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ALMOST CONTACT MANIFOLDS WITH KILLING STRUCTURES TENSORS. II

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1. Introduction

Almost contact manifolds with Killing structure tensors were defined in [2] as nearly cosymplectic manifolds, and it was shown normal nearly cosymplectic manifolds are cosymplectic (see also [4]). In this note we study a nearly cosymplectic structure (φ, ξ, η, g) on a manifold M^{2n+1} with η closed primarily from the topological viewpoint, and extend some of Gray's results for nearly Kähler manifolds [5] to this case. In particular on a compact manifold satisfying some curvature condition we are able to distinguish between the cosymplectic and non-cosymplectic cases. In addition, we show that if ξ is regular, M^{2n+1} is a principal circle bundle $S^1 \to M^{2n+1} \to K^{2n}$ over a nearly Kähler manifold K^{2n}, and moreover if M^{2n+1} has positive φ-sectional curvature, then M^{2n+1} is the product $K^{2n} \times S^1$.

2. Almost contact structures

A $(2n+1)$-dimensional C^∞ manifold M^{2n+1} is said to have an *almost contact structure* if there exist on M^{2n+1} a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \ \varphi\xi = 0, \ \eta \circ \varphi = 0, \ \varphi^2 = -I + \xi \otimes \eta ,$$

Moreover, there exists for such a structure a Riemannian metric g such that

$$\eta(X) = g(\xi, X) , \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) ,$$

where X and Y are vector fields on M^{2n+1} (see e.g., [14]). Now define on $M^{2n+1} \times R$ an almost complex structure J by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right) ,$$

where f is a C^∞ function on $M^{2n+1} \times R$, [15]. If this almost complex structure is integrable, we say that the almost contact structure is *normal*; the condition for normality in terms of φ, ξ and η is $[\varphi, \varphi] + \xi \otimes d\eta = 0$, where $[\varphi, \varphi]$ is the

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Nijenhuis torsion of φ. Finally the *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(X, \varphi Y)$.

An almost contact metric structure (φ, ξ, η, g) is said to be *cosymplectic*, if it is normal and both Φ and η are closed [1]. (Our notion of a cosymplectic manifold differs from the one given by P. Libermann [9].) The structure is said to be *nearly cosymplectic* if φ is Killing, i.e., if $(\nabla_X\varphi)Y + (\nabla_Y\varphi)X = 0$, where ∇ denotes the Riemannian connexion of g. The structure is said to be *closely cosymplectic* if φ is Killing and η is closed.

Proposition 2.1. *On a nearly cosymplectic manifold the vector field ξ is Killing.*

Proof. It suffices to show that $g(\nabla_X\xi, X) = 0$ for X belonging to an orthonormal basis. Clearly $g(\nabla_\xi\xi, \xi) = 0$, so we may assume that X is orthogonal to ξ. Thus

$$\begin{aligned} g(\nabla_X\xi, X) &= g(\varphi\nabla_X\xi, \varphi X) = -g((\nabla_X\varphi)\xi, \varphi X) = g((\nabla_\xi\varphi)X, \varphi X) \\ &= \tfrac{1}{2}(\xi g(\varphi X, \varphi X) - \xi g(X, X)) = 0 . \end{aligned}$$

Remark. (1) From Proposition 2.1 it is clear that on a closely cosymplectic manifold we have $\nabla_X\eta = 0$.

(2) If an almost contact metric structure is normal and $\nabla_X\varphi = 0$, then it is cosymplectic; conversely on a cosymplectic manifold $\nabla_X\varphi = 0$, [1].

(3) Since ξ is parallel on a closely cosymplectic manifold, it is clear that $(\nabla_X\varphi)\xi = 0$, from which, since φ is Killing, $\nabla_\xi\varphi = 0$.

A plane section of the tangent space M_m^{2n+1} at $m \in M^{2n+1}$ is called a *φ-section* if it is determined by a vector X orthogonal to ξ such that $\{X, \varphi X\}$ is an orthonormal pair spanning the section. The sectional curvature $K(X, \varphi X)$ is called a *φ-sectional curvature* [13].

