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A GENERALIZED HODGE THEORY

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1. Introduction

For a given nonsingular vector one-form $\underline{h}$ with vanishing Nijenhuis tensor, there is an associated exterior derivative d_h which satisfies a Poincaré lemma and hence provides an $\underline{h}$-dependent version of de Rham's theorem. The exterior derivative d_h also has an adjoint δ_h with respect to the usual global inner product. This fact permits one to define a strongly elliptic self-adjoint second order differential operator Δ_h which is a generalization of the Laplace-Beltrami operator. Consequently one can then obtain a generalization of the classical Hodge decomposition theorem.

2. Preliminaries

Let A denote the algebra of C^∞ functions on a compact orientable n-dimensional Riemannian manifold M without boundary. Let E denote the A-module of differential one-forms on M. A vector 1-form $\underline{h} \in \mathrm{End}_A(E)$ induces endmorphisms $h^{(q)} \in \mathrm{End}_A(\bigwedge^p E)$ for any nonnegative integer q, and the $h^{(q)}$ are defined by setting $h^{(q)} = 0$ if $q > p$, and

$$h^{(q)}(\varphi^1 \wedge \cdots \wedge \varphi^p) = \frac{1}{(p-q)!\,q!} \sum_\pi |\pi| \{\underline{h}\varphi^{\pi(1)} \wedge \cdots \wedge \underline{h}\varphi^{\pi(q)}\} \wedge \varphi^{\pi(q+1)} \wedge \cdots \wedge \varphi^{\pi(p)}$$

if $0 \leq q \leq p$, where $\varphi^i \in E$, π runs through all permutations of $(1, \cdots, p)$ and $|\pi|$ denotes the sign of the permutation. The transformation $h^{(0)}$ is taken to be the identity mapping on $\bigwedge^p E$.

In the case where $q = p \leq n$, the operator $h^{(p)}$ is locally represented by an $\binom{n}{p} \times \binom{n}{p}$ matrix $[h^{(p)}]$ relative to some local basis of p-forms. If $[\underline{h}]$ denotes an $n \times n$ matrix which locally represents $\underline{h}$, then it can be shown that $\det [h^{(p)}] = (\det [\underline{h}])^{\binom{n-1}{p-1}}$, and hence $\underline{h}$ is invertible on 1-forms if and only if $h^{(p)}$ is invertible in p-forms. This fact will be of use later in this section.

An alternating derivation $d_h\colon \bigwedge E \to \bigwedge E$ is obtained as in [3] from $\underline{h}$ and exterior derivation d by setting $d_h = h^{(1)}d - dh^{(1)}$. Thus when $\underline{h}$ is the identity,

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d_h reduces to d. The Nijenhuis tensor can be extended as a derivation on $\bigwedge E$. On p-forms one thus obtains the formula

$$[\underline{h}, \underline{h}] = -h^{(2)}d + d_h h^{(1)} + dh^{(2)} ,$$

which appears in [12]. A short calculation then yields the formula $d_h \circ d_h = d[\underline{h}, \underline{h}] + [\underline{h}, \underline{h}]d$, from which it is evident that a vanishing Nijenhuis tensor implies that d_h is an exterior derivation.

A Poincaré lemma may be obtained for the operator d_h when $\underline{h}$ is nonsingular and $[\underline{h}, \underline{h}] = 0$. In order to prove this fact, the following lemma is needed.

Lemma 2.1. *If $\underline{h}$ is a vector* 1*-form with vanishing Nijenhuis tensor, then $h^{(p+1)}d = d_h h^{(p)}$ on p-forms.*

Proof. Note that the result is trivial when $p = 0$ or $p \geq n$. If $1 \leq p < n$, one may proceed by induction on p. The case $p = 1$ follows immediately from the vanishing of the Nijenhuis tensor. Suppose now that the proposition is true for p-forms. Observe that if β is a $(p + 1)$-form, then it is sufficient to let $\beta = \varphi \wedge \alpha$, where φ is a 1-form and α is a p-form. Since $h^{(p+1)}\beta = h^{(p+1)}(\varphi \wedge \alpha) = \underline{h}\varphi \wedge h^{(p)}\alpha$ and d_h is an exterior derivation, one obtains

$$d_h h^{(p+1)}\beta = d_h(\underline{h}\varphi \wedge h^{(p)}\alpha) = (d_h \underline{h}\varphi) \wedge h^{(p)}\alpha - \underline{h}\varphi \wedge d_h h^{(p)}\alpha .$$

