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ENTIRE SOLUTIONS OF THE ALLEN-CAHN EQUATION AND COMPLETE EMBEDDED MINIMAL SURFACES OF FINITE TOTAL CURVATURE IN $\mathbb{R}^3$

Manuel del Pino, Michal Kowalczyk & Juncheng Wei

Abstract

We consider minimal surfaces M which are complete, embedded, and have finite total curvature in $\mathbb{R}^3$, and bounded, entire solutions with finite Morse index of the Allen-Cahn equation $\Delta u + f(u) = 0$ in $\mathbb{R}^3$. Here $f = -W'$ with W bi-stable and balanced, for instance $W(u) = \frac{1}{4}(1-u^2)^2$. We assume that M has $m \geq 2$ ends, and additionally that M is non-degenerate, in the sense that its bounded Jacobi fields are all originated from rigid motions (this is known for instance for a Catenoid and for the Costa-Hoffman-Meeks surface of any genus). We prove that for any small $\alpha > 0$, the Allen-Cahn equation has a family of bounded solutions depending on $m-1$ parameters distinct from rigid motions, whose level sets are embedded surfaces lying close to the blown-up surface $M_\alpha := \alpha^{-1}M$, with ends possibly diverging logarithmically from M_α. We prove that these solutions are L^∞-*non-degenerate* up to rigid motions, and find that their Morse index coincides with the index of the minimal surface. Our construction suggests parallels of De Giorgi conjecture for general bounded solutions of finite Morse index.

1. Introduction and main results

1.1. The Allen-Cahn equation and minimal surfaces. The Allen-Cahn equation in $\mathbb{R}^N$ is the semilinear elliptic problem

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $f(s) = -W'(s)$ and W is a "double-well potential," bi-stable and balanced, namely

$$\begin{aligned} &W(s) > 0 \text{ if } \ s \neq 1, -1, \\ &W(1) = 0 = W(-1), \quad W''(\pm 1) = f'(\pm 1) =: \sigma_\pm^2 > 0. \end{aligned} \tag{1.2}$$

A typical example of such a nonlinearity is

$$f(u) = (1-u^2)u \quad \text{ for } W(u) = \frac{1}{4}(1-u^2)^2, \tag{1.3}$$

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while we will not make use of the special symmetries enjoyed by this example.

Equation (1.1) is a prototype for the continuous modeling of phase transition phenomena. Let us consider the energy in a bounded region Ω of $\mathbb{R}^N$

$$J_\alpha(v) = \int_\Omega \frac{\alpha}{2}\,|\nabla v|^2 + \frac{1}{4\alpha}W(v),$$

whose Euler-Lagrange equation is a scaled version of (1.1),

$$\alpha^2\Delta v \,+\, f(v) \,=\, 0 \quad \text{in } \Omega\,. \tag{1.4}$$

We observe that the constant functions $u = \pm 1$ minimize J_α. They are idealized as two *stable phases* of a material in Ω. It is of interest to analyze stationary configurations in which the two phases coexist. Given any subset Λ of Ω, any discontinuous function of the form

$$v_* = \chi_\Lambda - \chi_{\Omega\setminus\Lambda} \tag{1.5}$$

minimizes the second term in J_α. The introduction of the gradient term in J_α makes an α-regularization of u_* a test function for which the energy gets bounded and proportional to the surface area of the *interface* $M = \partial\Lambda$, so that in addition to minimizing approximately the second term, stationary configurations should also asymptotically select interfaces M that are stationary for surface area, namely (generalized) minimal surfaces. This intuition on the Allen-Cahn equation gave important impulse to the calculus of variations, motivating the development of the theory of Γ-*convergence* in the 1970s. Modica [**31**] proved that a family of local minimizers u_α of J_α with uniformly bounded energy must converge in suitable sense to a function of the form (1.5) where $\partial\Lambda$ minimizes perimeter. Thus, intuitively, for each given $\lambda \in (-1, 1)$, the level sets $[v_\alpha = \lambda]$ collapse as $\alpha \to 0$ onto the interface $\partial\Lambda$. A similar result holds for critical points not necessarily minimizers, see [**26**]. For minimizers this convergence is known in a very strong sense; see [**2**, **3**].

If, on the other hand, we take such a critical point u_α and scale it around an interior point $0 \in \Omega$, setting $u_\alpha(x) = v_\alpha(\alpha x)$, then u_α satisfies equation (1.1) in an expanding domain,

$$\Delta u_\alpha + f(u_\alpha) = 0 \quad \text{in } \alpha^{-1}\Omega,$$

so that formally letting $\alpha \to 0$ we end up with equation (1.1) in entire space. The "interface" for u_α should thus be around the (asymptotically flat) minimal surface $M_\alpha = \alpha^{-1}M$. Modica's result is based on the intuition that if M happens to be a smooth surface, then the transition from the equilibria -1 to 1 of u_α along the normal direction should take place in the approximate form $u_\alpha(x) \approx w(z)$, where z designates the normal coordinate to M_α. Then w should solve the ODE problem

$$w'' + f(w) \,=\, 0 \quad \text{in } \mathbb{R}, \quad w(-\infty) = -1,\ w(+\infty) = 1. \tag{1.6}$$

This solution indeed exists thanks to assumption (1.2). It is strictly increasing and unique up to constant translations. We fix in what follows the unique w for which

$$\int_{\mathbb{R}} t\, w'(t)^2\, dt \;=\; 0. \tag{1.7}$$

For example (1.3), we have $w(t) = \tanh\left(t/\sqrt{2}\right)$. In general w approaches its limits at exponential rates,

$$w(t) - \pm 1 \;=\; O(\, e^{-\sigma_{\pm}|t|}\,) \quad \text{as } t \to \pm\infty.$$

Observe then that

$$J_\alpha(u_\alpha) \approx Area\,(M)\, \int_{\mathbb{R}} [\frac{1}{2} {w'}^2 + W(w)],$$

which is what makes it plausible that M is critical for its area, namely a minimal surface.

The above considerations led E. De Giorgi [**9**] to formulate in 1978 a celebrated conjecture on the Allen-Cahn equation (1.1), parallel to Bernstein's theorem for minimal surfaces: The level sets $[u = \lambda]$ of a bounded entire solution u to (1.1), which is also monotone in one direction, must be hyperplanes, at least for dimension $N \le 8$. Equivalently, up to a translation and a rotation, $u = w(x_1)$. This conjecture has been proven in dimensions $N = 2$ by Ghoussoub and Gui [**16**], $N = 3$ by Ambrosio and Cabré [**1**], and under a mild additional assumption by Savin [**38**]. A counterexample was recently built for $N \ge 9$ by us in [**11**, **12**]; see also [**6**, **27**]. See [**13**, **15**] for a recent survey on the state of the art of this question.

The assumption of monotonicity in one direction for the solution u in De Giorgi conjecture implies a form of stability, locally minimizing character for u when compactly supported perturbations are considered in the energy. Indeed, if $Z = \partial_{x_N} u > 0$, then the linearized operator $L = \Delta + f'(u)$ satisfies maximum principle. This implies stability of u, in the sense that its associated quadratic form, namely the second variation of the corresponding energy,

$$\mathcal{Q}(\psi,\psi) := \int |\nabla \psi|^2 - f'(u)\psi^2, \tag{1.8}$$

satisfies $\mathcal{Q}(\psi,\psi) > 0$ for all $\psi \ne 0$ smooth and compactly supported. Stability is a basic ingredient in the proof of the conjecture dimensions $2, 3$ in [**1**, **16**], based on finding a control at infinity of the growth of the Dirichlet integral. In dimension $N = 3$ it turns out that

$$\int_{B(0,R)} |\nabla u|^2 \;=\; O(R^2) \tag{1.9}$$

which intuitively means that the embedded level surfaces $[u = \lambda]$ must have a finite number of components outside a large ball, which are all "asymptotically flat." The question whether stability alone suffices for property (1.9) remains open. More generally, it is believed that this property is equivalent to the *finite Morse index* of the solution u (which means essentially that u is stable outside a bounded set). The Morse index $m(u)$ is defined as the maximal dimension of a vector space E of compactly supported functions such that

$$\mathcal{Q}(\psi,\psi) < 0 \quad \text{for all} \quad \psi \in E \setminus \{0\}.$$

Rather surprisingly, basically no examples of finite Morse index entire solutions of the Allen-Cahn equation seem known in dimension $N = 3$. Great progress has been achieved in the last decades, both in the theory of semilinear elliptic PDE like (1.1) and in minimal surface theory in $\mathbb{R}^3$. While this link traces back to the very origins of the study of (1.1) as discussed above, it has only been partially explored in producing new solutions.

In this paper we construct a new class of entire solutions to the Allen-Cahn equation in $\mathbb{R}^3$ which have the characteristic (1.9), and also finite Morse index, whose level sets resemble a large dilation of a given complete, embedded minimal surface M, asymptotically flat in the sense that it has *finite total curvature*, namely

$$\int_M |K|\, dV \;<\; +\infty$$

where K denotes Gauss curvature of the manifold, which is also *non-degenerate* in a sense that we will make precise below.

As pointed out by Dancer [**7**], the Morse index is a natural element to attempt classification of solutions of (1.1). Beyond De Giorgi conjecture, classifying solutions with a given Morse index should be a natural step toward understanding the structure of the bounded solutions of (1.1). Our main results show that, unlike the stable case, the structure of the set of solutions with finite Morse index is highly complex. On the other hand, we believe that our construction contains germs of generality, providing elements to extrapolate what may be true in general, in analogy with classification of embedded minimal surfaces. We elaborate on these issues in §10.

1.2. Embedded minimal surfaces of finite total curvature. The theory of embedded, minimal surfaces of finite total curvature in $\mathbb{R}^3$ has reached a notable development in the last 25 years. For more than a century, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa [**4**, **5**]. The *Costa surface* is a genus one minimal surface, complete and properly embedded, which outside a large ball has exactly three components (its *ends*), two of which are asymptotically catenoids with the same axis and opposite directions, the third one asymptotic to a plane perpendicular to that axis. The complete proof of embeddedness is due to Hoffman and Meeks [**21**]. In [**22**, **24**] these authors notably generalized Costa's example by exhibiting a class of three-end, embedded minimal surface, with the same look as Costa's far away, but with an array of tunnels that provides arbitrary genus $k \geq 1$. This is known as the Costa-Hoffman-Meeks surface with genus k.

Many other examples of multiple-end embedded minimal surfaces have been found since; see for instance [**29**, **40**] and references therein. In general all these surfaces look like parallel planes, slightly perturbed at their ends by asymptotically logarithmic corrections with a certain number of catenoidal links connecting their adjacent sheets. In reality this intuitive picture is not a coincidence. Indeed, Osserman [**35**] established that a complete minimal surface with finite total curvature can be described by a conformal diffeomorphism of a compact surface (actually of a Riemann surface), with a finite number of its points removed. These points correspond to the ends. Moreover, assuming that the ends are embedded, after a convenient rotation, they are asymptotically either catenoids or planes, all of them with parallel axes; see Schoen [**39**] and Jorge and Meeks [**28**]. The topology of the surface is thus characterized by the genus of the compact surface and the number of ends, having therefore "finite topology."

1.3. Main results. In what follows, M designates a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature (to which, below we, will make a further nondegeneracy assumption). As pointed out in [**25**], M is orientable and the set $\mathbb{R}^3 \setminus M$ has exactly two components S_+, S_-. In what follows we fix a continuous choice of unit normal field $\nu(y)$, which conventionally we take to point toward S_+.

For $x = (x_1, x_2, x_3) = (x', x_3) \in \mathbb{R}^3$, we denote

$$r = r(x) = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}.$$

After a suitable rotation of the coordinate axes, outside the infinite cylinder $r < R_0$ with sufficiently large radius R_0, M decomposes into a finite number m of unbounded components $M_1, \dots, M_m$, its *ends*. From a result in [**39**], we know that asymptotically each end of M_k either resembles a plane or a catenoid. More precisely, M_k can be represented as the graph of a function F_k of the first two variables,

$$M_k = \{\, y \in \mathbb{R}^3 \ / \ r(y) > R_0,\ y_3 = F_k(y') \,\}$$

where F_k is a smooth function which can be expanded as

$$F_k(y') = a_k \log r + b_k + b_{ik} \frac{y_i}{r^2} + O(r^{-3}) \quad \text{for} \quad r \geq R_0, \tag{1.10}$$

for certain constants a_k, b_k, b_{ik}, and this relation can also be differentiated. Here

$$a_1 \le a_2 \le \cdots \le a_m \ , \qquad \sum_{k=1}^m a_k \ = \ 0. \tag{1.11}$$

The direction of the normal vector $\nu(y)$ for large $r(y)$ approaches, on the ends, that of the x_3 axis, with alternate signs. We use the convention that for $r(y)$ large we have

$$\nu(y) \ = \ \frac{(-1)^k}{\sqrt{1+|\nabla F_k(y')|^2}}\,(\nabla F_k(y')\,,\,-1\,) \quad \text{if } y \in M_k. \tag{1.12}$$

Let us consider the Jacobi operator of M

$$\mathcal{J}(h) \ := \ \Delta_M h + |A|^2 h \tag{1.13}$$

where $|A|^2 = -2K$ is the Euclidean norm of the second fundamental form of M. $\mathcal{J}$ is the linearization of the mean curvature operator with respect to perturbations of M measured along its normal direction. A smooth function $z(y)$ defined on M is called a *Jacobi field* if $\mathcal{J}(z) = 0$. Rigid motions of the surface naturally induce some bounded Jacobi fields. Associated to, respectively, translations along coordinates axes and rotation around the x_3-axis, are the functions

$$z_i(y) = \nu(y)\cdot e_i, \quad y \in M, \quad i = 1,2,3,$$

$$z_4(y) = (-y_2, y_1, 0)\cdot \nu(y), \quad y \in M. \tag{1.14}$$

We assume that M is *non-degenerate* in the sense that these functions are actually *all* bounded Jacobi fields, namely

$$\{\ z \in L^\infty(M) \ / \ \mathcal{J}(z) = 0\,\} \ = \ \text{span}\,\{\,z_1, z_2, z_3, z_4\,\}. \tag{1.15}$$

In what follows we denote by J the dimension (≤ 4) of the above vector space.

This assumption, expected to be generic for this class of surfaces, is known in some important cases, most notably the catenoid and the Costa-Hoffmann-Meeks surface, which is an example of a three-ended M whose genus may be of any order. See Nayatani [**33**, **34**] and Morabito [**32**]. Note that for a catenoid, $z_{04} = 0$ so that $J = 3$. Non-degeneracy has been used as a tool to build new minimal surfaces for instance in Hauswirth and Pacard [**20**] and in Pérez and Ros [**37**]. It is also the basic element for building solutions to the singularly perturbed Allen-Cahn equation in compact manifolds in Pacard and Ritoré [**36**].

In this paper we will construct a solution to the Allen-Cahn equation whose zero level sets look like a large dilation of the surface M, with ends perturbed logarithmically. Let us consider a large dilation of M,

$$M_\alpha \ := \ \alpha^{-1} M.$$

This dilated minimal surface has ends parameterized as

$$M_{k,\alpha} = \{\, y \in \mathbb{R}^3 \ / \ r(\alpha y) > R_0,\ y_3 = \alpha^{-1} F_k(\alpha y') \,\}.$$

Let β be a vector of given m real numbers with

$$\beta = (\beta_1, \dots, \beta_m), \qquad \sum_{i=1}^{m} \beta_i = 0. \tag{1.16}$$

Our first result asserts the existence of a solution $u = u_\alpha$ defined for all sufficiently small $\alpha > 0$ such that given $\lambda \in (-1, 1)$, its level set $[u_\alpha = \lambda]$ defines an embedded surface lying at a uniformly bounded distance in α from the surface M_α, for points with $r(\alpha y) = O(1)$, while its k-th end, $k = 1, \dots, m$, lies at a uniformly bounded distance from the graph

$$r(\alpha y) > R_0,\ y_3 = \alpha^{-1} F_k(\alpha y') + \beta_k \log |\alpha y'| \,. \tag{1.17}$$

The parameters β must satisfy an additional constraint. It is clear that if two ends are parallel, say $a_{k+1} = a_k$, we need at least that $\beta_{k+1} - \beta_k \geq 0$, for otherwise the ends would eventually intersect. Our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate. We require

$$\beta_{k+1} - \beta_k > 4 \max \{\sigma_-^{-1}, \sigma_+^{-1}\} \quad \text{if} \quad a_{k+1} = a_k. \tag{1.18}$$

Let us consider the smooth map

$$X(y, z) = y + z\nu(\alpha y), \quad (y, t) \in M_\alpha \times \mathbb{R}. \tag{1.19}$$

$x = X(y, z)$ defines coordinates inside the image of any region where the map is one-to-one. In particular, let us consider a function $p(y)$ with

$$p(y) = (-1)^k \beta_k \log |\alpha y'| + O(1), \quad k = 1, \dots, m,$$

and β satisfying $\beta_{k+1} - \beta_k > \gamma > 0$ for all k with $a_k = a_{k+1}$. Then the map X is one-to-one for all small α in the region of points (y, z) with

$$|z - p(y)| < \frac{\delta}{\alpha} + \gamma \log(1 + |\alpha y'|)$$

provided that $\delta > 0$ is chosen sufficiently small.

Theorem 1. *Let $N = 3$ and M be an embedded minimal surface, complete with finite total curvature which is nondegenerate. Then, given β satisfying relations* (1.16) *and* (1.18)*, there exists a bounded solution u_α of equation* (1.1)*, defined for all sufficiently small α, such that*

$$u_\alpha(x) = w(z - q(y)) + O(\alpha) \quad \textit{for all} \quad x = y + z\nu(\alpha y), \quad |z - q(y)| < \frac{\delta}{\alpha}, \tag{1.20}$$

where the function q satisfies

$$q(y) = (-1)^k \beta_k \log |\alpha y'| + O(1) \quad y \in M_{k,\alpha}, \quad k = 1, \dots, m.$$

In particular, for each given $\lambda \in (-1, 1)$, the level set $[u_\alpha = \lambda]$ is an embedded surface that decomposes for all sufficiently small α into m

disjoint components (ends) outside a bounded set. The k-th end lies at $O(1)$ distance from the graph

$$y_3 \;=\; \alpha^{-1} F_k(\alpha y) + \beta_k \log|\alpha y'|.$$

The solution predicted by this theorem depends, for fixed α, on m parameters. Taking into account the constraint $\sum_{j=1}^m \beta_j = 0$, this gives $m-1$ independent parameters corresponding to logarithmic twisting of the ends of the level sets. Let us observe that, consistently, the combination $\beta \in \mathrm{Span}\,\{(a_1,\ldots,a_m)\}$ can be set in correspondence with moving α itself, namely with a dilation parameter of the surface. We are thus left with $m-2$ parameters for the solution in addition to α. Thus, besides the trivial rigid motions of the solution, translation along the coordinates' axes, and rotation about the x_3 axis, this family of solutions depends exactly on $m-1$ "independent" parameters. Part of the conclusion of our second result is that the bounded kernel of the linearization of equation (1.1) about one of these solutions is made up exactly of the generators of the rigid motions, so that in some sense the solutions found are L^∞-isolated, and the set of bounded solutions nearby is actually $m-1+J$-dimensional. A result parallel to this one, in which the *moduli space* of the minimal surface M is described by a similar number of parameters, is found in [**37**].

Next we discuss the connection of the Morse index of the solutions of Theorem 1 and the *index* of the minimal surface M, $i(M)$, which has a similar definition relative to the quadratic form for the Jacobi operator: The number $i(M)$ is the largest dimension for a vector space E of compactly supported smooth functions in M with

$$\int_M |\nabla k|^2 \, dV \;-\; \int_M |A|^2 k^2 \, dV \;<0 \quad \text{for all} \quad k \in E\setminus\{0\}.$$

We point out that for complete, embedded surfaces, finite index is equivalent to finite total curvature; see [**19**] and also §7 of [**25**] and references therein. Thus, for our surface M, $i(M)$ is indeed finite. Moreover, in the Costa-Hoffmann-Meeks surface it is known that $i(M) = 2l-1$ where l is the genus of M. See [**33**], [**34**], and [**32**].

Our second result is that the Morse index and non-degeneracy of M are transmitted into the linearization of equation (1.1).

Theorem 2. *Let u_α be the solution of problem* (1.1) *given by Theorem* 1*. Then for all sufficiently small α, we have*

$$m(u_\alpha) \;=\; i(M).$$

Besides, the solution is non-degenerate, in the sense that any bounded solution of

$$\Delta\phi + f'(u_\alpha)\phi = 0 \quad \text{in } \mathbb{R}^3$$

must be a linear combination of the functions $Z_i,\ i = 1, 2, 3, 4$ *defined as*

$$Z_i = \partial_i u_\alpha, \quad i = 1, 2, 3, \quad Z_4 = -x_2 \partial_1 u_\alpha + x_1 \partial_2 u_\alpha.$$

We will devote the rest of this paper to the proofs of Theorems 1 and 2. Before that, we discuss new ingredients and ideas in our approach, comparing with Pacard-Ritoré's [**36**].

1.4. A comparison with Pacard-Ritoré's work. In the seminal paper [**36**], Pacard and Ritoré considered the Allen-Cahn equation on an N-dimensional smooth and compact Riemannian manifold (M, g)

$$\alpha^2 \Delta_g u + u - u^3 = 0 \quad \text{in } M. \tag{1.21}$$

They constructed solutions concentrating on an $(N-1)-$dimensional embedded nondegenerate minimal submanifold of M, through an argument that shares some similarities with the one used here. Here nondegeneracy means that the Jacobi operator admits only a trivial Jacobi field. Our result in its existence part may be regarded as a counterpart of this one, but now in the entire Euclidean space. In the proof of our results we encounter the *loss of compactness* twice: First, the ambient manifold is the Euclidean space. Second, the minimal embedded surface is also non-compact. Thus, several difficulties due to this lack of compactness have to be overcome in the analysis, and a major part of our analysis is indeed devoted to design techniques to deal with this issue. Additionally, unlike the setting in [**36**], non-trivial Jacobi fields associated to rigid motions are present, and therefore we cannot speak of non-degeneracy of M.

To prove its existence, we follow the *infinite dimensional reduction* approach used in our earlier work [**11**]. (See also [**10**].) Namely, we split the problem into two steps: first we solve the problem in the orthogonal complement of the kernel of the linearized operator. This, by projection on the kernel, eventually reduces the full problem to one of solving a nonlinear, nonlocal equation which involves as a main term the Jacobi operator of the minimal surface. The solvability of the Jacobi operator with inhomogeneous and nonlocal terms is the main objective of the second step. Finally, establishing the *correspondence* between the Morse index of the solution constructed and that of the minimal surface M is fairly delicate and technically involved, and a major portion of the paper is precisely devoted to that. Our method is likely to adapt to the compact setting, with an easier proof. In [**36**], only existence of solutions was obtained. Here we obtain a complete spectral correspondence between a class of solutions of the Allen-Cahn equation and embedded, finite total curvature minimal surfaces.