Given two φ-sections determined, say by unit vectors X and Y, we define the *φ-bisectional curvature* $B(X, Y)$ by

$$B(X, Y) = g(R_{X\varphi X}Y, \varphi Y) ,$$

where R_{XY} denotes the curvature transformation of ∇.

A local orthonormal basis of the form $\{\xi, X_i, X_{i*} = \varphi X_i\}$, $i = 1, \cdots, n$ on an almost contact manifold M^{2n+1} is called a *φ-basis*. It is well known that such a basis always exists. Let $\{\eta, \omega_i, \omega_{i*}\}$ be the dual basis. A 2-form α is said to be of *tridegree* $(1, 1, 0)$ if α satisfies $\alpha(X, \varphi Y) + \alpha(\varphi X, Y) = 0$. For a more general discussion of p-forms of tridegree (λ, μ, ν), $\lambda + \mu + \nu = p$ on almost contact manifolds see [12]. We denote by $H^{110}(M^{2n+1})$ the space of harmonic 2-forms on M^{2n+1} of tridegree $(1, 1, 0)$.

3. Closely cosymplectic manifolds

Lemma 3.1. *On a closely cosymplectic manifold we have*

$$\|(\nabla_X\varphi)Y\|^2 = g(R_{XY}X, Y) - g(R_{XY}\varphi X, \varphi Y) .$$

The proof is a long but straightforward computation similar to the proof of the corresponding result on nearly Kähler manifolds [6].

Corollary 3.2. *On a closely cosymplectic manifold*

$$g(R_{XY}X, Y) = g(R_{\varphi X \varphi Y}\varphi X, \varphi Y) .$$

Corollary 3.3. *On a closely cosymplectic manifold $g(R_{\xi X}\xi, X) = 0$; in particular the sectional curvatures of plane sections containing ξ vanish.*

This last corollary generalizes the result for cosymplectic manifolds [1].

Lemma 3.4 [11]. *Let α be a 2-form on an almost contact manifold satisfying $\alpha(X, \varphi Y) + \alpha(\varphi X, Y) = 0$. Then for any $m \in M^{2n+1}$, there exists a φ-basis of M^{2n+1}_m such that $\alpha_{ii^*} = \alpha(X_i, X_{i^*})$ are the only nonzero components of α.*

Proof. For X orthogonal to ξ we have

$$\alpha(\xi, X) = -\alpha(\xi, \varphi^2 X) = \alpha(\varphi\xi, \varphi Y) = 0 .$$

Now let $S(X, Y) = \alpha(\varphi X, Y)$. Then $S(X, Y) = S(Y, X)$ and $S(\varphi X, \varphi Y) = S(X, Y)$, i.e., S is a symmetric bilinear form invariant under φ. If X_1 is an eigenvector of S orthogonal to ξ, then so is φX_1. Thus we can inductively choose a φ-basis $\{\xi, X_i, X_{i^*} = \varphi X_i\}$ such that the only nonvanishing components of S are of the form $S_{ii} = S_{i^*i^*} = \alpha_{i^*i}$.

Theorem 3.5. *Let M^{2n+1} be a compact closely cosymplectic manifold having nonnegative φ-bisectional curvature and satisfying $K(X, Y) + K(X, \varphi Y) > 0$ for linearly independant $X, Y, \varphi X, \varphi Y$ orthogonal to ξ. Then M^{2n+1} is cosymplectic or not cosymplectic according as* $\dim H^{110}(M^{2n+1}) = 1$ *or* 0.