Hence the induction hypothesis yields

$$\begin{aligned} d_h h^{(p+1)}\beta &= (h^{(2)}d\varphi) \wedge h^{(p)}\alpha - \underline{h}\varphi \wedge d_h h^{(p)}\alpha \\ &= h^{(p+2)}\{d\varphi \wedge \alpha - \varphi \wedge d\alpha\} = h^{(p+2)}d\beta , \end{aligned}$$

and the lemma is established.

A Poincaré lemma for the operator d_h can now be proven.

Proposition 2.2. *Let $\underline{h}$ be a nonsingular vector one-form with vanishing Nijenhuis tensor. If $d_h\alpha = 0$ for any p-form α on a contractible space U, then there is a $(p - 1)$-form β on U such that $\alpha = d_h\beta$, for $p \geq 1$.*

Proof. Since $\underline{h}$ is invertible on 1-forms if and only if $h^{(p)}$ is invertible on p-forms, it follows from Lemma 2.1 that $[\underline{h}, \underline{h}] = 0$ implies $h^{(p+1)}d(h^{-1})^{(p)}\alpha = 0$ and hence that $d(h^{-1})^{(p)}\alpha = 0$. Thus $(h^{-1})^{(p)}\alpha = d\lambda$ for some $(p - 1)$-form λ, and consequently $\alpha = h^{(p)}d\lambda = d_h h^{(p-1)}\lambda$. The form β is obtained by letting $\beta = h^{(p-1)}\lambda$, and thus $\alpha = d_h\beta$ as asserted.

The condition that $\underline{h}$ be nonsingular cannot be omitted from the statement of the proposition. For example, in R^2 a singular $\underline{h}$ whose Nijenhuis tensor vanishes can be defined by setting $\underline{h}(dx) = dy$, and $\underline{h}(dy) = 0$. Then clearly $d_h dx = 0$, and it is easily checked that there cannot exist a differentiable function f such that $dx = d_h f$, as otherwise $dx = 0$.

The result of Proposition 2.2 can then be combined with the standard sheaf theoretic proof (given in [5]) for de Rham's theorem to yield the following $\underline{h}$ dependent version.

Theorem 2.3. *If $\underline{h}$ is a nonsingular vector one-form with vanishing Nijenhuis tensor, then the real cohomology of the cochain complex $(\bigwedge^* E, d_h)$ is isomorphic to the real cohomology of the underlying manifold M.*

3. The h-Laplacian

Let $\omega^1, \cdots, \omega^n$ be a local orthonormal basis of differential 1-forms, and suppose that $\omega^1 \wedge \cdots \wedge \omega^n$ agrees with an orientation of M. Suppose also that the vector one-form $\underline{h}$ is locally specified by setting $\underline{h}\omega^j = h^j_i \omega^i$, where $h^j_i \in A$ and $1 \leq i, j \leq n$, and the Einstein summation convention has been invoked. Since the orientation is specified, the Hodge star operator $*: \bigwedge^p E \rightarrow \bigwedge^{n-p} E$ is determined by setting $*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) = \varepsilon_{i_1 \cdots i_p} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{n-p}}$, where $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_{n-p}$ are complementary sets of positive integers, and $\varepsilon_{i_1 \cdots i_p}$ is $+1$ or -1 according as the permutation $(i_1, \cdots, i_p, j_1, \cdots, j_{n-p})$ of the integers $(1, \cdots, n)$ is even or odd respectively. In the sequel the trace and the transpose of $\underline{h}$ will be denoted by tr $\underline{h}$ and $\underline{h}_t$ respectively. The following proposition states a result which is fundamental to the theory presented in this paper.

