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BALANCED METRICS ON NON-KÄHLER CALABI-YAU THREEFOLDS

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Abstract

We construct balanced metrics on the family of non-Kähler Calabi-Yau threefolds that are obtained by smoothing after contracting $(-1,-1)$-rational curves on a Kähler Calabi-Yau threefold. As an application, we construct balanced metrics on complex manifolds diffeomorphic to the connected sum of $k \geq 2$ copies of $S^3 \times S^3$.

1. Introduction

We construct balanced metrics on the class of complex threefolds that are obtained by conifold transitions of Kähler Calabi-Yau threefolds; this class includes complex structures on the connected sum of $k \geq 2$ copies of $S^3 \times S^3$.

A central problem in studying compact complex manifolds is to find special hermitian metrics on them. (All complex manifolds in this paper are compact, unless otherwise stated.) The most distinguished class of metrics on complex manifolds are Kähler metrics. A Kähler metric is a hermitian metric whose hermitian form ω satisfies $d\omega = 0$. Kähler metrics offer many advantages: their (hermitian) connections are torsionless; their d, ∂ and $\bar{\partial}$-harmonic forms coincide, which lead to the Hodge structure on their cohomology groups. The drawback is that many important complex manifolds do not admit Kähler metrics.

In search for a wider class of special metrics on an n-dimensional complex manifold X, since the vanishing $d\omega^k = 0$ automatically yields $d\omega = 0$ when $2 \leq k \leq n-2$ (see [**19**]), the only weaker condition along this line is the balanced condition

$$d\omega^{n-1} = 0.$$

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(One can generalize the Kähler condition along other directions, like the pluriclosed metric: $\partial\bar{\partial}\omega = 0$. In this paper, we concentrate on balanced metrics.)

The balanced metrics on X (an n-dimensional complex manifold) form an important class of hermitian metrics. First, the form ω^{n-1} defines a cohomology class in $H^{2n-2}(X,\mathbb{R})$, thus can be used to define the degree of vector bundles on X; the balanced metrics also occur as part of the Strominger system, a system that generalizes the complex Monge-Ampere equations and hermitian-Yang-Mills equations (see [**33**]). Paired with cohomologically Kähler requirement on the manifold X, (i.e. the validity of the $\partial\bar{\partial}$-Lemma on X), we expect that a balanced metric would yield properties resembling that of a Kähler metric.

The existence of balanced metrics is also more robust than that of Kähler metrics; more so when the base manifold is cohomologically Kähler. For a pair of birational complex manifolds, Alessandrini and Bassanelli [**2**, **3**] proved that one admits balanced metrics if the other admits balanced metrics; when X is cohomologically Kähler and has balanced metrics, then small deformations of the complex structure of X is also cohomologically Kähler [**34**, **35**] and admits balanced metrics [**35**].

This leads to the natural question whether balanced metrics are preserved under singular transitions of the underlying manifold. A singular transition of a complex manifold Y is a contraction $Y \to X_0$ followed by a smoothing $X_0 \rightsquigarrow X_t$, (i.e. X_t are small deformations of X_0 such that X_t are smooth for general t.) The simplest such case is the conifold transition:

Definition 1.1. A conifold transition consists of a smooth compact threefold Y, a holomorphic map to a singular complex space $\pi : Y \to X_0$ and an analytic family of complex spaces X_t, $t \in \Delta \subset \mathbb{C}$, such that

1) X_0 is compact and smooth away from a finite set $\Lambda = \{p_1, \cdots, p_\ell\}$;
2) $\pi^{-1}(p_i) \triangleq E_i$ are $(-1,-1)$-curves; i.e., they are smooth rational curves, and the normal bundles $N_{E_i/Y}$ are isomorphic to $\mathcal{O}_{E_i}(-1)^{\oplus 2}$;
3) $\pi|_{Y-\pi^{-1}(\Lambda)} : Y - \pi^{-1}(\Lambda) \to X_0 - \Lambda$ is a biholomorphism;
4) X_t are compact smooth complex manifolds for $t \neq 0$.

In this paper, we prove the existence of balanced metrics under conifold transitions.

Theorem 1.2. *Let Y be a smooth Kähler Calabi-Yau threefold and let $Y \to X_0 \rightsquigarrow X_t$ be a conifold transition. Then for sufficiently small t, X_t admits smooth balanced metrics.*

Here a smooth Calabi-Yau threefold is a three dimensional complex manifold with finite fundamental group and trivial canonical line bundle. There are plenty of conifold transitions of Kähler Calabi-Yau threefolds. Given such a threefold Y, let $\mathbf{E}$ be a union of mutually disjoint $(-1,-1)$-curves E_i. By contracting $\mathbf{E}$, we obtain a singular complex space X_0. When the homology classes $[E_i] \in H_2(Y, \mathbb{Z})$ satisfies the criterion of Friedman [**12**, **13**], X_0 can be smoothed to a family of Calabi-Yau threefolds X_t. The theorem states that for sufficiently small t, all X_t have balanced metrics.

The connected sum $\#_k(S^3 \times S^3)$ of $k \geq 2$ copies of $S^3 \times S^3$ can be given a complex structure in this way [**13**, **26**]. As a corollary of the Theorem,

Corollary 1.3. *The complex structures on $\#_k(S^3 \times S^3)$ for any $k \geq 2$ constructed from the conifold transitions admit balanced metrics.*

On the other hand, according to Lemma 2 in [**7**], any pluriclosed metric ω on $\#_k(S^3 \times S^3)$ can be written as $\omega = \partial\bar{\phi} + \bar{\partial}\phi$ for a $(1,0)$-form ϕ. We claim that in this case there is no balanced metric on it. Otherwise, a balanced metric $\tilde{\omega}$ would give $0 < \int_{\#_k(S^3\times S^3)} \omega \wedge \tilde{\omega}^{n-1} = 0$, a contradiction.

Combining Corollary 1.3 and the above discussion, we prove a result stated in [**7**]. (The proof of this statement in [**7**] was incomplete; the reason given in [**7**] for $T = 0$ is insufficient.)

Corollary 1.4. *There exists no pluriclosed metric on the complex structures on $\#_k(S^3 \times S^3)$ for any $k \geq 2$ constructed from the conifold transition.*

This shows that the balanced metrics are the *only* known special hermitian metrics on these manifolds (c.f. [**7**]). We add that in [**8**] it is proved that their holomorphic tangent bundles are stable with respect to any Gaudchon metric.

We believe that the theorem will play an important role in investigating the geometry of Calabi-Yau threefolds within the framework of Reid's conjecture. To shed lights on the immense collection of diverse Calabi-Yau threefolds, Reid conjectured that all Calabi-Yau threefolds are connected by deformations and singular transitions [**31**]. The current work is a step to study Calabi-Yau threefolds in the framework of metric geometry along Reid's conjecture.

Our proof of the Theorem is partially constructive in that we construct balanced metrics g_t on X_t with prescribed limiting behavior near the singularities of X_0. This helps to investigate the solutions to the Strominger system of supersymmetry with torsion under the conifold transition. Recall that the Strominger system is an elliptic system on a pair (g, h) of a hermitian metric g on a Calabi-Yau threefold Y and a hermitian metric h on a vector bundle V on Y (c.f. [**33**, **24**, **15**, **6**, **14**]). This system includes an equation on the hermitian form ω of g:

$$d^*\omega = \sqrt{-1}(\bar\partial - \partial)\ln\|\Omega\|_\omega,$$

which is equivalent to the balanced condition [**24**]:

$$d(\|\,\Omega\,\|_\omega\,\omega^2) = 0.$$

(Here Ω is a holomorphic 3-form of the Calabi-Yau threefold.) We hope that the solutions to the Strominger system for Y can be prolonged through conifold transitions. One can also consult the discussion on this point from CFT in [**1**].

We add that there are explicit existence results on balanced metrics. Goldstein and Prokushkin [**17**] constructed balanced metrics on torus bundles over $K3$ surfaces and over complex abelian surfaces (cf. [**11**] and [**5**]). Later, D. Grantcharov, G. Grantcharov and Poon [**18**] constructed CYT structures on torus bundles over more general compact Kähler surfaces; as a consequence they constructed CYT structures on complex manifolds of topological type $(k-1)(S^2\times S^4)\#k(S^3\times S^3)$ for $k\geq 1$. However, the canonical line bundles of these complex manifolds are non-trivial. Note that for compact complex manifolds with trivial canonical line bundles, the existence of CYT structures is equivalent to the existence of balanced metrics [**25**]. Along this line, our construction provides CYT structures on a large class of threefolds, including those of types $\#_{k\geq 2}(S^3\times S^3)$.

We now outline the proof of the theorem. Our first step is to modify a Kähler metric on Y near the contracted curves E_i to get a balanced metric ω_0 on the contraction X_0 so that near the singularities of X_0 the metric ω_0 coincides with the Kähler Ricci-flat metric of Candelas-de la Ossa's (see [**9**]).

After this, we deform ω_0 to smooth almost balanced hermitian metrics ω_t on X_t so that they are Kähler and Ricci-flat near the singular points of X_0. We achieve the Ricci-flatness by using the deformation of Candelas-de la Ossa's metric on the cone singularity to smooth Ricci-flat metrics on the smoothing of the cone singularity.

To get balanced metrics, we consider perturbation $\omega_t^2+\theta_t+\bar\theta_t$, with $\theta_t = i\partial\mu_t$ for μ_t a $(1,2)$-form on X_t solving

$$i\partial_t\bar\partial_t\mu_t = \bar\partial_t\omega_t^2 \quad \text{subject to} \quad \mu_t\perp_{\omega_t}\ker\partial_t\bar\partial_t.$$

This way, $d(\omega_t^2+\theta_t+\bar\theta_t)=0$ automatically. We then prove that the C^0-norms $\|\theta_t\|_{C^0,\omega_t}\to 0$ as $t\to 0$. Thus $\omega_t^2+\theta_t+\bar\theta_t$ is positive definite for small t; $(\tilde\omega_t)^2=\omega_t^2+\theta_t+\bar\theta_t$ is solvable, and $\tilde\omega_t$ is a family of balanced metrics on X_t.

The technical part is to control the norms $\|\,\theta_t\,\|_{C^0,\omega_t}$. To this end, we choose γ_t to be the solution to the Kodaira-Spencer equation [**22**] $E_t(\gamma_t)=\bar\partial\omega_t^2$ subject to $\gamma_t\perp_{\omega_t}\ker E_t$. The solution γ_t satisfies $\partial_t\gamma_t=0$

and $\mu_t = -i\bar{\partial}_t^*\partial_t^*\gamma_t$. Applying the elliptic estimates, the L^2-estimates and the vanishing theorem of L^2-cohomology groups, we prove that

$$\lim_{t\to 0} |t|^\kappa \cdot \|\theta_t\|^2_{C^0,\omega_t} = 0 \quad \text{for} \quad \kappa > -\frac{4}{3};$$

this is more than enough to get the desired bound on $\|\theta_t\|_{C^0,\omega_t}$. Section 3 and 4 are devoted to prove this estimate.

The above construction of the family of hermitian metrics ω_t and the estimate on the perturbation terms θ_t provide a precise control on the local behavior of the metrics $\tilde{\omega}_t$ near the singularities of X_0. Such information will be useful in the further study of the geometry of X_t. For instance, using this M.-T. Chuan [**10**] has proved certian existence of Hermitian-Yang-Mills metrics on bundles over X_t.

It is worthwhile to compare this approach with a possible approach using Michelsohn's existence criterion of balanced metrics [**28**]. Let $Y \to X_0 \rightsquigarrow X_t$ be a conifold transition of the Calabi-Yau threefold and suppose Y is cohomologically Kähler and has balanced metrics. In case X_t does not have balanced metrics, by Michelsohn's criterion we find a non-zero positive $(1,1)$-current T_t on X_t of the form $T_t = \bar{\partial}S_t + \partial\bar{S}_t$ with $(1,0)$-current S_t. Suppose X_{t_k} has no balanced metrics for a sequence $t_k \to 0$, then after normalization and passing to a subsequence, we find a non-zero positive $\partial\bar{\partial}$-closed $(1,1)$-current T_0 on $Y - E$ that is a weak limit of the T_{t_k} mentioned. If we can show that T_0 extends to a non-zero positive current $\tilde{T}_0$ on Y such that $\tilde{T}_0 = \partial\bar{S} + \bar{\partial}S$ for a $(1,0)$-current S on Y, we obtain a contradiction by applying Michelsohn's criterion to our assumption that Y has balanced metrics.

The extension is guaranteed if $T_0(\Phi) = 0$ *for any d-closed* $(2,2)$*-form* Φ *on* $Y - E$ *with compact support*. One possible approach to such proof is to establish estimates on a family of $(2,2)$-forms θ'_t (similar to the θ_t mentioned before) on X_t:

$$\|\theta'_t\|_{C^0,\omega'_t} \to 0 \quad \text{as} \quad t \to 0.$$

Here ω'_t are the hermitian metrics on X_t that are the restriction to X_t of a smooth hermitian metric on $\mathcal{X} = \coprod_t X_t$. (Note for conifold transitions, $\mathcal{X}$ is a smooth, non-compact four-fold). At the moment we are unable to prove this estimate. Though this is weaker than the estimate $t^\kappa \cdot \|\theta_t\|^2_{C^0,\omega_t} \to 0$ mentioned earlier, we can prove the stronger estimate because we use essentially the Ricci-flatness of ω_t near the singularities of X_0.

We hope that a refined version of this suggested approach will be useful to attack the question on balanced metrics via singular transitions.

Question 1.5. Let Y be a compact cohomologically Kähler complex manifold and let $Y \to X_0 \rightsquigarrow X_t$ be a singular transition. Suppose Y

has a balanced metric. Does X_t admit balanced metrics for sufficiently small t?

It will be too optimistic to believe that this question has an affirmative answer in general. The case of threefolds (or Calabi-Yau threefolds) holds more hope. Our theorem is the first step toward answering this question. A more detailed understanding of this question will be important to the metric geometry of threefolds.

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2. Balanced metrics with conifold singularity

Let (Y, ω) be a Kähler threefold. Let $E \subset Y$ be a $(-1,-1)$-curve. By contracting E we obtain a variety X_0 with ordinary double point singularity. In this section, by modifying the 4-form ω^2 we construct a balanced metric ω_0^2 on $Y - E$ that coincides with Candelas-de la Ossa's cone Ricci-flat metric near the singular point of X_0.

We begin with setting up the convention for the geometry of Y near the $(-1,-1)$-curve E. We let L be the degree -1 line bundle on E; we pick a neighborhood U of E in Y that is biholomorphic to a disk bundle in $L^{\oplus 2}$.

To give coordinates to U, we fix an isomorphism $E \cong \mathbf{P}^1$, pick an $\infty \in E$ and let $z \in E - \infty = \mathbb{C}$ be the standard coordinates of $\mathbb{C}$. Using $L^{\oplus 2}|_{E-\infty} \equiv \mathbb{C}^{\oplus 2}_{E-\infty}$, and taking e_1 and e_2 be the standard basis of $\mathbb{C}^{\oplus 2}_{E-\infty}$, we give $L^{\oplus 2}|_{E-\infty}$ the coordinates (z,u,v), meaning the point $ue_1 + ve_2$ over $z \in E - \infty$.

We let $\mathbf{r} \geq 0$ be the function

$$\mathbf{r}(z,u,v)^2 = (1+ \mid z \mid^2)(\mid u \mid^2 + \mid v \mid^2). \tag{2.1}$$

A direct check shows that this function extends to a smooth hermitian metric of $L^{\oplus 2}$. Using $\mathbf{r}$, we agree that $U \subset Y$ (containing E) is biholomorphic to the open unit disk in $L^{\oplus 2}$. For $1 \geq c > 0$, we let

$$U(c) = \{(z,u,v) \in U \mid \mathbf{r}(z,u,v) < c\} \subset U(1) = U. \tag{2.2}$$

As $U \subset Y$ is viewed as an open neighborhood of $E \subset Y$, using the above inclusion, $U(c)$ for $0 < c < 1$ are open neighborhoods of $E \subset Y$ as well.

We recall Candelas-de la Ossa's metric on U. To make the forthcoming manipulation more tractable, since both $L^{\oplus 2}$ and $\mathbf{r}^2$ are invariant under the transitive group $G = U(2) \leq \mathrm{Aut}(E)$, to study the G-invariant property we only need to work out its restriction to $z = 0$ in E.

Using (2.1) and the convention (2.2), we view $\mathbf{r}$ as a function on $U \subset Y$. We consider

$$\begin{aligned} i\partial\bar\partial\mathbf{r}^2 &= i(\mid u\mid^2 + \mid v\mid^2)\, dz\wedge d\bar z + i(1+\mid z\mid^2)(du\wedge d\bar u + dv\wedge d\bar v)\\ &\quad + iz\bar u\, du\wedge d\bar z + i\bar z u\, dz\wedge d\bar u + iz\bar v\, dv\wedge d\bar z + iv\bar z\, dz\wedge d\bar v. \end{aligned}$$

Restricting to 0, and introducing

$$\lambda_1 = dz,\quad \lambda_2 = \frac{\bar u du + \bar v dv}{\sqrt{\mid u\mid^2 + \mid v\mid^2}},$$
$$\lambda_3 = \frac{v du - u dv}{\sqrt{\mid u\mid^2 + \mid v\mid^2}}\quad \text{and}\quad \lambda_{k\bar l} = i\lambda_k\wedge\lambda_{\bar l}\,,$$

we obtain

$$i\partial\bar\partial\mathbf{r}^2|_{z=0} = \mathbf{r}^2\lambda_{1\bar 1} + \lambda_{2\bar 2} + \lambda_{3\bar 3}. \tag{2.3}$$

For the same reason, $i\partial\mathbf{r}^2\wedge\bar\partial\mathbf{r}^2$ is also G-invariant, and has the form

$$i\partial\mathbf{r}^2\wedge\bar\partial\mathbf{r}^2|_{z=0} = \mathbf{r}^2\lambda_{2\bar 2}. \tag{2.4}$$

Definition 2.1. Let $f_0 = \frac{3}{2}(\mathbf{r}^2)^{\frac{2}{3}}$. The two-form $i\partial\bar\partial f_0$ is the Kähler form of Candelas-de la Ossa's metric on $U\setminus E$. It is G-invariant.

We denote this metric by $\omega_{\mathrm{co},0}$; call it the CO-metric. In explicit form,

$$\omega_{\mathrm{co},0}|_{z=0} = (\mathbf{r}^2)^{\frac{2}{3}}\lambda_{1\bar 1} + 2/3\ (\mathbf{r}^2)^{-\frac{1}{3}}\lambda_{2\bar 2} + (\mathbf{r}^2)^{-\frac{1}{3}}\lambda_{3\bar 3}. \tag{2.5}$$

Our next step is to modify ω using the CO-metric near E. For this, we need to select a cut off function $\chi(s)$.

Lemma 2.2. *There is a constant C_1 such that for any sufficiently large n, we can find a smooth function $\chi : [0,\infty)\to\mathbb{R}$ such that*

1) $\chi(s) = s$ *when* $s\in[0,2^{\frac{4}{3}}]$;
2) $\chi'(s)\ge -C_1 n^{-\frac{11}{3}}$ *and* $2\chi'(s) + s\chi''(s)\ge -C_1 n^{-\frac{11}{3}}$ *when* $s\in[2^{\frac{4}{3}},(n-1)^{\frac{4}{3}}]$;
3) $\chi'(s)\ge -C_1 n^{-\frac{5}{3}}$ *and* $2\chi'(s) + s\chi''(s)\ge -C_1 n^{-\frac{5}{3}}$ *when* $s\in[(n-1)^{\frac{4}{3}}, n^{\frac{4}{3}}]$;
4) χ *is constant when* $s\ge n^{\frac{4}{3}}$.

Proof. We first construct a C^2-function χ that satisfies the required properties. We let $c_1 = 2^{\frac{4}{3}}$; we define

$$\chi(s) = s,\quad \text{for } s\in[0,c_1].$$

We consider $\phi(s) = c_1 + (s-c_1) - (s-c_1)^3$; χ and ϕ have identical derivatives up to second order at $s = c_1$.

We let c_2 to be the (unique) element in $[c_1,\infty)$ so that $2\phi'(c_2)+c_2\phi''(c_2)=0$. This way, $\phi'(s)>0$ and $2\phi'(s)+s\phi''(s)\ge 0$ for $s\in[c_1,c_2]$. We define

$$\chi(s)=\phi(s),\quad \text{for } s\in[c_1,c_2].$$

Next, we pick $c_3=(n-1)^{\frac{4}{3}}$; $c_3>c_2$ for n large. We define

$$\chi(s)=\chi(c_2)+c_2\chi'(c_2)-c_2^2\chi'(c_2)\cdot s^{-1},\quad \text{for } [c_2,c_3].$$

One checks that for $s\in[c_2,c_3]$, $\chi'(s)>0$ and $2\chi'(s)+s\chi''(s)=0$.

To extend χ to $[c_3,c_4]$ with $c_4=n^{\frac{4}{3}}$, we let

$$\psi(s)=a_0+a_1(s-c_3)+a_2(s-c_3)^2+a_3(s-c_3)^3;$$

we choose a_i so that $\psi(c_3)=\chi'(c_3)$, $\psi'(c_3)=\chi''(c_3)$ and $\psi(c_4)=\psi'(c_4)=0$. Solving explicitly and using $\tau=c_2^2\chi_2'(c_2)$, we get

$$a_0=\tau c_3^{-2},\ a_1=-2\tau c_3^{-3},\ a_2=\frac{\tau(4c_4-7c_3)}{c_3^3(c_4-c_3)^2},\ a_3=\frac{2\tau(2c_3-c_4)}{c_3^3(c_4-c_3)^3}.$$

Using the explicit form of c_3 and c_4, we see that there is a constant C_1 independent of n so that for large n, $-C_1n^{-\frac{10}{3}}\le a_2<0$ and $0<a_3\le C_1n^{-\frac{11}{3}}$. Therefore, over $[c_3,c_4]$ we have $\psi(s)\ge -C_1n^{-\frac{5}{3}}$ and $2\psi(s)+s\psi'(s)\ge -C_1n^{-\frac{5}{3}}$. We define

$$\chi(s)=\int_{c_3}^{s}\psi(\tau)d\tau+\chi(c_3),\quad \text{for } s\in[c_3,c_4],$$

and define χ to be a constant function over $[c_4,\infty)$.

