

ON THE HOLONOMIC RANK PROBLEM

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SRINIVAS & SHING-TUNG YAU**Abstract**

A tautological system, introduced in [17, 18], arises as a regular holonomic system of partial differential equations that govern the period integrals of a family of complete intersections in a complex manifold X , equipped with a suitable Lie group action. In this article, we introduce two formulas—one purely algebraic, the other geometric—to compute the rank of the solution sheaf of such a system for CY hypersurfaces in a generalized flag variety. The algebraic version gives the local solution space as a Lie algebra homology group, while the geometric one as the middle de Rham cohomology of the complement of a hyperplane section in X . We use both formulas to find certain degenerate points for which the rank of the solution sheaf becomes 1. These rank 1 points appear to be good candidates for the so-called large complex structure limits in mirror symmetry. The formulas are also used to prove a conjecture of Hosono, Lian, and Yau, on the completeness of the extended GKZ system when X is \mathbb{P}^n [12].

1. Introduction

Let X be a compact complex manifold, such that the complete linear system of anticanonical divisors in X is base point free. In [18], the period integrals of the corresponding universal family of CY hypersurfaces is studied. It is shown that they satisfy a certain system of partial differential equations defined on the affine space $V^* = \Gamma(X, \omega_X^{-1})$, which we call a *tautological system*. When X is a homogeneous manifold of a semi-simple Lie group G , such a system can be explicitly described. For example, one description says that the tautological system can be generated by the vector fields corresponding to the linear G action on V^* , together with a set of quadratic differential operators corresponding to the defining relations of X in $\mathbb{P}V$ under the Plücker embedding. The case where X is a Grassmannian has been worked out in detail [17].

Definition 1.1. [17, 18] *Let \hat{G} be a complex Lie group, let $Z : \hat{G} \rightarrow \text{Aut } V$ be a given holomorphic representation such that $Z(\hat{G})$ contains*

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$\mathbb{C}^\times 1_V$, and let $Z : \hat{\mathfrak{g}} \rightarrow \text{End } V$ be the corresponding Lie algebra representation. Let $\hat{X} \subset V$ be a \hat{G} -stable subvariety, and let $\beta : \hat{\mathfrak{g}} \rightarrow \mathbb{C}$ be a Lie algebra homomorphism. The tautological system $\tau(\hat{X}, V, \hat{G}, \beta)$ is the differential system generated by the operators

$$\begin{aligned} Z_\beta(x) &:= Z(x) + \beta(x), \quad x \in \hat{\mathfrak{g}}, \\ p(\partial_\zeta), \quad p(\zeta) &\in I(\hat{X}, V). \end{aligned}$$

Here $\partial_\zeta \in \text{Der } \mathbb{C}[V^*]$ is defined by $\partial_\zeta \cdot a = \langle \zeta, a \rangle$ ($a \in V$, $\zeta \in V^*$); $I(\hat{X}, V) \subset \mathbb{C}[V]$ is the defining ideal of $\hat{X} \subset V$.

Note that in the definition, we can view $Z(x) \in \text{End } V$ as a differential operator on V^* because $\text{End } V \subset \text{Der } (\text{Sym } V) = \text{Der } \mathbb{C}[V^*]$.

There are a number of important special cases of this definition that have been extensively studied in various contexts, and we shall now briefly discuss some examples.

Let X be a complete toric variety with a dense torus T , and let L be a base point free line bundle on X . Put $V = \Gamma(X, L)^*$, let $\varphi_L : X \rightarrow \mathbb{P}V^*$ be the natural T -equivariant map and \hat{X} the projective cone over its image. Put $\hat{T} = T \times \mathbb{G}_m$, and let $\beta : \hat{\mathfrak{t}} \rightarrow \mathbb{C}$ be any character of the Lie algebra $\hat{\mathfrak{t}}$ of \hat{T} . Then $\tau(\hat{X}, V, \hat{T}, \beta)$ is a GKZ hypergeometric system [10]. We can also replace T by $\text{Aut } X$, in which case our tautological system becomes an extended GKZ hypergeometric system [12]. We can further specialize either system to the case $L = \omega_X^{-1}$. It is well known that both of these systems are important tools for studying and computing period integrals of CY hypersurfaces in toric varieties, especially in the context of mirror symmetry [2, 13].

The theory of GKZ systems has been generalized by replacing the torus T with a reductive algebraic group G [16]. Let R be a holomorphic representation of G , and let $X \subset \mathbb{P}V$ be the projectivization of the closure of

$$\text{Im}(\rho : G \hookrightarrow \text{Aut } R \subset \text{End } R \cong V).$$

Then X is a spherical variety and $G \times G$ acts on X naturally. Let \hat{X} be the projective cone of X in V , and let β be any character of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathbb{C}$. In this case, our tautological system $\tau = \tau(\hat{X}, V, G \times G \times \mathbb{G}_m, \beta)$ specializes to Kapranov's hypergeometric system.

Let us return to the geometric context. Let $\pi : \mathcal{Y} \rightarrow B := \Gamma(X, \omega_X^{-1})_{sm}$ be the family of smooth CY hyperplane sections in X , and let \mathbb{H}^{top} be the Hodge bundle over B whose fiber at $f \in B$ is the line $\Gamma(Y_f, \omega_{Y_f}) \subset H^{n-1}(Y_f)$, where $n = \dim X$. In [18], the period integrals of this family are constructed by giving a canonical trivialization of \mathbb{H}^{top} . Let $\Pi = \Pi(X)$ be the period sheaf of this family, i.e., the locally constant sheaf generated by the period integrals [18, Definition 1.1].

Theorem 1.2. *The period integrals of the family are solutions to the tautological system $\mathcal{M} = \tau(\hat{X}, V, \hat{G}, (0; 1))$, where \hat{X} is the cone over X in $V = \Gamma(X, \omega_X^{-1})^*$ and $\hat{G} = \text{Aut } X \times \mathbb{G}_m$.*

This was proved in [17] for X a partial flag variety, and in full generality in [18], in which the result was also generalized to hyperplane sections of general type. Applying an argument of [16], it was also shown that if X has only a finite number of $G = \text{Aut } X$ -orbits, then \mathcal{M} is regular holonomic [17, Theorem 3.4]. In this case, if $X = \sqcup_{l=1}^r X_l$ is the decomposition into G -orbits, then the singular locus of \mathcal{M} is contained in $\cup_{l=1}^r X_l^\vee$. Here $X_l^\vee \subset V^*$ is the conical variety whose projectivization $\mathbb{P}(X_l^\vee)$ is the projective dual to the Zariski closure of X_l in X .

In the well-known applications of variation of Hodge structures in mirror symmetry, it is important to decide which solutions of our differential system come from period integrals. By Theorem 1.2, the period sheaf is a subsheaf of the solution sheaf of a tautological system. Thus an important problem is to decide when the two sheaves actually coincide. If they do not coincide, how much larger is the solution sheaf? From Hodge theory, we know that (see Proposition 6.3) the rank of the period sheaf is given by the dimension of the middle vanishing cohomology of the smooth hypersurfaces Y_f . Therefore, to answer those questions, it is desirable to know precisely the holonomic rank of our tautological system.

Let us recall what is known on these questions in a number of special cases. In the case of CY hypersurfaces in, say, a semipositive toric manifold X , it is known [10, 1] that the holonomic rank of the GKZ hypergeometric system in this case is the normalized volume of the polytope generated by the exponents of the monomial sections in $\Gamma(X, \omega_X^{-1})$. This number is also the same as the degree of the anticanonical embedding $X \hookrightarrow \mathbb{P}\Gamma(X, \omega_X^{-1})^*$. However, it is also known [12] that this number always exceeds (and is usually a lot larger than) the rank of the period sheaf. If one considers the extended GKZ hypergeometric system, where the torus T acting on X is replaced by the full automorphism group $\text{Aut } X$, one would expect that the rank of the extended system to be closer to that of the period sheaf. In fact, based on numerical evidence, it was conjectured [12] that for $X = \mathbb{P}^n$ (which lives in both the toric world and the homogeneous world), the rank of \mathcal{M} coincides with that of the period sheaf at generic points. In the case when $X = X_A$ is a spherical variety of a reductive group G corresponding to a given set of irreducible G -modules A , Kapranov [16] showed that the rank of his A -hypergeometric system is bounded above by the degree of embedding $X_A \subset \mathbb{P}M_A^*$, if the cone \hat{X}_A over X_A in M_A^* is assumed to be Cohen-Macaulay. This result was generalized to any smooth G -variety X with a finite number of G -orbits by Lian, Song, and Yau [17]. Note, however,

that the rank upper bound in each case cited above makes no assumptions about whether the underlying D-module arises from the variation of Hodge structures of CY varieties. Moreover, since the holonomic rank gives the number of independent solutions only away from the singular locus, the bound yields no information about solutions at singularities.

In this paper, we introduce two new formulas—one purely algebraic, and the other geometric—to compute the rank of the solution sheaf of a tautological system for CY hyperplanes sections in X . The algebraic formula expresses the solution space at any given point (singular or not), as the dual of a certain Lie algebra homology with coefficient in the coordinate ring of X (Theorem 2.9). The geometric formula uses the algebraic result to identify the solution space with the middle de Rham cohomology of the complement of the same CY hyperplane section in X . Based on much numerical evidence, it is conjectured that the geometric result holds for an arbitrary homogeneous variety. Our proof is valid for most familiar cases (e.g., projective spaces, Grassmannians, quadrics, spinor varieties, maximal Lagrangian Grassmannians, two exceptional varieties, full flag varieties G/B , and products of such).

We also use both formulas to find certain degenerate points for which the rank of the solution sheaf is 1. We conjecture that the rank 1 points in Theorem 8.1 in fact correspond to large complex structure limits (in the sense of [19, 11]), in the moduli space of CY hypersurfaces in X .

