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QUATERNION KÄHLERIAN MANIFOLDS

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A quaternion Kählerian manifold is defined as a Riemannian manifold whose holonomy group is a subgroup of $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$. Recently, several authors (Alekseevskii [1], [2], Gray [3], Ishihara [4], Ishihara and Konishi [5], Krainse [6] and Wolf [10]) have studied quaternion Kählerian manifolds and obtained many interesting results. In the present paper, we shall study those manifolds by using tensor calculus. To do so, it is rather convinient to define a quaternion Kählerian manifold as a Riemannian manifold which admits a bundle V of tensors of type $(1, 1)$ having some properties. The bundle V is 3-dimensional as a vector bundle and admits an algebraic structure which is isomorphic to that of pure imaginary quaternions.

In § 1, we define quaternion Kählerian manifolds in our fashion and give some results proved in [6]. § 2 is devoted to the establishment of some formulas required in the following sections. In § 3, it is proved among some other theorems that any quaternion Kählerian manifold is an Einstein space (Alekseevskii [1]). We prove in § 4 that a quaternion Kählerian manifold, which is of constant curvature or conformally flat, is of zero curvature, if the manifold is of dimension ≥ 8. In § 5, we define Q-sectional curvatures and determine the form of the curvature tensor of a quaternion Kählerian manifold when it has constant Q-sectional curvature (See Alekseevskii [1]).

Manifolds, mappings and geometric objects under discussion are assumed to be differentiable and of class C^∞. The indices $h, i, j, k, l, p, q, r, s, t, u, v$ run over the range $\{1, \cdots, n\}$, and the summation convention will be used with respect to this system of indices.

1. Quaternion Kählerian manifolds

Let M be a differentiable manifold of dimension n, and assume that there is a 3-dimensional vector bundle V consisting of tensors of type $(1, 1)$ over M satisfying the following condition:

(a) In any coordinate neighborhood U of M, there is a local base $\{F, G, H\}$ of V such that

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$$F^2 = -I\,, \quad G^2 = -I\,, \quad H^2 = -I\,, \tag{1.1}$$
$$GH = -HG = F\,, \quad HF = -FH = G\,, \quad FG = -GF = H\,,$$

I denoting the identify tensor field of type $(1, 1)$ in M.

Such a local base $\{F, G, H\}$ is called a *canonical local base* of the bundle V in U. Then the bundle V is called an *almost quaternion structure* in M, and (M, V) an *almost quaternion manifold*. Thus an almost quaternion manifold is necessarily of dimension $n = 4m$ $(m \geq 1)$.

In an almost quaternion manifold (M, V), we take intersecting coordinate neighborhoods U and U'. Let $\{F, G, H\}$ and $\{F', G', H'\}$ be canonical local bases of V in U and U' respectively. Then F', G' and H' are linear combinations of F, G and H in $U \cap U'$, that is,

$$\begin{aligned} F' &= s_{11}F + s_{12}G + s_{13}H\,, \\ G' &= s_{21}F + s_{22}G + s_{23}H\,, \\ H' &= s_{31}F + s_{32}G + s_{33}H \end{aligned} \tag{1.2}$$

with functions $s_{\gamma\beta}$ $(\gamma, \beta = 1, 2, 3)$ in $U \cap U'$. The coefficients $s_{\gamma\beta}$ appearing in (1.2) form an element $S_{U,U'} = (s_{\gamma\beta})$ of the proper orthogonal group $SO(3)$ of dimension 3, because both of $\{F, G, H\}$ and $\{F', G', H'\}$ satisfy (1.1). Thus *any almost quaternion manifold is orientable.*

If there is in an almost quaternion manifold (M, V) a global base $\{F, G, H\}$ of the bundle V which satisfies (1.1), then (M, V) is what is traditionally called almost quaternion manifold. Such a global base $\{F, G, H\}$ of V is called a *canonical global base* of V.

Let (M, V) be an almost quaternion manifold, and $\{F, G, H\}$ a canonical local base of V in a coordinate neighborhood U of M. We now assume that there is in each U a system of coordinates (x^h) with respect to which F, G and H have numerical components of the form

$$F\colon \begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & -E \\ 0 & 0 & E & 0 \end{pmatrix}, \qquad G\colon \begin{pmatrix} 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{pmatrix}, \tag{1.3}$$
$$H\colon \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & -E & 0 \\ 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix},$$

where E denotes the identity (m, m)-matrix $(\dim M = 4m)$. In such a case, the given almost quaternion structure V or such a canonical local base $\{F, G, H\}$ is said to be *integrable*.

We denote by Sp (m) the real representation of the symplectic unitary group acting on a unitary vector space C^{2m}, that is, Sp (m) is the subgroup of $SO(4m)$ which leaves the tensors F, G and H invariant, where F, G and H have respectively components given by (1.3) with respect to an orthogonal base of a real metric vector space R^{4m}. We denote by Sp $(m) \cdot$ Sp (1) the subgroup Sp $(m) \times$ Sp $(1)/\{\pm I\}$ of $SO(4m)$, and by Sp (m) the subgroup Sp $(m) \cdot I$, where I is the identity transformation of R^{4m}.

Remark. A $(4m, 4m)$-matrix $A = (A_i{}^h)$ belonging to the Lie algebra sp (m) $\times$ sp (1) of Sp $(m) \cdot$ Sp (1) is an element of the subalgebra sp $(m) \times 0$ of sp (m) $\times$ sp (1) if and only if

$$A_i{}^h F_h{}^i = 0 , \quad A_i{}^h G_h{}^i = 0 , \quad A_i{}^h H_h{}^i = 0 ,$$

where $(F_i{}^h)$, $(G_i{}^h)$ and $(H_i{}^h)$ are respectively the matrices F, G and H given by (1.3).

In any almost quaternion manifold (M, V), there is a Riemannian metric g such that $g(\phi X, Y) + g(X, \phi Y) = 0$ holds for any cross-section ϕ of V, X and Y being arbitrary vector fields. A pair $\{g, V\}$ of such a Riemannian metric and an almost quaternion structure V is called an *almost quaternion metric structure*, and $\{M, g, V\}$ an *almost quaternion metric manifold*. Thus a manifold M admits an almost quaternion structure if and only if the structure group of the tangent bundle $T(M)$ over M is reducible to Sp $(m) \cdot$ Sp (1), where $\dim M = 4m$. A manifold with almost quaternion structure admits a canonical global base of V if and only if the structure group of $T(M)$ is reducible to Sp (m).