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2. The Laplacian near M_α

2.1. The Laplace-Betrami Operator of M_α. Let D be the set

$$D \;=\; \{\mathtt{y} \in \mathbb{R}^2 \;/\; |\mathtt{y}| > R_0\}.$$

We can parameterize the end M_k of M as

$$\mathtt{y} \in D \longmapsto y := Y_k(\mathtt{y}) \;=\; \mathtt{y}_i e_i + F_k(\mathtt{y}) e_3, \tag{2.1}$$

and F_k is the function in (1.10). In other words, for $y = (y', y_3) \in M_k$ the coordinate $\mathtt{y}$ is just defined as $\mathtt{y} = y'$. We want to represent Δ_M—the Laplace-Beltrami operator of M—with respect to these coordinates. For the coefficients of the metric g_{ij} on M_k we have

$$\partial_{\mathtt{y}_i} Y_k = e_i + O\left(r^{-1}\right) e_3$$

so that

$$g_{ij}(\mathtt{y}) = \langle \partial_i Y_k, \partial_j Y_k \rangle = \delta_{ij} + O\left(r^{-2}\right), \tag{2.2}$$

where $r = |\mathtt{y}|$. The above relations "can be differentiated" in the sense that differentiation makes the terms $O(r^{-j})$ gain corresponding negative powers of r. Then we find the representation

$$\Delta_M = \frac{1}{\sqrt{\det g_{ij}}} \partial_i (\sqrt{\det g_{ij}}\, g^{ij} \partial_j) = \Delta_{\mathtt{y}} + O(r^{-2}) \partial_{ij} + O(r^{-3})\, \partial_i \;\text{ on } M_k\,. \tag{2.3}$$

The normal vector to M at $y \in M_k$ $k = 1, \ldots, m$ corresponds to

$$\nu(y) \;=\; (-1)^k \frac{1}{\sqrt{1 + |\nabla F_k(\mathtt{y})|^2}} \left(\partial_i F_k(\mathtt{y}) e_i - e_3 \right), \quad y = Y_k(\mathtt{y}) \in M_k$$

so that

$$\nu(y) \;=\; (-1)^k e_3 + \alpha_k r^{-2}\, \mathtt{y}_i e_i + O(r^{-2}), \quad y = Y_k(\mathtt{y}) \in M_k. \tag{2.4}$$

Let us observe for later reference that since $\partial_i \nu = O(r^{-2})$, the principal curvatures of M, k_1, k_2 satisfy $k_l = O(r^{-2})$. In particular, we have that

$$|A(y)|^2 = k_1^2 + k_2^2 = O(r^{-4}). \tag{2.5}$$

To describe the entire manifold M, we consider a finite number $N \geq m + 1$ of local parameterizations

$$\mathtt{y} \in \mathcal{U}_k \subset \mathbb{R}^2 \;\longmapsto\; y = Y_k(\mathtt{y}), \quad Y_k \in C^\infty(\bar{\mathcal{U}}_k), \quad k = 1, \ldots, N. \tag{2.6}$$

For $k = 1, \ldots, m$ we choose them to be those in (2.1), with $\mathcal{U}_k = D$, so that $Y_k(\mathcal{U}_k) = M_k$, and $\bar{\mathcal{U}}_k$ is bounded for $k = m+1, \ldots, N$. We require then that $M = \bigcup_{k=1}^{N} Y_k(\mathcal{U}_k)$. We remark that the Weierstrass representation of M implies that we can actually take $N = m + 1$, namely only one extra parameterization is needed to describe the bounded complement of the ends in M. We will not use this fact. In general, we represent for $y \in Y_k(\mathcal{U}_k)$,

$$\Delta_M = a^0_{ij}(y)\partial_{ij} + b^0_i(y)\partial_i, \quad y = Y_k(\mathtt{y}), \quad \mathtt{y} \in \mathcal{U}_k, \tag{2.7}$$

where a^0_{ij} is a uniformly elliptic matrix and the index k is not made explicit in the coefficients. For $k = 1, \ldots, m$ we have

$$a^0_{ij}(y) = \delta_{ij} + O(r^{-2}), \quad b^0_i = O(r^{-3}), \quad \text{as } r(y) = |\mathtt{y}| \to \infty. \tag{2.8}$$

The parameterizations set up above naturally induce a description of the expanded manifold $M_\alpha = \alpha^{-1}M$ as follows. Let us consider the functions

$$Y_{k\alpha} : \mathcal{U}_{k\alpha} := \alpha^{-1}\mathcal{U}_k \to M_\alpha,\ \mathtt{y} \mapsto Y_{k\alpha}(\mathtt{y}) := \alpha^{-1}Y_k(\alpha\mathtt{y}),\ k = 1, \ldots, N. \tag{2.9}$$

Obviously we have $M_\alpha = \bigcup_{k=1}^N Y_{k\alpha}(\mathcal{U}_{k\alpha})$. The computations above lead to the following representation for the operator Δ_{M_α}:

$$\Delta_{M_\alpha} = a^0_{ij}(\alpha y)\partial_{ij} + b^0_i(\alpha y)\partial_i, \quad y = Y_{k\alpha}(\mathtt{y}), \quad \mathtt{y} \in \mathcal{U}_{k\alpha}, \tag{2.10}$$

where a^0_{ij}, b^0_i are the functions in (2.7), so that for $k = 1, \ldots, m$ we have

$$a^0_{ij} = \delta_{ij} + O(r_\alpha^{-2}), \quad b^0_i = O(r_\alpha^{-3}), \quad \text{for } r_\alpha(y) := |\alpha\mathtt{y}| \geq R_0. \tag{2.11}$$

2.2. The Euclidean Laplacian near M_α. We will describe in coordinates relative to M_α the Euclidean Laplacian Δ_x, $x \in \mathbb{R}^3$, in a setting needed for the proof of our main results. Let us consider a smooth function $h : M \to \mathbb{R}$, and the smooth map X_h defined as

$$X_h \,:\, M_\alpha \times \mathbb{R} \to \mathbb{R}^3, \quad (y,t) \longmapsto X_h(y,t) := y + (t + h(\alpha y)\,)\,\nu(\alpha y) \tag{2.12}$$

where ν is the unit normal vector to M. Let us consider an open subset $\mathcal{O}$ of $M_\alpha \times \mathbb{R}$ and assume that the map $X_h|_{\mathcal{O}}$ is one to one, and that it defines a diffeomorphism onto its image $\mathcal{N} = X_h(\mathcal{O})$. Then $x = X_h(y,t), \quad (y,t) \in \mathcal{O}$, defines smooth coordinates to describe the open set $\mathcal{N}$ in $\mathbb{R}^3$. Moreover, the maps

$$x \;=\; X_h(\,Y_{k\alpha}(\mathtt{y})\,, t), \quad (\mathtt{y}, t) \in (\mathcal{U}_{k\alpha} \times \mathbb{R}) \cap \mathcal{O}, \quad k = 1, \ldots, N$$

define local coordinates $(\mathtt{y}, t)$ to describe the region $\mathcal{N}$. We shall assume in addition that for certain small number $\delta > 0$, we have

$$\mathcal{O} \subset \{(y,t) \;/\; |t + h(\alpha y)| < \frac{\delta}{\alpha} \log(2 + r_\alpha(y)\,)\,\}. \tag{2.13}$$

The Euclidean Laplacian Δ_x can be computed in a neighborhood of a region of M by the well-known formula in terms of the coordinates (y,z) with $x = y + z\nu(y)$ as

$$\Delta_x = \partial_{zz} + \Delta_{M_z} - H_{M_z}\partial_z, \tag{2.14}$$

where M_z is the manifold

$$M_z = \{y + z\nu(y) \ / \ y \in M\}.$$

Local coordinates $y = Y_k(\mathtt{y})$, $\mathtt{y} \in \mathbb{R}^2$ as in (2.1) induce natural local coordinates in M_z. The metric $g_{ij}(z)$ in M_z can then be computed as

$$g_{ij}(z) = \langle \partial_i Y, \partial_j Y\rangle + z(\langle \partial_i Y, \partial_j \nu\rangle + \langle \partial_j Y, \partial_i \nu\rangle) + z^2 \langle \partial_i \nu, \partial_j \nu\rangle \tag{2.15}$$

or

$$g_{ij}(z) = g_{ij} + z\, O(r^{-2}) + z^2 O(r^{-4}),$$

where these relations can be differentiated. Thus we find from the expression of Δ_{M_z} in local coordinates that

$$\Delta_{M_z} = \Delta_M + z a^1_{ij}(y,z)\partial_{ij} + z b^1_i(y,z)\partial_i, \quad y = Y(\mathtt{y}) \tag{2.16}$$

where a^1_{ij}, b^1_i are smooth functions of their arguments. We also find

$$g^{ij}(z) = g^{ij} + z\, O(r^{-2}) + z^2 O(r^4) + \cdots$$

Then we find that for large r,

$$\Delta_{M_z} = \Delta_M + z\, O(r^{-2})\partial_{ij} + z O(r^{-3})\partial_i. \tag{2.17}$$

We have the validity of the formula

$$H_{M_z} = \sum_{i=1}^{2} \frac{k_i}{1 - k_i z} = \sum_{i=1}^{2} k_i + k_i^2 z + k_i^3 z^2 + \cdots$$

where k_i, $i = 1, 2$ are the principal curvatures. Since M is a minimal surface, we have that $k_1 + k_2 = 0$. Thus

$$|A|^2 = k_1^2 + k_2^2 = -2k_1k_2 = -2K$$

where $|A|$ is the Euclidean norm of the second fundamental form, and K the Gauss curvature. As $r \to +\infty$ we have seen that $k_i = O(r^{-2})$ and hence $|A|^2 = O(r^{-4})$. More precisely, we find for large r,

$$H_{M_z} = |A|^2 z + z^2 O(r^{-6}).$$

Using the above considerations, a straightforward computation leads to the following expression for the Euclidean Laplacian operator in $\mathcal{N}$.

Lemma 2.1. *For $x = X_h(y,t)$, $(y,t) \in \mathcal{O}$ with $y = Y_{k\alpha}(\mathtt{y})$, $\mathtt{y} \in \mathcal{U}_{k\alpha}$, we have the validity of the identity*

$$\begin{aligned}\Delta_x = {}& \partial_{tt} + \Delta_{M_\alpha} - \alpha^2[(t+h)|A|^2 + \Delta_M h]\partial_t \ - 2\alpha\, a^0_{ij}\, \partial_j h \partial_{it} \\ & + \alpha(t+h)\,[a^1_{ij}\partial_{ij} - 2\alpha\, a^1_{ij}\, \partial_i h \partial_{jt} \ + \ \alpha\, b^1_i\, (\partial_i - \alpha \partial_i h \partial_t)\ ] \\ & + \alpha^3 (t+h)^2 b^1_3 \partial_t \ + \ \alpha^2 [\, a^0_{ij} + \alpha(t+h) a^1_{ij}\,]\partial_i h \partial_j h\, \partial_{tt}\ .\end{aligned} \tag{2.18}$$

Here, in agreement with (2.10), $\Delta_{M_\alpha} = a^0_{ij}(\alpha y)\partial_{ij} + b^0_i(\alpha y)\partial_i$. *The functions a^1_{ij}, b^1_i, b^1_3 in the above expressions appear evaluated at the pair*

$(\alpha y, \alpha(t+h(\alpha y)))$, *while the functions* h, $\partial_i h$, $\Delta_M h$, $|A|^2$, a^0_{ij}, b^0_i *are evaluated at* αy. *In addition, for* $k = 1, \dots, m$, $l = 0, 1$,

$$a^l_{ij} = \delta_{ij}\delta_{0l} + O(r_\alpha^{-2}), \quad b^l_i = O(r_\alpha^{-3}), \quad b^1_3 = O(r_\alpha^{-6}),$$

for $r_\alpha(y) = |\alpha \mathrm{y}| \geq R_0$, *uniformly in their second variables. The notation* $\partial_j h$ *refers to* $\partial_j[h \circ Y_k]$.

Actually, the coefficients a^1_{ij} and b^1_i can be further expanded as follows:

$$a^1_{ij} = a^1_{ij}(\alpha y, 0) + \alpha(t+h)\, a^{(2)}_{ij}(\alpha y, \alpha(t+h)) =: a^{1,0}_{ij} + \alpha(t+h)a^2_{ij},$$

with $a^{(2)}_{ij} = O(r_\alpha^{-3})$, and similarly

$$b^1_j = b^1_j(\alpha y, 0) + \alpha(t+h)\, b^{(2)}_j(\alpha y, \alpha(t+h)) =: b^{1,0}_j + \alpha(t+h)b^2_j,$$

with $b^{(2)}_j = O(r_\alpha^{-4})$. As an example of the previous formula, let us compute the Laplacian of a function that separates variables t and y, which will be useful in §3 and §11.

Lemma 2.2. *Let* $v(x) = k(y)\,\psi(t)$. *Then the following holds.*

$$\begin{aligned}\Delta_x v = {}& k\psi'' + \psi\,\Delta_{M_\alpha} k - \alpha^2[(t+h)|A|^2 + \Delta_M h]\,k\,\psi' - 2\alpha\, a^0_{ij}\,\partial_j h \partial_i k\,\psi' \\ & + \alpha(t+h)\,[a^{1,0}_{ij}\partial_{ij}k\psi - 2\alpha\, a^{1,0}_{ij}\,\partial_j h\partial_i k\,\psi' \\ & + \alpha(b^{1,0}_i \partial_i k\,\psi - \alpha b^{1,0}_i \partial_i h\,k\,\psi')\,] \\ & + \alpha^2(t+h)^2\,[a^2_{ij}\partial_{ij}k\psi - 2\alpha\, a^2_{ij}\,\partial_j h\partial_i k\,\psi' \\ & + \alpha(b^2_i\partial_i k\,\psi - \alpha b^2_i\partial_i h\,k\,\psi')\,] \\ & + \alpha^3(t+h)^2 b^1_3\,k\,\psi' + \alpha^2[\,a^0_{ij} + \alpha(t+h)a^1_{ij}\,]\partial_i h\partial_j h\,k\,\psi''.\end{aligned} \tag{2.19}$$

3. Approximation of the solution and preliminary discussion

3.1. Approximation of order zero and its projection. Let us consider a function h and sets $\mathcal{O}$ and $\mathcal{N}$ as in §2.2. Let $x = X_h(y,t)$ be the coordinates introduced in (2.12). At this point we shall make a more precise assumption about the function h. We need the following preliminary result, whose proof we postpone for §5.2.

We consider a fixed m-tuple of real numbers $\beta = (\beta_1, \dots, \beta_m)$ such that

$$\sum_{i=1}^m \beta_j = 0. \tag{3.1}$$

Lemma 3.1. *Given any real numbers* $\beta_1, \dots, \beta_m$ *satisfying* (3.1), *there exists a smooth function* $h_0(y)$ *defined on* M *such that*

$$\mathcal{J}(h_0) = \Delta_M h_0 + |A|^2 h_0 = 0 \quad \textit{in } M,$$

$$h_0(y) = (-1)^j \beta_j \log r + \theta \;\; \textit{as } r \to \infty \quad \textit{in } M_j \quad \textit{for all} \quad y \in M_j,$$

where θ *satisfies*

$$\|\theta\|_\infty + \|r^2 D\theta\|_\infty < +\infty \; . \tag{3.2}$$

We fix a function h_0 as in the above lemma and consider a function h in the form

$$h = h_0 + h_1.$$

We allow h_1 to be a parameter which we will adjust. For now we will assume that for a certain constant $\mathcal{K}$ we have

$$\|h_1\|_{L^\infty(M)} + \|(1+r^2)Dh_1\|_{L^\infty(M)} \;\le\; \mathcal{K}\alpha \, . \tag{3.3}$$

We want to find a solution to

$$S(u) := \Delta_x u + f(u) = 0.$$

We consider in the region $\mathcal{N}$ the approximation

$$u_0(x) := w(t) = w(z - h_0(\alpha y) - h_1(\alpha y))$$

where z designates the normal coordinate to M_α. Thus, whenever $\beta_j \neq 0$, the level sets $[u_0 = \lambda]$ for a fixed $\lambda \in (-1,1)$ depart logarithmically from the end $\alpha^{-1}M_j$, being still asymptotically catenoidal; more precisely, it is described as the graph

$$y_3 = (\alpha^{-1}a_j + \beta_j) \, \log r \, + \, O(1) \text{ as } r \to \infty.$$

Note that, just as in the minimal surface case, the coefficients of the ends are balanced in the sense that they add up to zero.

It is clear that if two ends are parallel, say $a_{j+1} = a_j$, we need at least that $\beta_{j+1} - \beta_j \ge 0$, for otherwise the ends of this zero level set would eventually intersect. We recall that our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate:

$$\beta_{j+1} - \beta_j \; > \; 4 \, \max\{\sigma_-^{-1}, \sigma_+^{-1}\} \quad \text{if} \quad a_{j+1} = a_j. \tag{3.4}$$

We will explain later the role of this condition. Let us evaluate the error of approximation $S(u_0)$. Using Lemma 2.2 and the fact that $w'' + f(w) = 0$, we find

$$\begin{aligned} S(u_0) :=& \Delta_x u_0 + f(u_0) \\ =& -\alpha^2[|A|^2 h_1 + \Delta_M h_1]\, w' \\ & + -\alpha^2 |A|^2 \, tw' + 2\,\alpha^2 a_{ij}^0 \, \partial_i h_0 \partial_j h_0 \, w'' \\ & + \alpha^2 \, a_{ij}^0 (2\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_1) w'' \\ & + 2\alpha^3 (t + h_0 + h_1) a_{ij}^1 \, \partial_i (h_0 + h_1) \partial_j (h_0 + h_1) \, w'' \\ & + \alpha^3 (t + h_0 + h_1) b_i^1 \partial_i (h_0 + h_1) \, w' + \alpha^3 (t + h_0 + h_1)^3 b_3^1 w' \end{aligned} \tag{3.5}$$

where the formula above has been broken into "sizes," keeping in mind that h_0 is fixed while $h_1 = O(\alpha)$. Since we want u_0 to be, as close as possible, a solution of (1.1), we would like to choose h_1 in such a way that the quantity (3.5) is as small as possible. Examining the above expression, it does not look like we can do that in absolute terms. However, part of the error could be made smaller by adjusting h_1. Let us consider the "L^2-projection" onto $w'(t)$ of the error for each fixed y, given by

$$\Pi(y) := \int_{-\infty}^{\infty} S(u_0)(y,t)\, w'(t)\, dt$$

where for now, and for simplicity we assume the coordinates are defined for all t; the difference with the integration taken in all the actual domain for t produces only exponentially small terms in α^{-1}. Then we find

$$\begin{aligned}\Pi(y) = {}& \alpha^2(\Delta_M h_1 + h_1|A|^2)\int_{-\infty}^{\infty} w'^2 dt + \alpha^3 \partial_i(h_0+h_1)\\ & \int_{-\infty}^{\infty} b_i^1(t+h_0+h_1)w'^2 dt\\ & + \alpha^3\partial_i(h_0+h_1)\partial_j(h_0+h_1)\int_{-\infty}^{\infty}(t+h_0+h)a_{ij}^1 w''w'\, dt + \alpha^3\\ & \int_{-\infty}^{\infty}(t+h_0+h_1)^3 b_3^1 w'^2 dt \end{aligned} \tag{3.6}$$

where we have used $\int_{-\infty}^{\infty} tw'^2\, dt = \int_{-\infty}^{\infty} w''w'\, dt = 0$ to get rid, in particular, of the terms of order α^2.

Making all these "projections" equal to zero amounts to a nonlinear differential equation for h of the form

$$\mathcal{J}(h_1) = \Delta_M h_1 + h_1|A(y)|^2 = G_0(h_1) \quad y \in M \tag{3.7}$$

where G_0 is easily checked to be a contraction mapping of small constant in h_1, in the ball radius $O(\alpha)$ with the C^1 norm defined by the expression in the left-hand side of inequality (3.3). This is where the nondegeneracy assumption on the Jacobi operator $\mathcal{J}$ enters, since we would like to invert it in such a way as to set up equation (3.7) as a fixed point problem for a contraction mapping of a ball of the form (3.3).

3.2. Improvement of approximation. The previous considerations are not sufficient since, even after optimally adjusting h, the error in absolute value does not necessarily decrease. As we observed, the "large" term in the error

$$-\alpha^2|A|^2 tw' + \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0\, w''$$

did not contribute to the projection. In order to eliminate or reduce the size of this remaining part $O(\alpha^2)$ of the error, we improve the approximation through the following argument. Let us consider the differential equation

$$\psi_0''(t) + f'(w(t))\psi_0(t) = tw'(t),$$

which has a unique bounded solution with $\psi_0(0) = 0$, given explicitly by the formula

$$\psi_0(t) = w'(t) \int_0^t (w'(s)^{-2}(\int_{-\infty}^{s} s^{'} w'(s^{'})^2 ds^{'}) ds) \,.$$

Observe that this function is well defined and that it is decaying exponentially since $\int_{-\infty}^{\infty} s w'(s)^2 ds = 0$ and $w'(t) \sim e^{-\sigma_\pm |t|}$ as $t \to \pm\infty$, with $\sigma_\pm > 0$. Note also that $\psi_1(t) = \frac{1}{2} t w'(t)$ solves

$$\psi_1''(t) + f'(w(t))\psi_1(t) = w''(t).$$

We consider as a second approximation

(3.8)

$$u_1 = u_0 + \phi_1, \quad \phi_1(y,t) := \alpha^2 |A(\alpha y)|^2 \psi_0(t) - \alpha^2 a^0_{ij} \partial_i h_0 \partial_j h_0(\alpha y)\, \psi_1(t) \ .$$

Let us observe that

$$S(u_0 + \phi) = S(u_0) + \Delta_x \phi + f'(u_0)\phi + N_0(\phi),$$
$$N_0(\phi) = f(u_0 + \phi) - f(u_0) - f'(u_0)\phi.$$

We have that

$$\partial_{tt}\phi_1 + f'(u_0)\phi_1 = \alpha^2 |A(\alpha y)|^2 t w' - \alpha^2 a^0_{ij} \partial_i h_0 \partial_j h_0(\alpha y)\, w''.$$

Hence we get that the largest remaining term in the error is canceled. Indeed, we have

$$S(u_1) = S(u_0) - (2\alpha^2 a^0_{ij} \partial_i h_0 \partial_j h_0\, w'' - \alpha^2 |A(\alpha y)|^2 t w') + [\Delta_x - \partial_{tt}]\phi_1 + N_0(\phi_1).$$

Since ϕ_1 has size of order α^2 and a smooth dependence in αy, and it is of size $O(r_\alpha^{-2} e^{-\sigma|t|})$ using Lemma 2.2, we readily check that the "error created"

$$[\Delta_x - \partial_{tt}]\phi_1 + N_0(\phi_1) := -\alpha^4 \,(\, |A|^2 t \psi_0' - a^0_{ij} \partial_i h_0 \partial_j h_0\, t \psi_1' \,)\, \Delta h_1 \ + R_0$$

satisfies

$$|R_0(y,t)| \le C\alpha^3 (1 + r_\alpha(y))^{-4} e^{-\sigma|t|}.$$

Hence we have eliminated the h_1-independent term $O(\alpha^2)$ that did not contribute to the projection $\Pi(y)$, and replaced it by one smaller and with faster decay. Let us be slightly more explicit for later reference. We have

$$\begin{aligned} S(u_1) := \Delta u_1 + f(u_1) = & \\ & - \alpha^2 [|A|^2 h_1 + \Delta_M h_1]\, w' \\ & + \alpha^2\, a^0_{ij}\, (\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_0 + \partial_i h_1 \partial_j h_1)\, w'' \\ & - \alpha^4 \,(\, |A|^2 t \psi_0' - a^0_{ij} \partial_i h_0 \partial_j h_0\, t \psi_1' \,)\, \Delta_M h_1 \\ & + 2\alpha^3 (t + h) a^1_{ij}\, \partial_i h \partial_j h\, w'' + R_1 \end{aligned} \tag{3.9}$$

where

$$R_1 = R_1(y, t, h_1(\alpha y), \nabla_M h_1(\alpha y))$$

with

$$\begin{aligned}|D_\imath R_1(y,t,\imath,\jmath)| + |D_\jmath R_1(y,t,\imath,\jmath)| \\ + |R_1(y,t,\imath,\jmath)| \le C\alpha^3 (1+r_\alpha(y))^{-4} e^{-\sigma|t|},\end{aligned}$$

and the constant C above possibly depends on the number $\mathcal{K}$ of condition (3.3).

The above arguments are in reality the way we will actually solve the problem: two separate but coupled steps are involved: (1) Eliminate the parts of the error that do not contribute to the projection Π, and (2) adjust h_1 so that the projection Π becomes identically zero.

3.3. The condition of diverging ends. Let us explain the reason to introduce condition (3.4) in the parameters β_j. To fix ideas, let us assume that we have two consecutive planar ends of M, M_j and M_{j+1}, namely with $a_j = a_{j+1}$ and with $d = b_{j+1} - b_j > 0$. Assuming that the normal in M_j points upward, the coordinate t reads approximately as

$$t = x_3 - \alpha^{-1} b_j - h \quad \text{near } M_{j\alpha}, \quad t = \alpha^{-1} b_{j+1} - x_3 - h \quad \text{near } M_{j+1\alpha}.$$

If we let $h_0 \equiv 0$ both on $M_{j\alpha}$ and $M_{(j+1)\alpha}$, which are separated at distance d/α, then a good approximation in the entire region between $M_{j\alpha}$ and $M_{(j+1)\alpha}$ that matches the parts of $w(t)$ coming both from M_j and M_{j+1} should read near M_j approximately as

$$w(t) + w(\alpha^{-1} d - t) - 1.$$

When computing the error of approximation, we observe that the following additional term arises near $M_{j\alpha}$:

$$\begin{aligned} E := f(\, w(t) + w(\alpha^{-1}d - t) - 1\,) \; - \; f(w(t)) - f(\, w(\alpha^{-1}d - t)\,) \\ \sim \; [f'(w(t)) - f'(1)\,]\,(\, w(\alpha^{-1}d - t) - 1\,). \end{aligned}$$

Now in the computation of the projection of the error, this would give rise to

$$\int_{-\infty}^{\infty} [\, f'(w(t)) - f'(1)\,]\,(\, w(\alpha^{-1}d - t) \; - \; 1\,)\, w'(t)\, dt \; \sim c_* e^{-\sigma_+ \frac{d}{\alpha}}$$

where $c_* \neq 0$ is a constant. Thus equation (3.7) for h_1 gets modified with a term which, even though very tiny, has no decay as $|y| \to +\infty$ on M_j, unlike the others involved in the operator G_0 in (3.7). That term eventually dominates, and the equation for h_1 for very large r would read in M_j as

$$\Delta_M h_1 \sim e^{-\frac{\sigma}{\alpha}} \neq 0,$$

which is inconsistent with the assumption that h is bounded. Worse yet, its solution would be quadratic, thus eventually intersecting another end. This problem is solved with the introduction of h_0 satisfying condition (3.4). In that case, the term E created above will now read near $M_{j\alpha}$ as

$$E \sim Ce^{-\sigma+\frac{d}{\alpha}}\, e^{-(\beta_{j+1}-\beta_j)\log r_\alpha}\, e^{-\sigma|t|} \;=\; O(e^{-\frac{\sigma}{\alpha}} r_\alpha^{-4} e^{-\sigma|t|})$$

which is qualitatively of the same type as the other terms involved in the computation of the error.