Conjecture 1.3. (*Holonomic rank conjecture*) *Let X be an n -dimensional projective homogeneous space of a semisimple Lie group G . Then the dimension of the solution space of the tautological system $\tau(\hat{X}, V, G \times \mathbb{G}_m, \beta)$, where $V = \Gamma(X, \omega_X^{-1})^*$, $\beta = (0; 1)$, at the point $f \in V^*$, coincides with*

$$\dim H_{dR}^n(X - Y_f).$$

In this paper, we will prove the following.

Theorem 1.4. *Assume that the natural map*

$$\mathfrak{g} \otimes \Gamma(X, \omega_X^{-r}) \rightarrow \Gamma(X, T_X \otimes \omega_X^{-r})$$

is surjective for each $r \geq 0$. Then Conjecture 1.3 holds for all $f \in V^$.*

The list of homogeneous spaces known to satisfy the condition in the theorem includes Grassmannians, full flag varieties G/B , quadrics, spinor varieties, maximal Lagrangian Grassmannians, and two exceptional X 's, as well as products of such; see Proposition 4.1. As one immediate consequence, we also deduce the following corollary.

Corollary 1.5. [12] *For $X = \mathbb{P}^n$, the tautological system*

$$\tau(\hat{X}, \Gamma(X, \omega_X^{-1})^*, SL_{n+1} \times \mathbb{G}_m, (0; 1))$$

(which is a special case of an extended GKZ system) is complete. In other words, the solutions at a generic point $f \in V^*$ are precisely the period integrals of CY hypersurfaces in X .

More generally, we will show as a corollary of Conjecture 1.3 that the question of completeness for a given X can be reduced to the vanishing of the middle primitive cohomology of X , which is essentially topological.

After the posting of this paper, Conjecture 1.3 has recently been solved by X. Zhu and two of the coauthors following a different approach [14]. Namely, they use Fourier transform and the Riemann–Hilbert correspondence to realize the de Rham cohomology as a part of the perverse sheaf corresponding to the tautological system under RH. Conjecture 1.3 then follows from a certain vanishing theorem. The conjecture has also inspired a generalization of the rank formula whereby the homogeneous variety X is replaced by an arbitrary G -variety with finitely many orbits.

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2. Solution sheaf and Lie algebra homology

We begin with the set up in [18] and consider the rank of the solution sheaf to the tautological system $\tau(\hat{X}, V, \hat{G}, \beta)$. We have a holomorphic representation

$$Z : \hat{\mathfrak{g}} \rightarrow \text{End } V$$

and its contragredient dual representation

$$Z^* : \hat{\mathfrak{g}} \rightarrow \text{End } V^*.$$

Since $\text{End } V \subset \text{Der}(\text{Sym } V) = \text{Der } \mathbb{C}[V^*]$ and $\text{End } V^* \subset \text{Der}(\text{Sym } V^*) = \text{Der } \mathbb{C}[V]$, we can view, for $x \in \hat{\mathfrak{g}}$,

$$Z(x) \in \text{Der } \mathbb{C}[V^*], \quad Z^*(x) \in \text{Der } \mathbb{C}[V].$$

Thus, by fixing a basis a_i for V and dual basis a_i^* for V^* , we can write

$$\mathbb{C}[V^*] = \mathbb{C}[a], \quad \mathbb{C}[V] = \mathbb{C}[a^*]$$

and

$$Z(x) = \sum_{i,j} x_{ji} a_j \frac{\partial}{\partial a_i}, \quad Z^*(x) = - \sum_{i,j} x_{ij} a_j^* \frac{\partial}{\partial a_i^*}.$$

Put

$$Z_\beta(x) = Z(x) + \beta(x) \quad (x \in \hat{\mathfrak{g}}).$$

Definition 2.1. Let $\mathcal{D} = \mathbb{C}[a][\partial_1, \partial_2, \dots]$ be the Weyl algebra on V^* , where $\partial_i = \frac{\partial}{\partial a_i}$, and consider the linear isomorphism

$$\Phi : \mathcal{D} \rightarrow \mathbb{C}[V \times V^*] = \mathbb{C}[a, a^*], \quad \sum_u g_u(a) \partial^u \mapsto \sum_u g_u(a) a^{*u}.$$

Let $\Psi : \mathcal{D} \rightarrow \text{End } \mathbb{C}[a, a^*]$ be the \mathcal{D} -module structure induced by Φ , i.e.,

$$\Psi(Q) \cdot q = \Phi(Q \cdot \Phi^{-1}(q)), \quad (Q \in \mathcal{D}, q \in \mathbb{C}[a, a^*]).$$

Observe that on the variables ∂_i , Φ is precisely the inverse of the Fourier transform we used to define the tautological system $\tau(\hat{X}, V, \hat{G}, \beta)$ in [17, 18]; cf. [1, Equation (4.5)].

Next, it is straightforward to check the following.

Lemma 2.2. We have $\Psi(a_i) = a_i$ (acting by left multiplication) and $\Psi(\partial_i) = \frac{\partial}{\partial a_i} + a_i^*$. Let $I = I(\hat{X}, V) \subset \mathbb{C}[a^*] = \mathbb{C}[V]$ be the vanishing ideal of X in $\mathbb{P}V$. Then the ideal $\mathbb{C}[a]I$ of the ring $\mathbb{C}[a, a^*]$ is a \mathcal{D} -submodule of $\mathbb{C}[a, a^*]$ under the action Ψ .

Lemma 2.3. The image under Φ of $\mathcal{D}\Phi^{-1}(I)$ is $\mathbb{C}[a]I$. In particular, Φ induces a \mathcal{D} -module isomorphism

$$\mathcal{D}/\mathcal{D}\Phi^{-1}(I) \cong R[a],$$

where $R = R_V := \mathbb{C}[V]/I(\hat{X}, V) = \mathbb{C}[a^*]/I$.

Proof. Since $\Phi^{-1}(I) \subset \mathbb{C}[\partial_1, \partial_2, \dots]$, we have $\Phi(\mathcal{D}\Phi^{-1}(I)) = \mathbb{C}[a, a^*]I$. The right side is $\mathbb{C}[a]I$, since I is an ideal in $\mathbb{C}[a^*]$. q.e.d.

Put

$$\mathbf{f} = \sum_i a_i a_i^* \in \mathbb{C}[V \times V^*] = \mathbb{C}[a, a^*],$$

which is the “generic” hyperplane section under the embedding $X \subset \mathbb{P}V$. Then by a straightforward calculation, we find that the following holds.

Lemma 2.4. The map $Z_{\mathbf{f}, \beta}^* : \hat{\mathfrak{g}} \rightarrow \text{End } \mathbb{C}[a, a^*]$ given by

$$x \mapsto Z_{\mathbf{f}, \beta}^*(x) = Z^*(x) + (Z^*(x)\mathbf{f}) - \beta(x) \quad (x \in \hat{\mathfrak{g}})$$

is a Lie algebra homomorphism. (Here $(Z^*(x)\mathbf{f})$ means the operator “multiplication by $Z^*(x)\mathbf{f}$.” This is not the same as the composition of the two operators $Z^*(x)$ and multiplication by \mathbf{f} .)

As we shall see later, in the case when $\hat{\mathfrak{g}}$ is a direct sum of Lie algebras $\mathfrak{g} \oplus \mathbb{C}$, the choice $\beta = (0; 1)$ and $Z^*(1)$ being the negative Euler operator on $\mathbb{C}[a^*]$ will be important for computing the holonomic rank using the method of Feynman measures. Note further that the lemma also holds true if we replace \mathbf{f} by a fixed section $f = \sum_i a_i^{(0)} a_i^* \in V^*$ and

$\mathbb{C}[a, a^*]$ by $\mathbb{C}[a^*]$ (i.e., evaluate the a_i at $a_i = a_i^{(0)} \in \mathbb{C}$), since the derivations $Z^*(x) \in \text{Der } \mathbb{C}[a^*]$ do not affect the variables a_i in the calculation leading to Lemma 2.4.

Lemma 2.5. *For $x \in \hat{\mathfrak{g}}$, $Z_{\mathbf{f},\beta}^*(x) \in \text{End } \mathcal{D}\mathbb{C}[a, a^*]$. In other words, the $\hat{\mathfrak{g}}$ -action and the \mathcal{D} -action on $\mathbb{C}[a, a^*]$ commute.*

Proof. It is obvious that $[Z_{\mathbf{f},\beta}^*(x), a_i] = 0$. We also have

$$[Z_{\mathbf{f},\beta}^*(x), \frac{\partial}{\partial a_i} + a_i^*] = 0.$$

By Lemma 2.2, it follows that $[Z_{\mathbf{f},\beta}^*(x), \Psi(\mathcal{D})] = 0$, i.e., the $\hat{\mathfrak{g}}$ -action and the \mathcal{D} -action on $\mathbb{C}[a, a^*]$ commute. q.e.d.

Lemma 2.6. *The \mathcal{D} -submodule $\mathbb{C}[a]I \subset \mathbb{C}[a, a^*]$ is also a $\hat{\mathfrak{g}}$ -submodule, where $\hat{\mathfrak{g}}$ acts via the operators $Z_{\mathbf{f},\beta}^*(x)$. Hence $\hat{\mathfrak{g}}$ acts on the quotient $R[a] = \mathbb{C}[a, a^*]/\mathbb{C}[a]I$.*

Proof. Since $I \subset \mathbb{C}[a^*] = \mathbb{C}[V]$ is the vanishing ideal of the \hat{G} invariant subvariety $\hat{X} \subset V$, and since the Lie algebra $\hat{\mathfrak{g}}$ of \hat{G} acts on $\mathbb{C}[V]$ by $Z^* : \hat{\mathfrak{g}} \rightarrow \text{Der } \mathbb{C}[V]$, it follows that $Z^*(x)I \subset I$. Since the $Z^*(x)\mathbf{f}$ acts on $\mathbb{C}[V \times V^*] = \mathbb{C}[a, a^*]$ by left multiplication, they also leave $\mathbb{C}[a]I$ stable. It follows that the $Z_{\mathbf{f},\beta}^*(x) = Z^*(x) + Z^*(x)\mathbf{f} - \beta(x)$ leave $\mathbb{C}[a]I$ stable. q.e.d.