Let $\{F, G, H\}$ be a canonical local base of V in a coordinate neighborhood U of an almost quaternion metric manifold (M, g, V). Since each of F, G and H is almost Hermitian with respect to g, putting

$$\begin{aligned} \Phi(X, Y) &= g(FX, Y) , \qquad \Psi(X, Y) = g(HX, Y) , \\ \Theta(X, Y) &= g(HX, Y) , \end{aligned} \tag{1.4}$$

for any vector fields X and Y, we see that Φ, Ψ and Θ are local 2-forms in U. However, by means of (1.2),

$$\Omega = \Phi \wedge \Phi + \Psi \wedge \Psi + \Theta \wedge \Theta \tag{1.5}$$

is a 4-form defined globally in M. Using (1.2), we easily see that

$$\Lambda = F \otimes F + G \otimes G + H \otimes M \tag{1.6}$$

is a global tensor field of type (2, 2) in M.

Now, let us assume that the Riemannian connection ∇ of (M, g, V) satisfies the following conditions:

(b) If ϕ is a cross-section (local or global) of the bundle V, then $\nabla_X \phi$ is also a cross-section (resp. local or global) of V, X being an arbitrary vector field in M.

By means of (1.1), the condition (b) is equivalent to the following condition:
(b)′ If $\{F, G, H\}$ is in U a canonical local base of V in U, then

$$\begin{aligned} \nabla_X F &= \qquad\qquad\quad r(X)G - q(X)H\,, \\ \nabla_X G &= -r(X)F \qquad\qquad\quad + p(X)H\,, \\ \nabla_X H &= \quad q(X)F - p(X)G \end{aligned} \tag{1.7}$$

for any vector field X, where p, q and r are certain local 1-forms defined in U.

If an almost quaternion metric manifold (M, g, V) satisfies the condition (b) or (b)′, then (M, g, V) or M is called a *quaternion Kählerian manifold* and (g, V) a *quaternion Kählerian structure*. Thus a Riemannian manifold is a quaternion Kählerian manifold if and only if the holonomy group is a subgroup of $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)$ (see [1], [2], [3], [4], [5], [6], [10]). In a quaternion Kählerian manifold, we easily have

$$\nabla\Omega = 0\,, \qquad \nabla\Lambda = 0 \tag{1.8}$$

because of (1.7). Conversely, if we have one of equations (1.8) in an almost quaternion metric manifold, then it is a quaternion Kählerian manifold. Thus we have

Theorem 1.1. *An almost quaternion metric manifold is a quaternion Kählerian manifold if and only if $\nabla\Omega = 0$ or $\nabla\Lambda = 0$* (See [4], for example).

From Theorem 1.1, we have $\nabla\Omega = 0$ in a quaternion Kählerian manifold. Thus Ω is a nonzero harmonic 4-form and hence Ω^k $(1 \leq k \leq m)$ is a nonzero harmonic $4k$-form, where $\dim M = 4m$. Therefore, when a quaternion Kählerian manifold M is compact, $B_{4k} \geq 1$ for $0 \leq k \leq m$, where B_r is the r-th Betti number of M, because M is orientable. Using the harmonic form Ω, Kraines [6] proved inequalities $B_r \leq B_{r+4} \leq \cdots \leq B_{r+4p}$ for $r + 4p \leq m + 1$, $r = 0, 1, 2$ or 3.

2. Some formulas

In this section we establish some formulas, which will be required in the sequel, concerning quaternion Kählerian manifolds. We denote a quaternion Kählerian manifold by (M, g, V), and assume that $\dim M = 4m$.

In a coordinate neighborhood $\{U; x^h\}$ of M we denote by g_{ji} the components of the metric tensor g, and put $(g^{ji}) = (g_{ji})^{-1}$. Take a canonical local base $\{F, G, H\}$ of V in U, and denote by $F_i{}^h, G_i{}^h$ and $H_i{}^h$ respectively the components of F, G and H in $\{U, x^h\}$, where $\{F, G, H\}$ is a canonical local base of V in U. Thus (1.7) is written as

$$\begin{aligned} \nabla_j F_i{}^h &= \qquad\quad r_j G_i{}^h - q_j H_i{}^h , \\ \nabla_j G_i{}^h &= -r_j F_i{}^h \qquad\quad + p_j H_i{}^h , \\ \nabla_j H_i{}^h &= q_j F_i{}^h - p_j G_i{}^h , \end{aligned} \tag{2.1}$$

where p_j, q_j and r_j are respectively the components of p, q and r in $\{U; x^h\}$. Using the Ricci formula, from (2.1) we have

$$\begin{aligned} K_{kjt}{}^h F_i{}^t - K_{kji}{}^s F_s{}^h &= \qquad\quad C_{kj} G_i{}^h - B_{kj} H_i{}^h , \\ K_{kjt}{}^h G_i{}^t - K_{kji}{}^s G_s{}^h &= -C_{kj} F_i{}^h \qquad\quad + A_{kj} H_i{}^h , \\ K_{kjt}{}^h H_i{}^t - K_{kji}{}^s H_s{}^h &= -B_{kj} F_i{}^h - A_{kj} G_i{}^h , \end{aligned} \tag{2.2}$$

$K_{kji}{}^h$ being the components of the curvature tensor K of the quaternion Kählerian manifold (M, g), and A, B, C being defined by

$$A = dp + q \wedge r , \quad B = dq + r \wedge p , \quad C = dr + q \wedge p , \tag{2.3}$$

where $A_{ji} = -A_{ij}$, $B_{ji} = -B_{ij}$, $C_{ij} = -C_{ji}$ and

$$A = \tfrac{1}{2} A_{ji} dx^j \wedge dx^i , \quad B = \tfrac{1}{2} B_{ji} dx^j \wedge dx^i , \quad C = \tfrac{1}{2} C_{ji} dx^j \wedge dx^i . \tag{2.4}$$

Thus A, B and C are local 2-forms defined in U.

We now obtain, from (2.2),

$$\begin{aligned} [K(X, Y), F] &= \qquad\quad C(X, Y)G - B(X, Y)H , \\ [K(X, Y), G] &= -C(X, Y)F \qquad\quad + A(X, Y)H , \\ [K(X, Y), H] &= B(X, Y)F - A(X, Y)G \end{aligned} \tag{2.5}$$

in a coordinate neighborhood U, X and Y being arbitrary vector fields in M. In another neighborhood U' we may have

$$\begin{aligned} [K(X, Y), F'] &= \qquad\quad C'(X, Y)G' - B'(X, Y)H' , \\ [K(X, Y), G'] &= -C'(X, Y)F' \qquad\quad + A'(X, Y)H' , \\ [K(X, Y), H'] &= B'(X, Y)F' - A'(X, Y)G' , \end{aligned}$$

where F', G', H' form a canonical local base of V in U'. Because $S_{U,U'} = (s_{j\beta}) \in SO(3)$, by means of (1.2) we thus find in $U \cap U'$

$$\begin{aligned} A' &= s_{11} A + s_{12} B + s_{13} C , \\ B' &= s_{21} A + s_{22} B + s_{23} C , \\ C' &= s_{31} A + s_{32} B + s_{33} C . \end{aligned} \tag{2.6}$$