3.4. The global first approximation. The approximation $u_1(x)$ in (3.2) will be sufficient for our purposes; however, it is so far defined only in a region of the type $\mathcal{N}$ which we have not made precise yet. Since we are assuming that M_α is connected, the fact that M_α is properly embedded implies that $\mathbb{R}^3 \setminus M_\alpha$ consists of precisely two components, S_- and S_+. Let us use the convention that ν points in the direction of S_+. Let us consider the function $\mathbb{H}$ defined in $\mathbb{R}^3 \setminus M_\alpha$ as

$$\mathbb{H}(x) \;:=\; \begin{cases} 1 & \text{if } x \in S_+ \\ -1 & \text{if } x \in S_- \end{cases}. \tag{3.10}$$

Then our approximation $u_1(x)$ approaches $\mathbb{H}(x)$ at an exponential rate $O(e^{-\sigma_\pm |t|})$ as $|t|$ increases. The global approximation we will use consists simply of interpolating u_1 with $\mathbb{H}$ sufficiently well inside $\mathbb{R}^3 \setminus M_\alpha$ through a cut-off in $|t|$. In order to avoid the problem described in §3.3 and having the coordinates (y,t) well-defined, we consider this cut-off to be supported in a y-dependent region that expands logarithmically in r_α. Thus we will actually consider a region $\mathcal{N}_\delta$ expanding at the ends, thus becoming wider as $r_\alpha \to \infty$ than the set $\mathcal{N}_\delta^\alpha$ previously considered, where the coordinates are still well-defined.

We consider the open set $\mathcal{O}$ in $M_\alpha \times \mathbb{R}$ to be defined as

$$\begin{aligned} \mathcal{O} =& \{(y,t) \in M_\alpha \times \mathbb{R},\ |t + h_1(\alpha y)| < \frac{\delta}{\alpha} \\ & + 4\max\{\sigma_-^{-1}, \sigma_+^{-1}\} \log(1 + r_\alpha(y)) =: \rho_\alpha(y)\} \end{aligned} \tag{3.11}$$

where δ is a small positive number. We consider the region $\mathcal{N} =: \mathcal{N}_\delta$ of points x of the form

$$x = X_h(y,t) = y + (t + h_0(\alpha y) + h_1(\alpha y))\, \nu(\alpha y), \quad (y,t) \in \mathcal{O},$$

namely $\mathcal{N}_\delta = X_h(\mathcal{O})$. The coordinates (y,t) are well-defined in $\mathcal{N}_\delta$ for any sufficiently small δ: indeed, the map X_h is one-to-one in $\mathcal{O}$ thanks to assumption (3.4) and the fact that $h_1 = O(\alpha)$. Moreover, Lemma 2.1 applies in $\mathcal{N}_\delta$.

Let $\eta(s)$ be a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$, and define

$$\eta_\delta(x) \;:=\; \begin{cases} \eta(\,|t + h_1(\alpha y)| - \rho_\alpha(y) - 3) & \text{if } x \in \mathcal{N}_\delta, \\ 0 & \text{if } x \notin \mathcal{N}_\delta \end{cases} \tag{3.12}$$

where ρ_α is defined in (3.11). Then we let our global approximation $\mathtt{w}(x)$ be defined simply as

$$\mathtt{w} := \eta_\delta u_1 + (1-\eta_\delta)\mathbb{H} \tag{3.13}$$

where $\mathbb{H}$ is given by (3.10) and $u_1(x)$ is just understood to be $\mathbb{H}(x)$ outside $\mathcal{N}_\delta$.

Since $\mathbb{H}$ is an exact solution in $\mathbb{R}^3 \setminus M_\delta$, the global error of approximation is simply computed as

$$S(\mathtt{w}) \;=\; \Delta\mathtt{w} + f(\mathtt{w}) \;=\; \eta_\delta S(u_1) \quad + E \tag{3.14}$$

where

$$E = 2\nabla\eta_\delta \nabla u_1 + \Delta\eta_\delta(u_1 - \mathbb{H}) \;+\; f(\eta_\delta u_1 + (1-\eta_\delta)\mathbb{H})\,) \;-\; \eta_\delta f(u_1).$$

The new error terms created are of exponentially small size $O(e^{-\frac{\sigma}{\alpha}})$ but have, in addition, decay with r_α. In fact, we have

$$|E| \le Ce^{-\frac{\delta}{\alpha}}\, r_\alpha^{-4}.$$

Let us observe that $|t + h_1(\alpha y)| = |z - h_0(\alpha y)|$ where z is the normal coordinate to M_α; hence η_δ does not depend on h_1, and in particular the term $\Delta\eta_\delta$ does involve second derivatives of h_1 on which we have not yet made assumptions.

4. The proof of Theorem 1

The proof of Theorem 1 involves various ingredients whose detailed proofs are fairly technical. In order to keep the presentation as clear as possible, in this section we carry out the proof, skimming it from several (important) steps, which we state as lemmas or propositions. Some proofs are postponed to subsequent sections. We also refer systematically to our work [**11**]. The reader may also consult a preliminary version of this paper in the preprint [**14**].

We look for a solution u of the Allen-Cahn equation (1.1) in the form

$$u = \mathtt{w} + \varphi \tag{4.1}$$

where $\mathtt{w}$ is the global approximation defined in (3.13) and φ, is in some suitable sense, small. Thus we need to solve the following problem:

$$\Delta\varphi + f'(\mathtt{w})\varphi = -S(\mathtt{w}) - N(\varphi) \tag{4.2}$$

where

$$N(\varphi) = f(\mathtt{w}+\varphi) - f(\mathtt{w}) - f'(\mathtt{w})\varphi.$$

Next we introduce various norms that we will use to set up a suitable functional analytic scheme for solving problem (4.2). For a function $g(x)$ defined in $\mathbb{R}^3$, $1 < p \le +\infty$, $\mu > 0$, and $\alpha > 0$, we write

$$\|g\|_{p,\mu,*} := \sup_{x\in\mathbb{R}^3} (1 + r(\alpha x))^\mu \|g\|_{L^p(B(x,1))}, \quad r(x', x_3) = |x'|.$$

On the other hand, given numbers $\mu \geq 0$, $0 < \sigma < \min\{\sigma_+, \sigma_-\}$, $p > 3$, and functions $g(y,t)$ and $\phi(y,t)$ defined in $M_\alpha \times \mathbb{R}$, we consider the norms

$$\|g\|_{p,\mu,\sigma} := \sup_{(y,t)\in M_\alpha\times\mathbb{R}} r_\alpha(y)^\mu\, e^{\sigma|t|} \left(\int_{B((y,t),1)} |f|^p \, dV_\alpha \right)^{\frac{1}{p}}. \tag{4.3}$$

Consistently we set

$$\|g\|_{\infty,\mu,\sigma} := \sup_{(y,t)\in M_\alpha\times\mathbb{R}} r_\alpha(y)^\mu\, e^{\sigma|t|} \, \|f\|_{L^\infty(B((y,t),1))} \tag{4.4}$$

and let

$$\|\phi\|_{2,p,\mu,\sigma} := \|D^2\phi\|_{p,\mu,\sigma} + \|D\phi\|_{\infty,\mu,\sigma} + \|\phi\|_{\infty,\mu,\sigma}\,. \tag{4.5}$$

We also consider for a function $g(y)$ defined in M the L^p-weighted norm (4.6)

$$\|f\|_{p,\beta} := \left(\int_M |f(y)|^p \, (1+|y|^\beta\,)^p \, dV(y) \right)^{1/p} = \|\,(1+|y|^\beta)\, f\,\|_{L^p(M)}$$

where $p > 1$ and $\beta > 0$.

We assume in what follows that for a certain constant $\mathcal{K} > 0$ and $p > 3$ we have that the parameter function $h_1(y)$ satisfies

$$\|h_1\|_* := \|h_1\|_{L^\infty(M)} + \|(1+r^2)Dh_1\|_{L^\infty(M)} + \|D^2h_1\|_{p,4-\frac{4}{p}} \leq \mathcal{K}\alpha. \tag{4.7}$$

Next we reduce problem (4.2) to solving one qualitatively similar (equation (4.20) below) for a function $\phi(y,t)$ defined in the whole space $M_\alpha \times \mathbb{R}$.

4.1. Step 1: The gluing reduction. We will follow the following procedure. Let us again consider $\eta(s)$, a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$, and define

$$\zeta_n(x) := \begin{cases} \eta(\,|t + h_1(\alpha y)| - \frac{\delta}{\alpha} + n) & \text{if } x \in \mathcal{N}_\delta \\ 0 & \text{if } x \notin \mathcal{N}_\delta \end{cases}. \tag{4.8}$$

We look for a solution $\varphi(x)$ of problem (4.2) of the following form:

$$\varphi(x) = \zeta_2(x)\phi(y,t) + \psi(x), \tag{4.9}$$

where ϕ is defined in the entire $M_\alpha \times \mathbb{R}$, $\psi(x)$ is defined in $\mathbb{R}^3$, and $\zeta_2(x)\phi(y,t)$ is understood as zero outside $\mathcal{N}_\delta$.

We compute, using that $\zeta_2 \cdot \zeta_1 = \zeta_1$,

$$\begin{aligned} S(\mathtt{w} + \varphi) &= \Delta\varphi + f'(\mathtt{w})\varphi + N(\varphi) + S(\mathtt{w}) \\ = \zeta_2 \,\big[\, \Delta\phi + f'(u_1)\phi \, &+ \zeta_1(f'(u_1) + H(t))\psi \, + \zeta_1 N(\psi+\phi) + S(u_1) \,\big] \\ &+ \Delta\psi - [\,(1-\zeta_1)f'(u_1) + \zeta_1 H(t)\,]\psi \\ &+ (1-\zeta_2)S(\mathtt{w}) + (1-\zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 \end{aligned} \tag{4.10}$$

where $H(t)$ is any smooth, strictly negative function satisfying

$$H(t) \;=\; \begin{cases} f'(+1) & \text{if } t > 1\,, \\ f'(-1) & \text{if } t < -1. \end{cases}$$

Thus, we will have constructed a solution $\varphi = \zeta_2\phi + \psi$ to problem (4.2) if we require that the pair (ϕ, ψ) satisfies the following coupled system:

(4.11)

$$\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi+\phi) + S(u_1) = 0 \text{ for } |t| < \frac{\delta}{\alpha} + 3$$

$$\Delta\psi + [\,(1-\zeta_1)f'(u_1) + \zeta_1 H(t)\,]\psi \,+$$

(4.12) $$(1-\zeta_2)S(\mathtt{w}) + (1-\zeta_1)N(\psi+\zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 \;=\; 0 \quad \text{in } \mathbb{R}^3\,.$$

In order to find a solution to this system, we will first extend equation (4.11) to the entire $M_\alpha \times \mathbb{R}$ in the following manner. Let us set

(4.13) $$\mathtt{B}(\phi) = \zeta_4[\Delta_x - \partial_{tt} - \Delta_{y,M_\alpha}\,]\,\phi$$

where Δ_x is expressed in (y,t) coordinates using expression (2.18) and $\mathtt{B}(\phi)$ is understood to be zero for $|t + h_1| > \frac{\delta}{\alpha} + 5$. The other terms in equation (4.11) are simply extended as zero beyond the support of ζ_1. Thus we consider the extension of equation (4.11) given by

$$\partial_{tt}\phi \,+\, \Delta_{y,M_\alpha}\phi \,+ \mathtt{B}(\phi) \,+\, f'(w(t))\phi = -\tilde S(u_1)$$

(4.14)

$$-\left\{[f'(u_1) - f'(w)]\phi \,+\, \zeta_1(f'(u_1) - H(t))\psi \,+ \zeta_1 N(\psi+\phi)\right\} \text{ in } \in M_\alpha \times \mathbb{R},$$

where we set, with reference to expression (3.9),

$$\tilde S(u_1) = -\alpha^2[|A|^2 h_1 + \Delta_M h_1]\,w' + \alpha^2\, a^0_{ij}\,(2\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_1\,)\,w''$$

(4.15)

$$-\alpha^4\,(\,|A|^2 t\psi_0' - a^0_{ij}\partial_i h_0 \partial_j h_0\, t\psi_1'\,)\,\Delta h_1 + \zeta_4\,[\,\alpha^3 (t+h) a^1_{ij}\,\partial_i h \partial_j h\, w'' + R_1(y,t)\,]$$

and, we recall,

$$R_1 = R_1(y, t, h_1(\alpha y), \nabla_M h_1(\alpha y))$$

with

(4.16)

$$|D_\imath R_1(y,t,\imath,\jmath)| + |D_\jmath R_1(y,t,\imath,\jmath)| + |R_1(y,t,\imath,\jmath)| \le C\alpha^3(1+r_\alpha(y))^{-4}e^{-\sigma|t|}.$$

In summary, $\tilde S(u_1)$ coincides with $S(u_1)$ if $\zeta_4 = 1$, while outside the support of ζ_4, their parts that are not defined for all t are cut-off.

To solve the resulting system (4.12)–(4.14), we first solve equation (4.12) in ψ for a given ϕ, a small function in absolute value. Noticing that the potential $[\,(1-\zeta_1)f'(u_1) + \zeta_1 H(t)\,]$ is uniformly negative, so that the linear operator is qualitatively like $\Delta - 1$ and using the contraction mapping principle, a solution $\psi = \Psi(\phi)$ is found according to the following lemma, whose proof is essentially contained in Lemma 4.1 of [**11**].

Lemma 4.1. *For all sufficiently small α the following holds. Given ϕ with $\|\phi\|_{2,p,\mu,\sigma} \le 1$, there exists a unique solution $\psi = \Psi(\phi)$ of problem* (4.12) *such that*

$$\|\psi\|_X := \|D^2\psi\|_{p,\mu,*} + \|\psi\|_{p,\mu,*} \le Ce^{-\frac{\sigma\delta}{\alpha}}. \tag{4.17}$$

Besides, Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_X \le C\, e^{-\frac{\sigma\delta}{\alpha}} \|\phi_1 - \phi_2\|_{2,p,\mu,\sigma}. \tag{4.18}$$

Thus we replace $\psi = \Psi(\phi)$ in the first equation (4.11) so that by setting

$$\mathtt{N}(\phi) := \mathtt{B}(\phi) + [f'(u_1) - f'(w)]\phi + \zeta_1(f'(u_1) - H(t))\Psi(\phi) + \zeta_1 N(\Psi(\phi) + \phi), \tag{4.19}$$

our problem is reduced to finding a solution ϕ to the following nonlinear, nonlocal problem in $M_\alpha \times \mathbb{R}$:

$$\partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) \quad \text{in } M_\alpha \times \mathbb{R}. \tag{4.20}$$

Thus, we concentrate in the remainder of this proof on solving equation (4.20). As we stated in §3.2, we will find a solution of (4.20) by considering two more steps: We improve the approximation, roughly solving for ϕ that eliminates the part of the error that does not contribute to the "projections" $\int[\tilde{S}(U_1) + \mathtt{N}(\phi)]w'(t)dt$, which amounts to a nonlinear problem in ϕ. Then we adjust h_1 in such a way that the resulting projection is actually zero.

Let us set up the scheme for the next step in a precise form.

4.2. Step 2: Eliminating terms not contributing to projections.

Let us consider the problem of finding a function $\phi(y,t)$ such that for a certain function $c(y)$ defined in M_α, we have

$$\begin{aligned} &\partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathtt{N}(\phi) + c(y)w'(t) \quad \text{in } M_\alpha \times \mathbb{R}, \\ &\int_{\mathbb{R}} \phi(y,t)\, w'(t)\, dt = 0, \quad \text{for all} \quad y \in M_\alpha . \end{aligned} \tag{4.21}$$

Solving this problem for ϕ amounts to "eliminating the part of the error that does not contribute to the projection" in problem (4.20). To justify this phrase, let us consider the associated linear problem in $M_\alpha \times \mathbb{R}$,

$$\begin{aligned} &\partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + f'(w(t))\phi = g(y,t) + c(y)w'(t), \text{ for all } (y,t) \in M_\alpha \times \mathbb{R}, \\ &\int_{-\infty}^{\infty} \phi(y,t)\, w'(t)\, dt = 0, \text{ for all } y \in M_\alpha \end{aligned} \tag{4.22}$$

where

$$c(y) = -\frac{\int_{\mathbb{R}} g(y,t) w' dt}{\int_{\mathbb{R}} w'^2 dt}. \tag{4.23}$$

In order to solve problem (4.21), we need to devise a theory to solve problem (4.22), where we consider a class of right-hand sides g with a qualitative behavior similar to that of the error $S(u_1)$. As we have seen in (4.15), typical elements in this error are of the type $O((1 + r_\alpha(y))^{-\mu} e^{-\sigma|t|})$. The following fact holds.

Proposition 4.1. *Given $p > 3$, $\mu \geq 0$, and $0 < \sigma < \min\{\sigma_-, \sigma_+\}$, there exists a constant $C > 0$ such that for all sufficiently small $\alpha > 0$, the following holds: Given f with $\|g\|_{p,\mu,\sigma} < +\infty$, Problem* (4.22) *with $c(y)$ given by* (4.23) *has a unique solution ϕ with $\|\phi\|_{\infty,\mu,\sigma} < +\infty$. This solution satisfies in addition that*

$$\|\phi\|_{2,p,\mu,\sigma} \leq C\|g\|_{p,\mu,\sigma}. \tag{4.24}$$

Proof. The proof consists just in a small variation of that for Proposition 6.1 in [**11**]. We omit the details. q.e.d.

After Proposition 4.1, solving Problem (4.21) for a small ϕ is easy, using the small Lipschitz character of the terms involved in the operator $\mathtt{N}(\phi)$ in (4.19) and the contraction mapping principle. The error term $\tilde{S}(u_1)$ satisfies

$$\|\tilde{S}(u_1) + \alpha^2 \Delta h_1 w'\|_{p,4,\sigma} \leq C\alpha^3. \tag{4.25}$$

Using this, and the fact that $\mathtt{N}(\phi)$ defines a contraction mapping in a ball center zero and radius $O(\alpha^3)$ in $\| \ \|_{2,p,4,\sigma}$, we conclude the existence of a unique small solution ϕ to problem (4.21) whose size is $O(\alpha^3)$ for this norm. This solution ϕ turns out to define an operator in h_1 $\phi = \Phi(h_1)$ which is Lipschitz in the norms $\| \ \|_*$ appearing in condition (4.7). In precise terms, we have the validity of the following result, whose proof is essentially that of Proposition 4.1 in [**11**]:

Proposition 4.2. *Assume $p > 3$, $0 \leq \mu \leq 3$, $0 < \sigma < \min\{\sigma_+, \sigma_-\}$. There exists a $K > 0$ such that problem* (4.21) *has a unique solution $\phi = \Phi(h_1)$, such that*

$$\|\phi\|_{2,p,\mu,\sigma} \leq K\alpha^3 .$$

Besides, Φ has a Lipschitz dependence on h_1 satisfying (4.7) *in the sense that*

$$\|\Phi(h_1) - \Phi(h_2)\|_{2,p,\mu,\sigma} \leq C\alpha^2 \|h_1 - h_2\|_*. \tag{4.26}$$

4.3. Step 3: Adjusting h_1 to make the projection zero.

In order to conclude the proof of the theorem, we have to carry out the next step, namely adjusting h_1, within a region of the form (4.7) for suitable $\mathcal{K}$ in such a way that the "projections" are identically zero, i.e. making zero the function $c(y)$ found for the solution $\phi = \Phi(h_1)$ of problem (4.21). Using expression (4.23) for $c(y)$, we find that

$$c(y) \int_{\mathbb{R}} w'^2 = \int_{\mathbb{R}} \tilde{S}(u_1) w' \, dt + \int_{\mathbb{R}} \mathtt{N}(\Phi(h_1)\,)\, w' \, dt\,. \tag{4.27}$$

Now, setting $c_* := \int_{\mathbb{R}} w'^2 dt$ and using the same computation employed to derive formula (3.6), we find from expression (4.15) that

$$\int_{\mathbb{R}} \tilde{S}(u_1)(y,t)\, w'(t)\, dt \;=\; -c_*\, \alpha^2(\Delta_M h_1 + h_1|A|^2) \;+\; c_*\alpha^2 G_1(h_1)$$

where

$$\begin{aligned} c_* G_1(h_1) = & -\alpha^2\, \Delta h_1\, (\, |A|^2 \int_{\mathbb{R}} t\psi_0' w'\, dt - a_{ij}^0 \partial_i h_0 \partial_j h_0 \int_{\mathbb{R}} t\psi_1' w'\, dt\,) \\ & + \alpha \partial_i (h_0 + h_1) \partial_j (h_0 + h_1) \int_{\mathbb{R}} \zeta_4 (t+h) a_{ij}^1 w'' w'\, dt \\ & + \alpha^{-2} \int_{\mathbb{R}} \zeta_4\, R_1(y,t,h_1,\nabla_M h_1\,)\, w'\, dt, \end{aligned} \tag{4.28}$$

and we recall that R_1 is of size $O(\alpha^3)$ in the sense of (4.16). Thus, setting

$$c_* G_2(h_1) \;:=\; \alpha^{-2} \int_{\mathbb{R}} \mathtt{N}(\Phi(h_1)\,)\, w'\, dt, \quad G(h_1) := G_1(h_1) + G_2(h_1), \tag{4.29}$$

we find that the equation $c(y) = 0$ is equivalent to the problem

$$\mathcal{J}(h_1) = \Delta_M h_1 + |A|^2 h_1 = G(h_1) \quad \text{ in } M. \tag{4.30}$$

Therefore, we will have proven Theorem 1 if we find a function h_1 defined on M satisfying constraint (4.7) for a suitable $\mathcal{K}$ that solves equation (4.30). Again, this is not so direct, since the operator $\mathcal{J}$ has a nontrivial bounded kernel. Rather than directly solving (4.30), we consider first a projected version of this problem, namely that of finding h_1 such that for certain scalars $c_1, \ldots, c_J$ we have

$$\mathcal{J}(h_1) = G(h_1) + \sum_{i=1}^{J} \frac{c_i}{1+r^4}\, \hat{z}_i \quad \text{ in } M,$$

$$\int_M \frac{\hat{z}_i h}{1+r^4}\, dV \;=\; 0, \quad i = 1, \ldots, J. \tag{4.31}$$

Here $\hat{z}_1, \ldots, \hat{z}_J$ is a basis of the vector space of bounded Jacobi fields.

In order to solve problem (4.31), we need a corresponding linear invertibility theory. This leads us to consider the linear problem

$$\mathcal{J}(h) = f + \sum_{i=1}^{J} \frac{c_i}{1+r^4}\, \hat{z}_i \quad \text{ in } M,$$

$$\int_M \frac{\hat{z}_i h}{1+r^4}\, dV \;=\; 0, \quad i = 1, \ldots J. \tag{4.32}$$

Here $\hat{z}_1, \ldots, \hat{z}_J$ are bounded, linearly independent Jacobi fields, and J is the dimension of the vector space of bounded Jacobi fields.

We will prove in §5.1 the following result.

Proposition 4.3. *Given $p > 2$ and f with $\|f\|_{p,4-\frac{4}{p}} < +\infty$, there exists a unique bounded solution h of problem* (4.32). *Moreover, there exists a positive number $C = C(p, M)$ such that*

$$\|h\|_* := \|h\|_\infty + \|\,(1+|y|^2)\,Dh\|_\infty + \|\,D^2 h\,\|_{p,4-\frac{4}{p}} \leq C\|f\|_{p,4-\frac{4}{p}}\,. \tag{4.33}$$

Using the fact that G is a small operator of size $O(\alpha)$ uniformly on functions h_1 satisfying (4.7), Proposition 4.3 and the contraction mapping principle yield the following result, whose detailed proof we carry out in §6.

Proposition 4.4. *Given $p > 3$, there exists a number $\mathcal{K} > 0$ such that for all sufficiently small $\alpha > 0$, there is a unique solution h_1 of problem* (4.31) *that satisfies constraint* (4.7).

4.4. Step 4: Conclusion. As the last step, we prove that the constants c_i found in equation (4.31) are in reality all zero, without the need of adjusting any further parameters, but rather as a consequence of the natural invariances of the full equation. The key point is to realize what equation has been solved so far.

First we observe the following. For each h_1 satisfying (4.7), the pair (ϕ, ψ) with $\phi = \Phi(h_1)$, $\psi = \Psi(\phi)$, solves the system

$$\begin{aligned}
&\Delta\phi + f'(u_1)\phi \,+\, \zeta_1(f'(u_1) - H(t))\psi \,+ \zeta_1 N(\psi+\phi) + S(u_1)\\
&\qquad = c(y)w'(t) \text{ for } |t| < \frac{\delta}{\alpha} + 3\\
&\Delta\psi + [\,(1-\zeta_1)f'(u_1) + \zeta_1 H(t)\,]\psi\\
&\qquad + (1-\zeta_2)S(\mathtt{w}) + (1-\zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 \,=\, 0 \quad \text{ in } \mathbb{R}^3.
\end{aligned}$$

Thus setting

$$\varphi(x) = \zeta_2(x)\phi(y,t) + \psi(x), \quad u = \mathtt{w} + \varphi,$$

we find from formula (4.10) that

$$\Delta u + f(u) = S(\mathtt{w} + \varphi) \,=\, \zeta_2 c(y)\, w'(t).$$

On the other hand, choosing h_1 as that given in Proposition 4.4 which solves problem (4.31) amounts precisely to making

$$c(y) = c_*\alpha^2 \sum_{i=1}^{J} c_i \frac{\hat{z}_i(\alpha y)}{1 + r_\alpha(y)^4}$$

for certain scalars c_i. In summary, we have found h_1 satisfying constraint (4.7) such that

$$u = \mathtt{w} + \zeta_2(x)\Phi(h_1) + \Psi(\Phi(h_1)\,) \tag{4.34}$$

solves the equation

$$\Delta u + f(u) = \sum_{j=1}^{J} \frac{\tilde{c}_i}{1+r_\alpha^4}\, \hat{z}_i(\alpha y) w'(t) \tag{4.35}$$

where $\tilde{c}_i = c_* \alpha^2 c_i$. Testing equation (4.35) against the generators of the rigid motions $\partial_i u$ $i = 1,2,3$, $-x_2\partial_1 u + x_1 \partial_2 u$, and using the balancing formula for the minimal surface and the zero average of the numbers β_j in the definition of h_0, we find a system of equations that leads us to $c_i = 0$ for all i, thus concluding the proof. We will carry out the details in §7.