Lemma 2.7. *For $x \in \hat{\mathfrak{g}}$, $\Phi Z_{\beta}(x) = -Z_{\mathbf{f},\beta}^*(x) \cdot 1$. Moreover, the image under Φ of $\mathcal{D}Z_{\beta}(\hat{\mathfrak{g}})$ is $Z_{\mathbf{f},\beta}^*(\hat{\mathfrak{g}}) \cdot \mathbb{C}[a, a^*]$.*

Proof. For $x \in \hat{\mathfrak{g}}$, we have

$$\begin{aligned} \Phi Z_{\beta}(x) &= \Phi\left(\sum x_{ji}a_j \frac{\partial}{\partial a_i} + \beta(x)\right) \\ &= \sum x_{ji}a_j a_i^* + \beta(x) \\ &= -Z^*(x)\mathbf{f} + \beta(x) = -Z_{\mathbf{f},\beta}^*(x) \cdot 1, \end{aligned}$$

which gives the first assertion. Since the \mathcal{D} -action on $\mathbb{C}[a, a^*]$ commutes with the $Z_{\mathbf{f},\beta}^*(x)$ by Lemma 2.5, it follows that

$$\Phi(\mathcal{D}Z_{\beta}(x)) \subset Z_{\mathbf{f},\beta}^*(x)\mathbb{C}[a, a^*].$$

Hence $\Phi(\mathcal{D}Z_{\beta}(\hat{\mathfrak{g}})) \subset Z_{\mathbf{f},\beta}^*(\hat{\mathfrak{g}})\mathbb{C}[a, a^*]$. To see the reverse inclusion, let $q \in \mathbb{C}[a, a^*]$, $x \in \hat{\mathfrak{g}}$. We have

$$\begin{aligned} Z_{\mathbf{f},\beta}^*(x)q &= Z_{\mathbf{f},\beta}^*(x)\Phi(\Phi^{-1}(q) \cdot 1) \\ &= Z_{\mathbf{f},\beta}^*(x)\Psi(\Phi^{-1}(q)) \cdot 1 \\ &= \Psi(\Phi^{-1}(q))Z_{\mathbf{f},\beta}^*(x) \cdot 1 \\ &= -\Psi(\Phi^{-1}(q))\Phi(Z_{\beta}(x)) \\ &= -\Phi(\Phi^{-1}(q)Z_{\beta}(x)) \in \Phi(\mathcal{D}Z_{\beta}(x)). \end{aligned}$$

Here the second, fourth, and last equalities follow from Definition 2.1, while the third equality follows from Lemma 2.5. This proves the reverse inclusion. q.e.d.

Recall Definition 8.1 of [18]:

$$\tau(\hat{X}, V, \hat{G}, \beta) := \mathcal{D}/(\mathcal{D}Z_\beta(\hat{\mathfrak{g}}) + \mathcal{D}\Phi^{-1}I(\hat{X}, V)).$$

Combining Lemmas 2.3, 2.6, and 2.7, we get the following.

Theorem 2.8. Φ induces a \mathcal{D} -module isomorphism

$$\tau(\hat{X}, V, \hat{G}, \beta) \cong R[a]/Z_{\mathfrak{f},\beta}^*(\hat{\mathfrak{g}})R[a],$$

where $R := \mathbb{C}[V]/I(\hat{X}, V)$ and $R[a] = R[V^*]$.

For $a^{(0)} \in \mathbb{C}^{\dim V}$, let $\hat{\mathcal{O}}_{a^{(0)}}$ be the \mathcal{D} -module of formal power series at $a^{(0)}$, and let $\mathcal{O}_{a^{(0)}}$ be the \mathcal{D} -module of convergent power series at $a^{(0)}$. Let $\mathbb{C}_{a^{(0)}} = \mathbb{C}$ be the one-dimensional $\mathbb{C}[a]$ -module such that a_i acts by $a_i^{(0)}$. As before, we put

$$f = \sum_i a_i^{(0)} a_i^* \in V^*.$$

Theorem 2.9. Suppose \hat{X} has only a finite number of \hat{G} -orbits. Put $\mathcal{M} = \tau(\hat{X}, V, \hat{G}, \beta)$. Then we have

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_{a^{(0)}}) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{a^{(0)}}) \cong H_0^{\mathrm{Lie}}(\hat{\mathfrak{g}}, R_f)^*,$$

where $\hat{\mathfrak{g}}$ acts on $R_f := \mathbb{C}[\hat{X}]$ by

$$\begin{aligned} Z_{f,\beta}^* : \hat{\mathfrak{g}} &\rightarrow \mathrm{End} R_f, \\ x &\mapsto Z^*(x) + Z^*(x)f - \beta(x). \end{aligned}$$

Part of the argument of Theorem 4.17 of [1] generalizes to our setting. The main point here is that even though the argument there which contained calculations that relied heavily on the assumptions that X is a toric variety and that the ideal I is binomial, when the argument is reinterpreted suitably, the assumptions turn out to be unnecessary. One further new observation here is that it is useful to interpret the space $R_f/Z_{f,\beta}^*(\hat{\mathfrak{g}})R_f$ as the Lie algebra homology of the $\hat{\mathfrak{g}}$ -module R_f .

Proof. By Theorem 3.4 of [17], \mathcal{M} is regular holonomic, so the first isomorphism holds [3, Proposition 14.8]. Since \mathcal{M} is a finitely generated \mathcal{D} -module, we have

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{a^{(0)}}) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_{a^{(0)}} \otimes_{\mathbb{C}[a]} \mathcal{M}, \mathbb{C}).$$

By the preceding theorem, the right side is

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_{a^{(0)}} \otimes_{\mathbb{C}[a]} R[a]/Z_{\mathfrak{f},\beta}^*(\hat{\mathfrak{g}})R[a], \mathbb{C}) &\cong \mathrm{Hom}_{\mathbb{C}}(R_f/Z_{f,\beta}^*(\hat{\mathfrak{g}})R_f, \mathbb{C}) \\ &\cong H_0^{\mathrm{Lie}}(\hat{\mathfrak{g}}, R_f)^*. \end{aligned}$$

This completes the proof. q.e.d.

Remark 2.10.

(a) Consider the linear isomorphism $R_f \rightarrow Re^f$, $\phi \mapsto \phi e^f$. Under this identification, the $\hat{\mathfrak{g}}$ -action by $Z_{f,\beta}^*(\hat{\mathfrak{g}})$ on R_f corresponds to the action

$$\hat{\mathfrak{g}} \otimes Re^f \rightarrow Re^f, \quad x \otimes \phi e^f \mapsto (Z^*(x) - \beta(x))(\phi e^f).$$

From now on, $H_*^{Lie}(\hat{\mathfrak{g}}, Re^f)$ will be understood to be the Lie algebra homology with respect to this action. We can do the same for the $\hat{\mathfrak{g}}$ -modules $I(\hat{X}, V)$ and $\mathbb{C}[V]$.

(b) We will see that writing the $\hat{\mathfrak{g}}$ -action as such allows us to use the idea of Feynman measures [4] to directly compute the holonomic rank of $\tau(\hat{X}, V, \hat{G}, \beta)$ in some cases.

(c) In a later section, we will reinterpret the Lie algebra homology in the theorem in terms of certain de Rham cohomology, in the case $X \hookrightarrow \mathbb{P}V$ where $V = \Gamma(X, \omega_X^{-1})^*$, $\hat{G} = G \times \mathbb{G}_m$ where G is semisimple and $\beta = (0; 1)$.

Example 2.11. As an application of Theorem 2.9, we will show that for $X = \mathbb{P}^n$ the period integrals of Calabi–Yau hypersurfaces form a complete set of solutions to the tautological system (Corollary 1.5). This will also turn out to be an easy consequence the geometric formula (Theorem 1.4) later.

We will eventually specialize to the Fermat case $f = x_0^{n+1} + \dots + x_n^{n+1}$, but for now f can be any smooth CY hyperplane section. First consider the period sheaf, i.e., the sheaf defined on $\Gamma(X, \omega_X^{-1})_{sm}$, that is generated by the period integrals of smooth CY hypersurfaces Y_f in X . By Proposition 6.3, it is locally constant of rank given by the dimension of the vanishing cohomology. That is

$$\nu_n := \dim H^{n-1}(Y_f) - \dim i^* H^{n-1}(X),$$

where $i : Y_f \hookrightarrow X$ is the inclusion map. By the Lefschetz hyperplane theorem, it is easy to show that for $X = \mathbb{P}^n$,

$$\nu_n = \frac{n}{n+1}(n^n - (-1)^n).$$

Since the period sheaf is a subsheaf of the solution sheaf $Sol(\mathcal{M})$ (Theorem 1.2), $\dim H_0^{Lie}(\hat{\mathfrak{g}}, Re^f) \geq \nu_n$. Thus it remains to show that

$$(2.1) \quad \dim H_0^{Lie}(\hat{\mathfrak{g}}, Re^f) \leq \nu_n.$$

Note that we can identify R with the subring of $\mathbb{C}[x] := \mathbb{C}[x_0, \dots, x_n]$ consisting of polynomials of degrees divisible by $n+1$.