Using (2.6) we easily see that the local 4-form

$$\sum = A \wedge A + B \wedge B + C \wedge C \tag{2.7}$$

determines in M a global 4-form, which is denoted also by $\sum$. This $\sum$ is, in some sense, the curvature tensor of a linear connection defined in the bundle V by means of (1.7). Now, using (2.3) and taking account of Theorem 3.3 proved in § 3, we can easily prove

Lemma 2.1. *Let (M, g, V) be a quaternion Kählerian manifold. A necessary and sufficient condition for the* 4*-form $\sum$ to vanish identically in M, provided that* $\dim M \geq 8$, *(or for all of the local* 2*-forms A, B and C to vanish identically in each coordinate neighborhood U) is that in each coordinate neighborhood U there be a canonical local base $\{F, G, H\}$ of V satisfying $\nabla F = \nabla G = \nabla H = 0$, i.e., that the bundle V be locally parallelizable.*

Assuming that a quaternion Kählerian manifold satisfies the condition stated in Lemma 2.1, we see that the functions $s_{\gamma\beta}$ appearing in (1.2) are all constant in a connected component of $U \cap U'$, U and U' being coordinate neighborhoods, if we take $\{F, G, H\}$ such that $\nabla F = \nabla G = \nabla H = 0$ in each U. In a quaternion Kählerian manifold (M, g, V), if M is simply connected, and the bundle V is locally parallelizable, then V has a canonical global base.

Transvecting the three equations of (2.2) respectively with $F_{hu} = F_h{}^t G_{tu}$, $G_{hu} = G_h{}^t g_{tu}$ and $H_{hu} = H_h{}^t g_{tu}$ and changing indices, we find respectively

$$\begin{aligned} -K_{kjts}F_i{}^tF_h{}^s + K_{kjih} &= C_{kj}H_{ih} + B_{kj}G_{ih} , \\ -K_{kjts}G_i{}^tG_h{}^s + K_{kjih} &= A_{kj}F_{ih} + C_{kj}H_{ih} , \\ -K_{kjts}H_i{}^tH_h{}^s + K_{kjih} &= B_{kj}G_{ih} + A_{kj}F_{ih} , \end{aligned} \tag{2.8}$$

where $K_{kjih} = K_{kji}{}^s g_{sh}$, and $F_{ih} = F_i{}^s g_{sh}$, $G_{ih} = G_i{}^s g_{sh}$, $H_{ih} = H_i{}^h g_{sh}$ are the components of Φ, Ψ, Θ defined by (1.4) respectively.

Transvecting the second equation of (2.8) with $F^{ih} = g^{is}F_s{}^h$, we have

$$-K_{khts}G_i{}^tG_h{}^sF^{ih} + K_{kjih}F^{ih} = 4mA_{kj} ,$$

from which it follows that

$$A_{kj} = \frac{1}{2m}K_{kjih}F^{ih} ,$$

where $\dim M = 4m$. Similarly, we obtain

$$\begin{aligned} A_{kj} &= \frac{1}{2m}K_{kjih}F^{ih} , \qquad B_{kj} = \frac{1}{2m}K_{kjih}G^{ih} , \\ C_{kj} &= \frac{1}{2m}K_{kjih}H^{ih} . \end{aligned} \tag{2.9}$$

Next, using (2.9) we have

$$K_{ktsh}F^{ts} = \tfrac{1}{2}(K_{ktsh} - K_{ksth})F^{ts} = \tfrac{1}{2}(K_{ktsh} + K_{skth})F^{ts}$$
$$= -\tfrac{1}{2}K_{khts}F^{ts} = -mA_{kh} ,$$

where we have used the identity $K_{kjih} + K_{jikh} + K_{ikjh} = 0$. Similarly we find

$$\begin{aligned} &K_{ktsh}F^{ts} = -mA_{kh} , \qquad K_{ktsh}G^{ts} = -mB_{kh} , \\ &K_{ktsh}H^{ts} = -mC_{kh} . \end{aligned} \tag{2.10}$$

On the other hand because of (2.10), transvecting (2.8) with g^{ji} gives $K_{kh} = -mA_{ks}F_h{}^s - B_{ks}G_h{}^s - C_{ks}H_h{}^s$ where $K_{kh} = K_{kjih}g^{ji}$ are the components of the Ricci tensor S of (M, g). Similarly we obtain

$$\begin{aligned} K_{kh} &= -mA_{ks}F_h{}^s - B_{ks}G_h{}^s - C_{ks}H_h{}^s , \\ K_{kh} &= -A_{ks}F_h{}^s - mB_{ks}G_h{}^s - C_{ks}H_h{}^s , \\ K_{kh} &= -A_{ks}F_h{}^s - B_{ks}G_h{}^s - mC_{ks}H_h{}^s , \end{aligned}$$

from which, it follows that for $m > 1$,

$$\begin{aligned} &K_{kh} = -(m+2)A_{ks}F_h{}^s , \qquad K_{kh} = -(m+2)B_{ks}G_h{}^s , \\ &K_{kh} = -(m+2)C_{ks}H_h{}^s , \end{aligned} \tag{2.11}$$

and for $m = 1$,

$$K_{kh} = -A_{ks}F_h{}^s - B_{ks}G_h{}^s - C_{ks}H_h{}^s . \tag{2.12}$$

We find from (2.11) that if $m > 1$, then

$$\begin{aligned} &A_{kh} = \frac{1}{m+2}K_{ks}F_h{}^s , \qquad B_{kh} = \frac{1}{m+2}K_{ks}G_h{}^s , \\ &C_{kh} = \frac{1}{m+2}K_{ks}H_h{}^s . \end{aligned} \tag{2.13}$$

Substituting (2.13) in (2.8) we have for $m > 1$,

$$\begin{aligned} -K_{kjts}F_i{}^tF_h{}^s + K_{kjih} &= \frac{1}{m+2}K_{kt}(G_j{}^tG_{ih} + H_j{}^tH_{ih}) , \\ -K_{kjts}G_i{}^tG_h{}^s + K_{kjih} &= \frac{1}{m+2}K_{kt}(H_j{}^tH_{ih} + F_j{}^tF_{ih}), \\ -K_{kjts}H_i{}^tG_h{}^s + K_{kjih} &= \frac{1}{m+2}K_{kt}(F_j{}^tF_{ih} + G_j{}^tG_{ih}) . \end{aligned} \tag{2.14}$$

Since A_{kj}, B_{kj} and C_{kj} are all skew-symmetric, using (2.13) we find, for $m > 1$,

$$(2.15)\qquad K_{ts}F_k{}^tF_j{}^s = K_{kj}\ ,\quad K_{ts}G_k{}^tG_j{}^s = K_{kj}\ ,\quad K_{ts}H_k{}^tH_j{}^s = K_{kj}\ .$$

