J. DIFFERENTIAL GEOMETRY
94 (2013) 267-300

GROWTH OF WEIL-PETERSSON VOLUMES AND RANDOM HYPERBOLIC SURFACES OF LARGE GENUS

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Abstract

In this paper, we investigate the geometric properties of random hyperbolic surfaces of large genus. We describe the relationship between the behavior of lengths of simple closed geodesics on a hyperbolic surface and properties of the moduli space of such surfaces. First, we study the asymptotic behavior of Weil-Petersson volume $V_{g,n}$ of the moduli spaces of hyperbolic surfaces of genus g with n punctures as $g \to \infty$. Then we discuss basic geometric properties of a random hyperbolic surface of genus g with respect to the Weil-Petersson measure as $g \to \infty$.

1. Introduction

The moduli space $\mathcal{M}_{g,n}$ of complete hyperbolic surfaces of genus g with n punctures is equipped with a natural notion of measure, which is induced by the *Weil-Petersson* symplectic form $\omega_{g,n}$ (§2). This is the symplectic form of a Kähler noncomplete metric on $\mathcal{M}_{g,n}$.

1.1. New results. First, we discuss the main results obtained in this paper:

I): Asymptotic behavior of Weil-Petersson volumes. Peter Zograf has developed a fast algorithm for calculating the Weil-Petersson volume $V_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$, and made several conjectures on the basis of the numerical data obtained by his algorithm [**Z2**].

Conjecture 1.1 (Zograf). *For any fixed $n \geq 0$,*

$$V_{g,n} = (4\pi^2)^{2g+n-3}(2g-3+n)!\frac{1}{\sqrt{g\pi}}\left(1+\frac{c_n}{g}+O\left(\frac{1}{g^2}\right)\right)$$

as $g \to \infty$.

Here

$$V_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \omega_{g,n}^{3g-3+n}/(3g-3+n)!.$$

Received 8/10/2011.

In §3, we show:

Theorem 1.2. *For any $n \geq 0$:*

$$\frac{V_{g,n+1}}{2gV_{g,n}} = 4\pi^2 + O(\frac{1}{g}),$$

and

$$\frac{V_{g,n}}{V_{g-1,n+2}} = 1 + O(\frac{1}{g})$$

as $g \to \infty$.

These estimates imply that for any $n \geq 0$ there exists $m > 0$ such that

$$g^{-m} \leq \frac{V_{g,n}}{(4\pi^2)^{2g+n-3}(2g-3+n)!} \leq g^m. \tag{1.1}$$

II): Geometric behavior of surfaces of high genus. To simplify the notation, let $\mathcal{M}_g = \mathcal{M}_{g,0}$ and $V_g = V_{g,0}$. Given a function $F : \mathcal{M}_g \to \mathbb{R}$, let

$$\mathbb{E}^g_{X\sim wp}(F(X)) = \frac{\int_{\mathcal{M}_g} F(X)dX}{V_g},$$

where the integral is taken with respect to the Weil-Petersson volume form. Also,

$$\mathrm{Prob}^g_{wp}(F(X) \leq C) = \mathbb{E}^g_{X\sim wp}(G(X)),$$

where $G(X) = 1$ iff $F(X) \leq C$ and $G(X) = 0$ otherwise. In §4, we prove that as $g \to \infty$ the following hold:

- The probability that a random Riemann surface has a short non-separating simple closed geodesic is asymptotically positive (§4.2). More precisely, let $\ell_{sys}(X)$ denote the length of the shortest simple closed geodesic on X. Then for any small (but fixed) $\epsilon > 0$, as $g \to \infty$,
$$\mathrm{Prob}^g_{wp}(\ell_{sys}(X) < \epsilon) \asymp \epsilon^2.$$
- However, *separating* simple closed geodesics tend to be much longer (§4.3). Let $\ell^s_{sys}(X)$ denote the length of the shortest *separating* simple closed geodesic on X. We show that
$$\mathrm{Prob}^g_{wp}(\ell^s_{sys}(X) < a\log(g)) = O(\log(g)^3 g^{(a/2-1)})),$$
and
$$\mathbb{E}^g_{X\sim wp}(\ell^s_{sys}(X)) \asymp \log(g)$$
as $g \to \infty$. In fact, one can obtain upper bounds for the expected length of the shortest simple closed geodesic of a given combinatorial type. In particular, we prove that the shortest simple closed geodesic separating the surface into two roughly equal areas has length at least linear in g.

- The Cheeger constant $h(X)$ of a random Riemann surface $X \in \mathcal{M}_g$ is bounded from below by a universal constant (§4.5). More precisely, given $C < C_h = \frac{\ln(2)}{2\pi+\ln(2)}$, we have
$$\mathrm{Prob}^g_{wp}(h(X) \leq C) \to 0$$
as $g \to \infty$. By Cheeger's theorem, the smallest positive eigenvalue of the Laplacian on a generic point $X \in \mathcal{M}_g$ is asymptotically $\geq \frac{1}{4}C^2$.
- Finally, a generic hyperbolic surface in $\mathcal{M}_g$ has a small diameter, with a large embedded ball (§4.6). More precisely, as $g \to \infty$,
$$\mathrm{Prob}^g_{wp}(\mathrm{diam}(X) \geq C_d \log(g)) \to 0$$
and
$$\mathbb{E}^g_{X\sim wp}(\mathrm{diam}(X)) \asymp \log(g).$$
Also,
$$\mathrm{Prob}^g_{wp}(\mathrm{Emb}(X) \leq C_E \log(g)) \to 0$$
and
$$\mathbb{E}^g_{X\sim wp}(\mathrm{Emb}(X)) \asymp \log(g),$$
where $\mathrm{Emb}(X)$ is the radius of the largest embedded ball in X. Here $C_E = \frac{1}{6}$, and $C_d = 40$.

We remark that none of the constants in these statements are sharp.

Notation. In this paper, $f_1(g) \asymp f_2(g)$ means that there exists a constant $C > 0$ independent of g such that
$$\frac{1}{C} f_2(g) \leq f_1(g) \leq C f_2(g).$$
Similarly, $f_1(g) = O(f_2(g))$ means there exists a constant $C > 0$ independent of g such that
$$f_1(g) \leq C f_2(g).$$

1.2. Moduli spaces of hyperbolic surfaces with geodesic boundary components. The universal cover of $\mathcal{M}_{g,n}$ is the Teichmüller space $\mathcal{T}_{g,n}$. Every isotopy class of a closed curve on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ contains a unique closed geodesic. Given a homotopy class of a closed curve α on a topological surface $S_{g,n}$ of genus g with n marked points and $X \in \mathcal{T}_{g,n}$, let $\ell_\alpha(X)$ be the length of the unique geodesic in the homotopy class of α on X. This defines a length function ℓ_α on the Teichmüller space $\mathcal{T}_{g,n}$. When studying the behavior of hyperbolic length functions, it proves fruitful to consider more generally bordered hyperbolic surfaces with geodesic boundary components. Given $L = (L_1, \ldots, L_n) \in \mathbb{R}^n_+$, we consider the Teichmüller space $\mathcal{T}_{g,n}(L)$ of hyperbolic structures with geodesic boundary components of length $L_1, \ldots, L_n$. Note that a geodesic of length zero is the same as a puncture. The space $\mathcal{T}_{g,n}(L)$ is naturally equipped with a symplectic form ω_{wp}

([**Go**], [**W2**]). The Weil-Petersson volume $V_{g,n}(L)$ of $\mathcal{M}_{g,n}(L_1,\dots,L_n)$ is a polynomial in $L_1^2,\dots,L_n^2$ of degree $3g-3+n$ and $V_{g,n}=V_{g,n}(0,\dots,0)$ [**M2**].

It is crucial for the applications in this paper to understand the behavior of $V_{g,1}(L)$ as $g\to\infty$. In section 3, we will discuss the asymptotics of the coefficients of the volume polynomials $V_{g,n}(L_1,\dots,L_n)$ (see Theorem 2.3); the coefficient of $L_1^{2d_1}\dots L_n^{2d_n}$ can be written in terms of

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\cdot\omega^{3g-3+n-|\mathbf{d}|},$$

where for $1\le i\le n$, $\psi_i\in H^2(\mathcal{M}_{g,n},\mathbb{Q})$ is the first Chern class of the tautological line bundle corresponding to the i-th puncture on $X\in\mathcal{M}_{g,n}$ (§2), and $|\mathbf{d}|=d_1+\cdots+d_n$ [**M1**]. In Section 3 we apply known recursive formulas for these numbers and obtain estimates for the intersection pairings of ψ_i classes on $\overline{\mathcal{M}}_{g,n}$ as $g\to\infty$. In section 4, we will prove bounds on the integrals of certain geometric functions over $\mathcal{M}_g$ by investigating the asymptotics of the polynomials $V_{g,n}(L)$ as $g\to\infty$. Our main tool in this section is the close relationship between the Weil-Petersson geometry of $\mathcal{M}_{g,n}$ and the lengths of simple closed geodesics on surfaces in $\mathcal{M}_{g,n}$ [**M2**]. Here we discuss one application of this relationship in the case of $n=0$. Let $\mathcal{S}_g$ denote the set of homotopy classes of non-trivial simple closed curves on a compact surface S_g of genus g. For any $\gamma\in\mathcal{S}_g$, let $S_g-\gamma$ denote the surface obtained by cutting the surface S_g along γ. Given $\alpha_1,\alpha_2\in\mathcal{S}_g$, we say $\alpha_1\sim\alpha_2$ if α_1 and α_2 are of the same *type*; that is, $S_g-\alpha_1$ is homeomorphic to $S_g-\alpha_2$. Given a connected simple closed curve $\gamma\in\mathcal{S}_g$ and $f:\mathbb{R}_+\to\mathbb{R}_+$, define $f_\gamma:\mathcal{M}_g\to\mathbb{R}_+$ by

$$f_\gamma(X)=\sum_{\alpha\sim\gamma} f(\ell_\alpha(X)).$$

Then we can integrate the function f_γ with respect to the Weil-Petersson volume form. If γ is non-separating, then $S_g-\gamma\cong S_{g-1,2}$, and we have

$$\int_{\mathcal{M}_g} f_\gamma(X)\,dX=\int_0^\infty t\cdot f(t)\,V_{g-1,2}(t,t)\,dt.$$

For the general case, see Theorem 2.2. By this formula, integrating f_γ, even for a compact Riemann surface, reduces to the calculation of volumes of moduli spaces of bordered Riemann surfaces.

1.3. Remarks.

- A recursive formula for the Weil-Petersson volume of the moduli space of punctured spheres was obtained by Zograf [**Z1**]. Moreover, Zograf and Manin have obtained generating functions for the Weil-Petersson volume of $\mathcal{M}_{g,n}$[**MZ**]. See also ([**KMZ**]). The following exact asymptotic formula was proved in [**MZ**].

Theorem 1.3. *There exists* $C > 0$ *such that for any fixed* $g \geq 0$,

$$V_{g,n} = n! C^n n^{(5g-7)/2}(a_g + O(1/n)) \tag{1.2}$$

as $n \to \infty$.

Penner has developed a different method for calculating the Weil-Petersson volume of the moduli spaces of curves with marked points by using decorated Teichmüller theory [**Pe**].