Lemma 2.12. *We have*

$$(2.2) \quad \hat{\mathfrak{g}} \cdot (Re^f) = Re^f \cap \sum_i \frac{\partial}{\partial x_i} (\mathbb{C}[x]e^f).$$

Proof. Consider the action $\hat{\mathfrak{g}} \rightarrow \text{End } Re^f$, $y \mapsto Z^*(y) - \beta(y)$. Here Z^* comes from the dual of the representation $Z : \hat{\mathfrak{g}} = \mathfrak{gl}_{n+1} \rightarrow \text{End } V$. We have

$$Z^*(1) = -\frac{1}{n+1} \sum_i x_i \frac{\partial}{\partial x_i} = -\frac{1}{n+1} \sum_i \frac{\partial}{\partial x_i} x_i + 1.$$

We also have $Z^*(X_{ij}) = -x_i \frac{\partial}{\partial x_j} = -\frac{\partial}{\partial x_j} x_i$ ($i \neq j$) and $Z^*(H_i) = -x_0 \frac{\partial}{\partial x_0} + x_i \frac{\partial}{\partial x_i} = -\frac{\partial}{\partial x_0} x_0 + \frac{\partial}{\partial x_i} x_i$, where the X_{ij}, H_i form the standard basis of $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Using $\beta(\mathfrak{g}) = 0$ and $\beta(1) = 1$ (this value is crucial!), it follows easily that

$$(Z^* - \beta)(\hat{\mathfrak{g}}) = \sum_{ij} \mathbb{C} \frac{\partial}{\partial x_i} x_j.$$

This shows that the left side of (2.2) is a subspace of the right side.

To see the reverse inclusion, we consider the $\mathbb{Z}/(n+1)\mathbb{Z}$ grading on $\mathbb{C}[x]e^f$: for $p(x) \in \mathbb{C}[x]$ a degree k polynomial, the grading of $p(x)e^f$ is $k \bmod (n+1)$. Since $R \subset \mathbb{C}[x]$ is the subring generated by polynomials of degree $0 \bmod (n+1)$, both sides of (2.2) have grading $0 \bmod (n+1)$. Let A be an element on the right side of (2.2), so that it has grading $0 \bmod (n+1)$ and it has the form

$$A = \sum_i \frac{\partial}{\partial x_i} (p_i(x)e^f),$$

where $p_i(x) \in \mathbb{C}[x]$. By grouping homogeneous terms, we may as well assume that the $p_i(x)$ have polynomial degree $1 \bmod (n+1)$, which means that $p_i(x) = \sum_j x_j p_{ji}(x)$ for some $p_{ji}(x)$ of degree $0 \bmod (n+1)$. This shows that $A = \sum_{ij} \frac{\partial}{\partial x_i} x_j (p_{ji}(x)e^f)$, which lies in the left side of (2.2). This proves the reverse inclusion. q.e.d.

To complete the proof of the Corollary 1.5, we now choose

$$f = x_0^{n+1} + \cdots + x_n^{n+1}.$$

Consider an element of the form $x_0^{k_0} \cdots x_n^{k_n} e^f$ in Re^f with $k_0 \geq n$. Then

$$\begin{aligned} \frac{\partial}{\partial x_0} (x_0^{k_0-n} x_1^{k_1} \cdots x_n^{k_n} e^f) &= (n+1) x_0^{k_0} \cdots x_n^{k_n} e^f \\ &+ \frac{\partial}{\partial x_0} (x_0^{k_0-n} x_1^{k_1} \cdots x_n^{k_n}) e^f. \end{aligned}$$

By the lemma, $x_0^{k_0} \cdots x_n^{k_n} e^f$ and $\frac{\partial}{\partial x_0} (x_0^{k_0-n} x_1^{k_1} \cdots x_n^{k_n}) e^f$ represent the same element in $H_0^{Lie}(\hat{\mathfrak{g}}, Re^f)$. The analogous statement also holds for each $k_i \geq n$. It follows that any element in $H_0^{Lie}(\hat{\mathfrak{g}}, Re^f)$ can be represented by a linear combination of elements of the form $x_0^{k_0} \cdots x_n^{k_n} e^f$, where the k_i are at most $n-1$ and

$$k_0 + \cdots + k_n \equiv 0 \pmod{(n+1)}.$$

For integer $0 \leq s \leq (n+1)(n-1)$, we denote by $a(s)$ the number of integer solutions to the equation

$$k_0 + \cdots + k_n = s,$$

where the k_i are between 0 and $n-1$. By the preceding paragraph,

$$(2.3) \quad \dim H_0^{Lie}(\hat{\mathfrak{g}}, Re^f) \leq \sum_{(n+1)|s} a(s).$$

To calculate the right side, we consider the Gauss sum

$$\sum_{\lambda=0}^n \sum_{0 \leq k_i \leq n-1} e^{\frac{2\pi i(k_0 + \cdots + k_n)\lambda}{n+1}},$$

which is equal to $(n+1) \sum_{(n+1)|s} a(s)$. On the other hand, by summing over the k_i individually, this sum is equal to $\sum_{\lambda=0}^n \left(\frac{1-\xi^{n\lambda}}{1-\xi^\lambda}\right)^{n+1}$, where $\xi = e^{\frac{2\pi i}{n+1}}$. The latter sum can be calculated, and it is equal to $(n+1)\nu_n$. Therefore,

$$(2.4) \quad \sum_{(n+1)|s} a(s) = \nu_n.$$

Finally, (2.3) and (2.4) yield (2.1).

3. Lie algebra homology in geometric terms

Let X be a smooth, projective variety over \mathbb{C} , and let $\pi_L : L \rightarrow X$ be a line bundle over X . Let G be a reductive Lie group acting on (L, X) . Let \mathfrak{g} be the Lie algebra of G . \mathfrak{g} acts by derivations on the function algebra $\text{Sym } L^*$. In particular, if ℓ is a local section of L^* and f is a function on X , then for $x \in \mathfrak{g}$ we have $x(f\ell) = fx(\ell) + x(f)\ell$; i.e., \mathfrak{g} acts by relative derivations. In addition, the action is linear; i.e., \mathfrak{g} preserves the grading on $\text{Sym } L^*$.

Let $U := L - \{0\}$ be the complement of the zero section, and write $\pi = \pi_L|_U : U \rightarrow X$. We have an exact sequence of tangent bundles

$$(3.1) \quad 0 \rightarrow \pi_L^* L \rightarrow T_L \rightarrow \pi_L^* T_X \rightarrow 0.$$

Here $\pi_L^* L$ is the sheaf of tangent vectors along the fibres. Restricting (3.1) over U yields

$$(3.2) \quad 0 \rightarrow \mathcal{O}_U \rightarrow T_U \rightarrow \pi^* T_X \rightarrow 0.$$

The sheaf \mathcal{O}_U in this context has a canonical generator that is the Euler operator E given by $E(f\ell^r) = rf\ell^r$. Assuming the action of \mathfrak{g} is faithful, we have

$$(3.3) \quad \mathfrak{g} + \mathbb{C} \cdot E \subset \Gamma(U, T_U).$$

Another way to think about this is to note that $G \times \mathbb{G}_m$ acts on L . It follows that E commutes with the action of \mathfrak{g} . Assuming that \mathfrak{g} acts

faithfully on X (note even in the homogeneous space case, this is an extra hypothesis if \mathfrak{g} is not simple), we will have a sub-Lie algebra $\mathfrak{g} \oplus \mathbb{C} \cdot E \hookrightarrow \Gamma(U, T_U)$.

Because of the \mathbb{G}_m -action on U , sheaves like $\pi_* \mathcal{O}_U$, $\pi_* T_U$, and $\pi_* \Omega_U^i$ will have a grading. The exterior derivative $d : \pi_* \Omega_U^i \rightarrow \pi_* \Omega_U^{i+1}$ has degree 0.

Lemma 3.1. *$\pi^* L$ has a unique (upto \mathbb{C}^\times) non-vanishing section of degree -1 for the \mathbb{G}_m -action.*

Proof. It suffices to check with $X = \mathbb{P}^n = \text{Proj } \mathbb{C}[t_0, \dots, t_n]$, $L = \mathcal{O}_{\mathbb{P}^n}(1)$. Write $\mathbb{P}^n = \bigcup U_i$ with $U_i = \text{Spec } \mathbb{C}[t_0/t_i, \dots, t_n/t_i]$. Identify $L|_{U_i} = \mathcal{O}_{U_i}$ with transition cocycle $\sigma_{ij} = t_j/t_i$ (so the sections t_k/t_i on U_i glue to global sections). We have $\pi^{-1}(U_i) = \text{Spec } \mathbb{C}[t_0, \dots, t_n, t_i^{-1}]$. The global section of $\pi^* L$ is given by t_i^{-1} on $\pi^{-1}(U_i)$. q.e.d.

Assume now that the canonical bundle $\omega_X \cong L^{-N}$ for some $N \geq 1$. We have $\Omega_{U/X}^1 = \mathcal{O}_U \frac{d\ell}{\ell}$, so there is an isomorphism that we denote by α ,

$$(3.4) \quad \alpha : \omega_U \cong \pi^* \omega_X \cong \mathcal{O}_U[-N].$$

Note that if α_1, α_2 are two choices for such an isomorphism, then $\alpha_2 \circ \alpha_1^{-1}$ is an isomorphism $\mathcal{O}_U \rightarrow \mathcal{O}_U$ of degree 0, and hence it lies in \mathbb{C}^\times . In particular, if we assume the group G is semi-simple and hence has no abelian characters, the isomorphism α is invariant under the action of G .

Let $\dim X = n$, so $\omega_U = \Omega_U^{n+1}$. Then we define a map θ ,

$$(3.5) \quad \theta : T_U = \text{Hom}(\Omega_U^1, \mathcal{O}_U) \cong \text{Hom}(\Omega_U^1, \omega_U)[N] = \Omega_U^n[N].$$

Under this identification, the exterior derivative $d : \Omega_U^n \rightarrow \omega_U$ is identified with a map (of degree 0)

$$(3.6) \quad D := \alpha \circ d \circ \theta : T_U \rightarrow \mathcal{O}_U.$$

Now suppose given $0 \neq f \in \Gamma(X, L^N) = \Gamma(X, \omega_X^{-1})$. We have a contraction operator

$$(3.7) \quad i_{df} : T_U \rightarrow \mathcal{O}_U[N],$$

and we may consider the composition (ignoring the grading)

$$(3.8) \quad (\mathfrak{g} \oplus \mathbb{C} \cdot E) \otimes_{\mathbb{C}} \mathcal{O}_U \rightarrow T_U \xrightarrow{D+i_{df}} \mathcal{O}_U.$$

To connect with Theorem 2.9, we will now assume that L is *very ample* and G is *semi-simple*. We claim the following.