Now using (2.3) we have the identities

$$(2.16)\qquad \begin{aligned} &dA + q\wedge C - r\wedge B = 0\ ,\qquad dB + r\wedge A - p\wedge C = 0\ ,\\ &dC + p\wedge B - q\wedge A = 0\ . \end{aligned}$$

Next by means of (2.13) we see that $K_{js}F_i{}^s$, $K_{js}G_i{}^s$ and $K_{js}H_i{}^s$ are all skew-symmetric if $\dim M = 4m > 4$. On the other hand, use of (2.1) gives

$$\nabla_k(K_{js}F_i{}^s) = (\nabla_k K_{js})F_i{}^s + r_k K_{js}G_i{}^s - q_k K_{js}H_i{}^s\ ,$$

from which it follows that $(\nabla_k K_{js})F_i{}^s + (\nabla_k K_{is})F_j{}^s = 0$. Thus, if $\dim M = 4m > 4$, then

$$(2.17)\qquad \nabla_k K_{ji} = (\nabla_k K_{ts})F_j{}^tF_i{}^s\ .$$

3. Some theorems

First, we prove

Lemma 3.1. *For any quaternion Kählerian manifold of dimension* ≥ 8, *the Ricci tensor is parallel.*

Proof. By means of (2.1) and (2.13), from the first identity of (2.16) it follows that if $\dim M \geq 8$, then

$$(3.1)\qquad (\nabla_k K_{js})F_i{}^s + (\nabla_j K_{is})F_k{}^s + (\nabla_i K_{ks})F_j{}^s = 0\ .$$

Transvecting (3.1) with $F_h{}^i$ gives

$$-\nabla_k K_{jh} + (\nabla_j K_{ts})F_h{}^tF_k{}^s + (\nabla_t K_{ks})F_h{}^tF_j{}^s = 0\ .$$

Substituting in this equation $(\nabla_j K_{ts})F_h{}^tF_k{}^s = \nabla_j K_{kh}$ which is a consequence of (2.17), we thus obtain

$$\nabla_j K_{kh} - \nabla_k K_{jh} = -(\nabla_t K_{ks})F_h{}^tF_j{}^s\ .$$

If we substitute in this equation $\nabla_t K_{ks} = (\nabla_t K_{ba})G_k{}^bG_s{}^a$ which will be proved in the same way as (2.17), then we find

$$\nabla_j K_{kh} - \nabla_k K_{jh} = (\nabla_c K_{ba})F_h{}^cG_k{}^bH_j{}^a\ .$$

Similarly we obtain

$$\nabla_j K_{kh} - \nabla_k K_{jh} = -(\nabla_c K_{ba})G_h{}^cH_k{}^bF_j{}^a = -(\nabla_c K_{ba})H_h{}^cF_k{}^bG_j{}^a\ .$$

Combining these equations gives

$$(3.2) \qquad (\nabla_c K_{ba}) F_k{}^c G_j{}^b H_i{}^a = (\nabla_c K_{ba}) G_k{}^c H_j{}^b F_i{}^a = (\nabla_c K_{ba}) H_k{}^c F_j{}^b G_i{}^a \ .$$

In particular, we have

$$(\nabla_c K_{ba}) F_k{}^c G_j{}^b H_c{}^a = (\nabla_c K_{ba}) G_k{}^c H_j{}^b F_i{}^a \ ,$$

from which by transvecting with $H_r{}^k G_q{}^j F_p{}^i$ we obtain

$$(3.3) \qquad -(\nabla_c K_{ba}) G_r{}^c H_q{}^b F_p{}^a = (\nabla_c K_{ba}) F_r{}^c G_q{}^b H_p{}^a \ .$$

Thus from (3.2) and (3.3) it follows that $(\nabla_c K_{ba}) F_k{}^c G_j{}^b H_i{}^a = 0$ which implies $\nabla_c K_{ba} = 0$ proving Lemma 3.1.

Next we prove

Lemma 3.2. *Let (M, g, V) be a quaternion Kählerian manifold of dimension ≥ 8. Then the Riemannian manifold (M, g) is irreducible if (M, g) has nonvanishing Ricci tensor.*

Proof. Suppose that (M, g) is reducible and has nonvanishing Ricci tensor. Since (M, g) is not flat and the Ricci tensor S is parallel because of Lemma 3.1, we can take a coordinate neighborhood U of M such that the Riemannian manifold (U, g) is decomposed into a Riemannian product of certain number of Riemannian manifolds $(W_1, g_1), \cdots, (W_p, g_p)$, $p \geq 2$, in such a way that

$$(3.4) \qquad g(X, Y) = \sum_{A=1}^{p} g_A(\pi_A X, \pi_A Y) \ , \qquad S(X, Y) = \sum_{A=1}^{p} \rho_A g_A(\pi_A X, \pi_A Y)$$

for any vector fields X and Y, where ρ_A are constants such that $\rho_1 < \cdots < \rho_p$ and $\pi_A \colon U \to W_A$ are the natural projections which denote at the same time their differential mappings.

On the other hand, since (F, g) is an almost Hermitian structure in U, we have $g(FX, FY) = g(X, Y)$ and $S(FX, FY) = S(X, Y)$ from (2.15). Thus using (3.4) we obtain in U

$$(3.5) \qquad \sum_{A=1}^{p} g_A(\pi_A FX, \pi_A FY) = \sum_{A=1}^{p} g_A(\pi_A X, \pi_A Y) \ ,$$

$$(3.6) \qquad \sum_{A=1}^{p} \rho_A g_A(\pi_A FX, \pi_A FY) = \sum_{A=1}^{p} \rho_A g_A(\pi_A X, \pi_A Y)$$

for any vector fields X and Y. Since $\rho_1 < \cdots < \rho_p$ and each of g_A is positive definite, we have, from (3.5) and (3.6),

$$(3.7) \qquad g_A(\pi_A FX, \pi_A FY) = g_A(\pi_A X, \pi_A Y) \qquad (A = 1, \cdots, p)$$

for any vector fields X and Y.