In the end, after a small perturbation of the function χ, we obtain a smooth function that satisfies the requirements stated. This proves the Lemma. q.e.d.

From now on, we let n be a large integer satisfying the conclusion of Lemma 2.2. We introduce some auxiliary functions depending on n. We will use subscript n to emphasize their dependence on n. Later we will drop the subscript n when n is fixed.

We set $\mathbf{s}_n=n^{\frac{4}{3}}(\mathbf{r}^2)^{\frac{2}{3}}$ and continue to denote $f_0=\frac{3}{2}(\mathbf{r}^2)^{\frac{2}{3}}$, both are functions of (z,u,v). Using the function χ, we construct a d-closed real $(2,2)$-form on $U\setminus E$:

$$\Phi_n=\frac{3}{2}\,i\partial\bar\partial\Big(n^{-\frac{4}{3}}\chi(\mathbf{s}_n)(i\partial\bar\partial f_0)\Big);$$

since $\mathbf{r}$ is smooth on $U\setminus E$, it is well-defined. Expanding,

$$\Phi_n=\chi'(\mathbf{s}_n)(i\partial\bar\partial f_0)\wedge(i\partial\bar\partial f_0)+2/3\,n^{\frac{4}{3}}(\mathbf{r}^2)^{-\frac{2}{3}}\chi''(\mathbf{s}_n)(i\partial\mathbf{r}^2\wedge\bar\partial\mathbf{r}^2)\wedge(i\partial\bar\partial f_0).$$

Restricting Φ_n to $z=0$ in E, from (2.4) and (2.5), we get

$$\begin{aligned} n^{\frac{2}{3}}\Phi_n|_{z=0} \;=\; & 2/3\ \big(2\chi'(\mathbf{s}_n)+\mathbf{s}_n\chi''(\mathbf{s}_n)\big)\mathbf{s}_n^{\frac{1}{2}}\lambda_{1\bar 1}\wedge\lambda_{2\bar 2}+2\chi'(\mathbf{s}_n)\mathbf{s}_n^{\frac{1}{2}}\lambda_{1\bar 1}\wedge\lambda_{3\bar 3} \\ & +2/3\ \big(2\chi'(\mathbf{s}_n)+\mathbf{s}_n\chi''(\mathbf{s}_n)\big)\mathbf{s}_n^{\frac{1}{2}}\mathbf{r}^{-2}\lambda_{2\bar 2}\wedge\lambda_{3\bar 3}. \end{aligned}$$

Lemma 2.3. *The $(2,2)$-form Φ_n satisfies:*

1) *over $U(\frac{2}{n})\setminus E$, $\Phi_n=\omega_{\mathrm{co},0}^2$ is positive;*
2) *over $U\setminus U(\frac{2}{n})$, there is a constant C_2 such that for sufficiently large n,*
$$n^{\frac{2}{3}}\Phi_n|_{z=0}\geq -C_2 n^{-1}\sum_{k\neq j}\lambda_{k\bar k}\wedge\lambda_{j\bar j};$$
3) *Φ_n has compact support (contained in) U.*

Proof. This follows from Lemma 2.2. q.e.d.

Since Φ_n has compact support (contained in) U; using extension by zero, we can view it as a global form on $Y\setminus E$.

Next we investigate the restriction of ω to U. We let $\iota: E\to Y$ be the inclusion and consider the restriction (pull back) $\omega|_E=\iota^*\omega$; it is a Kähler metric on E. With $\pi: U\to E$ the tautological projection induced by the bundle structure of $L^{\oplus 2}$, the form

$$\tilde\omega_E=\pi^*(\omega|_E)$$

is a closed semi-positive $(1,1)$-form on U.

Lemma 2.4. *There is a smooth function h of U such that $\omega|_U=\tilde\omega_E+i\partial\bar\partial h$.*

Proof. Since $[\omega|_U]=[\tilde\omega_E]\in H^2_{dR}(U,\mathbb{R})$, there exists a real 1-form α such that $\omega|_U-\tilde\omega_E=d\alpha$. Since α is real, we can write $\alpha=\beta+\bar\beta$ for β a $(0,1)$-form. Therefore from

$$\omega|_U-\tilde\omega_E=\partial\bar\beta+(\partial\beta+\bar\partial\bar\beta)+\bar\partial\beta,$$

we obtain $\bar\partial\beta=0$.

We now prove that the Dolbeault cohomology group $H^{0,1}_{\bar\partial}(U,\mathbb{C})=0$. Let $0,\infty\in E$ be the standard 0 and ∞ in $\mathbf{P}^1$ using the isomorphism $E\cong\mathbf{P}^1$ fixed at the beginning of this section. Continue to denote by $\pi: U\to E$ the projection, we introduce open subsets $U_+=U\setminus\pi^{-1}(\infty)$, $U_-=U\setminus\pi^{-1}(0)$ and $B=U_-\cap U_+$. Following the argument leading to (2.1), $U_+\subset\mathbb{C}^3$ is the domain $\{(z,u,v)\mid \mathbf{r}(z,u,v)^2<1\}$. Since $\mathbf{r}(z,u,v)^2$ is pluri-subharmonic, U_+ is Levi-pseudo-convex; thus is a domain of holomorphy. Applying the Dolbeault theorem [**23**, Thm 6.3.1], $H^{0,1}_{\bar\partial}(U_+,\mathbb{C})=0$. For the same reason, $H^{0,1}_{\bar\partial}(U_-,\mathbb{C})=0$.

Let $\gamma\in H^{0,1}_{\bar\partial}(U,\mathbb{C})$. Then there exist functions h_+ on U_+ and h_- on U_- such that $\gamma|_{U_+}=\bar\partial h_+$ and $\gamma|_{U_-}=\bar\partial h_-$. Thus $h_0=(h_+-h_-)|_B$ is holomorphic (on B). We claim that we can find holomorphic a_+ on U_+ and holomorphic a_- on U_- so that $(a_+ - a_-)|_B = h_0$; therefore $h_+ - a_+$ on U_+ and $h_- - a_-$ on U_- patch along B to form a smooth function $\tilde{h}$ on U so that $\bar{\partial}\tilde{h} = \gamma$. This will prove $H^{0,1}_{\bar{\partial}}(U, \mathbb{C}) = 0$.

We now prove the claim. We keep the embedding $U_+ \subset \mathbb{C}^3$ mentioned; using $B \subset U_+$ we have the induced embedding $B \subset U_+ \subset \mathbb{C}^3$. Let h_0 be the holomorphic function on B mentioned. Since for any $c \in \mathbb{C}^*$ the slice $B \cap \{z = c\}$ is a polydisk in $\mathbb{C}^2$, h_0 has a power series expansion $h_0 = \sum_{i,j\geq 0} a_{ij}(z)u^i v^j$, where $a_{ij}(z)$ are holomorphic on $\mathbb{C}^*$. Using the Laurent series expansions, we can write $a_{ij}(z) = a^+_{ij}(z) + a^-_{ij}(z^{-1})$ so that $a^{\pm}_{ij}(z)$ are holomorphic on $\mathbb{C}$ and $a^-_{ij}(0) = 0$. (Such decompositions are unique.) We let $h^+_0 = \sum_{i,j} a^+_{ij}(z)u^i v^j$ and $h^-_0 = \sum_{i,j} a^-_{ij}(z^{-1})u^i v^j$. Using the Cauchy integral formula and applying power series convergence criterion, one checks that h^+_0 extends to a holomorphic function on U_+.

It remains to show that a^- extends to a holomorphic function on U_-. For this, we use that $U \subset L^{\oplus 2}$ and L is the degree -1 line bundle on $\mathbf{P}^1 \cong E$. Thus we can embed $U_- \subset \mathbb{C}^3$ via coordinates (z', u', v') such that the transition function from U_+ to U_- is

$$(z', u', v') = (z^{-1}, uz, vz). \tag{2.6}$$

(Note that $u \equiv 1$ transforms to $u' = 1/z'$, which has a simple pole at $z' = 0$.) Thus $h^-_0 = \sum_{i,j} a^-_{ij}(z')(z')^{i+1}(u')^i(v')^j$. Since $a^-_{ij}(0) = 0$, h^-_0 converges on B implies that it extends to a holomorphic function on U_-. This proves the claim; hence $H^{0,1}_{\bar{\partial}}(U, \mathbb{C}) = 0$.

Because $H^{0,1}_{\bar{\partial}}(U, \mathbb{C}) = 0$, we can find a function g on U such that $\beta = \bar{\partial} g$. Therefore letting $h = -i(g - \bar{g})$, $\omega|_U - \tilde{\omega}_E = i\partial\bar{\partial}h$. q.e.d.

Since $i\partial\bar{\partial}h|_E = \iota^*(\omega - \tilde{\omega}_E) = 0$, the restriction $h|_E = \text{const.}$. Thus by subtracting a constant from h we can assume that $h|_E = 0$. Next, using the open $U_+ = U \setminus \pi^{-1}(\infty)$ and the embedding $U_+ \subset \mathbb{C}^3$, we introduce directional derivatives:

$$\mathbf{a} = \frac{\partial}{\partial u}(h|_{U_+})|_{E-\infty} \quad \text{and} \quad \mathbf{b} = \frac{\partial}{\partial v}(h|_{U_+})|_{E-\infty}. \tag{2.7}$$

Using the embedding $U_- \subset \mathbb{C}^3$ and the transition function (2.6), one sees that the smooth function on U_+ defined via

$$h_1|_{U_+} := \mathbf{a}u + \bar{\mathbf{a}}\bar{u} + \mathbf{b}v + \bar{\mathbf{b}}\bar{v} \tag{2.8}$$

extends to a smooth function on U that is $\mathbb{R}$-linear along the fibers of $\pi : U \to E$; we denote this extension by h_1.

Using h_1, we now introduce another $(2,2)$-form. We let $h_2 = h - h_1$. We pick a decreasing function $\sigma(s)$ that takes value 1 when $0 \leq s \leq 1$ and vanishes when $s \geq 4$. We set $\mathbf{t}_n = n^2\mathbf{r}^2$, which is a function of (z, u, v). Since $\sigma(\mathbf{t})$ has compact support (contained in) $\overline{U(\frac{2}{n^2})}$, using extension by zero, we can view it as a function on Y and then use it to define a real $(2,2)$-form on Y:

$$\Psi_n = \omega^2 - i\partial\bar\partial\big(\sigma(\mathbf{t}_n)\cdot h_2\cdot\big(2\tilde\omega_E + i\partial\bar\partial(2h_1+h_2)\big)\big) - i\partial\bar\partial\big(\sigma(\mathbf{t}_n)\cdot h_1\cdot i\partial\bar\partial h_1\big).$$

This form satisfies

$$d\Psi_n = 0, \qquad \Psi_n|_{Y\setminus U(\frac{2}{n})} = \omega^2 \quad \text{and} \quad \Psi_n|_{U(\frac{1}{n})} = 0.$$

Here the first and second follows from the definitions of Ψ_n and $\sigma(\mathbf{t}_n)$; the third follows from $i\partial\bar\partial h_1 \wedge \tilde\omega_E = 0$.

We now add a multiple of the compactly supported form Φ_n to Ψ_n:

$$\Omega_0 = \Psi_n + C_0 n^{\frac{2}{3}}\Phi_n, \quad C_0 > 0.$$

We emphasize that the form Ω_0 depends on the constant C_0 and the integer n. We shall specify their choices later.

Lemma 2.5. *The real $(2,2)$-form Ω_0 is d-closed;*

1) *restricting to $U(\frac{1}{n})\setminus E$, $\Omega_0|_{U(\frac{1}{n})\setminus E} = C_0 n^{\frac{2}{3}}\omega^2_{\mathrm{co},0}$;*
2) *restricting to $Y\setminus U$, $\Omega_0|_{Y\setminus U} = \omega^2$.*

Further, for sufficiently large C_0, there is a constant $n(C_0)$ such that for $n \ge n(C_0)$, $\Omega_0 > 0$.

Proof. Because Φ_n and Ψ_n are both d-closed, Ω_0 is d-closed. We show that $\Omega_0 > 0$. By the definitions of Φ_n and Ψ_n,

$$\Omega_0|_{X\setminus U} = \Psi_n|_{X\setminus U} = \omega^2 > 0 \quad \text{and}$$

$$\Omega_0|_{U(\frac{1}{n})} = C_0 n^{\frac{2}{3}}\Phi_n|_{U(\frac{1}{n})} = C_0 n^{\frac{2}{3}}\omega^2_{\mathrm{co},0} > 0.$$

So we only need to check the positivity of Ω_0 over $U\setminus U(\frac{1}{n})$. We first look at the region $U(\frac{2}{n})\setminus U(\frac{1}{n})$. Within this region,

$$\begin{aligned}\Psi_n = {}& (1-\sigma(\mathbf{t}_n))\omega^2 \\ & - i\big(h_1\partial\bar\partial\sigma(\mathbf{t}_n) + \partial\sigma(\mathbf{t}_n)\wedge\bar\partial h_1 + \partial h_1\wedge\bar\partial\sigma(\mathbf{t}_n)\big)\wedge i\partial\bar\partial h_1 \\ & - i\big(h_2\partial\bar\partial\sigma(\mathbf{t}_n) + \partial\sigma(\mathbf{t}_n)\wedge\bar\partial h_2 + \partial h_2\wedge\bar\partial\sigma(\mathbf{t}_n)\big) \\ & \quad\wedge\big(2\tilde\omega_E + i\partial\bar\partial(2h_1+h_2)\big).\end{aligned} \tag{2.9}$$

Since $1-\sigma(\mathbf{t}_n)\ge 0$, the first term is non-negative. For the other two terms, because E is covered by $D = \{|z|\le 2\}$ and $D' = \{|z|\ge 1\}$, we only need to investigate the positivity over D and D' separately. Because the discussion is similar, we shall deal with D now.

To begin with, we fix a small $\delta > 0$ (to be determined later). We consider $V_\delta = \pi^{-1}(D)\cap U(\delta)$. Over V_δ, the second term in (2.9) is

$$-i\big(h_1\partial\bar\partial\sigma(\mathbf{t}_n) + \partial\sigma(\mathbf{t}_n)\wedge\bar\partial h_1 + \partial h_1\wedge\bar\partial\sigma(\mathbf{t}_n)\big)\wedge i\partial\bar\partial h_1,$$

which, after expanding, becomes

$$-n^2 h_1\big(\sigma' i\partial\bar\partial \mathbf{r}^2 + \mathbf{t}_n \sigma'' r^{-2} i\partial \mathbf{r}^2 \wedge \bar\partial \mathbf{r}^2\big) \wedge i\partial\bar\partial h_1$$
$$-\mathbf{t}_n \sigma' \mathbf{r}^{-2}(i\partial \mathbf{r}^2 \wedge \bar\partial h_1 + i\partial h_1 \wedge \bar\partial \mathbf{r}^2) \wedge i\partial\bar\partial h_1.$$

Over the same region, we expand the relevant terms:

$$\partial \mathbf{r}^2 = \Gamma^{-2}\mathbf{r}^2 \bar z \lambda_1 + \Gamma \mathbf{r} \lambda_2$$

for $\Gamma \triangleq (1+ \mid z \mid^2)^{\frac{1}{2}}$, and

$$i\partial\bar\partial \mathbf{r}^2 = \Gamma^{-2}\mathbf{r}^2 \lambda_{1\bar 1} + \Gamma^2 \cdot (\lambda_{2\bar 2} + \lambda_{3\bar 3}) + \Gamma^{-1}\mathbf{r} \cdot (\bar z \lambda_{1\bar 2} + z\lambda_{2\bar 1}).$$

For simplicity, we use subindex z and $\bar z$ to denote the partial derivatives with respect to z and $\bar z$. For instance, $\mathbf{a}_z = \frac{\partial \mathbf{a}}{\partial z}$ and $\mathbf{b}_{z\bar z} = \frac{\partial^2 \mathbf{b}}{\partial z \partial \bar z}$. We introduce

$$c_{1\bar 1} = 2\mathrm{Re}\,\Big(\frac{\mathbf{a}_{z\bar z} u + \mathbf{b}_{z\bar z} v}{\mathbf{r}}\Big), \quad c_{2\bar 1} = \overline{c_{1\bar 2}} = \Gamma \cdot \frac{\mathbf{a}_{\bar z} u + \mathbf{b}_{\bar z} v}{\mathbf{r}},$$
$$c_{3\bar 1} = \overline{c_{1\bar 3}} = \Gamma \cdot \frac{\mathbf{a}_{\bar z} \bar v - \mathbf{b}_{\bar z} \bar u}{\mathbf{r}},$$

$$d_{1\bar 2} = \frac{\mathbf{a}_z u + \mathbf{b}_z v + \overline{\mathbf{a}_{\bar z}}\bar u + \overline{\mathbf{b}_{\bar z}}\bar v}{\mathbf{r}}, \quad d_{2\bar 2} = \Gamma \cdot \frac{\mathbf{a} u + \mathbf{b} v}{\mathbf{r}},$$
$$d_{3\bar 2} = \Gamma \cdot \frac{\mathbf{a}\bar v - \mathbf{b}\bar u}{\mathbf{r}},$$

where Re is the real part. Following such convention, we have

$$\partial h_1 = \mathbf{r} d_{1\bar 2}\lambda_1 + d_{2\bar 2}\lambda_2 + d_{3\bar 2}\lambda_3$$

and

$$i\partial\bar\partial h_1 = \mathbf{r} c_{1\bar 1}\lambda_{1\bar 1} + c_{2\bar 1}\lambda_{2\bar 1} + c_{1\bar 2}\lambda_{1\bar 2} + c_{3\bar 1}\lambda_{3\bar 1} + c_{1\bar 3}\lambda_{1\bar 3}.$$

To simplify further, we introduce

$$\alpha_{1\bar 2} = \overline{\alpha_{2\bar 1}} = -nh_1\sigma'\Gamma^2 c_{2\bar 1} + \mathbf{t}_n^{\frac{1}{2}}\sigma'\Gamma c_{3\bar 1}\overline{d_{3\bar 2}};$$
$$\alpha_{2\bar 2} = -\,nh_1\mathbf{t}_n^{\frac{1}{2}}\sigma'\Gamma^2 c_{1\bar 1} + 2\mathbf{t}_n\sigma'\Gamma^{-2}\mathrm{Re}(zc_{1\bar 3}d_{3\bar 2});$$
$$\alpha_{2\bar 3} = \overline{\alpha_{3\bar 2}} = nh_1\mathbf{t}_n^{\frac{1}{2}}(\sigma' + \mathbf{t}_n\sigma'')\Gamma^{-1}\bar z c_{3\bar 1}$$
$$+\,\mathbf{t}_n\sigma'\Gamma^{-2}\big(\bar z c_{3\bar 1}\overline{d_{2\bar 2}} + zc_{1\bar 2}d_{3\bar 2}\big) + \mathbf{t}_n\sigma'\Gamma(c_{3\bar 1}d_{1\bar 2} - c_{1\bar 1}d_{3\bar 2});$$
$$\alpha_{1\bar 3} = \overline{\alpha_{3\bar 1}} = -nh_1(\sigma' + \mathbf{t}_n\sigma'')\Gamma^2 c_{3\bar 1} - \mathbf{t}_n^{\frac{1}{2}}\sigma'\Gamma(2c_{3\bar 1}\mathrm{Re}d_{2\bar 2} - c_{2\bar 1}d_{3\bar 2});$$
$$\alpha_{3\bar 3} = -\,nh_1\mathbf{t}_n^{\frac{1}{2}}(\sigma' + \mathbf{t}_n\sigma'')\big(\Gamma^2 c_{1\bar 1} - 2\Gamma^{-1}\mathrm{Re}(zc_{1\bar 2})\big)$$
$$-\,2\mathbf{t}_n\sigma'\Gamma\big(c_{1\bar 1}\mathrm{Re}d_{2\bar 2} - \mathrm{Re}(c_{2\bar 1}d_{1\bar 2}) - \Gamma^{-3}\mathrm{Re}(zc_{1\bar 2}d_{2\bar 2})\big).$$

Because for $\mathbf{r}$ small, $|u|, |v| \le 2\mathbf{r}$, we can find a constant C_3 depending on δ so that

$$|c_{i\bar j}|,\ |d_{i\bar j}| \le C_3 \quad \text{for } (z, u, v) \in V_\delta.$$

To control the terms $\alpha_{i\bar{j}}$, we need to bound the term nh_1. For this, since $n^{-1} \le \mathbf{r} < 2n^{-1}$ over $U(\frac{2}{n}) \setminus U(\frac{1}{n})$, the term nh_1 is bounded from above uniformly over $V_\delta \cap \big(U(\frac{2}{n}) \setminus U(\frac{1}{n})\big)$ for $n > \frac{2}{\delta}$. Thus δ must be sufficiently small, which can be determined accordingly to Lemma 2.2. Therefore, enlarging C_3 if necessary and for $n > \frac{2}{\delta}$, we have

$$|\alpha_{i\bar{j}}| \le C_3, \quad (z,u,v) \in V_{[\frac{1}{n},\frac{2}{n}]} \triangleq \pi^{-1}(D) \cap \big(U(2/n) \setminus U(1/n)\big).$$

Finally, we introduce $\Lambda_{i\bar{j}}$

$$\Lambda_{i\bar{j}} = (i\lambda_k \wedge \lambda_{\bar{k}}) \wedge (i\lambda_{l_1} \wedge \lambda_{\bar{l}_2}) = \lambda_{k\bar{k}} \wedge \lambda_{l_1\bar{l}_2}$$

for $\{i,k,l_1\} = \{j,k,l_2\} = \{1,2,3\}$.