- In [**Gr**], it is shown that for a fixed $n > 0$ there are $c_1, c_2 > 0$ such that
$$c_2^g (2g)! < \mathrm{Vol}_{wp}(\mathcal{M}_{g,n}) < c_1^g (2g)!.$$
This result was extended to the case of $n = 0$ in [**ST**]. Note that these estimates do not give much information about the growth of $V_{g,n}/V_{g-1,n+2}$ and $V_{g,n+1}/(2gV_{g,n})$ when $g \to \infty$.
- In a recent joint work with P. Zograf, we show [**MZ**]:

Theorem 1.4. *There exists a universal constant* $\alpha \in (0, \infty)$ *such that for any given* $k \geq 1, n \geq 0$,

$$V_{g,n} = \alpha \, \frac{(2g-3+n)! \, (4\pi^2)^{2g-3+n}}{\sqrt{g}} \left(1 + \frac{c_n^{(1)}}{g} + \cdots + \frac{c_n^{(k)}}{g^k} + O\left(\frac{1}{g^{k+1}}\right)\right),$$

as $g \to \infty$ *Each term* $c_n^{(i)}$ *in the asymptotic expansion is a polynomial in* n *of degree* $2i$ *with coefficients in* $\mathbb{Q}[\pi^{-2}, \pi^2]$ *that are effectively computable.*

- In [**BM**], Brooks and Makover developed a method for the study of *typical* Riemann surfaces with large genus by using trivalent graphs. In this model the expected value of the systole of a random Riemann surface turns out to be bounded (independent of the genus) [**MM**]. (See also [**Ga**].) We will see in this note that a random Riemann surface with respect to the Weil-Petersson volume form has similar features. However, it is not clear how the model in [**BM**] is related to the one discussed in this paper.
- The distribution of hyperbolic surfaces of genus g produced randomly by gluing Riemann surfaces with long geodesic boundary components is closely related to the volume form induced by ω on $\mathcal{M}_{g,n}$. See [**M3**] for details.

1.4. Questions.

- In general,
$$\lim_{g+n\to\infty} \frac{\log(V_{g,n})}{(2g+n)\log(2g+n)} = 1,$$

but understanding the asymptotics of $V_{g,n}$ for arbitrary g, n seems to be more complicated. It would be useful to know the asymptotics of

$$\frac{V_{g,n(g)}}{V_{g-1,n(g)+2}},$$

where $n(g) \to \infty$ as $g \to \infty$. Note that by Theorem 1.3 and Theorem 1.2, we know the asymptotics of $V_g/V_{g-1,2}$ and $V_{1,2g-4}/V_{0,2g-2}$. However, we don't know much about the behavior of the sequence

$$V_g\ ,\ V_{g-1,2}, \dots, V_{0,2g}$$

as $g \to \infty$.

- As in Theorem 2.3, when $n = 1$ the volume polynomial can be written as

$$V_{g,1}(L) = \sum_{k=0}^{3g-2} \frac{a_{g,k}}{(2k+1)!} L^{2k},$$

where $a_{g,k}$ are rational multiples of powers of π. It would be helpful to understand the asymptotics of $a_{g,k}/a_{g,k+1}$ for an arbitrary k (which can grow with g). Note that $a_{g,0} = V_{g,1}$. In Theorem 3.5(1), we show that for given $i \geq 0$,

$$\lim_{g\to\infty} \frac{a_{g,i+1}}{a_{g,i}} = 1.$$

On the other hand, it is known that [**IZ**]

$$\int_{\overline{\mathcal{M}}_{g,}} \psi_1^{3g-2} = \frac{1}{24^g g!},$$

and hence

$$\frac{a_{g,3g-2}}{a_{g,0}} \to 0$$

as $g \to \infty$.

- The results obtained in this paper are only small steps toward understanding the geometry of random hyperbolic surfaces of large genus. Many interesting questions about such random surfaces are open. Investigating geometric properties of random Riemann surfaces could shed some light on the asymptotics geometry of $\mathcal{M}_g$ as $g \to \infty$. See [**CP**], [**T**], and [**Hu**] for some results in this direction.

Acknowledgments. I would like to thank P. Zograf for many illuminating discussions regarding the growth of Weil-Petersson volumes. I am grateful to Rick Schoen and Jan Vondrak for helpful remarks. I would also like to thank the referee for pointing out a mistake in section 3 of the previous version of this paper.

The author was partially supported by NSF grant DMS 0804136.

2. Background and notation

In this section, we recall definitions and known results about the geometry of hyperbolic surfaces and properties of their moduli spaces. For more details, see [**M2**], [**Bu**], and [**W3**].

2.1. Teichmüller space. A point in the *Teichmüller space* $\mathcal{T}(S)$ is a complete hyperbolic surface X equipped with a diffeomorphism $f : S \to X$. The map f provides a *marking* on X by S. Two marked surfaces, $f : S \to X$ and $g : S \to Y$, define the same point in $\mathcal{T}(S)$ if and only if $f \circ g^{-1} : Y \to X$ is isotopic to a conformal map. When ∂S is nonempty, consider hyperbolic Riemann surfaces homeomorphic to S with geodesic boundary components of fixed length. Let $A = \partial S$ and $L = (L_\alpha)_{\alpha \in A} \in \mathbb{R}_+^{|A|}$. A point $X \in \mathcal{T}_{g,n}(L)$ is a marked hyperbolic surface with geodesic boundary components such that for each boundary component $\beta \in \partial S$, we have

$$\ell_\beta(X) = L_\beta.$$

By convention, a geodesic of length zero is a cusp and we have

$$\mathcal{T}_{g,n} = \mathcal{T}_{g,n}(0, \ldots, 0).$$

Let $\mathrm{Mod}(S)$ denote the mapping class group of S, or the group of isotopy classes of orientation preserving self homeomorphisms of S leaving each boundary component setwise fixed. The mapping class group $\mathrm{Mod}_{g,n} = \mathrm{Mod}(S_{g,n})$ acts on $\mathcal{T}_{g,n}(L)$ by changing the marking. The quotient space

$$\mathcal{M}_{g,n}(L) = \mathcal{M}(S_{g,n}, \ell_{\beta_i} = L_i) = \mathcal{T}_{g,n}(L_1, \ldots, L_n)/\operatorname{Mod}_{g,n}$$

is the moduli space of Riemann surfaces homeomorphic to $S_{g,n}$ with n boundary components of length $\ell_{\beta_i} = L_i$. Also, we have

$$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0, \ldots, 0).$$

By the work of Goldman [**Go**], the space $\mathcal{T}_{g,n}(L_1, \ldots, L_n)$ carries a natural symplectic form invariant under the action of the mapping class group. This symplectic form is called the *Weil-Petersson symplectic form,* and is denoted by ω or ω_{wp}. When $L_1 = \cdots = L_n = 0$, this symplectic form is the Kähler form of a Kähler metric on $\mathcal{M}_{g,n}$ [**IT**].

The Fenchel-Nielsen coordinates. A *pants decomposition* of S is a set of disjoint simple closed curves that decompose the surface into pairs of pants. Fix a system of pants decomposition of $S_{g,n}$, $\mathcal{P} = \{\alpha_i\}_{i=1}^k$, where $k = 3g - 3 + n$. For a marked hyperbolic surface $X \in \mathcal{T}_{g,n}(L)$, the *Fenchel-Nielsen coordinates* associated with $\mathcal{P}$, $\{\ell_{\alpha_1}(X), \ldots, \ell_{\alpha_k}(X), \tau_{\alpha_1}(X), \ldots, \tau_{\alpha_k}(X)\}$ consist of the set of lengths of all geodesics used in the decomposition and the set of the *twisting* parameters used to glue the pieces. We have an isomorphism

$$\mathcal{T}_{g,n}(L) \cong \mathbb{R}_+^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}}$$

by the map

$$X \to (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X)).$$

See [**Bu**] for more details. By the work of Wolpert, over Teichmüller space the Weil-Petersson symplectic structure has a simple form in Fenchel-Nielsen coordinates [**W1**], [**Go**]:

Theorem 2.1 (Wolpert)**.** *The Weil-Petersson symplectic form is given by*

$$\omega_{wp} = \sum_{i=1}^{k} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

By Theorem 2.1, the natural twisting around α is the Hamiltonian flow of the length function of α.

2.2. Integrating geometric functions over moduli spaces. Here, we discuss a method for integrating certain geometric functions over $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson volume form [**M2**]. As in the introduction, let $\mathcal{S}_{g,n}$ denote the set of homotopy classes of non-trivial, non-peripheral, simple closed curves on the surface $S_{g,n}$. Let $\Gamma = (\gamma_1, \ldots, \gamma_k)$, where γ_i's are distinct and disjoint elements of $\mathcal{S}_{g,n}$. To each Γ, we associate the set

$$\mathcal{O}_\Gamma = \{(g \cdot \gamma_1, \ldots, g \cdot \gamma_k) \mid g \in \mathrm{Mod}_{g,n}\}.$$

Given a function $F : \mathbb{R}_+^k \to \mathbb{R}_+$, define

$$F^\Gamma : \mathcal{M}_{g,n} \to \mathbb{R}$$

by

$$F^\Gamma(X) = \sum_{(\alpha_1,\ldots,\alpha_k)\in\mathcal{O}_\Gamma} F(\ell_{\alpha_1}(X), \ldots, \ell_{\alpha_k}(X)). \tag{2.1}$$

Let $S_{g,n}(\Gamma)$ be the result of cutting the surface $S_{g,n}$ along $\gamma_1, \ldots, \gamma_k$. In fact, $S_{g,n}(\Gamma) \cong S_{g,n} - U_\Gamma$, where U_Γ is an open neighborhood of $\gamma_1 \cup \cdots \cup \gamma_k$ homeomorphic to $\cup_{i=1}^k \gamma_i \times (0,1)$. Thus $S_{g,n}(\Gamma)$ is a (possibly disconnected) compact surface with $n + 2k$ boundary components; each γ_i gives rise to two boundary components γ_i^1 and γ_i^2 of $S_{g,n}(\Gamma)$. Given $\mathbf{x} = (x_1, \ldots, x_k)$ with $x_i \geq 0$, we consider the moduli space $\mathcal{M}(S_{g,n}(\Gamma), \ell_\Gamma = \mathbf{x})$ of hyperbolic Riemann surfaces homeomorphic to $S_{g,n}(\Gamma)$ such that for $1 \leq i \leq k$, $\ell_{\gamma_i^1} = x_i$ and $\ell_{\gamma_i^2} = x_i$. Given $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}_+^k$, $V_{g,n}(\Gamma, \mathbf{x})$ is defined by

$$V_{g,n}(\Gamma, \mathbf{x}) = \mathrm{Vol}_{wp}(\mathcal{M}(S_{g,n}(\Gamma), \ell_\Gamma = \mathbf{x})).$$

In general,

$$V_{g,n}(\Gamma, \mathbf{x}) = \prod_{i=1}^{s} V_{g_i,n_i}(\ell_{A_i}),$$

where

$$S_{g,n}(\Gamma) = \bigcup_{i=1}^{s} S_i, \tag{2.2}$$

$S_i \cong S_{g_i,n_i}$, and $A_i = \partial S_i$. Then in terms of the above notation, we have ([**M2**]):

Theorem 2.2. *For any $\Gamma = (\gamma_1, \dots, \gamma_k)$, the integral of F^Γ over $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson volume form is given by*

$$\int\limits_{\mathcal{M}_{g,n}} F^\Gamma(X)\, dX = 2^{-M(\gamma)} \int\limits_{\mathbf{x}\in\mathbb{R}_+^k} F(x_1, \dots, x_k)\, V_{g,n}(\Gamma, \mathbf{x})\, \mathbf{x}\cdot d\mathbf{x},$$

where $\mathbf{x}\cdot d\mathbf{x} = x_1 \cdots x_k \cdot dx_1 \wedge \cdots \wedge dx_k$, and

$$M(\gamma) = |\{i|\gamma_i \quad \text{separates off a one-handle from } \ S_{g,n}\}|.$$

Remark. Given a multicurve $\gamma = \sum_{i=1}^k c_i\gamma_i$, the symmetry group of γ, $\mathrm{Sym}(\gamma)$, is defined by

$$\mathrm{Sym}(\gamma) = \mathrm{Stab}(\gamma)/\cap_{i=1}^k \mathrm{Stab}(\gamma_i).$$

When F is a symmetric function, we can define

$$F_\gamma : \mathcal{M}_{g,n} \to \mathbb{R}$$

$$F_\gamma(X) = \sum_{\sum_{i=1}^k c_i\alpha_i \in \mathrm{Mod}_{g,n}\cdot\gamma} F(c_1\ell_{\alpha_1}(X), \dots, c_k\ell_{\alpha_k}(X)).$$

Then it is easy to check that

$$F^\Gamma(X) = \mathrm{Sym}(\gamma)\cdot F_\gamma(X), \tag{2.3}$$

where $\Gamma = (c_1\gamma_1, \dots, c_k\gamma_k)$.