Lemma 3.2. *the resulting action*

$$(3.9) \quad (\mathfrak{g} \oplus \mathbb{C} \cdot E) \otimes_{\mathbb{C}} \Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(U, \mathcal{O}_U)$$

is given by

$$(3.10) \quad x \otimes \phi \mapsto \rho(x)\phi + \phi\rho(x)f, \quad E \otimes \phi \mapsto E(\phi) + \phi E(f) + N\phi,$$

where $x \in \mathfrak{g}$.

Proof. Let ϕ, δ be local sections of \mathcal{O}_U, T_U , respectively. We have

$$(3.11) \quad \alpha(d\phi \wedge \theta(\delta)) = \delta(\phi),$$

$$(3.12) \quad D(\phi\delta) = \phi D(\delta) + \delta(\phi).$$

Let $\rho : \mathfrak{g} \hookrightarrow \Gamma(U, T_U)$ be the map given by the action. Since the actions of G and \mathbb{G}_m on U commute, the image of ρ lies in the degree 0 part of $\Gamma(U, T_U)$. Since D has degree 0, we get $D \circ \rho(\mathfrak{g}) \subset \Gamma(U, \mathcal{O}_U)_{\deg 0} = \mathbb{C}$. Since D commutes with the action of G , we get that $D \circ \rho[\mathfrak{g}, \mathfrak{g}] = (0)$. But G semi-simple implies $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Thus $D \circ \rho = 0$.

Taking $\delta = \rho(x)$ in (3.12), where $x \in \mathfrak{g}$, we conclude that the diagram

$$(3.13) \quad \begin{array}{ccc} \Gamma(U, T_U) & \xrightarrow{D} & \Gamma(U, \mathcal{O}_U) \\ \uparrow & & \parallel \\ \mathfrak{g} \otimes \Gamma(U, \mathcal{O}_U) & \xrightarrow{\text{natural action}} & \Gamma(U, \mathcal{O}_U) \end{array}$$

commutes. This yields the first half of (3.10).

Next, we calculate $D(E)$. Let $S = \text{Spec } A$ be a non-empty open in X such that $L|_S \cong \mathcal{O}_S$. Let $U_S = \pi^{-1}(S) = \text{Spec } A[t, t^{-1}]$. (Here $t \in \mathcal{O}_U$ has degree 1.) Then $\omega_S = \mathcal{O}_S \cdot \eta$ for some n -form η , and $\alpha^{-1}(1)|_{U_S} = t^N dt/t \wedge \pi^* \eta$. Restricted to S , we have $E = td/dt$, and it is straightforward to check from (3.11) that $\theta(E) = t^N \pi^* \eta$. Thus

$$(3.14) \quad D(E) = \alpha d\theta(E) = \alpha(Nt^N dt/t \wedge \eta) = N,$$

$$(3.15) \quad D(\phi E) = N\phi + E(\phi).$$

Finally, we have $\alpha(d + df)\theta = D + \alpha(df\theta)$. For $\delta \in T_U$, it follows from (3.11) that $\alpha(df)\theta(\delta) = \delta(f)$. This yields the second half of (3.10).q.e.d.

Assume now that $\omega_X = \mathcal{O}_X(-N)$ for some $N \geq 1$. We view the various graded rings and modules as being graded mod N , and the subscript $0 \bmod N$ will refer to the sub-object of graded degree $0 \bmod N$. For example,

$$(3.16) \quad \Gamma(U, \mathcal{O}_U)_{0 \bmod N} \cong \bigoplus_{r \geq 0} \Gamma(X, \omega_X^{-r}).$$

Corollary 3.3. *Assume the following*

- (i) $\omega_X \cong L^{-N}$ for some $N \geq 1$.
- (ii) The map $\mathfrak{g} \otimes \mathcal{O}_X \rightarrow T_X$ is surjective.
- (iii) The maps $\mathfrak{g} \otimes \Gamma(X, \omega_X^{-r}) \rightarrow \Gamma(X, T \otimes \omega_X^{-r})$ are surjective for all $r \geq 0$.

Then the Lie algebra homology $H_0^{\text{Lie}}(\hat{\mathfrak{g}}, R_f)$ in Theorem 2.9 is isomorphic to

$$(3.17) \quad \text{Coker}(\Gamma(U, \Omega_U^n)_{0 \bmod N} \xrightarrow{d+df} \Gamma(U, \omega_U)_{0 \bmod N}).$$

Proof of corollary. It follows from (ii) together with (3.1) and (3.5) that $(\mathfrak{g} \oplus \mathbb{C}) \otimes \mathcal{O}_U \rightarrow T_U \cong \Omega_U^n[N]$. Moreover, the coordinate ring of $X \subset \mathbb{P}\Gamma(X, \omega_X^{-1})^*$ is

$$\mathbb{C}[\hat{X}] = \bigoplus_{r \geq 0} \Gamma(X, \omega_X^{-r}) \cong \Gamma(U, \mathcal{O}_U)_{0 \bmod N}.$$

Comparing the $\hat{\mathfrak{g}}$ -actions on both sides (Theorem 2.9 and Lemma 3.2), we see that this is an isomorphism of $\hat{\mathfrak{g}}$ -module.

Now consider the diagram of global sections viewed as graded mod N .

$$(3.18) \quad \begin{array}{ccc} \Gamma(U, \Omega_U^n)_{0 \bmod N} & \xrightarrow{d+df} & \Gamma(U, \omega_U)_{0 \bmod N} \\ \theta \uparrow \cong & & \alpha \downarrow \cong \\ \Gamma(U, T_U)_{0 \bmod N} & \xrightarrow{D+i_{df}} & \Gamma(U, \mathcal{O}_U)_{0 \bmod N} \\ \text{surj} \uparrow & & \parallel \\ (\mathfrak{g} \oplus \mathbb{C}) \otimes \Gamma(U, \mathcal{O}_U)_{0 \bmod N} & \longrightarrow & \Gamma(U, \mathcal{O}_U)_{0 \bmod N}. \end{array}$$

It follows from (iii) that the left vertical arrow in (3.18) is surjective, so the three horizontal arrows have isomorphic cokernels. q.e.d.

4. Homogeneous spaces having the surjectivity property

Concerning condition (iii) of the Corollary 3.3. Here is what we can say at the moment.

Proposition 4.1. *Let G be a semi-simple group over a field of characteristic 0, and let $P \subset G$ be a parabolic subgroup. Write $X = G/P$. Let $\mathcal{O}_X(1)$ on X be very ample with G action, and assume $\omega_X = \mathcal{O}_X(-N)$ for some $N > 0$. Write $S = \bigoplus_{r \geq 0} \Gamma(X, \mathcal{O}_X(rN))$, and let $M := \bigoplus_{r \geq 0} \Gamma(X, T_X(rN))$, so M is a graded S -module. Then M is generated in degree 0 in the following cases:*

(a) $\mathfrak{g} \cong \Gamma(X, T_X)$, and the unipotent $\mathfrak{u} \subset \mathfrak{p} := \text{Lie}(P)$ is abelian. (In [20], such X are referred to as Hermitian symmetric. Examples include Grassmannians, quadrics, spinor varieties, maximal Lagrangian Grassmannians, and two exceptional X 's, as well as products of such.)

(b) $P = B$ is a Borel.

Proof. In case (a), let $\mathfrak{l} \subset \mathfrak{p}$ be the Lie algebra of the Levi. The representation of \mathfrak{p} on $\mathfrak{g}/\mathfrak{p}$ factors through $\mathfrak{p} \twoheadrightarrow \mathfrak{l}$. In particular, this representation is completely reducible. It follows that the tangent bundle $T_X = G \times^P \mathfrak{g}/\mathfrak{p}$ breaks up as a direct sum of bundles $(T_X)_i$ associated to irreducible representations. The same will be true if we tensor with any abelian character of \mathfrak{p} . By Theorem IV, of [5], $\Gamma(X, (T_X)_i)$ is an

irreducible G -module. By assumption

$$(4.1) \quad \mathfrak{g} \cong \Gamma(X, T_X) \cong \bigoplus_i \Gamma(X, (T_X)_i),$$

so we get a decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ of the adjoint representation of G on \mathfrak{g} . Again by Bott, $\Gamma(X, (T_X)_i \otimes \omega_X^{-r})$ is G -irreducible for $r > 0$. Finally, the maps $\mathfrak{g}_i \otimes \Gamma(X, \omega_X^{-r}) \rightarrow \Gamma(X, (T_X)_i \otimes \omega_X^{-r})$ are non-zero and hence surjective since they are compatible with the G -action.

Suppose now G is arbitrary semi-simple and $P = B$ is a Borel with Lie algebra \mathfrak{b} , maximal torus T with Lie algebra \mathfrak{t} , and unipotent radical U with Lie algebra \mathfrak{u} .

Lemma 4.2. *Let M be a G -module. Then the vector bundle $G \times^P M \cong \mathcal{O}_X \otimes_{\mathbb{C}} M$.*

Proof. The bundle is obtained from $G \times M$ by identifying $(gb, m) \sim (g, bm)$. A section of $G \times M$ is given by a function $f : G \rightarrow M$. Since $(gb, f(gb)) \sim (g, bf(gb))$, this section descends to a section of $G \times^P M$ if and only if $f(gb) = b^{-1}f(g)$. If the representation of P on M lifts to G , we define for $m \in M$, $f_m(g) := g^{-1}m$. Then $f_m(gb) = b^{-1}f_m(g)$, so f_m descends to a section of the bundle. In this way, we obtain a trivialization. q.e.d.