Proposition 3.1. *For any vector one-form $\underline{h}$,*

$$h^{(1)}* + *h_t^{(1)} = (\mathrm{tr}\, \underline{h})* \ .$$

Proof. Because of linearity it will be sufficient to assume that any p-form φ has the decomposition $\varphi = \omega^1 \wedge \cdots \wedge \omega^p$ where the orthonormal basis $\omega^1, \cdots, \omega^n$ could be relabeled to fit this ordering if necessary. The proof then proceeds by calculation. Thus

$$\begin{aligned} h^{(1)}*\varphi &= h^{(1)}(\omega^{p+1} \wedge \cdots \wedge \omega^n) \\ &= \sum_{j=p+1}^{n} \omega^{p+1} \wedge \cdots \wedge \underline{h}\omega^j \wedge \cdots \wedge \omega^n \\ &= \sum_{j=p+1}^{n} h^j_\alpha \omega^{p+1} \wedge \cdots \wedge \omega^{j-1} \wedge \omega^\alpha \wedge \omega^{j+1} \wedge \cdots \wedge \omega^n \ ; \end{aligned}$$

and since for each j the sum on α produces a sum from 1 to p plus a term for $\alpha = j$, one obtains

$$\begin{aligned} h^{(1)}*\varphi = \sum_{j=p+1}^{n} \Big\{ & h^j_j \omega^{p+1} \wedge \cdots \wedge \omega^n \\ & + \sum_{i=1}^{p} h^j_i \omega^{p+1} \wedge \cdots \wedge \omega^{j-1} \wedge \omega^i \wedge \omega^{j+1} \wedge \cdots \wedge \omega^n \Big\} \ , \end{aligned}$$

and hence

$$\begin{aligned} h^{(1)}*\varphi = & \sum_{j=p+1}^{n} h^j_j \omega^{p+1} \wedge \cdots \wedge \omega^n \\ & + \sum_{i=1}^{p} \sum_{j=p+1}^{n} h^j_i \omega^{p+1} \wedge \cdots \wedge \omega^{j-1} \wedge \omega^i \wedge \omega^{j+1} \wedge \cdots \wedge \omega^n \ . \end{aligned}$$

In a similar fashion one may compute $*h_t^{(1)}\varphi$ and add it to the above result.

Corollary 3.2. *Any vector one-form $\underline{h}$ is skew symmetric if and only if $*$ and $h^{(1)}$ commute.*

Proof. If $*$ and $h^{(1)}$ commute, then it is clear that $h^{(1)} + h_t^{(1)} = (\mathrm{tr}\, \underline{h})h^{(0)}$ on p-forms. Hence, if $p = n$ the above formula implies that that $\mathrm{tr}\, \underline{h}$ vanishes, and consequently the skew symmetry of $\underline{h}$ is established by taking $p = 1$. The proof of the converse statement is obvious.

Since the manifold M is assumed to be compact, an inner product on $\bigwedge^p E$ is defined by setting

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta$$

for any p-forms α and β. With respect to this inner product the adjoint of $h^{(p)}$ can be found. Specifically one has the following result.

Proposition 3.3. *If $\underline{h}$ is a vector 1-form on M, then $h^{(q)}$ and $h_t^{(q)}$ are adjoints for $q = 0, 1, \cdots, n$.*

Proof. The proof will be carried out by an induction of q. Let α and β be p-forms; then as a consequence of Proposition 3.1 one has

$$(\alpha, h^{(1)}\beta) = \int_M \alpha \wedge *h^{(1)}\beta = -\int_M \alpha \wedge h_t^{(1)}*\beta + \int_M (\mathrm{tr}\, \underline{h})\alpha \wedge *\beta \,.$$