Simplifying using the notations introduced, the expression

$$\begin{aligned} & - i\big(h_1\partial\bar{\partial}\sigma(\mathbf{t}_n)\partial\sigma(\mathbf{t}_n) \wedge \bar{\partial}h_1 + \partial h_1 \wedge \bar{\partial}\sigma(\mathbf{t}_n)\big) \wedge i\partial\bar{\partial}h_1 \\ & = n\sum_{l=2,3}(\alpha_{1\bar{l}}\Lambda_{1\bar{l}} + \alpha_{l\bar{1}}\Lambda_{l\bar{1}}) + \sum_{k,l=2,3}\alpha_{k\bar{l}}\Lambda_{k\bar{l}}. \end{aligned} \tag{2.10}$$

We now look at the third term in (2.9). This time we consider

$$\begin{aligned} & -i(h_2\partial\bar{\partial}\sigma(\mathbf{t}_n) + \partial\sigma(\mathbf{t}_n) \wedge \bar{\partial}h_2 + \partial h_2 \wedge \bar{\partial}\sigma(\mathbf{t}_n))|_{V_\delta} \\ & = -\mathbf{r}^{-2}h_2(\mathbf{t}_n\sigma' i\partial\bar{\partial}\mathbf{r}^2 - \mathbf{t}_n^2\sigma''\mathbf{r}^{-2}i\partial\mathbf{r}^2 \wedge \bar{\partial}\mathbf{r}^2) \\ & \quad -\mathbf{t}_n\sigma'\mathbf{r}^{-2}(\partial\mathbf{r}^2 \wedge \bar{\partial}h_2 + \partial h_2 \wedge \bar{\partial}\mathbf{r}^2). \end{aligned}$$

Since restricting to E the partial derivatives of h_2 with respect to u and v are zero, when $\mathbf{r}$ is small, $|h_2| = O(\mathbf{r}^2)$ and $|\partial h_2| = O(\mathbf{r})$. Also notice that the mixed term such as $\lambda_{2\bar{3}}$ can be controlled by $\lambda_{2\bar{2}}$ and $\lambda_{3\bar{3}}$. Therefore for $n > \frac{2}{\delta}$, over $V_{[\frac{1}{n},\frac{2}{n}]}$ we have

$$C_3\sum_{k=1}^{3}\lambda_{k\bar{k}} \ge -i(h_2\partial\bar{\partial}\sigma(\mathbf{t}_n)+\partial\sigma(\mathbf{t}_n)\wedge\bar{\partial}h_2+\partial h_2\wedge\bar{\partial}\sigma(\mathbf{t}_n)) \ge -C_3\sum_{k=1}^{3}\lambda_{k\bar{k}}.$$

Therefore the third term in (2.9) can be controlled by $-C_3\sum_k \Lambda_{k\bar{k}}$. Inserting this and (2.10) into (2.9), we get

$$\Psi_n \ge n\sum_{l=2,3}(\alpha_{1\bar{l}}\Lambda_{1\bar{l}} + \alpha_{l\bar{1}}\Lambda_{l\bar{1}}) + \sum_{k,l=2,3}\alpha_{k\bar{l}}\Lambda_{k\bar{l}} - C_3\sum_{k=1}^{3}\Lambda_{k\bar{k}}.$$

On the other hand by a directly calculation, we have

$$\begin{aligned} n^{\frac{2}{3}}\Phi_n|_{V_{[\frac{1}{n},\frac{2}{n}]}} &= 4/3\ \mathbf{t}_n^{-\frac{2}{3}}\Gamma^4 n^2\Lambda_{1\bar{1}} + 4/3\ \mathbf{t}_n^{-\frac{1}{6}}n\Gamma\bar{z}\Lambda_{2\bar{1}} + 4/3\ \mathbf{t}_n^{-\frac{1}{6}}n\Gamma z\Lambda_{1\bar{2}} \\ &\quad + 2\mathbf{t}_n^{\frac{1}{3}}(1 - 3^{-1}\Gamma^{-2}|z|^2)\Lambda_{2\bar{2}} + 4/3\ \mathbf{t}_n^{\frac{1}{3}}(1 - \Gamma^{-2}|z|^2)\Lambda_{3\bar{3}}. \end{aligned}$$

Combining above two, over $V_{[\frac{1}{n},\frac{2}{n}]}$ we finally obtain

$$\begin{aligned}\Omega_0 \ \geq\ & \Big(\frac{4\Gamma^4 n^2}{3\mathbf{t}_n^{\frac{2}{3}}}C_0 - C_3\Big)\Lambda_{1\bar{1}} + n\Big(\frac{4\Gamma z}{3\mathbf{t}_n^{\frac{1}{6}}}C_0 + \alpha_{1\bar{2}}\Big)\Lambda_{1\bar{2}} + n\alpha_{1\bar{3}}\Lambda_{1\bar{3}} \\ & + n\Big(\frac{4\Gamma\bar{z}}{3\mathbf{t}_n^{\frac{1}{6}}}C_0 + \alpha_{2\bar{1}}\Big)\Lambda_{2\bar{1}} + \Big(2\mathbf{t}_n^{\frac{1}{3}}\Big(1-\frac{|z|^2}{3\Gamma^2}\Big)C_0 - C_3 + \alpha_{2\bar{2}}\Big)\Lambda_{2\bar{2}} \\ & + \alpha_{2\bar{3}}\Lambda_{2\bar{3}} + n\alpha_{3\bar{1}}\Lambda_{3\bar{1}} + \alpha_{3\bar{2}}\Lambda_{3\bar{2}} \\ & + \Big(\frac{4}{3}\mathbf{t}_n^{\frac{1}{3}}\Big(1-\frac{|z|^2}{\Gamma^2}\Big)C_0 - C_3 + \alpha_{3\bar{3}}\Big)\Lambda_{3\bar{3}}.\end{aligned}$$

We now prove that we can find a sufficiently large constant C_0 so that for any $n > \frac{2}{\delta}$, the right hand side of the above inequality is positive. We let e_{ij} be the coefficient of the term $\Lambda_{i\bar{j}}$ in the above inequality. To prove the mentioned positivity, we only need to check that under the stated constraint, the three minors of the 3×3 matrix $[e_{ij}]$ are positive:

$$e_{11} > 0, \quad \det[e_{ij}]_{1\leq i,j\leq 2} > 0, \quad \det[e_{ij}]_{3\times 3} > 0.$$

We recall that $\mathbf{t}_n = n^2\mathbf{r}^2$ and $\Gamma = (1+|z|^2)^{\frac{1}{2}}$. So in the region $V_{[\frac{1}{n},\frac{2}{n}]}$, $1 \leq \mathbf{t}_n < 2$ and $1 \leq \Gamma \leq \sqrt{5}$. Therefore by expanding the determinants, we see immediately that they are all positive for n positive and C_0 large enough. We fix such a C_0 in the definition of Ω_0. Therefore, for any $n > \frac{2}{\delta}$, the form Ω_0 is positive in the region $U(\frac{2}{n}) \setminus U(\frac{1}{n})$.

It remains to consider the region $U \setminus U(\frac{2}{n})$. Over this region, we shall prove that Ω_0 is positive when n is large enough. For this purpose, we will use the smooth homogenous Candelas-de la Ossa metric [**9**] on U:

$$\omega_{\mathrm{co}} = i\partial\bar{\partial} f(\mathbf{r}^2) + i\partial\bar{\partial}\log(1+|z|^2), \tag{2.11}$$

where f is defined via $f' = \mathbf{r}^{-2}\eta$ for $\eta^2(\eta+3/2) = \mathbf{r}^4$. Explicitly,

$$\omega_{\mathrm{co}}|_{z=0} = (\eta+1)\lambda_{1\bar{1}} + \frac{2}{3}\frac{(\eta+\frac{3}{2})^{\frac{1}{2}}}{\eta+1}\lambda_{2\bar{2}} + \frac{1}{(\eta+\frac{3}{2})^{\frac{1}{2}}}\lambda_{3\bar{3}}. \tag{2.12}$$

By simple estimate,

$$\omega_{\mathrm{co}}^2|_{z=0} \geq \frac{1}{3}\sum_{k\neq j}\lambda_{k\bar{k}}\wedge\lambda_{j\bar{j}}.$$

Comparing with Lemma 2.3 (2), since both ω_{co}^2 and Φ_n are homogeneous, over $U \setminus U(\frac{2}{n})$ we get

$$n^{\frac{2}{3}}\Phi_n \geq -3C_2 n^{-1}\omega_{\mathrm{co}}^2.$$

Therefore, over $U \setminus U(\frac{2}{n})$,

$$\Omega_0 \geq \omega^2 - 3C_0C_2n^{-1}\omega_{\mathrm{co}}^2.$$

This proves that for the fixed C_0 and C_2, we can choose n big enough so that the real $(2,2)$-form Ω_0 is positive over $U \setminus U(\frac{2}{n})$. This proves the lemma. q.e.d.

The closed $(2,2)$-form Ω_0 is positive $(2,2)$-form on $Y \setminus E$. From [**28**], there is a positive $(1,1)$-form ω_0 on $Y \setminus E$ such that $\omega_0^2 = \Omega_0$. This proves

Proposition 2.6. *For the open subset $E \subset U \subset Y$ chosen and for sufficiently large C_0 and n, we can find a balanced metric ω_0 over $Y \setminus E$ such that*

1) *restricting to $Y \setminus U(1)$: $\omega_0 = \omega$;*
2) *restricting to $U(\frac{1}{n}) \setminus E$: $\omega_0 = C_0^{\frac{1}{2}} n^{\frac{1}{3}} \omega_{\mathrm{co},0}$;*
3) *and restricting to $U(1) \setminus U(\frac{1}{n})$: ω_0^2 is $\partial\bar{\partial}$-exact.*

Let Y be a Calabi-Yau manifold and ω its Kähler metric. Let $\mathbf{E} \subset Y$ be a union of mutually disjoint $(-1,-1)$-curves $E_i \subset Y$, $1 \le i \le l$. For each $E_i \subset Y$ we choose an open neighborhood $E_i \subset U_i \subset Y$ as given by Proposition 2.6, and form $U_i(c)$ accordingly. We let $\mathbf{U} = \cup_{i=1}^l U_i$ and let $\mathbf{U}(c) = \cup_{i=1}^l U_i(c)$. We have

Corollary 2.7. *Let the notation be as stated. Then the Proposition* 2.6 *holds true with U and $U(c)$ replaced by $\mathbf{U}$ and $\mathbf{U}(c)$, respectively.*

Proof. Since the proof of Lemma 2.5 is by modifying ω^2 within the open neighborhood $E \subset U \subset Y$, if we choose U_i to be mutually disjoint, then we can modify ω^2 the same way within $\mathbf{U}$ to obtain the desired metric ω_0. Note that from the proof of Lemma 2.5, we can choose a common n and C_0 that work for all i. q.e.d.

Because $Y - \mathbf{E} = X_{0,\mathrm{sm}}$ (the smooth part of X_0), ω_0 is a smooth balanced metric on $X_{0,\mathrm{sm}}$ that is Candelas-de la Ossa's metric near the singular points of X_0.

3. constructing balanced metrics on the smoothings

Assuming the threefold X_0 can be smoothed to a family of smooth Calabi-Yau threefolds X_t, in this section we shall show that we can deform the metric ω_0 to a family of smooth balanced metrics on X_t.

Definition 3.1. We say X_t is a smoothing of X_0 if there is a smooth four dimensional complex manifold $\mathcal{X}$ and a proper holomorphic projection $\mathcal{X} \to \Delta$ to the unit disk Δ in $\mathbb{C}$ so that the general fibers $X_t = \mathcal{X} \times_\Delta t$ are smooth and the central fiber $\mathcal{X} \times_\Delta 0$ is X_0.

Let X_0 be a singular space that is a construction of disjoint $(-1,-1)$-curves; let ω_0 be the balanced metric on $X_{0,\mathrm{sm}}$ constructed in the previous section. We suppose X_t is a smoothing of X_0 with $\mathcal{X}$ the total space of the smoothing.

We begin with the local geometry of $\mathcal{X}$ near a singular point of X_0. Let $p \in X_0$ be any singular point that is the contraction of $E = \pi^{-1}(p)$. Since X_0 is a contraction of $(-1,-1)$-curves in Y, from the classification of singularities of threefolds, a neighborhood of p in X_0 is isomorphic to a neighborhood of 0 in

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0.$$

Applying the theorem in [**32**], a neighborhood of p in the total family $\mathcal{X}$ is isomorphic to a neighborhood of 0 in

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 - t = 0 \quad (\text{in } \mathbb{C}^4 \times \Delta),$$

as a family over $t \in \Delta$. (Here the t is linear because $\mathcal{X}$ is smooth.) More precisely, for some $\epsilon > 0$ and for

$$\tilde{U} = \left\{ (w,t) \in \mathbb{C}^4 \times \Delta_\epsilon \mid |t| < \epsilon,\ \| w \| < 1,\ w_1^2 + w_2^2 + w_3^2 + w_4^2 - t = 0 \right\},$$

$(\| w \|^2 = \sum_{i=1}^4 |w_i|^2,)$ there is a holomorphic map

$$\xi : \tilde{U} \longrightarrow \mathcal{X},$$

commuting with the projections $\tilde{U} \to \Delta_\epsilon$ and $\mathcal{X} \to \Delta$, so that $\mathcal{U} = \xi(\tilde{U})$ is an open neighborhood of $p \in \mathcal{X}$ and ξ induces an isomorphism from $\tilde{U}$ to $\mathcal{U} \subset \mathcal{X}$.

We fix such an isomorphism ξ; we denote by $\tilde{U}_t$ the fiber of $\tilde{U}$ over $t \in \Delta_\epsilon$, and denote $U_t = \xi(\tilde{U}_t)$, which is an open subset of $X_t \cap \mathcal{U}$. For any $1 > c > 0$, we let

$$\tilde{U}(c) = \{ (w,t) \in \tilde{U} \mid \| w \| < c \} \quad \text{and} \quad U_t(c) = \xi(\tilde{U}(c)) \cap X_t.$$

This way, for fixed t, $U_t(c)$ forms an increasing sequence of open subsets of X_t; the variables $(w_1, \cdots, w_4)$ can be viewed as coordinate functions with the constraint $\sum w_i^2 = t$ understood.

In case $t = 0$, we can choose ξ so that the $(w_1, \cdots, w_4)$ relates to the coordinate (z, u, v) of (2.2) by

$$w_1 = \frac{v - zu}{R},\ w_2 = \frac{v + zu}{iR},\ w_3 = \frac{u - zv}{iR},\ w_4 = \frac{u + zv}{R},$$

where the constant R is to be determined momentarily. Hence under ξ the function $\mathbf{r}$ introduced in Section 2 coincides with the function $R \cdot (\| w \|)|_{U_0}$. We then define $\mathbf{r}$ on $\mathcal{U}$ to be $\mathbf{r} = r \circ \xi^{-1}$; they are extensions of the similarly denoted $\mathbf{r}$ on X_0 used in the previous section. Also, the punctured opens $U_0(c)^* = U_0(c) - p$ are isomorphic to the opens $U(c) - E$ used in the previous section under ξ as well. Since we need to work with different fibers X_t simultaneously, we shall reserve the subscript $U_t(c)$ to denote open subsets in X_t.

We now choose R. By choosing R large and rescaling ω_0, we can assume that for $f_0 = \frac{3}{2}(\mathbf{r}^2)^{\frac{2}{3}}$

$$\Omega_0|_{U_0(1)} = \omega_0^2|_{U_0(1)} = i\partial\bar{\partial}(f_0 \cdot i\partial\bar{\partial} f_0). \tag{3.1}$$

Here since f_0 is understood as a function on $U_0(1) \subset X_0$, the partials ∂ and $\bar\partial$ are holomorphic and anti-holomorphic differentials of X_0.

One more convention we need to introduce before we move on. Note that X_0 has several singular points, say $p_1, \cdots, p_l$, corresponding to contracting $E_i \subset Y$. For each such p_i, we will go through the same procedure as we did for a general singular $p \in X_0$ moments earlier to pick an open $p_i \in \mathcal{U}_i \subset \mathcal{X}$, an isomorphism $\xi_i : \tilde U \to \mathcal{U}_i \subset \mathcal{X}$ and the open subsets $U_{i,t}(c) \subset X_t$, etc. In fixing the ξ_i for various p_i, since we can choose a common C_0 and n for all $i \in \{1, \cdots, l\}$ in Corollary 2.7, we can pick a single large enough R that works for all $U_{i,0}$ so that (3.1) holds over $U_{i,0}$.

We then form $V = \cup_{i=1}^l \mathcal{U}_i \subset \mathcal{X}$ and $V(c) = \cup_{i=1}^l \mathcal{U}_i(c) \subset \mathcal{X}$. Accordingly, we let $V_t = V \cap X_t$, let $V_t(c) = V(c) \cap X_t$, and let $\mathbf{r}$ be the function on V whose restriction to each $\mathcal{U}_i \subset V$ is the $\mathbf{r} = r \circ \xi_i^{-1}$ defined moment earlier. This procedure gives us a $(2,2)$-form Ω_0 on X_0 such that

$$\Omega_0|_{V_0(1)} = \omega_0^2|_{V_0(1)} = i\partial\bar\partial(f_0 \cdot i\partial\bar\partial f_0). \tag{3.2}$$

With these preparations, we now study the deformation X_t near the singular points $p_i \in X_0$. For $c \in (0, 1]$, we introduce

$$X_t[c] = X_t \setminus V_t(c).$$

For small t, $X_t[\frac{1}{2}]$ are diffeomorphic to each other. We fix diffeomorphisms $\psi_t : X_t[\frac{1}{2}] \to X_0[\frac{1}{2}]$ that depend smoothly on t and that $\psi_0 = \mathrm{id}$. The diffeomorphisms ψ_t pull back the form on $X_0[\frac{1}{2}]$ to form on $X_t[\frac{1}{2}]$.

We then let $\varrho(s)$ be a (decreasing) cut-off function such that $\varrho(s) = 1$ when $s \le \frac{5}{8}$ and $\varrho(s) = 0$ when $s \ge \frac{7}{8}$. This function defines a cut off function ϱ_0 on X_0 by rule $\varrho_0|_{X_0[1]} = 0$, $\varrho_0|_{V_0(\frac{1}{2})} = 1$ and $\varrho_0|_{V_0(1)\setminus V_0(\frac{1}{2})} = \varrho(\mathbf{r})$. Then

$$\Omega_0 - i\partial\bar\partial\big(\varrho_0 \cdot f_0 \cdot i\partial\bar\partial f_0\big)$$

is a smooth $(2,2)$-form on X_0 with compact support contained in $X_0[\frac{1}{2}]$. In particular, for small t,

$$\psi_t^*\big(\Omega_0 - i\partial\bar\partial(\varrho_0 \cdot f_0 \cdot i\partial\bar\partial f_0)\big)$$

is a form on $X_t[\frac{1}{2}]$ which compact support lies in it. So we can view this form as the form defined on X_t by defining it to be 0 on $V_t(\frac{1}{2})$.

Momentarily, we will use ∂ and $\bar\partial$ over X_t. In the remainder of this paper, we will take holomorphic and anti-holomorphic differentials of functions on X_t for either $t \ne 0$ or $t = 0$. To keep the notation simple, we use the same ∂f and $\bar\partial f$ to mean $\partial|_{X_t} f$ and $\bar\partial|_{X_t} f$ on either $t = 0$ or $t \ne 0$, depending on whether f is a function on X_0 or X_t. We hope the meaning of ∂ and $\bar\partial$ will cause no confusions.

In order to construct a positive $(2,2)$-form on X_t, we need to extend the function $f_0(\mathbf{r}^2) = \frac{3}{2}\mathbf{r}^{\frac{4}{3}}$ defined in Definition 2.1. For $t \ne 0 \in \triangle_\epsilon$, we define $f_t(s)$ to be the function

$$f_t(s) = 2^{-\frac{1}{3}}|t|^{\frac{2}{3}} \int_0^{\cosh^{-1}(\frac{s}{|t|})} (\sinh 2\tau - 2\tau)^{\frac{1}{3}} d\tau, \quad s \in (0,1). \tag{3.3}$$

The functions f_t give Candelas-de la Ossa's metrics (CO-metric).

Definition 3.2. The two form $\omega_{\mathrm{co},t} = i\partial\bar{\partial} f_t(\mathbf{r}^2)$ is the Ricci-flat Kähler form on $V_t(1)$ constructed by Candelas and de la Ossa.

For later application, we need to confirm the smooth dependence of the metrics $\omega_{oc,t}$ on t. We denote by $f_t^{(k)}(s)$ the k-th derivative in s of $f_t(s)$.

Lemma 3.3. *Let $f_0(s) = \frac{3}{2}s^{\frac{2}{3}}$. Then*
(1). for any $\delta > 0$ and k, restricting to $s \in [\delta, 1]$ the functions $f_t^{(k)}(s)$ converges uniformly to $f_0^{(k)}(s)$ when t goes to zero;
(2). For any pair $0 < \delta' < \delta < \frac{1}{4}$, there exists a $\alpha_{\delta'}$ such that when $\mid t \mid < \alpha_{\delta'}$ and $s \in [\delta', \delta]$, $\frac{1}{2} \le \frac{f_t'(s)}{f_0'(s)} \le 2$ and $\frac{1}{2} \le \frac{f_t''(s)}{f_0''(s)} \le 2$.

Proof. Since the dependence on $t \in \Delta_\epsilon$ is via its norm, we shall substitute $|t|$ by the positive real variable u and define $f_u(s)$ as in (3.3) with t replaced by $u > 0$.

At first we consider the convergence of $f_u(s)$. Using (3.3), we get

$$\lim_{u\to 0} f_u(s) = \frac{3}{2}s^{\frac{2}{3}} \lim_{u\to 0} g_u(s),$$

where

$$g_u(s) = \left(\left(1 - \frac{u^2}{s^2}\right)^{\frac{1}{2}} - \frac{u^2}{s^2}\cosh^{-1}\left(\frac{s}{u}\right)\right)^{\frac{1}{3}} \left(1 - \frac{u^2}{s^2}\right)^{-\frac{1}{2}}.$$

Since $\frac{u^2}{s^2}\cosh^{-1}(\frac{s}{u}) \sim \frac{u^2}{s^2}|\ln u|$ when $s \in [\delta, 1)$, $g_u(s)$ converges uniformly to 1 in $[\delta, 1)$; so $f_u(s)$ converges uniformly to $f_0(s) = \frac{3}{2}s^{\frac{2}{3}}$.

For the first and the second derivatives. by (3.3), we compute

$$f_u'(s) = s^{-\frac{1}{3}} g_u(s) \quad \text{and}$$
$$f_u''(s) = \left(-s^{-1} f_u'(s) + \tfrac{2}{3}s^{-2}(f_u'(s))^{-2}\right)\left(1 - \tfrac{u^2}{s^2}\right)^{-1}.$$

So by inspection, over $[\delta, 1)$, $f_u'(s)$ converges uniformly to $f_0'(s) = s^{-\frac{1}{3}}$ and $f_u''(s)$ converges uniformly to $f_0''(s) = -\frac{1}{3}s^{-\frac{4}{3}}$.