2.3. Connection with the intersection pairings of tautological line bundles. The moduli space $\mathcal{M}_{g,n}$ is endowed with natural cohomology classes. When $n > 0$, there are n tautological line bundles defined on $\overline{\mathcal{M}}_{g,n}$ as follows. We can define $\mathcal{L}_i$ in the orbifold sense whose fiber at the point $(\mathcal{C}, x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space of $\mathcal{C}$ at x_i. Then $\psi_i = c_1(\mathcal{L}_i) \in H_2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Note that although the complex curve $\mathcal{C}$ may have nodes, x_i never coincides with the singular points. See [**HM**] and [**AC**] for more details. Then we have [**M1**]:

Theorem 2.3. *In terms of the above notation,*

$$\mathrm{Vol}_{wp}(\mathcal{M}_{g,n}(L_1, \dots, L_n)) = \sum_{|\mathbf{d}|\leq 3g-3+n} C_g(\mathbf{d})\ \ L_1^{2d_1} \dots L_n^{2d_n},$$

where $\mathbf{d} = (d_1, \dots, d_n)$, and $C_g(\mathbf{d})$ is equal to

$$\frac{2^{m(g,n)|\mathbf{d}|}}{2^{|\mathbf{d}|}\ |\mathbf{d}|!\ (3g-3+n-|\mathbf{d}|)!} \int\limits_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \cdot \omega^{3g-3+n-|\mathbf{d}|}.$$

Here $m(g,n)=\delta(g-1)\times\delta(n-1)$, $\mathbf{d}!=\prod_{i=1}^{n} d_i!$, *and* $|\mathbf{d}|=\sum_{i=1}^{n} d_i$.

Remark. We warn the reader that there are some small differences in the normalization of the Weil-Petersson volume form in the literature; in this paper,

$$V_{g,n}=V_{g,n}(0,\dots,0)=\frac{1}{(3g-3+n)!}\int_{\mathcal{M}_{g,n}}\omega^{3g-3+n},$$

which is slightly different from the notation used in [**Z2**] and [**ST**]. Also, in [**Z1**] the Weil-Petersson Kähler form is 1/2 the imaginary part of the Weil-Petersson pairing, while here the factor 1/2 does not appear. So our answers are different by a power of 2.

3. Asymptotic behavior of Weil-Petersson volumes

In this section, we study the asymptotics of $V_{g,n}(\mathbf{L})=\mathrm{Vol}_{wp}(\mathcal{M}_{g,n}(L_1\dots,L_n))$ as $g\to\infty$.

Notation. For $\mathbf{d}=(d_1,\dots,d_n)$ with $d_i\in\mathbb{N}\cup\{0\}$ and $|\mathbf{d}|=d_1+\cdots+d_n\le 3g-3+n$, let $d_0=3g-3-|\mathbf{d}|$ and define

$$\begin{aligned}[\prod_{i=1}^{n}\tau_{d_i}]_{g,n}&=\frac{\prod_{i=1}^{n}(2d_i+1)!2^{|\mathbf{d}|}}{\prod_{i=0}^{n}d_i!}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\omega^{d_0}\\ &=\frac{\prod_{i=1}^{n}(2d_i+1)!!2^{2|\mathbf{d}|}(2\pi^2)^{d_0}}{d_0!}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_1^{d_0},\end{aligned}$$

where $\kappa_1=\omega/(2\pi^2)$ is the first Mumford class on $\overline{\mathcal{M}}_{g,n}$ [**AC**]. By Theorem 2.3 for $L=(L_1,\dots,L_n)$, we have:

$$V_{g,n}(2L)=\sum_{|\mathbf{d}|\le 3g-3+n}[\tau_{d_1},\dots,\tau_{d_n}]_{g,n}\,\frac{L_1^{2d_1}}{(2d_1+1)!}\cdots\frac{L_n^{2d_n}}{(2d_n+1)!}. \tag{3.1}$$

3.1. Recursive formulas for the intersection pairings. Given $\mathbf{d}=(d_1,\dots,d_n)$ with $|\mathbf{d}|\le 3g-3+n$, the following recursive formulas hold:

I.

$$\begin{aligned}[\tau_0\tau_1\prod_{i=1}^{n}\tau_{d_i}]_{g,n+2}&=[\tau_0^4\prod_{i=1}^{n}\tau_{d_i}]_{g-1,n+4}\\ &+6\sum_{\substack{g_1+g_2=g\\ \{1,\dots,n\}=I\amalg J}}[\tau_0^2\prod_{i\in I}\tau_{d_i}]_{g_1,|I|+2}\cdot[\tau_0^2\prod_{i\in J}\tau_{d_i}]_{g_2,|J|+2},\end{aligned}$$

II.

$$(2g-2+n)[\prod_{i=1}^{n}\tau_{d_i}]_{g,n}=\frac{1}{2}\sum_{L=0}^{3g-3+n}(-1)^L(L+1)\frac{\pi^{2L}}{(2L+3)!}[\tau_{L+1}\prod_{i=1}^{n}\tau_{d_i}]_{g,n+1}.$$

III. Let $a_0=1/2$, and for $n\ge 1$,

$$a_n=\zeta(2n)(1-2^{1-2n}).$$

Then we have

$$[\tau_{d_1}, \ldots, \tau_{d_n}]_{g,n} = \sum_{j=2}^{n} \mathcal{A}^j_{\mathbf{d}} + \mathcal{B}_{\mathbf{d}} + \mathcal{C}_{\mathbf{d}},$$

where

$$\mathcal{A}^j_{\mathbf{d}} = 8 \sum_{L=0}^{d_0} (2d_j + 1)\ a_L [\tau_{d_1+d_j+L-1}, \prod_{i \neq 1,j} \tau_{d_i}]_{g,n-1}, \tag{3.2}$$

$$\mathcal{B}_{\mathbf{d}} = 16 \sum_{L=0}^{d_0} \sum_{k_1+k_2=L+d_1-2} a_L [\tau_{k_1}\tau_{k_2} \prod_{i \neq 1} \tau_{d_i}]_{g-1,n+1}, \tag{3.3}$$

and

$$\mathcal{C}_{\mathbf{d}} = 16 \sum_{\substack{I \amalg J = \{2,\ldots,n\} \\ 0 \leq g' \leq g}} \sum_{L=0}^{d_0} \sum_{k_1+k_2=L+d_1-2} a_L\ [\tau_{k_1} \prod_{i \in I} \tau_{d_i}]_{g',|I|+1} \times [\tau_{k_2} \prod_{i \in J} \tau_{d_i}]_{g-g',|J|+1}. \tag{3.4}$$

References.

- For results on the relationship between the Weil-Petersson volumes and the intersections of ψ classes on $\overline{\mathcal{M}}_{g,n}$, see [**Wi**] and [**AC**]. An explicit formula for the volumes in terms of the intersection of ψ classes was developed in [**KMZ**].
- Formula (**I**) is a special case of Proposition 3.3 in [**LX1**].
- For different proofs of (**II**), see [**DN**] and [**LX1**]. The proof presented in [**DN**] uses the properties of moduli spaces of hyperbolic surfaces with cone points.
- For a proof of (**III**), see [**M2**]; in view of Theorem 2.3, (**III**) can be interpreted as a recursive formula for the volume of $\mathcal{M}_{g,n}(L)$ in terms of volumes of moduli spaces of Riemann surfaces that we get by removing a pair of pants containing at least one boundary component of $S_{g,n}$. See also [**Mc**] and [**LX2**].
- If $d_1 + \cdots + d_n = 3g - 3 + n$, (**III**) gives rise to a recursive formula for the intersection pairings of ψ_i classes which is the same as the Virasoro constraints for a point. See also [**MS**]. For different proofs and discussions related to these relations, see [**Wi**], [**Ko**], [**OP**], [**M1**], [**KL**], and [**EO**].

Remarks.

- In terms of the volume polynomials, equation (**II**) can be written as ([**DN**]):

$$\frac{\partial V_{g,n+1}}{\partial L}(L, 2\pi i) = 2\pi i (2g - 2 + n) V_{g,n}(L).$$

When $n = 0$,

$$V_{g,1}(2\pi i) = 0$$

and

$$\frac{\partial V_{g,1}}{\partial L}(2\pi i) = 2\pi i(2g-2)V_g. \tag{3.5}$$

- Note that (**III**) applies only when $n > 0$. In the case of $n = 0$, (3.5) allows us to prove necessary estimates for the growth of $V_{g,0}$.
- Although (**III**) has been described in purely combinatorial terms, it is closely related to the topology of different types of pairs of pants in a surface.
- In this paper, we are mainly interested in the intersection pairings only containing κ_1 and ψ_i classes. For generalizations of (**III**) to the case of higher Mumford's κ classes, see [**LX1**] and [**E**].
- We will show that when n is fixed and $g \to \infty$, both terms $\mathcal{A}_{\mathbf{d}}$, and $\mathcal{B}_{\mathbf{d}}$ in (**III**) contribute to $V_{g,n} = [\tau_0, \ldots, \tau_0]_g$. More precisely, for $\mathbf{d} = (0, \ldots, 0)$,
$$\frac{\mathcal{B}_{\mathbf{d}}}{\mathcal{A}_{\mathbf{d}}} \asymp 1.$$
On the other hand, for $\mathbf{d} = (0, \ldots, 0)$ the contribution of $\mathcal{C}_{\mathbf{d}}$ in (**III**) is negligible. More precisely, we will see that $\frac{\mathcal{C}_{\mathbf{d}}}{V_{g,n}} = O(1/g)$.

3.2. Basic estimates for the intersection pairings. The main advantage of using (**III**) is that all the coefficients are positive. Moreover, it is easy to check that

$$a_n = \zeta(2n)(1-2^{1-2n}) = \frac{1}{(2n-1)!}\int_0^\infty \frac{t^{2n-1}}{1+e^t}\,dt.$$

Hence,

$$a_{n+1} - a_n = \int_0^\infty \frac{1}{(1+e^t)^2}\left(\frac{t^{2n+1}}{(2n+1)!} + \frac{t^{2n}}{2n!}\right)dt.$$

As a result, we have:

Lemma 3.1. *In terms of the above notation, $\{a_n\}_{n=1}^\infty$ is an increasing sequence. Moreover, $\lim_{n\to\infty} a_n = 1$, and*

$$a_{n+1} - a_n \asymp 1/2^{2n}. \tag{3.6}$$

Using this observation and (3.1), one can prove the following general estimates:

Lemma 3.2. *In terms of the above notation, the following estimates hold:*

1)

$$[\tau_{d_1+1}, \tau_0, \ldots, \tau_0]_{g,n} \le [\tau_{d_1}, \tau_0, \ldots, \tau_0]_{g,n} \le [\tau_0, \ldots, \tau_0]_{g,n} = V_{g,n}.$$

2) *More generally,*

$$[\tau_{d_1}, \dots, \tau_{d_n}]_{g,n} \leq (2d_1+1)\cdots(2d_n+1)V_{g,n}$$

and

$$V_{g,n}(2L_1, \dots, 2L_n) \leq e^L V_{g,n}, \tag{3.7}$$

where $L = L_1 + \cdots + L_n$.

3) *For any* $g, n \geq 0$ *with* $2g-2+n > 0$,

$$V_{g-1,n+4} \leq V_{g,n+2} \tag{3.8}$$

and

$$b_0 < \frac{(2g-2+n)V_{g,n}}{V_{g,n+1}} < b_1, \tag{3.9}$$

where b_1 *and* b_0 *are universal constants independent of* g *and* n.

4) *Given* $d_1, \dots, d_k \geq 0$, *we have*

$$[\tau_{d_1}, \tau_{d_2}, \dots, \tau_{d_k}, \tau_0, \dots, \tau_0]_{g,n} \asymp V_{g,n},$$

where the implied constants are independent of g.