Proof. Let α be a 2-form of tridegree $(1, 1, 0)$. Then by Lemma 3.4 there exists a φ-basis such that the only nonzero components of α are $\alpha_{ii^*} = \alpha(X_i, \varphi X_i)$. Thus using Lemma 3.1 we have for the Bochner-Lichnerowicz form:

$$\begin{aligned} F(\alpha) &= R_{\mu\nu}\alpha^{\mu\lambda_2\cdots\lambda_p}\alpha^{\nu}{}_{\lambda_2\cdots\lambda_p} - \frac{p-1}{2}R_{\kappa\lambda\mu\nu}\alpha^{\kappa\lambda\lambda_3\cdots\lambda_p}\alpha^{\mu\nu}{}_{\lambda_3\cdots\lambda_p} \\ &= 2\sum_{i<j}(R_{ii^*jj^*}(\alpha_{ii^*} - \alpha_{jj^*})^2 + 2\|(\nabla_{X_i}\varphi)X_j\|^2(\alpha_{ii^*}^2 + \alpha_{jj^*}^2)) , \end{aligned}$$

where $\kappa, \lambda, \cdots$ range over $1, \cdots, 2n+1$. Now as $R_{ii^*jj^*} \geq 0$, we have $F(\alpha) \geq 0$; hence if α is harmonic, then $F(\alpha) = 0$ giving

$$(*) \qquad R_{ii^*jj^*}(\alpha_{ii^*} - \alpha_{jj^*})^2 + 2\|(\nabla_{X_i}\varphi)X_j\|^2(\alpha_{ii^*}^2 + \alpha_{jj^*}^2) = 0 .$$

If now M^{2n+1} is not cosymplectic, it is clear that $\nabla_{X_i}\varphi \neq 0$ for some i, and one can then check that $(\nabla_{X_i}\varphi)X_j \neq 0$ for some j. Thus $\alpha_{ii^*} = 0$ and $\alpha_{jj^*} = 0$. But if $(\nabla_{X_i}\varphi)X_k = 0$, then by Lemma 3.1, $R_{ii^*kk^*} = R_{ikik} + R_{ik^*ik^*} > 0$ giving $\alpha_{kk^*} = \alpha_{ii^*}$. Thus $\alpha = 0$ and we have $\dim H^{110}(M^{2n+1}) = 0$.

In the cosymplectic case, the fundamental 2-form $\Phi \in H^{110}(M^{2n+1})$, so that $\dim H^{110}(M^{2n+1}) \geq 1$. Therefore, if $\alpha \in H^{110}(M^{2n+1})$, then by a decomposition theorem of [3], $\alpha = \beta + f\Phi$, where $\sum_i (\iota(\omega_{i*})\iota(\omega_i))\beta = 0$ and f is a function. Thus $\sum \beta_{ii*} = 0$, and by equation $(*)$ we have $\beta_{ii*} = \beta_{jj*}$ giving $\beta = 0$. Hence $\alpha = f\Phi$, and $\dim H^{110}(M^{2n+1}) = 1$.

4. Fibration of closely cosymplectic manifolds

Let M^{2n+1} be a compact almost contact metric manifold on which ξ is regular, i.e., every point $m \in M^{2n+1}$ has a neighborhood through which the integral curve of ξ through m passes only once. Since M^{2n+1} is compact, the integral curves of ξ are homeomorphic to circles. If now ξ is parallel, then its integral curves are geodesics, and it follows from a result of Hermann [8] that M^{2n+1} is a principal circle bundle over an even-dimensional manifold $K^{2n}(S^1 \longrightarrow M^{2n+1} \longrightarrow K^{2n})$.

Theorem 4.1. *Let M^{2n+1} be a compact almost contact metric manifold on which ξ is regular. If M^{2n+1} is closely cosymplectic (respectively cosymplectic), then K^{2n} is nearly Kähler (respectively Kähler).*

Proof. As M^{2n+1} is closely cosymplectic, ξ is parallel and we have the fibration $S^1 \longrightarrow M^{2n+1} \longrightarrow K^{2n}$. Again since ξ is parallel and $\nabla_\xi\varphi = 0$, we have

$$(\mathscr{L}_\xi\varphi)X = \nabla_\xi\varphi X - \nabla_{\varphi X}\xi - \varphi\nabla_\xi X + \varphi\nabla_X\xi = (\nabla_\xi\varphi)X = 0 .$$