Since for any $k > 0$, the k-th derivative of $(1 - \frac{u^2}{s^2})^{-1}$ converges uniformly to the zero function over $s \in [\delta, 1)$, applying induction proves the remainder cases of (1).

The second part of the Lemma follows form the explicit expressions of $f_u'(s)$ and $f_u''(s)$. This proves the Lemma. q.e.d.

Our next step is to extend Ω_0 to nearby fibers so that near the singular point it equals to the CO-metrics $\omega_{\mathrm{co},t}$. To this end, we define

$$\varrho_t(z)=\begin{cases}(\psi_t^*\varrho_0)(z) & z\in X_t[\frac12],\\ 1 & z\in V_t(\frac12);\end{cases}$$

we define

(3.4)

$$\Theta_t=\psi_t^*\big(\Omega_0-i\partial\bar\partial(\varrho_0\cdot f_0(\mathbf{r}^2)\cdot i\partial\bar\partial f_0(\mathbf{r}^2))\big)+i\partial\bar\partial\big(\varrho_t\cdot f_t(\mathbf{r}^2)\cdot i\partial\bar\partial f_t(\mathbf{r}^2)\big).$$

It is well-defined and is a d-closed 4-form on X_t. It decomposes

$$\Theta_t=\Theta_t^{3,1}+\Theta_t^{2,2}+\Theta_t^{1,3}.$$

We claim that for t sufficiently small, $\Theta_t^{2,2}$ is positive definite. Indeed, over $V_t(\frac12)$, the first term in (3.4) is trivial and, since $\rho_t=1$,

$$\Theta_t^{2,2}|_{V_t(\frac12)}=\Theta_t|_{V_t(\frac12)}=\omega_{\mathrm{co},t}^2>0.$$

Over $X_t[\frac12]$, we argue that

$$\lim_{t\to 0}\Theta_t|_{X_t[\frac12]}=\Omega_0|_{X_0[\frac12]} \tag{3.5}$$

uniformly. From the expression of Θ_t it is clear that Θ_t only involves $f_u(s)$ and its derivatives up to second order. Hence by (1) of the previous Lemma, we see that over $V_t(1)\setminus V_t(\frac12)$, $f_t(\mathbf{r})$ and its partial derivatives up to second order all converges uniformly to that of $f_0(\mathbf{r})$. Hence since $X_0[\frac12]$ is compact and is disjoint from the singular points, the limit (3.5) holds uniformly. In the end, since the part $\Theta_t^{3,1}$ and $\Theta_t^{1,3}$ are trivial over $V_t(\frac12)$ and that the complex structure of X_t varies smoothly in t, the part $\Theta_t^{1,3}$ and $\Theta_t^{3,1}$ converges to zero uniformly as $t\to 0$. Consequently, for sufficiently small ϵ, $\Theta_t^{2,2}$ is positive on $X_t[\frac12]$ for $|t|<\epsilon$. Combined with the positivity of $\Theta_t^{2,2}$ over $V_t(\frac12)$, we obtained the desired positivity of $\Theta_t^{2,2}$ for t sufficiently small.

We let ω_t be the hermitian form on X_t such that $(\omega_t)^2=\Theta_t^{2,2}$. Note that for small t, these metrics have uniform geometry on $X_t[\frac12]$ and are Ricci-flat Kähler metric over $V_t(\frac12)$. In the following we will use ω_t as our background metric on X_t. Therefore objects such as norms and volume forms on X_t are all taken with respect to ω_t.

Recall that our goal is to find balanced metrics on X_t. We will achieve this by modifying the form $\Theta_t^{2,2}$ to make it both closed and positive definite.

Since Θ_t is d-closed on X_t,

$$\bar\partial\Theta_t^{2,2}=-\partial\Theta_t^{1,3}.$$

We claim that for sufficiently small t, $H^{1,3}(X_t,\mathbb{C})=0$. By the Dolbeault theorem and Serre duality, $H^{1,3}(X_t,\mathbb{C})=H^3(X_t,T^*_{X_t})=H^0(X_t,T_{X_t})$.

Thus $H^{1,3}(X_t, \mathbb{C}) = 0$ is equivalent to $H^0(T_{X_t}) = 0$. We consider the total family of the deformation $\pi : \mathcal{X} \to \Delta$ (cf. Definition 3.1). Let $\mathcal{X}^* = \mathcal{X} - \{p_1, \cdots, p_l\}$. Then $\mathcal{X}^* \to \Delta$ is smooth and the relative tangent bundle $T_{\mathcal{X}^*/\Delta}$ is a vector bundle on $\mathcal{X}^*$.

Now suppose for infinitely many $t \in \Delta$ approaching $0 \in \Delta$, $H^0(T_{X_t}) \neq 0$, then either by using elliptic estimations or applying the work of log-differential, one concludes that $H^0(T_{X_{0,\mathrm{sm}}}) \neq 0$, where $X_{0,\mathrm{sm}} = X_0 - \{p_1, \cdots, p_l\}$. Let $\mathbf{E} \subset Y$ be the union of the contracted rational curves under the projection $Y \to X_0$; then $Y - \mathbf{E} = X_{0,\mathrm{sm}}$, and thus $H^0(T_{X_{0,\mathrm{sm}}}) = H^0(T_{Y-\mathbf{E}})$. On the other hand, since Y is smooth and $\mathbf{E} \subset Y$ is a codimension 2 complex submanifold, by Hartogs' Lemma, $H^0(T_Y) = H^0(T_{Y-\mathbf{E}}) \neq 0$.

Since Y is Kähler and its fundamental group is finite, we have the following vanishing results. First, using $H^1(Y, \mathbb{C}) = 0$, we obtain $H^{0,1}(Y, \mathbb{C}) = H^{1,0}(Y, \mathbb{C}) = 0$. Since the canonical line bundle of Y is trivial, this implies $H^{3,1}(Y, \mathbb{C}) = H^{1,3}(Y, \mathbb{C}) = 0$. Applying the Serre duality, we get $H^0(T_Y) = H^{1,3}(Y, \mathbb{C}) = 0$, contradicting to $H^0(T_Y) \neq 0$ stated earlier. Thus $H^0(T_{X_t}) = 0$ for sufficiently small t.

Therefore there is a $(1, 2)$-form ν_t on X_t such that $\bar\partial \nu_t = -\Theta_t^{1,3}$. We let μ_t be a $(1, 2)$-form on X_t such that

$$i\partial\bar\partial\mu_t = -\partial\Theta_t^{1,3} = \bar\partial\Theta_t^{2,2} \quad \text{and} \quad \mu_t \perp_{\omega_t} \ker \partial\bar\partial. \tag{3.6}$$

We define

$$\Omega_t = \Theta_t^{2,2} + \theta_t + \bar\theta_t, \quad \theta_t = i\partial\mu_t. \tag{3.7}$$

Then (3.6) implies

$$\bar\partial\Omega_t = \bar\partial\Theta_t^{2,2} + \bar\partial(i\partial\mu_t) + \bar\partial(-i\bar\partial\bar\mu_t) = 0,$$

and since Ω_t is real, Ω_t is d-closed.

Proposition 3.4. *For sufficiently small t, Ω_t is positive.*

Once this is proved, then the hermitian form $\tilde\omega_t$ defined via $(\tilde\omega_t)^2 = \Omega_t$ is a balanced metric on X_t.

4. The positivity of Ω_t

To prove the Proposition, we first show that for the C^0-norm $\|\cdot\|_{C^0}$ measured using ω_t, we have

Lemma 4.1. *Suppose $\lim_{t\to 0} \| \theta_t \|_{C^0} = 0$, then Ω_t is positive for small t.*

Proof. We let $*_t$ be the Hodge operator associated to the hermitian metric ω_t. Then

$$*_t\Omega_t = \omega_t + *_t(\theta_t + \bar\theta_t)$$

and Ω_t is positive if $\omega_t + *_t(\theta_t + \bar\theta_t)$ is positive.

Now let q_t be any closed point of X_t and let (z_α) be a local chart of X_t at q_t so that

$$\omega_t(q_t) = \sqrt{-1}\delta_{\alpha\beta} dz_\alpha \wedge d\bar z_\beta \quad \text{and} \quad *_t(\theta_t + \bar\theta_t)(q_t) = \vartheta = \sqrt{-1}\vartheta_{\alpha\bar\beta} dz_\alpha \wedge d\bar z_\beta.$$

Thus $\omega_t + *_t(\theta_t + \bar\theta_t)$ is positive at q_t if and only if the matrix $\big(\delta_{\alpha\beta} + \vartheta_{\alpha\bar\beta}\big)_{1\le\alpha,\beta\le 3}$ is positive. Since $\omega_t(q_t) = \sqrt{-1}\delta_{\alpha\beta} dz_\alpha \wedge d\bar z_\beta$,

$$\sum_{\alpha,\beta} |\vartheta_{\alpha\bar\beta}|^2 = |*_t(\theta_t + \bar\theta_t)(q_t)|^2 = |(\theta_t + \bar\theta_t)(q_t)|^2 \le 4|\theta(q_t)|^2.$$

Thus if $|\theta(q_t)|^2$ is small, the matrix $\big(\delta_{\alpha\beta} + \bar\vartheta_{\alpha\bar\beta}\big)$ is positive. This proves that if the C^0-norm $\|\theta_t\|_{C^0}$ is small, the form $*_t\Omega_t$, and hence the form Ω_t, is positive. q.e.d.

The estimate of $\|\theta_t\|_{C^0}$ will be achieved in the remainder part of this section.

To estimate θ_t, we use the 4^{th}-order differential operator E_t (first introduced in [**22**]) on $\Lambda^{2,3}(X_t)$:

$$E_t = \partial\bar\partial\bar\partial^*\partial^* + \partial^*\bar\partial\bar\partial^*\partial + \partial^*\partial.$$

Here the adjoint operators $\partial^* = -*\bar\partial*$ and $\bar\partial^* = -*\partial*$ (the same as $\bar\vartheta$ and ϑ in [**29**]) are defined using the hermitian metric ω_t on X_t. In [**22**], Kodaira and Spencer proved that E_t are self-adjoint, strongly elliptic of order 4, and a form $\phi \in \Omega^{2,3}(X_t)$ satisfying $E_t\phi = 0$ if and only if

$$\partial\phi = 0 \quad \text{and} \quad \bar\partial^*\partial^*\phi = 0. \tag{4.1}$$

We now let γ_t be a solution of

$$E_t(\gamma_t) = -\partial\Theta_t^{1,3}. \tag{4.2}$$

We first check that $-\partial\Theta_t^{1,3} \perp \ker E_t$. Let $\phi \in \ker E_t$, from (4.1) we have $\bar\partial^*\partial^*\phi = 0$; from (3.6) we have

$$(-\partial\Theta_t^{1,3}, \phi) = (i\partial\bar\partial\mu_t, \phi) = \big(i\mu_t, \bar\partial^*\partial^*\phi\big) = 0.$$

This implies $-\partial\Theta_t^{1,3} \perp \ker E_t$. By the theory of elliptic operators, there is a unique smooth solution $\gamma_t \perp \ker E_t$ of (4.2).

We claim that the γ_t and the μ_t defined in (3.6) are related by

$$i\mu_t = \bar\partial^*\partial^*\gamma_t \quad \text{and} \quad \partial\gamma_t = 0. \tag{4.3}$$

This can be seen as follows. From (3.6) and (4.2), we get $E_t(\gamma_t) - i\partial\bar\partial\mu_t = 0$, which, from the definition of the operator E_t, is equivalent to

$$\partial\bar\partial(\bar\partial^*\partial^*\gamma_t - i\mu_t) + \partial^*(\bar\partial\bar\partial^* + 1)\partial\gamma_t = 0.$$

By taking the L^2-norm of the left hand side, we get

$$\partial\bar\partial(\bar\partial^*\partial^*\gamma_t - i\mu_t) = 0 \quad \text{and} \quad \partial^*(\bar\partial\bar\partial^* + 1)\partial\gamma_t = 0. \tag{4.4}$$

On the other hand, for any $\phi \in \ker \partial\bar\partial$, we have $(\bar\partial^*\partial^*\gamma_t, \phi) = (\gamma_t, \partial\bar\partial\phi) = 0$. Since $\mu_t \perp \ker \partial\bar\partial$,

$$(\bar\partial^*\partial^*\gamma_t - i\mu_t) \perp \ker \partial\bar\partial. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain $\bar\partial^*\partial^*\gamma_t - i\mu_t = 0$, which is the first identity in (4.3). The second in (4.3) follows from the second equality of (4.4), since

$$0 = \int_{X_t} \langle \partial^*(\bar\partial\bar\partial^* + 1)\partial\gamma_t, \gamma_t\rangle = \int_{X_t} (|\bar\partial^*\partial\gamma_t|^2 + |\partial\gamma_t|^2).$$

Summarizing

Lemma 4.2. *We let γ_t be the unique solution to $E_t(\gamma_t) = -\partial\Theta_t^{1,3}$ subject to the constraint $\gamma_t \perp \ker E_t$. Then γ_t satisfies $\partial\gamma_t = 0$ and the θ_t defined in* (3.7) *is of the form $\theta_t = \partial\bar\partial^*\partial^*\gamma_t$.*

Because of this Lemma, we can apply elliptic estimate to bound the norm of γ_t by that of $\partial\Theta_t^{1,3}$. We first check that for any given $0 < c < \frac{1}{2}$, E_t converges uniformly to the E_0 on $X_0[c]$. Since E_t depends on the complex structure of X_t and the Hodge star operator of the background metric ω_t, it depends on the derivatives of the components of ω_t of order at most four. By Lemma 3.3 and the discussion following the Lemma, for $c > 0$, over $X_t[c]$ the Hermition forms ω_t converges to ω_0 in C^4. Then because for any $0 < c < \frac{1}{2}$ and t sufficiently small, the Riemannian manifolds with boundaries $(X_t[c], \omega_t)$ have uniform geometry, there is a constant C independent of t small so that

$$\|\gamma_t\|_{L_4^2\left(X_t[2c]\right)} \le C\Big(\|\gamma_t\|_{L^2\left(X_t[c]\right)} + \|\partial\Theta_t^{1,3}\|_{L^2\left(X_t[c]\right)}\Big). \tag{4.6}$$

To proceed, we argue that the quantity $\int_{X_t} |\partial\Theta_t^{1,3}|^2$ is bounded by $C|t|^2$ for some constant C. Indeed, using the explicit expression (3.4), $\Theta_t^{1,3}$ has compact support contained in $X_t[\frac{1}{2}]$ and only depend on ψ_t and the complex structure of X_t. Because ψ_t are smooth in t and the complex structure are also smooth on $X_t[\frac{1}{2}]$, $\partial\Theta_t^{1,3}$ are smooth in t. Then because $\Theta_0 = \Omega_0$ is of type $(2,2)$ and d-closed, we have $\partial\Theta_0^{1,3} = (\partial\Theta_0)^{2,3} = 0$ and therefore

$$\sup_{z\in X_t} \|\partial\Theta_t^{1,3}(z)\| < C|t|. \tag{4.7}$$

This provides a bound on the last term in the inequality (4.6).

Proposition 3.4 will follow from the following stronger estimate.

Proposition 4.3. *For any $\kappa > -\frac{4}{3}$,*

$$\lim_{t\to 0}\big(|t|^\kappa \sup_{X_t} |\theta_t|^2_{\omega_t}\big) = 0.$$

Proof. First, according to the Sobolev imbedding theorem, since $X_t[\frac{1}{8}]$ have uniform geometry and E_t converges uniformly to E_0 on $X_0[\frac{1}{8}]$, there is a constant C independent of t so that for $p > 6$,

$$\|\gamma_t\|_{C^3(X_t[1/4])} \le C\big(\|\gamma_t\|_{L^2(X_t[1/8])} + \|\partial\Theta_t^{1,3}\|_{L^p(X_t[1/8])}\big).$$

Because of the identities in Lemma 4.2 and the inequality (4.7), there is a constant C independent of t so that

$$\sup_{X_t[\frac{1}{4}]} |\theta_t|^2 \le C\Big(|t|^2 + \int_{X_t[\frac{1}{8}]} |\gamma_t|^2\Big).$$

Multiplying by $|t|^\kappa$ on both sides, we get

$$\lim_{t\to 0}\Big(|t|^\kappa \sup_{X_t[\frac{1}{4}]} |\theta_t|^2\Big) \le C \lim_{t\to 0} |t|^\kappa \int_{X_t} |\gamma_t|^2. \tag{4.8}$$

This provides us the bound we need over $X_t[\frac{1}{4}]$.

To control that over its complement, namely that inside $V_t(\frac{1}{4})$, we need the following two Lemmas whose proofs will be postponed until the next section.

Lemma 4.4. *There is a constant C independent of t such that*

$$\sup_{V_t(\frac{1}{4})} |\theta_t|^2 \le C \int_{V_t(\frac{1}{4})} |\theta_t|^2 \mathbf{r}^{-4} + C \sup_{X_t[\frac{1}{4}]} |\theta_t|^2.$$

Lemma 4.5. *There is a constant C independent of t such that*

$$\int_{V_t(\frac{1}{4})} |\theta_t|^2 \mathbf{r}^{-\frac{8}{3}} \le C \int_{X_t[\frac{1}{8}]} (|\gamma_t|^2 + |\partial\Theta_t^{1,3}|^2).$$

We continue the proof of Proposition 4.3. Until the end of this section, all constant C's are independent of t; also when it depends on some other data, like a choice of $\delta > 0$, we shall use $C(\delta)$ to indicate so.

Since $\mathbf{r}^2 \ge |t|$ over $V_t(1)$, Lemma 4.5 implies

$$\int_{V_t(\frac{1}{4})} |\theta_t|^2 \mathbf{r}^{-4} \le C_1 |t|^{-\frac{2}{3}} \int_{X_t[\frac{1}{8}]} (|\gamma_t|^2 + |\partial\Theta_t^{1,3}|^2).$$

Combined with Lemma 4.4, we have

$$\sup_{V_t(\frac{1}{4})} |\theta_t|^2 \le C_2 |t|^{-\frac{2}{3}} \int_{X_t[\frac{1}{8}]} (|\gamma_t|^2 + |\partial\Theta_t^{1,3}|^2) + C \sup_{X_t[\frac{1}{4}]} |\theta_t|^2.$$

Then multiplying $|t|^\kappa$ to both sides and taking limit $t \to 0$, we find that by (4.7) the second term on the right hand vanishes since $-\frac{2}{3} + \kappa > -2$, and the third one can be controlled by the first one in view of (4.8). So we get

$$\lim_{t\to 0}\Big(|t|^\kappa \sup_{V_t(\frac{1}{4})} |\theta_t|^2\Big) \le C_3 \lim_{t\to 0} |t|^{-\frac{2}{3}+\kappa} \int_{X_t} |\gamma_t|^2.$$

Therefore by (4.8) and the above inequality, should Proposition 4.3 fail we must have

$$\overline{\lim}_{t\to 0}|t|^{-\frac{2}{3}+\kappa}\int_{X_t}|\gamma_t|^2>0.$$

In this case, there is a positive $\alpha>0$ and a sequence $t_i\to 0$ such that

$$|t_i|^{-\frac{2}{3}+\kappa}\int_{X_{t_i}}|\gamma_{t_i}|^2=\alpha_i^2\ge\alpha^2.$$

Normalizing $\tilde\gamma_{t_i}=t_i^{-\frac{1}{3}+\frac{\kappa}{2}}\alpha_i^{-1}\gamma_{t_i}$, it satisfies

$$E_{t_i}(\tilde\gamma_{t_i})=-t_i^{-\frac{1}{3}+\frac{\kappa}{2}}\alpha_i^{-1}\partial\Theta^{1,3}_{t_i} \tag{4.9}$$

and

$$\int_{X_{t_i}}|\tilde\gamma_{t_i}|^2=1. \tag{4.10}$$

Since $-\frac{1}{3}+\frac{\kappa}{2}>-1$, (4.7) implies that the C^0-norm of the right hand side of (4.9) uniformly goes to zero when $i\to\infty$. Therefore by passing to a subsequence, there exists a smooth $(1,3)$-form $\tilde\gamma_0$ on $X_{0,\mathrm{sm}}$ ($X_{0,\mathrm{sm}}$ is the smooth loci of X_0) such that $E_0(\tilde\gamma_0)=0$ and $\tilde\gamma_{t_i}\to\tilde\gamma_0$ pointwise.

To make sure that the limit is non-trivial, we check that $\|\tilde\gamma_0\|_{L^2}>0$. For this, we need the following estimate that will be proved in the next section.

Lemma 4.6. *For any $0<\iota<\frac{1}{3}$, there is a constant C such that for any $0<\delta<\frac{1}{4}$ and $|t|<\delta$,*

$$\int_{V_t(\delta)}|\gamma_t|^2\mathbf{r}^{-\frac{4}{3}}\le C\delta^{2\iota}\int_{X_t[\frac{1}{8}]}(|\gamma_t|^2+|\partial\Theta^{1,3}_t|^2).$$

We continue our proof of $\|\tilde\gamma_0\|_{L^2}>0$. By (4.7) and (4.10), for large i

$$\int_{V_{t_i}(\delta)}|\tilde\gamma_{t_i}|^2\mathbf{r}^{-\frac{4}{3}}\le C_4\delta^{2\iota}\int_{X_{t_i}[\frac{1}{8}]}(|\tilde\gamma_{t_i}|^2+t_i^{-\frac{2}{3}+\kappa}\alpha_i^{-2}|\partial_{t_i}\Theta^{1,3}_{t_i}|^2)\le C_5\delta^{2\iota}. \tag{4.11}$$

Letting $i\to\infty$ and using Lemma 3.3(2), we get

$$\int_{V_0(\delta)^*}|\tilde\gamma_0|^2\mathbf{r}^{-\frac{4}{3}}\le C_5\delta^{2\iota}, \tag{4.12}$$

where $V_0(\delta)^*=V_0(\delta)\setminus\{p_1,\cdots,p_l\}$. Because of (4.10) and the pointwise convergence $\tilde\gamma_{t_i}\to\tilde\gamma_0$ over $X_{0,\mathrm{sm}}$, we have

$$\int_{X_{0,\mathrm{sm}}}|\tilde\gamma_0|^2\ge 1-C_5\delta^{2\iota};$$

since δ is arbitrary, we obtain

$$\int_{X_{0,\mathrm{sm}}}|\tilde\gamma_0|^2=1. \tag{4.13}$$

To obtain a contradiction to complete the proof of Proposition 4.3, we now show that $\tilde\gamma_0 = 0$. We first show that $\partial^*\tilde\gamma_0 = 0$. Since $\partial\gamma_t = 0$,

$$E_t(\gamma_t) = \partial\bar\partial\bar\partial^*\partial^*\gamma_t;$$

consequently,

$$\int_{X_t} |\bar\partial^*\partial^*\gamma_t|^2 = \int_{X_t} \langle E_t(\gamma_t), \gamma_t\rangle.$$

Substituting $\tilde\gamma_{t_i}$ and applying the Hölder inequality, (4.10), (4.9) and (4.7), we obtain

$$\int_{X_{t_i}} |\bar\partial^*\partial^*\tilde\gamma_{t_i}|^2 \le \Big(\int_{X_{t_i}} |\tilde\gamma_{t_i}|^2\Big)^{\frac12}\Big(\int_{X_{t_i}} |E_{t_i}(\tilde\gamma_{t_i})|^2\Big)^{\frac12} \le C_6|t_i|^{\frac23+\frac{\kappa}{2}}.$$

Taking $i\to\infty$ and noticing $\kappa > -\frac43$, we get $\bar\partial^*\partial^*\tilde\gamma_0 = 0$.