From now on for simplicity we assume that $p = 2$, i.e., that $(U, g) = (W_1, g_1) \times (W_2, g_2)$. Let $(y^\alpha) = (y^1, \cdots, y^r)$ and $(z^\lambda) = (z^{r+1}, \cdots, z^n)$ be coordinates systems in W_1 and W_2 respectively. Then $(x^h) = (y^\alpha, z^\lambda)$ are naturally coordinates in $U = W_1 \times W_2$ (the indices α, β, γ run over the range $\{1, \cdots, r\}$ and the indices λ, μ, ν over the rang $\{r+1, \cdots, n\}$). With respect to $(x^h) = (y^\alpha, z^\lambda)$, g has components of the form

$$(g_{ji}) = \begin{pmatrix} g_{\gamma\beta} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix}$$

where $(g_{\gamma\beta})$ and $(g_{\mu\nu})$ are respectively the components of g_1 and g_2, $(g_{\gamma\beta})$ and $(g_{\mu\nu})$ being independent of the variables z^λ and y^α respectively. From (3.7) we have $F_\mu{}^\alpha = 0$ and hence $A_{\mu\lambda} = B_{\mu\lambda} = C_{\mu\lambda} = 0$ by putting $h = \alpha$, $i = \beta$, $j = \lambda$ and $k = \mu$ in (2.2). Similarly, we find $A_{\beta\alpha} = C_{\beta\alpha} = D_{\beta\alpha} = 0$ and $A_{\lambda\alpha} = B_{\lambda\alpha} = C_{\lambda\alpha} = 0$. Therefore we have $A = B = C = 0$, which implies $S = 0$ because of (2.11). Since this is a contradiction, the Riemannian manifold (M, g) is necessarily irreducible and Lemma 3.2 is proved.

From Lemmas 3.1 and 3.2, we have the following theorems.

Theorem 3.3. *Any quaternion Kählerian manifold of dimension* ≥ 8 *is an Einstein space* (see Alekseevskii [1]).

Theorem 3.4. *When a quaternion Kählerian manifold* (M, g, V) *of dimension* ≥ 8 *has nonvanishing scalar curvature, the Riemannian manifold* (M, g) *is irreducible* (see Alekseevskii [1]).

Theorem 3.5. *When a quaternion Kählerian manifold* (M, g, V) *of dimension* ≥ 8 *has zero scalar curvature,* (M, g) *is locally a Riemannian product of a flat quaternion Kählerian manifold and irreducible quaternion Kählerian manifolds with vanishing Ricci tensor* (see Alekseevskii [1]).

Now from (2.9) and (2.13) we have

$$\tag{3.8} \begin{aligned} &K_{kjih}F^{ih} = \frac{2m}{m+2}K_{ks}F_j{}^s\,, \qquad K_{kjih}G^{ih} = \frac{2m}{m+2}K_{ks}G_j{}^s\,, \\ &K_{kjih}H^{ih} = \frac{2m}{m+2}K_{ks}H_j{}^s\,, \end{aligned}$$

if $\dim M = 4\mathrm{m} \geq 8$. Thus, if the Ricci tensor vanishes identically, then we obtain, for succesive covariant derivatives of the curvature tensor,

$$\tag{3.9} \begin{aligned} &K_{kjih}F^{ih} = K_{kjih}G^{ih} = K_{kjih}H^{ih} = 0\,, \\ &(\nabla_l K_{kjih})F^{ih} = (\nabla_l K_{kjih})G^{ih} = (\nabla_l K_{kjih})H^{ih} = 0\,, \\ &\cdots \qquad \cdots \qquad \cdots \\ &(\nabla_s \cdots \nabla_l K_{kjih})F^{ih} = (\nabla_s \cdots \nabla_l K_{kjih})G^{ih} = (\nabla_s \cdots \nabla_l K_{kjih})H^{ih} = 0\,, \\ &\cdots \qquad \cdots \qquad \cdots \end{aligned}$$

provided that $\dim M \geq 8$. Therefore by means of the remark stated in § 1 we see that the restricted holonomy group of (M, g, V) is a subgroup of $\mathrm{Sp}\,(m)$.

Conversely, if the restricted holonomy group is a subgroup of Sp(m), then we have (3.9) and hence $K_{kh} = 0$ by taking account of (3.8). Summing up, we have

Theorem 3.6. *The restricted holonomy group of a quaternion Kählerian manifold of dimension* ≥ 8 *is a subgroup of* Sp(m) *if and only if the Ricci tensor vanishes identically* (see Alekseevskii [1]).

When $\dim M = 4$, taking account of (2.10) and (2.12), we have

Theorem 3.7. *If the restricted holonomy group of a quaternion Kählerian manifold of dimension* 4 *is a subgroup of* Sp(m), *then the Ricci tensor vanishes identically.*

Taking account of (2.13), from Lemma 2.1 we have

Theorem 3.8. *For a quaternion Kählerian manifold* (M, g, V) *of dimension* ≥ 8, *the bundle* V *is locally parallizable if and only if the Ricci tensor vanishes identically. When a quaternion Kählerian manifold* (M, g, V) *is of dimension* 4, *the Ricci tensor vanishes identically if the bundle* V *is parallelizable.*

Now suppose that a quaternion Kählerian manifold (M, g, V) of dimension ≥ 8 is of zero curvature. Then by Theorem 3.8 the bundle V is locally parallizable. Thus, if (M, g, V) is further simply connected, then the bundle V admits a cannonical global base (i.e., M admits an almost quaternion structure in traditional sense). If we now take account of a theorem proved in [7] and [9], concerning the integrability of almost quaternion structure in traditional sense, then we have

Theorem 3.9. *If a quaternion Kählerian manifold* (M, g, V) *of dimension* ≥ 8 *is complete, simply connected and of zero curvature, then* (M, g, V) *is a Euclidean space with standard quaternion structure* V.

Let (M, V) be an almost quaternion manifold. Consider a linear connection $\bar{D}$ in the vector bundle V. If we take a canonical local base $\{F, G, H\}$ of V in a coordinate neighborhood U of M, then taking account of (1.1) we can put in U

$$\bar{D}_x F = \bar{p}(X)G - \bar{q}(X)H\ , \qquad \bar{D}_x G = \bar{p}(X)H - \bar{r}(X)F\ ,$$
$$\bar{D}_x H = \bar{q}(X)F - \bar{p}(X)G$$

for any vector field X in M, where $\bar{p}, \bar{q}$ and $\bar{r}$ are local 1-forms defined in U, which are called *connection forms* of $\bar{D}$ in U with respect to $\{F, G, H\}$. Now in U we put $\bar{\pi} = \bar{A} \wedge \bar{A} + \bar{B} \wedge \bar{B} + \bar{C} \wedge \bar{C}$, where $\bar{A} = d\bar{p} + \bar{q} \wedge \bar{r}$, $\bar{B} = d\bar{q} + \bar{r} \wedge \bar{p}$, $\bar{C} = d\bar{r} + \bar{p} \wedge \bar{q}$. Then $\bar{\pi}$ is a global 4-form in M, which is brieafly called the *curvature form* of $\bar{D}$. We can easily verify that $d\bar{\pi} = 0$, so that the curvature form $\bar{\pi}$ is closed.