Proof. Parts (1) and (2) follow by comparing the contributions of $\mathcal{A}_{\mathbf{d}}$, $\mathcal{B}_{\mathbf{d}}$, and $\mathcal{C}_{\mathbf{d}}$ for $(d_1, d_2, \dots, d_n)$, $(d_1, 0, \dots, 0)$, and $(0, \dots, 0)$ in (**III**). Moreover, since $1/2 \leq \min_i\{a_i/a_{i+1}\}$, we have

$$\frac{1}{2} \leq \frac{[\tau_1\,\tau_0^{n-1}]_{g,n}}{V_{g,n}}.$$

See (3.2), (3.3), and (3.4). Also, (3.1) implies (3.7). Note that equation (**I**) for $\mathbf{d} = (0)$ implies that for any $n \geq 0$, $V_{g,n+2} \geq V_{g-1,n+4}$. Also, since for $l \geq 1$

$$\frac{l\,\pi^{2l-2}}{(2l+1)!} \geq \frac{(l+1)\pi^{2l}}{(2l+3)!},$$

part (1) and equation (**II**) imply that

$$b_0 \leq \frac{(2g-2+n)V_{g,n}}{V_{g,n+1}} \leq b_1, \tag{3.10}$$

where

$$b_0 = \frac{1}{2}\cdot\left(\frac{1}{6} - \frac{\pi^2}{60}\right), \qquad b_1 = \sum_{l=1}^{\infty} \frac{l\,\pi^{2l-2}}{(2l+1)!}.$$

Note that

$$\frac{V_{g-1,n+2}}{V_{g,n}} = \frac{V_{g-1,n+2}}{V_{g-1,n+3}} \cdot \frac{V_{g-1,n+3}}{V_{g-1,n+4}} \cdot \frac{V_{g-1,n+4}}{V_{g,n+2}} \cdot \frac{V_{g,n+2}}{V_{g,n+1}} \cdot \frac{V_{g,n+1}}{V_{g,n}}.$$

So (3.8) and (3.9) imply that

$$V_{g-1,n+2} = O(V_{g,n}). \tag{3.11}$$

As a result, in view of part (3), (**I**) implies that

$$\sum_{\substack{g_1+g_2=g \\ \{2,\dots,n\}=I\amalg J}} V_{g_1,|I|+1}\cdot V_{g_2,|J|+1} = O\left(\frac{V_{g,n}}{g}\right). \tag{3.12}$$

In (3.11) and (3.12), the implied constants are independent of g.

Finally, we prove part (4) for $(d_1, 0, \dots, 0)$. Here we compare the contributions of $\mathcal{A}_{\mathbf{d}}$, $\mathcal{B}_{\mathbf{d}}$, and $\mathcal{C}_{\mathbf{d}}$ for $(d_1, 0, \dots, 0)$ and $(0, \dots, 0)$ in (**III**) and write $V_{g,n} = \mathcal{V}_1 + \mathcal{V}_2$, where $\mathcal{V}_1$ is the sum of terms in (3.2), (3.3), and (3.4) which also contribute to the expansion of $[\tau_{d_1}, \tau_0, \dots, \tau_0]_{g,n}$. It is easy to check that for $i, j \geq 1$, $a_j/a_i \leq 2$. Hence, we have

$$\mathcal{V}_1 \leq 2\,[\tau_{d_1}, \tau_0, \dots, \tau_0]_{g,n}.$$

Next, we show that

$$\mathcal{V}_2 \leq\ C_2(d_1, n)\,\frac{V_{g,n}}{g},$$

where C_2 is a constant independent of g, but it can depend on d_1 and n. There are $O(d_1^2)$ terms in $\mathcal{V}_2$ (from (3.2) and (3.3)); by (3.10) and (3.11) each one of these terms is bounded above by $\frac{V_{g,n}}{2g+n}$. We can use (3.12) to bound the contribution of (3.4) to $\mathcal{V}_2$. Hence, we have

$$\frac{[\tau_{d_1}, \dots, \dots, \tau_0]_{g,n}}{V_{g,n}} \geq \frac{1}{2}(1 - \frac{C_2(d_1, n)}{g})$$

where C_2 is a constant independent of g, but it can depend on d_1 and n. A similar argument can be applied to prove that in general

$$\frac{[\tau_{d_1}, \dots, \tau_{d_k}, \tau_0, \dots, \tau_0]_{g,n}}{V_{g,n}} \geq \frac{1}{2}(1 - \frac{C_2(\mathbf{d}, n)}{g}). \tag{3.13}$$

□

Remarks.

- A stronger lower bound for $\frac{V_{g,n+1}}{(2g-2+n)V_{g,n}}$ was obtained in [**ST**]. But in this paper, we will use only (3.9).
- We will show that given $n \geq 0$, (3.8) is asymptotically sharp as $g \to \infty$. However, (1.2) implies that when g is fixed and n is large, this inequality is far from being sharp; in fact, given $g \geq 1$ as $n \to \infty$,
$$V_{g,n+2} \asymp \sqrt{n}\, V_{g-1,n+4}.$$

3.3. The following lemma will be used in the proof of Theorem 1.2.

Lemma 3.3. *Let $n_1, n_2 \geq 0$. In terms of the above notation, we have*

$$\sum_{\substack{g_1+g_2=g \\ g_2\geq g_1\geq 0}} V_{g_1,n_1+1}\times V_{g_2,n_2+1} = O\left(\frac{V_{g,n}}{g}\right), \tag{3.14}$$

where $n = n_1 + n_2$. Here the implied constant is independent of g.

To simplify the notation, let

$$[\mathbf{x}]_{g,n} := [\tau_{x_1}, \ldots, \tau_{x_n}]_{g,n},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$. Also,

$$|\mathbf{x}| = x_1 + \cdots + x_n.$$

Our proof of Lemma 3.3 relies on the following statement:

Lemma 3.4. *Given $r \geq 1$ and $n \geq 0$, there exists $C = C(r, n) > 0$ such that for any $(g_1, m), (g_2, n)$ with $2g_1 - 2 + m \geq r$, we have*

$$[x_1, \ldots, x_m]_{g_1,m} \times V_{g_2,n+r} \leq C(r, n) \times [x_1, \ldots, x_m, 0, \ldots, 0]_{g_1+g_2,m+n}. \tag{3.15}$$

Sketch of proof of Lemma 3.4. We prove (3.15) by induction on $2g_1 + m$. Note that $[x_1, \ldots, x_n]_{g,n} \neq 0$ only if $x_1 + \cdots + x_n \leq 3g - 3 + n$. The recursive relation (**III**) implies that if $g \leq g'$ and $n \leq n'$, then

$$[x_1, \ldots, x_n]_{g,n} \leq [x_1, \ldots, x_n, 0, \ldots, 0]_{g',n'}. \tag{3.16}$$

First, by part (3) of Lemma 3.2, if $2g_1 - 2 + m \geq r$,

$$\frac{V_{g_2,n+r}}{V_{g_1+g_2,n+m}} = O(1),$$

where the implied constant is independent of g_2. So in view of (3.13), there exist $C_0, c(r, n) > 0$ such that

$$[x_1, \ldots, x_m]_{g_1,m} \times V_{g_2,n+r} \leq C_0 \times [x_1, \ldots, x_m, 0, \ldots, 0]_{g_1+g_2,m+n}$$

holds for any $(g_1, m), (g_2, n)$ with $r \leq 2g_1 - 2 + m \leq 3r$, and $g_2 \geq c(r, n)$. Let

$$C(r, n) = C_0 + \max\{V_{g_2,n+r}\}_{g_2 \leq c(r,n)}.$$

Then (3.16) yields that (3.15) holds for any $(g_1, m), (g_2, n)$ with $r \leq 2g_1 - 2 + m \leq 3r$.

The main idea for the rest of the proof is using the recursive formula (**III**) to expand both $[\mathbf{x}]_{g_1,m}$ and $[\mathbf{y}]_{g_1+g_2,m+n}$. Assume that the result holds for (g, n) with $2g + n < 2g_1 + m$, and $3r < 2g_1 - 2 + m$. To simplify the notation, let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (x_1, \ldots, x_m, 0, \ldots, 0)$. Expand both $[\mathbf{x}]_{g_1,m}$ and $[\mathbf{x}, 0, \ldots, 0]_{g_1+g_2,m+n}$ in (3.15) using the relation (**III**). Since all the terms in equation (**III**) and (3.15) are positive, it is enough to check that after expanding both sides every term in the expansion of $[\mathbf{x}]_{g_1,m}$ has a corresponding term on the right hand side (in the expansion of $[\mathbf{y}]_{g_1+g_2,m+n}$). More precisely, following (3.2), (3.3), and (3.4), we can write

$$[\mathbf{x}]_{g_1,m} = \sum_{j=2}^{m} \mathcal{A}^j_{\mathbf{x}} + \mathcal{B}_{\mathbf{x}} + \mathcal{C}_{\mathbf{x}}$$

and

$$[\mathbf{y}]_{g_1+g_2,m+n} = \sum_{j=2}^{n+m} \mathcal{A}^j_{\mathbf{y}} + \mathcal{B}_{\mathbf{y}} + \mathcal{C}_{\mathbf{y}}.$$

Then we have:

- For $2 \leq j \leq m$, each term in $\mathcal{A}^j_{\mathbf{x}}$ is of the form $a_l \cdot [\mathbf{x}']_{g_1,m-1}$, where $l = |\mathbf{x}'| - |\mathbf{x}| + 1$. In this case, the induction hypothesis for $[\mathbf{x}']_{g_1,m-1}$ implies that

$$a_l \cdot [\mathbf{x}']_{g_1,m-1} \times V_{g_2,n+r} \leq C(r,n)\ a_l \cdot [\mathbf{x}', 0, \dots, 0]_{g_1+g_2,m-1+n}.$$

  In this case, $a_l \cdot [\mathbf{x}', 0, \dots, 0]_{g_1+g_2,m-1+n}$ appears in the expansion of $\mathcal{A}^j_{\mathbf{y}}$.
- Similarly, each term in $\mathcal{B}_{\mathbf{x}}$ is of the form $a_l \cdot [\mathbf{x}']_{g_1-1,m+1}$, where $l = |\mathbf{x}'| - |\mathbf{x}| + 2$. The induction hypothesis for $[\mathbf{x}']_{g_1-1,m+1}$ implies that

$$a_l \cdot [\mathbf{x}']_{g_1-1,m+1} \times V_{g_2,n+r} \leq C(r,n)\ a_l \cdot [\mathbf{x}', 0, \dots, 0]_{g_1-1+g_2,m+1+n}.$$

  In this case, $a_l \cdot [\mathbf{x}', 0, \dots, 0]_{g_1-1+g_2,m+1+n}$ appears in the expansion of $\mathcal{B}_{\mathbf{y}}$.
- Finally, each term in $\mathcal{C}_{\mathbf{x}}$ is of the form $a_l \cdot [\mathbf{y}_1]_{h_1,m_1} \cdot [\mathbf{y}_2]_{h_2,m_2}$ where $l = |\mathbf{y}_1| + |\mathbf{y}_2| - |\mathbf{x}| + 2$, $m_1 + m_2 = m - 1$, $h_1 + h_2 = g_1$ and $2h_2 + m_2 \leq 2h_1 + m_1$. In this case, we can apply the induction hypothesis for $[\mathbf{y}_1]_{h_1,m_1}$ since $r < 2h_1 + m_1 < 2g_1 + m$. Then we have

$$a_l \cdot [\mathbf{y}_1]_{h_1,m_1} \cdot [\mathbf{y}_2]_{h_2,m_2} \times V_{g_2,n+r} \leq C(r,n)\ a_l \cdot [\mathbf{y}_1, 0, \dots, 0]_{h_1+g_2,m_1+n} \cdot [\mathbf{y}_2]_{h_2,m_2}.$$

  Note that all the terms of the form $a_l \cdot [\mathbf{y}_1, 0, \dots, 0]_{h_1+g_2,m_1+n} \cdot [\mathbf{y}_2]_{h_2,m_2}$ appear in the expansion of $\mathcal{C}_{\mathbf{y}}$.