We next pick a cut-off function $\tau(s)$ that vanishes when $s \le 0$ and $\tau(s) = 1$ when $s \ge 1$. For any $0 < \delta < 1$, we put $\mathbf{s}_\delta = \frac{2\mathbf{r}-\delta}{\delta}$. (Note that $\mathbf{r}$ is a function on $V_0(1)$ defined in (2.1) and is equal to $r\circ\xi_i^{-1}$.) We define

$$\tau_\delta = \tau(\mathbf{s}_\delta).$$

It vanishes in a small neighborhood of $\{p_1,\cdots,p_l\} \subset X_0$; it takes value 1 near the boundary of $V_0(1)$. We then extend to a function on X_0 by assigning value 1 elsewhere. We still denote this extension by τ_δ.

Using (2.4) and (2.5), over $V_0(\delta)\setminus V_0(\frac{\delta}{2})$ and for a constant C_7 independent of δ, we have

$$|\partial\tau_\delta|^2_{\omega_{\mathrm{co},0}} = \frac{4}{\delta^2}|\tau'(s)|^2|\partial\mathbf{r}|^2_{\omega_{\mathrm{co},0}} \le C_7\mathbf{r}^{-\frac43}. \tag{4.14}$$

We now fix a $\delta_1 < \frac18$. Since $\tau_{\delta_1}\partial^*\tilde\gamma_0$ has compact support, we can view $\tau_{\delta_1}\partial^*\tilde\gamma_0$ as a $(1,3)$-form on Y. Since $H^{1,3}(Y,\mathbb{C}) = 0$, (cf. discussion preceding (3.6)), there exists a smooth $(1,2)$-form ς_{δ_1} on Y such that

$$\tau_{\delta_1}\partial^*\tilde\gamma_0 = \bar\partial\varsigma_{\delta_1}.$$

Then for any $\delta < \frac{\delta_1}{2}$, by integration by parts and using $\bar\partial^*\partial^*\tilde\gamma_0 = 0$,

$$\begin{aligned}\int_{X_0} \tau_{\delta_1}|\partial^*\tilde\gamma_0|^2 &= \int_{X_0}\tau_\delta\tau_{\delta_1}|\partial^*\tilde\gamma_0|^2\\ &= \int_{X_0}\tau_\delta\langle\partial^*\tilde\gamma_0, \bar\partial\varsigma_{\delta_1}\rangle = \int_{X_0}\langle *(\partial\tau_\delta\wedge *\partial^*\tilde\gamma_0), \varsigma_{\delta_1}\rangle.\end{aligned} \tag{4.15}$$

By the Hölder inequality and the definition of τ_δ, the right hand side obeys

$$\begin{aligned}&\int_{X_0}\langle *(\partial\tau_\delta\wedge *\partial^*\tilde\gamma_0), \varsigma_{\delta_1}\rangle\\ &\le \Big(\int_{V(\delta)\setminus V(\frac{\delta}{2})} |\partial\tau_\delta|^2|\partial^*\tilde\gamma_0|^2\Big)^{\frac12}\Big(\int_{V(\delta)\setminus V(\frac{\delta}{2})} |\varsigma_{\delta_1}|^2\Big)^{\frac12}.\end{aligned} \tag{4.16}$$

We then apply the following estimate to be proved in the next section:

Lemma 4.7. *For any* $0 < \iota < \frac{1}{3}$*, there is a constant* C *such that for any* $0 < \delta < \frac{1}{4}$ *and any* $|t| < \delta$*,*

$$\int_{V_t(\delta)} |\partial^* \gamma_t|^2 \mathbf{r}^{-\frac{4}{3}} < C\delta^{2\iota} \int_{X_t[\frac{1}{8}]} (|\gamma_t|^2 + |\partial \Theta_t^{1,3}|^2).$$

From this Lemma, (4.7) and (4.10), we obtain for large i,

$$\int_{V_{t_i}(\delta)} |\partial^* \tilde{\gamma}_{t_i}|^2 \mathbf{r}^{-\frac{4}{3}} < C_8 \delta^{2\iota} \int_{X_{t_i}[\frac{1}{8}]} (|\tilde{\gamma}_{t_i}|^2 + \alpha_i^{-2} |t_i|^{-\frac{2}{3}+\kappa} |\partial \Theta_{t_i}^{1,3}|^2) \leq C_8 \delta^{2\iota},$$

where C_8 is independent of δ. Taking limit $i \to \infty$ and using Lemma 3.3(2), we get

$$\int_{V_0(\delta)^*} |\partial^* \tilde{\gamma}_0|^2 \mathbf{r}^{-\frac{4}{3}} \leq C_8 \delta^{2\iota}.$$

This inequality and (4.14) imply

$$\int_{V_0(\delta) \setminus V_0(\frac{\delta}{2})} |\partial \tau_\delta|^2 |\partial^* \tilde{\gamma}_0|^2 \leq C_9 \delta^{2\iota}. \tag{4.17}$$

Next, we denote by $\mathbf{U}(\delta)$ the union of all neighborhoods $U_i(\delta)$ of E_i in Y, defined in the previous section for $0 < \delta < 1$. Over $V_0(1)^* = V_0(1) - \{p_1, \cdots, p_l\} \cong \mathbf{U}(1) \setminus \mathbf{E}$ we have three metrics:

$$\omega_e = i\partial\bar{\partial}\mathbf{r}^2, \quad \omega_{\mathrm{co},0} = i\frac{3}{2}\partial\bar{\partial}(\mathbf{r}^2)^{\frac{2}{3}} \quad \text{and} \quad \omega_{\mathrm{co}}.$$

(Recall that $\omega_{\mathrm{co},0}$ is the cone Ricci-flat metric and ω_{co} is the Ricci-flat metric on $\mathbf{U}(1)$ (see (2.11)). Via isomorphism ξ, $\xi^*(\omega_e) = i\partial\bar{\partial}r^2$ is a metric induced from the Euclidean metric on $\mathbb{C}^4$.) Since all these metrics are homogeneous, to compare them we only need to compare their restrictions over a single point in one E_i, say at $z = 0$.

We now compare the metrics $\omega_{\mathrm{co},0}$ and ω_e by (2.5) and (2.3); compare the metrics ω_e and ω_{co} by (2.3) and (2.12). Since ς_{δ_1} is a $(1,2)$-form, the second factor in (4.16) fits into the inequalities

$$\begin{aligned} \int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})} |\varsigma_{\delta_1}|^2 &\leq C \int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})} |\varsigma_{\delta_1}|^2_{\omega_e} \mathrm{vol}_{\omega_e} \\ &\leq C_{10} \int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})} |\varsigma_{\delta_1}|^2_{\omega_{\mathrm{co}}} \mathbf{r}^{-4} \mathrm{vol}_{\omega_e}. \end{aligned} \tag{4.18}$$

Since ς_{δ_1} and ω_{co} are smooth on $\mathbf{U}(\delta_1)$, there exists a constant $C_{11}(\delta_1)$, possibly depending on δ_1, such that

$$\max_{\mathbf{U}(\delta_1)} |\varsigma_{\delta_1}|^2_{\omega_{\mathrm{co}}} \leq C_{11}(\delta_1).$$

Therefore

$$\int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})} |\varsigma_{\delta_1}|^2_{\omega_{\mathrm{co}}} \mathbf{r}^{-4} \mathrm{vol}_{\omega_e} \leq C_{11}(\delta_1) \int_{\{\mathbf{r}=1\}} \int_{\frac{\delta}{2}}^{\delta} \mathbf{r} d\mathbf{r} dS \leq C_{12}(\delta_1)\delta^2,$$

where dS is the volume element of the surface $\{\mathbf{r}=1\}$. Combined with (4.18), we get

$$\int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})} |\varsigma_{\delta_1}|^2 \leq C_{13}(\delta_1)\delta^2.$$

Then combined with (4.17) and (4.16), we obtain

$$\int_{X_0} \langle *(\partial\tau_\delta \wedge *\partial^*\tilde{\gamma}_0), \varsigma_{\delta_1}\rangle \leq C_{14}(\delta_1)\delta^{1+\iota},$$

and with (4.15),

$$\int_{X_0} \tau_{\delta_1}|\partial^*\tilde{\gamma}_0|^2 \leq C_{15}(\delta_1)\delta^{1+\iota}.$$

Taking $\delta \to 0$ and then $\delta_1 \to 0$, we get $\int_{X_{0,\mathrm{sm}}} |\partial^*\tilde{\gamma}_0|^2 = 0$; hence $\partial^*\tilde{\gamma}_0 = 0$.

It remains to show that $\tilde{\gamma}_0 = 0$. Since $\partial\gamma_t = 0$, we have $\partial\tilde{\gamma}_0 = 0$. Let $\varphi_0 \triangleq \bar{\tilde{\gamma}}_0|_{V_0(\frac{1}{4})^*}$ be the complex conjugate. Then $\bar{\partial}\varphi_0 = \bar{\partial}^*\varphi_0 = 0$. On the other hand, comparing the metrics (2.3) and (2.5), and using (4.12), we have

$$\int_{V_0(\frac{1}{4})^*} |\varphi_0|^2_{\omega_e}\mathrm{vol}_{\omega_e} \leq C\int_{V_0(\frac{1}{4})^*} |\varphi_0|^2\mathbf{r}^{-\frac{4}{3}} < +\infty.$$

Therefore, $\varphi_0 \in H^{3,2}_{(2)}\big(V_0(\frac{1}{4})^*, \omega_e\big)$ is an L^2-Dolbeault cohomology class of $V_0(\frac{1}{4})^*$, with respect to ω_e.

We claim that this cohomology group vanishes. First, for any $0 < \delta < \frac{1}{4}$, $V_0(\delta)^* = V_0(\delta)\setminus\{p_1,\cdots,p_l\}$. If we let $\tilde{V}_0(\delta) = \xi^{-1}(V_0(\delta))$, then $\tilde{V}_0(\delta)$ is a disjoint union of l copies of $\tilde{U}_0(\delta)$,

$$\tilde{U}_0(\delta) = \Big\{(w_1,\cdots,w_4)\in\mathbb{C}^4 \mid w_1^2+\cdots+w_4^2 = 0, r<\delta\Big\}.$$

Let $\tilde{\omega}_e = i\partial\bar{\partial}r^2$ on $\tilde{U}_0(\delta)^* = \tilde{U}_0(\delta)\setminus\{0\}$ be the metric induced by the flat metric on $\mathbb{C}^4$. From [**30**], we have $\lim_{\delta\to 0} H^{3,2}_{(2)}\big(\tilde{U}_0(\delta)^*, \tilde{\omega}_e\big) = 0$. Since $\omega_e = \xi^*(\tilde{\omega}_e)$ via the isomorphism ξ and since $V_0(\delta)^*$ is a disjoint union of l connected open sets each of which is isomorphic to $\tilde{U}_0(\delta)^*$, we also have $\lim_{\delta\to 0} H^{3,2}_{(2)}\big(V_0(\delta)^*, \omega_e\big) = 0$. Therefore, there exists a $\delta_2 < \frac{1}{4}$ and a $(3,1)$-form ν_0 on $V_0(\delta_2)^*$ such that $\bar{\partial}\nu_0 = \varphi_0$ and

$$\int_{V_0(\delta_2)^*} |\nu_0|^2_{\omega_e}\mathrm{vol}_{\omega_e} < +\infty. \tag{4.19}$$

Let

$$\varphi_{\delta_2} = \varphi_0 - \bar{\partial}\big((1-\tau_{\delta_2})\nu_0\big).$$

Then φ_{δ_2} has compact support in $X_{0,\mathrm{sm}}$ and $\bar{\partial}\varphi_{\delta_2} = 0$. By extension by 0, we view φ_{δ_2} as a $(3,2)$-form on Y. Since $H^{3,2}(Y,\mathbb{C}) = 0$, which follows from $H^{0,1}(Y,\mathbb{C})=0$ and the Serre duality, there exists a smooth function ν_{δ_2} on Y such that $\varphi_{\delta_2}=\bar\partial\nu_{\delta_2}$.

Now for any $\delta<\delta_2$, since $\bar\partial^*\varphi_0=0$,

(4.20)
$$\begin{aligned}\int_{X_{0,\mathrm{sm}}}\tau_\delta|\varphi_0|^2&=\int_{X_{0,\mathrm{sm}}}\tau_\delta\langle\varphi_0,\varphi_0-\bar\partial((1-\tau_{\delta_2})\nu_0)+\bar\partial((1-\tau_{\delta_2})\nu_0)\rangle\\&=\int_{X_{0,\mathrm{sm}}}\tau_\delta\langle\varphi_0,\bar\partial(\nu_{\delta_2}+(1-\tau_{\delta_2})\nu_0)\rangle\\&=\int_{X_{0,\mathrm{sm}}}\langle *(\partial\tau_\delta\wedge *\varphi_0),\nu_{\delta_2}+(1-\tau_{\delta_2})\nu_0\rangle\\&\le C\Big(\int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})}|\partial\tau_\delta|^2|\varphi_0|^2\Big)^{\frac12}\Big(\int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})}|\nu_{\delta_2}|^2+|\nu_0|^2\Big)^{\frac12}.\end{aligned}$$

Applying (4.12) and (4.14), and adding $\varphi_0=\bar{\tilde\gamma}_0|_{V_0(\frac14)^*}$, we obtain

$$\int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})}|\partial\tau_\delta|^2|\varphi_0|^2\le C_{16}\delta^{2\iota},$$

where C_{16} is independent of δ. On the other hand, since ν_{δ_2} is a smooth form in Y and ω_{co} is a smooth metric on $\mathbf{U}(\delta_2)$, there exists a constants $C_{17}(\delta_2)$, possibly depending on δ_2, such that $\max_{\mathbf{U}(\delta_2)}|\nu_{\delta_2}|^2_{\omega_{\mathrm{co}}}\le C_{17}(\delta_2)$. This, (2.3), (2.5) and (2.12) imply that

$$\int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})}|\nu_{\delta_2}|^2\le C_{18}\int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})}|\nu_{\delta_2}|^2_{\omega_{\mathrm{co}}}\mathbf{r}^{-\frac{10}{3}}\mathrm{vol}_{\omega_e}\le C_{19}(\delta_2)\delta^{\frac73}.$$

Applying (4.19), we get

$$\int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})}|\nu_0|^2\le C\int_{V_0(\delta)\setminus V_0(\frac{\delta}{2})}|\nu_0|^2_{\omega_e}\mathbf{r}^{\frac23}\mathrm{vol}_{\omega_e}\le C_{20}\delta^{\frac23}.$$

Substituting the above three inequalities into (4.20), we get

$$\int_{X_{0,\mathrm{sm}}}\tau_\delta|\varphi_0|^2\le C_{21}(\delta_2)\delta^{\iota+\frac13}.$$

Taking $\delta\to0$, since $0<\iota<\frac13$ and we have δ_2 fixed, we obtain $\int_{X_{0,\mathrm{sm}}}|\varphi_0|^2=0$. This proves $\tilde\gamma_0=0$, a contradiction that proves Proposition 4.3, and hence Proposition 3.4. q.e.d.

5. Proofs of Lemmas 4.4 to 4.7

We keep the notations introduced in the previous section. Among other things, we have the subsets $\tilde U_i\subset\mathbb{C}^4$, the biholomorphic maps $\xi_i:\tilde U_i\to\mathcal{U}_i=\xi_i(\tilde U_i)\subset\mathcal{X}$ and $V=\cup_{i=1}^l\mathcal{U}_i\subset\mathcal{X}$. Using the fiber X_t of $\mathcal{X}$ over $t\in\Delta$, we have biholomorphisms $\xi_{i,t}:\tilde U_{i,t}\to U_{i,t}\subset X_t$ and $V_t=\cup_{i=1}^lU_{i,t}\subset X_t$.

Looking at the statements of Lemmas 4.4 to 4.7, they are of the form that terms of the form $\int_{V_t(\frac{1}{2})}\cdot$ are bounded by a constant multiple of terms of the form $\int_{X_t[c]}\cdot$. Since $V_t(\frac{1}{2})$ is a disjoint union of l copies of $U_{i,t}(\frac{1}{2})$, by increase the multiple by l-fold, the Lemmas are consequence of similar statement with $V_t(\frac{1}{2})$ replaced by $U_{i,t}(\frac{1}{2})$.

But then since all geometry of $U_{i,t}$ is alike, we only need to prove the case where $V_t(\frac{1}{2})$ is replace by $U_{1,t}(\frac{1}{2})$. For notational simplicity, we use $\tilde{U}_t$ and U_t to denote $\tilde{U}_{1,t}$ and $U_{1,t}$, respectively.

Over $\tilde{U}_t$ $(=\tilde{U}_{1,t})$ we have the CO-metric $\tilde{\omega}_{\mathrm{co},t}\triangleq i\partial\bar{\partial}f_t(r^2)$, where $f_t(s)$ is defined in (3.3). The CO-metric $\omega_{\mathrm{co},t}$ on U_t is such that $\xi_t^*(\omega_{\mathrm{co},t})=\tilde{\omega}_{\mathrm{co},t}$ (we use ξ_t to denote $\xi_{1,t}=\xi|_{U_{1,t}}$); the CO-metric on V_t is $\omega_{\mathrm{co},t}$ on each U_t. The metrics ω_t on X_t are deformation of ω_0 away from the singularities of X_0, and coincide with $\omega_{\mathrm{co},t}$ over $U_t(\frac{1}{2})$.

One property of $\tilde{\omega}_{\mathrm{co},t}$ we need is the following. For any c such that $|t|^{\frac{1}{2}}<c<1$, the surface $\{r=c\}\subset\tilde{U}_t$ is diffeomorphic to $S^2\times S^3$ and $q=(\frac{\sqrt{c^2-t}}{\sqrt{2}},\frac{i\sqrt{c^2-t}}{\sqrt{2}},0,t^{\frac{1}{2}})$ lies in this surface. In the appendix, we prove that we can find a holomorphic coordinates (z_1,z_2,z_3) at q such that the CO-metric has the form

$$\tilde{\omega}_{\mathrm{co},t}|_q=i\partial\bar{\partial}f_t(r^2)|_q=i\sum_{\alpha=1}^{3}dz_\alpha\wedge d\bar{z}_\alpha; \tag{5.1}$$

letting $\eta_t(s)=sf_t'(s)$, we have

$$\partial\bar{\partial}r^2|_q=(r^2)^{\frac{1}{3}}\Big(\frac{r^4}{\eta_t^3(r^2)}\Big)^{\frac{1}{3}}\Big(dz_1\wedge d\bar{z}_1+\frac{3}{2}\frac{\eta_t^3(r^2)}{r^4}dz_2\wedge d\bar{z}_2+dz_3\wedge d\bar{z}_3\Big) \tag{5.2}$$

and

$$\partial r^2\wedge\bar{\partial}r^2|_q=\frac{3}{2}(r^2)^{\frac{4}{3}}\Big(\frac{\eta_t^3(r^2)}{r^4}\Big)^{\frac{2}{3}}\Big(1-\frac{t^2}{r^4}\Big)dz_2\wedge d\bar{z}_2. \tag{5.3}$$

In the appendix, we also prove that $r^{-4}\eta_t^3$ is increasing over $[|t|,+\infty)$ and

$$\lim_{r^2\to|t|}r^{-4}\eta_t^3=\frac{2}{3},\quad \lim_{r^2\to\infty}r^{-4}\eta_t^3=1. \tag{5.4}$$

We let $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ be the curvature tensor of $\tilde{\omega}_{\mathrm{co},t}$ at q in coordinate (z_1,z_2,z_3). Let $|R_{\alpha\bar{\beta}\gamma\bar{\delta}}|$ be its norm measured via the metric ω_t. In the appendix, we prove

Lemma 5.1. *There exists a constant C independent of t and r such that*

$$|R_{\alpha\bar{\beta}\gamma\bar{\delta}}|\le Cr^{-\frac{4}{3}}.$$

Let $\tilde{\omega}_e\triangleq i\partial\bar{\partial}r^2$ on $\tilde{U}_t$ be the metric induced by the Euclidean metric on $\mathbb{C}^4$. Let the norm and volume form defined by this metric be $|\cdot|_{\tilde{\omega}_e}$

and $\mathrm{vol}_{\tilde{\omega}_e}$. Comparing (5.1) with (5.2), since $\tilde{\omega}_{\mathrm{co},t}$ and $\tilde{\omega}_e$ are both homogeneous, we have the relation at any point in $\tilde{U}_t$:

$$\mathrm{vol}_{\tilde{\omega}_{\mathrm{co},t}} = \frac{2}{3} r^{-2} \mathrm{vol}_{\tilde{\omega}_e}, \tag{5.5}$$

and by (5.1), (5.2) and the estimate (5.4), for any smooth function f on $\tilde{U}_t$,

$$|\nabla f|^2_{\tilde{\omega}_e} \le C r^{-\frac{2}{3}} |\nabla f|^2_{\tilde{\omega}_{\mathrm{co},t}}. \tag{5.6}$$

Here $|\nabla f|^2_\omega = g^{\bar{\beta}\alpha} \frac{\partial f}{\partial z_\alpha} \frac{\partial f}{\partial \bar{z}_\beta}$ for ω a hermitian metric having the form $\omega = g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$, and $(g^{\bar{\beta}\alpha})$ is the inverse of $(g_{\alpha\bar{\beta}})$, that is $\sum_\alpha g^{\bar{\beta}\alpha} g_{\alpha\bar{\gamma}} = \delta^{\bar{\beta}}_{\bar{\gamma}}$.

