10**], we obtain:

Proposition 6.5 (Chen and Zhu [**10**]). *Let p_m be a sequence of points going to infinity. Then $|\langle X, \nabla R\rangle| \leq O(1)\, R^2$ at the point p_m. Moreover, if $d(p_0, p_m)^2\, R(p_m) \to \infty$, then we have $|\nabla R| \leq o(1)\, R^{\frac{3}{2}}$ and $|\langle X, \nabla R\rangle + \frac{2}{3}\, R^2| \leq o(1)\, R^2$ at the point p_m.*

Proof. The first statement follows immediately from Proposition 3.3 in [**10**]. To prove the second statement, we consider a sequence of points p_m such that $d(p_0, p_m)^2\, R(p_m) \to \infty$. Combining the compactness theorem for ancient solutions (cf. [**10**, Corollary 3.7]) with the splitting theorem (cf. [**10**, Lemma 3.1]), we conclude that $|\nabla R| \leq o(1)\, R^{\frac{3}{2}}$, $|\Delta R| \leq o(1)\, R^2$, and $3\,|\mathrm{Ric}|^2 = (1 + o(1))\, R^2$. From this, we deduce that $-\langle X, \nabla R\rangle = \Delta R + 2\,|\mathrm{Ric}|^2 = \big(\frac{2}{3} + o(1)\big)\, R^2$, as claimed. q.e.d.

Using Proposition 6.5, it is not difficult to show that $R \to 0$ at infinity. Consequently, there exists a unique point $p_0 \in M$ where the scalar curvature attains its maximum. The point p_0 must be a critical point of the function f. Since f is strictly convex, we conclude that f grows linearly near infinity. If we integrate the inequality $|\langle X, \nabla R\rangle| \leq O(1)\, R^2$ along integral curves of X, we obtain $R \geq \frac{\Lambda_1}{d(p_0,p)}$ outside a compact set, where Λ_1 is a positive constant. Hence, Proposition 6.5 gives $|\langle X, \nabla R\rangle + \frac{2}{3}\, R^2| \leq o(1)\, R^2$. Integrating this inequality along integral curves of X, we obtain $R \leq \frac{\Lambda_2}{d(p_0,p)}$ outside a compact set. Using Lemma 3.1 in [**10**] again, we conclude that (M, g) is asymptotically cylindrical. Hence, (M, g) must be rotationally symmetric by Theorem 1.2.

Appendix A. The eigenvalues of some elliptic operators on S^{n-1}

In this section, we analyze the eigenvalues of certain elliptic operators on S^{n-1}. In the following, $g_{S^{n-1}}$ will denote the standard metric on S^{n-1} with constant sectional curvature 1.

Proposition A.1. *Let σ be a one-form on S^{n-1} satisfying*

$$\Delta_{S^{n-1}}\sigma + \mu\, \sigma = 0,$$

where $\Delta_{S^{n-1}}$ *denotes the rough Laplacian and* $\mu \in (-\infty, 1)$ *is a constant. Then* $\sigma = 0$.

Proof. For any smooth function u, we have

$$\begin{aligned}\int_{S^{n-1}} u\,\Delta_{S^{n-1}}(d^*\sigma) &= \int_{S^{n-1}} \langle d(\Delta_{S^{n-1}}u), \sigma\rangle \\ &= \int_{S^{n-1}} \langle \Delta_{S^{n-1}}(du), \sigma\rangle - (n-2)\int_{S^{n-1}} \langle du, \sigma\rangle \\ &= -(n-2+\mu)\int_{S^{n-1}} \langle du, \sigma\rangle \\ &= -(n-2+\mu)\int_{S^{n-1}} u\,d^*\sigma.\end{aligned}$$

Since u is arbitrary, we conclude that

$$\Delta_{S^{n-1}}(d^*\sigma) + (n-2+\mu)\,d^*\sigma = 0.$$

Since $n-2+\mu < n-1$, it follows that $d^*\sigma$ is constant. Consequently, $d^*\sigma = 0$ by the divergence theorem.

We next consider the tensor $S_{ik} = D_i\sigma_k + D_k\sigma_i$. Then

$$(n-2-\mu)\,\sigma_i = \Delta_{S^{n-1}}\sigma_i + (n-2)\,\sigma_i = D^k S_{ik} - \frac{1}{2}\,D_i(\operatorname{tr} S).$$

Using the identity $d^*\sigma = 0$, we obtain

$$(n-2-\mu)\int_{S^{n-1}} |\sigma|^2 = \int_{S^{n-1}} \Big(D^k S_{ik} - \frac{1}{2}\,D_i(\operatorname{tr} S)\Big)\,\sigma^i = -\frac{1}{2}\int_{S^{n-1}} |S|^2.$$

Since $n-2-\mu > 0$, we conclude that $\sigma = 0$, as claimed. q.e.d.

Proposition A.2. *Let* χ *be a symmetric* $(0,2)$*-tensor on* S^{n-1} *satisfying*

$$\Delta_{S^{n-1}}\chi - 2(n-1)\,\overset{\circ}{\chi} + \mu\,\chi = 0,$$

where $\overset{\circ}{\chi}$ *denotes the trace-free part of* χ *and* $\mu \in (-\infty, n-1)$ *is a constant. Then* χ *is a constant multiple of* $g_{S^{n-1}}$.

Proof. The trace of χ satisfies

$$\Delta_{S^{n-1}}(\operatorname{tr}\chi) + \mu\,(\operatorname{tr}\chi) = 0.$$

Since $\mu < n-1$, we conclude that $\operatorname{tr}\chi$ is constant. Moreover, the trace-free part of χ satisfies

$$\Delta_{S^{n-1}}\overset{\circ}{\chi} + (\mu - 2(n-1))\,\overset{\circ}{\chi} = 0.$$

Since $\mu - 2(n-1) < 0$, it follows that $\overset{\circ}{\chi} = 0$. Putting these facts together, the assertion follows. q.e.d.

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