$\square$

Proof of Lemma 3.3. In view of Lemma 3.2, we have

$$\sum_{\substack{g_1+g_2=g \\ g_1 \geq g_2 \geq 0}} V_{g_1,n_1+1} \times V_{g_2,n_2+1} = O\left(\sum_{\substack{g_1+g_2=g \\ g_1 \geq g_2 \geq 0}} \frac{V_{g_1,n_1+4} \times V_{g_2,n_2+1}}{g^3} \right). \tag{3.17}$$

We have

$$\sum_{\substack{g_1+g_2=g \\ g_1 \geq g_2 \geq 0}} V_{g_1,n_1+4} \times V_{g_2,n_2+1} = V_{g,n_1+4} \cdot V_{0,n_2+1} + V_{g-1,n_1+4} \cdot V_{1,n_2+1} + V_{g-2,n_1+4} \cdot V_{2,n_2+1} + \sum_{\substack{g_1+g_2=g \\ g_1 \geq g_2 \geq 3}} V_{g_1,n_1+4} \times V_{g_2,n_2+1}.$$

Note that $V_{0,n_2+1} \neq 0$ only if $n_2 \geq 2$, which implies that $V_{g,n_1+4} \cdot V_{0,n_2+1} = O(V_{g,n+2})$. Similarly, by part (3) of Lemma 3.2, we have

$$V_{g-1,n_1+4} \cdot V_{1,n_2+1} + V_{g-2,n_1+4} \cdot V_{2,n_2+1} = O(V_{g,n+2}). \tag{3.18}$$

Also, by Lemma 3.4 for $\mathbf{x} = (0, \ldots, 0)$ and $r = 4$, if $i \geq 3$

$$V_{g-i,n_1+4} \times V_{i,n_2+1} \leq C_4 \cdot V_{g,n+1}.$$

Hence, (3.18) implies that

$$\sum_{\substack{g_1+g_2=g \\ g_1 \geq g_2 \geq 0}} V_{g_1,n_1+4} \times V_{g_2,n_2+1} = O\left(V_{g,n+2} + g \cdot V_{g,n+1}\right).$$

Now Equation (3.15) follows from part (3) of Lemma 3.2 and (3.17). □.

Remark.

- In particular, when $n_1 = n_2 = 0$, we have
$$\sum_{i=1}^{g-1} V_{i,1} \times V_{g-i,1} = O\left(\frac{V_g}{g}\right),$$
as $g \to \infty$. More generally, Lemma 3.4 implies that given $r \geq 0$,
$$\sum_{i=r+1}^{\lceil g/2 \rceil} V_{i,1} \times V_{g-i,1} = O\left(\frac{V_g}{g^{2r+1}}\right) \tag{3.19}$$
as $g \to \infty$, where the implied constants only depend on r. We prove a stronger statement in Corollary 3.7.
- The implied constant in Lemma 3.3 depends on n_1 and n_2.

Now we can prove the main result of this section:

Theorem 3.5. *Let $n, k \geq 0$. As $g \to \infty$, we have:*

1)
$$\frac{[\tau_k, \tau_0, \ldots, \tau_0]_{g,n+1}}{V_{g,n+1}} = 1 + O(1/g),$$

2)
$$\frac{V_{g,n+1}}{2gV_{g,n}} = 4\pi^2 + O(1/g),$$

and

3)
$$\frac{V_{g,n}}{V_{g-1,n+2}} = 1 + O(1/g).$$

Remark.

- These estimates are consistent with the conjectures on the growth of Weil-Petersson volumes in [**Z2**]; we remark that the statements had been predicted by Peter Zograf.

- By part (3) of Theorem 3.5 and (3.8),

$$\frac{V_{g-i,k}}{V_{g,k-2i}} = O(1), \tag{3.20}$$

where the implied constant is independent of g, k, and i. Also, given $i \geq k/2$, we have

$$\frac{V_{g-i,k}}{V_g} = \frac{V_{g-i,k}}{V_{g-i,k+1}} \cdots \frac{V_{g-i,2i-1}}{V_{g-i,2i}} \times \frac{V_{g-i,2i}}{V_{g-i+1,2i-2}} \cdots \frac{V_{g-1,2}}{V_g}.$$

Hence Theorem 3.5 and Lemma 3.3 imply that

$$V_{g-i,k} = O\left(\frac{V_g}{g^{2i-k}}\right) \tag{3.21}$$

as $g \to \infty$.
- Following the ideas used in the proof of Theorem 3.5, one can show that

$$\frac{V_{g,n+1}}{2gV_{g,n}} = 4\pi^2 + \frac{a_{1,n}}{g} + \cdots + \frac{a_{k,n}}{g^k} + O\left(\frac{1}{g^{k+1}}\right)$$

and

$$\frac{V_{g,n}}{V_{g-1,n+2}} = 1 + \frac{b_{1,n}}{g} + \cdots + \frac{b_{k,n}}{g^k} + O(\frac{1}{g^{k+1}}).$$

However, in general it is not easy to calculate $a_{i,n}$ and $b_{i,n}$'s explicitly (see [**MZ**]).

Proof of Theorem 3.5. Fix $n \geq 0$. By Lemma 3.3,

$$\sum_{\substack{g_1+g_2=g \\ \{1,\ldots,n\}=I_1 \amalg I_2}} V_{g_1,|I_1|+1} \times V_{g_2,|I_2|+1} = O\left(\frac{V_{g,n+1}}{g^2}\right). \tag{3.22}$$

Now we compare the contributions of $\mathcal{A}_{\mathbf{d}}$, $\mathcal{B}_{\mathbf{d}}$, and $\mathcal{C}_{\mathbf{d}}$ for $\mathbf{d} = (k, 0, \ldots, 0)$ and $(0, \ldots, 0)$ in (**III**). We can expand the difference $[\tau_k\, \tau_0^{n-1}]_{g,n} - [\tau_{k+1}\, \tau_0^{n-1}]_{g,n}$ in terms of the intersection numbers on $\overline{\mathcal{M}}_{g-1,n+1}$, $\overline{\mathcal{M}}_{g,n-1}$, and $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2}$. Following (3.2), the numbers

$$[\tau_{k-1}\, \tau_0^{n-2}]_{g,n-1}, \ldots, [\tau_{3g+n-4}\, \tau_0^{n-2}]_{g,n-1}$$

contribute to $[\tau_k\, \tau_0^{n-1}]_{g,n}$ and $[\tau_{k+1}\, \tau_0^{n-1}]_{g,n}$. It is easy to check that the contribution of $[\tau_{k-1+i}\, \tau_0^{n-2}]_{g,n-1}$ to $[\tau_k\, \tau_0^{n-1}]_{g,n} - [\tau_{k+1}\, \tau_0^{n-1}]_{g,n}$ is equal to $(a_{i+1} - a_i)[\tau_{k-1+i}\, \tau_0^{n-2}]_{g,n-1}$. Similarly, the numbers $[\tau_i\, \tau_j\, \tau_0^{n-1}]_{g-1,n+1}$ contribute to $[\tau_k\, \tau_0^{n-1}]_{g,n}$ (resp. to $[\tau_{k+1}\, \tau_0^{n-1}]_{g,n}$) whenever $i + j \geq k - 2$ (resp. $i + j \geq k - 1$). In view of Lemma 3.1, part (3) of Lemma 3.2, and (3.22), we get

$$[\tau_k\, \tau_0^{n-1}]_{g,n} - [\tau_{k+1}\, \tau_0^{n-1}]_{g,n} \leq c_0 \cdot k\, \frac{V_{g,n}}{g},$$

where c_0 is a universal constant independent of g and k. Therefore, we have

$$0 \leq 1 - \frac{[\tau_k, \tau_0, \ldots, \tau_0]_{g,n}}{V_{g,n}} \leq c_0 \frac{k^2}{g}. \tag{3.23}$$

We use the following elementary observation to prove part (2) for $n \geq 1$:

Elementary fact. *Let $\{r_i\}_{i=1}^{\infty}$ be a sequence of real numbers and $\{k_g\}_{g=1}^{\infty}$ be an increasing sequence of positive integers. Assume that for $g \geq 1$ and $i \in \mathbb{N}$, $0 \leq c_{g,i} \leq c_i$ and $\lim_{g\to\infty} c_{g,i} = c_i$. If $\sum_{i=1}^{\infty} |c_i r_i| < \infty$, then*

$$\lim_{g\to\infty} \sum_{i=1}^{k_g} r_i c_{g,i} = \sum_{i=1}^{\infty} r_i c_i. \tag{3.24}$$

Now, let

$$r_i = (-1)^i \frac{\pi^{2i}(i+1)}{(2i+3)!},\ k_g = 3g-3+n\ , c_i = 1 \text{ and } c_{g,i} = \frac{[\tau_{i+1}, \tau_0, \ldots, \tau_0]_{g,n}}{V_{g,n+1}}.$$

By (3.24), and (**II**) for $\mathbf{d} = 0$, we get

$$\lim_{g\to\infty} \frac{2(2g-2+n)V_{g,n}}{V_{g,n+1}} = \frac{1}{3!} - \frac{2\pi^2}{5!} + \cdots + (-1)^L (L+1) \frac{\pi^{2L}}{(2L+3)!} + \cdots = \frac{1}{2\pi^2}.$$

In fact, (3.23) similarly implies that

$$\frac{2(2g-2+n)V_{g,n}}{V_{g,n+1}} = \frac{1}{2\pi^2} + O(\frac{1}{g}).$$

On the other hand, from (**I**) and (3.21) we get that for $n \geq 2$:

$$\lim_{g\to\infty} \frac{V_{g,n}}{V_{g-1,n+2}} = 1 + O(1/g).$$

Now it is easy to check that

$$V_{g,1} = \frac{1}{g} V_{g,2} (\frac{1}{4\pi^2}(1 - O(1/g)))\ ,\ V_{g-1,3} = \frac{1}{g} V_{g-1,4} (\frac{1}{4\pi^2}(1 - O(1/g)))$$

and

$$V_{g,2} = V_{g-1,4}(1 + O(1/g))$$

imply

$$\frac{V_{g,1}}{V_{g-1,3}} = 1 + O(1/g).$$

In other words, (**b**) for $n = 1$ and $n = 2$ proves part (3) for $n = 1$. Also, (3.5) implies part (2) for $n = 0$, and part (2) for $n = 0$ and $n = 1$ implies part (3) for $n = 0$. $\square$

Remark. More generally, the argument used in the proof of (3.23) implies that given $n \geq 1$,

$$0 \leq 1 - \frac{[\tau_{d_1}, \dots, \tau_{d_n}]_{g,n}}{V_{g,n}} \leq c_0 \frac{(d_1 + \ldots + d_n)^2}{g},$$

where c_0 is a constant independent of g and $d_1, \dots, d_n$.

For any bounded sequence $\{k_i\}_{i=1}^{\infty}$ with $|k_i| < a$, we have:

$$\frac{1}{c} g^{-a} < \prod_{i=1}^{g} (1 + \frac{k_i}{i}) < c\, g^a,$$

where c is independent of g. Hence, parts (3) and (4) of Theorem 3.5 imply that:

Corollary 3.6. *Given $n \geq 0$, there exists $m \geq 0$ such that:*

$$g^{-m} \mathcal{F}_{g,n} < V_{g,n} < g^m \mathcal{F}_{g,n},$$

where

$$\mathcal{F}_{g,n} = (4\pi^2)^{2g+n-3} (2g - 3 + n)! \frac{1}{\sqrt{g\pi}}.$$

As a result, we get the following estimate which will be used in the next section:

Corollary 3.7. *Let $b, k \geq 0$ and $C < C_0 = 2\ln(2)$,*

$$\sum_{\substack{g_1+g_2=g+1-k \\ r+1 \leq g_1 \leq g_2}} e^{Cg_1} \cdot g_1^b \cdot V_{g_1,k} \cdot V_{g_2,k} \asymp \frac{V_g}{g^{2r+k}},$$

as $g \to \infty$.

We remark that following (3.20), it is enough to prove the statement for $k = 1$ and $k = 2$.