In sections §5–7 we will complete the proofs of the intermediate steps of the program designed in this section.

5. The Jacobi operator

We consider in this section the problem of finding a function h such that for certain constants $c_1, \dots, c_J$,

$$\mathcal{J}(h) = \Delta_M h + |A|^2 h = f + \sum_{j=1}^{J} \frac{c_i}{1+r^4}\, \hat{z}_i \quad \text{in } M, \tag{5.1}$$

$$\int_M \frac{\hat{z}_i h}{1+r^4} \;=\; 0, \quad i = 1, \dots, J \tag{5.2}$$

and prove the result of Proposition 4.3. We will also deduce the existence of Jacobi fields of logarithmic growth as in Lemma 3.1. We recall the definition of the norms $\|\ \|_{p,\beta}$ in (4.6).

Outside of a ball of sufficiently large radius R_0, it is natural to parameterize each end of M, $y_3 = F_k(y_1, y_2)$ using the Euclidean coordinates $\mathrm{y} = (y_1, y_2) \in \mathbb{R}^2$. The requirement in f on each end amounts to $\tilde{f} \in L^p(B(0, 1/R_0))$ where

$$\tilde{f}(\mathbf{y}) := |\mathbf{y}|^{-4} f(|\mathbf{y}|^{-2}\mathbf{y}). \tag{5.3}$$

Indeed, observe that

$$\begin{aligned} \|\tilde{f}\|^p_{L^p(B(0,1/R_0))} &= \int_{B(0,1/R_0)} |\mathbf{y}|^{-4p} |\, f(|\mathbf{y}|^{-2}\mathbf{y})\,|^p \, d\mathbf{y} \\ &= \int_{\mathbb{R}^2 \setminus B(0,R_0)} |\mathrm{y}|^{4(p-1)} |f(\mathrm{y})|^p \, d\mathrm{y}. \end{aligned}$$

In order to prove the proposition, we need some a priori estimates.

Lemma 5.1. *Let $p > 2$. For each $R_0 > 0$ sufficiently large, there exists a constant $C > 0$ such that if*

$$\|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(M)} < +\infty$$

and h solves

$$\Delta_M h + |A|^2 h = f, \quad y \in M, \quad |y| > R_0,$$

then

$$\|h\|_{L^\infty(|y|>2R_0)} + \| \, |y|^2 Dh\|_{L^\infty(|y|>2R_0)} + \| \, |y|^{4-\frac{4}{p}} D^2 h\|_{L^p(|y|>2R_0)} \\ \le C \, [\, \|f\|_{p,4-\frac{4}{p}} \, + \, \|h\|_{L^\infty(R_0<|y|<3R_0)} \,].$$

Proof. Along each end M_k of M, Δ_M can be expanded in the coordinate $\mathtt{y}$ as

$$\Delta_M = \Delta + O(|\mathtt{y}|^{-2})D^2 + O(|\mathtt{y}|^{-3})D.$$

A solution of h of equation (5.1) satisfies

$$\Delta_M h + |A|^2 h = f, \quad |\mathtt{y}| > R_0$$

for a sufficiently large R_0. Let us consider a Kelvin's transform

$$h(\mathtt{y}) = \tilde{h}(\mathtt{y}/|\mathtt{y}|^2).$$

Then $\tilde{h}$ satisfies the equation

$$\Delta\tilde{h} + O(|\mathbf{y}|^2)D^2\tilde{h} + O(|\mathbf{y}|)D\tilde{h} + O(1)h = \tilde{f}(\mathbf{y}), \quad 0 < |\mathbf{y}| < \frac{1}{R_0}$$

where $\tilde{f}$ is given by (5.3). The operator above satisfies the maximum principle in $B(0, \frac{1}{R_0})$ if R_0 is fixed large enough. This, the fact that $\tilde{h}$ is bounded, and L^p-elliptic regularity for $p > 2$ in two dimensional space imply that

$$\|\tilde{h}\|_{L^\infty(B(0,1/2R_0))} + \|D\tilde{h}\|_{L^\infty(B(0,1/2R_0))} + \|D^2\tilde{h}\|_{L^p(B(0,1/2R_0))} \\ \le C \, [\, \|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(B(R_0<|\mathtt{y}|<3R_0))} \,].$$

From this it directly follows that

$$\|h\|_{L^\infty(|\mathtt{y}|>2R_0)} + \| \, |\mathtt{y}|^2 Dh\|_{L^\infty(|\mathtt{y}|>2R_0)} + \| \, |\mathtt{y}|^{4-\frac{4}{p}} D^2 h\|_{L^p(|\mathtt{y}|>2R_0)} \\ \le C \, [\, \|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(B(R_0<|\mathtt{y}|<3R_0))} \,] \, .$$

Since this estimate holds at each end, the result of the lemma follows, after possibly changing slightly the value R_0. q.e.d.

Lemma 5.2. *Under the conditions of Lemma* 5.1*, assume that h is a bounded solution of problems* (5.1)–(5.2)*. Then the a priori estimate (*4.33*) holds.*

Proof. Let us observe that this a priori estimate in Lemma 5.1 implies in particular that the Jacobi fields $\hat{z}_i$ satisfy

$$\nabla\hat{z}_i(y) = O(|y|^{-2}) \quad \text{ as } |y| \to +\infty.$$

Using $\hat{z}_i$ as a test function in a ball $B(0,\rho)$ in M, we obtain

$$\int_{\partial B(0,\rho)} (h\partial_\nu\hat{z}_i - \hat{z}_i\partial_\nu\hat{z}_i) \, + \int_{|y|<\rho} (\Delta_M\hat{z}_i + |A|^2\hat{z}_i) \, h =$$

$$\int_{|y|<\rho} f\hat z_i \; + \sum_{j=1}^J c_j \int_M \frac{\hat z_i \hat z_j}{1+r^4}.$$

Since the boundary integral in the above identity is of size $O(\rho^{-1})$, we get

$$\int_M f\hat z_i \; + \sum_{j=1}^J c_j \int_M \frac{\hat z_i \hat z_j}{1+r^4} = 0 \tag{5.4}$$

so that in particular

$$|c_j| \le C\|f\|_{p,4-\frac{4}{p}} \quad \text{for all} \quad j = 1,\dots,J. \tag{5.5}$$

In order to prove the desired estimate, we assume by contradiction that there are sequences h_n, f_n with $\|h_n\|_\infty = 1$ and $\|f_n\|_{p,4-\frac{4}{p}} \to 0$, such that

$$\Delta_M h_n + |A|^2 h_n = f_n + \sum_{j=1}^J \frac{c_i^n \hat z_i}{1+r^4}$$

$$\int_M \frac{h_n \hat z_i}{1+r^4} = 0 \quad \text{for all} \quad i = 1,\dots,J.$$

Thus, according to estimate (5.5), we have that $c_i^n \to 0$. From Lemma 5.1 we find

$$\|h_n\|_{L^\infty(|y|>2R_0)} \le C[o(1) + \|h_n\|_{L^\infty(B(0,3R_0))}]\,.$$

The latter inequality implies that

$$\|h_n\|_{L^\infty(B(0,3R_0))} \ge \gamma > 0.$$

Local elliptic estimates imply a C^1 bound for h_n on bounded sets. This implies the presence of a subsequence h_n which we denote the same way such that $h_n \to h$ uniformly on compact subsets of M, where h satisfies

$$\Delta_M h + |A|^2 h = 0\,.$$

h is bounded; hence, by the nondegeneracy assumption, it is a linear combination of the functions $\hat z_i$. Besides, $h \ne 0$ and satisfies

$$\int_M \frac{h\hat z_i}{1+r^4} = 0 \quad \text{for all} \quad i = 1,\dots,J\,.$$

The latter relations imply $h = 0$, and hence a contradiction that proves the validity of the a priori estimate. q.e.d.

5.1. Proof of Proposition 4.3. Thanks to Lemma 5.2, it only remains to prove existence of a bounded solution to problems (5.1)–(5.2). Let f be as in the statement of the proposition. Let us consider the Hilbert space H of functions $h \in H^1_{loc}(M)$ with

$$\|h\|_H^2 := \int_M |\nabla h|^2 + \frac{1}{1+r^4}|h|^2 < +\infty,$$

$$\int_M \frac{1}{1+r^4} h\hat{z}_i = 0 \quad \text{for all} \quad i = 1, \dots, J.$$

Problems (5.1)–(5.2) can be formulated in weak form as finding $h \in H$ with

$$\int_M \nabla h \nabla \psi - |A|^2 h\, \psi = -\int_M f\psi \quad \text{for all} \quad \psi \in H.$$

In fact, a weak solution $h \in H$ of this problem must be bounded, thanks to elliptic regularity. This weak problem can be written as an equation of the form

$$h - T(h) = \tilde{f}$$

where T is a compact operator in H and $\tilde{f} \in H$ depends linearly on f. When $f = 0$, the a priori estimates found yield that, necessarily, $h = 0$. Existence of a solution then follows from Fredholm's alternative. The proof is complete. q.e.d.

5.2. Jacobi fields of logarithmic growth and the proof of Lemma 3.1. Let us consider an m-tuple of numbers $\beta_1, \dots, \beta_m$ with $\sum_{j=1}^m \beta_j = 0$, and any smooth function $p(y)$ in M such that on each end M_j we have that for sufficiently large $r = r(y)$,

$$p(y) = (-1)^j \beta_j \log r(y), \quad y \in M_j$$

for certain numbers $\beta_1, \dots, \beta_m$ that we will choose later. To prove the result of Lemma 3.1, we need to find a solution h_0 of the equation $\mathcal{J}(h_0) = 0$ of the form $h_0 = p + h$ where h is bounded. This amounts to solving

$$\mathcal{J}(h) = -\mathcal{J}(p). \tag{5.6}$$

Let us consider the cylinder $C_R = \{x \in \mathbb{R}^3 \ / \ r(x) < R\}$ for a large R. Then

$$\int_{M \cap C_R} \mathcal{J}(p)\, z_3 dV = \sum_{j=1}^m \int_{\partial C_R \cap M_j} (z_3 \partial_n p - p \partial_n z_3)\, d\sigma(y).$$

Thus, using the graph coordinates on each end, we find

$$\int_{M \cap C_R} \mathcal{J}(p)\, z_3 dV \ =$$

$$\sum_{j=1}^m (-1)^j \left[\frac{\beta_j}{R} \int_{|\mathrm{y}|=R} \nu_3 d\sigma(\mathrm{y}) - \beta_j \log R \int_{|\mathrm{y}|=R} \partial_r \nu_3 d\sigma(\mathrm{y}) \right] + O(R^{-1}).$$

We have that, on each end M_j,

$$\nu_3(\mathtt{y}) = \frac{(-1)^j}{\sqrt{1+|\nabla F_k(\mathtt{y})|^2}} = (-1)^j + O(r^{-2}), \quad \partial_r \nu_3(\mathtt{y}) = O(r^{-3}).$$

Hence we get

$$\int_{M\cap C_R} \mathcal{J}(p)\, z_3 dV \ = \ 2\pi \sum_{j=1}^m \beta_j + O(R^{-1})\,.$$

It is easy to see, using the graph coordinates, that $\mathcal{J}(p) = O(r^{-4})$ and it is hence integrable. We pass to the limit $R \to +\infty$ and get

$$\int_M \mathcal{J}(p)\, z_3 dV \ = \ 2\pi \sum_{j=1}^m \beta_j = 0. \tag{5.7}$$

We make a similar integration for the remaining bounded Jacobi fields. For $z_i = \nu_i(y)$ $i = 1,2$ we find

$$\int_{M\cap C_R} \mathcal{J}(p)\, z_2 dV = \sum_{j=1}^m (-1)^j$$

$$\left[\frac{\beta_j}{R} \int_{|\mathtt{y}|=R} \nu_2 d\sigma(\mathtt{y}) \ - \ \beta_j \log R \int_{|\mathtt{y}|=R} \partial_r \nu_2 d\sigma(\mathtt{y}) \right] + O(R^{-1}).$$

Now, on M_j,

$$\nu_2(\mathtt{y}) = \frac{(-1)^j}{\sqrt{1+|\nabla F_k(\mathtt{y})|^2}} = (-1)^j a_j \frac{x_i}{r^2} + O(r^{-3}), \quad \partial_r \nu_2(\mathtt{y}) = O(r^{-2}).$$

Hence

$$\int_M \mathcal{J}(p)\, z_i dV \ = \ 0 \ i = 1,2.$$

Finally, for $z_4(y) = (-y_2, y_1, 0)\cdot \nu(y)$, we find on M_j,

$$(-1)^j z_4(\mathtt{y}) = -\mathbf{y}_2 \partial_2 F_j + \mathbf{y}_1 \partial_1 F_j = b_{j1}\frac{\mathtt{y}_2}{r^2} - b_{j2}\frac{\mathtt{y}_1}{r^2} + O(r^{-2}),\ \partial_r z_4 = O(r^{-2})$$

and hence again

$$\int_M \mathcal{J}(p)\, z_4 dV \ = \ 0.$$

From the solvability theory developed, we can then find a bounded solution to the problem

$$\mathcal{J}(h) = -\mathcal{J}(p) + \sum_{j=1}^J q c_j \hat{z}_j\,.$$

Since $\int_M \mathcal{J}(p) z_i dV \ = \ 0$ and hence $\int_M \mathcal{J}(p)\hat{z}_i dV = 0$, relations (5.4) imply that $c_i = 0$ for all i.

We have thus found a bounded solution to equation (5.6) and the proof is concluded. q.e.d.

Remark 5.1. *Observe that, in particular, the explicit Jacobi field* $z_0(y) = y \cdot \nu(y)$ *satisfies that*

$$z(y) = (-1)^j a_j \log r + O(1) \quad \textit{for all} \quad y \in M_j$$

and we have indeed $\sum_j a_j = 0$. *Besides this one, we thus have the presence of another* $m-2$ *linearly independent Jacobi fields with* $|z(y)| \sim \log r$ *as* $r \to +\infty$, *where* m *is the number of ends.*

These are in reality all *Jacobi fields with exact logarithmic growth. In fact, if* $\mathcal{J}(z) = 0$ *and*

$$|z(y)| \leq C \log r \,, \tag{5.8}$$

then the argument in the proof of Lemma 5.1 *shows that the Kelvin's inversion* $\tilde{z}(\mathtt{y})$ *as in the proof of Lemma* 5.2 *satisfies near the origin* $\Delta \tilde{z} = \tilde{f}$ *where* $\tilde{f}$ *belongs to any* L^p *near the origin, so it must equal a multiple of* $\log |\mathtt{y}|$ *plus a regular function. It follows that on* M_j *there is a number* β_j *with*

$$z(\mathtt{y}) = (-1)^j \beta_j \log |\mathtt{y}| + h$$

where h *is smooth and bounded. The computations above force* $\sum_j \beta_j = 0$. *It follows from Lemma* 3.1 *that then* z *must be equal to one of the elements predicted there, plus a bounded Jacobi field. We conclude in particular that the dimension of the space of Jacobi fields satisfying* (5.8) *must be at most* $m-1+J$, *thus recovering a fact stated in Lemma 5.2 of* [**37**].

6. The reduced problem: Proof of Proposition 4.4

In this section we prove Proposition 4.4, based on the linear theory provided by Proposition 4.3. Thus, we want to solve the problem

$$\mathcal{J}(h_1) = \Delta_M h_1 + h_1 |A|^2 = G(h_1) + \sum_{i=1}^{J} \frac{c_i}{1+r^4} \hat{z}_i \quad \text{in } M, \tag{6.1}$$

$$\int_M \frac{h_1 \hat{z}_i}{1+r^4} \, dV = 0 \quad \text{for all} \quad i = 1, \ldots, J,$$

where the linearly independent Jacobi fields $\hat{z}_i$ will be as chosen in (7.1) and (7.2) of §8, and $G = G_1 + G_2$ is as defined in (4.28) and (4.29). We will use the contraction mapping principle to determine the existence of a unique solution h_1 for which constraint (4.7), namely

$$\|h_1\|_* := \|h_1\|_{L^\infty(M)} + \|(1+r^2) D h_1\|_{L^\infty(M)} + \|D^2 h_1\|_{p, 4-\frac{4}{p}} \leq \mathcal{K} \alpha, \tag{6.2}$$

is satisfied after fixing $\mathcal{K}$ sufficiently large.

To analyze the size of the operator G, we make use of the following estimate, whose lengthy but rather straightforward proof we omit.

Lemma 6.1. *Let $\psi(y,t)$ be a function defined in $M_\alpha \times \mathbb{R}$ such that*

$$\|\psi\|_{p,\mu,\sigma} := \sup_{(y,t)\in M_\alpha\times\mathbb{R}} e^{\sigma|t|}(1+r_\alpha^\mu)\,\|\psi\|_{L^p(B((y,t),1)} < +\infty$$

for $\sigma,\mu \geq 0$. The function defined in M as

$$q(y) := \int_{\mathbb{R}} \psi(y/\alpha,t)\,w'(t)\,dt$$

satisfies

$$\|q\|_{p,a} \leq C\,\|\psi\|_{p,\mu,\sigma} \tag{6.3}$$

provided that

$$\mu > \frac{2}{p} + a\,.$$

In particular, for any $\tau > 0$,

$$\|q\|_{p,2-\frac{2}{p}-\tau} \leq C\,\|\psi\|_{p,2,\sigma} \tag{6.4}$$

and

$$\|q\|_{p,4-\frac{4}{p}} \leq C\,\|\psi\|_{p,4,\sigma}\,. \tag{6.5}$$

Let us apply this result to $\psi(y,t) = \mathtt{N}(\Phi(h_1))$ to estimate the size of the operator G_2 in (4.29). For $\phi = \Phi(h_1)$, we have that

$$G_2(h_1)(y) := c_*^{-1}\alpha^{-2}\int_{\mathbb{R}} \mathtt{N}(\phi)(y/\alpha,t)\,w'\,dt$$

satisfies

$$\|G_2(h_1)\|_{p,4-\frac{4}{p}} \leq C\alpha^{-2}\|\mathtt{N}(\phi)\|_{p,4,\sigma} \leq C\,\alpha^2.$$

On the other hand, we have that, similarly, for $\phi_l = \Phi(h_l)$, $l = 1,2$,

$$\|G_2(h_1) - G_2(h_2)\|_{p,4-\frac{4}{p}} \leq C\alpha^{-2}\|\mathtt{N}(\phi_1,h_1) - \mathtt{N}(\phi_2,h_2)\|_{p,4,\sigma}.$$

Now,

$$\|\mathtt{N}(\phi_1,h_1) - \mathtt{N}(\phi_1,h_2)\|_{p,4,\sigma} \leq C\alpha^2\|h_1-h_2\|_*\|\phi_1\|_{2,p,3,\sigma}, \leq C\alpha^5\|h_1-h_2\|_*,$$

and

$$\|\mathtt{N}(\phi_1,h_1) - \mathtt{N}(\phi_2,h_1)\|_{p,4,\sigma} \leq C\alpha^2\|\phi_1-\phi_2\|_{p,3,\sigma} \leq C\alpha^4\|h_1-h_2\|_*\,.$$

We conclude then that

$$\|G_2(h_1) - G_2(h_2)\|_{p,4-\frac{4}{p}} \leq C\,\alpha^2\|h_1-h_2\|_*.$$

In addition, we also have that

$$\|G_2(0)\|_{p,4-\frac{4}{p}} \leq C\alpha^2$$

for some $C > 0$ possibly dependent of $\mathcal{K}$. On the other hand, it is similarly checked that the remaining small operator $G_1(h_1)$ in (4.28) satisfies

$$\|G_1(h_1) - G_1(h_2)\|_{p,4-\frac{4}{p}} \leq C_1\,\alpha\|h_1-h_2\|_*.$$

A simple but crucial observation we make is that

$$c_* G_1(0) = \alpha \partial_i h_0 \partial_j h_0 \int_{\mathbb{R}} \zeta_4(t+h_0) a^1_{ij} w'' w' \, dt + \alpha^{-2} \int_{\mathbb{R}} \zeta_4 R_1(y,t,0,0) w' \, dt$$

so that for a constant C_2 independent of $\mathcal{K}$ in (6.2), we have

$$\|G_1(0)\|_{p,4-\frac{4}{p}} \le C_2 \alpha \, .$$

In all, we have that the operator $G(h_1)$ has an $O(\alpha)$ Lipschitz constant, and in addition satisfies

$$\|G(0)\|_{p,4-\frac{4}{p}} \le 2C_2\alpha.$$

Let $h = T(g)$ be the linear operator defined by Proposition 4.3. Then we consider the problem (6.1) written as the fixed point problem

$$h_1 \;=\; T(\, G(h_1)\,), \quad \|h\|_* \le \mathcal{K}\alpha. \tag{6.6}$$

We have

$$\|T(\,G(h_1)\,)\|_* \le \|T\| \, \|G(0)\|_{p,4-\frac{4}{p}} + C\alpha \|h_1\|_* \, .$$

Hence, fixing $\mathcal{K} > 2C_2\|T\|$, we find that for all α sufficiently small, the operator $T\,G$ is a contraction mapping of the ball $\|h\|_* \le \mathcal{K}\alpha$ into itself. We thus have the existence of a unique solution of the fixed problem (6.6), namely a unique solution h_1 to problem (6.1) satisfying (6.2), and the proof of Proposition 4.4 is concluded. q.e.d.

7. Conclusion of the proof of Theorem 1

We denote in what follows

$$r(x) = \sqrt{x_1^2 + x_2^2}, \quad \hat{r} = \frac{1}{r}(x_1, x_2, 0), \quad \hat{\theta} = \frac{1}{r}(-x_2, x_1, 0)\, .$$

We consider the four Jacobi fields associated to rigid motions, $z_1, \dots, z_4$ introduced in (1.14). Let J be the number of bounded, linearly independent Jacobi fields of $\mathcal{J}$. By our assumption and the asymptotic expansion of the ends (1.12), $3 \le J \le 4$. (Note that when M is a catenoid, $z_4 = 0$ and $J = 3$.) Let us choose

$$\hat{z}_j = \sum_{l=1}^{4} d_{jl} z_{0l}, j = 1, \dots, J, \tag{7.1}$$

normalized such that

$$\int_M q(y) \hat{z}_i \hat{z}_j = 0, \text{ for } i \ne j, \int_M q(y) \hat{z}_i^2 = 1, i, j = 1, \dots, J. \tag{7.2}$$

In what follows we fix the function q as

$$q(y) \;:=\; \frac{1}{1 + r(y)^4} \, . \tag{7.3}$$

So far we have built, for certain constants $\tilde{c}_i$, a solution u of equation (4.35), namely

$$\Delta u + f(u) = \sum_{j=1}^{J} \tilde{c}_i \hat{z}_i(\alpha y) w'(t) q(\alpha y) \zeta_2$$

where u, defined in (4.34), satisfies the following properties:

$$u(x) = w(t) + \phi(y,t) \tag{7.4}$$

near the manifold, meaning $x = y + (t + h(\alpha y)\,)\,\nu(\alpha y)$ with

$$y \in M_\alpha, \quad |t| \le \frac{\delta}{\alpha} + \gamma \log(2 + r(\alpha y)).$$

The function ϕ satisfies in this region the estimate

$$|\phi| + |\nabla\phi| \le C\alpha^2 \frac{1}{1 + r^2(\alpha y)} e^{-\sigma|t|}\,. \tag{7.5}$$

Moreover, we have the validity of the global estimate

$$|\nabla u(x)| \le \frac{C}{1 + r^3(\alpha x)} e^{-\sigma\frac{\delta}{\alpha}}.$$

We introduce the functions

$$Z_i(x) = \partial_{x_i} u(x),\ i = 1,2,3, \quad Z_4(x) = -\alpha x_2 \partial_{x_2} u + \alpha x_1 \partial_{x_2} u\,.$$

From the expansion (7.4) we see that

$$\nabla u(x) = w'(t)\,\nabla t \ + \ \nabla\phi.$$

Now, $t = z - h(\alpha y)$, where z designates a normal coordinate to M_α. Since $\nabla z = \nu = \nu(\alpha y)$, we then get

$$\nabla t = \nu(\alpha y) - \alpha \nabla h(\alpha y).$$

Let us recall that h satisfies $h = (-1)^k \beta_k \log r + O(1)$ along the k-th end, and

$$\nabla h = (-1)^k \frac{\beta_k}{r} \hat{r} + O(r^{-2})\,.$$

From estimate (7.5) we find that

$$\nabla u(x) = w'(t)(\nu - \alpha(-1)^k \frac{\beta_k}{r_\alpha} \hat{r}) + O(\alpha r_\alpha^{-2} e^{-\sigma|t|}). \tag{7.6}$$

From here we get that, near the manifold,

$$Z_i(x) = w'(t)\,(z_i(\alpha y) - \alpha(-1)^k \frac{\beta_k}{r_\alpha} \hat{r} e_i) + O(\alpha r_\alpha^{-2} e^{-\sigma|t|}), \quad i = 1,2,3, \tag{7.7}$$

$$Z_4(x) = w'(t)\, z_{04}(\alpha y) + O(\alpha r_\alpha^{-1} e^{-\sigma|t|}). \tag{7.8}$$

Using the characterization (4.35) of the solution u and barriers (in exactly the same way as in Lemma 9.4 below, which estimates eigenfunctions of the linearized operator), we find the following estimate for $r_\alpha(x) > R_0$:

$$|\nabla u(x)| \le C \sum_{k=1}^{m} e^{-\sigma|x_3-\alpha^{-1}(F_k(\alpha x')+\beta_j\alpha\log|\alpha x'|)\,|}\,. \tag{7.9}$$

We claim that

$$\int_{\mathbb{R}^3} (\Delta u + f(u))Z_i(x)\,dx = 0 \quad \text{for all} \quad i = 1, \dots, 4 \tag{7.10}$$

so that

$$\sum_{j=1}^{J} \tilde c_j \int_{\mathbb{R}^3} q(\alpha x)\hat z_j(\alpha y)w'(t)Z_i(x)\,\zeta_2\,dx \;=\; 0 \quad \text{for all} \quad i = 1, \dots, 4. \tag{7.11}$$

Let us accept this for the moment. Let us observe that from estimates (7.7) and (7.8),

$$\begin{aligned}\alpha^2 \int_{\mathbb{R}^3} q(\alpha x)\hat z_j(\alpha y)w'(t)\sum_{l=1}^{4} d_{il}Z_l(x)\,\zeta_2\,dx \\ = \int_{-\infty}^{\infty} w'(t)^2 dt \int_M q\,\hat z_j \hat z_i dV + o(1)\end{aligned}$$

with $o(1)$ small with α. Since the functions $\hat z_i$ are linearly independent on any open set, because they solve an homogeneous elliptic PDE, we conclude that the matrix with the above coefficients is invertible. Hence from (7.11) and (7.2), all $\tilde c_i$'s are necessarily zero. We have thus found a solution to the Allen-Cahn equation (1.1) with the properties required in Theorem 1.