Since $h_t^{(1)}$ is a derivation of degree zero, one also has

$$\begin{aligned}(\alpha, h^{(1)}\beta) &= -\int_M h_t^{(1)}(\alpha \wedge *\beta) + \int_M (h_t^{(1)}\alpha) \wedge *\beta + \int_M (\mathrm{tr}\, \underline{h})\alpha \wedge *\beta \\ &= \int_M (h_t^{(1)}\alpha) \wedge *\beta = (h_t^{(1)}\alpha, \beta) \,,\end{aligned}$$

because $h_t^{(1)}(\alpha \wedge *\beta) = (\mathrm{tr}\, \underline{h}_t)(\alpha \wedge *\beta) = (\mathrm{tr}\, \underline{h})(\alpha \wedge *\beta)$. Thus $h^{(1)}$ and $h_t^{(1)}$ are adjoints. Now assume that $h^{(j)}$ and $h_t^{(j)}$ are adjoints for $j < q$. If $q > p$, then $h^{(q)}$ and $h_t^{(q)}$ vanish, and the result is trivially true. If $q \leq p$, then Lemma 2.1 of [12] provides the formula

$$h^{(q)} = \frac{1}{q} \sum_{j=0}^{q-1} (-1)^{q+j-1} (h^{q-j})^{(1)} h^{(j)} \,,$$

from which the inductive result is obvious.

The codifferential δ is defined to be the adjoint of the exterior derivative d. The adjoint of $d_h = h^{(1)}d - dh^{(1)}$ is then easily seen to be $\delta_h = \delta h_t^{(1)} - h_t^{(1)}\delta$. The operator δ_h is called the $\underline{h}$-codifferential. As a consequence of Stokes' theorem one obtains the relation $\delta = (-1)^{np+n+1}*d*$ on p-forms. The corresponding expression for δ_h is obtained in the following proposition.

Proposition 3.4. *If $\underline{h}$ is a vector 1-form, then on p-forms*

$$\delta_h = (-1)^{np+n+1} * \{d_h + d(\mathrm{tr}\,\underline{h}) \wedge\} * \,.$$

Proof. On p-forms one obtains

$$\delta_h = \delta h_t^{(1)} - h_t^{(1)}\delta = (-1)^{np+n+1}\{*d*h_t^{(1)} - h_t^{(1)}*d*\}\,,$$

and thus Proposition 3.1 may then be applied to yield

$$\begin{aligned}\delta_h &= (-1)^{np+n+1}\{*d[\mathrm{tr}\,\underline{h}* - h^{(1)}*] - [*\mathrm{tr}\,\underline{h} - *h^{(1)}]d*\}\\ &= (-1)^{np+n+1}\{*d(\mathrm{tr}\,\underline{h})* - *dh^{(1)}* + *h^{(1)}d*\}\\ &= (-1)^{np+n+1}*\{d_h + d(\mathrm{tr}\,\underline{h}) \wedge\}*\,.\end{aligned}$$

For any given vector one-form $\underline{h}$ a generalization Δ_h of the classical Laplace-Beltrami operator $\Delta = d\delta + \delta d$ can be defined by setting $\Delta_h = d_h\delta_h + \delta_h d_h$. The operator Δ_h will be called the $\underline{h}$-Laplace Beltrami operator. It is self-adjoint and reduces to the classical Laplace-Beltrami operator when $\underline{h}$ is the identity.

4. $\underline{h}$-Hodge Theory

In this section a Hodge theory is developed in a way which depends on a vector 1-form $\underline{h}$ through the $\underline{h}$-Laplace Beltrami operator of the previous section. Since a standard decomposition theorem for elliptic operators will be used, the immediate task is to determine when the operator Δ_h is elliptic. A necessary and sufficient condition for (strong) ellipticity is thus given in Proposition 4.2. The proof of this proposition is modeled on the proof, which appears in [13], of the ellipticity of Δ.

Definition 4.1. Let $D\colon \bigwedge E \to \bigwedge E$ be a q-th order differential operator. The *symbol* σ_D of D at a point $m \in M$ is a linear transformation of $(\bigwedge E)_m$, the exterior algebra of differential forms at m, defined by

$$\sigma_D(dg)(\alpha(m)) = D(g^q\alpha)(m)$$

for any p-form α and any C^∞ real valued function g on M such that $g(m) = 0$ and $dg(m) \neq 0$. The operator D is said to be *elliptic* whenever σ_D is an isomorphism, and *strongly elliptic* whenever σ_D is either positive definite or negative definite and when q is even.