Proof of Corollary 3.7. Note that for $1 \leq i \leq N$, $\binom{N}{i} \geq (N/i)^i$. Also, for $0 < s \leq 1/2$, $s^s (1-s)^{1-s} \leq \frac{1}{4^s}$. Then a simple calculation using Stirling's formula implies that for $2i \leq N$ we have

$$\binom{N}{i} > \frac{4^i}{2e^2 \sqrt{N}}.$$

Hence, for $C < 2\ln(2)$, there exists a constant $c_0 = c_0(r, k, b)$ such that

$$\sum_{\substack{g_1+g_2=g+1-k, \\ c_0(r,k,b) \leq g_1 \leq g_2}} e^{Cg_1} g_1^{b+1} \cdot \mathcal{F}_{g_1,k} \cdot \mathcal{F}_{g_2,k} = O\left(\frac{\mathcal{F}_g}{g^{2m+3r}}\right), \tag{3.25}$$

where $\mathcal{F}_g = \mathcal{F}(g, 0)$ (see Corollary 3.7). In view of Corollary 3.6, we have

$$\sum_{g_1=c_0+1}^{\lceil g/2 \rceil} g_1^b \cdot e^{C\, g_1} \times V_{g_1,k} \times V_{g-g_1-k+1,k} = O\left(\frac{V_g}{g^{3r}}\right). \tag{3.26}$$

Now, it is enough to bound the terms $V_{i,k} \times V_{g-i-k+1,k}$ for $r+1 \leq i \leq c_0(r,k,b)$. Recall that in general by (3.21),

$$V_{g-i-k+1,k} = O\left(\frac{V_g}{g^{2i+k-2}}\right).$$

When $r+1 \leq i$, we have $2i+k-2 \geq 2r+k$, and hence

$$\sum_{g_1=r+1}^{c_0(r,k)} g_1^b \cdot e^{C\, g_1} \times V_{g_1,k} \times V_{g-g_1-k+1,k} = O\left(\frac{V_g}{g^{2r+k}}\right). \tag{3.27}$$

The result follows from (3.26) and (3.27). In this proof, the implied constants can depend on C, b, k and r. □

4. Random Riemann surfaces of high genus

In this section, we apply the asymptotic estimates on the volume polynomials to study the geometric properties of random hyperbolic surfaces; in particular, we are interested in the length of the shortest simple closed geodesic of a given combinatorial type, diameter, and the Cheeger constant of a random surface. See [**BM**] for more in the case of random hyperbolic surfaces constructed by random trivalent graphs.

4.1. Notation. Recall that the mapping class group $\mathrm{Mod}_{g,n}$ acts naturally on the set $\mathcal{S}_{g,n}$ of isotopy classes of simple closed curves on $S_{g,n}$: Two simple closed curves α_1 and α_2 are of the same *type* if and only if there exists $g \in \mathrm{Mod}_{g,n}$ such that $g \cdot \alpha_1 = \alpha_2$. The type of a simple closed curve is determined by the topology of $S_{g,n} - \alpha$, the surface that we get by cutting $S_{g,n}$ along α.

Given a multicurve $\alpha = \sum_{i=1}^{k} c_i \alpha_i$ on S_g, define

$$N_\alpha(\cdot,\cdot) : \mathcal{M}_g \times \mathbb{R}_+ \to \mathbb{R}_+$$

by

$$N_\alpha(X, L) = |\{\gamma | \gamma \in \mathrm{Mod}_g \cdot \alpha, \ell_\gamma(X) \leq L\}|.$$

Similarly, define

$$N_\alpha^c(\cdot,\cdot) : \mathcal{M}_g \times \mathbb{R}_+ \to \mathbb{R}_+$$

by

$$N_\alpha^c(X, L) = |\{\gamma = \sum_{i=1}^{k} c_i \gamma_i | \gamma \in \mathrm{Mod}_g \cdot \alpha, \forall i : 1 \leq i \leq k\ , \ell_{\gamma_i}(X) \leq L\}|.$$

By the definition, for any connected simple closed curve, $N_\alpha(X,L) = N_\alpha^c(X,L)$. To simplify the notation,

- For $k \leq g$, let $\beta_k = \gamma_1 + \cdots + \gamma_k$ be a multicurve on S_g with k connected components so that $S_g - \beta_k$ is connected. That is,
$$S_g - \beta_k \cong S_{g-k,2k}.$$
- For $1 \leq k \leq m \leq g-1$, let $\eta_{m,k} = \gamma_1' + \cdots + \gamma_k'$ be a separating multicurve with k connected components on S_g such that
$$S_g - \eta_{m,k} \cong S_{g_1,k} \cup S_{g_2,k}$$
and
$$m = |\chi(S_{g_1,k})| = 2g_1 - 2 + k,$$
where $g_1 + g_2 + k - 1 = g$.

We consider the following counting functions:

$$N_0(X,L) := N_{\beta_1}(X,L) = |\{\gamma \,|\ell_\gamma(X) \leq L, \gamma \in \mathcal{S}_g \text{ is non-separating}\}|$$

and

$$N_i(X,L) := N_{\eta_{2i-1,1}}(X,L)$$

for $i \geq 1$; that is, $N_i(X,L)$ is the number of connected simple closed geodesics of length $\leq L$ which divide X into a surface of genus i and a surface of genus $g-i$. Then we have:

Lemma 4.1. *As $g \to \infty$,*

$$\int_{\mathcal{M}_g} N_{\beta_k}^c(X,L)\, dX = O((e^L-1)^k \, \frac{L^k}{k!}\, V_g), \tag{4.1}$$

$$\int_{\mathcal{M}_g} N_1(X,L)\, dX = O(\frac{(e^{L/2}-1)(L^3+L)V_g}{g}), \tag{4.2}$$

and

$$\int_{\mathcal{M}_g} N_k(X,L)\, dX = O((e^L-1)L\; V_{k,1} \times V_{g-k,1}), \tag{4.3}$$

where $k>0$. Here the implied constants are independent of k, L, and g.

Proof of Lemma 4.1. Recall that by Lemma 3.2,

$$V_{g,n}(2L_1,\ldots,2L_n) \leq e^L \times V_{g,n}.$$

Let $C_k = [0,L]^k \subset \mathbb{R}^k$. Since $|\operatorname{Sym}(\beta_k)| = k!$, Theorem 2.2 for $F = \chi(C_k) : \mathbb{R}^k \to \mathbb{R}$ and (2.3) imply that

$$\int_{\mathcal{M}_g} N_{\beta_k}^c(X,L)\, dX \leq (\int_0^L t\cdot e^t dt)^k \, \frac{V_{g-k,2k}}{k!}. \tag{4.4}$$

On the other hand,

$$\int_0^L t \cdot e^t dt = O((e^L-1)L).$$

Also, by (3.8) and Theorem 3.5 we have

$$V_{g-k,2k} = O(V_g).$$

Hence (4.4) implies (4.1). Similar arguments imply (4.2) and (4.3). $\square$

Remark. In fact, when $L > 1$ is fixed,

$$\frac{\int_{\mathcal{M}_g} N_0(X, L)\, dX}{V_g} \asymp \frac{e^L}{L}$$

as $g \to \infty$. Similar estimates hold when L is *much smaller* than g. On the other hand, when L is very large compared to g, $\int_{\mathcal{M}_g} N_0(X, L)\, dX$ behaves like a polynomial of degree $6g - 6$ in L (see [**M4**]).

We recall that the number of all closed geodesics of length $\leq L$ on $X \in \mathcal{M}_g$ is at most $e^{L+6}(g-1)$ (see Lemma 6.6.4 in [**Bu**]).

4.2. Injectivity radius. As in the introduction, let $\ell_{sys}(X)$ denote the length of the shortest simple closed geodesic on X. Given $\epsilon > 0$, let

$$\mathcal{M}^{\epsilon}_{g,n} = \{X \mid \ell_{sys}(X) \leq \epsilon\} \subset \mathcal{M}_{g,n}.$$

The set $\mathcal{M}_{g,n} - \mathcal{M}^{\epsilon}_{g,n}$ of hyperbolic surfaces with lengths of closed geodesics bounded below by ϵ is a compact subset of the moduli space $\mathcal{M}_{g,n}$.

Theorem 4.2. *Let $n \geq 0$. There exists $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$,*

$$\mathrm{Vol}_{wp}(\mathcal{M}^{\epsilon}_{g,n}) \asymp \epsilon^2\, \mathrm{Vol}_{wp}(\mathcal{M}_{g,n})$$

as $g \to \infty$.

Proof. Here we sketch the proof for the case of $n = 0$. Fix ϵ_0 such that no two simple closed geodesics of length $\leq \epsilon_0$ on a hyperbolic surface could meet, and choose $\epsilon < \epsilon_0$. Consider the function

$$N(X, \epsilon) = N_0(X, \epsilon) + \cdots + N_{\lceil g/2 \rceil}(X, \epsilon),$$

as defined in §4.1. Then, in view of Theorem 2.2, we have

$$\mathrm{Vol}_{wp}(\mathcal{M}^{\epsilon}_g) \leq \int_{\mathcal{M}_g} N(X, \epsilon)\, dX$$

$$= \sum_{i=1}^{\lfloor g/2 \rfloor} \int_0^{\epsilon} t\, \mathrm{Vol}_{wp}(\mathcal{M}(S_{i,1} \times S_{g-i,1}, t, t))\, dt + \int_0^{\epsilon} t\, \mathrm{Vol}_{wp}(\mathcal{M}_{g-1,2}(t,t))\, dt.$$

On the other hand, (3.7) implies that when t is small enough, for $i \geq 1$,

$$\mathrm{Vol}_{wp}(\mathcal{M}_{i,1}(t)) \times \mathrm{Vol}_{wp}(\mathcal{M}_{i,1}(t)) \leq 2V_{i,1} \times V_{g-i,1}$$

and

$$\mathrm{Vol}_{wp}(\mathcal{M}_{g-1,2}(t,t)) \leq 2V_{g-1,2}.$$

Hence, when ϵ is small (independent of g), from (3.19) and (3.20) we get

$$\mathrm{Vol}_{wp}(\mathcal{M}_g^{\epsilon}) = O(\epsilon^2(\sum_{i=1}^{\lceil g/2 \rceil} V_{i,1}V_{g-i,1} + V_{g-1,2})) = O(\epsilon^2 V_g).$$

Next, we prove that the probability that a surface has a *non-separating* simple closed geodesic of length $\leq \epsilon$ is asymptotically positive. More precisely, we show that as $g \to \infty$,

$$\mathrm{Vol}_{wp}(\{X | N_0(X,\epsilon) \geq 1\}) \asymp \epsilon^2 V_g. \tag{4.5}$$

Recall that $V_{g-1,2} \asymp V_g$ and for $t < \epsilon_0$, $V_{g-1,2}(t,t) \asymp V_{g-1,2}$. Therefore, for $\epsilon < \epsilon_0$,

$$\int_{\mathcal{M}_g} N_0(X,\epsilon)\, dX \asymp \epsilon^2 V_g. \tag{4.6}$$

Given $1 \leq i \leq g-1$, let $\mathcal{U}_\epsilon^i \subset \mathcal{M}_g$ denote the set of points X in $\mathcal{M}_g$ such that $N^c_\alpha(X,\epsilon) \geq 1$ for $\alpha = \eta_{i_1,k_1}$ with $i_1 \geq i$ (see §4.1), and let

$$\mathcal{U}_\epsilon = \mathcal{U}_\epsilon^1.$$

Note that if $X \in \mathcal{M}_g - \mathcal{U}_\epsilon$, then the union of all closed geodesics of length $\leq \epsilon$ on X is a non-separating multicurve. Now (4.5) is a corollary of the following claims:

Claim 1. For any $k \geq 1$,

$$\mathrm{Vol}_{wp}(\{X | N_0(X,\epsilon) \geq k\} - \mathcal{U}_\epsilon) \leq c\frac{\epsilon^{2k} e^{\epsilon k}}{k!} V_g, \tag{4.7}$$

where c is a constant independent of g and k. Therefore, as $g \to \infty$,

$$\sum_{k=2}^{\infty} \mathrm{Vol}_{wp}(\{X | N_0(X,\epsilon) \geq k\} - \mathcal{U}_\epsilon) = O(\epsilon^4 V_g). \tag{4.8}$$

Claim 2. We have

$$\int_{\mathcal{U}_\epsilon} N_0(X,\epsilon)\, dX = O\left(\frac{\epsilon^2 V_g}{g}\right). \tag{4.9}$$

Therefore, by (4.6),

$$\int_{\mathcal{M}_g - \mathcal{U}_\epsilon} N_0(X,\epsilon)\, dX \asymp \epsilon^2 V_g \tag{4.10}$$

as $g \to \infty$.