It remains to prove identities (7.10). The idea is to use the invariance of $\Delta u + f(u)$ under rigid translations and rotations. This type of Pohozaev identity argument has been used in a number of places; see for instance [**18**].

In order to prove that the identity (7.10) holds for $i = 3$, we consider a large number $R >> \frac{1}{\alpha}$ and the infinite cylinder

$$C_R = \{x \,/\, x_1^2 + x_2^2 < R^2\}.$$

Since in C_R the quantities involved in the integration approach zero at an exponential rate as $|x_3| \to +\infty$ uniformly in (x_1, x_2), we have that

$$\int_{C_R} (\Delta u + f(u))\partial_{x_3} u - \int_{\partial C_R} \nabla u \cdot \hat r\,\partial_{x_3} u = \int_{C_R} \partial_{x_3}\left(F(u) - \frac{1}{2}|\nabla u|^2 \right) = 0.$$

We claim that

$$\lim_{R\to+\infty}\int_{\partial C_R} \nabla u\cdot \hat r\ \partial_{x_3} u \ =\ 0.$$

Using estimate (7.6), we have that near the manifold,

$$\partial_{x_3} u \nabla u(x)\cdot \hat r = w'(t)^2((\nu-\alpha(-1)^k\frac{\beta_k}{r_\alpha}\hat r)\cdot\hat r)\nu_3 + O(\alpha e^{-\sigma|t|}\frac{1}{r^2}).$$

Let us consider the k-th end, which for large r is expanded as

$$x_3 = F_{k,\alpha}(x_1,x_2) = \alpha^{-1}(a_k\log\alpha r + b_k + O(r^{-1}))$$

so that

$$(-1)^k\nu = \frac{1}{\sqrt{1+|\nabla F_{k,\alpha}|^2}}(\nabla F_{k,\alpha},-1) = \frac{a_k}{\alpha}\frac{\hat r}{r} - e_3 \ +\ O(r^{-2}). \tag{7.12}$$

Then on the portion of C_R near this end we have that

$$(\nu-\alpha(-1)^k\frac{\beta_k}{r_\alpha}\hat r)\cdot\hat r\,\nu_3 = -\alpha^{-1}\frac{a_k+\alpha\beta_k}{R} + O(R^{-2}). \tag{7.13}$$

In addition, for $x_1^2+x_2^2=R^2$ we have the expansion

$$t = (x_3 - F_{k,\alpha}(x_1,x_2) - \beta_k\log\alpha r + O(1))(1+O(R^{-2}))$$

with the same order valid after differentiation in x_3, uniformly in such (x_1,x_2). Let us choose $\rho=\gamma\log R$ for a large, fixed γ. Observe that on ∂C_R the distance between ends is greater than 2ρ whenever α is sufficiently small. We get

$$\int_{F_{k,\alpha}(x_1,x_2)+\beta_k\log\alpha r-\rho}^{F_{k,\alpha}(x_1,x_2)+\beta_k\log\alpha r+\rho} w'(t)^2dx_3 = \int_{-\infty}^{\infty} w'(t)^2dt + O(R^{-2}).$$

Because of estimate (7.9) we conclude, fixing γ, appropriately that

$$\int_{\bigcap_k\{|x_3-F_{k,\alpha}|>\rho\}} \partial_{x_3}u\nabla u(x)\cdot\hat r\,dx_3 = O(R^{-2})\,.$$

As a conclusion,

$$\int_{-\infty}^{\infty}\partial_{x_3}u\nabla u\cdot\hat r\,dx_3 = -\frac{1}{\alpha R}\sum_{k=1}^m(a_k+\alpha\beta_k)\int_{-\infty}^{\infty} w'(t)^2\,dt \ +O(R^{-2})$$

and hence

$$\int_{\partial C_R}\partial_{x_3}u\nabla u(x)\cdot\hat r = -\frac{2\pi}{\alpha}\sum_{k=1}^m(a_k+\alpha\beta_k) + O(R^{-1})\,.$$

But $\sum_{k=1}^m a_k = \sum_{k=1}^m\beta_k = 0$ and hence (7.10) for $i=3$ follows after letting $R\to\infty$.

Let us prove the identity for $i = 2$. We now need to carry out the integration against $\partial_{x_2} u$. In this case we get

$$\int_{C_R} (\Delta u + f(u))\partial_{x_2} u = \int_{\partial C_R} \nabla u \cdot \hat{r}\, \partial_{x_2} u + \int_{C_R} \partial_{x_2} \left(F(u) - \frac{1}{2}|\nabla u|^2 \right).$$

We have that

$$\int_{C_R} \partial_{x_2} \left(F(u) - \frac{1}{2}|\nabla u|^2 \right) = \int_{\partial C_R} \left(F(u) - \frac{1}{2}|\nabla u|^2 \right) n_2$$

where $n_2 = x_2/r$. Now, near the ends, estimate (7.6) yields

$$|\nabla u|^2 = |w'(t)|^2 + O(e^{-\sigma|t|}\frac{1}{r^2}),$$

and arguing as before, we get

$$\int_{-\infty}^{\infty} |\nabla u|^2 dx_3 = m \int_{-\infty}^{\infty} |w'(t)|^2 dt + O(R^{-2}).$$

Hence

$$\int_{\partial C_R} |\nabla u|^2 n_2 = m \int_{-\infty}^{\infty} |w'(t)|^2 dt \int_{[r=R]} n_2 + O(R^{-1})\,.$$

Since $\int_{[r=R]} n_2 = 0$, we conclude that

$$\lim_{R\to+\infty} \int_{\partial C_R} |\nabla u|^2\, n_2 = 0.$$

In a similar way we get

$$\lim_{R\to+\infty} \int_{\partial C_R} F(u)\, n_2 = 0.$$

Since near the ends we have

$$\partial_{x_2} u = w'(t)(\nu_2 - \alpha(-1)^k \frac{\beta_k}{r_\alpha} \hat{r} e_2) + O(\alpha r^{-2} e^{-\sigma|t|}),$$

and from (7.12) $\nu_2 = O(R^{-1})$, completing the computation as previously done yields

$$\int_{\partial C_R} \nabla u \cdot \hat{r}\, \partial_{x_2} u = O(R^{-1}).$$

As a conclusion of the previous estimates, letting $R \to +\infty$ we finally find the validity of (7.10) for $i = 2$. Of course, the same argument holds for $i = 1$.

Finally, for $i = 4$ it is convenient to compute the integral over C_R using cylindrical coordinates. Let us write $u = u(r, \theta, z)$. Then

$$\begin{aligned}
&\int_{C_R} (\Delta u + f(u))\,(x_2\partial_{x_1} u - x_1\partial_{x_1} u) \\
=&\int_0^{2\pi}\int_0^R\int_{-\infty}^{\infty}[u_{zz} + r^{-1}(ru_r)_r + f(u)]\,u_\theta\, r\,d\theta\,dr\,dz \\
=&-\frac{1}{2}\int_0^{2\pi}\int_0^R\int_{-\infty}^{\infty}\partial_\theta\,[u_z^2 + u_r^2 - 2F(u)]\,r\,d\theta\,dr\,dz \\
&+ R\int_{-\infty}^{\infty}\int_0^{2\pi} u_r\,u_\theta(R,\theta,z)\,d\theta\,dz \\
=&0 + \int_{\partial C_R} u_r u_\theta.
\end{aligned}$$

On the other hand, on the portion of ∂C_R near the ends we have

$$u_r\,u_\theta = w'(t)^2 R(\nu\cdot\hat{r})(\nu\cdot\hat{\theta}) + O(R^{-2}e^{-\sigma|t|}).$$

From (7.12) we find

$$(\nu\cdot\hat{r})(\nu\cdot\hat{\theta}) = O(R^{-3}),$$

and hence

$$u_r\,u_\theta = w'(t)^2 O(R^{-2}) + O(R^{-2}e^{-\sigma|t|}),$$

and finally

$$\int_{\partial C_R} u_r\,u_\theta = O(R^{-1}).$$

Letting $R \to +\infty$, we obtain relation (7.10) for $i = 4$. The proof is concluded. q.e.d.

8. Negative eigenvalues and their eigenfunctions for the Jacobi operator

For the proof of Theorem 2, we need to translate the information on the index of the minimal surface M into spectral features of the Jacobi operator. Since M has finite total curvature, the index $i(M)$ of the minimal surface M is finite. We will translate this information into an eigenvalue problem for the operator $\mathcal{J}$. Let

$$\mathtt{Q}(k,k) := \int_M |\nabla k|^2\,dV \;-\; \int_M |A|^2 k^2\,dV.$$

The number $i(M)$ is, by definition, the largest dimension for a vector space E of compactly supported smooth functions in M such that

$$\mathtt{Q}(z,z) < 0 \quad \text{for all} \quad z \in E\setminus\{0\}.$$

The number $i(M)$, when finite, has the following convenient characterization, whose proof is straightforward. In what follows, we fix the function q as

$$p(y) \ := \ \frac{1}{1+r(y)^4}\ . \tag{8.14}$$

Let us consider, for a large number R, the region

$$M^R = \{y \in M \ / \ r(y) < R\}$$

and the eigenvalue problem

$$\Delta_M k + |A|^2 k + \lambda p(y)\, k = 0 \quad \text{ in } M^R, \tag{8.15}$$
$$k = 0 \quad \text{ on } \partial M^R.$$

Let $m_R(p)$ denote the number of negative eigenvalues (counting multiplicities) for this problem. Then we have

$$i(M) \ = \ \sup_{R>0} m_R(p). \tag{8.16}$$

Let us also consider the eigenvalue problem in entire space

$$\Delta_M k + |A|^2 k + \lambda p(y)\, k = 0 \quad \text{ in } M, \quad k \in L^\infty(M)\,. \tag{8.17}$$

We will prove the following result.

Lemma 8.1. *Problem* (8.17) *has exactly $i(M)$ negative eigenvalues, counting multiplicities.*

8.0.1. A priori estimates in M^R. For the proof of Lemma 8.1, and for later purposes, it is useful to have a priori estimates uniform in large $R > 0$ for the linear problem

$$\Delta_M k + |A|^2 k - \gamma p(y)\, k = f \quad \text{ in } M^R, \tag{8.18}$$
$$k = 0 \quad \text{ on } \partial M^R\ .$$

We have the following result.

Lemma 8.2. *Let $p > 1$, $\sigma > 0$. Then for $R_0 > 0$ large enough and fixed and $\gamma_0 > 0$, there exists a $C > 0$ such that for all $R > R_0 + 1$, $0 \le \gamma < \gamma_0$, any f, and any solution k of problem* (8.18)*, we have that*

(*a*) *If $\|f\|_{p,4-\frac{4}{p}} < +\infty$, then*

$$\|k\|_\infty \ \le \ C\,[\, \|f\|_{p,4-\frac{4}{p}} \ + \ \|k\|_{L^\infty(|y|<3R_0)}\ ]\,. \tag{8.19}$$

(*b*) *If $\|f\|_{p,2-\frac{2}{p}-\sigma} < +\infty$, then*

$$\|D^2 k\|_{p,2-\frac{2}{p}-\sigma} + \|\, Dk\,\|_{p,1-\frac{2}{p}-\sigma} \le C\,[\, \|f\|_{p,2-\frac{2}{p}-\sigma} \ + \ \|k\|_\infty\ ]. \tag{8.20}$$

If $p > 2$, we have in addition

$$\|\,(1+|y|)^{1-\sigma}\, Dk\,\|_\infty \le C\,[\, \|f\|_{p,2-\frac{2}{p}-\sigma} \ + \ \|k\|_\infty\ ]\,. \tag{8.21}$$

Proof. Let us consider the equation in M

$$\Delta_M \psi + |A|^2\psi = -|f|\chi_{|y|<R}, \quad |y| > R_0\,, \tag{8.22}$$

$$\psi(y) = 0, \quad |y| = R_0\,. \tag{8.23}$$

For a large and fixed R_0, solving this problem amounts to doing it on each separate end. As in Lemma 5.1, after a Kelvin's transform the problem reduces in each end to solving in a ball in $\mathbb{R}^2$ an equation of the form

$$\Delta\tilde\psi + O(|\mathbf{y}|^2)D^2\tilde\psi + O(|\mathbf{y}|)D\tilde\psi + O(1)\tilde\psi = -|\tilde f|\chi_{|\mathrm{y}|>\frac{1}{R}}, \quad |\mathbf{y}| < \frac{1}{R_0}\,,$$

$$\tilde\psi(\mathbf{y}) = 0, \quad |\mathbf{y}| = \frac{1}{R_0}\,.$$

Enlarging R_0 if necessary, this problem has a unique solution, which is also positive. This produces a bounded, positive solution ψ of (8.22)–(8.23) with

$$\|\psi\|_\infty \le C\|f\|_{p,4-\frac{4}{p}}.$$

On the other hand, on this end the Jacobi field $z_3 = \nu\cdot e_3$ can be taken as positive with $z_3 \ge 1$ on $|\mathrm{y}| > R_0$. Thus the function $\psi + \|k\|_{L^\infty(|y|=R_0)} z_3$ is a positive, bounded supersolution for the problem (8.18) on this end, where $|\mathrm{y}| > R_0$, and estimate (8.19) then readily follows.

Let us now prove estimate (8.20). Fix a large number $R_0 > 0$ and another number $R >> R_0$. Consider also a large $\rho > 0$ with $3\rho < R$. On a given end, we parameterize with Euclidean coordinates $\mathrm{y} \in \mathbb{R}^2$ and get that the equation satisfied by $k = k(\mathrm{y})$ reads

$$\Delta k + O(|\mathrm{y}|^{-2})D^2k + O(|\mathrm{y}|^{-3})Dk + O(|\mathrm{y}|^{-4})k = f, \quad R_0 < |\mathrm{y}| < R.$$

Consider the function $k_\rho(z) = k(\rho z)$ wherever it is defined. Then

$$\Delta k_\rho + O(\rho^{-2}|z|^{-2})D^2k_\rho + O(\rho^{-2}|z|^{-3})Dk_\rho + O(\rho^{-2}|z|^{-4})k_\rho = f_\rho$$

where $f_\rho(z) = \rho^2 f(\rho z)$. Then interior elliptic estimates (see Theorem 9.11 of [**17**]) yield the existence of a constant $C = C(p)$ such that for any sufficiently large ρ,

$$\begin{aligned}\|Dk_\rho\|_{L^p(1<|z|<2)} + \|D^2k_\rho\|_{L^p(1<|z|<2)} \;\le\; & C\,(\,\|k_\rho\|_{L^\infty(\frac12<|z|<3)} \\ & + \|f_\rho\|_{L^p(\frac12<|z|<3)}\,).\end{aligned} \tag{8.24}$$

Now,

$$\begin{aligned}\|f_\rho\|^p_{L^p(\frac12<|z|<3)} &= \rho^{2p}\int_{(\frac12<|z|<3)} |f(\rho z)|^p\,dz \\ &\le C\rho^{p\sigma}\int_{(\frac12<|z|<3)} |\rho z|^{2p-2-p\sigma}|f(\rho z)|^p\,\rho^2 dz \\ &= C\,\rho^{p\sigma}\int_{(\frac{\rho}{2}<|\mathrm{y}|<3\rho)} |\mathrm{y}|^{2p-2-p\sigma}|f(\mathrm{y})|^p\,d\mathrm{y}.\end{aligned}$$

Similarly,

$$\|D^2 k_\rho\|^p_{L^p(1<|z|<2)} \geq C\,\rho^{p\sigma} \int_{(\rho<|\mathrm{y}|<2\rho)} |\mathrm{y}|^{2p-2-p\sigma} |D^2 k(\mathrm{y})|^p \, d\mathrm{y}.$$

Thus

$$\int_{(\rho<|\mathrm{y}|<2\rho)} |\mathrm{y}|^{2p-2-p\sigma}|D^2 k(\mathrm{y})|^p \, d\mathrm{y} \leq C$$
$$\int_{(\frac{\rho}{2}<|\mathrm{y}|<4\rho)} |\mathrm{y}|^{2p-2-p\sigma}|f(\mathrm{y})|^p \, d\mathrm{y} \; + \rho^{-p\sigma}\|k\|^p_\infty.$$

Take $\rho = \rho_j = 2^j$. Then

$$\int_{(\rho_j<|\mathrm{y}|<\rho_{j+1})} |\mathrm{y}|^{2p-2-p\sigma}|D^2 k(\mathrm{y})|^p \, d\mathrm{y} \; \leq$$
$$C \int_{(\rho_{j-1}<|\mathrm{y}|<\rho_{j+2})} |\mathrm{y}|^{2p-2-p\sigma}|f(\mathrm{y})|^p \, d\mathrm{y} \; + 2^{-jp\sigma}\|k\|^p_\infty \, .$$

Then, adding up these relations wherever they are defined, also taking into account boundary elliptic estimates which give that for $\rho = \frac{R}{2}$,

$$\|D^2 k_\rho\|_{L^p(1<|z|<2)} \; \leq \; C \, \left(\|k_\rho\|_{L^\infty(\frac{1}{2}<|z|<2)} + \|f_\rho\|_{L^p(\frac{1}{2}<|z|<2)} \right) ,$$

plus a local elliptic estimate in a bounded region, we obtain that for some $C>0$ independent of R,

$$\|D^2 k\|_{p,2-\frac{2}{p}-\sigma} \leq \; C\,(\,\|k\|_\infty + \|f\|_{p,2-\frac{2}{p}-\sigma}\,).$$

The corresponding estimate for the gradient follows immediately from (8.24). We have proven (8.20). In the case $p>2$, we can use Sobolev's embedding to include $\|Dk_\rho\|_{L^\infty(1<|z|<2)}$ on the left-hand side of (8.24), and estimate (8.21) follows. The proof is complete. q.e.d.

8.0.2. Proof of Lemma 8.1. We will prove first that problem (8.17) has at least $i(M)$ linearly independent eigenfunctions associated to negative eigenvalues in $L^\infty(M)$. For all $R>0$ sufficiently large, problem (8.15) has $n = i(M)$ linearly independent eigenfunctions $k_{1,R}, \ldots, k_{n,R}$ associated to negative eigenvalues

$$\lambda_{1,R} \leq \lambda_{2,R} \leq \cdots \leq \lambda_{n,R} < 0.$$

Through the min-max characterization of these eigenvalues, we see that they can be chosen to define decreasing functions of R. On the other hand, $\lambda_{1,R}$ must be bounded below. Indeed, for a sufficiently large $\gamma>0$, we have that

$$|A|^2 - \gamma p \; < \; 0 \quad \text{in } M$$

and by maximum principle, we must have $\lambda_{1,R} > -\gamma$. The eigenfunctions can be chosen orthogonal in the sense that

$$\int_{M^R} p\, k_{i,R}\, k_{j,R}\, dV \;\; = \;\; 0 \quad \text{for all} \quad i \neq j. \tag{8.25}$$

Let us assume that $\|k_{i,R}\|_\infty = 1$. By (8.19) of Lemma 8.2, we obtain

$$1 = \|k_{i,R}\|_\infty \leq \ C\,[\,\|(\lambda_{i,R}+\gamma)pk_{i,R}\|_{p,4-\frac{4}{p}} + \|k_{i,R}\|_{L^\infty(|y|<3R_0)}\,]\,. \tag{8.26}$$

Then the a priori estimate in Lemma 8.2 implies, passing to a subsequence in $R \to +\infty$, that we may assume that

$$\lambda_{i,R} \downarrow \lambda_i < 0, \qquad k_{i,R}(y) \to k_i(y),$$

uniformly on compact subsets of M, where $k_i \not\equiv 0$ (by (8.26)) is a bounded eigenfunction of (8.17) associated to the negative eigenvalue λ_i. Moreover, relations (8.25) pass to the limit and yield

$$\int_M p\, k_i\, k_j\, dV \ = \ 0 \quad \text{for all} \quad i \neq j. \tag{8.27}$$

Thus, problem (8.17) has at least $n = i(M)$ negative eigenvalues. Let us assume there is a further bounded eigenfunction k_{n+1}, linearly independent of $k_1, \dots, k_n$, say with

$$\int_M p\, k_i\, k_{n+1}\, dV \ = \ 0 \quad \text{for all} \quad i = 1, \dots, n, \tag{8.28}$$

associated to a negative eigenvalue λ_{n+1}. Then the a priori estimate of Lemma 5.1 implies that

$$\|(1 + r^2)\nabla k_{n+1}\| \ < +\infty.$$

The same of course holds for the remaining k_i's. It follows that

$$\mathtt{Q}(k, k) < 0 \quad \text{for all} \quad k \in \operatorname{span}\{k_1, \dots, k_{n+1}\} \setminus \{0\}.$$

However, again since ∇k_j decays fast, the same relation above will hold true for the k_i's replaced by suitable smooth truncations far away from the origin. This implies, by definition, $i(M) \geq n+1$, and we have reached a contradiction. The proof is concluded. q.e.d.

9. The proof of Theorem 2

In this section we will prove that the Morse index $m(u_\alpha)$ of the solution we have built in Theorem 1 coincides with the index of the surface M, as stated in Theorem 2. We recall that this number is defined as the supremum of all dimensions of vector spaces E of compactly supported smooth functions for which

$$\mathcal{Q}(\psi, \psi) \ = \ \int_{\mathbb{R}^3} |\nabla \psi|^2 - f'(u_\alpha)\psi^2 < 0 \quad \text{for all} \quad \psi \in E \setminus \{0\}.$$

We provide next a more convenient characterization of this number, analogous to that for the Jacobi operator of §8. Let us consider a smooth function $p(x)$ defined in $\mathbb{R}^3$ such that

$$p(\alpha x) = \frac{1}{1 + r_\alpha(y)^4} \quad \text{if} \quad x = y + (t + h(\alpha y))\nu(\alpha y) \ \in \mathcal{N}_\delta,$$

and such that for positive numbers a, b,

$$\frac{a}{1+|\alpha x'|^4} \ \le\ p(\alpha x) \ \le\ \frac{b}{1+|\alpha x'|^4} \qquad \text{for all} \quad x = (x', x_3) \in \mathbb{R}^3.$$

For each $R > 0$, we consider the eigenvalue problem in the cylinder

$$\mathcal{C}_R \ =\ \{\, (x', x_3) \ / \ |x'| < R\alpha^{-1},\ |x_3| < R\alpha^{-1}) \,\},$$

$$\Delta\phi + f'(u_\alpha)\phi + \lambda p(\alpha x)\phi = 0 \quad \text{in } \mathcal{C}_R, \tag{9.1}$$

$$\phi = 0 \quad \text{on } \partial\mathcal{C}_R.$$

We also consider the problem in entire space

$$\Delta\phi + f'(u_\alpha)\phi + \lambda p(\alpha x)\phi = 0 \quad \text{in } \mathbb{R}^3, \quad \phi \in L^\infty(\mathbb{R}^3). \tag{9.2}$$

Let $m_R(u_\alpha)$ be the number of negative eigenvalues λ (counting multiplicities) of problem (9.1). Then we readily check that

$$m(u_\alpha) = \sup_{R>0} m_R(u_\alpha).$$

On the other hand, we have seen in §8 that the index $i(M)$ of the minimal surface can be characterized as the number of linearly independent eigenfunctions associated to negative eigenvalues of the problem

$$\Delta z + |A|^2 z + \lambda p(y) z = 0 \quad \text{in } M, \quad z \in L^\infty(M)\,, \tag{9.3}$$

which corresponds to the maximal dimension of the negative subspace in $L^\infty(M)$ for the quadratic form

$$\mathtt{Q}(z,z) \ =\ \int_M |\nabla_M z|^2 - |A|^2 z^2 \, dV.$$

We shall prove in this section that $m(u_\alpha) = i(M)$ for any sufficiently small α.