Let there be given another linear connection D' in the vector bundle V, and denote by π' its curvature form which is obviously closed. Then, according to a well known theorem due to A. Weil, the two curvature forms $\bar{\pi}$ and π' determine the same cohomology class $\mathscr{C}$. We now denote, in the vector bundle V, by D the linear connection defined by $D_X F = \nabla_X F$, $D_X G = \nabla_X G$ and $D_X H = \nabla_X H$, X being an arbitrary vector field in M, when a quaternion Kählerian structure (g, V) is given in M. Then, in a quaternion Kählerian manifold (M, g, V) of dimension ≥ 8, the curvature from $\pi = A \wedge A + B \wedge B + C \wedge C = a(\Phi \wedge \Phi + \Psi \wedge \Psi + \Theta \wedge \Theta)$ determines the cohomology class $\mathscr{C}$, where a is a constant and π is harmonic, since (M, g, V) is an Einstein space.

When, for an almost quaternion manifold (M, V), there is a canonical local base $\{F, G, H\}$ of V in each coordinate neighborhood U of M in such a way that the coefficients $s_{\gamma\beta}$ appearing in (1.2) are all constant in every connected components of $U \cap U'$, we say that the almost quaternion structure V is *trivial*. If we assume that the bundle V is trivial, then there is a linear connection D in V satisfying $D_X F = 0$, $D_X G = 0$, $D_X H = 0$ for any vector field X in M, and the curvature form π of D is trivially zero. Thus, in such a case, the cohomology class $\mathscr{C}$ is zero.

Now we suppose that for a quaternion Kählerian manifold (M, g, V) the bundle V is trivial. Then the cohomology class $\mathscr{C}$ is zero and therefore the harmonic 4-form π is cohomologous to zero. Thus, if M is compact, then π is necessarily equal to zero, and hence the scalar curvature vanishes identically. Conversely, if (M, g, V) has vanishing scalar curvature, then the Ricci tensor vanishes, and hence the vector bundle V is locally parallelizable. Consequently, in such a case, V is trivial. Summing up the arguments developed above, we have

Theorem 3.10. *A necessary and sufficient condition for a compact quaternion Kählerian manifold (M, g, V) of dimension ≥ 8 to have a trivial almost quaternion structure V is that the scalar curvature vanish identically.*

4. Quaternion Kählerian manifolds of constant curvature

Let (M, g, V) be a $4m$-dimensional quaternion Kählerian manifold, and assume that (M, g) is of constant curvature c. Then the curvature tensor of (M, g) has components of the form

$$K_{kjih} = c(g_{kh}g_{ji} - g_{jh}g_{ki}) . \tag{4.1}$$

Thus we obtain

$$K_{kjts}F_i{}^tF_h{}^s = c(F_{kh}F_{ji} - F_{jh}F_{ki}) . \tag{4.2}$$

Substituting (4.1) in (2.9), we find

$$A_{kj} = -\frac{c}{m}F_{kj} .$$

Similary,

$$(4.3) \qquad A_{kj} = -\frac{c}{m}F_{kj}\,, \quad B_{kj} = -\frac{c}{m}G_{kj}\,, \quad C_{kj} = -\frac{c}{m}H_{kj}\,.$$

Substituting (4.1), (4.2) and (4.3) in the first equation of (2.8) we have

$$\begin{aligned} -c(F_{kh}F_{ji} - F_{jh}F_{ki}) &+ c(g_{kh}g_{ji} - g_{jh}g_{ki}) \\ &= -\frac{c}{m}(H_{kj}H_{ih} + G_{kj}G_{ih})\,, \end{aligned}$$

from which by transvecting with g^{ji} we obtain $c(m-1)(2m+1)g_{kh}/m = 0$. Thus, if $c \neq 0$, we get $m = 1$. Consequently, we have

Theorem 4.1. *If a quaternion Kählerian manifold (M, g, V) is of nonzero constant curvature, then (M, g, V) is necessarily of dimension* 4.

By Theorem 3.3, any quaternion Kählerian manifold is an Einstein space. If a quaternion Kählerian manifold is conformally flat, then it is of constant curvature. Thus from Theorem 4.1 we have

Theorem 4.2. *If a quaternion Kählerian manifold of dimension ≥ 8 is conformally flat, then it is of zero curvature.*

5. Q-sectional curvatures

We have already known that the curvature tensor of a quaternion projective space $HP(m)$ of dimension $4m$, which is the base space of the Hopf fibering $S^{4m+3} \to HP(m)$, where S^{4m+3} is a unit sphere of dimension $4m+3$ $(m \geq 1)$ (See [1], [4], [5]), is given by

$$(5.1) \qquad \begin{aligned} K_{kjih} = g_{kh}g_{ji} - g_{jh}g_{ki} &+ F_{kh}F_{ji} - F_{jh}F_{ki} - 2F_{kj}F_{ih} \\ &+ G_{kh}G_{ji} - G_{jh}G_{ki} - 2G_{kj}G_{ih} \\ &+ H_{kh}H_{ji} - H_{jh}H_{ki} - 2H_{kj}H_{ih}\,. \end{aligned}$$

Let X be a tangent vector of $HP(m)$ at a point P of $HP(m)$, and Y a linear combination of X, FX, GX and HX, which is linearly independent of X. Then by means of (5.1) we obtain $\sigma(X, Y) = 4$, where $\sigma(X, Y)$ denotes the sectional curvature of $HP(m)$ with respect to X and Y. In this connection we shall define Q-sections and Q-sectional curvatures in a quaternion Kählerian manifold.

We take a point P in a quaternion Kählerian manifold (M, g, V) of dimension $4m$ and a tangent vector X of M at P. Then the 4-dimensional subspace $Q(X)$ of the tangent space of M at P defined by

$$Q(X) = \{Y \mid Y = aX + bFX + cGX + dHX\}\,,$$

a, b, c and d being arbitrary real numbers, is called the *Q-section* determined by X.

If we denote by $\sigma(X, Y)$ the sectional curvature of (M, g, V) with respect to the section spanned by X and Y at a point, then by definition we have

$$\sigma(X, Y) = -\frac{K_{kjih}X^k Y^j X^i Y^h}{\|X\|^2 \|Y\|^2 - g(X, Y)^2},$$

where $\|X\|$ and $\|Y\|$ are lengths of X and Y respectively. Since (M, g, V) is an Einstein space, using (2.14) we obtain, for a unit vector X,

$$(5.2)\qquad \begin{aligned} \sigma(X, FX) &= \frac{k}{4m(m+2)}\|X\|^2 - K(X, FX, GX, HX), \\ \sigma(X, GX) &= \frac{k}{4m(m+2)}\|X\|^2 - K(X, GX, HX, FX), \\ \sigma(X, HX) &= \frac{k}{4m(m+2)}\|X\|^2 - K(G, HX, FX, GX), \end{aligned}$$

k being the scalar curvature of (M, g, V), where we have put $K(X, Y, Z, W) = g(K(X, Y)Z, W)$ for any tangent vectors X, Y, Z and W of M at P.