Proof of Claim 1. As in §4.1, let β_k be a multicurve with k connected components such that $S_g - \beta_k \cong S_{g-k,2k}$. For $X \notin \mathcal{U}_\epsilon$, $N_0(X,\epsilon) = k$ implies that $N^c_{\beta_k}(X,\epsilon) \geq 1$. That is,

$$\mathrm{Vol}_{wp}(\{N_0(X,\epsilon) \geq k\} - \mathcal{U}_\epsilon) \leq \mathrm{Vol}_{wp}(\{X | N_{\beta_k}(X,\epsilon) \geq 1\}).$$

Since for small t, $e^t - 1 = O(t)$, by Lemma 4.1 we have

$$\mathrm{Vol}_{wp}(\{X | N_{\beta_k}(X,\epsilon) \geq 1\}) \leq c\frac{\epsilon^{2k} e^{\epsilon k}}{k!} V_g, \tag{4.11}$$

where c is a constant independent of g and k. When ϵ is small enough, (4.11) implies (4.7).

Proof of Claim 2. For $1 \leq k \leq i \leq g-1$, let $\eta_{i,k}$ be a multicurve with k connected components defined in §4.1. Then we have

$$\mathrm{Vol}_{wp}(\mathcal{U}_\epsilon^{2i} - \mathcal{U}_\epsilon^{2i+1}) \leq \sum_{k=1}^{i} N^c_{\eta_{2i,2k}}(X,\epsilon).$$

Since $|\operatorname{Sym}(\eta_{i,2k})| = 2k!$, using Theorem 2.2 and (3.20) we get

$$\mathrm{Vol}_{wp}(\mathcal{U}_\epsilon^{2i} - \mathcal{U}_\epsilon^{2i+1}) = O\left((2i)\sum_{k=1}^{i} \frac{(e^\epsilon \epsilon)^{2k}}{2k!} \cdot V_{i,2} \times V_{g-i-1,2}\right).$$

Similarly, we have

$$\mathrm{Vol}_{wp}(\mathcal{U}_\epsilon^{2i+1} - \mathcal{U}_\epsilon^{2i+2}) = O\left((2i+1)\sum_{k=1}^{i} \frac{(e^\epsilon \epsilon)^{2k+1}}{(2k+1)!} \cdot V_{i+1,1} \times V_{g-i-1,1}\right).$$

Since $\sum_{k=1}^{\infty} \frac{(e^\epsilon \epsilon)^{2k}}{k!} < \epsilon^2 e^{2\epsilon}$, from Corollary 3.7 we get

$$\mathrm{Vol}_{wp}\left(\mathcal{U}_\epsilon^2\right) = O\left(\frac{\epsilon^2 V_g}{g^2}\right).$$

Let $\mathcal{S}_g^{1,1}$ be the set of multicurves $\gamma = \gamma_1' + \cdots + \gamma_j'$ on S_g such that $S_g - \gamma \cong S_1 \cup S_2 \cup S_3$, where $|\chi(S_1)| = |\chi(S_2)| = 1$, and S_3 is homeomorphic to $S_{g-2,2}$, $S_{g-3,4}$, or $S_{g-4,6}$. Define

$$\mathcal{U}_\epsilon^{1,1} = \{X \,| N_\gamma^c(X,\epsilon) \geq 1, \gamma \in \mathcal{S}_g^{1,1}\} \subset \mathcal{U}_\epsilon.$$

It is easy to check that

$$\mathrm{Vol}_{wp}\left(\mathcal{U}_\epsilon^{1,1}\right) = O\left(\epsilon^2 \frac{V_g}{g^2}\right).$$

Let $\alpha_\epsilon = \sum_{\ell_\alpha(X) \leq \epsilon} \alpha$. If $X \in \mathcal{U}_\epsilon - (\mathcal{U}_\epsilon^2 \cup \mathcal{U}_\epsilon^{1,1})$, then $X - \alpha_\epsilon$ has exactly two connected components. Moreover, for $k \geq 2$ we have:

$$\mathrm{Vol}_{wp}(\{X \,| N_0(X,\epsilon) = k\} \cap (\mathcal{U}_\epsilon - (\mathcal{U}_\epsilon^2 \cup \mathcal{U}_\epsilon^{1,1}))) = O\left(\frac{V_g \epsilon^{2k-2}}{(k-1)!g}\right).$$

Note that by the choice of ϵ, $N_0(X,\epsilon) \leq 3g-3$. Hence, we have

$$\int_{\mathcal{U}_\epsilon} N_0(X,\epsilon)\, dX = \int_{\mathcal{U}_\epsilon^2 \cup \mathcal{U}_\epsilon^{1,1}} N_0(X,\epsilon)\, dX + \int_{\mathcal{U}_\epsilon - (\mathcal{U}_\epsilon^2 \cup \mathcal{U}_\epsilon^{1,1})} N_0(X,\epsilon)\, dX$$

$$= O\left((3g-3)\cdot\frac{\epsilon^2 V_g}{g^2} + \frac{\epsilon^2 V_g}{g}\sum_{k=1}^{3g-3} k^2\frac{\epsilon^{k-1}}{k!}\right),$$

which proves the claim.

Finally, note that

$$\mathrm{Vol}_{wp}(\{X\ |N_0(X,\epsilon)\geq 1\}) \geq \mathrm{Vol}_{wp}(\{X|N_0(X,\epsilon)\geq 1\} - \mathcal{U}_\epsilon)$$
$$= \int_{\mathcal{M}_g-\mathcal{U}_\epsilon} N_0(X,\epsilon)\, dX - \sum_{k=2}^{\infty}\mathrm{Vol}_{wp}(\{X|N_0(X,\epsilon)\geq k\} - \mathcal{U}_\epsilon).$$

Hence, the lower bound in (4.5) follows from (4.10) and (4.8). □

Remark. It is easy to see from the first part of the proof of Theorem 4.2 that for any sequence $\{\epsilon_g\}_g$ with $\epsilon_g < \epsilon_0$, we have

$$\mathrm{Vol}_{wp}(\mathcal{M}_{g,n}^{\epsilon_g}) = O(\epsilon_g^2\,\mathrm{Vol}_{wp}(\mathcal{M}_{g,n}))$$

as $g\to\infty$.

Define $f:\mathcal{M}_g\to\mathbb{R}_+$ by

$$f(X) = \sum_{\ell_\alpha(X)\leq 1}\frac{1}{\ell_\alpha(X)}.$$

Then in view of Theorem 2.2, (3.20) and (3.19) imply that

$$\int_{\mathcal{M}_g} f(X)\, dX = \int_0^1 V_{g-1,2}(t,t)dt + \sum_{i=1}^{\lfloor g/2\rfloor}\int_0^1 V_{g-i,1}(t)V_{i,1}(t)dt \asymp V_g$$

and hence Theorem 4.2 implies that:

Corollary 4.3. *As $g\to\infty$,*

$$\int_{\mathcal{M}_g}\frac{1}{\ell_{sys}(X)}\, dX \asymp V_g.$$

4.3. Behavior of separating simple closed geodesics. Let $\ell^s_{sys}(X)$ denote the length of the shortest homologically trivial simple closed geodesic on X. We show that $\ell^s_{sys}(X)$ is generically at least of $(2-\epsilon)\log(g)$ as $g\to\infty$. In fact, there exists $C>0$ such that (as $g\to\infty$) for most points $X\in\mathcal{M}_g$, if a separating simple closed geodesic γ satisfies $\ell_\gamma(X) < C\log(g)$, then $S_g-\gamma = S_{g_1}\cup S_{g_2}$ with $\min\{g_1,g_2\} = O(1)$.

Theorem 4.4. *Let $0<a<2$. Then*

$$\mathrm{Prob}^g_{wp}(\ell^s_{sys}(X) < a\log(g)) = O\left(\frac{\log(g)^3 g^{a/2}}{g}\right),$$

and

$$\mathbb{E}^g_{X\sim wp}(\ell^s_{sys}(X)) \asymp \log(g)$$

as $g\to\infty$.

Proof. Note that in terms of the notation in §4.1, we have

$$\mathrm{Vol}_{wp}(\{X \mid \ell^s_{sys}(X) < L\}) \leq \sum_{i=1}^{\lceil g/2 \rceil} \int_{\mathcal{M}_g} N_i(X, L)\, dX.$$

Let $L > 1$. Then by Lemma 4.1,

$$\mathrm{Prob}^g_{wp}(\ell^s_{sys}(X) < L) \leq \frac{e^{L/2} L^3}{g} + \sum_{i=2}^{\lceil g/2 \rceil} \frac{e^L V_{i,1} \times V_{g-i,1}}{V_g}. \tag{4.12}$$

On the other hand, by (3.19),

$$\sum_{i=2}^{\lceil g/2 \rceil} V_{i,1} \times V_{g-i,1} = O\left(\frac{V_g}{g^3}\right).$$

Using these bounds for $L = a \log(g)$ implies the first part of the theorem.

By Theorem 1.3 of [**SS**], there exists a positive constant $C > 0$ such that every closed surface X of genus $g \geq 2$, $\ell^s_{sys}(X) \leq C \log(g)$, and hence

$$\mathbb{E}^g_{X \sim wp}(\ell^s_{sys}(X)) < C \log(g).$$

On the other hand, we have

$$\mathbb{E}^g_{X \sim wp}(\ell^s_{sys}(X)) \geq \frac{\log(g)}{10} \left(1 - \mathrm{Prob}^g_{wp}(\ell^s_{sys}(X) < \log(g)/10)\right),$$

which implies the second part of the theorem. □

4.4. Injectivity radius and embedded balls. Let $\mathrm{Inj}(x)$ denote the injectivity radius of $x \in X$. We show that on a generic $X \in \mathcal{M}_g$ most points $x \in X$ (with respect to the hyperbolic volume form on X) satisfy $\mathrm{Inj}(x) \geq \frac{1}{6} \log(g)$. Note that, corresponding to each x, there exists a simple closed curve γ_x of length $\leq 2\,\mathrm{Inj}(x)$ such that the distance of x from the geodesic representative of γ_x is at most $2\,\mathrm{Inj}(x)$.

- Let

  $$\mathcal{A}_g = \{X \mid N(X, \log(g)/3) \leq g^{1/3+1/4}\} \subset \mathcal{M}_g, \tag{4.13}$$

  where $N(X, L)$ is the number of connected simple closed geodesics of length $\leq L$ on X. Then by (4.1), (4.2), and (4.12)

  $$\frac{\mathrm{Vol}_{wp}(\mathcal{M}_g - \mathcal{A}_g)}{V_g} = O(g^{-1/4}) \tag{4.14}$$

  as $g \to \infty$.
- A simple calculation shows that given a simple closed geodesic γ of length $\leq \log(g)/3$, the (hyperbolic) volume of the locus on X with $\gamma_x = \gamma$ is at most $g^{1/3} \log(g)$.

Therefore, for any point in $X \in \mathcal{A}_g$ (defined by (4.13)),

$$\mathrm{Vol}(\{x \in X \mid \mathrm{Inj}(x) \leq \frac{1}{6}\log(g)\}) = O(g^{11/12}\log(g)),$$

where the volume is with respect to the hyperbolic volume form on $X \in \mathcal{M}_g$. In particular, for $X \in \mathcal{A}_g$ the radius $\mathrm{Emb}(X)$ of the largest embedded ball in X is $\geq \log(g)/6$ and (4.14) implies that:

Theorem 4.5. *As $g \to \infty$,*

$$\mathrm{Prob}^g_{wp}(\mathrm{Emb}(X) < C_E \log(g)) \to 0,$$

$$\mathbb{E}^g_{X\sim wp}(\mathrm{Emb}(X)) \asymp \log(g),$$

where $C_E = \frac{1}{6}$.