Applying the lemma to the exact sequence of \mathfrak{b} -modules $0 \rightarrow \mathfrak{b} \rightarrow \mathfrak{g}/\mathfrak{b} \rightarrow 0$ yields an exact sequence of bundles

$$(4.2) \quad 0 \rightarrow \mathcal{V} \rightarrow \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow T_X \rightarrow 0.$$

The exact sequence $0 \rightarrow \mathfrak{u} \rightarrow \mathfrak{b} \rightarrow \mathfrak{t} \rightarrow 0$ yields another exact sequence of bundles

$$(4.3) \quad 0 \rightarrow \Omega_X^1 \rightarrow \mathcal{V} \rightarrow \mathfrak{t} \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow 0$$

(Note that the vector bundle associated to \mathfrak{u} is Ω_X^1 . Also note that \mathfrak{t} has trivial \mathfrak{b} -action and hence by the lemma the corresponding equivariant bundle is trivial.)

For the proof of proposition 4.1 we need

$$(4.4) \quad H^1(X, \mathcal{V}(rN)) = (0), \quad r \geq 0.$$

When $r = 0$, this is true since $\mathfrak{g} \cong \Gamma(X, T_X)$. Assume $r \geq 1$. The character c of \mathfrak{b} associated to $\omega_X^{-1} = \mathcal{O}_X(N)$ is the sum of all the positive roots of \mathfrak{g} . (We follow the notation of [15]: (α, β) denotes the Killing form, and $\langle \alpha, \beta \rangle := 2(\alpha, \beta)/(\beta, \beta)$.) One knows [15, p. 50] that $\langle c, \alpha \rangle = 2$ for any simple positive root α . For α, β both positive, we have $\langle \beta, \alpha \rangle \leq 3$ [15, p. 45] so $\langle c - \beta, \alpha \rangle \geq 2 - 3 = -1$. Let $\mathcal{L}(c - \beta)$ be the line bundle on G/B associated to the character $c - \beta$. We have the following consequences of Borel-Bott-Weil theory [7, corollaire 8],

- (i) If $\langle c - \beta, \alpha \rangle \geq 0$ for all positive simple α , then $H^*(X, \mathcal{L}(c - \beta)) = (0)$, $* \geq 1$.
- (ii) If there exists a simple positive α for which $\langle c - \beta, \alpha \rangle = -1$, then $H^*(X, \mathcal{L}(c - \beta)) = (0)$ for all $* \geq 0$.
- (iii) For $n \geq 2$, the character $nc - \beta$ is dominant, so $H^*(X, \mathcal{L}(nc - \beta)) = (0)$, $* \geq 1$.

Start with the identity $\Omega_X^1 = G \times^B \mathfrak{u}$; the vector bundle $\Omega_X^1(rN)$ has a filtration with quotients that are line bundles of the form $\mathcal{L}(rc - \beta)$ as above. Since all these have vanishing cohomology in degrees ≥ 1 , it follows that the same will be true for $\Omega_X^1(rN)$. Since the line bundle $\mathcal{O}_X(rN)$ corresponds to a dominant weight for $r \geq 1$, the desired vanishing (4.4) now follows from (4.3). q.e.d.

5. From Lie algebra homology to de Rham cohomology

In this section, we will prove Theorem 1.4.

With notation as above, define

$$(5.1) \quad W = X - \mathcal{V}(f).$$

It remains to interpret the cokernel in (3.17) in terms of the cohomology of W in middle degree n . For this we adapt a method of Dimca [8]. $X \subset \mathbb{P}^n$ will be a smooth, projective variety with cone $\text{Spec } R$, so $R = \mathbb{C}[x_1, \dots, x_{n+1}]/I$ for a homogeneous ideal I . Let $B = \mathbb{C}[x_1, \dots, x_{n+1}]$. $\Delta = \sum x_i \frac{\partial}{\partial x_i}$ will be the Euler operator that we view as acting by contraction $\Delta : \Omega_B^i \rightarrow \Omega_B^{i-1}$. The following properties of Δ are elementary:

Lemma 5.1. (i) $\Delta^2 = 0$. The complex $\Omega_B^{n+1} \xrightarrow{\Delta} \Omega_B^n \rightarrow \dots \rightarrow \Omega_B^1 \rightarrow B$ can be identified with the Koszul complex associated to the ideal $(x_1, \dots, x_{n+1}) \subset B$. It is acyclic away from $0 \in \text{Spec } B$.

(ii) The B -module $\Omega_B^i = \bigoplus_{r \geq i} \Omega_{B,r}^i$ is graded, with x_i and dx_i of degree 1. Δ and the exterior differential d have degree 0 for this grading, and $d\Delta + \Delta d = \mu$ is the number operator for this grading, acting by multiplication by r on $\Omega_{B,r}^i$. In particular, if $f \in B$ is homogeneous of degree N , then $\Delta df = Nf$.

(iii) For $u \in \Omega_B^i, v \in \Omega_B^j$, we have $\Delta(u \wedge v) = \Delta(u) \wedge v + (-1)^i u \wedge \Delta(v)$.

(iv) Let $I \subset B$ be a homogeneous ideal, and let $\mathcal{I}^* \subset \Omega_B^*$ be the differential ideal generated by I . Then $\Delta(\mathcal{I}^*) \subset \mathcal{I}^*[-1]$. In particular, Δ induces a map of graded sheaves $\Delta_R : \Omega_R^* \rightarrow \Omega_R^*[-1]$, where $R := B/I$.

Proof. (ii) and (iii) are proved in detail in [9, section 2.1.3 lemma]. (i) is immediate, and (iv) is clear from the last assertion in (ii). q.e.d.

Let $U = \text{Spec } R - \{0\}$ be the punctured cone. We have a \mathbb{G}_m -bundle $\pi : U \rightarrow X$. As a consequence of Lemma 5.1, Δ induces a surjection

$\Delta_U : \Omega_U^1 \rightarrow \mathcal{O}_U$, and the induced complex

$$(5.2) \quad \Omega_U^{r+1} \xrightarrow{\Delta_U} \dots \rightarrow \mathcal{O}_U$$

is the corresponding Koszul complex and is acyclic. Note $\Delta_U(\pi^*\Omega_X^1) = (0)$ (it suffices to remark for z homogeneous of degree 0 that $\Delta_U(dz) = 0 \cdot z = 0$). It follows by looking at ranks that the sequence

$$(5.3) \quad 0 \rightarrow \pi^*\Omega_X^1 \rightarrow \Omega_U^1 \xrightarrow{\Delta_U} \mathcal{O}_U \rightarrow 0$$

is exact, and from (5.2) and (5.3) that

$$(5.4) \quad \pi^*\Omega_X^i = \text{Image}(\Delta_U : \Omega_U^{i+1} \rightarrow \Omega_U^i).$$

(Alternatively, we can identify $\Delta : \Omega_U^1 \rightarrow \mathcal{O}_U$ in (5.3) with $\Omega_U^1 \rightarrow \Omega_{U/X}^1 \cong \mathcal{O}_U \cdot dt/t$. The complex (5.2) then becomes the Koszul complex on this arrow.)

We get exact sequences of sheaves on U and on X

$$(5.5) \quad 0 \rightarrow \pi^*\Omega_X^i \rightarrow \Omega_U^i \rightarrow \pi^*\Omega_X^{i-1} \rightarrow 0,$$

$$(5.6) \quad 0 \rightarrow \bigoplus_{\mathbb{Z}} \Omega_X^i(n) \rightarrow \pi_*\Omega_U^i \rightarrow \bigoplus_{\mathbb{Z}} \Omega_X^{i-1}(n) \rightarrow 0.$$

Note that $\pi_*\Omega_U^i$ is \mathbb{Z} -graded (locally $U \cong X \times \text{Spec } \mathbb{C}[t, t^{-1}]$ and we give t degree 1 and dt/t degree 0. The resulting grading is independent of the choice of t . (Better said, there is a \mathbb{G}_m -action on U/X .) The exact sequence (5.6) is compatible with the grading. For convenience we will assume $\Gamma(X, \Omega_X^{>0}(n)) = (0)$ for $n \leq 0$. It follows that for $i \geq 1$, $\Gamma(U, \Omega_U^i)$ is graded in degrees > 0 . For $\omega \in \Gamma(U, \Omega_U^i)$ homogeneous, we write $|\omega|$ for the homogeneous degree.

Let $f \in R_N$ be a non-zero homogeneous function of degree $N \geq 1$. For integers a, s, t with $0 \leq a \leq N - 1$, we define

$$(5.7) \quad B_a^{s,t} := \Gamma(U, \Omega_U^{s+t+1})_{Nt-a}.$$

(The subscript on the right refers to the homogeneous degree of the form.) Let $d' : B_a^{s,t} \rightarrow B_a^{s+1,t}$ be the exterior derivative, and define $d_a''(\omega) = -tdf \wedge \omega$ for $\omega \in B_a^{s,t}$. This defines $d_a''(\omega)$ for ω homogeneous, and we extend the definition to all forms by linearity. We have $d_a'' : B_a^{s,t} \rightarrow B_a^{s,t+1}$. Note $d'd_a'' = -d_a''d'$, so the graded vectorspace $B_a^* = \bigoplus_{s+t=*} B_a^{s,t}$ is a complex with differential $\delta_a := d' + d_a'' : B_a^* \rightarrow B_a^{*+1}$. We write

$$(5.8) \quad D_f = \delta_0 : B_0^* \rightarrow B_0^{*+1},$$

$$(5.9) \quad \sigma : B_a^{s,t} \rightarrow \Gamma(W, \Omega_W^{s+t}(-a)); \quad \sigma(\omega) = \frac{\Delta\omega}{f^t}.$$

Lemma 5.2. *Let $d_W : \Gamma(W, \Omega_W^{s+t}(-a)) \rightarrow \Gamma(W, \Omega_W^{s+t+1}(-a))$ be exterior differentiation. Then $(d_W \circ \sigma + \sigma \circ \delta_a)(\omega) = \frac{-a\omega}{f^t}$. In particular, when $a = 0$, σ induces a map on cohomology $\sigma : H^*(B_0^*, D_f) \rightarrow H_{dR}^*(W)$.*

Proof. We have for $\omega \in B_a^{s,t}$,

$$(5.10) \quad \begin{aligned} d_W \sigma(\omega) &= d_W(\Delta(\omega)/f^t) = (f d_W \Delta(\omega) - t df \wedge \Delta\omega)/f^{t+1}, \\ \sigma \delta_a(\omega) &= \sigma(d\omega - t df \wedge \omega) = \frac{f \Delta d\omega - t N f \omega + t df \wedge \Delta\omega}{f^{t+1}}. \end{aligned}$$

Combining these, and using Lemma 5.1(ii),

$$(5.11) \quad (d_W \sigma + \sigma \delta_a)(\omega) = \frac{-a\omega}{f^t}.$$

q.e.d.