We comment that the prior discussion applies to metrics $\omega_{\mathrm{co},t}$ on $U_t(\frac{1}{2})$ since our chosen background metric ω_t restricted to $U_t(\frac{1}{2})$ is the CO-metric $\tilde{\omega}_{\mathrm{co},t}$ under the isomorphism ξ_t. By abuse of notation, we shall also view (z_1, z_2, z_3) as a local coordinate of the point $\xi_t(q) \in U_t(\frac{1}{2})$.

For simplicity, in the following we shall adopt the following convention. Since we will work primarily over X_t, we will omit the subscript t in all the functions and forms that was used to indicate the domain of definition. For instance, the form θ_t on X_t will be abbreviated to θ when the domain manifold X_t is clear from the context. Also, we shall continue to use ω_t to be our metric on X_t; thus all norms and integrations without specification are with respect to the metric ω_t and by the volume form of ω_t. In case we need to use a different metric, say with ω_e, we will use $|\cdot|_{\omega_e}$ and vol_{ω_e} to mean the associated norm and volume form.

We let $\tau(\mathbf{r})$ be a cut-off function defined on $V_t(1)$ such that $\tau(\mathbf{r}) = 1$ when $\mathbf{r} \le \frac{1}{4}$ and $\tau(\mathbf{r}) = 0$ when $\mathbf{r} \ge \frac{1}{2}$. We then extend it to X_t by zero and denote by the same notation $\tau(\mathbf{r})$.

Proof of Lemma 4.4. As commented, we only need to prove the statement of Lemma 4.4 with $V_t(\frac{1}{2})$ replaced by $U_t(\frac{1}{2}) \triangleq U_{1,t}(\frac{1}{2})$. We fix a t of small $|t|$. As commented, we use θ to denote the θ_t in Lemma 4.4.

We introduce a sequence $\beta_k = (\frac{3}{2})^k$. By the definition of τ and (5.5), we have

$$\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_k} \mathbf{r}^{-4} \le \frac{2}{3} \int_{U_t(\frac{1}{2})} |\theta|^{2\beta_k} \mathbf{r}^{-6} \tau^3 \mathrm{vol}_{\omega_e}. \tag{5.7}$$

The function $|\theta|^{2\beta_k} \mathbf{r}^{-6} \tau^3$ is a non-negative C^∞-function with compact support contained in $U_t(\frac{1}{2})$. Via ξ_t, $U_t(\frac{1}{2})$ is identified with a minimal submanifold in $\mathbb{C}^4$ endowed with the Euclidean metric.

We quote Michael-Simon's Sobolev inequality [**27**] (independently by Allard [**4**]): Let $M \subset \mathbb{R}^m$ be an n-dimensional submanifold in the Euclidean m-space $\mathbb{R}^m$, let H be its mean curvature vector, and let

$f \in C_0^\infty(M)$ be a nonnegative functions with compact support, then

$$\Big(\int_M f^{\frac{n}{n-1}}\mathrm{vol}\Big)^{\frac{n-1}{n}} \leq C(n)\int_M (|\nabla f| + |H|\cdot f)\mathrm{vol}.$$

Applying this to the minimal submanifold $\tilde{U}_t \subset \mathbb{C}^4$, and then applying the standard skill in page 156 of [**16**], for any nonnegative function f on $U_t(\frac{1}{2})$ with compact support, we see

$$\Big(\int_{U_t(\frac{1}{2})} f^3 \mathrm{vol}_{\omega_e}\Big)^{\frac{1}{3}} \leq C\Big(\int_{U_t(\frac{1}{2})} |\nabla f|^2_{\omega_e}\mathrm{vol}_{\omega_e}\Big)^{\frac{1}{2}},$$

where C is a constant depending only on the dimension of $U_t(\frac{1}{2})$.

Applying the volume comparison (5.5) and the norm comparison (5.6) to the right hand side of the above inequality, for C_1 a constant independent of t we get

$$\Big(\int_{U_t(\frac{1}{2})} f^3 \mathrm{vol}_{\omega_e}\Big)^{\frac{1}{3}} \leq C_1\Big(\int_{U_t(\frac{1}{2})} |\nabla f|^2_{\omega_{\mathrm{co},t}}\mathbf{r}^{\frac{4}{3}}\mathrm{vol}_{\omega_{\mathrm{co},t}}\Big)^{\frac{1}{2}}. \tag{5.8}$$

We remark that in this section we shall use C_i to denote constants independent of t and k. Since the exact sizes of these constants are irrelevant, we will be very loose in keeping track of them.

Applying (5.8) to the right hand side of (5.7) for $f = |\theta|^{\frac{2\beta_k}{3}}\mathbf{r}^{-2}\tau$, we have

$$\begin{aligned}
&\Big(\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_k}\mathbf{r}^{-4}\Big)^{\frac{2}{3}} \leq \Big(\int_{U_t(\frac{1}{2})} |\theta|^{2\beta_k}\mathbf{r}^{-6}\tau^3\mathrm{vol}_{\omega_e}\Big)^{\frac{2}{3}}\\
&\leq C_1^2\int_{U_t(\frac{1}{2})} \big|\nabla(|\theta|^{\frac{2\beta_k}{3}}\mathbf{r}^{-2}\tau)\big|^2\mathbf{r}^{\frac{4}{3}}\\
&\leq 3C_1^2\int_{U_t(\frac{1}{2})} \big|\nabla|\theta|^{\frac{2\beta_k}{3}}\big|^2\mathbf{r}^{-\frac{8}{3}}\tau^2\\
&\quad + 3C_1^2\int_{U_t(\frac{1}{2})} |\theta|^{2\beta_{k-1}}|\nabla\mathbf{r}^{-2}|^2\mathbf{r}^{\frac{4}{3}}\tau^2\\
&\quad + 3C_1^2\int_{U_t(\frac{1}{2})} |\theta|^{2\beta_{k-1}}\mathbf{r}^{-\frac{8}{3}}|\nabla\tau|^2.
\end{aligned} \tag{5.9}$$

We use (5.3) to estimate the second term after the third "$\leq$" in (5.9):

$$\begin{aligned}
&\int_{U_t(\frac{1}{2})} |\theta|^{2\beta_{k-1}}|\nabla\mathbf{r}^{-2}|^2\mathbf{r}^{\frac{4}{3}}\tau^2\\
&\leq C_2\int_{U_t(\frac{1}{2})} |\theta|^{2\beta_{k-1}}\mathbf{r}^{-4}\tau^2\\
&\leq C_2\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_{k-1}}\mathbf{r}^{-4} + 4^4C_2\int_{X_t[\frac{1}{4}]} |\theta|^{2\beta_{k-1}}.
\end{aligned} \tag{5.10}$$

From the definition of τ, the third term has an estimate:

$$\int_{U_t(\frac{1}{2})} |\theta|^{2\beta_{k-1}} \mathbf{r}^{-\frac{8}{3}} |\nabla\tau|^2 \leq C_3 \int_{X_t[\frac{1}{4}]} |\theta|^{2\beta_{k-1}}. \tag{5.11}$$

For the first term after the third "$\leq$" in (5.9), we claim that for any $k \geq 1$,

(5.12)
$$\begin{aligned}\int_{U_t(\frac{1}{2})} \big|\nabla|\theta|^{\frac{2\beta_k}{3}}\big|^2 \mathbf{r}^{-\frac{8}{3}}\tau^2 \leq& -C_4 \int_{U_t(\frac{1}{2})} |\theta|^{2\beta_{k-1}} \triangle_{\bar\partial} (\mathbf{r}^{-\frac{8}{3}}\tau^2) \\ &- \beta_{k-1} \int_{U_t(\frac{1}{2})} |\theta|^{2(\beta_{k-1}-1)} g^{\bar\beta\alpha} (\langle \nabla_\alpha \nabla_{\bar\beta}\theta, \theta\rangle + \langle \theta, \nabla_{\bar\alpha}\nabla_\beta\theta\rangle) \mathbf{r}^{-\frac{8}{3}}\tau^2.\end{aligned}$$

Here we denote $\omega_t = \sum g_{\alpha\bar\beta} dz_\alpha \wedge d\bar z_\beta$, (we have omitted the subscript t in $g_{\alpha\bar\beta}$,) and denote the inverse $(g_{\alpha\bar\beta})^{-1}$ of $(g_{\alpha\bar\beta})$ by $(g^{\bar\alpha\beta})$. We also denote $\triangle_{\bar\partial} = -g^{\bar\beta\alpha}\frac{\partial^2}{\partial z_\alpha \partial \bar z_\beta}$ and $\nabla_\alpha = \nabla_{\frac{\partial}{\partial\alpha}}$ and so on.

We first prove the case $k \geq 3$. By direct calculation, we have

$$\int_{U_t(\frac{1}{2})} \big|\nabla|\theta|^{\frac{2\beta_k}{3}}\big|^2 \mathbf{r}^{-\frac{8}{3}}\tau^2 = \frac{\beta_{k-1}^2}{4} \int_{U_t(\frac{1}{2})} |\theta|^{2(\beta_{k-1}-2)} \big|\nabla|\theta|^2\big|^2 \mathbf{r}^{-\frac{8}{3}}\tau^2. \tag{5.13}$$

We compute

$$\begin{aligned}&\beta_{k-1}(\beta_{k-1}-1)|\theta|^{2(\beta_{k-1}-2)}\big|\nabla|\theta|^2\big|^2 \\ &\quad= \beta_{k-1}|\theta|^{2(\beta_{k-1}-1)} \triangle_{\bar\partial} |\theta|^2 - \triangle_{\bar\partial}|\theta|^{2\beta_{k-1}} \\ &\quad\leq -\beta_{k-1}|\theta|^{2(\beta_{k-1}-1)} g^{\bar\beta\alpha}(\langle \nabla_\alpha\nabla_{\bar\beta}\theta, \theta\rangle + \langle\theta, \nabla_{\bar\alpha}\nabla_\beta\theta\rangle) - \triangle_{\bar\partial}|\theta|^{2\beta_{k-1}}.\end{aligned}$$

Multiplying $\mathbf{r}^{-\frac{8}{3}}\tau^2$ to both sides of the above inequality and then integrating over $U_t(\frac{1}{2})$, since the CO-metric is Kähler and τ^2 vanishes outside $U_t(\frac{1}{2})$, we get

$$\begin{aligned}&\beta_{k-1}(\beta_{k-1}-1)\int_{U_t(\frac{1}{2})} |\theta|^{2(\beta_{k-1}-2)}\big|\nabla|\theta|^2\big|^2 \mathbf{r}^{-\frac{8}{3}}\tau^2 \\ &\quad\leq -\beta_{k-1}\int_{U_t(\frac{1}{2})} |\theta|^{2(\beta_{k-1}-1)} g^{\bar\beta\alpha}(\langle \nabla_\alpha\nabla_{\bar\beta}\theta, \theta\rangle + \langle\theta, \nabla_{\bar\alpha}\nabla_\beta\theta\rangle)\mathbf{r}^{-\frac{8}{3}}\tau^2 \\ &\qquad - \int_{U_t(\frac{1}{2})} |\theta|^{2\beta_{k-1}} \triangle_{\bar\partial} \big(\mathbf{r}^{-\frac{8}{3}}\tau^2\big).\end{aligned}$$

This and (5.13) prove (5.12).

For $k = 2$, from $\triangle_{\bar\partial}|\theta|^3 = \frac{3}{2}|\theta| \triangle_{\bar\partial} |\theta|^2 - 3|\theta|\big|\nabla|\theta|\big|^2$, a computation gives

$$\begin{aligned}&\int_{U_t(\frac{1}{2})} \big|\nabla|\theta|^{\frac{2\beta_2}{3}}\big|^2 \mathbf{r}^{-\frac{8}{3}}\tau^2 \\ &\quad\leq \beta_1 \int_{U_t(\frac{1}{2})} |\theta|^{2(\beta_1-1)} \triangle_{\bar\partial} |\theta|^2 \mathbf{r}^{-\frac{8}{3}}\tau^2 - \int_{U_t(\frac{1}{2})} |\theta|^{2\beta_1} \triangle_{\bar\partial} (\mathbf{r}^{-\frac{8}{3}}\tau^2).\end{aligned}$$

This implies (5.12) in case of $k=2$.

For $k=1$, we need to estimate $\big|\nabla|\theta|\big|^2$. When $|\theta|\neq 0$,

$$\big|\nabla|\theta|\big|^2=\frac{1}{4}|\theta|^{-2}\big|\nabla|\theta|^2\big|^2\leq 2^{-1}g^{\bar\beta\alpha}\langle\nabla_\alpha\theta,\nabla_\beta\theta\rangle+2^{-1}g^{\bar\beta\alpha}\langle\nabla_{\bar\beta}\theta,\nabla_{\bar\alpha}\theta\rangle$$
$$=-2^{-1}\triangle_{\bar\partial}|\theta|^2-2^{-1}g^{\bar\beta\alpha}\langle\nabla_\alpha\nabla_{\bar\beta}\theta,\theta\rangle-2^{-1}g^{\bar\beta\alpha}\langle\theta,\nabla_{\bar\alpha}\nabla_\beta\theta\rangle.$$

When $|\theta|=0$, $|\bigtriangledown|\theta\,\|=0$ and $-\triangle_{\bar\partial}\mid\theta\mid^2\geq 0$. We still have above inequality. So (5.12) is valid for $k=1$ and $\beta_0=1$.

Next we estimate the second term in (5.12) by using Kodaira's Bochner formula ([**29**] p.119): for any (p,q)-form $\psi=\frac{1}{p!q!}\sum\psi_{\alpha_1\cdots\bar\beta_q}dz_{\alpha_1}\wedge\cdots\wedge d\bar z_{\beta_q}$,

$$(\triangle_{\bar\partial}\psi)_{\alpha_1\cdots\bar\beta_q}=-\sum_{\alpha,\beta}g^{\bar\beta\alpha}\nabla_\alpha\nabla_{\bar\beta}\psi_{\alpha_1\cdots\bar\beta_q}$$
$$+\sum_{i=1}^{p}\sum_{k=1}^{q}\sum_{\alpha,\beta}R^{\alpha}{}_{\alpha_i\bar\beta_k}{}^{\bar\beta}\psi_{\alpha_1\cdots\alpha_{i-1}\alpha\alpha_{i+1}\cdots\bar\beta_{k-1}\bar\beta\bar\beta_{k+1}\cdots\bar\beta_q}\tag{5.14}$$
$$-\sum_{k=1}^{q}\sum_{\beta}R_{\bar\beta_k}{}^{\bar\beta}\psi_{\alpha_1\cdots\bar\beta_{k-1}\bar\beta\bar\beta_{k+1}\cdots\bar\beta_q}.$$

We use above formula to $\psi=\theta$ over $U_t(\frac{1}{2})$. Since $\theta=\partial\bar\partial^*\partial^*\gamma_t$, $\partial\gamma_t=0$ and $\Theta_t^{1,3}|_{U_t(\frac{1}{2})}=0$,

$$\triangle_{\bar\partial}\theta|_{U_t(\frac{1}{2})}=(\bar\partial\bar\partial^*+\bar\partial^*\bar\partial)\theta|_{U_t(\frac{1}{2})}=-\bar\partial^*E_t(\gamma_t)|_{U_t(\frac{1}{2})}=-\bar\partial^*\partial\Theta_t^{1,3}|_{U_t(\frac{1}{2})}=0.\tag{5.15}$$

Then (5.14) and Lemma 5.1 imply

$$-g^{\bar\beta\alpha}(\langle\nabla_\alpha\nabla_{\bar\beta}\theta,\theta\rangle+\langle\theta,\nabla_{\bar\alpha}\nabla_\beta\theta\rangle)$$
$$=-g^{\bar\beta\alpha}(\langle\nabla_\alpha\nabla_{\bar\beta}\theta,\theta\rangle+\langle\theta,\nabla_\beta\nabla_{\bar\alpha}\theta\rangle+\langle\theta,[\nabla_{\bar\alpha},\nabla_\beta]\theta\rangle)\leq C_5\mathbf{r}^{-\frac{4}{3}}|\theta|^2,$$

where $[\nabla_{\bar\alpha},\nabla_\beta]=\nabla_{\bar\alpha}\nabla_\beta-\nabla_\beta\nabla_{\bar\alpha}$ is the curvature operator.

From the above inequality, we can estimate the second term after the inequality in (5.12):

$$-\beta_{k-1}\int_{U_t(\frac{1}{2})}|\theta|^{2(\beta_{k-1}-1)}g^{\bar\beta\alpha}(\langle\nabla_\alpha\nabla_{\bar\beta}\theta,\theta\rangle+\langle\theta,\nabla_{\bar\alpha}\nabla_\beta\theta\rangle)\mathbf{r}^{-\frac{8}{3}}\tau^2$$
$$\leq C_5\beta_{k-1}\int_{U_t(\frac{1}{2})}|\theta|^{2\beta_{k-1}}\mathbf{r}^{-4}\tau^2\tag{5.16}$$
$$\leq C_5\beta_{k-1}\int_{U_t(\frac{1}{4})}|\theta|^{2\beta_{k-1}}\mathbf{r}^{-4}+4^4C_5\beta_{k-1}\int_{X_t[\frac{1}{4}]}|\theta|^{2\beta_{k-1}}.$$

From (5.1), (5.2) and (5.3),

$$-\triangle_{\bar\partial}\mathbf{r}^{-\frac{8}{3}}\leq C_6\mathbf{r}^{-4}.$$

Thus the first term after the inequality in (5.12) has estimate

$$
\begin{aligned}
&-\int_{U_t(\frac{1}{2})} |\theta|^{2\beta_{k-1}} \triangle_{\bar\partial} \left(\mathbf{r}^{-\frac{8}{3}}\tau^2\right) \\
&\quad \le C_6 \int_{U_t(\frac{1}{4})} |\theta|^{2\beta_{k-1}}\mathbf{r}^{-4} + 4^4 C_6 \int_{X_t[\frac{1}{4}]} |\theta|^{2\beta_{k-1}}.
\end{aligned} \tag{5.17}
$$

Inserting (5.16) and (5.17) into (5.12), we get

$$
\begin{aligned}
&\int_{U_t(\frac{1}{4})} \left|\nabla|\theta|^{\frac{2\beta_k}{3}}\right|^2 \mathbf{r}^{-\frac{8}{3}}\tau^2 \\
&\quad \le C_7\beta_{k-1}\Big(\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_{k-1}}\mathbf{r}^{-4} + \int_{X_t[\frac{1}{4}]} |\theta|^{2\beta_{k-1}}\Big);
\end{aligned} \tag{5.18}
$$

inserting (5.18), (5.10) and (5.11) into (5.9), we obtain

$$
\begin{aligned}
&\Big(\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_k}\mathbf{r}^{-4}\Big)^{\frac{1}{\beta_k}} \\
&\quad \le (C_7\beta_{k-1})^{\frac{1}{\beta_{k-1}}}\Big(\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_{k-1}}\mathbf{r}^{-4} + \int_{X_t[\frac{1}{4}]} |\theta|^{2\beta_{k-1}}\Big)^{\frac{1}{\beta_{k-1}}}.
\end{aligned}
$$

So for any $k \ge 1$, the above inequality implies that either

$$
\Big(\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_k}\mathbf{r}^{-4}\Big)^{\frac{1}{\beta_k}} \le \big(2C_7\beta_{k-1}\big)^{\frac{1}{\beta_{k-1}}}\Big(\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_{k-1}}\mathbf{r}^{-4}\Big)^{\frac{1}{\beta_{k-1}}}
$$

or

$$
\Big(\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_k}\mathbf{r}^{-4}\Big)^{\frac{1}{\beta_k}} \le \big(2C_7\beta_{k-1}\big)^{\frac{1}{\beta_{k-1}}}\big(\mathrm{vol}(X_t[1/4])\big)^{\frac{1}{\beta_{k-1}}} \sup_{X_t[\frac{1}{4}]} |\theta|^2.
$$

Since the volume of $X_t[\frac{1}{4}]$ can be controlled by a constant independent of t, these two inequalities imply that for a constant C_8 independent of t and k,

$$
\Big(\int_{U_t(\frac{1}{4})} |\theta|^{2\beta_k}\mathbf{r}^{-4}\Big)^{\frac{1}{\beta_k}} \le \prod_{i=1}^{k-1}(C_8\beta_{i-1})^{\frac{1}{\beta_{i-1}}}\Big(\int_{U_t(\frac{1}{4})} |\theta|^2\mathbf{r}^{-4} + \sup_{X_t[\frac{1}{4}]}|\theta|^2\Big).
$$

Taking limit $k \to \infty$, we get the inequality in the statement of Lemma 4.4. q.e.d.