Let $\xi\colon \bigwedge E \to \bigwedge E$ denote the map which is left exterior multiplication by a one-form ξ. An inner product on $\bigwedge^p E$ is given by $\langle\alpha, \beta\rangle = *(\alpha \wedge *\beta)$ for p-forms α and β. With respect to this inner product it can be shown as in [13] that

$$(\xi + \operatorname{adj} \xi) \operatorname{adj} (\xi + \operatorname{adj} \xi) = (-1)^{n(p-1)}\xi*\xi* + (-1)^{np}*\xi*\xi$$

is an isomorphism and hence positive definite on $\bigwedge^p E$ if and only if ξ is a nonzero 1-form.

Proposition 4.2. *A vector* 1*-form* $\underline{h}$ *is nonsingular on* M *if and only if the second order operator* Δ_h *is strongly elliptic on* M.

Proof. Select any point $m \in M$, and consider any C^∞ real valued function g which vanishes at m and has nonvanishing exterior derivative at m. Let $\xi = (d_h g)(m) = (\underline{h}dg)(m)$. If ω is a p-form, then Proposition 3.4 may be applied to obtain

$$\begin{aligned}\{\delta_h d_h g^2\omega\}(m) &= (-1)^{np+1}\{*[d_h + d(\operatorname{tr} \underline{h}) \wedge]*[2g(d_h g) \wedge \omega + g^2 d_h\omega]\}(m) \\ &= (-1)^{np+1}2*(d_h g)(m) \wedge *(d_h g)(m) \wedge \omega(m) \\ &= (-1)^{np+1}2*\xi*\xi\omega(m) ,\end{aligned}$$

and similarly

$$\{d_h\delta_h g^2\omega\}(m) = (-1)^{np+n+1}2\xi*\xi*\omega(m) .$$

Thus

$$\sigma_{\Delta_h}(dg)(\omega(m)) = -2[(-1)^{n(p-1)}\xi*\xi* + (-1)^{np}*\xi*\xi]\omega(m) ,$$

which is negative definite if and only if ξ is a nonnegative one-form for all such g and consequently if and only if $\underline{h}$ is nonsingular.

Corollary 4.3. *Let* $\underline{h}$ *be a vector* 1*-form. Then the first order differential operator* $d_h + \delta_h$ *which maps even forms to odd forms and odd forms to even forms is elliptic if and only if* $\underline{h}$ *is nonsingular.*

Proof. By the last corollary of § 4 of Chapter 4 in [11] it follows that $d_h + \delta_h$ is elliptic if and only if $(d_h + \delta_h)^2 = (d_h + \delta_h) \operatorname{adj} (d_h + \delta_h)$ is strongly elliptic. But it is easy to check that the symbol of $(d_h + \delta_h)^2$ is equal to the symbol of Δ_h even if d_h^2 does not vanish. Consequently $d_h + \delta_h$ is elliptic if and only if $\underline{h}$ is nonsingular.

Proposition 4.4. *Let* $L\colon \bigwedge^p E \to \bigwedge^p E$ *be an elliptic operator on the space of smooth p-forms on M. Then the space* Ker $\{L\colon \bigwedge^p E \to \bigwedge^p E\}$ *is finite dimensional and one has an orthogonal direct sum decomposition of* $\bigwedge^p E$ *given by*

$$\bigwedge^p E = \operatorname{Im}\{\operatorname{adj} L\colon \bigwedge^p E \to \bigwedge^p E\} \oplus \operatorname{Ker}\{L\colon \bigwedge^p E \to \bigwedge^p E\} .$$

A proof of this proposition can be obtained from a modification of the proof of the Hodge decomposition theorem given in [13].

Definition 4.5. For any vector one-form $\underline{h}$, elements of the space $K_h^p \equiv \operatorname{Ker}\{\Delta_h\colon \bigwedge^p E \to \bigwedge^p E\}$ are called *h-harmonic p-forms*. Similarly, elements of $d_h \bigwedge^{p-1} E$ and $\delta_h \bigwedge^{p+1} E$ are called *h-exact* and *h-coexact* p-forms respectively.