The idea of the proof is to put in correspondence eigenfunctions for negative eigenvalues of problem (9.1) for large R with those of problem (9.3). This correspondence comes roughly as follows. If z is such an eigenfunction for problem (9.3), then the function defined near M_α as

$$k(y)w'(t), \quad k(y) = z(\alpha y) \tag{9.4}$$

defines after truncation a negative direction for the quadratic form $\mathcal{Q}$ on any large ball. Reciprocally, an eigenfunction for a negative eigenvalue of problem (9.1) will look for any sufficiently small α and all large R like a function of the form (9.4). In the following two lemmas, we clarify the action of the operator L on functions of this type, and the corresponding connection at the level of the quadratic forms $\mathcal{Q}$ and $\mathtt{Q}$.

Lemma 9.1. *Let $k(y)$ be a function of class C^2 defined in some open subset $\mathcal{V}$ of M_α. Let us consider the function $v(x)$ defined for $x \in \mathcal{N}_\delta$, $y \in \mathcal{V}$ as*

$$v(x) \ =\ v(y,t) \ :=\ k(y)\, w'(t)\,, \quad y \in \mathcal{V}, \quad |t + h_1(\alpha y)| \ <\ \rho_\alpha(y)$$

where ρ_α is the function in the definition of $\mathcal{N}_\alpha$, (3.11). Then $L(v) := \Delta_x v + f'(u_\alpha)v$ can be expanded as in (9.8) below. Besides, we have

$$\int_{|t+h_1|<\rho_\alpha} L(v)\, w'\, dt \;=\; (\,\Delta_{M_\alpha} k + \alpha^2|A|^2 k \,+\, \alpha h a_{ij}^{1,0}\partial_{ij} k\,) \int_{\mathbb{R}} w'^2\, dt$$

$$+\, O(\alpha^2 r_\alpha^{-2})\,\partial_{ij}k \,+\, O(\alpha^2 r_\alpha^{-3})\,\partial_i k \,+ O(\alpha^3 r_\alpha^{-4})k. \tag{9.5}$$

Here

$$a_{ij}^{1,0} = a_{ij}^{1,0}(\alpha y) = O(r_\alpha^{-2})\ .$$

The same conclusions hold for the function

$$v(x) \,=\, v(y,t) \;:=\; k(y)\, w'(t)\, \eta_\delta(y,t), \quad y \in \mathcal{V}, \quad |t + h_1(\alpha y)| \,<\, \rho_\alpha(y)$$

where the cut-off function η_δ is defined as in (3.12).

Proof. Using Lemma 2.2, we get

$$\begin{aligned}
\Delta_x v + f'(u_\alpha)v &= k(w''' + f'(w)w') + [f'(u_\alpha) - f'(w)]\, kw' \\
&\quad + w'\,\Delta_{M_\alpha} k \,-\, \alpha^2[(t+h_1)|A|^2 + \Delta_M h_1]\, k\, w'' \,-\, 2\alpha\, a_{ij}^0 \partial_j h \partial_i k\, w'' \\
&\quad + \alpha(t+h)\,[a_{ij}^1 \partial_{ij} k w' - \alpha\, a_{ij}^1(\,\partial_j h \partial_i k + \partial_i h \partial_j k)\, w'' \\
&\quad + \alpha(b_i^1 \partial_i k\, w' - \alpha b_i^1 \partial_i h w'')\,] \\
&\quad + \alpha^3(t+h)^2 b_3^1\, k\, w'' \,+\, \alpha^2[\,a_{ij}^0 + \alpha(t+h)a_{ij}^1)\,]\partial_i h \partial_j h\, k\, w'''.
\end{aligned} \tag{9.6}$$

We can expand

$$a_{ij}^1 = a_{ij}^1(\alpha y, 0) + \alpha(t+h)\, a_{ij}^2(\alpha y, \alpha(t+h)) =: a_{ij}^{1,0} + \alpha(t+h) a_{ij}^2$$

with $a_{ij}^2 = O(r_\alpha^{-2})$, and similarly

$$b_j^1 = b_j^1(\alpha y, 0) + \alpha(t+h)\, b_j^2(\alpha y, \alpha(t+h)) =: b_j^{1,0} + \alpha(t+h) b_j^2$$

with $b_j^2 = O(r_\alpha^{-3})$. On the other hand, let us recall that

$$u_\alpha - w = \phi_1 + O(\alpha^3 r_\alpha^{-4} e^{-\sigma|t|})$$

where ϕ_1 is given by (3.2),

$$\phi_1(y,t) = \alpha^2|A(\alpha y)|^2 \psi_0(t) - \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0(\alpha y)\, \psi_1(t), \tag{9.7}$$

and ψ_0, ψ_1 decay exponentially as $|t| \to +\infty$. Hence

$$[f'(u_\alpha) - f'(w)]\, w' \;=\; f''(w)\, \phi_1\, w' \;+\; O(\alpha^3 e^{-\sigma|t|} r_\alpha^{-4}).$$

Using these considerations and expression (9.6), we can write

$$Q \;:=\; \Delta_x v + f'(u_\alpha)v \;=$$

$$\underbrace{\Delta_{M_\alpha} k\, w' \,-\, \alpha^2|A|^2\, k\, t w'' + \alpha^2\, a_{ij}^0 \partial_i h_0 \partial_j h_0\, k\, w''' + \alpha h a_{ij}^{1,0} \partial_{ij} k w'}_{Q_1}$$

$$+ \underbrace{f''(w)\, \phi_1\, kw'}_{Q_2}$$

$$\underbrace{-w''\left[\alpha a^0_{ij}(\partial_j h\partial_i k+\partial_i h\partial_j k)+\alpha^2 k\Delta_M h_1+\alpha^2 h a^{1,0}_{ij}(\partial_j h\partial_i k+\partial_i h\partial_j k)\right]}_{Q_3}$$

$$+\ \underbrace{\alpha t w'\left[a^{1,0}_{ij}\partial_{ij}k+\alpha b^{1,0}_i\partial_i k\right]}_{Q_4}$$

$$+\underbrace{\alpha^2(t+h)^2 a^2_{ij}\partial_{ij}k w'+\alpha^2(t+h)a^2_{ij}(\partial_j h\partial_i k+\partial_i h\partial_j k)w''}_{Q_5}+\underbrace{O(\alpha^3 e^{-\sigma|t|}r_\alpha^{-2})}_{Q_6}. \tag{9.8}$$

The precise meaning of the remainder Q_6 is

$$Q_6\ =\ O(\alpha^3 e^{-\sigma|t|}r_\alpha^{-2})\,\partial_{ij}k\ +\ O(\alpha^3 e^{-\sigma|t|}r_\alpha^{-3})\,\partial_j k\,.$$

We will integrate the above relation against $w'(t)$ in the region $|t+h_1(\alpha y)|<\rho_\alpha(y)$. Let us observe that the terms Q_i for $i=1,\dots,4$ are in reality defined for all t and that

$$\int_{|t+h_1|<\rho_\alpha} Q_i\,w'\,dt\ =\ \int_{\mathbb{R}} Q_i\,w'\,dt\ +O(\alpha^3 r_\alpha^{-4}) \tag{9.9}$$

where the remainder means

$$O(\alpha^3 r_\alpha^{-4})\ :=\ O(\alpha^3 r_\alpha^{-4})\,\partial_{ij}k+O(\alpha^3 r_\alpha^{-4})\,\partial_i k+\ O(\alpha^3 r_\alpha^{-4})\,k.$$

Let us observe that

$$\int_{\mathbb{R}}(Q_3\ +\ Q_4)\,w'\,dt\ =\ 0. \tag{9.10}$$

On the other hand, since

$$\int_{\mathbb{R}} t w'' w'\,dt\ =\ -\frac{1}{2}\int_{\mathbb{R}} w'^2\,dt,$$

we get that

$$\begin{aligned}\int_{\mathbb{R}} Q_1\,w'\,dt &= (\,\Delta_{M_\alpha}k+\frac{1}{2}|A|^2k+\alpha h a^{1,0}_{ij}\partial_{ij}k)\int_{\mathbb{R}} w'^2\,dt\\ &\quad + a^0_{ij}\partial_i h_0\partial_j h_0\int_{\mathbb{R}} w'''\,w'\,dt.\end{aligned} \tag{9.11}$$

Next we will compute $\int_{\mathbb{R}} Q_2\,w'\,dt$. We recall that, setting $L_0(\psi)=\psi''+f'(w)\psi$, the functions ψ_0 and ψ_1 in (9.7) satisfy

$$L_0(\psi_0)=tw'(t),\quad L_0(\psi_1)=w''.$$

Differentiating these equations, we get

$$L_0(\psi_0')+f''(w)w'\psi_0=(tw')',\quad L_0(\psi_1')+f''(w)w'\psi_1=w'''.$$

Integrating by parts against w', using $L_0(w')=0$, we obtain

$$\int_{\mathbb{R}} f''(w)w'^2\psi_0=-\int_{\mathbb{R}} tw''w'=\frac{1}{2}\int w'^2,\quad \int_{\mathbb{R}} f''(w)w'^2\psi_1=\int_{\mathbb{R}} w'''w'.$$

Therefore

$$\int_{\mathbb{R}} Q_1\, w'\, dt \;=\; \int_{\mathbb{R}} f''(w)\phi_1\, kw'^2\, dt$$

$$= \alpha^2 k|A|^2 \int_{\mathbb{R}} f''(w)\psi_0\, w'^2\, dt \,-\, \alpha^2 a^0_{ij}\partial_i h_0 \partial_j h_0\, k \int_{\mathbb{R}} f''(w)\psi_1\, w'^2\, dt$$

$$= \alpha^2 k|A|^2 \frac{1}{2}\int_{\mathbb{R}} w'^2 - \alpha^2 a^0_{ij}\partial_i h_0 \partial_j h_0\, k \int_{\mathbb{R}} w''' w'. \tag{9.12}$$

Thus, combining relations (9.10)–(9.12), we get

$$\int_{\mathbb{R}} (\, Q_1 + \cdots + Q_4)\, w'\, dt \;=\; (\, \Delta_{M_\alpha} k + |A|^2 k \,+\, \alpha h a^{1,0}_{ij}\partial_{ij} k\,) \int_{\mathbb{R}} w'^2\, dt. \tag{9.13}$$

On the other hand, we observe that

$$\int_{|t+h_1|<\rho_\alpha} (Q_5 + Q_6)\, w'\, dt \;=\; O(\alpha^2 r_\alpha^{-2})\, \partial_{ij} k \,+\, O(\alpha^2 r_\alpha^{-3})\, \partial_i k. \tag{9.14}$$

Combining relations (9.13), (9.14), and (9.9), expansion (9.5) follows. Finally, for v replaced by $\eta_\delta k\, w'$, we have that

$$\int L(kw\eta_\delta)\eta_\delta\, kw'\, dt = \int \eta_\delta^2 L(kw)kw\, dt$$
$$+ \int \eta_\delta(\Delta\eta_\delta\, kw' + 2\nabla\eta_\delta \nabla(kw'))\, kw\, dt.$$

The arguments above apply to obtain the desired expansion for the first integral in the right-hand side of the above decomposition. The second integral produces only smaller order operators in k since $\Delta\eta_\delta$, $\nabla\eta_\delta$ are both of order $O(r_\alpha^{-4}\alpha^4)$ inside their supports. The proof is concluded. q.e.d.

Let us now consider the region

$$\mathcal{W} := \{x \in \mathcal{N}_\delta \;/\; r_\alpha(y) < R\},$$

where R is a given large number.

Lemma 9.2. *Let $k(y)$ be a smooth function in M_α that vanishes when $r_\alpha(y) = R$, and set $v(y,t) := \eta_\delta(y,t)\, k(y)\, w'(t)$. Then the following estimate holds.*

$$\mathcal{Q}(v,v) = \int_{\mathcal{W}} |\nabla v\,|^2 \,-\, f'(u_\alpha)\, v^2\, dx$$

$$= \int_{r_\alpha(y)<R} \left[\, |\nabla_{M_\alpha} k|^2 - \alpha^2 |A(\alpha y)|^2\, k^2\,\right]\, dV_\alpha \int_{\mathbb{R}} w'^2\, dt$$

$$+\, O\left(\alpha \int_{r_\alpha(y)<R} [\, |\nabla k|^2 \,+\, \alpha^2\, (1 + r_\alpha^4)^{-1}\, k^2\,]\, dV_\alpha\right). \tag{9.15}$$

Proof. The proof follows from computations in a similar spirit to those in Lemma 9.1, and the facts that mean curvature of M vanishes and the Gauss curvature equals $|A|^2$, so that

$$dx \;=\; (\,1+\alpha^2(t+h)^2\,|A(\alpha y)|^2\,)\;dV_\alpha(y)\,dt.$$

After Lemma 9.2, the inequality

$$m(u_\alpha)\;\geq\; i(M) \tag{9.16}$$

for small α follows at once. Indeed, we showed in §8 that the Jacobi operator has exactly $i(M)$ linearly independent bounded eigenfunctions $\hat{z}_i$ associated to negative eigenvalues λ_i of the weighted problem in entire space M. According to the theory developed in §5, we also find that $\nabla\hat{z}_i = O(r^{-2})$; hence we may assume

$$\mathtt{Q}(\hat{z}_i,\hat{z}_j)\;=\;\lambda_i\int_M q\,\hat{z}_i\,\hat{z}_j\,dV. \tag{9.17}$$

Let us set $k_i(y) := \hat{z}_i(\alpha y)$. According to Lemma 9.2, setting $v_i(x) = k_i(y)\,w'(t)\eta_\delta$ and changing variables, we get

$$\mathcal{Q}(v_i,v_j) = \alpha^2\mathtt{Q}(\hat{z}_i,\hat{z}_j)\int_{\mathbb{R}} w'^2 + O(\alpha^3)\sum_{l=i,j}\int_M |\nabla\hat{z}_l|^2 + (1+r^4)^{-1}\hat{z}_l^2\,dV. \tag{9.18}$$

From here and relation (9.17), we find that the quadratic form $\mathcal{Q}$ is negative on the space spanned by the functions $v_1,\dots,v_{i(M)}$. The same remains true for the functions v_i smoothly truncated around $r_\alpha(y) = R$, for very large R. We have thus proven inequality (9.16).

In what remains of this section, we will carry out the proof of the inequality

$$m(u_\alpha)\;\leq\; i(M). \tag{9.19}$$

Relation (9.18) suggests, that associated to a negative eigenvalue λ_i of problem (9.3), there is an eigenvalue of (9.1) approximated by $\sim\lambda_i\alpha^2$. We will show next that negative eigenvalues of problem (9.1) cannot exceed a size $O(\alpha^2)$.

Lemma 9.3. *There exists a $\mu > 0$ independent of $R > 0$ and all small α such that if λ is an eigenvalue of problem* (9.1)*, then*

$$\lambda \geq -\mu\,\alpha^2.$$

Proof. Let us denote

$$\mathcal{Q}_\Omega(\psi,\psi)\;:=\;\int_\Omega |\nabla\psi|^2 - f'(u_\alpha)\psi^2.$$

Then if $\psi(x)$ is any function that vanishes for $|x'| > R\alpha^{-1}$, we have

$$\mathcal{Q}(\psi,\psi)\;\geq\;\mathcal{Q}_{\mathcal{N}_\delta\cap\{r_\alpha(y)<R\}}\,(\psi,\psi)\;+\;\gamma\int_{\mathbb{R}^3\setminus\mathcal{N}_\delta}\psi^2$$

where $\gamma > 0$ is independent of α and R. We want to prove that, for some $\mu > 0$, we have in $\Omega = \mathcal{N}_\delta \cap \{r_\alpha(y) < R\}$ that

$$\mathcal{Q}_\Omega(\psi,\psi) \geq -\mu\alpha^2 \int_\Omega \frac{\psi^2}{1+r_\alpha^4}\,dx. \tag{9.20}$$

Equivalently, let us consider the eigenvalue problem

$$L(\psi) + \lambda p(\alpha x)\psi = 0 \quad \text{in } \Omega, \tag{9.21}$$

$$\psi = 0 \quad \text{on } r_\alpha = R, \quad \partial_n\psi = 0 \quad \text{on } |t+h_1| = \rho_\alpha.$$

Then we need to show that for any eigenfunction ψ associated to a negative eigenfunction, inequality (9.20) holds. Here ∂_n denotes the normal derivative. Let us express this boundary operator in terms of the coordinates (t,y). Let us consider the portion of $\partial\mathcal{N}_\delta$ where

$$t + h_1(\alpha y) = \rho_\alpha(y). \tag{9.22}$$

We recall that for some $\gamma > 0$, $\rho_\alpha(y) = \rho(\alpha y) = \gamma\log(1+r_\alpha(y))$. Relation (9.22) is equivalent to

$$z - h_0(\alpha y) - \rho_\alpha(y) = 0 \tag{9.23}$$

where z denotes the normal coordinate to M_α. Then, for $\nabla = \nabla_x$, we have that a normal vector to the boundary at a point satisfying (9.23) is

$$n = \nabla z - \nabla_{M_\alpha}(h_0+\rho) = \nu(\alpha y) - \alpha\nabla_M(h_0+\rho)(\alpha y).$$

Now we have that $\partial_t\psi = \nabla_x\psi\cdot\nu(\alpha y)$. Hence, on points (9.22), condition $\partial_n\psi = 0$ is equivalent to

$$\partial_t\psi - \alpha\nabla_M(h_0+\rho)\cdot\nabla_{M_\alpha}\psi = 0, \tag{9.24}$$

and similarly, for

$$t + h_1(\alpha y) = \rho_\alpha(y), \tag{9.25}$$

it corresponds to

$$\partial_t\psi - \alpha\nabla_M(h_0-\rho)\cdot\nabla_{M_\alpha}\psi = 0. \tag{9.26}$$

Let us consider a solution ψ of problem (9.21). We decompose

$$\psi = k(y)w'(t)\eta_\delta + \psi^\perp$$

where η_δ is the cut-off function (3.12) and

$$\int_{|\tau+h_1(\alpha y)|<\rho_\alpha(y)} \psi^\perp(y,\tau)\,w'(\tau)\,d\tau = 0 \quad \text{for all} \quad y \in M_\alpha \cap \{r_\alpha(y) < R\},$$

namely

$$k(y) = \frac{\int_{|\tau+h_1(\alpha y)|<\rho_\alpha(y)} \psi(y,\tau)\,w'(\tau)\,d\tau}{\int_{\mathbb{R}} w'(t)^2\eta_\delta\,dt}. \tag{9.27}$$

Then we have

$$\mathcal{Q}_\Omega(\psi,\psi) = \mathcal{Q}_\Omega(\psi^\perp,\psi^\perp) + \mathcal{Q}_\Omega(kw'\eta_\delta,kw'\eta_\delta) + 2\mathcal{Q}_\Omega(kw'\eta_\delta,\psi^\perp).$$

Since $\psi^\perp$ satisfies the same boundary conditions as ψ, we have that

$$\mathcal{Q}_\Omega(\psi^\perp,\psi^\perp) = -\int_\Omega (\,\psi^\perp \Delta_x \psi^\perp + f'(u_\alpha){\psi^\perp}^2)\,dx.$$

Thus,

$$\mathcal{Q}_\Omega(\psi^\perp,\psi^\perp) = -\int_{r_\alpha<R}\int_{|t+h_1|<\rho_\alpha} [\psi^\perp\,\Delta_x\psi^\perp \\ + f'(u_\alpha){\psi^\perp}^2]\,(1+\alpha^2(t+h)^2|A|^2)\,dV_\alpha\,dt\,.$$

Let us fix a smooth function $H(t)$ with $H(t)=+1$ if $t>1$, $H(t)=-1$ for $t<-1$. Let us write

$$-\Delta_x\psi^\perp - f'(u_\alpha)\psi^\perp = -\partial_{tt}\psi^\perp \\ - f'(w)\psi^\perp + \alpha\partial_t\left[\nabla_M(h_0+H(t)\rho)\cdot\nabla_{M_\alpha}\psi^\perp\right] \\ - \Delta_{M_\alpha}\psi^\perp + B(\psi^\perp).$$

Then, integrating by parts in t, using the Neumann boundary condition, we get that the integral

$$I := \\ -\int_{|t+h_1|<\rho_\alpha}\psi^\perp\left[\partial_{tt}\psi^\perp + f'(w)\psi^\perp\right]\,dt \\ +\int_{|t+h_1|<\rho_\alpha}\alpha\partial_t\left(\nabla_M(h_0+H(t)\rho)\cdot\nabla_{M_\alpha}\psi^\perp\right)\,\psi^\perp\,(1+\alpha^2(t+h)^2|A|^2)\,dt \\ = \int_{|t+h_1|<\rho_\alpha}[\,|\partial_t\psi^\perp|^2 - f'(w)|\psi^\perp|^2\,]\,(1+o(1)\,)\,dt \\ + \int_{|t+h_1|<\rho_\alpha}[\alpha O(r_\alpha^{-1})\nabla_{M_\alpha}\psi^\perp\,\partial_t\psi^\perp + o(1)\partial_t\psi^\perp\,\psi^\perp]\,dt\ .$$

Now we need to make use of the following standard fact: there is a $\gamma>0$ such that if $a>0$ is a sufficiently large number, then for any smooth function $\xi(t)$ with $\int_{-a}^a \xi\,w'\,dt=0$, we have that

$$\int_{-a}^a {\xi'}^2 - f'(w)\xi^2\,dt \ \ge\ \gamma\int_{-a}^a {\xi'}^2+\xi^2\,dt\,. \tag{9.28}$$

Hence

(9.29)

$$I\ \ge\ \frac{\gamma}{2}\int_{|t+h_1|<\rho_\alpha}[\,|\partial_t\psi^\perp|^2+|\psi^\perp|^2\,]\,dt + \int_{|t+h_1|<\rho_\alpha}\alpha O(r_\alpha^{-1})\nabla_M\psi^\perp\,\partial_t\psi^\perp\,dt\,.$$

On the other hand, for the remaining part, integrating by parts in the y variable the terms that involve two derivatives of $\psi^\perp$, we get that

$$II := -\int_{|t+h_1|<\rho_\alpha}dt\int_{r_\alpha(y)<R}(\Delta_{M_\alpha}\psi^\perp + B\psi^\perp)\,\psi^\perp\,(1+\alpha^2(t+h)^2|A|^2)\,dV_\alpha(y)\ \ge$$

(9.30)
$$\int_{|t+h_1|<\rho_\alpha} dt \int_{r_\alpha(y)<R} |\nabla_{M_\alpha}\psi^\perp|^2 \,+\, o(1)\,(\,{\psi^\perp}^2 + |\partial_t\psi^\perp|^2|\nabla_{M_\alpha}\psi^\perp|^2\,).$$

Using estimates (9.29) and (9.30), we finally get

$$\mathcal{Q}_\Omega(\psi^\perp,\psi^\perp) \;\ge\; 3\mu\int_\Omega (|\partial_t\psi^\perp|^2 + |\nabla_{M_\alpha}\psi^\perp|^2 + {\psi^\perp}^2)\,dx, \tag{9.31}$$

for some $\mu > 0$.