Proof of Lemma 4.5. We keep the convention introduced in the proof of Lemma 4.4. To streamline the notation, we assign the symbol Λ_t to

$$
\Lambda_t := \int_{X_t[\frac{1}{8}]} (|\gamma|^2 + |\partial\Phi^{1,3}|^2).
$$

The Lemma 4.5 is then equivalent to that for a constant C independent of t,

$$
\int_{U_t(\frac{1}{4})} |\theta_t|^2\mathbf{r}^{-\frac{8}{3}} \le C\Lambda_t.
$$

To begin with, for a smooth positive function ϕ, we define $\bar\partial^*_\phi\zeta = \bar\partial^*\zeta - *(\partial\log\phi\wedge *\zeta)$, $\nabla^\phi_\alpha = \nabla_\alpha + \partial_\alpha\log\phi$ and $X_{\phi\ \bar\beta_k}^{\ \bar\beta} = -g^{\bar\beta_k\alpha}\partial_{\bar\beta}\partial_\alpha\log\phi$. We need another Kodaira's Bochner formula ([**29**], p.124): For any (p,q)-form $\zeta = \frac{1}{p!q!}\sum\zeta_{\alpha_1\cdots\bar\beta_q}dz_{\alpha_1}\wedge\cdots\wedge d\bar z_{\beta_q}$,

$$
\begin{aligned}
((\bar\partial\bar\partial^*_\phi + \bar\partial^*_\phi\bar\partial)\zeta)_{\alpha_1\cdots\bar\beta_q} &= -\sum_{\alpha,\beta} g^{\bar\beta\alpha}\nabla^\phi_\alpha\nabla_{\bar\beta}\zeta_{\alpha_1\cdots\bar\beta_q}\\
&\quad + \sum_{i=1}^{p}\sum_{k=1}^{q}\sum_{\alpha,\beta} R^{\alpha}{}_{\alpha_i\bar\beta_k}{}^{\bar\beta}\zeta_{\alpha_1\cdots\alpha_{i-1}\alpha\alpha_{i+1}\cdots\bar\beta_{k-1}\bar\beta\bar\beta_{k+1}\cdots\bar\beta_q}\\
&\quad + \sum_{k=1}^{q}\sum_{\beta}(X_{\phi\ \bar\beta_k}^{\ \bar\beta} - R_{\bar\beta_k}{}^{\bar\beta})\zeta_{\alpha_1\cdots\bar\beta_{k-1}\bar\beta\bar\beta_{k+1}\cdots\bar\beta_q}.
\end{aligned}\tag{5.19}
$$

We let $\psi = \partial\partial^*\gamma$. Over $U_t(\frac12)$, since the CO-metric is Kähler, we have $\theta = \partial\bar\partial^*\partial^*\gamma = -\bar\partial^*\psi$. We apply the Kodaira formula for $\phi = \phi_1 = \mathbf{r}^{-\frac83}$ and $\zeta = \psi$. Since ψ is a $(2,3)$-form and the CO-metric is Ricci flat,

$$
\begin{aligned}
&\int_{U_t(\frac12)}\langle\bar\partial\bar\partial^*_{\phi_1}\psi,\psi\rangle\phi_1\tau\\
&\quad = -\int_{V_t(\frac12)}\langle g^{\bar\beta\alpha}\nabla^{\phi_1}_\alpha\nabla_{\bar\beta}\psi,\psi\rangle\phi_1\tau + \int_{U_t(\frac12)}\sum_{\beta=1}^{3}X_{\phi_1\ \bar\beta}^{\ \bar\beta}|\psi|^2\phi_1\tau.
\end{aligned}\tag{5.20}
$$

Since τ has a compact support in $U_t(\frac12)$, we compute

$$
\begin{aligned}
&\int_{U_t(\frac12)}\langle\bar\partial\bar\partial^*_{\phi_1}\psi,\psi\rangle\phi_1\tau\\
&\quad = \int_{U_t(\frac12)}\langle\bar\partial^*_{\phi_1}\psi,\bar\partial^*_{\phi_1}\psi\rangle\phi_1\tau + \int_{V_t(\frac12)}\langle\bar\partial\tau\wedge\bar\partial^*_{\phi_1}\psi,\psi\rangle\phi_1\\
&\quad \le \int_{U_t(\frac12)}|\theta|^2\phi_1\tau + \int_{U_t(\frac12)}|\partial\log\phi_1\wedge *\psi|^2\phi_1\tau\\
&\qquad - 2\mathrm{Re}\int_{U_t(\frac12)}\langle *(\partial\log\phi_1\wedge *\psi),\bar\partial^*\psi\rangle\phi_1\tau + \cdots,
\end{aligned}
$$

where the dots denote terms that are integrations over $X_t[\frac14]$ of smooth function including the derivatives of τ. By (4.6), the dotted terms are bounded by a fixed multiple, independent of t, of $\Lambda_t = \int_{X_t[\frac18]}(|\gamma|^2 + |\partial\Phi^{1,3}|^2)$. In the remainder of this section, the term $C\Lambda_t$ will appear in various places for the same reason.

On the other hand, since $\bar\partial\bar\partial^*\psi=-\partial\Phi^{1,3}=0$ on $U_t(\frac12)$,

$$\begin{aligned}\int_{U_t(\frac12)}|\theta|^2\phi_1\tau &= \mathrm{Re}\int_{U_t(\frac12)}\langle\bar\partial^*_{\phi_1}\psi,\bar\partial^*\psi\rangle\phi_1\tau\\ &\quad+\mathrm{Re}\int_{U_t(\frac12)}\langle *(\partial\log\phi_1\wedge*\psi),\bar\partial^*\psi\rangle\phi_1\tau\\ &\le \mathrm{Re}\int_{U_t(\frac12)}\langle *(\partial\log\phi_1\wedge*\psi),\bar\partial^*\psi\rangle\phi_1\tau + C_1\Lambda_t.\end{aligned}$$

We remark that the C_1 and the C_i to appear later are all independent of t. Combining the above two inequalities, we get

$$\begin{aligned}&\int_{U_t(\frac12)}\langle\bar\partial\bar\partial^*_{\phi_1}\psi,\psi\rangle\phi_1\tau\\ &\quad\le -\int_{U_t(\frac12)}|\theta|^2\phi_1\tau+\int_{U_t(\frac12)}|\partial\log\phi_1\wedge*\psi|^2\phi_1\tau+C_2\Lambda_t.\end{aligned}$$

Inserting the above inequality into (5.20) and applying divergence theorem to the first term on the right hand side (5.20), since ψ is a $(2,3)$-form, we get

$$\begin{aligned}&\int_{U_t(\frac12)}|\theta|^2\phi_1\tau\\ &\quad\le\int_{U_t(\frac12)}\big(|\partial\log\phi_1|^2-\sum_{\beta=1}^3 X_{\phi_1}{}^{\bar\beta}{}_{\bar\beta}\big)|\psi|^2\phi_1\tau+C_3\Lambda_t.\end{aligned}\tag{5.21}$$

According to (5.1)-(5.4), we have

$$|\partial\log\phi_1|^2\le\frac83\mathbf{r}^{-\frac43}\quad\text{and}\quad\sum_{\beta=1}^3 X_{\phi_1}{}^{\bar\beta}{}_{\bar\beta}\ge\frac83\mathbf{r}^{-\frac43}.$$

So from (5.21), we get

$$\int_{U_t(\frac14)}|\theta|^2\mathbf{r}^{-\frac43}\le C_3\Lambda_t=C_3\int_{X_t[\frac18]}(|\gamma|^2+|\partial\Phi^{1,3}|)^2.$$

This proves Lemma 4.5. q.e.d.

Proof of Lemma 4.6. For any $0<\iota<\frac13$ and any $0<\delta<\frac14$, by the Hölder inequality,

$$\int_{U_t(\delta)}|\gamma|^2\mathbf{r}^{-\frac43}\le\Big(\int_{U_t(\frac14)}|\gamma|^3\mathbf{r}^{-3\iota}\Big)^{\frac23}\Big(\int_{U_t(\delta)}\mathbf{r}^{-4+6\iota}\Big)^{\frac13}.$$

Clearly,

$$\Big(\int_{U_t(\delta)}\mathbf{r}^{-4+6\iota}\Big)^{\frac13}=\Big(\frac23\int_{U_t(\delta)}\mathbf{r}^{-6+6\iota}\mathrm{vol}_{\omega_e}\Big)^{\frac13}\le C\delta^{2\iota},$$

where the constant C is independent on t and δ. So to prove the Lemma we only need to prove that for a constant C independent of t,

$$\Big(\int_{U_t(\frac{1}{4})} |\gamma|^3 \mathbf{r}^{-3\iota}\Big)^{\frac{2}{3}} \le C \int_{X_t[\frac{1}{8}]} (|\gamma|^2 + |\partial\Phi^{1,3}|^2). \tag{5.22}$$

We will prove the above inequality in three steps. Our first step is to establish the inequality

$$\begin{aligned} &\Big(\int_{U_t(\frac{1}{4})} |\gamma|^3 \mathbf{r}^{-3\iota}\Big)^{\frac{2}{3}} \\ &\le\ 8 \int_{U_t(\frac{1}{2})} |\bar\partial^* \gamma|^2 \mathbf{r}^{-2\iota}\tau^2 + C_1 \int_{U_t(\frac{1}{2})} |\gamma|^2 \mathbf{r}^{-2\iota-\frac{4}{3}}\tau^2 + C_1\Lambda_t. \end{aligned} \tag{5.23}$$

We now prove this inequality. Using the method in deriving (5.9) and (5.12) for $k=1$, we get

$$\begin{aligned} &\Big(\int_{U_t(\frac{1}{4})} |\gamma|^3 \mathbf{r}^{-3\iota}\Big)^{\frac{2}{3}} \\ &\le C_2 \int_{U_t(\frac{1}{2})} \big|\nabla|\gamma|\big|^2 \mathbf{r}^{-2\iota}\tau^2 + \\ &\quad + C_2 \int_{U_t(\frac{1}{2})} |\gamma|^2 |\nabla \mathbf{r}^{-\iota-\frac{2}{3}}|^2 \mathbf{r}^{\frac{4}{3}}\tau^2 + C_2 \int_{U_t(\frac{1}{2})} |\gamma|^2 \mathbf{r}^{-2\iota}|\nabla\tau|^2 \\ &\le C_3 \int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha}(\langle \nabla_\alpha\gamma, \nabla_\beta\gamma\rangle + \langle \nabla_{\bar\beta}\gamma, \nabla_{\bar\alpha}\gamma\rangle)\mathbf{r}^{-2\iota}\tau^2 \\ &\quad + C_3 \int_{U_t(\frac{1}{2})} |\gamma|^2 \mathbf{r}^{-2\iota-\frac{4}{3}}\tau^2 + C_3\Lambda_t. \end{aligned} \tag{5.24}$$

Let $\phi_2 = \mathbf{r}^{-2\iota}$ and $\phi_3 = \mathbf{r}^{-2\iota-\frac{4}{3}}$. By divergence theorem,

$$\begin{aligned} &\int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha}\big(\langle \nabla_\alpha\gamma, \nabla_\beta\gamma\rangle + \langle \nabla_{\bar\beta}\gamma, \nabla_{\bar\alpha}\gamma\rangle\big)\phi_2\tau^2 \\ &\le -2 \int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha} \langle \nabla^{\phi_2}_\alpha \nabla_{\bar\beta}\gamma, \gamma\rangle \phi_2\tau^2 \\ &\quad + \int_{U_t(\frac{1}{2})} \langle g^{\bar\beta\alpha}[\nabla_{\bar\beta}, \nabla_\alpha]\gamma, \gamma\rangle\phi_2\tau^2 + C_4\Lambda \\ &\quad + \int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha}\langle \partial_\alpha \log\phi_2 \cdot \nabla_{\bar\beta}\gamma, \gamma\rangle\phi_2\tau^2 \\ &\quad - \int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha}\langle \nabla_\alpha\gamma, \partial_\beta(\phi_2\tau^2)\gamma\rangle. \end{aligned} \tag{5.25}$$

To bound the four terms after the inequality, we use that the curvature is bounded by $C\mathbf{r}^{-\frac{4}{3}}$ to the second item, and apply the Hölder inequality to the last two items. After simplification, we get

(5.26)
$$\begin{aligned}
&\int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha}(\langle\nabla_\alpha\gamma,\nabla_\beta\gamma\rangle+\langle\nabla_{\bar\beta}\gamma,\nabla_{\bar\alpha}\gamma\rangle)\phi_2\tau^2\\
&\quad\le -4\int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha}\langle\nabla^{\phi_2}_\alpha\nabla_{\bar\beta}\gamma,\gamma\rangle\phi_2\tau^2+C_4\int_{U_t(\frac{1}{2})}|\gamma|^2\phi_3\tau^2+C_4\Lambda_t.
\end{aligned}$$

We now deal with the first term after "$\le$" in the above inequality. We use Kodaira's formula (5.19) to the case $\zeta=\gamma$ and $\phi=\phi_2$. Since γ is a $(2,3)$-form and CO-metric is Ricci flat, (5.19) reduces to

$$-\sum_{\alpha,\beta} g^{\bar\beta\alpha}\nabla^{\phi_2}_\alpha\nabla_{\bar\beta}\gamma=\bar\partial\bar\partial^*_{\phi_2}\gamma-\sum_{\beta=1}^3 X_{\phi_2}{}^{\bar\beta}{}_{\bar\beta}\gamma.$$

So we get

$$\begin{aligned}
&-\int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha}\langle\nabla^{\phi_2}_\alpha\nabla_{\bar\beta}\gamma,\gamma\rangle\phi_2\tau^2\\
&\qquad=\int_{U_t(\frac{1}{2})}\langle\bar\partial\bar\partial^*_{\phi_2}\gamma,\gamma\rangle\phi_2\tau^2-\int_{V_t(\frac{1}{2})}\sum_{\beta=1}^3 X_{\phi_2}{}^{\bar\beta}{}_{\bar\beta}|\gamma|^2\phi_2\tau^2.
\end{aligned}$$

By the Hölder inequality, we estimate

$$\begin{aligned}
&\int_{U_t(\frac{1}{2})}\langle\bar\partial\bar\partial^*_{\phi_2}\gamma,\gamma\rangle\phi_2\tau^2\\
&\quad=\int_{U_t(\frac{1}{2})}\langle\bar\partial^*_{\phi_2}\gamma,\bar\partial^*_{\phi_2}\gamma\rangle\phi_2\tau^2-\int_{U_t(\frac{1}{2})}\langle\bar\partial\tau^2\wedge\bar\partial^*_{\phi_2}\gamma,\gamma\rangle\phi_2\\
&\quad\le 2\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2+2\int_{U_t(\frac{1}{2})}|\partial\log\phi_2|^2|\gamma|^2\phi_2\tau^2+C_5\Lambda_t.
\end{aligned}$$

Putting together, we get

$$\begin{aligned}
&-\int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha}\langle\nabla^{\phi_2}_\alpha\nabla_{\bar\beta}\gamma,\gamma\rangle\phi_2\tau^2\\
&\quad\le\int_{U_t(\frac{1}{2})}\big(2|\partial\log\phi_2|^2-\sum X_{\phi_2}{}^{\bar\beta}{}_{\bar\beta}\big)|\gamma|^2\phi_2\tau^2\\
&\qquad+2\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2+C_5\Lambda_t.
\end{aligned}\tag{5.27}$$

On the other hand, by direct calculation,

$$|\partial\log\phi_2|^2\le\frac{3}{2}\iota^2\mathbf{r}^{-\frac{4}{3}}\quad\text{and}\quad\sum X_{\phi_2}{}^{\bar\beta}{}_{\bar\beta}\ge 2\iota\mathbf{r}^{-\frac{4}{3}}.$$

So inequality (5.27) implies

$$\begin{aligned} &-\int_{U_t(\frac{1}{2})} g^{\bar\beta\alpha}\langle\nabla^{\phi_2}_{\alpha}\nabla_{\bar\beta}\gamma,\gamma\rangle\phi_2\tau^2 \\ &\qquad\le 2\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2+\int_{U_t(\frac{1}{2})}(3\iota^2-2\iota)|\gamma|^2\phi_3\tau^2+C_5\Lambda_t. \end{aligned} \tag{5.28}$$

Finally, inserting (5.28) into (5.26) and then inserting (5.26) into (5.24), since $0<\iota<\frac{1}{3}$, we complete our first step in establishing the inequality (5.23).

Our second step is to prove

$$\int_{U_t(\frac{1}{2})}|\gamma|^2\phi_3\tau^2\le\frac{2}{2\iota-3\iota^2}\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2+C_6\Lambda_t. \tag{5.29}$$

For this, we first apply the divergence theorem to the left had side of (5.28):

$$(2\iota-3\iota^2)\int_{U_t(\frac{1}{2})}|\gamma|^2\phi_3\tau^2\le 2\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2+C_6\Lambda_t. \tag{5.30}$$

This inequality implies (5.29) since when $\iota<\frac{1}{3}$, $2\iota-3\iota^2>0$.

Our third step is to prove

$$\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2\le C_7\int_{X_t[\frac{1}{8}]}(|\gamma|^2+|\ \partial\Phi^{1,3}\ |^2)). \tag{5.31}$$

To achieve this, we write

$$\begin{aligned} 2\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2 &= 2\mathrm{Re}\int_{U_t(\frac{1}{2})}\langle\bar\partial^*_{\phi_2}\gamma,\bar\partial^*\gamma\rangle\phi_2\tau^2 \\ &\qquad + 2\mathrm{Re}\int_{U_t(\frac{1}{2})}\langle *(\partial\log\phi_2\wedge *\gamma),\bar\partial^*\gamma\rangle\phi_2\tau^2. \end{aligned} \tag{5.32}$$

The first term after the equal sign in (5.32) is bounded from above by

$$2\mathrm{Re}\int_{U_t(\frac{1}{2})}\langle\gamma,\bar\partial\bar\partial^*\gamma\rangle\phi_2\tau^2+C_8\Lambda_t,$$

which is bounded from above by

$$\le 2b\int_{U_t(\frac{1}{2})}|\gamma|^2\phi_3\tau^2+\frac{1}{2b}\int_{U_t(\frac{1}{2})}|\bar\partial\bar\partial^*\gamma|^2\phi_4\tau^2+C_8\Lambda_t,$$

for some $b>0$, and for $\phi_4=\mathbf{r}^{-2\iota+\frac{4}{3}}$ and $\phi_3=\mathbf{r}^{-2\iota-\frac{4}{3}}$. By (5.3), the second item after the equal sign in (5.32) is bounded by

$$\le\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2+\frac{3}{2}\iota^2\int_{U_t(\frac{1}{2})}|\gamma|^2\phi_3\tau^2.$$

Therefore (5.32) implies

$$\int_{U_t(\frac{1}{2})} |\bar\partial^*\gamma|^2\phi_2\tau^2 \\ \leq \left(\frac{3}{2}\iota^2+2b\right)\int_{U_t(\frac{1}{2})}|\gamma|^2\phi_3\tau^2+\frac{1}{2b}\int_{U_t(\frac{1}{2})}|\partial\bar\partial^*\gamma|^2\phi_4\tau^2+C_8\Lambda_t.$$

Inserting (5.29) into the above inequality and simplifying, we obtain

$$\frac{\iota(2-6\iota)-4b}{\iota(2-3\iota)}\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2 \leq \frac{1}{2b}\int_{U_t(\frac{1}{2})}|\partial\bar\partial^*\gamma|^2\phi_4\tau^2+C_9\Lambda_t. \tag{5.33}$$

Since $\iota<\frac{1}{3}$, for any given ι we can choose b such that $\iota(1-3\iota)-2b>0$. Then the above inequality implies

$$\int_{U_t(\frac{1}{2})}|\bar\partial^*\gamma|^2\phi_2\tau^2 \leq \frac{\iota(2-3\iota)}{2b\big(\iota(1-3\iota)-2b\big)}\int_{U_t(\frac{1}{2})}|\partial\bar\partial^*\gamma|^2\phi_4\tau^2+C_{10}\Lambda_t. \tag{5.34}$$

Finally, we need to estimate $\int_{U_t(\frac{1}{2})}|\partial\bar\partial^*\gamma|^2\phi_4\tau^2$. Since the CO-matric is Kähler and $\partial\gamma=\bar\partial\gamma=0$, then $\partial\partial^*\gamma=\bar\partial\bar\partial^*\gamma$. When restricted to $U_t(\frac{1}{2})$, $\bar\partial\bar\partial^*\partial\partial^*\gamma=-\partial\Phi^{1,3}=0$. From these identities, since $0<\iota<\frac{1}{3}$,

$$\int_{U_t(\frac{1}{2})}|\partial\bar\partial^*\gamma|^2 r^{-2\iota+\frac{4}{3}}\tau^2 \leq \int_{U_t(\frac{1}{2})}|\partial\bar\partial^*\gamma|^2\tau^2 \\ = \int_{U_t(\frac{1}{2})}\langle\partial\partial^*\gamma,\bar\partial\bar\partial^*\gamma\rangle\tau^2 \leq \int_{U_t(\frac{1}{2})}\langle\bar\partial\bar\partial^*\partial\partial^*\gamma,\gamma\rangle\tau^2+C_{11}\Lambda_t = C_{11}\Lambda_t. \tag{5.35}$$

Combining the above two inequalities, we prove the inequality (5.31). This completes our third step.

In the end, we insert (5.31) into (5.29), insert (5.29) and (5.31) into (5.23); we get (5.22). This proves Lemma 4.6. q.e.d.

Proof of Lemma 4.7. The proof is parallel to that of the previous Lemma, except that in Lemma 4.6 the form γ is a $(2,3)$-form while in this Lemma $\partial^*\gamma$ is a $(1,3)$-form. Replacing γ by $\partial^*\gamma$, we find that all inequalities up to (5.34) are valid. So to prove Lemma 4.7 we only need to estimate $\int_{U_t(\frac{1}{2})}|\bar\partial\bar\partial^*\partial^*\gamma|^2\mathbf{r}^{-2\iota+\frac{4}{3}}\tau^2$. Since $\bar\partial\bar\partial^*\partial^*\gamma=\partial^*\partial\partial^*\gamma$, by the same method in proving (5.35), we get

$$\int_{U_t(\frac{1}{2})}|\bar\partial\bar\partial^*\partial^*\gamma|^2\mathbf{r}^{-2\iota+\frac{4}{3}}\tau^2 \leq C_{12}\int_{X_t[\frac{1}{8}]}(|\gamma|^2+|\partial\Phi^{1,3}|^2).$$

This proves Lemma 4.7. q.e.d.

Appendix A. The geometry of Candelas-de la Ossa's metrics

We first recall some notations from Candelas-de la Ossa's paper [**9**]. We consider the family V_t:

$$V_t = \left\{ (w_1, \cdots, w_4) \mid \sum_{i=1}^{4} (w_i)^2 = t \right\} \subset \mathbb{C}^4.$$

Since the individual V_t only depend on $|t|$, in the following we shall work with $t > 0$. We let $r^2 = \sum_{i=1}^4 |w_i|^2$ be the radial coordinate. We set

$$\tilde{\omega}_{\mathrm{co},t} = i\partial\bar{\partial} f_t(r^2)$$

The condition that the metric be Ricci-flat is

$$r^2(r^4 - t^2)(\eta_t^3)' + 3t^2\eta_t^3 = 2r^8 \quad \text{with} \quad \eta_t(r^2) = r^2 f_t'(r^2). \tag{A.1}$$

The scale has been chosen so η_t has the same asymptotic behavior as $r^{\frac{4}{3}}$ for large r. After setting

$$r^2 = t\cosh\tau, \quad \text{for} \quad \tau \geq 0$$

and integrating, we pick the solution

$$\eta_t = \frac{2^{-1/3} t^{2/3}}{\tanh\tau} (\sinh 2\tau - 2\tau)^{1/3}. \tag{A.2}$$

Note that this choice of η_t makes the metric regular at $r^2 = t$. Also note that from (A.1), $f_t'(s) = s^{-1}\eta_t(s)$, and that $f_t(s)$ defined in (3.3) is a solution of this equation.