Since $(\Delta_h\alpha, \alpha) = (d_h\alpha, d_h\alpha) + (\delta_h\alpha, \delta_h\alpha)$ for any p-form α, it is obvious that α is $\underline{h}$-harmonic if and only if $d_h\alpha = \delta_h\alpha = 0$. Hence $\underline{h}$-harmonic forms are orthogonal to $\underline{h}$-exact and $\underline{h}$-coexact forms, since d_h and δ_h are adjoints. If the Nijenhuis tensor of $\underline{h}$ vanishes, then for p-forms α and β one has

$$(d_h\alpha, \delta_h\beta) = (d_h^2\alpha, \beta) = (\{[\underline{h}, \underline{h}]d + d[\underline{h}, \underline{h}]\}\alpha, \beta) = 0 .$$

Thus $\underline{h}$-exact and h-coexact forms are orthogonal when $[\underline{h}, \underline{h}] = 0$. One is then led to the following generalization of the classical Hodge decomposition theorem.

Theorem 4.6. *Let h be a nonsingular vector 1-form with vanishing Nijenhuis tensor. Then for each integer p with $0 \leq p \leq n$, the space K_h^p of $\underline{h}$-harmonic forms is finite dimensional and the space $\bigwedge^p E$ of smooth p-forms on M has a unique orthogonal direct sum decomposition*

$$\bigwedge{}^p E = d_h(\bigwedge{}^{p-1}E) \oplus \delta_h(\bigwedge{}^{p+1}E) \oplus K_h^p$$

into $\underline{h}$-exact, $\underline{h}$-coexact, and $\underline{h}$-harmonic spaces respectively.

Proof. Since $\underline{h}$ is nonsingular on M, the operator Δ_h is elliptic. Consequently an application of Proposition 4.4 implies that the space K_h^p of harmonic p-forms is finite dimensional and that one has the orthogonal direct sum decomposition $\bigwedge^p E = \Delta_h \bigwedge^p E \oplus K_n^p$. It is then obvious that $\bigwedge^p E$ is spanned by $\underline{h}$-exact, $\underline{h}$-coexact, and $\underline{h}$-harmonic forms respectively. The mutual orthogonality of the summands then implies the uniqueness of the direct sum decomposition.

Corollary 4.7. *The equation $\alpha = \Delta_h\omega$ has a solution $\omega \in \bigwedge^p E$ if and only if the p-form α is orthogonal to the space K_h^p of $\underline{h}$-harmonic forms.*

A standard argument using the orthogonality of the forms concerned can be used to establish the following theorem.

Theorem 4.8. *If $\underline{h}$ is a nonsingular vector 1-form with vanishing Nijenhuis tensor, then there is precisely one $\underline{h}$-harmonic form in each cohomology class of the cochain complex $(\bigwedge^* E, d_h)$.*

Corollary 4.9. *If $\underline{h}$ is a nonsingular vector 1-form with vanishing Nijenhuis tensor, then the number of linearly independent $\underline{h}$-harmonic p-forms is equal to the p-th betti number of M for $0 \leq p \leq n$.*

Remark 1. With precisely one $\underline{h}$-harmonic form in each cohomology class of $(\bigwedge^* E, d_h)$, the cohomology of the underlying manifold M can be obtained in as many ways as there are nonsingular endomorphisms $\underline{h}$ with vanishing Nijenhuis tensor $[\underline{h}, \underline{h}]$. This fact allows one to study the topology of manifolds which admit certain one-one tensor fields. The study of manifolds which admit a prescribed polynomial structure can be topologically analyzed with these tools. Some structures, such as the f-structure studied by Blair, Goldberg, Ludden, Yano and others, are singular, and consequently the methods presented here can give only information concerning the topology of submanifolds where f is nonsingular.

The theory described here for a nonsingular $\underline{h}$ is applicable to complex structures, Kähler structures, structures defined by the condition that $\underline{h}$ be covariant constant, and so forth. In addition the complex Laplacians studied by K. Kodaira, D. C. Spencer, Y. Ogawa, C. C. Hsiung and J. J. Levko III can all be extended to h-complex Laplacians.

Remark 2. For nonsingular $\underline{h}$ with vanishing Nijenhuis tensor it is also easy to see that the analytic index i_a (cf. [11]) of the operator $d_h + \delta_h$ (from Corollary 4.3) is equal to the Euler characteristic of M.

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