Lemma 5.3. *Assume that $B_0^{i,0} = (0)$ for $i \geq 0$ (i.e., $\Gamma(X, \Omega^j) = (0)$ for $j \geq 1$.) Then B_0^* with differential $D_f = d - tdf$ is quasi-isomorphic to B_0^* with differential $d - df$.*

Proof. Define constants $\mu_{s,t}$ recursively by $\mu_{s,1} = 1$ and $t\mu_{s,t+1} = \mu_{s,t}$. (More simply, $\mu_{s,t} = 1/(t-1)!$.) The diagrams

$$(5.12) \quad \begin{array}{ccc} B_0^{s,t} & \xrightarrow{d-tdf} & B_0^{s+1,t} \oplus B_0^{s,t+1} \\ \mu_{s,t} \downarrow & & \mu_{s+1,t} \oplus \mu_{s,t+1} \downarrow \\ B_0^{s,t} & \xrightarrow{d-df} & B_0^{s+1,t} \oplus B_0^{s,t+1} \end{array}$$

all commute. Note our assumption means we need only consider the case $t \geq 1$. q.e.d.

Theorem 5.4. *We continue to assume $\Gamma(X, \Omega^i) = (0)$ for $i \geq 1$. Then the map σ induces an isomorphism on cohomology $H^*(B_0^*, D_f) \rightarrow H_{dR}^*(W)$.*

Proof. We have a decreasing filtration

$$(5.13) \quad F^p B_0^k := \bigoplus_{s \geq p} B_0^{s,k-s}; \quad D_f(F^p B_0^k) \subset F^p B_0^{k+1}.$$

We define a filtration on $\Gamma(W, \Omega_W^*)$ by

$$(5.14) \quad F^p \Gamma(W, \Omega_W^j) = \begin{cases} \{\omega/f^{j-p} \mid \omega \text{ has no pole along } f=0\} & j \geq p \\ 0 & p > j. \end{cases}$$

Again, $dF^p \subset F^p$. We have $\sigma(F^p B_0^k) \subset F^p \Gamma(W, \Omega_W^k)$ and a map of spectral sequences

$$(5.15) \quad \sigma : {}^l E_1^{p,q} := H^{p+q}(gr_F^p B_0^*) \rightarrow E_1^{p,q} := H^{p+q}(gr_F^p \Gamma(W, \Omega_W^*)).$$

Let $\eta/f^{j-p} \in F^p \Gamma(W, \Omega_W^j)$ represent a class in $H^j(gr_F^p \Gamma(W, \Omega_W^*))$. (By our hypothesis, $j > p$.) Then

$$(5.16) \quad d(\eta/f^{j-p}) = (f d\eta - (j-p)df \wedge \eta)/f^{j-p+1}$$

is divisible by f , i.e., $\theta := -\frac{|\eta|}{N} \frac{df}{f} \wedge \eta \in B_0^j$. $H^j(gr_F^p B_0^*)$ is the cohomology of the complex

$$(5.17) \quad B_0^{p,j-1-p} \xrightarrow{df \wedge} B_0^{p,j-p} \xrightarrow{df \wedge} B_0^{p,j+1-p}.$$

The form η is well defined up to a multiple of f and a form $df \wedge x$, so θ is well defined up to a form $df \wedge \xi$. Since $df \wedge \theta = 0$, we see that θ represents a well-defined class in $H^j(gr_F^p B_0^*)$, and $\sigma(\theta) = |\eta| \eta / f^{j-p}$. It follows that σ induces an isomorphism on E_1 terms, and hence on E_r -terms for any finite r . To conclude, we remark that $B_0^* = \varinjlim_{p \rightarrow -\infty} F^p B_0^*$ and $\Gamma(W, \Omega_W^*) = \varinjlim_{p \rightarrow -\infty} F^p \Gamma(W, \Omega_W^*)$. For any finite value of p , there are induced spectral sequences on F^p . For any one of these, we have $E_1^{u,v}, 'E_1^{u,v}$ vanishing for $u \ll 0$. Again σ will induce isomorphisms on E_1 . It follows that σ is a direct limit of isomorphisms and hence is an isomorphism. q.e.d.

Corollary 5.5. *We have (notation as in Corollary 3.3)*

$$(5.18) \quad H_{dR}^n(W) \cong H_0^{Lie}(\mathfrak{g} \oplus \mathbb{C} \cdot E, \Gamma(U, \mathcal{O}_U)_{0 \bmod N}) \cong H_0^{Lie}(\hat{\mathfrak{g}}, R_f).$$

Proof of Theorem 1.4. This follows immediately from Corollary 5.5 and Theorem 2.9. \square

6. Solution sheaf vs. period sheaf

As an application, we will compare the solution sheaf of our tautological system and the period sheaf of smooth CY hyperplane sections in X by giving a completeness criterion for the tautological system. We will then deduce Conjecture 1.5 as a special case.

Definition 6.1. *We say that $\mathcal{M} = \tau(\hat{X}, \Gamma(X, \omega_X^{-1})^*, G \times \mathbb{G}_m, (0; 1))$ is complete if its solutions sheaf coincides with the period sheaf $\Pi(X)$ on $\Gamma(X, \omega_X^{-1})_{sm}$.*

Corollary 6.2. *The tautological system*

$$\mathcal{M} = \tau(\hat{X}, \Gamma(X, \omega_X^{-1})^*, G \times \mathbb{G}_m, (0; 1))$$

is complete iff the primitive cohomology $H^n(X)_{prim}$ is zero.

Proof. We have the exact sequence

$$0 \rightarrow H^n(X)_{prim} \xrightarrow{j^*} H_{dR}^n(W) \xrightarrow{\text{Res}} H^{n-1}(Y_f)_{van} \rightarrow 0,$$

where $H^{n-1}(Y_f)_{van}$ is the vanishing cohomology of $Y_f = \mathcal{V}(f)$. Thus Res is an isomorphism iff $H^n(X)_{prim} = 0$. By Proposition 6.3, the dimension of the vanishing cohomology coincides with the rank of the period sheaf. By Theorem 2.9 and Corollary 5.5, this agrees with the generic rank of our system \mathcal{M} iff $H^n(X)_{prim} = 0$. q.e.d.

Proposition 6.3. *The rank of $\Pi(X)$ is equal to the dimension of the vanishing cohomology of a smooth CY hypersurface Y_f .*

Proof. Fix a smooth CY hyperplane section f . We know that the monodromy representation on the vanishing cohomology $H^{n-1}(Y_f)_{van}$ is irreducible. It follows that the monodromy action on $H_{n-1}(Y_f)/H^{n-1}(Y_f)_{van}^\perp$ is also irreducible. We have a non-zero homomorphism of representations from $H_n(Y_f)$ to the stalk of Π at f ,

$$H_{n-1}(Y_f) \rightarrow \Pi_f, \gamma \mapsto \int_\gamma \text{Res } \Omega_f,$$

where $\text{Res } \Omega_f$ is the Poincaré residue of a meromorphic form on X with pole along Y_f [18, Theorem 6.6]. Since $\text{Res } \Omega_f \in H^{n-1}(Y_f)_{van}$, it follows that $H^{n-1}(Y_f)_{van}^\perp \subset H_{n-1}(Y_f)$ lies in the kernel of map. By irreducibility, the map induces

$$H_{n-1}(Y_f)/H^{n-1}(Y_f)_{van}^\perp \cong \Pi_f.$$

q.e.d.

Corollary 6.4. *Conjecture 1.5 holds. Therefore, the generic rank of the solution sheaf of the tautological system in this case is*

$$\frac{n}{n+1}(n^n - (-1)^n).$$

Proof. For $X = \mathbb{P}^n$, Corollary 5.5 holds in this case (see Proposition 4.1), and we obviously have $H^n(X)_{prim} = 0$. So the tautological system \mathcal{M} in the preceding corollary is complete, proving Conjecture 1.5. The last assertion is an easy calculation of $\dim H^{n-1}(Y_f)$ using the Lefschetz hyperplane theorem. q.e.d.

7. A chain map

Corollary 5.5 suggests that there might be a similar relation between Lie algebra homology groups and de Rham cohomology groups in degrees other than 0 and n . In this section, we define a chain map between the complexes defining those (co)homology groups.

Recall that the Lie algebra homology of $\hat{\mathfrak{g}}$ with coefficient in R_f can be given as the homology of the chain complex $(C_*(\hat{\mathfrak{g}}, R_f), d_{CE})$, where d_{CE} is the Chevelley–Eilenberg homology differential and

$$C_p(\hat{\mathfrak{g}}, R_f) := (U\hat{\mathfrak{g}} \otimes_{\mathbb{C}} \wedge^p \hat{\mathfrak{g}}) \otimes_{U\hat{\mathfrak{g}}} R_f \cong \wedge^p \hat{\mathfrak{g}} \otimes_{\mathbb{C}} R_f.$$

We will define a chain map

$$(7.1) \quad \varphi : (C_*(\hat{\mathfrak{g}}, R_f), d_{CE}) \rightarrow (\Gamma(U, \Omega_U^{n+1-*})_{0 \bmod N}, d + df \wedge -)$$

(where N will be 1) that induces the isomorphism in (co)homology in one degree given by Corollary 3.3. Here the second complex extends (horizontally) the top row of (3.18).