Now we suppose that for any $Y, Z \in Q(X)$ the sectional curvature $\sigma(Y, Z)$ is a constant $\rho(X)$, which will be called the *Q-sectional curvature* of (M, g, V) with respect to X at P. Then, putting $Y = X$, $Z = FX$; $Y = X$, $Z = GX$ and $Y = X$, $Z = HX$, we have respectively

$$(5.3)\qquad \sigma(X, FX) = \sigma(X, GX) = \sigma(X, HX) = \rho(X).$$

Thus, if we take account of the identity

$$K(X, Y, Z, W) + K(X, Z, W, Y) + K(X, W, Y, Z) = 0$$

and use (5.2), we obtain

$$(5.4)\qquad \rho(X) = \frac{k}{4m(m+2)}\|X\|^2,$$

$$(5.5)\qquad \begin{aligned} K(X, FX, GX, HX) &= K(X, GX, HX, FX) \\ &= K(X, HX, FX, GX) = 0. \end{aligned}$$

Next, from the assumption we find $\sigma(X, aFX + bGX) = \rho(X)$, $a^2 + b^2 \neq 0$, which together with (5.3) implies $K(X, FX, X, GX) = 0$. Similarly,

$$(5.6)\qquad \begin{aligned} K(X, FX, X, GX) &= K(X, GX, X, HX) \\ &= K(X, HX, X, FX) = 0. \end{aligned}$$

Thus, using (5.5) and (5.6), we find

$$(5.7)\qquad K(Y, Z)Y - \rho(X)Z \in Q^{\perp}(X)$$

for any $Y, Z \in Q(X)$, where $Q^{\perp}(X)$ denotes the orthogonal complement of $Q(X)$ in the tangent space of M at P.

Conversely, if we assume that (5.7) is established for any $Y, Z \in Q(X)$ at P of M, then we easily see that the sectional curvature $\sigma(Y, Z)$ is constant for any $Y, Z \in Q(X)$. Thus we have

Lemma 5.1. *Let $Q(X)$ be a Q-section at a point P of a quaternion Kählerian manifold (M, g, V). Then the sectional curvature $\sigma(Y, Z)$ with respect to the section spanned by any $Y, Z \in Q(X)$ is a constant $\rho(X)$ if and only if $K(Y, Z) - \rho(X)Z \in Q^{\perp}(X)$ for any $Y, Z \in Q(X)$, and in such a case we have*

$$\rho(X) = \frac{k}{4m(m+2)} \|X\|^2 , \tag{5.8}$$

where k is the scalar curvature of (M, g, V), and $\dim M = 4m$.

In Lemma 5.1, $\rho(X)$ is called the *Q-sectional curvature* of (M, g, V) with respect to X at P, and the Q-section is said to *have Q-sectional curvature $\rho(X)$*.

It is easily verified that a Q-section $Q(X)$ at P has Q-sectional curvature $\rho(X)$ if in (M, g, V) there is a totally geodesic submanifold N of dimension 4 and constant curvature $\rho(X)$ such that N is tangent to $Q(X)$ at P. When $\dim M = 4$, there is only one Q-section $Q(X)$ at each point P of M, X being anarbitrary tangent vector of M at P. Thus, when $\dim M = 4$, this $Q(X)$ has Q-sectional curvature c if and only if (M, g, V) is of constant curvature c at P.

Let (M, g, V) be a quaternion Kählerian manifold and assume that each Q-section $Q(X)$ at each point P of M, X being an arbitrary tangent vector to M at P, has a Q-sectional curvature $\rho(X)$. Moreover, if we suppose that the sectional curvature $\rho(X)$ is a constant $c = c(P)$ independent of X at each point P, then we say that the quaternion Kählerian manifold (M, g, V) is of *constant Q-sectional curvature $c(P)$*. Taking account of (5.1) we see that a quaternion projective space $HP(m)$ is a quaternion Kählerian manifold of constant Q-sectional curvature 1. In the sequel we shall determine the form of the curvature tensor K_{kjih} of a quaternion Kählerian manifold of constant Q-section curvature $c(P)$.

Let (M, g, V) be a quaternion Kählerian manifold of constant Q-sectional curvature $c = c(P)$ of dimension ≥ 8. Since (M, g, V) is an Einstein space, we may put

$$A_{kj} = -4aF_{kj} , \quad B_{kj} = -4aG_{kj} , \quad C_{kj} = -4aH_{kj} ,$$

a being a certain constant. Therefore substituting these equations in (2.8) we have

$$-K_{kjts}F_i{}^tF_h{}^s + K_{kjih} = -4a(G_{kj}G_{ih} + H_{kj}H_{ih}) . \tag{5.9}$$

Transvecting (5.9) with $F_r{}^kF_p{}^h$ and $F_q{}^jF_p{}^h$ we obtain

$$(5.10)\qquad -K_{vjsh}F_k{}^vF_i{}^s - K_{vjsi}F_k{}^vF_h{}^s = 4a(G_{kj}G_{ih} + H_{kj}H_{ih}) ,$$

$$(5.11)\qquad -K_{ktsh}F_j{}^tF_i{}^s - K_{ktsi}F_j{}^tF_h{}^s = -4a(G_{kj}G_{ih} + H_{kj}H_{ih})$$

respectively, where we have changed indices.

From the assumption we have $\sigma(X, FX) = c$ for any tangent vector X, taking account of the definition of sectional curvatures, so that

$$\begin{aligned} K_{vjsh}F_k{}^vF_i{}^sX^kX^jX^iX^h &= c(g_{vh}g_{js} - g_{jh}g_{vs})F_k{}^vF_i{}^sX^kX^jX^iX^h \\ &= -cg_{jh}g_{ki}X^kX^jX^iX^h , \end{aligned}$$

X^h being the components of X. Since X is arbitrarily taken, from the above equation we obtain

$$\begin{aligned} &K_{vjsh}F_k{}^vF_i{}^s + K_{vish}F_j{}^vF_k{}^s + K_{vksh}F_i{}^vF_j{}^s + K_{vjsi}F_k{}^vF_h{}^s \\ &\quad + K_{vhsi}F_j{}^vF_k{}^s + K_{vksi}F_h{}^vF_j{}^s + K_{vksh}F_j{}^vF_i{}^s + K_{vish}F_k{}^vF_j{}^s \\ &\quad + K_{vjsh}F_i{}^vF_k{}^s + K_{vksi}F_j{}^vF_h{}^s + K_{vhsi}F_k{}^vF_j{}^s + K_{vjsi}F_h{}^vF_k{}^s \\ &\qquad = -4c(g_{kj}g_{ih} + g_{ji}g_{kh} + g_{ik}g_{jh}) , \end{aligned}$$

which, together with (5.10), implies

$$\begin{aligned} &K_{vjsh}F_k{}^vF_i{}^s + K_{vish}F_j{}^vF_k{}^s + K_{vksh}F_i{}^vF_j{}^s \\ &\quad = -c(g_{kj}g_{ih} + g_{ji}g_{kh} + g_{ik}g_{jh}) - 4a(G_{kj}G_{ih} + H_{kj}H_{ih}) \\ &\qquad - 2a(G_{kj}G_{ih} + G_{ji}G_{kh} + G_{ik}G_{jh} + H_{kj}H_{ih} + H_{ji}H_{kh} + H_{ik}H_{jh}) . \end{aligned}$$