In this appendix, we want to estimate the curvature of the CO metric. Since it is homogeneous (see [**9**]), we only need to perform our calculation at points $q = (\frac{\sqrt{r^2-t}}{\sqrt{2}}, \frac{i\sqrt{r^2-t}}{\sqrt{2}}, 0, t^{\frac{1}{2}})$. At first we pick some orthogonal coordinate at this point. Since $dw_1 \wedge dw_2 \wedge dw_3 \neq 0$ near q, we can take (w_1, w_2, w_3) as a (holomorphic) coordinate in a neighborhood of the point q. By directly calculation, we get

$$\begin{aligned} \partial\bar{\partial} r^2|_q = {} & \frac{r^2+t}{2t}\big(dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2\big) + dw_3 \wedge d\bar{w}_3 \\ & + i\frac{r^2-t}{2t}\big(dw_2 \wedge d\bar{w}_1 - dw_1 \wedge d\bar{w}_2\big) \end{aligned}$$

and

$$\partial r^2 \wedge \bar{\partial} r^2|_q = 2(r^2 - t) dw_2 \wedge d\bar{w}_2.$$

To simplify them, we introduce a new coordinate (u_1, u_2, u_3) at the point q:

$$w_1 = \frac{2t}{r_q^2 + t} u_1 - i\frac{r_q^2 - t}{r_q^2 + t} u_2, \quad w_2 = u_2, \quad w_3 = u_3,$$

where $r_q^2 \triangleq r^2(q)$. Under this coordinate, the $\partial\bar{\partial}r^2$ and $\partial r^2 \wedge \bar{\partial} r^2$ are expressed as

$$\partial\bar{\partial}r^2|_q = \frac{2t}{r^2+t}du_1 \wedge d\bar{u}_1 + \frac{2r^2}{r^2+t}du_2 \wedge d\bar{u}_2 + du_3 \wedge d\bar{u}_3 \tag{A.3}$$

and

$$\partial r^2 \wedge \bar{\partial} r^2|_q = 2(r^2-t)du_2 \wedge d\bar{u}_2.$$

Combined with (A.1),

$$f_t' + r^2 f_t'' = \eta_t' = \frac{2r^8 - 3t^2\eta_t^3}{3r^2(r^4-t^2)\eta_t^2}; \tag{A.4}$$

so at the point q the CO metric is

$$i\partial\bar{\partial}f_t|_q = \frac{2t\eta_t}{r^2(r^2+t)}idu_1 \wedge d\bar{u}_1 + \frac{4r^4}{3\eta_t^2(r^2+t)}idu_2 \wedge d\bar{u}_2 + \frac{\eta_t}{r^2}idu_3 \wedge d\bar{u}_3.$$

At last we introduce a new coordinate (z_1, z_2, z_3) near the point q as:

$$z_1 = \Big(\frac{2t\eta_t(q)}{r_q^2(r_q^2+t)}\Big)^{\frac{1}{2}}u_1, \quad z_2 = \Big(\frac{4r_q^4}{3\eta_t^2(q)(r_q^2+t)}\Big)^{\frac{1}{2}}u_2, \quad z_3 = \Big(\frac{\eta_t(q)}{r_q^2}\Big)^{\frac{1}{2}}u_3.$$

The CO metric at this point is then expressed as

$$i\partial\bar{\partial}f_t(r^2)|_q = i\sum_{j=1}^{3} dz_j \wedge d\bar{z}_j.$$

Under this coordinate, we can rewrite (A.3) as

$$\partial\bar{\partial}r^2|_q = (r^2)^{\frac{1}{3}}\Big(\frac{r^4}{\eta_t^3}\Big)^{\frac{1}{3}}\Big(dz_1 \wedge d\bar{z}_1 + \frac{3}{2}\frac{\eta_t^3}{r^4}dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3\Big)$$

and

$$\partial r^2 \wedge \bar{\partial}r^2|_q = \frac{3}{2}(r^2)^{\frac{4}{3}}\Big(\frac{\eta_t^3}{r^4}\Big)^{\frac{2}{3}}\Big(1 - \frac{t^2}{r^4}\Big)dz_2 \wedge d\bar{z}_2.$$

To estimate the curvature of the CO metric, we need to investigate the asymptotic behavior of $\frac{\eta_t^3}{r^4}$.

Lemma A.1. *Over $[t, +\infty)$, the function $r^{-4}\eta_t^3$ is an increasing function and*

$$\lim_{r^2 \to t} r^{-4}\eta_t^3 = \frac{2}{3}, \quad \lim_{r^2 \to \infty} r^{-4}\eta_t^3 = 1.$$

Proof. Let $h(\tau) = r^{-4}\eta_t^3$. From (A.2),

$$h(\tau) = \frac{1}{2}\frac{\cosh\tau(\sinh 2\tau - 2\tau)}{\sinh^3\tau}.$$

Differentiating,

$$h'(\tau) = \frac{1}{2\sinh^4\tau}h_1(\tau) \quad \text{where} \quad h_1(\tau) = 4\tau + e^{2\tau}(\tau - 3/2) + e^{-2\tau}(\tau + 3/2)$$

and

$$h_1'(\tau) = 2\tau e^{2\tau} - 2e^{2\tau} - 2\tau e^{-2\tau} - 2e^{-2\tau} + 4.$$

Thus for any $\tau > 0$,

$$h_1'(\tau) = 4\tau \sum_{n=1}^{\infty} \frac{(2\tau)^{2n+1}}{(2n+1)!} \cdot \frac{n}{n+1} > 0.$$

Since $h_1(0) = 0$, $h_1(\tau) > 0$ and $h'(\tau) > 0$. So over $[0, +\infty)$, the function $r^{-4}\eta_t^3$ is a increasing function of τ. Since $r^2 = t\cosh\tau$ for $\tau \geq 0$ is an increasing function in τ, $r^{-4}\eta^3$ is increasing in r^2. Since $\tau \to 0$ when $r^2 \to t$, and $\tau \to \infty$ when $r^2 \to \infty$, we obtain the two desired limits by applying the L'Hospital rule. This proves the Lemma. q.e.d.

We next investigate η_t'. From $\eta_t^3 = r^4 h(\tau)$,

$$3\eta_t^2\eta_t' = 2r^2 h(\tau) + r^4 h'(\tau)\frac{d\tau}{dr^2} = 2r^2 h(\tau) + t^{-1} r^4 h'(\tau)\sinh^{-1}\tau > 0,$$

hence $r^{\frac{2}{3}}\eta_t' > 0$. On the other hand by (A.4), we get

$$r^{\frac{2}{3}}\eta_t' = \frac{2 - 3\frac{t^2}{r^4}\frac{\eta_t^3}{r^4}}{3(1-\frac{t^2}{r^4})(\frac{\eta_t^3}{r^4})^{\frac{2}{3}}} = \big(\frac{\eta_t^3}{r^4}\big)^{\frac{1}{3}} + \frac{2 - 3\frac{\eta_t^3}{r^4}}{3(1-\frac{t^2}{r^4})(\frac{\eta_t^3}{r^4})^{\frac{2}{3}}}.$$

Then from Lemma A.1, we see that

$$0 < r^{\frac{2}{3}}\eta_t' < 1.$$

In the following for two functions $\alpha(r,t)$ and $\beta(r,t)$ in r and t we shall use $\alpha \lesssim \beta$ to mean that there is a constant C independent on r and t such that

$$|\alpha(r,t)| \leq C|\beta(r,t)|.$$

Under this convention, the previous Lemma and the last inequality can be abbreviated as

$$\eta_t \lesssim r^{\frac{4}{3}} \quad \text{and} \quad \eta_t' \lesssim r^{-\frac{2}{3}}.$$

For higher derivatives, by introducing $\epsilon = \frac{t}{r^2}$, the identities (A.1) and (A.4) imply

$$\eta_t'' \lesssim r^{-\frac{8}{3}}(1-\epsilon)^{-1}, \quad \eta_t^{(3)} \lesssim r^{-\frac{14}{3}}(1-\epsilon)^{-2}.$$

Therefore, by the second identity of (A.1), we obtain the following asymptotic estimates

$$f_t' \lesssim r^{-\frac{2}{3}}, \quad f_t'' \lesssim r^{-\frac{8}{3}}, \quad f_t^{(3)} \lesssim r^{-\frac{14}{3}}(1-\epsilon)^{-1} \quad \text{and} \quad f_t^{(4)} \lesssim r^{-\frac{20}{3}}(1-\epsilon)^{-2}.$$

To proceed, we need to the partial derivatives of r^2 with respect to z_i and $\bar{z}_i$. For simplicity, we shall use the subscript i to denote the partial derivative with respect to z_i, and use $\bar{i}$ for derivatives with respect to $\bar{z}_i$. Thus, for instance, $\frac{\partial^2 r^2}{\partial z_i \partial \bar{z}_j} = (r^2)_{i\bar{j}}$.

Under this convention, we compute directly that at the point q, the first order partial derivatives

(A.5)

$$(r^2)_1 = (r^2)_3 = 0 \quad \text{and} \quad (r^2)_2 = -\frac{\sqrt{6}}{2} i (r^2-t)^{\frac{1}{2}} (r^2+t)^{\frac{1}{2}} \frac{\eta_t}{r^2} \lesssim r^{\frac{4}{3}} (1-\epsilon)^{\frac{1}{2}};$$

the second order derivatives $(r^2)_{i\bar{j}} = 0$ except the following

$$(r^2)_{1\bar{1}} = (r^2)_{3\bar{3}} = \frac{r^2}{\eta_t} \lesssim r^{\frac{2}{3}}, \quad (r^2)_{2\bar{2}} = \frac{3}{2} \frac{\eta_t^2}{r^2} \lesssim r^{\frac{2}{3}}; \tag{A.6}$$

the second derivatives $(r^2)_{ij} = 0$ except the following

$$(r^2)_{11} = (r^2)_{33} = -\frac{r^2}{\eta_t} \lesssim r^{\frac{2}{3}}, \quad (r^2)_{22} = -\frac{3}{2} \frac{\eta_t^2}{r^2} \epsilon \lesssim r^{\frac{2}{3}}.$$

For the third order partial derivatives of type $i\bar{j}k$, we have the vanishing $(r^2)_{i\bar{j}k} = 0$ except the following

$$\begin{aligned}
(r^2)_{1\bar{1}1} &= \frac{r^3 (1-\epsilon)^{\frac{1}{2}}}{t^{\frac{1}{2}} \eta_t^{\frac{3}{2}} (1+\epsilon)^{\frac{1}{2}}} \lesssim \epsilon^{-\frac{1}{2}} (1-\epsilon)^{\frac{1}{2}}, \\
(r^2)_{1\bar{2}1} &= -i \frac{\sqrt{6}}{2} \frac{(1-\epsilon)^{\frac{1}{2}}}{(1+\epsilon)^{\frac{1}{2}}} \lesssim (1-\epsilon)^{\frac{1}{2}}, \\
(r^2)_{2\bar{1}2} &= \frac{3}{2} \frac{t^{\frac{1}{2}} \eta_t^{\frac{3}{2}} (1-\epsilon)^{\frac{1}{2}}}{r^3 (1+\epsilon)^{\frac{1}{2}}} \lesssim \epsilon^{\frac{1}{2}} (1-\epsilon)^{\frac{1}{2}}, \\
(r^2)_{2\bar{2}2} &= -i \frac{3\sqrt{6}}{4} \frac{t \eta_t^3 (1-\epsilon)^{\frac{1}{2}}}{r^6 (1+\epsilon)^{\frac{1}{2}}} \lesssim \epsilon (1-\epsilon)^{\frac{1}{2}}, \\
(r^2)_{3\bar{1}3} &= \frac{r^3 (1-\epsilon)^{\frac{1}{2}}}{t^{\frac{1}{2}} \eta_t^{\frac{3}{2}} (1+\epsilon)^{\frac{1}{2}}} \lesssim \epsilon^{-\frac{1}{2}} (1-\epsilon)^{\frac{1}{2}}, \\
(r^2)_{3\bar{2}3} &= -i \frac{\sqrt{6}}{2} \frac{(1-\epsilon)^{\frac{1}{2}}}{(1+\epsilon)^{\frac{1}{2}}} \lesssim (1-\epsilon)^{\frac{1}{2}}.
\end{aligned}$$

For the fourth order partial derivatives, we still have the vanishing except the following

$$\begin{aligned}
&(r^2)_{1\bar{1}1\bar{1}} = \frac{r^4}{t\eta_t^2} \lesssim r^{\frac{4}{3}} t^{-1}, \quad (r^2)_{1\bar{2}1\bar{2}} = (r^2)_{2\bar{1}2\bar{1}} = \frac{3\eta_t}{2r^2} \lesssim r^{-\frac{2}{3}}, \\
&(r^2)_{2\bar{2}2\bar{2}} = \frac{9t\eta_t^4}{4r^8} \lesssim t r^{-\frac{8}{3}}, \quad (r^2)_{1\bar{3}1\bar{3}} = (r^2)_{3\bar{1}3\bar{1}} = \frac{r^4}{t\eta_t^2} \lesssim r^{\frac{4}{3}} t^{-1}, \\
&(r^2)_{3\bar{3}3\bar{3}} = \frac{r^4}{t\eta_t^2} \lesssim r^{\frac{4}{3}} t^{-1}, \quad (r^2)_{2\bar{3}2\bar{3}} = (r^2)_{3\bar{2}3\bar{2}} = \frac{3\eta_t}{2r^2} \lesssim r^{-\frac{2}{3}}.
\end{aligned}$$

We now use these asymptotic estimate of the partial derivatives of r^2 to prove Lemma 5.1. Since (z_1, z_2, z_3) is the orthogonal coordinate at the point p, we only need to prove that *there is a constant C independent of t and r such that the curvature tensor $R_{i\bar{j}k\bar{l}}$ of the CO metric $\tilde{\omega}_{\mathrm{co},t}$* at q has bound

$$R_{i\bar{j}k\bar{l}} \lesssim r^{-\frac{4}{3}}. \tag{A.7}$$

The curvature at q has the form

$$R_{i\bar{j}k\bar{l}} = -(f_t)_{i\bar{j}k\bar{l}} + (f_t)_{ik\bar{q}}(f_t)_{q\bar{j}\bar{l}}.$$

One group of terms appearing in $(f_t)_{i\bar{j}k\bar{l}}$ are of the type

$$f_t^{(4)} \cdot (r^2)_i(r^2)_{\bar{j}}(r^2)_k(r^2)_{\bar{l}}, \quad f_t^{(3)} \cdot \sum (r^2)_{i_1i_2}(r^2)_{i_3}(r^2)_{i_4},$$

$$f_t'' \cdot \sum (r^2)_{i_1i_2}(r^2)_{i_3i_4},$$

where the summations are taken over all possible permutation $\{i_1, i_2, i_3, i_4\} = \{i, \bar{j}, k, \bar{l}\}$. For such type of terms, using the previous estimate, we check directly that they are bounded by $Cr^{-\frac{4}{3}}$.

The other group of terms in appearing $(f_t)_{i\bar{j}k\bar{l}}$ are of the type:

$$f_t' \cdot (r^2)_{i\bar{j}k\bar{l}} \tag{A.8}$$

$$f_t'' \cdot \big((r^2)_{i\bar{j}k}(r^2)_{\bar{l}} + (r^2)_{i\bar{l}k}(r^2)_{\bar{j}}\big), \tag{A.9}$$

$$f_t'' \cdot \big((r^2)_{\bar{j}k\bar{l}}(r^2)_i + (r^2)_{\bar{j}i\bar{l}}(r^2)_k\big). \tag{A.10}$$

Of these, (A.8) vanishes when $i \neq k$ or $j \neq l$, (A.9) vanishes when $i \neq k$ and (A.10) vanishes when $j \neq l$. The remaining cases in (A.8)-(A.10) may be not vanishing, and will be treated separately momentarily.

We now look at the product term $(f_t)_{ik\bar{q}}(f_t)_{q\bar{j}\bar{l}}$. First in the expression of $(f_t)_{ik\bar{q}}$, the following two types of terms

$$f_t^{(3)} \cdot (r^2)_i(r^2)_k(r^2)_{\bar{q}} \quad \text{and} \quad f_t'' \cdot \sum_{i_1,i_2,i_3} (r^2)_{i_1i_2}(r^2)_{i_3}$$

are bounded by $Cr^{-\frac{2}{3}}$; therefore corresponding product terms

$$\Big(f_t^{(3)} \cdot (r^2)_i(r^2)_k(r^2)_{\bar{q}} + f_t'' \cdot \sum_{i_1,i_2,i_3} (r^2)_{i_1i_2}(r^2)_{i_3}\Big)$$
$$\times \Big(f_t^{(3)} \cdot (r^2)_{\bar{j}}(r^2)_{\bar{l}}(r^2)_q + f_t'' \cdot \sum_{j_1,j_2,j_3} (r^2)_{j_1j_2}(r^2)_{j_3}\Big)$$

in the expansion of $(f_t)_{ik\bar{q}}(f_t)_{q\bar{j}\bar{l}}$ are also bounded by $Cr^{-\frac{4}{3}}$. Here the summations are over all possible permutation $\{i_1, i_2, i_3\} = \{i, k, \bar{q}\}$ and $\{j_1, j_2, j_3\} = \{\bar{j}, \bar{l}, q\}$.

The remaining terms in $(f_t)_{ik\bar{q}}(f_t)_{q\bar{j}\bar{l}}$ are of the following types:

$$(f_t')^2 \cdot (r^2)_{ik\bar{q}}(r^2)_{\bar{j}\bar{l}q}, \tag{A.11}$$

$$f_t' \cdot (r^2)_{ik\bar{q}} \cdot \big(f_t'' \sum (r^2)_{i_1 i_2}(r^2)_{i_3} + f_t^{(3)} \cdot (r^2)_{\bar{j}}(r^2)_{\bar{l}}(r^2)_q\big), \tag{A.12}$$

$$f_t' \cdot (r^2)_{\bar{j}\bar{l}q} \cdot \big(f_t'' \sum (r^2)_{i_1 i_2}(r^2)_{i_3} + f_t^{(3)} \cdot (r^2)_{i}(r^2)_{k}(r^2)_{\bar{q}}\big), \tag{A.13}$$

where the summation in the second line is taken over all possible permutation $\{i_1, i_2, i_3\} = \{\bar{j}, \bar{l}, q\}$ and summation in the last line is taken over all possible permutation $\{i_1, i_2, i_3\} = \{i, k, \bar{q}\}$. Like before, they vanish when $i \neq k$ in case (A.11) and (A.12) or when $j \neq l$ in case (A.11) and (A.13).

Combining the above discussion, we see that when $i \neq k$ and $j \neq l$, the bound (A.7) follow immediately. For others, we need to treat case by case.

For the case $i \neq k$ and $j = l$, we have

$$R_{i\bar{j}k\bar{j}} \lesssim r^{-\frac{4}{3}} - f_t'' \cdot (r^2)_{\bar{j}\bar{j}i}(r^2)_k\big(1 - f_t' \cdot (r^2)_{i\bar{i}}\big) - f_t'' \cdot (r^2)_{\bar{j}\bar{j}k}(r^2)_i\big(1 - f_t' \cdot (r^2)_{k\bar{k}}\big).$$

We claim that the last term is always zero. Indeed, because of (A.5), if $i \neq 2$, then the last term is equal to zero. If $i = 2$, then $k \neq 2$, and by (A.1) and (A.6) we have $1 - f_t' \cdot (r^2)_{k\bar{k}} = 0$. This proves the claim that the last item always vanishes. For the same reason, the second item vanishes. Therefore, when $i \neq k$ and $j = l$, the estimate (A.7) holds.

For the case $i = k$ and $j \neq l$, we have $R_{i\bar{j}i\bar{l}} = \overline{R_{j\bar{i}l\bar{i}}} \lesssim r^{-\frac{4}{3}}$. This proves the bound (A.7) in this case.

Finally, we need to consider the cases $i = k$, $j = l$ and $i \neq j$. We should consider these cases individually. In case $i\bar{j}k\bar{l} = 1\bar{3}1\bar{3}$, we have

$$\begin{aligned} R_{1\bar{3}1\bar{3}} \lesssim\ & r^{-\frac{4}{3}} - f_t' \cdot (r^2)_{1\bar{3}1\bar{3}} + (f_t')^2 \cdot (r^2)_{11\bar{1}}(r^2)_{\bar{3}\bar{3}1} \\ & + \big(f_t' \cdot (r^2)_{11\bar{2}} + f_t'' \cdot (r^2)_{11}(r^2)_{\bar{2}}\big) \\ & \cdot \big(f_t' \cdot (r^2)_{\bar{3}\bar{3}2} + f_t'' \cdot (r^2)_{\bar{3}\bar{3}}(r^2)_2\big). \end{aligned}$$

The last term is clearly $\lesssim r^{-\frac{4}{3}}$; the second and third items combined give

$$\begin{aligned} & - f_t' \cdot (r^2)_{1\bar{3}1\bar{3}} + (f_t')^2 \cdot (r^2)_{11\bar{1}}(r^2)_{\bar{3}\bar{3}1} \\ & = -f_t' \frac{r^4}{t\eta_t^2} + (f_t')^2 \frac{r^6(r^2 - t)}{t\eta_t^3(r^2 + t)} = \frac{-2r^2}{\eta_t(r^2 + t)} \lesssim r^{-\frac{4}{3}}. \end{aligned}$$

This proves the bound (A.7) for $R_{1\bar{3}1\bar{3}}$. By similar method, we obtain desired bound (A.7) for $R_{1\bar{2}1\bar{2}}$ and $R_{2\bar{3}2\bar{3}}$.

Finally, we consider the case $i = j = k = l$. Since the metric is Ricci-flat, we have

$$R_{i\bar{i}i\bar{i}} = -\sum_{j \neq i} R_{i\bar{i}j\bar{j}} \lesssim r^{-\frac{4}{3}}.$$

This completes the proof of Lemma 5.1.

Remark A.2. We remark that when $r \to t$, the induced metric on the surface $r^2 = \text{const.}$ approaches

$$\frac{1}{2}\Big(\frac{2t^2}{3}\Big)^{\frac{1}{3}} ds^2|_{S^3},$$

where $ds^2|_{S^3}$ is the standard metric on S^3. The curvature of the limiting metric is $Ct^{-\frac{2}{3}}$ for some constant C.

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