Thus φ is projectable, and we define J on K^{2n} by $JX = \pi_*\varphi\tilde{\pi}X$, where $\tilde{\pi}$ denotes the horizontal lift with respect to the Riemannian connexion on M^{2n+1}. It is easy to see that $J^2 = -I$ on K^{2n}. Now as ξ is also Killing, the metric g is projectable to a metric g' on K^{2n}, i.e., $g'(X, Y) \circ \pi = g(\tilde{\pi}X, \tilde{\pi}Y)$. Letting ∇' denote the Riemannian connexion on K^{2n}, by a direct computation we obtain $(\nabla'_X J)Y = \pi_*(\nabla_{\tilde{\pi}X}\varphi)\tilde{\pi}Y$, from which the result follows.

Theorem 4.2. *Let $S^1 \longrightarrow M^{2n+1} \xrightarrow{\pi} K^{2n}$ be the above fibration with M^{2n+1} closely cosymplectic. If M^{2n+1} has positive φ-sectional curvature, then M^{2n+1} is the product space $K^{2n} \times S^1$.*

Proof. Since η is harmonic on M^{2n+1}, we have $H^1(M^{2n+1}, Z) \neq 0$. Secondly, by a direct computation positive φ-sectional curvature on M^{2n+1} implies positive holomorphic sectional curvature on K^{2n}, and hence $\pi_1(K^{2n}) = 0$ by a result of Gray [5]. We claim a principal circle bundle $S^1 \to M \to K$ with $\pi_1(K) = 0$ and $H^1(M) \neq 0$ is necessarily trivial. Let x be a base point of M, and S^1_x the fibre over x. Then the sequence

$$\cdots \longrightarrow H^1(M, S^1_x) \to H^1(M) \xrightarrow{\iota^*} H^1(S^1_x) \longrightarrow H^2(M, S^1_x) \longrightarrow \cdots$$

is exact. First note that $H^1(S^1_x) \approx Z$. Now by the universal coefficient theorem $H^1(M)$ is a free abelian group, and $H^1(M, S^1_x) \approx \text{free}\, H^1(M, S^1_x) \approx \text{free}\, H_1(M, S^1_x)$

$\approx$ free $H_1(K) = 0$ where the identification of $H_1(M, S^1_x)$ and $H_1(K)$ is made by the Serre sequence of the fibration (see for example, Mosher and Tangora [10]). Hence ι^* is a nontrivial monomorphism. Moreover torsion $H^2(M, S^1_x) \approx$ torsion $H_1(M, S^1_x) \approx$ torsion $H_1(K) = 0$. Thus ι^* is an isomorphism, and hence the characteristic class of the bundle is zero.

5. Examples

It is well known that S^6 carries a nearly Kähler structure, so let J denote such an almost complex structure on S^6 and let θ be a coordinate function on S^1. On $S^6 \times S^1$ define φ, ξ, η by

$$\varphi\left(X, f\frac{d}{d\theta}\right) = (JX, 0) \ , \quad \xi = \frac{d}{d\theta} \ , \quad \eta = d\theta \ ,$$

where X is tangent to S^6. Then as J is not parallel on S^6 (i.e., S^6 is not Kählerian), $\nabla\varphi \neq 0$ with respect to the product metric. However it is easy to check that the structure defined on $S^6 \times S^1$ is closely cosymplectic.

On the other hand, Gray [6] showed that every 4-dimensional nearly Kähler manifold is Kählerian. We now give the corresponding result for closely cosymplectic manifolds.

Theorem 5.1. *Every* 5*-dimensional closely cosymplectic manifold is cosymplectic.*

Proof. As the manifold is closely cosymplectic, a direct computation shows that $(\nabla_X\varphi)Y = \varphi(\nabla_X\varphi)\varphi Y$. Now let $\{\xi, X_1, \varphi X_1, X_2, \varphi X_2\}$ be a φ-basis. Then computing $\nabla\varphi$ on this basis we obtain $\nabla\varphi = 0$ and hence that the manifold is cosymplectic.