If we transvect the above equation with $F_q{}^kF_p{}^i$, then we obtain

$$\begin{aligned} &K_{qjpk} - K_{vuqh}F_p{}^vF_j{}^u - K_{pvuh}F_q{}^vF_j{}^u \\ &\quad = -c(F_{qj}F_{ph} - F_{qh}F_{jp} + g_{qp}g_{jh}) \\ &\qquad - 2a(G_{qj}G_{ph} + G_{qh}G_{jp} + G_{qp}G_{jh} + H_{qj}H_{ph} + H_{qh}H_{jp} + H_{qp}H_{jh}) . \end{aligned}$$

Substituting

$$K_{vuqh}F_q{}^vF_j{}^u = K_{jpqh} + 4A(G_{jp}G_{qh} + H_{jp}H_{qh}) ,$$

which is a consequence of (5.9), in the above equation we get

$$(5.12)\qquad \begin{aligned} &K_{qiph} - K_{jqph} - K_{pvuh}F_q{}^vF_j{}^u - 4a(G_{jp}G_{qh} + H_{jp}H_{qh}) \\ &\quad = -c(F_{qj}F_{ph} - F_{qh}F_{jp} + g_{qp}g_{jh}) \\ &\qquad - 2a(G_{qj}G_{ph} + G_{qh}G_{jp} + G_{qp}G_{jh} + H_{qj}H_{ph} \\ &\qquad\qquad + H_{qh}H_{jp} + H_{qp}H_{jh}) . \end{aligned}$$

On the other hand, using (5.9) we find

$$\begin{aligned}(5.13)\qquad & K_{pvuh}F_q{}^vF_j{}^u - K_{puvh}F_q{}^vF_j{}^u \\ & = (K_{pvuh} - K_{puvh})F_q{}^vF_j{}^u = -K_{vuph}F_q{}^vF_j{}^u \\ & = -K_{qjph} - 4a(G_{qj}G_{ph} + H_{qj}H_{ph}) .\end{aligned}$$

Taking the skew-symmetric parts of both sides of (5.12) with respect to q and j, and substituting (5.13) in the resulting equation we obtain

$$\begin{aligned} & 2K_{qjph} - K_{jpqh} + K_{qpjh} + K_{qjph} + 4a(G_{qj}G_{ph} + H_{qj}H_{ph}) \\ & \quad - 4a(G_{jp}G_{qh} - G_{qp}G_{jh} + H_{jp}H_{qh} - H_{qp}H_{jh}) \\ & = -c(2F_{qj}F_{ph} - F_{qh}F_{jp} + F_{jh}F_{qp} + g_{qp}g_{jh} - g_{jp}g_{qh}) \\ & \quad - 4a(G_{qj}G_{ph} + H_{qj}H_{ph}) ,\end{aligned}$$

from which it follows that

$$\begin{aligned} K_{qjph} = {} & \tfrac{1}{4}c(g_{qh}g_{jp} - g_{jh}g_{qp} + F_{qh}F_{jp} - F_{jh}F_{qp} - 2F_{qj}F_{ph}) \\ & + a(G_{qh}G_{jp} - G_{jh}G_{qp} - 2G_{qj}G_{ih} + H_{qh}H_{jp} \\ & - H_{jh}H_{qp} - 2H_{qj}H_{ih}) .\end{aligned}$$

If we now take account of $\sigma(X, FX) = c$, then we have $a = \frac{1}{4}c$ and therefore

$$\begin{aligned}(5.14)\qquad K_{kjih} = {} & \tfrac{1}{4}c(g_{kh}g_{ji} - g_{jh}g_{ki} + F_{kh}F_{ji} - F_{jh}F_{ki} - 2F_{kj}F_{ih} \\ & + G_{kh}G_{ji} - G_{jh}G_{ki} - 2G_{kj}G_{ih} + H_{kh}H_{ji} \\ & - H_{jh}H_{ki} - 2H_{kj}H_{ih}) .\end{aligned}$$

Summing up, we have

Therem 5.2. *A quaternion Kählerian manifold of dimension $n \geq 8$ is of constant Q-sectional curvature $c = c(P)$ if and only if its curvature tensor has components of the form* (5.14) (see Alekseevskii [1]).

Transvecting (5.14) with g^{ji} we find $K_{ji} = c(m + 2)g_{ji}$ which implies $k = 4m(m + 2)c$, so that $c = c(P)$ is necessarily constant in M. Hence we have

Theorem 5.3. *For a quaternion Kählerian manifold (M, g, V) of constant Q-sectional curvature $c = c(P)$, $P \in M$, the function $c(P)$ is constant in M if* $\dim M \geq 8$.

Let X and Y be two mutually orthogonal unit tangent vectors of (M, g, V) at a point P. If (M, g, V) is of constant Q-sectional curvature c, then by (5.14) we obtain

$$(X, Y) = \tfrac{1}{4}c[1 + 3(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)] ,$$

where $\cos\alpha = g(FX, Y)$, $\cos\beta = g(GX, Y)$, $\cos\gamma = g(HX, Y)$. Thus we have the inequality $0 \leq \cos^2\alpha + \cos^2\beta + \cos^2\gamma \leq 1$, where the first equality holds if and only if $Y \in Q(X)$, and the second equality holds if and only if $Y \in Q^{\perp}(X)$. Therefore we arrive at (see [6])

Theorem 5.5. *Denote by K a general sectional curvature of a quaternion Kählerian manifold of constant Q-sectional curvature c ($c \neq 0$). Then*

$$\tfrac{1}{4}c \leq K \leq c \quad \textit{for } c > 0\,, \quad \textit{and} \quad \tfrac{1}{4}c \geq K \geq c \quad \textit{for } c < 0\,.$$

If we denote by $\Lambda_{ki}{}^{jh}$ the components of the the tensor field Λ defined by (1.6) and put $\Lambda_{kjih} = \Lambda_{ki}{}^{ts} g_{tj} g_{sh}$, then (5.14) reduces to

$$K_{kjih} = \tfrac{1}{4}c(g_{kh}g_{ji} - g_{jh}g_{ki} + \Lambda_{khji} - \Lambda_{jhki} - 2\Lambda_{kjih})\,.$$

Since $\nabla\Lambda = 0$, we get $\nabla_l K_{kjih} = 0$. Thus *a quaternion Kählerian manifold of constant Q-sectional curvature is locally symmetric.*

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