As in Section 3, we choose $L = \omega_X^{-1}$, $U = L - \{0\}$, so that we have the identification $R_f = \Gamma(U, \mathcal{O}_U)$. The space U has a unique (up to scalar) G -invariant non-vanishing holomorphic top form ω_1 such that

$$h \cdot \omega_1 = h^{-1} \omega_1$$

for $h \in \mathbb{G}_m$ [18, Theorem 3.3]. Then a straightforward calculation yields the following.

Proposition 7.1. *Define (7.1) by*

$$\varphi(x_1 \wedge \cdots \wedge x_p \otimes g) = g^{i_{x_1}} \cdots i_{x_p} \omega_1,$$

where $g \in R_f$, $x_j \in \hat{\mathfrak{g}}$, and i_{x_j} denotes the contraction with the vector field generated by x_j . Then φ is a chain map. Moreover, φ is surjective iff the contraction map

$$\hat{\mathfrak{g}} \otimes \Gamma(U, \Omega_U^{n+1-p}) \rightarrow \Gamma(U, \Omega_U^{n-p}), \quad x \otimes \lambda \mapsto i_x \lambda$$

is surjective for each $p \geq 0$.

Note that the $p = 0$ surjectivity condition above is equivalent to condition (iii) of Corollary 3.3. We expect that φ is surjective in general. However, it need not induce an isomorphism on all (co)homology groups. In any case, the subcomplex $\ker(\varphi) \subset C_*(\hat{\mathfrak{g}}, R_f)$ can be described as follows.

First, note that the $C_p(\hat{\mathfrak{g}}, R_f) = \wedge^p \hat{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[\hat{X}]$ as vector spaces and φ as a linear map are both independent of f . The dependence on f is through the differential of the complex. Put $S_0 := 0$, and for $p \geq 1$ define $S_p \subset C_p(\hat{\mathfrak{g}}, R_f)$ inductively by

$$(7.2) \quad S_p := \bigcap_{g \in \Gamma(X, \omega_X^{-1})} d_g^{-1}(S_{p-1}),$$

where $d_g : C_p(\hat{\mathfrak{g}}, R_g) \rightarrow C_{p-1}(\hat{\mathfrak{g}}, R_g)$ denotes the Chevellay–Eilenberg differential for a given $g \in \Gamma(X, \omega_X^{-1})$. In other words, given $c \in C_p(\hat{\mathfrak{g}}, R_f)$, we have $c \in S_p$ iff $d_g c \in S_{p-1}$ for all $g \in \Gamma(X, \omega_X^{-1})$. Clearly, $S_* \subset C_*(\hat{\mathfrak{g}}, R_f)$ is a subcomplex. (Again, as a subspace it is clearly independent of f .)

We claim that $S_* = \ker(\varphi)$. This follows from the following standard argument:

At degree $p = 0$, $\ker(\varphi) = 0$ is obvious. To see that at degree p , $\ker(\varphi) = S_p$, note that $\ker(\varphi) \subseteq S_p$ is easy. On the other hand, for any $\beta \in S_p$, by induction $\varphi(\beta)$ is in the kernel of $d + dg \wedge \cdot$ for any $g \in H^0(X, -K_X)$, then it is easy to show that $\varphi(\beta)$ has to be zero, and thus $S_p \subseteq \ker(\varphi)$.

8. Rank 1 points for $G(2, N)$

In this section, we give an example of a rank 1 point for $G(2, N)$.

Theorem 8.1. *Let $X = G(2, N)$, $\hat{G} = SL_N \times \mathbb{G}_m$. At the hyperplane section $f = x_{1,2} \cdots x_{N-1,N} x_{N,1}$ (where the x_{ij} are the Plücker coordinates of X), the rank of the solution sheaf to $\mathcal{M} := \tau(\hat{X}, \Gamma(X, \omega_X^{-1})^*, \hat{G}, (0; 1))$ is 1.*

Proof. Write $\mathbb{C}^N = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_N$. We have $X \hookrightarrow \mathbb{P}(\wedge^2 \mathbb{C}^N)$, and X is identified with the space of decomposable tensors $v \wedge w$ up to scale. Write $v = \sum a_i e_i$, $w = \sum b_i e_i$. Then $x_{ij}(v \wedge w) = a_i b_j - a_j b_i$. We identify

$$(8.1) \quad X - \{x_{12} = 0\} \cong \text{Hom}(\mathbb{C}e_1 \oplus \mathbb{C}e_2, \bigoplus_{3 \leq i \leq N} \mathbb{C}e_i)$$

in the standard way, which amounts to taking decomposable elements $v \wedge w$ with $v = e_1 + \sum_{i \geq 3} a_i e_i$ and $w = e_2 + \sum_{i \geq 3} b_i e_i$. We want to compute

$$(8.2) \quad H^{2N-4}(X - \{x_{12} \cdots x_{N-1,N} x_{N,1} = 0\}).$$

We have

$$(8.3) \quad X - \{x_{12} \cdots x_{N-1,N} x_{N,1} = 0\} \cong \text{Spec } \mathbb{C}[a_3, b_3, \dots, a_N, b_N, \frac{1}{a_3}, \frac{1}{a_3 b_4 - a_4 b_3}, \dots, \frac{1}{a_{N-1} b_N - a_N b_{N-1}}, \frac{1}{b_N}].$$

Define $V_3 = \text{Spec } \mathbb{C}[a_3, b_3, 1/a_3] \cong \mathbb{G}_m \times \mathbb{G}_a$, where I write $\mathbb{G}_m = \mathbb{C}^\times$ for the multiplicative group and $\mathbb{G}_a = \mathbb{C}$ for the additive group. More generally, for $p \geq 4$

$$(8.4) \quad V_p := \text{Spec } \mathbb{C}[a_3, b_3, \dots, a_p, b_p, \frac{1}{a_3}, \frac{1}{a_3 b_4 - a_4 b_3}, \dots, \frac{1}{a_{p-1} b_p - a_p b_{p-1}}].$$

Let $\mathcal{G} := \mathbb{G}_m \rtimes \mathbb{G}_a$ be the group of affine transformations $x \mapsto ux + v$. Let $\pi_p : V_p \rightarrow V_{p-1}$ be the evident projection. We have

$$(8.5) \quad \pi_p^{-1}(\alpha_3, \beta_3, \dots, \alpha_{p-1}, \beta_{p-1}) = \{(\alpha_3, \beta_3, \dots, \alpha_p, \beta_p) \mid \det \begin{pmatrix} \alpha_{p-1} & \alpha_p \\ \beta_{p-1} & \beta_p \end{pmatrix} \neq 0\}.$$

The action of \mathcal{G} on V_p/V_{p-1} given by

$$(8.6) \quad (u, v) \cdot (\dots, \alpha_{p-1}, \beta_{p-1}, \alpha_p, \beta_p) = (\dots, \alpha_{p-1}, \beta_{p-1}, u\alpha_p + v\alpha_{p-1}, u\beta_p + v\beta_{p-1})$$

makes V_p a principal \mathcal{G} -bundle over V_{p-1} . But any such \mathcal{G} -bundle is split, because V_{p-1} affine implies $H^1(V_{p-1}, \mathbb{G}_a) = (0)$, and $H^1(V_{p-1}, \mathbb{G}_m) = (0)$ implies the set of \mathcal{G} -bundles on V_{p-1} which split when pushed out to \mathbb{G}_a has one element. Thus $V_p \cong V_{p-1} \times \mathbb{G}_m \times \mathbb{G}_a$ as a variety. We conclude

$$(8.7) \quad V_p \cong \mathbb{G}_m^{p-2} \times \mathbb{G}_a^{p-2}.$$

In particular,

$$(8.8) \quad H^i(V_p, \mathbb{Z}) = (0), \quad i \geq p - 1.$$

Next, define $W_p \hookrightarrow V_p$ to be the closed subvariety defined by $b_p = 0$. One gets a diagram of bundles

$$(8.9) \quad \begin{array}{ccc} W_p & \longrightarrow & V_p \\ \downarrow \mathbb{G}_m & & \downarrow \mathcal{G} \\ V_{p-1} - W_{p-1} & \longrightarrow & V_{p-1}. \end{array}$$

These are open subvarieties of affine space, so the Picard groups vanish and we have

$$(8.10) \quad W_p \cong (V_{p-1} - W_{p-1}) \times \mathbb{G}_m.$$

We prove by induction on $p \geq 3$ that

$$(8.11) \quad H^i(V_p - W_p, \mathbb{Z}) = (0); \quad i \geq 2p - 3; \quad H^{2p-4}(V_p - W_p) = \mathbb{Z}.$$

For $p = 3$, the assertions are $H^i(\mathbb{G}_m^2) = (0), i \geq 3$ and $H^2(\mathbb{G}_m^2) = \mathbb{Z}$, both of which are true. For $p > 3$, we have the Gysin sequence

$$(8.12) \quad H^i(V_p) \rightarrow H^i(V_p - W_p) \rightarrow H^{i-1}(W_p) \rightarrow H^{i+1}(V_p).$$

Since $2p - 4 \geq p - 1$ in our case, we see from (8.8), (8.10), and (8.12) that

$$(8.13) \quad H^i(V_p - W_p) \cong H^{i-1}(W_p) \cong H^{i-1}((V_{p-1} - W_{p-1}) \times \mathbb{G}_m) \cong H^{i-1}((V_{p-1} - W_{p-1}) \oplus H^{i-2}(V_{p-1} - W_{p-1})).$$

By induction we get the desired vanishing for $i \geq 2p - 3$. For $i = 2p - 4$, the same argument yields

$$(8.14) \quad H^{2p-4}(V_p - W_p) \cong H^{2p-6}(V_{p-1} - W_{p-1}) \cong \mathbb{Z}.$$

Again, we conclude by induction.

In the case $p = N$, we get from (8.14) that $H^{2N-4}(V_N - W_N) \cong \mathbb{Z}$ as desired, completing the proof. q.e.d.

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