In [2] one of the authors showed that besides its usual normal contact metric structure, S^5 carries a nearly cosymplectic structure which is not cosymplectic. Consider S^5 as a totally geodesic hypersurface of S^6; then the nearly Kähler structure induces an almost contact metric structure (φ, ξ, η, g) with φ and hence η Killing. In view of Theorem 5.1 this nearly cosymplectic structure is not closely cosymplectic.

Moreover this almost constact structure on S^5 is also not contact as the following theorem shows.

Theorem 5.2. *There are no nearly cosymplectic structures which are contact metric structures.*

Proof. Let M^{2n+1} be a nearly cosymplectic manifold, and suppose that its (almost) contact form η is a contact structure (i.e., $\eta \wedge (d\eta)^n \neq 0$ everywhere). Since the structure is contact and ξ is Killing, M^{2n+1} is *K-contact* and $-\varphi X = \nabla_X\xi$. Now on a K-contact manifold the sectional curvature of a plane section containing ξ is equal to 1, [7]. Thus if X is a unit vector orthogonal to ξ, then

$$
\begin{aligned}
-1 &= g(\nabla_\xi \nabla_X \xi - \nabla_X \nabla_\xi \xi - \nabla_{[\xi,X]}\xi, X) \\
&= -g(\nabla_\xi \varphi X - \varphi[\xi, X], X) = -g((\nabla_\xi \varphi)X + \varphi \nabla_X \xi, X) \\
&= g((\nabla_X \varphi)\xi, X) + g(\varphi^2 X, X) = g((\nabla_X \varphi)\xi, X) - 1 .
\end{aligned}
$$

Therefore

$$
0 = g((\nabla_X \varphi)\xi, X) = -g(\varphi \nabla_X \xi, X) = -g(\varphi^2 X, X) = g(X, X) ,
$$

and hence $X = 0$, a contradiction.

References

[1] D. E. Blair, *The theory of quasi-Sasakian structures, J. Differential Geometry* **1** (1967) 331–345.
[2] ——, *Almost contact manifolds with Killing structure tensors,* Pacific J. Math. **39** (1971) 285–292.
[3] D. E. Blair & S. I. Goldberg, *Topology of almost contact manifolds,* J. Differential Geometry **1** (1967) 347–354.
[4] D. E. Blair & K. Yano, *Affine almost contact manifolds and f-manifolds with affine Killing structure tensors,* Kōdai Math. Sem. Rep. **23** (1971) 473–479.
[5] A. Gray, *Nearly Kähler manifolds,* J. Differential Geometry **4** (1970) 283–309.
[6] ——, *Almost complex submanifolds of the six-sphere,* Proc. Amer. Math. Soc. **20** (1969) 277–279.
[7] Y. Hatakeyama, Y. Ogawa & S. Tanno, *Some properties of manifolds with contact metric structure,* Tôhoku Math. J. **15** (1963) 42–48.
[8] R. Hermann, *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle,* Proc. Amer. Math. Soc. **11** (1960) 236–242.
[9] P. Libermann, *Sur les automorphisms infinitésimaux des structures symplectiques et des structures de contact,* Colloq. Géométrie Différentielle Globale (Bruxelles, 1958), Louvain, 1959, 37–59.
[10] P. Mosher & M. Tangora, *Cohomology operations and applications in homotopy theory,* Harper and Row, New York, 1968.
[11] E. M. Moskal, *Contact manifolds of positive curvature,* Thesis, University of Illinois, 1966.
[12] ——, *On the tridegree of forms on f-manifolds with applications,* to appear.
[13] K. Ogiue, *On almost contact manifolds admitting axiom of planes or axiom of free mobility,* Kōdai Math. Sem. Rep. **16** (1964) 223–232.
[14] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure.* I, Tôhoku Math. J. **12** (1960) 459–476.
[15] S. Sasaki & Y. Hatakeyama, *On differentiable manifolds with certain structures which are closely related to almost contact structure.* II, Tôhoku Math. J. **13** (1961) 281–294.

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