4.5. Cheeger constants and isoperimetric inequalities. Recall that the Cheeger constant of a Riemann surface X is defined by

$$h(X) = \inf_{\alpha} \frac{\ell(\alpha)}{\min\{\mathrm{Area}(X_1), \mathrm{Area}(X_2)\}} \tag{4.15}$$

where the infimum is taken over all smooth 1-dimensional submanifolds of X which divide it into two disjoint submanifolds X_1 and X_2 such that $X - \alpha = X_1 \cup X_2$ and $\alpha \subset \partial(X_1) \cap \partial(X_2)$.

We remark that:

- In fact, by an observation due to Yau, we may restrict A to a family of curves for which X_1 and X_2 are connected. See Lemma 8.3.6 in [**Bu**].
- By a result of Cheng [**C**], for any compact hyperbolic surface X

$$h^2(X) \leq 1 + \frac{16\pi^2}{\mathrm{diam}(X)}, \tag{4.16}$$

  Therefore, there is an upper bound for the Cheeger constant which tends to 1 as $g(X) \to \infty$. See also §III and §X in [**Ch**].

Given $i \leq g-1$, let

$$H_i(X) = \inf_{\alpha} \frac{\ell_\alpha(X)}{\min\{\mathrm{Area}(X_1), \mathrm{Area}(X_2)\}}, \tag{4.17}$$

where $\alpha = \cup_{j=1}^{s} \alpha_j$ is a union of simple closed *geodesics* on X with $X - \alpha = X_1 \cup X_2$, and X_1 and X_2 are connected subsurfaces of X such that $|\chi(X_1)| = i \leq |\chi(X_2)|$. So $\min\{\mathrm{Area}(X_1), \mathrm{Area}(X_2)\} = 2\pi \cdot i$. We define the *geodesic* Cheeger constant of X by

$$H(X) = \min_{i \leq g-1} H_i(X).$$

In general, by the definition

$$h(X) \leq H(X),$$

but the inequality is not sharp. Using basic isoperimetric inequalities for hyperbolic surfaces, we will obtain a lower bound for $h(X)$ in terms of $H(X)$.

Recall that in a compact hyperbolic surface, there exists a perimeter minimizer among regions of prescribed area bounded by embedded rectifiable curves; it consists of curves of equal constant curvature. Moreover, by a result of Adams and Morgan [**AM**]:

Theorem 4.6. *For given area $0 < A < 4\pi g$, a perimeter-minimizing system of embedded rectifiable curves bounding a region R of area A consists of a set of curves of one of the following four types:*

1) *a circle,*
2) *horocycles around cusps,*
3) *two neighboring curves at constant distance from a closed geodesic, bounding an annulus or complement,*
4) *geodesics or single neighboring curves.*

All curves in the set have the same constant curvature.

On the other hand, one can easily check that:

- If α is a circle or a union of two neighboring curves at constant distance from a closed geodesic then the ratio $\frac{\ell(\alpha)}{\min\{\text{Area}(X_1),\text{Area}(X_2)\}}$ in (4.15) is strictly bigger than one.
- For a neighboring curve of length L and curvature κ at distance s from a closed geodesic of length ℓ enclosing area A, we have
$$A = \ell \sinh(s), \quad L = \ell \cosh(s), \quad \text{and} \quad \kappa = \tanh(s).$$
See Lemma 2.3 in [**AM**].

Therefore, a simple calculation shows that:

Proposition 4.7. *Let $X \in \mathcal{M}_g$ be a hyperbolic surface of genus g. Then*
$$\frac{H(X)}{H(X)+1} \leq h(X) \leq H(X).$$

Now we can show:

Theorem 4.8. *Let $C_1 < \frac{\ln(2)}{2\pi+\ln(2)}$, and $0 \leq \beta < 2$. As $g \to \infty$*
$$\text{Prob}^g_{wp}(h(X) \leq C_1) \to 0$$
and
$$\int_{\mathcal{M}_g} \left(\frac{1}{h(X)}\right)^\beta dX \asymp V_g.$$

Our proof relies on the following lemma:

Lemma 4.9. *Let $m = 2g_1 - 2 + n_1 \leq g - 1$, where $1 \leq n_1 \leq 2$. Then given $0 < C_1 < C_2$,*
$$\text{Vol}_{wp}(\{X | X \in \mathcal{M}_g, H_m(X) \leq C_1\}) = O(m^2 \cdot e^{2\pi \cdot m \cdot C_2} \cdot V_{g_1,n_1} \times V_{g-g_1-n_1+1,n_1}),$$

as $g \to \infty$. Here the implied constant is independent of m and g, but it might depend on C_1 and C_2.

Proof of Lemma 4.9. We prove the statement for $n_1 = 2$. In this case, m is even and $g_1 = m/2$. The proof for $n_1 = 1$ is similar. The argument is similar to the one we used in the proof of Claim 2 in Theorem 4.2. For $1 \leq k \leq m$, consider the multicurve $\eta_{m,k}$ defined in §4.1.

Let

$$\mathcal{W}_k^m(L) = \mathrm{Vol}_{wp}(\{X \in \mathcal{M}_g \mid N_{\eta_{m,k}}(X, L) \geq 1\}),$$

and

$$\mathcal{W}^m(L) = \mathrm{Vol}_{wp}(\{X \in \mathcal{M}_g \mid \sum_{k=1}^{m} N_{\eta_{m,k}}(X, L) \geq 1\}).$$

Then, by the above definitions,

$$\mathcal{W}_k^m(L) \leq \int_{\mathcal{M}_g} N_{\eta_{m,k}}(X, L)\, dX$$

and

$$\mathcal{W}^m(L) \leq \sum_{k=1}^{m} \mathcal{W}_k^m(L).$$

Recall that by Lemma 3.2, for $1 \leq i \leq g_1$ and $1 \leq j \leq g - 1 - g_1$

$$V_{g_1 - i, 2+2i} \leq V_{g_1, 2}$$

and

$$V_{g-1-g_1-j, 2+2j} \leq V_{g-1-g_1, 2}.$$

Since $|\operatorname{Sym}(\eta_{m,k})| = k!$, Theorem 2.2 and (3.7) yield that:

$$\mathcal{W}_k^m(L) = O\left(e^L \times V_{g_1,2} \times V_{g-g_1-1,2} \times \int_{L_1+\cdots+L_k \leq L} \frac{1}{k!} L_1 \cdots L_k\, dL_1 \cdots dL_k\right).$$

Hence,

$$\mathcal{W}^m(L) = O\left(e^L \times V_{g_1,2} \times V_{g-1-g_1,2} \times \sum_{k=1}^{m} \int_{L_1+\cdots+L_k \leq L} \frac{1}{k!} L_1 \cdots L_k\, dL_1 \cdots dL_k\right).$$

Since

$$\int_{L_1+\ldots L_s \leq L} L_1 \cdots L_s\, dL_1 \cdots dL_s = \frac{L^{2s}}{(2s)!}$$

and

$$\sum_{s=1}^{\infty} \frac{L^{2s}}{s!(2s)!} = O(L^2 e^{3L^{2/3}}),$$

we get

$$\mathcal{W}^m(L) = O(L^2 \times e^{L+3L^{2/3}} \times V_{g_1,2} \times V_{g-1-g_1,2}), \tag{4.18}$$

where the implied constant is independent of g, m, and L. By definition (4.17), if $H_m(X) < C$, then $\sum_{k=1}^{m} N_{\eta_{m,k}}(X, C \cdot 2\pi \cdot m) \geq 1$. Therefore, we have

$$\mathrm{Vol}_{wp}(\{X | X \in \mathcal{M}_g, H_m(X) \leq C_1\}) \leq \mathcal{W}^m(C_1 \cdot 2\pi \cdot m). \tag{4.19}$$

Hence, (4.18) implies the lemma for $n_1 = 2$. □

Proof of Theorem 4.8. Lemma 4.9 implies that for any $H_0 > 0$, as $g \to \infty$

$$\mathrm{Prob}_{wp}^{g}\left(H(X) \leq H_0\right) = O\left(\sum_{2g_1 \leq g} e^{4\pi \cdot H_0 \cdot g_1} g_1^3 (V_{g_1,2} \times V_{g-1-g_1,2} + V_{g_1,1} \times V_{g-g_1,1})\right).$$

For $H_0 < \ln(2)/2\pi$, by Corollary 3.7 we have:

$$\mathrm{Prob}_{wp}^{g}\left(H(X) \leq H_0\right) = O\left(\frac{V_g}{g}\right).$$

Therefore, Proposition 4.7 implies the first part of the theorem.

In view of (4.18) and (4.19), and similar statements for $n_1 = 1$, the argument in the proof of Lemma 4.9 implies that there exists $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$, we have

$$\mathrm{Vol}_{wp}(\{X |\ H(X) \leq \epsilon\}) = O\left(\frac{\epsilon^2 V_g}{g}\right), \tag{4.20}$$

where the implied constant is independent of g and ϵ. Note that following Proposition 4.7, if $\epsilon \leq 1/2$, and $h(X) \leq \epsilon$, then $H(X) \leq 2\epsilon$. Hence, the second part of the theorem is a corollary of (4.20). □

Remark. Let s_g be a sequence such that $\lim_{g\to\infty} \frac{s_g}{g} = 0$. Given $M > 0$,

$$\mathrm{Prob}_{wp}^{g}(H_{s_g}(X) \leq M) \to 0$$

as $g \to \infty$.

4.6. Diameter.

It is known that the diameter of a Riemannian manifold of constant curvature -1 satisfies:

$$\mathrm{diam}(X) \leq 2(r_0 + \frac{1}{h(X)} \log(\frac{\mathrm{Vol}(X)}{2B(r_0)})), \tag{4.21}$$

where $r_0 > 0$ and $B(r_o)$ is the infimum of the volume of a ball of radius r_0 in X (see [**B**]).

We show:

Theorem 4.10. *As $g \to \infty$,*

$$\mathrm{Prob}_{wp}^{g}(\mathrm{diam}(X) \geq C_d \log(g)) \to 0$$

and

$$\mathbb{E}^g_{X\sim wp}(\mathrm{diam}(X)) \asymp \log(g),$$

where $C_d = 40$.

Proof. Following (4.21) for $r_0 = \ell_{sys}(X)/2$, we have

$$\mathrm{diam}(X) = O\left(\ell_{sys}(X) + \frac{\log(g) + |\log(\ell_1(X))|}{h(X)}\right), \tag{4.22}$$

where $\ell_1(X) = \min\{\ell_{sys}(X), 1\}$. Note that $\ell_{sys}(X) = O(\log(g))$. Also, by the proof of Theorem 4.2 we have

$$\mathrm{Prob}^g_{wp}(\{X \mid \ell_{sys}(X) \leq \frac{1}{g}\}) = O\left(\frac{1}{g^2}\right).$$

Therefore, the first part of the theorem is a direct consequence of (4.22) and the first part of Theorem 4.8. In order to prove the second part of the theorem, it is enough to show that:

$$\mathbb{E}^g_{X\sim wp}(\mathrm{diam}(X)) = O(\log(g)).$$

By Hölder's inequality, we have

$$\int_{\mathcal{M}_g} \frac{|\log(\ell_1(X))|}{h(X)}\, dX \leq \left(\int_{\mathcal{M}_g} \frac{1}{h(X)^{3/2}} dX\right)^{2/3} \times \left(\int_{\mathcal{M}_g} |\log(\ell_1(X))|^3 dX\right)^{1/3}.$$

The second part of Theorem 4.8 (for $\beta = 3/2$) and Corollary 4.3 imply that:

$$\int_{\mathcal{M}_g} \frac{|\log(\ell_1(X))|}{h(X)}\, dX = O(V_g).$$

Hence, the second part of the theorem follows from (4.22). $\square$

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