Now we estimate the crossed term. We have

$$-\mathcal{Q}_\Omega(\psi^\perp, kw'\eta_\delta) \;=\; \int_\Omega L(kw'\eta_\delta)\,\psi^\perp\,(1+\alpha^2(t+h)^2|A|^2)\,dV_\alpha\,dt\,.$$

Let us consider expression (9.8) for $L(kw')$, and let us also consider the fact that

$$L(\eta_\delta kw') = \eta_\delta L(kw') + 2\nabla\eta_\delta\nabla(kw') + \Delta\eta_\delta\, kw',$$

with the last two terms producing a first order operator in k with exponentially small size, at the same time with decay $O(r_\alpha^{-4})$. Thus all main contributions come from the integral

$$I = \int_\Omega \eta_\delta L(kw')\,\psi^\perp\,(1+\alpha^2(t+h)^2|A|^2)\,dV_\alpha\,dt.$$

Examining the expression (9.8), integrating by parts once in the y variable those terms involving two derivatives in k, we see that most of the terms obtained straightforwardly produce quantities of the type

$$\theta := o(1)\int_{M_\alpha}(|\nabla k|^2 + \alpha^2|A|^2k^2\,)\,dV_\alpha + o(1)\int_\Omega(|\psi^\perp|^2 + |\nabla\psi^\perp|^2).$$

In fact, we have

$$I \;=\; \underbrace{\int_\Omega \Delta_{M_\alpha}k\,w'\,\eta_\delta\,\psi^\perp\,dV_\alpha\,dt}_{I_1} + \underbrace{\int_\Omega \alpha^2\,a^0_{ij}\partial_i h_0\partial_j h_0\,k\,w'''\,\psi^\perp dV_\alpha\;dt}_{I_2} \;+$$
$$\underbrace{\int_\Omega f''(w)\,\phi_1\,kw'\,\psi^\perp dV_\alpha\;dt}_{I_3} + \theta.$$

On the other hand, the orthogonality definition of $\psi^\perp$ essentially eliminates I_1. Indeed,

$$I_1 = -\int_\Omega \Delta_{M_\alpha}k\,w'\,(1-\eta_\delta)\,\psi^\perp\,dV_\alpha\,dt$$
$$= \int_\Omega \nabla_{M_\alpha}k\,w'\,[(1-\eta_\delta)\,\nabla_{M_\alpha}\psi^\perp - \nabla\eta_\delta\psi^\perp)\,dV_\alpha\,dt = \theta\,.$$

On the other hand, for a small, fixed number $\nu > 0$, we have

$$|I_2| \le C\alpha^2\int_\Omega \frac{1}{1+r_\alpha^2}\,|k|\,|w'''|\,|\psi^\perp|\,dV_\alpha\;dt \;\le\; C\nu^{-1}\alpha^2\int_{M_\alpha}\frac{1}{1+r_\alpha^4}\,k^2\,dV_\alpha$$

$$+\nu \int_\Omega |\psi|^2\, dx\ .$$

A similar estimate is valid for I_3, since $\phi_1 = O(\alpha^2 r_\alpha^{-2})$. We then get

$$I\ \ge\ -C\nu^{-1}\alpha^2 \int_{M_\alpha} \frac{1}{1+r_\alpha^4}\, k^2\, dV_\alpha - \nu \int_\Omega |\psi^\perp|^2. \tag{9.32}$$

Finally, we recall that from Lemma 9.2,

$$\mathcal{Q}_\Omega(kw'\eta_\delta, kw'\eta_\delta) = \int_{r_\alpha(y)<R} \left[\, |\nabla_{M_\alpha} k|^2 - \alpha^2 |A(\alpha y)|^2\, k^2\, \right]\, dV_\alpha \int_{\mathbb{R}} w'^2\, dt + \theta. \tag{9.33}$$

From estimates (9.31), (9.32), and (9.33), we obtain that if ν is chosen sufficiently small, then

$$\mathcal{Q}_\Omega(\psi,\psi) \ge -C\,\alpha^2 \int_{M_\alpha} \frac{1}{1+r_\alpha^4}\, k^2\, dV_\alpha\ \ \ge -\mu\,\alpha^2 \int_\Omega \frac{1}{1+r_\alpha^4}\, |\psi|^2\, dx$$

for some $\mu > 0$, and inequality (9.20) follows. q.e.d.

In the next result, we show that an eigenfunction with negative eigenvalue of problem (9.1) or (9.2) decays exponentially, away from the interface of u_α.

Lemma 9.4. *Let ϕ be a solution of either* (9.1) *or* (9.2) *with $\lambda \le 0$. Then, in the subregion of $\mathcal{N}_\alpha$ where it is defined, ϕ satisfies that*

$$|\phi(y,t)|\ \le\ C\, \|\phi\|_\infty\, e^{-\sigma|t|} \tag{9.34}$$

where $\sigma > 0$ can be taken arbitrarily close to $\min\{\sigma_+,\sigma_-\}$. *The number C depends on σ, but it is independent of small α and large R. We have, moreover, that for $|\alpha x'| > R_0$,*

$$|\phi(x)| \le C \sum_{j=1}^m e^{-\sigma|x_3 - \alpha^{-1}(F_k(\alpha x') + \beta_j \alpha \log|\alpha x'|\,)\,|} \tag{9.35}$$

where R_0 is independent of α. Finally, we have that

$$|\phi(x)| \le C\, e^{-\sigma\frac{\delta}{\alpha}} \quad \textit{for } \operatorname{dist}(x, M_\alpha) > \frac{\delta}{\alpha}. \tag{9.36}$$

Proof. Let ϕ solve problem (9.1) for a large R. Let us consider the region between two consecutive ends $M_{j,\alpha}$ and $M_{j+1,\alpha}$. For definiteness, we assume that this region lies inside S_+ so that $f'(u_\alpha)$ approaches σ_+^2 inside it. So let us consider the region S of points $x = (x', x_3)$ such that $r_\alpha(x) > R_0$ for a sufficiently large but fixed $R_0 > 0$, and

$$(a_j + \alpha\beta_j)\log\alpha|x'| + b_j + \alpha\gamma\ <\ \alpha x_3\ <\ (a_{j+1} + \alpha\beta_{j+1}\,)\,\log\alpha|x'| + b_{j+1} - \alpha\gamma.$$

In terms of the coordinate t near $M_{j,\alpha}$, saying that

$$\alpha x_3 \sim (a_j + \alpha\beta_j)\log\alpha|x'| + b_j + \alpha\gamma$$

is, up to lower order terms, the same as saying $t \sim \gamma$, and similarly near $M_{j+1,\alpha}$. Thus, given any small number $\tau > 0$, we can choose γ

sufficiently large but fixed, independently of all R_0 sufficiently large and any small α, such that

$$f'(u_\alpha) \,<\, -(\sigma_+ - \tau)^2 \quad \text{in } S.$$

Let us consider, for $x \in S$ and $\sigma = \sigma_+ - 2\tau$, the function

$$v_1(x) := e^{-\sigma[\, x_3 - \alpha^{-1}(a_j + \alpha\beta_j)\log\alpha|x'| + b_j]} \\ + e^{-\sigma(\alpha^{-1}[a_{j+1} + \alpha\beta_{j+1})\log\alpha|x'| + b_{j+1}) - x_3]}.$$

Then v has the form

$$v_1 = A_1 e^{-\sigma x_3} r^{A_2} + B_1 e^{\sigma x_3} r^{-B_2}, \quad r = |x'|,$$

so that

$$\Delta v_1 = A_2^2 r^{-2} r^{A_2}\, A_1 e^{-\sigma x_3} + B_2^2 r^{-2}\, B_1 r^{-B_2} e^{\sigma x_3} + \sigma^2 v_1 < \\ [\,\alpha^2 A_2^2 R_0^{-2} + \alpha^2 B_2^2 R_0^{-2} \,+\, \sigma^2\,]\, v_1\,.$$

Here

$$A_2 = \sigma\alpha^{-1}(a_j + \alpha\beta_j), \quad B_2 = \sigma\alpha^{-1}(a_{j+1} + \alpha\beta_{j+1}).$$

Hence, enlarging R_0 if necessary, we achieve

$$\Delta v_1 + f'(u_\alpha) v_1 < 0 \quad \text{in } S.$$

Therefore, v so chosen is a positive supersolution of

$$\Delta v + f'(u_\alpha) v + \lambda p(\alpha x) v \le 0 \quad \text{in } S. \tag{9.37}$$

Observe that the definition of v also achieves that

$$\inf_{\partial S \setminus \{r_\alpha = R_0\}} \;\ge \gamma \,>\, 0$$

where γ is independent of α. Now, let us observe that the function $v_2 = e^{-\sigma(|x'| - \frac{R_0}{\alpha})}$ also satisfies, for small α, inequality (9.37). As a conclusion, for ϕ, a solution of (9.1), we have that

$$|\phi(x)| \;\le\; C\,\|\phi\|_\infty\,[v_1(x) + v_2(x)] \quad \text{for all} \quad x \in S, r_\alpha(x) < R. \tag{9.38}$$

Using the form of this barrier, we then obtain the validity of estimate (9.35), in particular that of (9.34), in the subregion of $\mathcal{N}_\delta$ in the positive t direction of $M_{j,\alpha}$ and $M_{j+1,\alpha}$ when $r_\alpha(y) > R_0$. The remaining subregions of $\mathcal{N}_\delta \cap \{r_\alpha(y) > R_0\}$ are dealt with in a similar manner. Finally, to prove the desired estimate for $r_\alpha(y) < r_0$, we consider the region where $|t| < \frac{2\delta}{\alpha}$, assuming that the local coordinates are well defined there. In this case we use, for instance in the region

$$\nu < t < \frac{2\delta}{\alpha}$$

for $\nu > 0$ large and fixed, a barrier of the form

$$v(y,t) = e^{-\sigma t} + e^{-\sigma(\frac{2\delta}{\alpha} - t)}\,.$$

It is easily seen that, for small α, this function indeed satisfies

$$\Delta_x v - f'(u_\alpha)v < 0$$

where σ can be taken arbitrarily close to σ_+. We conclude that

$$|\phi(y,t)| \ \le\ C\|\phi\|_\infty e^{-\sigma t} \quad \text{for } \nu < t < \frac{\delta}{\alpha}\ .$$

Thus estimate (9.34) holds true. Inequality (9.36) follows from the maximum principle.

Finally, for a solution of problem (9.2), the same procedure works, with only minor differences introduced. Estimate (9.38) can be obtained after adding a growing barrier. Indeed, we obtain

$$|\phi(x)| \ \le\ C\,\|\phi\|_\infty\,[v_1(x) + v_2(x) + \varepsilon v_3(x)] \quad \text{for all} \quad x \in S$$

with $v_3(x) = \varepsilon e^{\sigma|x'|}$, and then we let $\varepsilon \to 0$. We should also use $\varepsilon e^{\sigma x_3}$ to deal with the region above the last end M_m, and similarly below M_1. We then use the controls far away to deal with the comparisons at the second step. The proof is concluded. q.e.d.

9.1. The proof of inequality (8.23). Let us assume by contradiction that there is a sequence $\alpha = \alpha_n \to 0$ along which

$$m(u_\alpha)\ >\ i(M) =: N.$$

This implies that for some sequence $R_n \to +\infty$, we have that, for all $R > R_n$, problem (9.1) has at least $N+1$ linearly independent eigenfunctions

$$\phi_{1,\alpha,R}, \dots, \phi_{N+1,\alpha,R}$$

associated to negative eigenvalues

$$\lambda_{1,\alpha,R} \le \lambda_{2,\alpha,R} \le \cdots \le \lambda_{N+1,\alpha,R}\ < 0\,.$$

We may assume that $\|\phi_{i,\alpha,R}\|_\infty = 1$ and that

$$\int_{\mathbb{R}^3} p(\alpha x)\,\phi_{i,\alpha,R}\,\phi_{j,\alpha,R}\,dx \ =\ 0 \quad \text{for all} \quad i,j = 1,\dots,N+1, \quad i \ne j.$$

Let us observe that then the estimates in Lemma 9.4 imply that the contribution to the above integrals of the region outside $\mathcal{N}_\delta$ is small. We have at most

(9.39)
$$\int_{\mathcal{N}_\delta} p(\alpha y)\,\phi_{i,\alpha,R}\,\phi_{j,\alpha,R}\,dx \ =\ O(\alpha^3) \quad \text{for all} \quad i,j = 1,\dots,N{+}1, \quad i \ne j.$$

From the variational characterization of the eigenvalues, we may also assume that $\lambda_{i,\alpha,R}$ defines a decreasing function of R. On the other hand, from Lemma 9.3 we know that $\lambda_{i,\alpha,R} = O(\alpha^2)$, uniformly in R, so that we write for convenience

$$\lambda_{i,\alpha,R} \ =\ \mu_{i,\alpha,R}\,\alpha^2, \quad \mu_{i,\alpha,R} < 0.$$

We may assume $\mu_{i,\alpha,R} \to \mu_{i,\alpha} < 0$ as $R \to +\infty$. We will prove that $\phi_{i,\alpha,R}$ converges, up to subsequences, uniformly over compacts to a nonzero bounded limit $\phi_{i,\alpha}$, which is an eigenfunction with eigenvalue $\mu_{i,\alpha}\alpha^2$ of problem (9.2). We will then take limits when $\alpha \to 0$ and find a contradiction with the fact that $\mathcal{J}$ has at least $i(M)$ negative eigenvalues.

We fix an index i and consider the corresponding pair $\phi_{i,\alpha,R}$, $\mu_{i,\alpha,R}$, to which temporarily we drop the subscripts i, α, R.

Note that by maximum principle, $|\phi|$ can have values that stay away from zero only inside $\mathcal{N}_\delta$. Besides, from Lemma 9.4, $\phi = O(e^{-\sigma|t|})$ in $\mathcal{N}_\delta$. We observe then that since λ remains bounded, local elliptic estimates imply the stronger assertion

$$|D^2\phi| + |D\phi| + |\phi| \ \le\ C\, e^{-\sigma|t|} \quad \text{in } \mathcal{N}_\delta. \tag{9.40}$$

In particular, considering its dependence in R, ϕ approaches a limit, locally uniformly in $\mathbb{R}^3$, up to subsequences. We will prove by suitable estimates that that limit is nonzero. Moreover, we will show that $\phi \approx z(\alpha y)w'(t)$ in $\mathcal{N}_\delta$ where z is an eigenfunction with negative eigenvalue $\approx \mu$ of the Jacobi operator $\mathcal{J}$.

First, let us localize ϕ inside $\mathcal{N}_\delta$. Let us consider the cut-off function η_δ in (3.12), and the function

$$\tilde\phi = \eta_\delta \phi.$$

Then $\tilde\phi$ satisfies

$$L(\tilde\phi) \,+\, \mu\alpha^2 q(\alpha x)\tilde\phi \,=\, E_\alpha := -2\nabla\eta_\delta\nabla\phi - \Delta\eta_\delta\phi \tag{9.41}$$

with $L(\tilde\phi) = \Delta\tilde\phi + f'(u_\alpha)\tilde\phi$. Then from (9.40) we have that for some $\sigma > 0$,

$$|E_\alpha| \le C\alpha^3 e^{-\sigma|t|}(1 + r_\alpha^4)^{-1}.$$

Inside $\mathcal{N}_\delta$, we write equation (9.41) in (y, t) coordinates as

$$L_*(\tilde\phi) + B(\tilde\phi) + \lambda p(\alpha y)\tilde\phi = E_\alpha \tag{9.42}$$

where

$$L_*(\tilde\phi) = \partial_{tt}\tilde\phi + \Delta_{M_\alpha}\tilde\phi + f'(w(t))\tilde\phi\ .$$

Extending $\tilde\phi$ and E_α as zero, we can regard equation (9.42) as the solution of a problem in entire $M_\alpha \times \mathbb{R}$ for an operator $\mathtt{L}$ that interpolates L inside $\mathcal{N}_\delta$ with L_* outside. More precisely, $\tilde\phi$ satisfies

$$\mathtt{L}(\tilde\phi) \,:=\, L_*(\tilde\phi) + \mathtt{B}(\tilde\phi) + \lambda p(\alpha y)\tilde\phi = E_\alpha \quad \text{in } M_\alpha^R \times \mathbb{R}, \tag{9.43}$$

where for a function $\psi(y, t)$ we denote

$$\mathtt{B}(\psi) \,:=\, \begin{cases} \chi\, B(\psi) & \text{if } |t + h_1(\alpha y)| < \rho_\alpha(y) + 3 \\ 0 & \text{otherwise} \end{cases} \tag{9.44}$$

and

$$\chi(y, t) = \zeta_1(y + (t + h)\nu_\alpha(y))$$

with ζ_1 the cut off function defined by (4.8) for $n = 1$. In particular, $L = \mathtt{L}$ in $\mathcal{N}_\delta$.

Now, we decompose

$$\tilde\phi(y,t) \;=\; \varphi(y,t) \;+\; k(y)\,\eta_\delta\, w'(t) \tag{9.45}$$

where

$$k(y) \;=\; -\,w'(t)\,\frac{\int_{\mathbb{R}} \tilde\phi(y,\cdot)\, w'\, d\tau}{\int_{\mathbb{R}} \eta_\delta w'^2\, d\tau}$$

so that

$$\int_{\mathbb{R}} \varphi(y,t)\, w'(t)\, dt \;=\; 0 \quad \text{for all} \quad y \in M_\alpha^R.$$

From (9.40), k is a bounded function of class C^2 defined on M_α^R with first and second derivatives uniformly bounded independently of large R. A posteriori we expect that k also has bounded smoothness as a function of αy, which means in particular that $Dk = O(\alpha)$. We will see that this is indeed the case.

The function φ satisfies the equation

$$\mathtt{L}(\varphi) + \mu\alpha^2 p(\alpha y)\,\varphi \;=\; -\mathtt{L}(kw') + E_\alpha - \mu\alpha^2 p\, k\, w' \qquad \text{in } M_\alpha^R \times \mathbb{R}. \tag{9.46}$$

We observe that the expansion (9.8) holds true globally in $M_\alpha^R \times \mathbb{R}$ for $\mathtt{L}(kw')$ replacing $L(kw')$. We also have the validity of expansion (9.5) for the corresponding projection, namely

$$\begin{aligned}\int_{\mathbb{R}} \mathtt{L}(kw')\, w'\, dt \;=\;& \left(\,\Delta_{M_\alpha} k + \alpha^2 |A|^2 k\,\right) \int_{\mathbb{R}} w'^2\, dt \\ &+ O(\alpha r_\alpha^{-2})\,\partial_{ij} k \;+\; O(\alpha^2 r_\alpha^{-3})\,\partial_i k \;+ O(\alpha^3 r_\alpha^{-4})\, k\,.\end{aligned} \tag{9.47}$$

Thus, integrating equation (9.46) against w', we find that k satisfies

$$\begin{aligned}&\Delta_{M_\alpha} k + \alpha^2 |A|^2 k \;+\; \mu\alpha^2\, p(\alpha y)\, k \\ &+ O(\alpha r_\alpha^{-2})\,\partial_{ij} k \;+\; O(\alpha^2 r_\alpha^{-3})\,\partial_i k \;+ O(\alpha^3 r_\alpha^{-4})\, k \\ &= O(\alpha^3 r_\alpha^{-4}) - \frac{1}{\int_{\mathbb{R}} w'^2} \int_{\mathbb{R}} \mathtt{B}(\varphi)\, w'\, dt, \quad y \in M_\alpha^R\,.\end{aligned} \tag{9.48}$$

Let us consider the function $z(y)$ defined in M by the relation $k(y) = z(\alpha y)$. Then (9.48) translates in terms of z as

$$\begin{aligned}&\Delta_M z \;+\; |A(y)|^2 z \;+\; \mu\, q(y)\, z \;= \\ &\alpha\left[O(r^{-2})\,\partial_{ij} z \;+\; O(r^{-3})\,\partial_i z \;+ O(r^{-4})\, z + O(r^{-4})\right] + \mathcal{B} \quad y \in M^R,\end{aligned} \tag{9.49}$$

where

$$\mathcal{B}(y) \;:=\; \frac{1}{\int_{\mathbb{R}} w'^2}\,\alpha^{-2} \int_{\mathbb{R}} \mathtt{B}(\varphi)(\alpha^{-1} y, t)\, w'\, dt, \quad y \in M^R\,. \tag{9.50}$$

In other words, we have that $k(y) = z(\alpha y)$, where z solves “a perturbation” of the eigenvalue equation for the Jacobi operator that we treated in §8. We need to make this assertion precise, the basic element being to prove that the operator $\mathcal{B}[z]$ is "small." For this we will derive estimates for φ from equation (9.46).

We shall refer to the decomposition $Q_1+\cdots+Q_6$ in (9.8) to identify different terms in $\mathtt{L}(kw')$. Let us consider the decomposition

$$\varphi=\varphi_1+\varphi_2,$$

where φ_1 solves the linear problem for the operator L_* and the part of $\mathtt{L}(kw')$ that "does not contribute to projections," namely

$$Q_3+Q_4=-w'' \Big[\alpha a^0_{ij}(\partial_j h\partial_i k+\partial_i h\partial_j k)+\alpha^2 k\Delta_M h_1+\alpha^2 h a^{1,0}_{ij}(\partial_j h\partial_i k+\partial_i h\partial_j k)\Big] \tag{9.51}$$

$$+\,\alpha t w'\,\Big[a^{1,0}_{ij}\partial_{ij}k+\alpha b^{1,0}_i\partial_i k\Big]. \tag{9.52}$$

More precisely, φ_1 solves the equation

$$L_*(\varphi_1)\,+\,\alpha^2\mu\,p\,\varphi_1=Q_3+Q_4\quad\text{in } M^R_\alpha\times\mathbb{R}. \tag{9.53}$$

This problem can indeed be solved: according to the linear theory developed, there exists a unique solution to the problem

$$L_*(\varphi_1)+\mu\alpha^2 p\,\varphi_1=Q_3+Q_4+c(y)w'(t)\quad\text{in } M^R_\alpha\times\mathbb{R},$$

such that

$$\int_{\mathbb{R}}\varphi_1\,w'\,dt=0\quad\text{for all}\quad y\in M^R_\alpha$$

and

$$\|D^2\varphi_1\|_{p,1,\sigma}\,+\,\|\,D\varphi_1\|_{\infty,1,\sigma}+\,\|\varphi_1\|_{\infty,1,\sigma}\,\le\,\|Q_3+Q_4\|_{p,1,\sigma}\,\le\,C\alpha\,. \tag{9.54}$$

But since

$$\int_{\mathbb{R}}(Q_3+Q_4)\,w'\,dt\;=\;0\quad\text{for all}\quad y\in M^R_\alpha$$

it follows that actually $c(y)\equiv 0$; namely, φ_1 solves equation (9.53).

We claim that φ_2 actually has a smaller size than φ_1. Indeed, φ_2 solves the equation

$$\begin{aligned}L_*(\varphi_2)+\mathtt{B}(\varphi_2)+\mu\alpha^2 p\varphi_2=E_\alpha-\mathtt{B}(\varphi_1)-(Q_1&\\ +Q_2+Q_5+Q_6)-\mu\alpha^2 q\,kw'\quad\text{in } M^R_\alpha\times\mathbb{R}.&\end{aligned} \tag{9.55}$$

Now we have that

$$\begin{aligned}&Q_1+Q_2+Q_5+Q_6\\ &=\Big[\Delta_{M_\alpha}k\,+\,\alpha h a^{1,0}_{ij}\partial_{ij}k\Big]\,w'\,+\,\alpha^2\,\big[-|A|^2\,k\,tw''+\,a^0_{ij}\partial_i h_0\partial_j h_0\,k\,w'''\\ &\quad+\alpha^{-2}f''(w)\,\phi_1\,kw'\,+\,(t+h)^2a^2_{ij}\partial_{ij}kw'+2(t+h)a^2_{ij}\,\partial_i h\partial_j k)\,w''\,\big]\end{aligned}$$

$$+\alpha^3\left[O(e^{-\sigma|t|}r_\alpha^{-2})\,\partial_{ij}k\,+\,O(e^{-\sigma|t|}r_\alpha^{-3})\,\partial_j k\right] \tag{9.56}$$
$$=\,O(\alpha^2 r_\alpha^{-2}\,\log^2 r_\alpha\,e^{-\sigma|t|})\,+\,\rho(y)w'(t),$$

for a certain function $\rho(y)$. On the other hand, let us recall that

$$B\,=\,(f'(u_\alpha)-f'(w))-\alpha^2[(t+h_1)|A|^2+\Delta_M h_1]\partial_t\,-\alpha\,a^0_{ij}(\,\partial_j h\partial_{it}+\partial_i h\partial_{jt})\,+$$
$$\alpha(t+h)\,[a^1_{ij}\partial_{ij}-2\alpha\,a^1_{ij}\partial_i h\partial_{jt}+\alpha(b^1_i\partial_i-\alpha b^1_i\partial_i h\partial_t)\,]\,+$$
$$\alpha^3(t+h)^2 b^1_3\partial_t\,+\,\alpha^2[\,a^0_{ij}+\alpha(t+h)a^1_{ij}\,]\partial_i h\partial_j h\,\partial_{tt} \tag{9.57}$$

Thus the order of $\mathtt{B}(\varphi_1)$ carries both an extra α and an extra r_α^{-1} over those of φ_1, in the sense that

$$\|\mathtt{B}(\varphi_1)\|_{p,2,\sigma}\le C\alpha^2. \tag{9.58}$$

From relations (9.56) and (9.58), we find that φ_2 satisfies an equation of the form

$$L_*(\varphi_2)+\mathtt{B}(\varphi_2)+\mu\alpha^2 q\varphi_2=g+c(y)\,w'\quad\text{in } M^R_\alpha\times\mathbb{R} \tag{9.59}$$

where for arbitrarily small $\sigma'>0$ we have

$$\|g\|_{p,2-\sigma',\sigma}\le C\alpha^2.$$

Since φ_2 satisfies $\int_{\mathbb{R}}\varphi_2\,w'\,dt\,\equiv\,0$, the linear theory for the operator L_* yields then that

$$\|D^2\varphi_2\|_{p,2-\sigma',\sigma}\,+\,\|\,D\varphi_2\|_{\infty,2-\sigma',\sigma}+\,\|\,\varphi_2\|_{\infty,2-\sigma',\sigma}\,\le\,C\alpha^2\,, \tag{9.60}$$

which compared with (9.54) gives us the claimed extra smallness:

$$\|\mathtt{B}(\varphi_2)\|_{p,3-\sigma',\sigma}\le C\alpha^3. \tag{9.61}$$

Let us decompose in (9.50)

$$\mathcal{B}=\mathcal{B}_1+\mathcal{B}_2$$

where

$$\mathcal{B}_l:=\frac{1}{\int_{\mathbb{R}}w'^2}\,\alpha^{-2}\int_{\mathbb{R}}\mathtt{B}(\varphi_l)(\alpha^{-1}y,t)\,w'\,dt,\qquad l=1,2. \tag{9.62}$$

From Lemma 6.1 we get that

$$\|\mathcal{B}_1\|_{p,2-\frac{2}{p}-\sigma'}\le C\alpha^{-2}\|\mathtt{B}(\varphi_1)\|_{p,2,\sigma}\le C \tag{9.63}$$

and

$$\|\mathcal{B}_2\|_{p,3-\frac{2}{p}-2\sigma'}\le C\alpha^{-2}\|\mathtt{B}(\varphi_2)\|_{p,3-\sigma',\sigma}\le C\alpha. \tag{9.64}$$

Now we apply the estimate in part (b) of Lemma 8.2 to equation (9.49) and then get for $z(y)=k(\frac{y}{\alpha})$ the estimate

(9.65)
$$\|D^2 z\|_{p,2-\frac{2}{p}-2\sigma'}+\|\,(1+|x|)^{1-2\sigma'}\,Dz\,\|_\infty\le C\,[\,\|f\|_{p,2-\frac{2}{p}-2\sigma'}\,+\,\|z\|_\infty\,]$$

where

$$f = \alpha \left[O(r^{-2})\, \partial_{ij} z \,+\, O(r^{-3})\, \partial_i z \,+ O(r^{-4})\, z + O(r^{-4}) \right] + \mathcal{B}.$$

Then from estimate (9.65) it follows that for small α,

$$\|D^2 z\|_{p,2-\frac{2}{p}-2\sigma'} + \|\,(1+|x|)^{1-2\sigma'}\, Dz\,\|_\infty \le C\alpha. \tag{9.66}$$

Using this new information, let us go back to equation (9.53) and to the expression (9.52) for Q_3+Q_4. The terms contributing the largest sizes in this function can be bounded by

$$C\,\alpha\, e^{-\sigma|t|}\, \left[\frac{|Dk|}{1+r_\alpha} + \frac{|D^2 k|}{1+r_\alpha^2} \right].$$

Now we compute

$$(1+r_\alpha(y)^2)^p \int_{B(y,1)} \frac{|D^2 k|^p}{(1+r_\alpha^2)^p}\, dV_\alpha \;\le$$

$$C\alpha^{2p-2} \int_{B(y,\alpha)} |D^2 z|^p\, dV \;\le\; C\alpha^{2p-2} \|D^2 z\|_{p,2-\frac{2}{p}-2\sigma'} \;\le\; C\alpha^{2p-2}$$

and

$$(1+r_\alpha(y)^{2-2\sigma'})^p \int_{B(y,1)} \frac{|Dk|^p}{(1+r_\alpha)^p}\, dV_\alpha \;\le$$

$$C\|\,|Dk|\,(1+r_\alpha)^{1-2\sigma'}\,\|_\infty^p \;=\; C\,\alpha^p\,\|\,|Dz|\,(1+r)^{1-2\sigma'}\,\|_\infty^p \;\le\; C\alpha^p\,.$$

As a conclusion, from expression (9.52) we obtain that

$$\|Q_3+Q_4\|_{p,2-2\sigma',\sigma} \le C\alpha^2,$$

and therefore a substantial reduction of the size of φ_1, compared with (9.54). We have

$$\|D^2\varphi_1\|_{p,2-2\sigma',\sigma} \,+\, \|\,D\varphi_1\|_{\infty,2-2\sigma',\sigma} \;\le\; C\alpha^2\ , \tag{9.67}$$

and hence, again using Lemma 6.1, we get

$$\|\mathcal{B}_1\|_{p,3-\frac{2}{p}-3\sigma'} \le C\alpha^{-2} \|\mathsf{B}(\varphi_1)\|_{p,3-2\sigma',\sigma} \le C\alpha, \tag{9.68}$$

which matches the size we initially found for $\mathcal{B}_2$ in (9.64).

We recall that $\phi = \phi_{i,\alpha,R}$ has a uniform C^1 bound (9.40). Thus, passing to a subsequence if necessary, we may assume that

$$\phi_{i,\alpha,R} \to \phi_{i,\alpha} \quad \text{as } R \to +\infty$$

locally uniformly, where $\phi_{i,\alpha}$ is bounded and solves

$$\Delta\phi_{i,\alpha} + f'(u_\alpha)\phi_{i,\alpha} + \mu_{i,\alpha}\,\alpha^2\, p(\alpha x)\,\phi_{i,\alpha} \;=\; 0 \quad \text{in } \mathbb{R}^3. \tag{9.69}$$

Let us return to equation (9.49), including the omitted subscripts. Thus $k = k_{i,\alpha,R}$ satisfies the local uniform convergence in M^α,

$$k_{i,\alpha,R}(y) = c \int_{|t+h_1|<\rho_\alpha} \phi_{i,\alpha,R}\, w'\, dt \to c \int_{|t+h_1|<\rho_\alpha} \phi_{i,\alpha}\, w'\, dt \;=:\; k_{i,\alpha}(y).$$

We have that $z = z_{i,\alpha,R}$ satisfies

$$\Delta_M z_{i,\alpha,R} + |A(y)|^2 z_{i,\alpha,R} + \mu_{i,\alpha,R}\, q(y)\, z_{i,\alpha,R} =$$
$$\alpha\left[O(r^{-2})\,\partial_{ij} z_{i,\alpha,R} + O(r^{-3})\,\partial_i z_{i,\alpha,R} + O(r^{-4})\, z_{i,\alpha,R} + O(r^{-4})\right] +$$
$$\mathcal{B}_{i,\alpha,R}, \quad y \in M^R,$$

where

$$\|\mathcal{B}_{i,\alpha,R}\|_{p,3-\frac{2}{p}-3\sigma'} \le C\alpha \tag{9.70}$$

with arbitrarily small $\sigma' > 0$ and C independent of R. We now apply the estimates in Lemma 8.2 for some $1 < p < 2$ and find that for C independent of R we have

$$\|z_{i,\alpha,R}\|_{L^\infty(M^R)} \le C\left[\, \|z_{i,\alpha,R}\|_{L^\infty(r<R_0)} + O(\alpha)\,\right]$$

or equivalently

$$\|k_{i,\alpha,R}\|_{L^\infty(M^R_\alpha)} \le C\left[\, \|k_{i,\alpha,R}\|_{L^\infty(r_\alpha<R_0)} + O(\alpha)\,\right]. \tag{9.71}$$

Since from (9.45) we have that

$$\phi_{i,R,\alpha}(y,t) = \varphi_{i,R,\alpha}(y,t) + k_{i,R,\alpha}(y)\, w'(t) \quad \text{in } \mathcal{N}_\delta, \quad r_\alpha(y) \le R, \tag{9.72}$$

where we have uniformly in R $|\varphi_{i,R,\alpha}(y,t)| = O(\alpha\, e^{-\sigma|t|} r_\alpha^{-2})$, while $\phi_{i,R,\alpha} = O(e^{-\frac{a}{\alpha}})$ outside $\mathcal{N}_\delta$, and $\|\phi_{i,R,\alpha}\|_\infty = 1$, then $\|k_{i,\alpha,R}\|_{L^\infty(M^R_\alpha)} \ge \gamma > 0$ uniformly in R. Thus from (9.71), the limit $k_{i,\alpha}$ as $R \to +\infty$ cannot be zero. We have thus found that $\phi_{i,\alpha}$ is non-zero. Moreover, we observe the following: Since the functions

$$Z_i := \partial_i u_\alpha, \quad i = 1,2,3, \quad Z_4 := -x_2 \partial_1 u_\alpha + x_1 \partial_2 u_\alpha$$

are bounded solutions of (9.2) for $\lambda = 0$, we necessarily have that

$$\int_{\mathbb{R}^3} p(\alpha x)\, Z_j \phi_{i,\alpha}\, dx = 0, \quad j = 1,2,3,4. \tag{9.73}$$

Let $\hat{Z}_i = \sum_{l=1}^4 d_{il} Z_l, i = 1,\dots,J$. Then we also have

$$\int_{\mathbb{R}^3} p(\alpha x)\, \hat{Z}_i \phi_{i,\alpha}\, dx = 0, \quad i = 1,\dots,J. \tag{9.74}$$

Now we want to let $\alpha \to 0$. $z_{i,\alpha}$ satisfies

$$\begin{aligned} \Delta_M z_{i,\alpha} &+ |A(y)|^2 z_{i,\alpha} + \mu_{i,\alpha}\, p(y)\, z_{i,\alpha} \\ &= \alpha\left[O(r^{-2})\,\partial_{ij} z_{i,\alpha} + O(r^{-3})\,\partial_i z_{i,\alpha} + O(r^{-4})\, z_{i,\alpha} + O(r^{-4})\right] \\ &+ \mathcal{B}_{i,\alpha}, \quad y \in M, \end{aligned}$$

with

$$\|\mathcal{B}_{i,\alpha}\|_{p,3-\frac{2}{p}-3\sigma'} \le C\alpha. \tag{9.75}$$

Moreover, $\|z_{i,\alpha}\|_{L^\infty(M)} \le C\left[\, \|z_{i,\alpha,R}\|_{L^\infty(r<R_0)} + O(\alpha)\,\right]$. Since we also have that $\|D^2 z_{i,\alpha}\|_{L^p(M)} \le C$, Sobolev's embedding implies that passing

to a subsequence in α, $z_{i,\alpha}$ converges as $\alpha \to 0$, uniformly over compact subsets of M to a non-zero bounded solution $\bar z_i$ of the equation

$$\Delta_M \bar z_i \,+\, |A(y)|^2 \bar z_i \,+\, \mu_i\, q(y)\, \bar z_i \;=\; 0 \quad \text{in } M,$$

with $\mu_i \le 0$.

Now we have that $\phi_{i,\alpha} \;=\; z_{i,\alpha}(\alpha y)\, w'(t) \;+\; \varphi_i(y,t)$ in $\mathcal{N}_\delta$ where $|\varphi_i(y,t)| \le C\alpha e^{-\sigma|t|}$. We recall that

$$\int_{\mathcal{N}_\alpha} q(\alpha y)\phi_{i,\alpha}\, \phi_{j,\alpha}\, dx \;= O(\alpha) \quad \text{for all} \quad i \ne j\,.$$

Since on $\mathcal{N}_\delta$, $dx = (1+\alpha^2|A|^2(t+h))dV_\alpha\, dt$, we get then that $\int_{M_\alpha} q(\alpha y)\, z_{i,\alpha}(\alpha y)\, z_{j,\alpha}(\alpha y)\, dV_\alpha = O(\alpha)$ or

$$\int_M q(y) z_{i,\alpha}(y)\, z_{j,\alpha}(y)\, dV = O(\alpha^3) \quad \text{for all} \quad i \ne j.$$

We conclude, passing to the limit, that the z_i's $i = 1, \dots, N+1$ satisfy

$$\int_M q\, \bar z_i \bar z_j\, dV \;=\; 0 \quad \text{for all} \quad i \ne j.$$

Since, as we have seen in §8, this problem has exactly $N = i(M)$ negative eigenvalues, it follows that $\mu_{N+1} = 0$, so that z_{N+1} is a bounded Jacobi field. But we also recall that

$$Z_i = z_i(\alpha y) w'(t) + O(\alpha e^{-\sigma|t|}) \quad \text{for all} \quad i = 1, \dots, J,$$

and hence the orthogonality relations (9.74) pass to the limit to yield

$$\int_M q\, \hat z_i \cdot \bar z_{N+1}\, dV \;=\; 0, \quad i = 1, \dots, J$$

where $\hat z_i$'s are the J linearly independent Jacobi fields. We have thus reached a contradiction with the non-degeneracy assumption for M, and the proof of $m(u_\alpha) = i(M)$ is concluded.

Finally, the proof of the non-degeneracy of u_α for all small α goes along the same lines. Indeed, the above arguments are also valid for a bounded eigenfunction in entire space, in particular for $\mu = 0$. If we assume that a bounded solution Z_5 of equation (9.2) is present, linearly independent from $Z_1, \dots, Z_4$, then we assume that

$$\int_{\mathbb{R}^3} p(\alpha x)\, Z_5\, \hat Z_i\, dx \;=\; 0 \quad i = 1, \dots, J. \tag{9.76}$$

Thus, in the same way as before, we have that in $\mathcal{N}_\delta$, $Z_5 = z_5(\alpha y) w'(t) + \varphi$ with φ orthogonal to $w'(t)$ for all y and φ small with size α and uniform exponential decay in t, the function z_α solves an equation of the form of (9.75), but now for $\mu = 0$. In the same way as before, it converges uniformly on compacts to a non-zero limit which is a bounded Jacobi field. But the orthogonality (9.76) passes to the limit, thus implying the existence of at least $J+1$ linearly independent Jacobi fields. We have reached a contradiction that finishes the proof of Theorem 2. q.e.d.

10. Further comments and open questions

10.1. Symmetries. As is natural, the invariances of the surface are at the same time inherited from the construction. If M is a catenoid, revolved around the x_3 axis, the solution in Theorem 1 is radial in the first two variables,

$$u_\alpha(x) = u_\alpha\left(\,|x'|,\, x_3\right).$$

This is a consequence of the construction. The invariance of the Laplacian under rotations and the autonomous character of the nonlinearity imply that the entire proof can be carried out in spaces of functions with this radial symmetry. More generally, if M is invariant under a group of linear isometries, so will be the solution found, at least in the case that $f(u)$ is odd. This assumption allows for odd reflections. The Costa-Hoffmann-Meeks surface is invariant under a discrete group constituted of a combination of dihedral symmetries and reflections to which this remark applyies.

10.2. Toward a classification of finite Morse index solutions. Understanding bounded, entire solutions of nonlinear elliptic equations in $\mathbb{R}^N$ is a problem that has always been at the center of PDE research. This is the context of various classical results in PDE literature such as the Gidas-Ni-Nirenberg theorems on radial symmetry of one-signed solutions, Liouville-type theorems, or the achievements around the De Giorgi conjecture. In those results, the geometry of level sets of the solutions turns out to be a posteriori very simple (planes or spheres). The problem of classifying solutions with finite Morse index, though, seems more challenging, even in a model as simple as the Allen-Cahn equation. While the solutions predicted by Theorem 1 are generated in an asymptotic setting, it seems plausible that they contain germs of generality, in view of parallel facts in the theory of minimal surfaces. In particular, we believe that the following two statements hold true for a bounded solution u to equation (1.1) in $\mathbb{R}^3$.

(1) *If u has finite Morse index and $\nabla u(x) \neq 0$ outside a bounded set, then, outside a large ball, each level set of u must have a finite number of components, each of them asymptotic either to a plane or to a catenoid. After a rotation of the coordinate system, all these components are graphs of functions of the same two variables.*

(2) *If u has Morse index equal to one, then u must be axially symmetric: namely, after a rotation and a translation, u is radially symmetric in two of its variables. Its level sets have two ends, both of them catenoidal.*

It is worth mentioning that a *balancing formula* for the "ends" of level sets to the Allen-Cahn equation is available in $\mathbb{R}^2$; see [**18**]. An extension of such a formula to $\mathbb{R}^3$ should involve the configuration (1) as its basis.

The condition of finite Morse index can probably be replaced by the energy growth (1.9).

On the other hand, (1) should not hold if the condition $\nabla u \neq 0$ outside a large ball is violated. For instance, let us consider the octant $\{x_1, x_2, x_3 \geq 0\}$ and the odd nonlinearity $f(u) = (1-u^2)u$. Problem (1.1) in the octant with zero boundary data can be solved by a super-subsolution scheme (similar to that in [**8**]) yielding a positive solution. Extending by successive odd reflections to the remaining octants, one generates an entire solution (likely to have finite Morse index), whose zero level set does not have the characteristics above: the condition $\nabla u \neq 0$ far away corresponds to *embeddedness of the ends.*

Various rather general conditions on a minimal surface imply that it is a catenoid. For example, R. Schoen [**39**] proved that a complete embedded minimal surface in $\mathbb{R}^3$ with two ends must be catenoid (and hence it has index one). One may wonder if a bounded solution to (1.1) whose zero level set has only two ends is radially symmetric in two variables. On the other hand a one-end minimal surface is forced to be a plane [**23**]. We may wonder: if the zero level set lies in a half space, then the solution depends on only one variable.

The case of infinite topology may also give rise to very complicated patterns; we refer to Hauswirth and Pacard [**20**] and references therein for recent result on construction of minimal surfaces in this scenario.

References

[1] L. Ambrosio & X. Cabré, *Entire solutions of semilinear elliptic equations in* $\mathbb{R}^3$ *and a conjecture of De Giorgi,* Journal Amer. Math. Soc. **13** (2000), 725–739, MR 1843784, Zbl0968.35041.

[2] L. Caffarelli & A. Córdoba, *Uniform convergence of a singular perturbation problem,* Comm. Pure Appl. Math. XLVII (1995), 1–12, MR 1310848, Zbl 0829.49013.

[3] L. Caffarelli & A. Córdoba, *Phase transitions: uniform regularity of the intermediate layers,* J. Reine Angew. Math. **593** (2006), 209–235, MR 2227143, Zbl 1090.49019.

[4] C.J. Costa, *Imersoes minimas en* $\mathbb{R}^3$ *de genero un e curvatura total finita,* PhD thesis, IMPA, Rio de Janeiro, Brasil (1982).

[5] C.J. Costa, *Example of a complete minimal immersions in* $\mathbb{R}^3$ *of genus one and three embedded ends*, Bol. Soc. Bras. Mat. **15**(1-2) (1984), 47–54, MR 0794728, Zbl 0613.53002.

[6] X. Cabré & J. Terra *Saddle-shaped solutions of bistable diffusion equations in all of* $\mathbb{R}^{2m}$*,* J. Eur. Math. Soc. **11** (2009), no. 4, 819–843, MR 2538506, Zbl 1182.35110.

[7] E.N. Dancer, *Stable and finite Morse index solutions on* $\mathbf{R}^n$ *or on bounded domains with small diffusion,* Trans. Amer. Math. Soc. **357** (2005), no. 3, 1225–1243, MR 2110438, Zbl 1145.35149.

[8] H. Dang, P.C. Fife & L.A. Peletier, *Saddle solutions of the bistable diffusion equation,* Z. Angew. Math. Phys. **43** (1992), no. 6, 984–998, MR 1198672, Zbl 0764.35048.

[9] E. De Giorgi, *Convergence problems for functionals and operators,* Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188, Pitagora, Bologna (1979), MR 0533166, Zbl 0405.49001.

[10] M. del Pino, M. Kowalczyk & J. Wei, *Concentration on curves for nonlinear Schrödinger equations,* Comm. Pure Appl. Math. **60** (2007), no. 1, 113–146, MR 2270164, Zbl 1123.35003.

[11] M. del Pino, M. Kowalczyk & J. Wei, *On De Giorgi's Conjecture in Dimensions* $N \geq 9$, Annals of Mathematics **174** (2011), no. 3, 1485–1569, MR 2846486, Zbl 1238.35019.

[12] M. del Pino, M. Kowalczyk & J. Wei, *A counterexample to a conjecture by De Giorgi in large dimensions*, Comp. Rend. Mathematique **346** (2008), 23–24, 1261–1266, MR 2473304, Zbl 1161.35023.

[13] M. del Pino, M. Kowalczyk & J. Wei, *On De Giorgi conjecture and beyond*, PNAS vol. **109** no. 18 (2012), 6845-6850.

[14] M. del Pino, M. Kowalczyk & J. Wei, *Entire Solutions of the Allen-Cahn equation and Complete Embedded Minimal Surfaces of Finite Total Curvature in* $\mathbb{R}^3$ *(preliminary version)*, Preprint arXiv:0902.2047v1 (2009).

[15] A. Farina and E. Valdinoci, *The state of art for a conjecture of De Giorgi and related questions*, to appear in "Reaction-Diffusion Systems and Viscosity Solutions," World Scientific, 2008, MR 2528756, Zbl 1183.35137.

[16] N. Ghoussoub & C. Gui, *On a conjecture of De Giorgi and some related problems,* Math. Ann. **311** (1998), 481–491, MR 1637919, Zbl 0918.35046.

[17] D. Gilbarg & N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd edition, Springer-Verlag, 1983, MR 0737190, Zbl 0562.35001.

[18] C. Gui, *Hamiltonian identities for elliptic partial differential equations*, J. Funct. Anal. **254** (2008), no. 4, 904–933, MR 2381198, Zbl 1148.35023

[19] R. Gulliver. *Index and total curvature of complete minimal surfaces.* Proc. Symp. Pure Math., **44** (1986), 207–242, MR 0840274, Zbl 0592.53006.

[20] L. Hauswirth & F. Pacard, *Higher genus Riemann minimal surfaces*, Invent. Math. **169** (2007), 569–620, MR 2336041, Zbl 1129.53009.

[21] D. Hoffman & W.H. Meeks III, *A complete embedded minimal surface in* $\mathbb{R}^3$ *with genus one and three ends*, J. Diff. Geom. **21** (1985), 109–127, MR 0806705, Zbl 0604.53002.

[22] D. Hoffman & W.H. Meeks III, *The asymptotic behavior of properly embedded minimal surfaces of finite topology*, J. Am. Math. Soc. **4**(2) (1989), 667–681, MR 1002088, Zbl 0683.53005.

[23] D. Hoffman & W.H. Meeks III, *The strong halfspace theorem for minimal surfaces,* Inventiones Math. (101) 1990, 373–377, MR 1062966, Zbl 0722.53054.

[24] D. Hoffman & W.H. Meeks III, *Embedded minimal surfaces of finite topology*, Ann. Math. **131** (1990), 1–34, MR 1038356, Zbl 0695.53004.

[25] D. Hoffman & H. Karcher, *Complete embedded minimal surfaces of finite total curvature.* In Geometry V, Encyclopaedia Math. Sci., vol. **90**, pp. 5–93, 262-272, Springer Berlin (1997), MR 1490038, Zbl 0890.53001.

[26] J.E. Hutchinson & Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. Partial Differential Equations **10** (2000), no. 1, 49–84, MR 1803974, Zbl 1070.49026.

[27] D. Jerison & R. Monneau, *Towards a counter-example to a conjecture of De Giorgi in high dimensions,* Ann. Mat. Pura Appl. **183** (2004), 439–467, MR 2140525, Zbl 1105.35037.

[28] L.P. Jorge, W.H. Meeks III, *The topology of complete minimal surfaces of finite total Gaussian curvature,* Topology **22** (1983), no. 2, 203–221, MR 0683761, Zbl 0517.53008.

[29] N. Kapouleas, *Complete embedded minimal surfaces of finite total curvature*, J. Diff. Geom. **45** (1997), 95–169, MR 1601434, Zbl 0936.53006.

[30] F.J. López & A. Ros, *On embedded complete minimal surfaces of genus zero*, J. Diff. Geom. **33** (1991), 293–300, MR 1085145, Zbl 0719.53004.

[31] L. Modica, *Convergence to minimal surfaces problem and global solutions of* $\Delta u = 2(u^3 - u)$, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 223–244, Pitagora, Bologna (1979), MR 0533169, Zbl 0408.49041.

[32] F. Morabito, *Index and nullity of the Gauss map of the Costa-Hoffman-Meeks surfaces,* Indiana Univ. Math. J. **58** (2009), no. 2, 677–707, MR 2514384, Zbl 1176.58010.

[33] S. Nayatani, *Morse index and Gauss maps of complete minimal surfaces in Euclidean* $3-$*space*, *Comm. Math. Helv.* **68** (4)(1993), 511–537, MR 124171, Zbl 0797.58018.

[34] S. Nayatani, *Morse index of complete minimal surfaces*, in Rassis, T.M. (ed.) The Problem of Plateau, pp. 181–189 (1992), MR 1209216, Zbl 0794.58011.

[35] R. Osserman, *A survey of minimal surfaces*, Math. Studies **25**, Van Nostrand, New York 1969, MR 0256278, Zbl 0209.52901.

[36] F. Pacard & M. Ritoré, *From the constant mean curvature hypersurfaces to the gradient theory of phase transitions,* J. Differential Geom. **64** (2003), no. 3, 359–423, MR 2032110, Zbl 1070.58014.

[37] J. Pérez & A. Ros, *The space of properly embedded minimal surfaces with finite total curvature.* Indiana Univ. Math. J. **45** (1996), no. 1, 177–204, MR 1406689, Zbl 0864.53008.

[38] O. Savin, *Regularity of flat level sets in phase transitions*, Annals of Mathematics, **169** (2009), 41–78, MR 2480601, Zbl 1180.35499.

[39] R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Diff. Geom. **18** (1983), 791–809, MR 0730928, Zbl 0575.53037.

[40] M. Traizet, *An embedded minimal surface with no symmetries*, J. Diff. Geom. **60** (2002) 103–153, MR 1924593, Zbl 1054.53014.

Departamento de Ingeniería Matemática and CMM
Universidad de Chile
Casilla 170 Correo 3
Santiago, Chile

E-mail address: delpino@dim.uchile.cl

Departamento de Ingeniería Matemática and CMM
Universidad de Chile
Casilla 170 Correo 3
Santiago, Chile

E-mail address: kowalczy@dim.uchile.cl

Department of Mathematics
Chinese University of Hong Kong
Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk