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ON THE EXTENDABILITY OF PROJECTIVE SURFACES AND A GENUS BOUND FOR ENRIQUES-FANO THREEFOLDS

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To the memory of Giulia Cerutti, Olindo Ado Lopez and Saúl Sánchez

Abstract

We introduce a technique based on Gaussian maps to study whether a surface can lie on a threefold as a very ample divisor with given normal bundle. We give applications, among which one to surfaces of general type and another to Enriques surfaces. In particular, we prove the genus bound $g \leq 17$ for Enriques-Fano threefolds. Moreover we find a new Enriques-Fano threefold of genus 9 whose normalization has canonical but not terminal singularities and does not admit $\mathbb{Q}$-smoothings.

1. Introduction

One of the most important contributions in algebraic geometry is the scheme of classification of higher dimensional varieties proposed by Mori theory. The latter is particularly clear in dimension three: starting with a threefold with terminal singularities and using contractions of extremal rays, the Minimal Model Program predicts to arrive either at a threefold X with K_X nef or at a Mori fiber space. Arguably the simplest case of such spaces is when X is a Fano threefold. As is well known, *smooth* Fano threefolds have been classified [**18**, **19**, **27**], while, in the singular case, a classification, or at least a search for the numerical invariants, is still underway.

In [**7**, **8**] a good part of the classification, in the smooth case, was recovered, using the point of view of Gaussian maps. The starting step of the latter method is Zak's theorem [**32**], [**24**, Thm. 0.1]: If $Y \subset \mathbb{P}^r$ is a smooth variety of codimension at least two with $h^0(N_{Y/\mathbb{P}^r}(-1)) \leq r+1$, then the only variety $X \subset \mathbb{P}^{r+1}$ that has Y as hyperplane section is a

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cone over Y. When this happens $Y \subset \mathbb{P}^r$ is said to be *nonextendable.* The key point in the application of this theorem is to calculate the cohomology of the normal bundle. It is here that Gaussian maps enter the picture by giving a big help in the case of curves [**31**, Prop. 1.10]: if Y is a curve then

$$h^0(N_{Y/\mathbb{P}^r}(-1)) = r + 1 + \operatorname{cork} \Phi_{H_Y,\omega_Y} \tag{1}$$

where Φ_{H_Y,ω_Y} is the Gaussian map associated to the canonical and hyperplane bundle H_Y of Y. For example when $X \subset \mathbb{P}^{r+1}$ is a smooth anticanonically embedded Fano threefold and Y is a general hyperplane section, $h^0(N_{Y/\mathbb{P}^r}(-1))$ was computed in [**7**] by considering the general curve section C of Y. That proof was strongly based on the fact that C is a general curve on a general K3 surface and that the Hilbert scheme of K3 surfaces is essentially irreducible. As the latter fact is peculiar to K3 surfaces, we immediately realized that if one imposes different hyperplane sections to a threefold, for example Enriques surfaces, it gets quite difficult to rely on the curve section. To study this and other cases one therefore needs an analogue of the formula (1) in higher dimension. We accomplish this in Section 2 by proving the following:

Theorem 1.1. *Let $Y \subset \mathbb{P}^r$ be a smooth irreducible linearly normal surface and let H be its hyperplane bundle. Assume there is a base-point free and big line bundle D_0 on Y with $H^1(H - D_0) = 0$ and such that the general element $D \in |D_0|$ is not rational and satisfies*

(i) *the Gaussian map $\Phi_{H_D,\omega_D(D_0)}$ is surjective;*
(ii) *the multiplication maps μ_{V_D,ω_D} and $\mu_{V_D,\omega_D(D_0)}$ are surjective*

where $V_D := \operatorname{Im}\{H^0(Y, H - D_0) \to H^0(D, (H - D_0)_{|D})\}$. Then

$$h^0(N_{Y/\mathbb{P}^r}(-1)) \leq r + 1 + \operatorname{cork} \Phi_{H_D,\omega_D}.$$

The above theorem is a flexible instrument to study threefolds whose hyperplane sections have large Picard group, since, if both D_0 and $H - D_0$ are base-point free and the degree of D is large with respect to its genus, the hypotheses are fulfilled unless D is hyperelliptic (note that the case where Y is a K3 and H has no moving decomposition has been considered by Mukai [**26**]).

As we will see in Section 3, Theorem 1.1 has several applications. A sample of this is for pluricanonical embeddings of surfaces of general type:

Corollary 1.2. *Let $Y \subset \mathbb{P}V_m$ be a minimal surface of general type with base-point free and nonhyperelliptic canonical bundle and $V_m \subseteq H^0(\mathcal{O}_Y(mK_Y + \Delta))$, where $\Delta \geq 0$ and either Δ is nef or Δ is reduced* *and K_Y is ample. Suppose that Y is regular or linearly normal and that*

$$m \geq \begin{cases} 9 & \text{if } K_Y^2 = 2; \\ 7 & \text{if } K_Y^2 = 3; \\ 6 & \text{if } K_Y^2 = 4 \text{ and the general curve in } |K_Y| \text{ is trigonal or if} \\ & K_Y^2 = 5 \text{ and the general curve in } |K_Y| \text{ is a plane quintic;} \\ 5 & \text{if either the general curve in } |K_Y| \text{ has Clifford index 2 or} \\ & 5 \leq K_Y^2 \leq 9 \text{ and the general curve in } |K_Y| \text{ is trigonal;} \\ 4 & \text{otherwise.} \end{cases}$$

Then Y is nonextendable.

In general, the conditions on K_Y^2 and m are optimal (see Remark 3.4).

Besides the applications in Section 3, we will concentrate our attention on the following threefolds:

Definition 1.3. An **Enriques-Fano threefold** is an irreducible three-dimensional variety $X \subset \mathbb{P}^N$ having a hyperplane section S that is a smooth Enriques surface, and such that X is not a cone over S. We will say that X has genus g if g is the genus of its general curve section.

Fano [**13**] claimed a classification of such threefolds, but his proof contains several gaps. Conte and Murre [**9**] remarked that an Enriques-Fano threefold must have some isolated singularities and, under special assumptions on the singularities, recovered some of the results of Fano, but not enough to give a classification, nor to bound the numerical invariants. Assuming that the Enriques-Fano threefold is a quotient of a *smooth* Fano threefold by an involution (this corresponds to having only cyclic quotient terminal singularities), a list was given by Bayle [**2**, Thm. A] and Sano [**29**, Thm. 1.1]. Moreover, by [**25**, MainThm. 2], any Enriques-Fano threefold with at most terminal singularities admits a $\mathbb{Q}$-smoothing, that is, it appears as central fiber of a small deformation over the 1-parameter unit disk such that a general fiber has only cyclic quotient terminal singularities. This, together with the results of Bayle and Sano, gives the bound $g \leq 13$ for Enriques-Fano threefolds with at most terminal singularities. Bayle and Sano recovered all of the known examples of Enriques-Fano threefolds. Therefore it has been conjectured that this list is complete or, at least, that the genus is bounded, in analogy with smooth Fano threefolds [**18**, **19**]. In Section 13, we will show that the list of known Enriques-Fano threefolds *is not complete* (not even after specialization), by finding a new Enriques-Fano threefold enjoying several peculiar properties (for a more precise version, see Proposition 13.1):

Proposition 1.4. *There exists an Enriques-Fano threefold* $X \subset \mathbb{P}^9$ *of genus* 9 *such that neither* X *nor its normalization belong to the list of Fano-Conte-Murre-Bayle-Sano.*

Moreover, X *does not have a* $\mathbb{Q}$*-smoothing and in particular* X *is not in the closure of the component of the Hilbert scheme made of Fano-Conte-Murre-Bayle-Sano's examples. Its normalization* $\widetilde{X}$ *has canonical but not terminal singularities and does not admit* $\mathbb{Q}$*-smoothings.*

Observe that $\widetilde{X}$ is a $\mathbb{Q}$-Fano threefold of Fano index 1 with canonical singularities not having a $\mathbb{Q}$-smoothing, showing that [**25**, MainThm. 2] cannot be extended to the canonical case.

In the present article (and [**20**]) we apply Theorem 1.1 to get a genus bound on Enriques-Fano threefolds, under no assumption on their singularities:

Theorem 1.5. *Any Enriques-Fano threefold has genus* $g \leq 17$*.*

A more precise result for $g = 15$ and 17 is proved in Proposition 12.1.

We remark that simultaneously and independently, Prokhorov [**28**] proved the same genus bound at the same time constructing an example of a genus 17 Enriques-Fano threefold, thus showing that the bound $g \leq 17$ is optimal. His methods, relying on the MMP, are completely different from ours.

Now a few words on our method of proof. In Section 4 we review some basic results. In Section 5 we apply Theorem 1.1 to Enriques surfaces and obtain the main results on nonextendability needed in the rest of the article. In Section 6 we prove Theorem 1.5 for all Enriques-Fano threefolds except for some concrete embedding line bundles on the Enriques surface section. These are handled case by case in Sections 7-11 by finding divisors satisfying the conditions of Theorem 1.1, thus allowing us to prove our theorem and a more precise statement for $g = 15$ and 17 in Section 12. A part of the proof for a special class of line bundles has been moved to the note [**20**]. This part involves no new ideas compared to the parts treated in the present article.

To prove our results we use criteria for the surjectivity of Gaussian maps on curves on Enriques surfaces from [**22**, **23**] and of multiplication maps of linear systems on such curves, which we obtain in Lemma 5.6 (which holds on any surface) in the present article.

2. Proof of Theorem 1.1

Let L and M be line bundles on a smooth projective variety. Given $V \subseteq H^0(L)$ we denote by $\mu_{V,M} : V \otimes H^0(M) \longrightarrow H^0(L \otimes M)$ the multiplication map of sections, $\mu_{L,M}$ when $V = H^0(L)$, and by $\Phi_{L,M}$: $\operatorname{Ker} \mu_{L,M} \longrightarrow H^0(\Omega^1_X \otimes L \otimes M)$ the Gaussian map [**31**, 1.1].

Proof of Theorem 1.1. To prove the bound on $h^0(N_{Y/\mathbb{P}^r}(-1))$, we use the short exact sequence

$$0 \longrightarrow N_{Y/\mathbb{P}^r}(-D_0-H) \longrightarrow N_{Y/\mathbb{P}^r}(-H) \longrightarrow N_{Y/\mathbb{P}^r}(-H)_{|D} \longrightarrow 0$$

and prove that

$$h^0(N_{Y/\mathbb{P}^r}(-D_0-H)) = 0, \text{ and} \tag{2}$$

$$h^0(N_{Y/\mathbb{P}^r}(-H)_{|D}) \le r+1+\operatorname{cork}\Phi_{H_D,\omega_D}. \tag{3}$$

To prove (2), note that since $\dim|D_0| \ge 1$, it is enough to have

$$h^0(N_{Y/\mathbb{P}^r}(-D_0-H)_{|D}) = 0 \quad \text{for a general } D \in |D_0|. \tag{4}$$

Now, setting $D_1 := D_0 + H$, (4) follows from the exact sequence

$$0 \longrightarrow N_{D/Y}(-D_1) \xrightarrow{\alpha} N_{D/\mathbb{P}^r}(-D_1) \longrightarrow N_{Y/\mathbb{P}^r}(-D_1)_{|D} \longrightarrow 0 \tag{5}$$

and the facts, proved below, that $h^0(N_{D/\mathbb{P}^r}(-D_1)) = 0$ and $H^1(\alpha)$ is injective.

To see that $h^0(N_{D/\mathbb{P}^r}(-D_1)) = 0$, we note that $\mu_{H_D,\omega_D(D_0)}$ is surjective by the H^0-lemma [**15**, Thm. 4.e.1], since $|D_{0|D}|$ is base-point free, whence $D_0^2 \ge 2$, therefore $h^1(\omega_D(D_0-H)) = h^0((H-D_0)_{|D}) \le h^0(H_D)-2$, as H_D is very ample. Now let $\mathbb{P}^k \subseteq \mathbb{P}^r$ be the linear span of D. Then

$$0 \longrightarrow N_{D/\mathbb{P}^k}(-D_1) \longrightarrow N_{D/\mathbb{P}^r}(-D_1) \longrightarrow (-D_0)_{|D}^{\oplus(r-k)} \longrightarrow 0 \tag{6}$$

implies that $h^0(N_{D/\mathbb{P}^r}(-D_1)) = h^0(N_{D/\mathbb{P}^k}(-D_1))$, as $D_0^2 > 0$. Since Y is linearly normal and $H^1(H-D_0) = 0$, also D is linearly normal. As $\mu_{H_D,\omega_D(D_0)}$ is surjective, $h^0(N_{D/\mathbb{P}^k}(-D_1)) = \operatorname{cork}\Phi_{H_D,\omega_D(D_0)} = 0$ by [**31**, Prop. 1.10] because of (i). Hence $h^0(N_{D/\mathbb{P}^r}(-D_1)) = 0$. As for the injectivity of $H^1(\alpha)$, we prove the surjectivity of $H^1(\alpha)^*$ with the help of the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{I}_{D/\mathbb{P}^r}(H)) \otimes H^0(\omega_D(D_0)) & \longrightarrow & H^0(N^*_{D/\mathbb{P}^r} \otimes \omega_D(D_1)) \\ \downarrow f & & \downarrow H^1(\alpha)^* \\ H^0(\mathcal{I}_{D/Y}(H)) \otimes H^0(\omega_D(D_0)) & \xrightarrow{h} & H^0(N^*_{D/Y} \otimes \omega_D(D_1)). \end{array} \tag{7}$$

Here f is surjective by linear normality of Y, while h is surjective by (ii) since it factors as the composition of the (surjective) restriction map $H^0(\mathcal{J}_{D/Y}(H)) \otimes H^0(\omega_D(D_0)) \to V_D \otimes H^0(\omega_D(D_0))$ and the multiplication map $\mu_{V_D,\omega_D(D_0)} : V_D \otimes H^0(\omega_D(D_0)) \to H^0(N^*_{D/Y} \otimes \omega_D(D_1))$.

Finally, to prove (3), recall that μ_{H_D,ω_D} is surjective by [**1**, Thm. 1.6] since D is not rational, whence $h^0(N_{D/\mathbb{P}^k}(-H)) = k+1+\operatorname{cork}\Phi_{H_D,\omega_D}$

by [**31**, Prop. 1.10]. Therefore, twisting (6) by $(D_0)_{|D}$, we find that $h^0(N_{D/\mathbb{P}^r}(-H)) \leq r+1+\operatorname{cork}\Phi_{H_D,\omega_D}$ and (3) will follow by the sequence (5) tensored by $(D_0)_{|D}$ and injectivity of $H^1(\alpha\otimes(D_0)_{|D})$, which is proved exactly as the injectivity of $H^1(\alpha)$ above, using now the surjectivity of μ_{V_D,ω_D}. q.e.d.

Remark 2.1. In the above proposition and also in Corollary 2.2 below, the surjectivity of $\mu_{V_D,\omega_D(D_0)}$ can be replaced by either one of the following conditions: (i) $\mu_{\omega_D(H-D_0),D_{0|D}}$ is surjective; (ii) $h^0((2D_0-H)_{|D}) \leq h^0(D_{0|D})-2$; (iii) $H.D_0 > 2D_0^2$. Indeed, condition (iii) implies $h^0((2D_0-H)_{|D}) = 0$, whence (ii), while (ii) implies (i) by the H^0-lemma [**15**, Thm. 4.e.1]. Finally, (i) is enough by surjectivity of μ_{V_D,ω_D} and the commutative diagram

$$\begin{array}{ccc} V_D \otimes H^0(\omega_D) \otimes H^0(D_{0|D}) & \xrightarrow{\mu_{V_D,\omega_D}\otimes \mathrm{Id}} & H^0(\omega_D(H-D_0)) \otimes H^0(D_{0|D}) \\ \downarrow & & \downarrow{\scriptstyle \mu_{\omega_D(H-D_0),D_{0|D}}} \\ V_D \otimes H^0(\omega_D(D_0)) & \xrightarrow{\mu_{V_D,\omega_D(D_0)}} & H^0(\omega_D(H)). \end{array}$$

Whereas the upper bound in Theorem 1.1 can be applied to control how many times Y can be extended to higher dimensional varieties, we will concentrate on the case of *one* simple extension.

Corollary 2.2. *Let $Y \subset \mathbb{P}^r$ be a smooth irreducible surface which is linearly normal or regular and let H be its hyperplane bundle. Assume there is a base-point free and big line bundle D_0 on Y with $H^1(H-D_0) = 0$ and such that the general element $D \in |D_0|$ is not rational and satisfies*

(i) *the Gaussian map Φ_{H_D,ω_D} is surjective;*
(ii) *the multiplication maps μ_{V_D,ω_D} and $\mu_{V_D,\omega_D(D_0)}$ are surjective,*

where $V_D := \operatorname{Im}\{H^0(Y,H-D_0) \to H^0(D,(H-D_0)_{|D})\}$.

Then Y is nonextendable.

Proof. Note that $g(D) \geq 2$, as Φ_{H_D,ω_D} is surjective. Since $\mu_{V_D,\omega_D(D_0)}$ is surjective, we have that V_D (whence also $|(H-D_0)_{|D}|$) is base-point free, as $|\omega_D(H)|$ is such. Therefore $2g(D)-2+(H-D_0).D > 0$, whence $h^1(\omega_D^2(H-D_0)) = 0$, and the H^0-lemma [**15**, Thm. 4.e.1] implies that $\mu_{\omega_D^2(H),D_{0|D}}$ is surjective. Now $\Phi_{H_D,\omega_D(D_0)}$ is surjective by (i) and the commutative diagram

$$\begin{array}{ccc} \operatorname{Ker}\mu_{H_D,\omega_D} \otimes H^0(D_{0|D}) & \xrightarrow{\Phi_{H_D,\omega_D}\otimes \mathrm{Id}} & H^0(\omega_D^2(H)) \otimes H^0(D_{0|D}) \\ \downarrow & & \downarrow{\scriptstyle \mu_{\omega_D^2(H),D_{0|D}}} \\ \operatorname{Ker}\mu_{H_D,\omega_D(D_0)} & \xrightarrow{\Phi_{H_D,\omega_D(D_0)}} & H^0(\omega_D^2(H+D_0)). \end{array}$$

If Y is linearly normal, we are done by Zak's theorem [**32**] and Theorem 1.1.

Assume now that $h^1(\mathcal{O}_Y) = 0$ and that $Y \subset \mathbb{P}^r$ is extendable, that is, that Y is a hyperplane section of some nondegenerate threefold $X \subset \mathbb{P}^{r+1}$ which is not a cone over Y. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities and let $L = \pi^*\mathcal{O}_X(1)$ and $\widetilde{Y} = \pi^{-1}(Y) \cong Y$, as Y is smooth, so that $Y \cap \operatorname{Sing} X = \emptyset$. Using $H^1(\mathcal{O}_{\widetilde{Y}}) = H^1(\mathcal{O}_Y) = 0$ and Kawamata-Viehweg vanishing, one easily deduces the surjectivity of the restriction map $H^0(\widetilde{X}, L) \to H^0(\widetilde{Y}, L_{|\widetilde{Y}})$. Consider the birational map $\varphi_L : \widetilde{X} \to \mathbb{P}^N$ where $N \geq r+1$. Then $\overline{Y} := \varphi_L(\widetilde{Y}) \cong Y$ is a hyperplane section of $\varphi_L(\widetilde{X})$ that is linearly normal and extendable and we reduce to the linearly normal case above. q.e.d.

3. Absence of Veronese embeddings on threefolds

It was known to Scorza [**30**] that the Veronese varieties $v_m(\mathbb{P}^n)$ are nonextendable for $m, n > 1$. For an arbitrary Veronese embedding we can use Zak's theorem [**32**], [**24**, Thm. 0.1] as follows:

Remark 3.1. Let $X \subset \mathbb{P}^r$ be a smooth irreducible nondegenerate n-dimensional variety, $n \geq 2$, $L = \mathcal{O}_X(1)$ and let $\varphi_{mL}(X) \subset \mathbb{P}^N$ be the m-th Veronese embedding of X.

If $H^1(T_X(-mL)) = 0$ then $\varphi_{mL}(X)$ is nonextendable. In particular the latter holds if $m > \max\left\{2, n+2+\frac{K_X.L^{n-1}-2r+2n+2}{L^n}\right\}$.

Proof. Set $Y = \varphi_{mL}(X)$. From standard sequences and Kodaira vanishing one gets $h^0(N_{Y/\mathbb{P}^N}(-1)) \leq h^0(T_{\mathbb{P}^N}(-1)_{|Y}) + h^1(T_Y(-1)) = N+1+h^1(T_X(-mL)) = N+1$, and we just apply Zak's theorem [**32**].

To see the last assertion, observe that since $n \geq 2$ and $m \geq 3$ we have, as is well-known, $h^1(T_X(-mL)) = h^0(N_{X/\mathbb{P}^r}(-mL))$. If the latter were not zero, the same would hold for a general curve section $C \subset \mathbb{P}^{r-n+1}$. Taking $r-n-1$ general points $x_j \in C$, we get from the exact sequence [**4**, (2.7)] that $h^0(N_{C/\mathbb{P}^{r-n+1}}(-m)) = 0$ for reasons of degree, a contradiction. q.e.d.

In the case of surfaces, Corollary 2.2 yields an extension of this remark:

Definition 3.2. Let Y be a smooth surface and let L be an effective line bundle on Y such that the general divisor $D \in |L|$ is smooth and irreducible. We say that L is **hyperelliptic, trigonal**, etc., if D is such. We denote by $\operatorname{Cliff}(L)$ the Clifford index of D. Moreover, when $L^2 > 0$, we set

$$\varepsilon(L) = 3 \text{ if } L \text{ is trigonal; } \varepsilon(L) = 5 \text{ if } \operatorname{Cliff}(L) \geq 3;$$

$$\varepsilon(L) = 0 \text{ if } \operatorname{Cliff}(L) = 2;$$

$$m(L)=\begin{cases}\frac{16}{L^2} & \text{if } L.(L+K_Y)=4;\\ \frac{25}{L^2} & \text{if } L.(L+K_Y)=10 \text{ and the general}\\ & \text{divisor in } |L| \text{ is a plane quintic;}\\ \frac{3L.K_Y+18}{2L^2}+\frac{3}{2} & \text{if } 6\le L.(L+K_Y)\le 22 \text{ and } L \text{ is trigonal;}\\ \frac{2L.K_Y-\varepsilon(L)}{L^2}+2 & \text{otherwise.}\end{cases}$$

Corollary 3.3. *Let $Y \subset \mathbb{P}V$ be a smooth surface with $V \subseteq H^0(L^{\otimes m} \otimes \mathcal{O}_Y(\Delta))$, where L is a base-point free, big, nonhyperelliptic line bundle on Y with $L.(L+K_Y) \geq 4$ and $\Delta \geq 0$ is a divisor. Suppose that Y is regular or linearly normal and that m is such that $H^1(L^{\otimes(m-2)} \otimes \mathcal{O}_Y(\Delta)) = 0$ and $m > \max\{m(L) - \frac{L.\Delta}{L^2}, \lceil \frac{L.K_Y+2-L.\Delta}{L^2} \rceil + 1\}$. Then Y is nonextendable.*

Proof. We apply Corollary 2.2 with $D_0 = L$ and $H = L^{\otimes m} \otimes \mathcal{O}_Y(\Delta)$. By hypothesis the general $D \in |L|$ is smooth and irreducible of genus $g(D) = \frac{1}{2}L.(L+K_Y)+1$. Since $H^1(H-2L) = 0$, we have $V_D = H^0((H-L)_{|D})$. Also $(H-L).D = (m-1)L^2 + L.\Delta \geq L.(L+K_Y)+2 = 2g(D)$ by hypothesis, whence $|(H-L)_{|D}|$ is base-point free and birational (as D is not hyperelliptic) and μ_{V_D,ω_D} is surjective by [**1**, Thm. 1.6]. Moreover $H^1((H-L)_{|D}) = 0$, whence also $H^1(H-L) = 0$.

The surjectivity of $\mu_{V_D,\omega_D(L)}$ follows by [**15**, Cor. 4.e.4], as $\deg \omega_D(L) \geq 2g(D)+1$ because $L^2 \geq 3$. Indeed, if $L^2 \leq 2$ we have that $h^0(L_{|D}) \leq 1$ as D is not hyperelliptic, whence $h^0(L) \leq 2$, contradicting the hypotheses on L. The surjectivity of Φ_{H_D,ω_D} follows by the inequality $m > m(L) - \frac{L.\Delta}{L^2}$ and well-known results about Gaussian maps (see e.g. [**31**, Prop. 1.10], [**23**, Prop. 2.9, Prop. 2.11 and Cor. 2.10], [**4**, Thm. 2]). q.e.d.

We can be a little bit more precise in the case of pluricanonical embeddings:

Proof of Corollary 1.2. We apply Corollary 3.3 with $L = \mathcal{O}_Y(K_Y)$ and $H = \mathcal{O}_Y(mK_Y + \Delta)$ and prove that $H^1(\mathcal{O}_Y((m-2)K_Y + \Delta)) = 0$. If Δ is nef this follows by Kawamata-Viehweg vanishing. Now suppose that Δ is reduced and K_Y is ample. Again $H^1(\mathcal{O}_Y((m-2)K_Y)) = 0$, whence $H^1(\mathcal{O}_Y((m-2)K_Y + \Delta)) = 0$, since $h^1(\mathcal{O}_\Delta((m-2)K_Y + \Delta)) = h^0(\mathcal{O}_\Delta(-(m-3)K_Y)) = 0$. q.e.d.

Remark 3.4. Consider the 5-uple embedding X of $\mathbb{P}^3$ into $\mathbb{P}^{55}$ (respectively, the 4-uple embedding of a smooth quadric hypersurface in $\mathbb{P}^4$ into $\mathbb{P}^{54}$). A general hyperplane section Y of X is embedded with $5K_Y$ (resp. $4K_Y$) and $K_Y^2 = 5$ (resp. $K_Y^2 = 8$). Thus, in Corollary 1.2, the conditions on K_Y^2 and m cannot, in general, be weakened.

We can be even more precise in the case of adjoint embeddings.

Corollary 3.5. *Let $Y \subset \mathbb{P}V$ be a minimal surface of general type with base-point free and nonhyperelliptic canonical bundle and $V \subseteq H^0(L \otimes \mathcal{O}_Y(K_Y + \Delta))$, where L is a line bundle on Y and $\Delta \geq 0$ is a divisor. Suppose that Y is regular or linearly normal, that $H^1(L \otimes \mathcal{O}_Y(\Delta - K_Y)) = 0$ and that*

$$L.K_Y + K_Y.\Delta > \begin{cases} 14 & \text{if } K_Y^2 = 2; \\ 20 & \text{if } K_Y^2 = 5 \text{ and the general divisor} \\ & \text{in } |K_Y| \text{ is a plane quintic;} \\ 2K_Y^2 + 9 & \text{if } 3 \leq K_Y^2 \leq 11 \text{ and } K_Y \text{ is trigonal;} \\ 3K_Y^2 - \varepsilon(K_Y) & \text{otherwise.} \end{cases}$$

Then Y is nonextendable.

Proof. Similar to the proof of Corollary 3.3 with $D_0 = \mathcal{O}_Y(K_Y)$ and $H = L \otimes \mathcal{O}_Y(K_Y + \Delta)$. q.e.d.

To state the pluriadjoint case, given a big line bundle L on a smooth surface Y, we define

$$\nu(L) = \begin{cases} \frac{12}{L^2} + 1 & \text{if } L.(L + K_Y) = 4; \\ \frac{15}{L^2} + 1 & \text{if } L.(L + K_Y) = 10 \text{ and the general divisor} \\ & \text{in } |L| \text{ is a plane quintic;} \\ \frac{L.K_Y + 18}{2L^2} + \frac{3}{2} & \text{if } 6 \leq L.(L + K_Y) \leq 22 \text{ and } L \text{ is trigonal;} \\ \frac{L.K_Y - \varepsilon(L)}{L^2} + 2 & \text{otherwise.} \end{cases}$$

Corollary 3.6. *Let $Y \subset \mathbb{P}V$ be a smooth surface with $V \subseteq H^0(L^{\otimes m} \otimes \mathcal{O}_Y(K_Y + \Delta))$ where L is a base-point free, big and nonhyperelliptic line bundle on Y with $L.(L + K_Y) \geq 4$ and $\Delta \geq 0$ is a divisor such that $H^1(L^{\otimes(m-2)} \otimes \mathcal{O}_Y(K_Y + \Delta)) = 0$. Suppose that Y is regular or linearly normal and that $m > \max\{2 + \frac{1}{L^2}, \nu(L)\} - \frac{L.\Delta}{L^2}$. Then Y is nonextendable.*

Proof. Similar to the proof of Corollary 3.3 with $D_0 = L$ and $H = L^{\otimes m} \otimes \mathcal{O}_Y(K_Y + \Delta)$. q.e.d.

4. Basic results on line bundles on Enriques surfaces

Definition 4.1. Let S be an Enriques surface. If D is a divisor on S we will denote by $H^i(D)$ the cohomology $H^i(\mathcal{O}_S(D))$. We denote by $\sim$ (respectively $\equiv$) the linear (respectively numerical) equivalence of divisors (or line bundles) on S. A line bundle L is **primitive** if $L \equiv hL'$ for some line bundle L' and some integer h, implies $h = \pm 1$. An effective line bundle L is **quasi-nef** [**21**] if $L^2 \geq 0$ and $L.\Delta \geq -1$ for every Δ such that $\Delta > 0$ and $\Delta^2 = -2$.

A **nodal** curve is a smooth rational curve. A **nodal cycle** is a divisor $R > 0$ such that $(R')^2 \leq -2$ for any $0 < R' \leq R$. An **isotropic divisor**

F is a divisor such that $F^2 = 0$ and $F \not\equiv 0$. An **isotropic k-sequence** is a set $\{f_1, \ldots, f_k\}$ of isotropic divisors such that $f_i.f_j = 1$ for $i \neq j$.

We will often use the fact that if R is a nodal cycle, then $h^0(\mathcal{O}_S(R)) = 1$ and $h^0(\mathcal{O}_S(R + K_S)) = 0$.

Let L be a line bundle on S with $L^2 > 0$. Following [**11**] we define $\phi(L) = \inf\{|F.L| \ : \ F \in \operatorname{Pic} S, F^2 = 0, F \not\equiv 0\}$. One has $\phi(L)^2 \leq L^2$ [**11**, Cor. 2.7.1] and, if L is nef, then there exists a genus one pencil $|2E|$ such that $E.L = \phi(L)$ [**10**, 2.11]. Moreover we will extensively use the fact that if L is nef, then it is base-point free if and only if $\phi(L) \geq 2$ [**11**, Prop. 3.1.6, 3.1.4 and Thm. 4.4.1].

A line bundle $L > 0$ with $L^2 \geq 0$ on S has a (nonunique) decomposition $L \equiv a_1E_1 + \ldots + a_nE_n$, where a_i are positive integers, and each E_i is primitive, effective and isotropic, cf. e.g. [**22**, Lemma 2.12]. We will call such a decomposition an **arithmetic genus 1 decomposition**.

Definition 4.2. An effective line bundle L with $L^2 \geq 0$ is said to be of **small type** if either $L = 0$ or if in every arithmetic genus 1 decomposition of L as above, all $a_i = 1$.

The next result is an easy (computational) consequence of [**21**, Lemma 2.1] and [**22**, Lemma 2.4].

Lemma 4.3. *Let L be an effective line bundle on an Enriques surface with $L^2 \geq 0$. Then L is of small type if and only if it is of one of the following types (where $E_i > 0$, $E_i^2 = 0$ and E_i primitive):* (a) $L = 0$; (b) $L^2 = 0$, $L \sim E_1$; (c) $L^2 = 2$, $L \sim E_1 + E_2$, $E_1.E_2 = 1$; (d) $L^2 = 4$, $\phi(L) = 2$, $L \sim E_1 + E_2$, $E_1.E_2 = 2$; (e) $L^2 = 6$, $\phi(L) = 2$, $L \sim E_1 + E_2 + E_3$, $E_1.E_2 = E_1.E_3 = E_2.E_3 = 1$; (f) $L^2 = 10$, $\phi(L) = 3$, $L \sim E_1 + E_2 + E_3$, $E_1.E_2 = 1$, $E_1.E_3 = E_2.E_3 = 2$.

Among all arithmetic genus 1 decompositions of an effective line bundle L with $L^2 > 0$, we want to choose the most convenient for our purposes. For any line bundle $L > 0$ which is *not of small type* with $L^2 > 0$ and $\phi(L) = F.L$ for some $F > 0$ with $F^2 = 0$, define

(8) $\alpha_F(L) = \min\{k \geq 2 \mid (L - kF)^2 \geq 0$ and if $(L - kF)^2 > 0,$ then there exists $F' > 0$ with $(F')^2 = 0, F'.F > 0$ and $F'.(L - kF) \leq \phi(L)\}$.

By [**22**, Lemma 2.4], it is easy to see that $\alpha_F(L)$ exists and that one obtains an equivalent definition by replacing the last inequality by $F'.(L - kF) = \phi(L - kF)$.

If $L^2 = 0$ and L is *not of small type*, then we define $\alpha_F(L)$ to be the maximal integer $k \geq 2$ such that there exists an isotropic F such that $L \equiv kF$. The next result is an easy computation.

Lemma 4.4. *Let L be an effective line bundle not of small type with $L^2 > 0$ and $(L^2, \phi(L)) \neq (16, 4), (12, 3), (8, 2), (4, 1)$. Then $(L - \alpha_F(L)F)^2 > 0$.*

We will also use the following consequence of [**11**, Prop. 3.1.4], [**21**, Cor. 2.5] and [**22**, Lemma 2.3]:

Lemma 4.5. *Let L be a nef and big line bundle on an Enriques surface and let F be a divisor satisfying $F.L < 2\phi(L)$ (respectively $F.L = \phi(L)$ and L is ample). Then $h^0(F) \leq 1$ and if $F > 0$ and $F^2 \geq 0$ we have $F^2 = 0$, $h^0(F) = 1$, $h^1(F) = 0$ and F is primitive and quasi-nef (resp. nef).*

5. Main results on extendability of Enriques surfaces

It is well-known that abelian and hyperelliptic surfaces are nonextendable [**14**, Rmk. 3.12]. The extendability problem is open for $K3$'s, but answers are known for general $K3$'s [**7**, **8**, **3**]. Let us deal now with Enriques surfaces.

We start with a simplification of Corollary 2.2 that will be central to us.

Proposition 5.1. *Let $S \subset \mathbb{P}^r$ be an Enriques surface and H its hyperplane bundle. Suppose there is a nef and big (whence effective) line bundle D_0 on S with $\phi(D_0) \geq 2, H^1(H-D_0) = 0$ and such that the following conditions are satisfied by the general element $D \in |D_0|$:*

(i) *the Gaussian map Φ_{H_D,ω_D} is surjective;*
(ii) *the multiplication map μ_{V_D,ω_D} is surjective, where* $V_D := \mathrm{Im}\{H^0(S, H-D_0) \to H^0(D, (H-D_0)_{|D})\}$;
(iii) $h^0((2D_0 - H)_{|D}) \leq \frac{1}{2}D_0^2 - 2$.

Then S is nonextendable.

Proof. Apply Corollary 2.2 and Remark 2.1, using that D_0 is base-point free since $\phi(D_0) \geq 2$. q.e.d.

Our first observation will be that, for many line bundles H, a line bundle D_0 satisfying the conditions of Proposition 5.1 can be found with the help of Ramanujam's vanishing theorem.

Proposition 5.2. *Let $S \subset \mathbb{P}^r$ be an Enriques surface such that its hyperplane section H is not 2-divisible in* Num S. *Suppose there exists an effective divisor B on S satisfying:*

(i) $B^2 \geq 4$ *and* $\phi(B) \geq 2$,
(ii) $(H-2B)^2 \geq 0$ *and* $H - 2B \geq 0$,
(iii) $H^2 \geq 64$ *if* $B^2 = 4$ *and* $H^2 \geq 54$ *if* $B^2 = 6$.

Then S is nonextendable.

Proof. We first claim that there is a *nef* divisor $D' > 0$ satisfying (i)-(iii) and with $D' \leq B$, $(D')^2 = B^2, \phi(D') = \phi(B)$. Indeed, if Γ is a nodal curve, define the Picard-Lefschetz reflection on Pic S as $\pi_\Gamma(L) := L + (L.\Gamma)\Gamma$. Then π_Γ preserves intersections, effectiveness

[**5**, Prop. VIII.16.3] and the function ϕ. Now if B is not nef, there is a nodal Γ such that $\Gamma.B < 0$. Since $0 < \pi_\Gamma(B) < B$, we see that $\pi_\Gamma(B)$ satisfies (i)-(iii). If $\pi_\Gamma(B)$ is not nef, we repeat the process, which must end, as $\pi_\Gamma(B) < B$, and we get the desired nef D'. Since $H - D' \geq H - B > H - 2B \geq 0$ and $(D')^2 > 0$, we have $D'.(H - D') > 0$. Now define the following set, which is nonempty, by what we just saw,

$$\Omega(D') = \{M \in \operatorname{Pic} S : M \geq D', M \text{ is nef, satisfies (i)-(ii) and } M.(H - M) \leq D'.(H - D')\}.$$

For any $M \in \Omega(D')$ we have $H - 2M > 0$, whence $H.M$ is bounded. Let then D_0 be a *maximal* divisor in $\Omega(D')$, that is, such that $H.D_0$ is maximal. We want to show that $h^1(H - 2D_0) = 0$.

Set $R := H - 2D_0$. If $h^1(R) > 0$, then by Ramanujam vanishing [**5**, Cor. II.12.3] we could write $R + K_S \sim R_1 + R_2$, for $R_1 > 0$ and $R_2 > 0$ with $R_1.R_2 \leq 0$. We can assume that $R_1.H \leq R_2.H$. If $D_1 := D_0 + R_1$ is nef, then $\phi(D_1)$ is calculated by a nef divisor, whence $\phi(D_1) \geq \phi(D') \geq 2$ and $D_1^2 \geq D_0^2 \geq (D')^2 \geq 4$ (since $D_1 \geq D_0 \geq D'$). Moreover $(H - 2D_1)^2 = R^2 - 4R_1.R_2 \geq R^2 \geq 0$, and since $(H - 2D_1).H = (R_2 - R_1).H \geq 0$, we get by Riemann-Roch and the fact that H is not 2-divisible in $\operatorname{Num} S$, that $H - 2D_1 > 0$. Furthermore, $D_1.(H - D_1) = D_0.(H - D_0) + R_1.R_2 \leq D_0.(H - D_0)$, whence $D_1 \in \Omega(D')$ with $H.D_1 > H.D_0$, contradicting the maximality of D_0.

Hence D_1 cannot be nef and there exists a nodal curve Γ with $\Gamma.D_1 < 0$ (whence $\Gamma.R_1 < 0$). Since H is ample, we have $\Gamma.(H - D_1) \geq -\Gamma.D_1 + 1 \geq 2$. Since $\Gamma.R_1 < 0$, we have $D_2 := D_1 - \Gamma \geq D_0$, whence, if D_2 is nef, we have as above that $\phi(D_2) \geq \phi(D') \geq 2$ and $D_2^2 \geq D_0^2 \geq (D')^2 \geq 4$. Moreover $H - 2D_2 > H - 2D_1 > 0$ and $(H - 2D_2)^2 = (H - 2D_1)^2 - 8 + 4(H - 2D_1).\Gamma \geq (H - 2D_1)^2 + 4 > 0$. Furthermore $D_2.(H - D_2) < D_1.(H - D_1) \leq D_0.(H - D_0)$, whence $D_2 \neq D_0$. If D_2 is nef, then $D_2 \in \Omega(D')$ with $H.D_2 > H.D_0$, a contradiction. Hence D_2 is not nef, and we repeat the process, with a nodal $\Gamma_1 \leq R_1 - \Gamma$. As the process must end, we get $h^1(H - 2D_0) = 0$.

Note that since $D_0^2 \geq (D')^2 = B^2$, then D_0 also satisfies (iii) above. Furthermore D_0 is base-point free since it is nef with $\phi(D_0) \geq \phi(D') \geq 2$. Let $D \in |D_0|$ be a general smooth curve. We have $\deg(H - D_0)_{|D} = D_0^2 + (H - 2D_0).D_0 \geq D_0^2 + \phi(D_0) \geq 2g(D)$. As D is not hyperelliptic, $(H - D_0)_{|D}$ is base-point free and birational, whence $\mu_{(H-D_0)_{|D},\omega_D}$ is surjective by [**1**, Thm. 1.6].

Since $h^1(H - 2D_0) = 0$ and $h^1(\mathcal{O}_D(H - D_0)) = 0$ for reasons of degree, we find $h^1(H - D_0) = 0$. To prove the proposition, we only have left to show, by Proposition 5.1, that Φ_{H_D,ω_D} is surjective.

From $(H - 2D_0).D_0 \geq 2$ again, we get $\deg H_D \geq 4g(D) - 2$, whence Φ_{H_D,ω_D} is surjective if $\operatorname{Cliff}(D) \geq 2$ by [**4**, Thm. 2]. This is satisfied if $D_0^2 \geq 8$ by [**22**, Cor. 1.5 and Prop. 4.13].

If $D_0^2 = 6$, then $g(D) = 4$, whence Φ_{H_D,ω_D} is surjective if we have $h^0(\mathcal{O}_D(3D_0 + K_S - H)) = 0$ by [**31**, Prop. 1.10]. Since $H^2 \geq 54$, we get by Hodge index that $H.D \geq 18$ with equality if and only if $H \equiv 3D_0$. If $H.D_0 > 18$, we get $\deg \mathcal{O}_D(3D_0 + K_S - H) < 0$. If $H \equiv 3D_0$, then we may assume $H \sim 3D_0$, possibly after exchanging D_0 with $D_0 + K_S$, so that $h^0(\mathcal{O}_D(3D_0 + K_S - H)) = h^0(\mathcal{O}_D(K_S)) = 0$. If $D_0^2 = 4$, then $g(D) = 3$, whence Φ_{H_D,ω_D} is surjective by [**31**, Prop. 1.10] as $h^0(\mathcal{O}_D(4D_0 - H)) = 0$. Indeed, since $H^2 \geq 64$, we get by Hodge index that $H.D \geq 17$, whence $\deg \mathcal{O}_D(4D_0 - H) < 0$. q.e.d.

We now improve Proposition 5.2 in the cases $B^2 = 4$ and 6, using [**23**].

Proposition 5.3. *Let $S \subset \mathbb{P}^r$ be an Enriques surface such that its hyperplane section H is not* 2*-divisible in* $\operatorname{Num} S$*. Suppose there exists an effective divisor B on S satisfying:* (i) $B^2 = 6$ *and* $\phi(B) = 2$*,* (ii) $(H - 2B)^2 \geq 0$ *and* $H - 2B \geq 0$*,* (iii) $h^0(3B - H) = 0$ *or* $h^0(3B + K_S - H) = 0$*. Then S is nonextendable.*

Proof. Argue exactly as in the proof of Proposition 5.2 and let D', D_0 and D be as in that proof, so that, in particular, $D_0^2 \geq (D')^2 = 6$. If $D_0^2 \geq 8$, we are done by Proposition 5.2. If $D_0^2 = 6$ write $D_0 = D' + M$ with $M \geq 0$. Since both D_0 and D' are nef we find $6 = D_0^2 = (D')^2 + D'.M + D_0.M \geq 6$, whence $D'.M = D_0.M = 0$, so that $M^2 = 0$. Therefore $M = 0$ and $D_0 = D'$, whence $3D_0 - H \sim 3D' - H \leq 3B - H$. It follows that either $h^0(3D_0 - H) = 0$ or $h^0(3D_0 + K_S - H) = 0$. Possibly after exchanging D_0 with $D_0 + K_S$, we can assume that $h^0(3D_0 + K_S - H) = 0$. As $h^1(2D_0 + K_S - H) = h^1(H - 2D_0) = 0$, we get $h^0(\mathcal{O}_D(3D_0 + K_S - H)) = 0$, whence Φ_{H_D,ω_D} is surjective by [**23**, Thm(ii)]. The map μ_{V_D,ω_D} is surjective as in the previous proof. q.e.d.

Proposition 5.4. *Let $S \subset \mathbb{P}^r$ be an Enriques surface such that its hyperplane section H is not* 2*-divisible in* $\operatorname{Num} S$*. Suppose there exists an effective divisor B on S satisfying:* (i) *B is nef,* $B^2 = 4$ *and* $\phi(B) = 2$*,* (ii) $(H - 2B)^2 \geq 0$ *and* $H - 2B \geq 0$*,* (iii) $H.B > 16$*. Then S is nonextendable.*

Proof. Argue as in the proof of Proposition 5.2 and let D', D_0 and D be as in that proof. By (i) we have $D' = B$, and since $D_0 \geq D'$, we get $H.D_0 > 16$. If $D_0^2 \geq 8$, we are done by Proposition 5.2. If $D_0^2 = 6$, then $D_0 > D' = B$, so that $H.D_0 \geq 18$ whence $(3D_0 - H).D_0 \leq 0$. If $3D_0 - H > 0$, it is a nodal cycle, whence either $h^0(3D_0 - H) = 0$ or $h^0(3D_0 + K_S - H) = 0$ and we are done by Proposition 5.3. If $D_0^2 = 4$, then $D_0 = D' = B$ and $\deg \mathcal{O}_D(4D_0 - H) < 0$ as in the proof of Proposition 5.3, whence Φ_{H_D,ω_D} is surjective by [**23**, Thm(i)] and so is μ_{V_D,ω_D}, as in the proof of Proposition 5.2. q.e.d.

In several cases the following will be very useful:

Lemma 5.5. *Let $S \subset \mathbb{P}^r$ be an Enriques surface with hyperplane section $H \sim 2B + A$, for B nef, $B^2 \geq 2$, $A^2 = 0$, $A > 0$ primitive, $H^2 \geq 28$ and satisfying one of the following conditions:*

(i) *A is quasi-nef and $(B^2, A.B) \notin \{(4,3),(6,2)\}$;*
(ii) *$\phi(B) \geq 2$ and $(B^2, A.B) \notin \{(4,3),(6,2)\}$;*
(iii) *$\phi(B) = 1$, $B^2 = 2l$, $B \sim lF_1 + F_2$, $l \geq 1$, $F_i > 0$, $F_i^2 = 0$, $i = 1,2$, $F_1.F_2 = 1$, and either*
  (a) *$l \geq 2$, $F_i.A \leq 3$ for $i = 1,2$ and $(l, F_1.A, F_2.A) \neq (2,1,1)$; or*
  (b) *$l = 1$, $5 \leq B.A \leq 8$, $F_i.A \geq 2$ for $i = 1,2$ and $(\phi(H), F_1.A, F_2.A) \neq (6,4,4)$.*

Then S is nonextendable.

Proof. Possibly after replacing B with $B + K_S$ if $B^2 = 2$ we can, without loss of generality, assume that B is base-component free.

We first prove the lemma under hypothesis (i).

One easily sees that $D_0 := B + A$ is nef, since A is quasi-nef and H is ample. Moreover, $D_0^2 = B^2 + 2B.A \geq 6$, as $2A.B = A.H \geq \phi(H) \geq 3$, since H is very ample. If $\phi(D_0) = 1 = F.D_0$ for some $F > 0$ with $F^2 = 0$, we get $F.B = 1, F.A = 0$ and the contradiction $F.H = 2$. Hence $\phi(D_0) \geq 2$.

One easily checks that (i) implies $D_0^2 \geq 12$. Since $h^0(2D_0 - H) = h^0(A) = 1$ by [**21**, Cor. 2.5], we have that Φ_{H_D,ω_D} is surjective by [**23**, Thm(iv)]. Also $h^1(H - 2D_0) = h^1(-A) = 0$, again by [**21**, Cor. 2.5], so that $V_D = H^0(\mathcal{O}_D(H - D_0))$. As $H - D_0 = B$ is base-component free and $|D_0|$ is base-point free and birational by [**11**, Lemma 4.6.2, Thm. 4.6.3 and Prop. 4.7.1], also V_D is base-point free and is either a complete pencil or birational. Hence μ_{V_D,ω_D} is surjective by the base-point free pencil trick [**1**, §1] and [**1**, Thm. 1.6]. Then S is nonextendable by Proposition 5.1.

Therefore the lemma is proved under the assumption (i) and, in particular, the whole lemma is proved with the additional assumption that A is quasi-nef.

Now assume that A is not quasi-nef. Then there is a $\Delta > 0$ with $\Delta^2 = -2$ and $\Delta.A \leq -2$. We have $\Delta.B \geq 2$ by the ampleness of H. Furthermore, among all such Δ's we will choose a minimal one, that is, such that no $0 < \Delta' < \Delta$ satisfies $(\Delta')^2 = -2$ and $\Delta'.A \leq -2$. Then one easily proves that $B_0 := B + \Delta$ is nef. Moreover, $B_0^2 \geq 2 + B^2$, and $\phi(B_0) \geq \phi(B)$. We also note that $H - 2B_0 \sim A - 2\Delta > 0$ and is primitive by [**22**, Lemma 2.3] with $(H - 2B_0)^2 \geq 0$.

Under the assumptions (ii), we have $\phi(B_0) \geq 2$. Then S is nonextendable by Proposition 5.2 if $B_0^2 \geq 8$. If $B_0^2 = 6$, we have $B^2 = 4$ and $\Delta.B = 2$, so that $\Delta.A = -2$ or -3 by the ampleness of H. Hence $H \sim 2B_0 + A'$, with $B_0^2 = 6$ and $A' \sim A - 2\Delta$ satisfies $(A')^2 = 0$ or 4. In the first case we are done by conditions (i) if A' is quasi-nef, and if not we can just repeat the process and find that S is nonextendable by Proposition 5.2. In the case $(A')^2 = 4$ we have $A'.B_0 \geq 5$ by Hodge index. Therefore $(3B_0 - H).B_0 = (B_0 - A').B_0 \leq 1 < \phi(B_0)$, so that if $3B_0 - H > 0$, then it is a nodal cycle. Hence either $h^0(3B_0 - H) = 0$ or $h^0(3B_0 + K_S - H) = 0$ and S is nonextendable by Proposition 5.3. We have therefore shown that S is nonextendable under conditions (ii).

Now assume (iii). Set $k := -A.\Delta$. By [**22**, Lemma 2.3] we have that $A_0 := A - k\Delta$ is primitive, effective and isotropic. In case (iii-a) we deduce $k = 2$ and $F_1.\Delta = F_1.A_0 = 1$. Then $H \sim 2B_0 + A_0$ satisfies conditions (ii) and S is nonextendable. Now consider case (iii-b), so that $F_i.A \leq 6$ for $i = 1, 2$. If $\Delta.F_1 \leq 0$, then $F_2.\Delta \geq 2$. As $6 \geq F_2.A = F_2.A_0 + kF_2.\Delta$, we get $k = F_2.\Delta = 2$, so that $\Delta.F_1 = 0$ and $F_2.A \geq 4$. Then $F_1.B_0 = 1$, so that $B_0 \sim 2F_1 + F_2'$, where $F_2' \sim F_2 + \Delta - F_1 > 0$ and $(F_2')^2 = 0$. Also $F_1.A_0 = F_1.A \leq 4$, and equality implies $F_2.A = 4, F_2 \equiv A_0$ and the contradiction $F_1.A_0 = F_1.F_2 = 1$. Hence $F_1.A_0 \leq 3$. Moreover $F_2'.A_0 = (F_2 + \Delta - F_1).A_0 = (F_2 - F_1).A - 2 \leq 2$ and it cannot be that $(F_1.A_0, F_2'.A_0) = (1, 1)$, for then $F_1.A = 1$. Then $H \sim 2B_0 + A_0$ satisfies the conditions in (iii-a) and S is nonextendable. We can therefore assume $\Delta.F_1 > 0$, and by symmetry, also $\Delta.F_2 > 0$. Hence $\phi(B_0) \geq 2$. If $k \geq 3$, then $F_i.A = F_i.A_0 + kF_i.\Delta \geq 4$ for $i = 1, 2$, and we get $k = 3$, $F_i.A = 4$ and $F_i.\Delta = F_i.A_0 = 1$. Then $B.A = 8$ and $H^2 = 40$, so that $\phi(H) \leq 5$ by hypothesis. Let F be isotropic with $F.H = \phi(H)$. Now $(A')^2 = 4$ and $5 \geq F.H = 2F.B_0 + F.A' \geq 5$, so that $F.H = 5$, $F.A' = 1$, $(A' - 2F)^2 = 0$, $A' - 2F > 0$ and $(A' - 2F).H = (A - 2\Delta - 2F).H = 4$, a contradiction. Hence $k = 2, A_0^2 = 0$ and $B_0.A_0 = (B + \Delta).A_0 \geq 3$. Then the conditions (ii) are satisfied and S is nonextendable, unless possibly if $B_0^2 = 4$ and $B.A_0 = 1$. But then $B.\Delta = 2$ and $A_0 \equiv F_i$, for $i = 1$ or 2. Hence $\Delta.B = \Delta.(F_1 + F_2) = 3$, a contradiction. q.e.d.

We also have the following helpful tools to check surjectivity of μ_{V_D,ω_D} when $h^1(H - 2D_0) \neq 0$. The first lemma holds on any smooth surface.

Lemma 5.6. *Let S be a smooth surface, L a line bundle on S and $D_1 > 0$ and $D_2 > 0$ divisors on S not intersecting the base locus of $|L|$, such that $h^0(\mathcal{O}_{D_1}) = 1$ and $h^0(\mathcal{O}_{D_1}(-L)) = h^0(\mathcal{O}_{D_2}(-D_1)) = 0$. For any divisor $B > 0$ on S set $V_B := \operatorname{Im}\{H^0(S, L) \to H^0(B, L_{|B})\}$. If $\mu_{V_{D_1},\omega_{D_1}}$ and $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ are surjective, then μ_{V_D,ω_D} is surjective for general $D \in |D_1 + D_2|$.*

Proof. Let $D' = D_1 + D_2$. We have two surjective maps $\pi_i : V_{D'} \to V_{D_i}$, for $i = 1, 2$, and an exact sequence

$$0 \longrightarrow H^0(\omega_{D_1}) \longrightarrow H^0(\omega_{D'}) \xrightarrow{\psi} H^0(\omega_{D_2}(D_1)) \longrightarrow 0,$$

whence a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & V_{D'} \otimes H^0(\omega_{D'}) & \xrightarrow{\pi_2 \otimes \psi} & V_{D_2} \otimes H^0(\omega_{D_2}(D_1)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mu_{V_{D'},\omega_{D'}}} & & \downarrow{\scriptstyle \mu_{V_{D_2},\omega_{D_2}(D_1)}} & & \\ 0 & \longrightarrow & H^0(\omega_{D_1}(L)) & \xrightarrow{\chi} & H^0(\omega_{D'}(L)) & \longrightarrow & H^0(\omega_{D_2}(D_1+L)) & \longrightarrow & 0 \end{array}$$

where $W := \operatorname{Ker} \pi_2 \otimes H^0(\omega_{D'}) + V_{D'} \otimes \operatorname{Ker} \psi$ and φ is the restriction of $\mu_{V_{D'},\omega_{D'}}$. The surjectivity of $\mu_{V_{D_1},\omega_{D_1}}$ and the injectivity of χ show that $H^0(\omega_{D_1}(L)) = \operatorname{Im} \mu_{V_{D_1},\omega_{D_1}} = \operatorname{Im} \varphi_{|V_{D'} \otimes \operatorname{Ker} \psi}$. Hence φ is surjective and so is $\mu_{V_{D'},\omega_{D'}}$. By semicontinuity, μ_{V_D,ω_D} is surjective for general $D \in |D_1 + D_2|$. q.e.d.

Lemma 5.7. *Let S be an Enriques surface, L a very ample divisor on S and D_0 a nef and big divisor on S such that $\phi(D_0) \geq 2$. Let $E > 0$ be such that $E^2 = 0$ and $E.L = \phi(L)$.*

If $|L - D_0 - 2E|$ is base-component free, $h^1(D_0 + K_S - 2E) = h^2(D_0 + K_S - 4E) = 0$ and

$$(9) \quad h^0(L - 2D_0 - 2E) + h^0(\mathcal{O}_D(L - D_0 - 4E)) \leq \frac{1}{2}(L - D_0 - 2E)^2 - 1,$$

then μ_{V_D,ω_D} surjects for general $D \in |D_0|$, where $V_D = \operatorname{Im}\{H^0(\mathcal{O}_S(L - D_0)) \to H^0(\mathcal{O}_D(L - D_0))\}$.

Proof. Set $N = L - D_0 - 2E$. We have a commutative diagram

$$\begin{array}{ccc} H^0(2E) \otimes H^0(N) \otimes H^0(D_0 + K_S) & \xrightarrow{\operatorname{Id} \otimes \mu} & H^0(N) \otimes H^0(2E + D_0 + K_S) \\ {\scriptstyle \mu' \otimes \operatorname{Id}}\downarrow & & \downarrow{\scriptstyle r_D \otimes r'_D} \\ H^0(L - D_0) \otimes H^0(D_0 + K_S) & & W_D \otimes H^0(\omega_D(2E)) \\ {\scriptstyle p_D \otimes p'_D}\downarrow & & \downarrow{\scriptstyle \mu_{W_D,\omega_D(2E)}} \\ V_D \otimes H^0(\omega_D) & \xrightarrow{\mu_{V_D,\omega_D}} & H^0(\mathcal{O}_D(L + K_S)), \end{array}$$

where p_D, p'_D, r_D, r'_D are restriction maps, $W_D := \operatorname{Im} r_D$, $\mu = \mu_{2E, D_0 + K_S}$ and $\mu' = \mu_{2E,N}$. Since $H^1(D_0 + K_S - 2E) = H^2(D_0 + K_S - 4E) = 0$, the map μ is surjective by Castelnuovo-Mumford's lemma, and so is r'_D since $h^1(2E + K_S) = 0$. To conclude we need the surjectivity of $\mu_{W_D,\omega_D(2E)}$. As D is general, by [**15**, Thm. 4.e.1] we need $h^1(\omega_D(2E - N)) \leq h^0(N) - h^0(L - 2D_0 - 2E) - 2$, which is equivalent to (9) by Riemann-Roch and Serre duality. q.e.d.

6. Strategy of the proof of Theorem 1.5

In this section we prove Theorem 1.5 except for some concrete cases, and then we give the main strategy of the proof in these remaining cases, which will be carried out in Sections 7-11.

Let $S \subset \mathbb{P}^r$ be an Enriques surface of sectional genus g and let H be its hyperplane divisor. As we will prove a result also for $g = 15$ and 17 (Proposition 12.1) we will henceforth assume $g \geq 17$ or $g = 15$, so that $H^2 = 2g - 2 \geq 32$ or $H^2 = 28$, and, as H is very ample, $\phi(H) \geq 3$. We choose a genus one pencil $|2E|$ such that $E.H = \phi(H)$ and, as H is not of small type by Lemma 4.3, we define $\alpha := \alpha_E(H)$ as in (8) and $L_1 := H - \alpha E$. By [**22**, Lemma 2.4] and Lemma 4.4 we have that $L_1 > 0$ and $L_1^2 > 0$. Now suppose that L_1 is not of small type. Starting with $L_0 := H$ and $E_0 := E$ we continue the process inductively until we reach a line bundle of small type, as follows. Suppose given, for $i \geq 1$, $L_i > 0$ not of small type with $L_i^2 > 0$. We choose $E_i > 0$ such that $E_i^2 = 0$, $E_i.E_{i-1} > 0$, $E_i.L_i = \phi(L_i)$ and define $\alpha_i = \alpha_{E_i}(L_i)$ and $L_{i+1} = L_i - \alpha_i E_i$. Again $L_{i+1} > 0$. If $L_{i+1}^2 = 0$ we write $L_{i+1} \equiv \alpha_{i+1} E_{i+1}$ and define $L_{i+2} = 0$. We also have $E_{i+1}.E_i > 0$ because $L_i^2 > 0$. If $L_{i+1}^2 > 0$ then either L_{i+1} is of small type or we carry on. We then get

(10) $H = \alpha E + \alpha_1 E_1 + \ldots + \alpha_{n-1} E_{n-1} + L_n$, for some positive integer n

with $\alpha \geq 2$, $\alpha_i \geq 2$ for $1 \leq i \leq n-1$ and L_n is of small type. Moreover $E.E_1 \geq 1$, $E_i.E_{i+1} \geq 1$, E and E_i are primitive for all i, $L_i^2 > 0$ and $E_i.L_i = \phi(L_i)$ for $0 \leq i \leq n-2$ and $L_{n-1}^2 \geq 0$.

We record for later the following fact, which follows immediately from the definitions:

(11)
$E_1.(H - \alpha E) \leq \phi(H)$ and if $\alpha \geq 3$, then $E_1.(H - \alpha E) \geq \phi(H) + 1 - E.E_1$.

We now claim that $\alpha_i = 2$ for $1 \leq i \leq n-1$. If $(L_1 - 2E_1)^2 = 0$ then $\alpha_1 = 2$ by definition. If $(L_1 - 2E_1)^2 > 0$ we need $E_0.(L_1 - 2E_1) \leq \phi(L_1)$, that is $\phi(L_0) \leq E_1.L_0 + (2 - \alpha_0)E_1.E_0$. The latter holds if $\alpha_0 = 2$ and, by (11), if $\alpha_0 \geq 3$. By induction and the proof for $i = 1$ we get that $\alpha_i = 2$ for $1 \leq i \leq n-2$ and also for $i = n-1$ if $L_{n-1}^2 > 0$. If $L_{n-1}^2 = 0$ we have $L_{n-2} \equiv 2E_{n-2} + \alpha_{n-1} E_{n-1}$, whence $(\alpha_{n-1} E_{n-2}.E_{n-1})^2 = \phi(L_{n-2})^2 \leq L_{n-2}^2 = 4\alpha_{n-1} E_{n-2}.E_{n-1}$. Now if $\alpha_{n-1} \geq 3$ we get $E_{n-2}.E_{n-1} = 1$, giving the contradiction $\alpha_{n-1} = \phi(L_{n-2}) \leq E_{n-1}.L_{n-2} = 2$ and the claim is proved.

We now search for a divisor B as in Proposition 5.2 to show that $S \subset \mathbb{P}^r$ is nonextendable. Assume first that H is not 2-divisible in $\operatorname{Num} S$ and that $n \geq 2$ (that is L_1 is not of small type). If $n \geq 4$, then $B := E + E_1 + E_2 + E_3$ satisfies the conditions in Proposition 5.2 and S is nonextendable. If $n = 3$, then $H = \alpha E + 2E_1 + 2E_2 + L_3$. In this case $B := \lfloor \frac{\alpha}{2} \rfloor E + E_1 + E_2$ satisfies the conditions in Proposition 5.2, whence S is nonextendable, unless

(I-A) $n = 3, E_2 \equiv E$, $E.E_1 = 1$.
(II) $n = 3, E.E_1 = E.E_2 = E_1.E_2 = 1$, $\alpha \in \{2, 3\}$, $H^2 \leq 52$.

If $n = 2$, then $H = \alpha E + 2E_1 + L_2$. Set $B = \lfloor \frac{\alpha}{2} \rfloor E + E_1$. Then B satisfies the conditions in Proposition 5.2, whence S is nonextendable, unless

(I-B) $n = 2, E.E_1 = 1$.
(III) $n = 2, E.E_1 = 2$, $\alpha \in \{2, 3\}$, $H^2 \leq 62$,

or $E.E_1 = 3$, $\alpha \in \{2, 3\}$ and $H^2 \leq 52$. But the latter case does not occur. Indeed, then $E.H = \phi(H) = 6$ by [**22**, Prop. 1.4], whence $E.L_2 = 0$, so that $L_2 = 0$ or $L_2 \equiv E$. Since we can write $E + E_1 \sim A_1 + A_2 + A_3$ with $A_i > 0$, $A_i^2 = 0$ by [**22**, Lemma 2.4], we get $18 = 3\phi(H) \leq (E+E_1).H = 6 + 3\alpha + E_1.L_2$, whence $\alpha = E_1.L_2 = 3$ and $E_1.(H - 2E) = 6 = \phi(H)$, contradicting $\alpha = 3$.

Now $L_n \geq 0$ and $L_n^2 \geq 0$ so that, if $L_n > 0$, it has (several) arithmetic genus 1 decompositions. We want to extract from them any divisors numerically equivalent to E or to E_1, if possible. If, for example, we give priority to E, we will write $L_n \equiv E + L'_n$ and then, if L'_n has an arithmetic genus 1 decomposition with E_1 present, we write $L'_n \equiv E_1 + M_n$. If the priority is given to E_1 we do it first with E_1 and then with E. Moreover, to unify notation in the two cases (I-A) and (I-B), we will set $M_2 = M_3$ in the case (I-A), where only M_3 is defined. To avoid treating the same cases more times, we make the following choice of **"removing conventions"**:

(I-A) Remove E and E_1 from L_3, the one with lowest intersection number with L_3 first, giving priority to E_1 in case $E.L_3 = E_1.L_3$.
(I-B) Remove E and E_1 from L_2, the one with lowest intersection number with L_2 first, giving priority to E in case $E.L_2 = E_1.L_2$.
(II) Remove E, E_1 and E_2 from L_3, the one with lowest intersection number with L_3 first, giving priority to E first and then to E_2.
(III) Remove E and E_1 from L_2, the one with lowest intersection number with L_2 first, giving priority to E in case $E.L_2 = E_1.L_2$.

Then the extendability of S remains to be checked only in the following cases, where $\gamma, \delta \in \{2, 3\}$:

(I) $H \equiv \beta E + \gamma E_1 + M_2$, $E.E_1 = 1$, $H^2 \geq 32$ or $H^2 = 28$,
(II) $H \equiv \beta E + \gamma E_1 + \delta E_2 + M_3, E.E_1 = E.E_2 = E_1.E_2 = 1, \beta \in \{2, 3\}, 32 \leq H^2 \leq 52$ or $H^2 = 28$,
(III) $H \equiv \beta E + \gamma E_1 + M_2$, $E.E_1 = 2$, $\beta \in \{2, 3\}$, $32 \leq H^2 \leq 62$ or $H^2 = 28$,

(where the limitations on β are obtained using the same B's as above), in addition to:

(D) $H \equiv 2H_1$ for some $H_1 > 0$, $H_1^2 \geq 8$,
(S) L_1 is of small type and $H^2 \geq 32$ or $H^2 = 28$.

We call such decompositions as in (I)-(III), obtained by the inductive process and removing conventions above, a **ladder decomposition** of H.

Note that $M_n \geq 0$, $M_n^2 \geq 0$ and M_n is of small type, for $n = 2, 3$. Moreover, when $M_n > 0$, we will replace M_n with $M_n + K_S$ that has the same properties, to avoid to study the two different numerically equivalent cases for H. Also note that $\beta \geq \alpha \geq 2$ and $\beta \geq \alpha + 2$ in (I-A).

We will treat all these cases separately in the next sections.

The next three lemmas will be useful.

Lemma 6.1. *If $E.E_1 \leq 2$, then $E + E_1$ is nef.*

Proof. Let Γ be a nodal curve with $\Gamma.(E + E_1) < 0$. As E is nef, we must have $k := -\Gamma.E_1 \geq 1$ and $A := E_1 - k\Gamma$ is primitive, effective and isotropic by [**22**, Lemma 2.3]. Since $A.L_1 \geq \phi(L_1) = E_1.L_1$, we get $k\Gamma.L_1 = (E_1 - A).L_1 \leq 0$, whence $\Gamma.E > 0$, because H is ample. This yields $k \geq \Gamma.E + 1 \geq 2$. Hence $E.E_1 = E.A + k\Gamma.E \geq 2\Gamma.E$, and we get $k = 2$, $\Gamma.E = 1$ and $E.A = 0$. Then $A \equiv E$ by [**21**, Lemma 2.1], contradicting $\Gamma.A = -\Gamma.E_1 = 2$. q.e.d.

From [**11**, Prop. 3.1.6, 3.1.4 and Thm. 4.4.1] and the lemma, $E + E_1$ is base-point free when $E.E_1 = 2$, and $E + E_1$ is base-component free when $E.E_1 = 1$, unless $E_1 \sim E + R$, for a nodal curve R such that $E.R = 1$. But since we are free to choose between E_1 and $E_1 + K_S$, **we adopt the convention of choosing E_1 such that $E + E_1$ is base-component free**. Thus we have

Lemma 6.2. *If $E.E_1 = 2$, then $E + E_1$ is base-point free.*

If $E.E_1 = 1$, then $E + E_1$ is base-component free. Furthermore if there exists $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.E_1 < 0$, then Δ is a nodal curve and $E_1 \sim E + \Delta + K_S$.

Moreover in both cases we have $H^1(E_1) = H^1(E_1 + K_S) = 0$.

Proof. We need to prove the last two assertions. If $\Delta > 0$ satisfies $\Delta^2 = -2$ and $\Delta.E_1 < 0$, then similarly to the previous proof one obtains $\Delta.E_1 = -1$, so that E_1 is quasi-nef and primitive and the desired vanishings follow by [**21**, Cor. 2.5]. Now if $E.E_1 = 1$ we obtain that $E_1 \equiv E + \Delta$ by [**21**, Lemma 2.1]. Since E_1 is not nef, by [**11**, Prop. 3.1.4, Prop. 3.6.1 and Cor. 3.1.4] there is a nodal curve R such that $E_1 \sim E + R + K_S$, whence $\Delta = R$. q.e.d.

Lemma 6.3. *Let $H \sim \beta E + \gamma E_1 + M_2$ be of type* (I) *or* (III)*, with $M_2 > 0$ and $M_2^2 \leq 4$. Let $i = 2$ and $M_2 \sim E_2$ or $i = 2, 3$ and $M_2 \sim E_2 + E_3$ be genus* 1 *decompositions of M_2 (note that, by construction, $E.E_j \geq 1$ for $j = 1, 2$). Assume that E_i is quasi-nef. Then:*

(a) *$|2E + E_1 + E_i|$ is base-point free.*
(b) *$|E + E_1 + E_i|$ is base-point free if $\beta = 2$ or if $E.E_1 = 1$ and $E_1.E_i \neq E.E_i - 1$.*
(c) *Assume $\gamma = 2$ and $E.E_1 = E_1.E_i = 1$. Then $E + E_i$ is nef if either $E.E_i \geq 2$ or if $M_2^2 \geq 2$ and $E_1.M_2 \geq 4$.*

(d) *Assume* $\gamma = 2$, $M_2^2 = 2$, $E.E_1 = E_1.E_2 = E_1.E_3 = 1$ *and that both* E_2 *and* E_3 *are quasi-nef. Then either* $E + E_2$ *or* $E + E_3$ *is nef.*
(e) *If* $E.E_1 = E.E_i = 1$ *and* $E_1.E_i \neq 1$ *then* $E_1 + E_i$ *is nef.*

Proof. Assume R is a nodal curve with $R.(E+E_1+E_i) < 0$. Arguing as above, using Lemma 6.1, [**22**, Lemma 2.3] and [**21**, Lemma 2.1], we find that $R.E_1 = R.E_i = -1$ and $R.E = 1$, so that $2E + E_1 + E_i$ is nef, whence base-point free, as $\phi(2E + E_1 + E_i) \geq 2$, and (a) is proved. Similarly, if $E.E_1 = 1$, then $E_1 \equiv E + R$ by Lemma 6.2, whence $E_1.E_i = E.E_i - 1$, and (b) is proved.

The remaining assertions are proved similarly. q.e.d.

The general strategy to prove the nonextendability of S in the remaining cases (I), (II), (III), (D) and (S), will be as follows: We will first use the ladder decomposition and Propositions 5.2-5.4 to reduce to genus one decompositions of M_2 or M_3 where we know all the intersections involved. Then we will find a big and nef divisor D_0 on S such that $\phi(D_0) \geq 2$ and $H - D_0$ is base-component free with $(H - D_0)^2 > 0$. Then $H^1(H - D_0) = H^1(D_0 - H) = 0$. In some cases this D_0 will satisfy the conditions of B in Lemma 5.5, so that S will be nonextendable. In the remaining cases we will apply Proposition 5.1, mostly without reference, in the following way: We denote by D a general smooth curve in $|D_0|$; we will do this without further mentioning. The surjectivity of Φ_{H_D,ω_D} will be proved using [**23**, Thm], and in all cases therein, with the exception of (v), we will have that $h^0(\mathcal{O}_D(2D_0 - H)) \leq 1$ if $D_0^2 \geq 6$ and $h^0(\mathcal{O}_D(2D_0 - H)) = 0$ if $D_0^2 = 4$. Therefore the hypothesis (iii) of Proposition 5.1 will always be satisfied and we will skip its verification. To study the surjectivity of μ_{V_D,ω_D} we will use several tools, outlined below. In several cases we will find an effective decomposition $D \sim D_1 + D_2$ and use Lemma 5.6. We remark that **except possibly for the one case in** (15) **below where** D_1 **is primitive of canonical type, both** D_1 **and** D_2 **will always be smooth curves** by [**11**, Prop. 3.1.4 and Thm. 4.10.2]. Furthermore the spaces V_D, V_{D_1} and V_{D_2} will always be base-point free. This is immediately clear for V_D, as $|D_0|$ is base-point free. As for V_{D_1} and V_{D_2}, one only has to make sure that, when $|H - D_0|$ has base points (that is, $\phi(H - D_0) = 1$), in which case it has precisely two distinct base points [**11**, Prop. 3.1.4 and Thm. 4.4.1], they do not intersect the possible base points of $|D_1|$ and $|D_2|$. This will always be satisfied and we will not repeatedly mention this.

Here are the criteria we will use to verify that the desired multiplication maps are surjective:

The map μ_{V_D,ω_D} is surjective in any of the following cases:

(12)
$H^1(H - 2D_0) = 0$ and $|D_0|$ or $|H - D_0|$ is birational (see Rem. 6.4).

(13) $$H^1(H-2D_0)=0 \text{ and } |H-D_0| \text{ is a pencil.}$$

If V_{D_1} is base-point free, $\mu_{V_{D_1},\omega_{D_1}}$ is surjective in any of the following cases:

(14) $H^1(H-D_0-D_1)=0$, D_1 is smooth and $(H-D_0).D_1 \geq D_1^2+3$;

(15) $$H^1(H-D_0-D_1)=0 \text{ and } D_1 \text{ is nef and isotropic.}$$

If D_2 is smooth and V_{D_2} is base-point free, then $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective if

(16) $$h^0(H-D_0-D_2)+h^0(\mathcal{O}_{D_2}(H-D_0-D_1)) \leq \tfrac{1}{2}(H-D_0)^2-1 \text{ (see Rem. 6.5 below).}$$

To see (12)-(13) note that $V_D = H^0(\mathcal{O}_D(H-D_0))$ if $H^1(H-2D_0)=0$, whence (13) is the base-point free pencil trick, while (12) follows using [**1**, Thm. 1.6] in addition, since $\mathcal{O}_D(H-D_0)$ is base-point free and is either a pencil or birational. The same proves (14). As for (15) the hypotheses imply $V_{D_1} = H^0(\mathcal{O}_{D_1}(H-D_0))$ and $\omega_{D_1} \cong \mathcal{O}_{D_1}$ by [**11**, III, §1], and surjectivity is immediate. For (16), the H^0-lemma [**15**, Thm. 4.e.1] gives surjectivity if $\dim V_{D_2}-2 = h^0(H-D_0)-h^0(H-D_0-D_2)-2 \geq h^1(\omega_{D_2}(D_1-(H-D_0))) = h^0(\mathcal{O}_{D_2}(H-D_0-D_1))$. This is equivalent to (16) by Riemann-Roch.

Remark 6.4. A complete linear system $|B|$ is birational if it defines a birational map. By [**11**, Prop. 3.1.4, Lemma 4.6.2, Thm. 4.6.3, Prop. 4.7.1 and Thm. 4.7.1] a nef divisor B with $B^2 \geq 8$ defines a birational morphism if $\phi(B) \geq 2$ and B is not 2-divisible in $\operatorname{Pic} S$ when $B^2 = 8$.

Remark 6.5. The inequality in (16) will be verified by giving an upper bound on $h^0(H-D_0-D_2)$ and using Riemann-Roch and Clifford's theorem on D_2 to bound $h^0(\mathcal{O}_{D_2}(H-D_0-D_1))$.

7. Case (D)

We have $H \equiv 2H_1$ whence H_1 is ample with $H_1^2 \geq 8$ and $\phi(H) = 2\phi(H_1) \geq 3$ gives $\phi(H_1) \geq 2$.

If $H \sim 2H_1 + K_S$, we set $D_0 := H_1$ and apply Proposition 5.1. Note that Φ_{H_D,ω_D} is surjective by [**23**, Thm(iii)] and, as $H^1(H-2D_0) = 0$, the map μ_{V_D,ω_D} is just μ_{ω_D,ω_D}, which is surjective since D is not hyperelliptic.

If $H \sim 2H_1$ we divide the treatment in various cases:

7.1. $\phi(H_1) = 2$ **and** $H_1^2 = 8$**.** Using [**22**, Lemma 2.4], we obtain the cases (a1) and (a2) in the proof of Proposition 12.1.

7.2. $\phi(H_1) = 2$ **and** $H_1^2 = 10$**.** By [**22**, Lemma 2.4] we can write $H \sim 4E + 2E_1 + 2E_2$ and one easily sees, by Lemma 6.2, that either E_1 or E_2 is nef. We can assume that E_1 is nef and, possibly adding K_S to E_2, that $E + E_2$ is base-component free. We set $D_0 := E + 2E_1 + E_2$ and apply Proposition 5.1. Now Φ_{H_D,ω_D} is surjective by [**23**, Thm(iii)]. As for μ_{V_D,ω_D}, consider the commutative diagram, with $N := E + 2E_1 + E_2 + K_S$,

$$\begin{array}{ccc}
H^0(2E) \otimes H^0(E+E_2) \otimes H^0(N) & \xrightarrow{r_D} & W_D \otimes H^0(\mathcal{O}_D(E+E_2)) \otimes H^0(\omega_D) \\
\downarrow{\scriptstyle \mu_{2E,E+E_2}} & & \downarrow{\scriptstyle \mathrm{Id} \otimes \mu_{\mathcal{O}_D(E+E_2),\omega_D}} \\
H^0(H-D_0) \otimes H^0(D_0+K_S) & & W_D \otimes H^0(\omega_D(E+E_2)) \\
\downarrow{\scriptstyle p_D} & & \downarrow{\scriptstyle \mu_{W_D,\omega_D(E+E_2)}} \\
V_D \otimes H^0(\omega_D) & \xrightarrow{\mu_{V_D,\omega_D}} & H^0(\mathcal{O}_D(H+K_S)),
\end{array}$$

where p_D and r_D are the natural restriction maps, which are easily seen to be surjective, and $W_D := \mathrm{Im}\{H^0(2E) \to H^0(\mathcal{O}_D(2E))\}$. The map $\mu_{\mathcal{O}_D(E+E_2),\omega_D}$ is surjective by the base-point free pencil trick. To prove that μ_{V_D,ω_D} is surjective it suffices to show that $\mu_{W_D,\omega_D(E+E_2)}$ is surjective. The latter follows by the H^0-lemma [**15**, Thm. 4.e.1], as $\dim W_D = 2$ and W_D is base-point free, and one computes $h^1(\omega_D(E_2 - E)) = h^0(\mathcal{O}_D(E - E_2)) = 0$.

7.3. $\phi(H_1) = 2$ **and** $H_1^2 \geq 12$**.** We set $D_0 := H_1$ and apply Proposition 5.1. The map Φ_{H_D,ω_D} is onto by [**23**, Thm(iv)] and μ_{V_D,ω_D} is onto by (12).

7.4. $\phi(H_1) \geq 3$**.** As S is regular, if it is extendable, it can be reembedded so that it is linearly normal and extendable, as in the proof of Corollary 2.2. Hence we can assume that $S \subset \mathbb{P}H^0(2H_1)$. Now H_1 is very ample [**11**, Cor. 2, Appendix Ch.IV], whence S is nonextendable by [**14**, Thm. 1.2].

8. Case (I)

If $M_2 = 0$, then $H \equiv \beta E + \gamma E_1$, $E.E_1 = 1, \beta \geq 2, \gamma \in \{2,3\}$ and $H^2 \geq 32$ or $H^2 = 28$. Now $\gamma = E.H = \phi(H) \geq 3$ so that $\gamma = 3$ and $\beta \geq 6$. We set $D_0 := H - \lfloor \frac{\beta+1}{2} \rfloor E - E_1$, which is nef by Lemma 6.1, and use Proposition 5.1. By Lemma 6.2, we have $h^0(2D_0 - H) \leq 1$, whence Φ_{H_D,ω_D} is surjective by [**23**, Thm(iii)-(iv)]. To see the surjectivity of μ_{V_D,ω_D} we apply Lemma 5.7. By Lemma 6.2 we get $H^1(D_0 + K_S - 2E) = 0$ and $H - D_0 - 2E = \lfloor \frac{\beta-3}{2} \rfloor E + E_1$ is base-component free. Also $H^2(D_0 + K_S - 4E) = 0$ and $h^0(H - 2D_0 - 2E) = 0$ by the nefness of E. Since $H^1(H - 2D_0 - 4E) = 0$, we get $h^0(\mathcal{O}_D(H - D_0 - 4E)) \leq h^0(H - D_0 - 4E)$. Now $H - D_0 - 4E = \lfloor \frac{\beta-7}{2} \rfloor E + E_1$, whence $h^0(H - D_0 - 4E) = \lfloor \frac{\beta-5}{2} \rfloor$ by Lemma 6.2 and (9) is satisfied.

Hence S is nonextendable if $M_2 = 0$.

Assume next that $M_2 > 0$ and $\gamma = 3$. We also have $\beta \geq 3$. Indeed, if $\beta = 2$ we have $L_2 \sim E_1 + M_2$ and $E.L_2 = 1 + E.M_2 = \phi(H) - 2 \leq E_1.H - 2 = E_1.M_2 = E_1.L_2$, contradicting the removing conventions of Section 6 (because then $(L_2 - E)^2 \geq (L_2 - E_1)^2 \geq 0$, therefore we could find E in a genus 1 decomposition of L_2, but then $\beta \geq 3$).

Lemma 8.1. *If* $|H - 2(E+E_1)|$ *has base points or* $h^1(H - 3(E+E_1)) \neq 0$*, then* S *is nonextendable.*

Proof. Set $N = E + E_1$. We have that $H - 2N$ is not base-point free if and only if it is not nef, in which case $H - 3N$ is not quasi-nef by ampleness of H, whence $h^1(H - 3N) \neq 0$ by [**21**, Cor. 2.5]. Hence it suffices to show that S is nonextendable if $H - 3N$ is not quasi-nef.

Let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta.(H - 3N) \leq -2$. We have $\Delta.N > 0$ since H is ample. Also note that $\Delta.E_1 \geq 0$, for if not, we would have $\Delta.E \geq 2$, whence the contradiction $(E + \Delta)^2 \geq 2$ and $E_1.(E + \Delta) \leq 0$. Hence $M_2.\Delta \leq -2$ and by [**22**, Lemma 2.4] we can write $M_2 \sim A + k\Delta$, with $A > 0$, primitive, $A^2 = M_2^2$ and $k := -\Delta.M_2 = \Delta.A \geq 2$. Now if $E.\Delta > 0$ we find that $E.M_2 \geq k$ and if equality holds, then $E.A = 0$ and $E.\Delta = 1$, whence $E \equiv A$ by [**21**, Lemma 2.1], a contradiction. We get the same contradiction if $E_1.\Delta > 0$. Therefore

$$\begin{aligned} E.M_2 &\geq -\Delta.M_2 + 1 \geq 3 \ \text{ if } \ E.\Delta > 0 \ \text{ and} \\ E_1.M_2 &\geq -\Delta.M_2 + 1 \geq 3 \ \text{ if } \ E_1.\Delta > 0. \end{aligned} \tag{17}$$

We first consider the case $E.\Delta > 0$. If $\beta = 3$ then H is of type (I-B) in Section 6 and $L_2 \sim (3 - \alpha)E + E_1 + M_2$ is of small type, whence $E_1.M_2 \leq 5$ by Lemma 4.3, so that $E_1.(H - 2E) = E_1.(E + 3E_1 + M_2) \leq 6$. Since $\phi(H) = E.H = 3 + E.M_2 \geq 6$ by (17), we get $\alpha = 2$ and $E_1.H = 3 + E_1.M_2 \geq 6$, so that $E_1.M_2 \geq 3$. Hence $L_2 \sim E + E_1 + M_2$ and $L_2^2 \geq 14$, a contradiction.

Therefore $\beta \geq 4$, whence $\Delta.M_2 \leq -2 - (\beta - 3)\Delta.E \leq -3$, so that $E.M_2 \geq 4$ by (17) and $\phi(H) \geq 7$, whence $H^2 \geq 54$ by [**22**, Prop. 1.4]. Now one easily verifies that $B := 2E + E_1 + \Delta$ satisfies the conditions in Proposition 5.2, so that S is nonextendable.

Now consider the case $\Delta.E = 0$, where $E_1.\Delta > 0$, so that $E_1.M_2 \geq 3$ by (17). Then $L_2 \sim (\beta - \alpha)E + E_1 + M_2$ if H is of type (I-B) in Section 6 and $L_3 \sim (\beta - \alpha - 2)E + E_1 + M_2$ if H is of type (I-A). We claim that the removing conventions of Section 6 now imply that $E_1.M_2 \leq E.M_2 + 1$ and, if $\beta = 3$, that $E_1.M_2 \leq E.M_2$. In fact if the latter inequalities do not hold we have that $E.L_2 \leq E_1.L_2$, $E.L_3 < E_1.L_3$ and $(E_1 + M_2 - E)^2 \geq 0$, contradicting the fact that L_2 and L_3 are of small type. Therefore $E.M_2 \geq 2$, and $E.M_2 \geq 3$ if $\beta = 3$, so that $H^2 \geq 54$. Now one easily verifies that $B := E + 2E_1 + \Delta$ satisfies the conditions in Proposition 5.2. q.e.d.

Now set $D_0 := 2(E+E_1)$, which is nef by Lemma 6.1. By Lemma 8.1 we can assume that $H-D_0$ is base-point free. Note that $H.D_0 = 2(\beta + 3 + (E+E_1).M_2) \geq 16$ with equality only if $\beta = 3$ and $E.M_2 = 1$. But in the latter case, since M_2 does not contain E in its arithmetic genus 1 decompositions, we have that $M_2^2 = 0$ and $H^2 = 30$, a contradiction. Hence $(2D_0 - H).D_0 < 0$, so that Φ_{H_D,ω_D} is surjective by [**23**, Thm(iii)]. The map μ_{V_D,ω_D} is surjective by Lemma 5.6, using general $D_1, D_2 \in |E+E_1|$. Indeed, by Lemma 8.1 we can assume $h^1(H-D_0-D_i) = 0$, whence $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by (14), and $\mu_{V_{D_2},\omega_{D_2}(D_1)} = \mu_{\mathcal{O}_{D_2}(H-D_0),\omega_{D_2}(D_1)}$ is surjective by [**15**, Cor. 4.e.4]. Therefore, in the case with $M_2 > 0$ and $\gamma = 3$, we have that S is nonextendable by Proposition 5.1.

Now we deal with the case $\gamma = 2$ and $M_2 > 0$. We have $E_1.M_2 \leq E_1.M_2 + \beta - \alpha = E_1.L_1 = \phi(L_1) \leq \phi(H) = 2 + E.M_2 \leq E_1.H = \beta + E_1.M_2$. Moreover, since by construction M_2 neither contains E nor E_1 in its arithmetic genus 1 decompositions, we have $(M_2 - E)^2 < 0$ and $(M_2 - E_1)^2 < 0$. Hence

$$\frac{1}{2}M_2^2 + 1 \leq \quad E.M_2 \quad \leq E_1.M_2 + \beta - 2, \quad \text{and} \tag{18}$$

$$\frac{1}{2}M_2^2 + 1 \leq \quad E_1.M_2 \quad \leq E.M_2 + 2 - \beta + \alpha \leq E.M_2 + 2. \tag{19}$$

Proposition 8.2. *Let H be of type* (I) *with $\gamma = 2$ and $M_2 > 0$. Then S is nonextendable if $\beta \geq 5$.*

Proof. We first prove that S is nonextendable if M_2 or $E_1 + M_2$ is not quasi-nef. The removing conventions in Section 6 imply $E_1.M_2 \geq E.M_2$. Assume first there is a $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.M_2 \leq -2$. By [**22**, Lemma 2.3], [**21**, Lemma 2.1] and Lemma 6.2 we must have $\Delta.E > 0$ and $\Delta.E_1 \geq 0$. Then $B := \lfloor \frac{\beta}{2} \rfloor E + E_1 + \Delta$ satisfies the conditions in Proposition 5.2 and we are done (some work is required to check that $H^2 \geq 54$ if $B^2 = 6$).

Assume similarly that there is a $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.(E_1 + M_2) \leq -2$. By what we have just proved and Lemma 6.2, we can assume that $\Delta.E_1 = \Delta.M_2 = -1$, but then we get $E_1 \equiv E + \Delta$, whence $E_1.M_2 = (E+\Delta).M_2 < E.M_2$, a contradiction.

We next prove that S is nonextendable if $M_2^2 \geq 4$.

Indeed, if $M_2^2 \geq 4$, we write $M_2 \sim E_2 + \ldots + E_{k+1}$ as in Lemma 4.3 with $k = 2$ or 3. Moreover we can assume that $1 \leq E.E_2 \leq \ldots \leq E.E_{k+1}$, whence that $E.M_2 \geq kE.E_2$. Set $B := E + E_1 + E_2$. Using (18) and (19), one easily verifies that B satisfies the conditions in Propositions 5.2 or 5.3, and S is nonextendable, except when $M_2^2 = 4$ and $E.E_2 = E.E_3$. In this case we can assume $1 \leq E_1.E_2 \leq E_1.E_3$. By (18), (19), Lemma 6.3(c) and Lemma 4.5, one verifies that $B := E + E_2$ satisfies the conditions in Propositions 5.2 or 5.4.

We can henceforth assume that E_1+M_2 and M_2 are quasi-nef, whence that $E+E_1+M_2$ is nef, and that $M_2^2 \leq 2$. Set $D_0 := \lfloor \frac{\beta-1}{2} \rfloor E+E_1+M_2$. Then $H-D_0$ and $H-D_0-2E$ are base-component free by Lemma 6.1 and μ_{V_D,ω_D} surjects by Lemma 5.7, using [**21**, Cor. 2.5] and Lemma 6.1 to verify (9).

To end the proof we deal with Φ_{H_D,ω_D}. By [**21**, Cor. 2.5] one gets $h^0(M_2-E) \leq 1$. Then Φ_{H_D,ω_D} is onto by [**23**, Thm(iii)-(iv)] (whence S is nonextendable by Proposition 5.1) unless possibly if $\beta = 5$, $E.M_2 = E_1.M_2 = 1, M_2^2 = 0$ and $h^0(M_2-E) > 0$. We now treat this case, setting $E_2 = M_2$. We first need two auxiliary results.

Claim 8.3. *Set $E_0 = E$. Let $F > 0$ be a divisor such that $F^2 = 0$ and $F.E = F.E_1 = F.E_2 = 1$. If F is not nef there exists a nodal curve R such that $F \equiv E_i + R$ and $E_i.R = 1$ for some $i \in \{0,1,2\}$.*

Proof. Let R be a nodal curve such that $R.F < 0$. Now $A := F + (R.F)R$ is primitive, effective and isotropic by [**22**, Lemma 2.3]. Since H is ample, there is an $i \in \{0,1,2\}$ such that $E_i.R \geq 1$. As $1 = E_i.F$, the only possibility is $E_i.R = -R.F = 1$, and $A \equiv E_i$ by [**21**, Lemma 2.1].
q.e.d.

Claim 8.4. *There is an isotropic effective 10-sequence $\{F_1, \ldots, F_{10}\}$ such that $F_1 = E$, $F_2 = E_1$, $F_3 = E_2$. For $4 \leq i \leq 10$ set $F'_i = E + E_1 + E_2 - F_i$. Then $F'_i > 0$, $(F'_i)^2 = 0$ and $F'_i.E = F'_i.E_1 = F'_i.E_2 = 1$. Moreover the following conditions are satisfied:* (i) *F_i is nef for $7 \leq i \leq 10$;* (ii) *$E + F'_i$ is nef for $9 \leq i \leq 10$;* (iii) *if $E_2 > E$ then $h^0(2F_{10} + E - E_2 + K_S) = 0$.*

Proof. The 10-sequence exists by completing the isotropic 3-sequence $\{E, E_1, E_2\}$, cf. [**11**, Cor. 2.5.6].

To see (i), suppose that $F_4, \ldots, F_7$ are not nef. By Claim 8.3 there is an $i \in \{0,1,2\}$ and $j, k \in \{4, \ldots, 7\}$, $j \neq k$, such that $F_j \equiv E_i + R_j$ and $F_k \equiv E_i + R_k$. Therefore $R_j.R_k = (F_j - E_i).(F_k - E_i) = -1$, a contradiction. Upon renumbering we can assume that F_i is nef for $7 \leq i \leq 10$.

Now $(F'_i)^2 = 0$ and $F'_i.E = F'_i.E_1 = F'_i.E_2 = 1$, whence $F'_i > 0$ by Riemann-Roch. To see (ii) suppose that $E + F'_7$, $E + F'_8$ and $E + F'_9$ are not nef. By Claim 8.3 there is an $i \in \{1,2\}$ and $j, k \in \{7,8,9\}$, $j \neq k$, such that $F'_j \equiv E_i + R_j$ and $F'_k \equiv E_i + R_k$, giving a contradiction as above. Upon renumbering we can assume that $E + F'_i$ is nef for $9 \leq i \leq 10$.

To see (iii), let F be either F_9 or F_{10} and suppose that $2F + E - E_2 + K_S \geq 0$. Let Γ be a nodal component of $E_2 - E$. Since $2F + K_S \geq E_2 - E \geq \Gamma$ and $h^0(2F + K_S) = 1$, we get that Γ must be either a component of F or of $F + K_S$. Therefore Γ is, for example, a component of both F_9 and F_{10}. This is not possible since $F_9.F_{10} = 1$ and F_9 and F_{10} are nef and primitive. q.e.d.

Conclusion of the proof of Proposition 8.2. By Claim 8.4(ii) we know that $E + F'_{10}$ is nef, whence, using [**11**, Prop. 3.1.6 and Cor. 3.1.4], we can choose $F \equiv F_{10}$ so that, setting $F' = E + E_1 + E_2 - F$, we have that $E + F'$ is a base-component free pencil. Let $D_0 := 3E + E_1 + F$. Then D_0 is nef by Lemma 6.1 and Claim 8.4(i) and $H - D_0 = E + F'$ is a base-component free pencil. Moreover, one can check that $h^0(2D_0 - H) \leq 1$, so Φ_{H_D,ω_D} is surjective by [**23**, Thm(iii)-(iv)].

We have $h^0(H - 2D_0) = 0$ as $E.(H - 2D_0) = -1$. By Riemann-Roch and Claim 8.4(iii), we have $h^1(H - 2D_0) = h^0(2F_{10} + E - E_2 + K_S) = 0$. Therefore μ_{V_D,ω_D} is surjective by (13). q.e.d.

The cases left to treat of Case (I) are therefore the ones with $\beta \leq 4$ (and $\gamma = 2$ and $M_2 > 0$). This involves a detailed case-by-case study, in particular of the various intersection properties of the components in the genus one decompositions of M_2. The proof of the following result involves no new ideas and is therefore left to the note [**20**]:

Proposition 8.5. *Let H be of type* (I) *with $\beta \leq 4$, $\gamma = 2$ and $M_2 > 0$ and such that $H^2 \geq 32$ or $H^2 = 28$. Then S is nonextendable, except possibly for the following two cases, where $H^2 = 28$ and $E_2 > 0, E_2^2 = 0$:*

(i) $H \sim 3E + 2E_1 + E_2$, $E.E_1 = E_1.E_2 = 1$, $E.E_2 = 2$,

(ii) $H \sim 4E + 2E_1 + E_2$, $E.E_1 = E.E_2 = E_1.E_2 = 1$.

9. Case (II)

As M_3 does not contain E, E_1 or E_2 in its genus 1 decompositions, we have:

(20) If $M_3 > 0$, then $E.M_3 \geq \frac{1}{2}M_3^2 + 1$, $E_i.M_3 \geq \frac{1}{2}M_3^2 + 1$, $i = 1, 2$.

Using Lemma 6.2 and [**22**, Lemma 2.3], it is easy to check that $B := E + E_1 + E_2$ is nef. If

(21) $$2(\beta + \gamma + \delta) + (E + E_1 + E_2).M_3 \geq 17,$$

then $(3B - H).B \leq 1$, whence if $3B - H > 0$, the nefness of B gives that it is a nodal cycle. Thus either $h^0(3B - H) = 0$ or $h^0(3B + K_S - H) = 0$ and S is nonextendable by Proposition 5.3.

We now deal with (21). Assume first that $M_3 > 0$. Then, in view of (20), the condition (21) is satisfied unless $M_3^2 = 0$, in which case S is nonextendable by Lemma 5.5(ii).

Assume now that $M_3 = 0$. Then (21) is satisfied unless $6 \leq \beta + \gamma + \delta \leq 8$. Since $E.H = \gamma + \delta$ and $E_1.H = \beta + \delta$, we get $\gamma \leq \beta$, and since $E_1.L_1 = \beta - \alpha + \delta$ and $E_2.L_1 = \beta - \alpha + \gamma$, we get $\gamma \geq \delta$. As we assume that H is not 2-divisible in $\operatorname{Num} S$, we end up with $(\beta, \gamma, \delta) = (3, 2, 2)$ or $(3, 3, 2)$.

The first case is case (a3) in the proof of Proposition 12.1. In the second case, set $D_0 := 2E + E_1 + E_2 = E + B$. Now E_1 is nef by

Lemma 4.5, so that $H - D_0 \equiv B + E_1$ is nef, whence base-point free. We have $(H-2D_0)^2 = -2$ and $(H-2D_0).H = 0$. Thus $h^i(H-2D_0) = h^i(H-2D_0+K_S) = 0$ for all $i = 0,1,2$. Then Φ_{H_D,ω_D} is onto by [**23**, Thm(iii)] and μ_{V_D,ω_D} is onto by (12).

10. Case (III)

Since $H^2 \leq 62$ and L_2 is of small type, we have

(22) $\phi(H) = E.H = 2\gamma + E.M_2 \leq 7$ and either $M_2 > 0$ or $\beta = \gamma = 3$.

As M_2 contains neither E nor E_1 in its genus 1 decompositions, we have:

(23) If $M_2 > 0$, then $E.M_2 \geq \frac{1}{2}M_2^2 + 1$ and $E_1.M_2 \geq \frac{1}{2}M_2^2 + 1$.

By Proposition 5.4 and Lemma 6.1 we can assume

(24) $(E+E_1).H = 2(\beta+\gamma) + (E+E_1).M_2 \leq 16.$

10.1. The case $\beta = 2$. We have $M_2 > 0$ by (22) and $E.M_2 \geq 1$ by (23).

If $\gamma = 3$, then $E.M_2 = 1$ and $\phi(H) = 7$ by (22), so that $M_2^2 = 0$ by (23). As $L_2 \equiv E_1 + M_2$, the removing conventions of Section 6 require that $E_1.L_2 < E.L_2$. Hence $E_1.M_2 \leq 2$, giving the contradiction $49 = \phi(H)^2 \leq H^2 \leq 40$. Therefore $\gamma = 2$, so that $E.M_2 \leq 3$ by (22), whence $M_2^2 \leq 4$ by (23). Moreover $(E+E_1).M_2 \leq 8$ by (24), whence

(25) $\phi(H)^2 = (4+E.M_2)^2 \leq H^2 = 16+M_2^2+4(E+E_1).M_2 \leq 48+M_2^2.$

Combining with [**22**, Prop. 1.4], we get $E.M_2 \leq 2$, whence $M_2^2 \leq 2$ by (23).

If $M_2^2 = 2$, then $E.M_2 = 2$ by (23) and since $(E_1.M_2)^2 = \phi(L_1)^2 \leq L_1^2 = 4E_1.M_2+2$, we get $E_1.M_2 \leq 4$. Writing $M_2 \sim E_2+E_3$ for isotropic $E_2 > 0$ and $E_3 > 0$ with $E_2.E_3 = 1$, we have $E.E_2 = E.E_3 = 1$. As $E_i.H \geq \phi(H) = E.H = 6$ for $i = 2,3$, we find $E_1.E_2 = E_1.E_3 = 2$. By Lemma 4.5, both E_2 and E_3 are quasi-nef, whence $E+E_1+E_i$ is nef for $i = 1,2$ by Lemma 6.3(b). Set $D_0 := E+E_1+E_2$. Now $(H-2D_0)^2 = -2$ with $(H-2D_0).H = 0$, whence $h^i(2D_0-H) = h^i(2D_0-H+K_S) = 0$ for $i = 0,1,2$. Then Φ_{H_D,ω_D} is surjective by [**23**, Thm(iii)] and μ_{V_D,ω_D} is surjective by (12).

Finally, if $M_2^2 = 0$, then S is nonextendable by Lemmas 6.1 and 5.5(ii) unless $(E+E_1).M_2 \leq 3$. In the latter case, by (25), we get $E.M_2 = 1$ and $E_1.M_2 = 2$. Set $E_2 := M_2$.

Claim 10.1. *There is an isotropic effective* 10*-sequence* $\{f_1,\dots,f_{10}\}$, *with* $f_1 = E$, $f_{10} = E_2$, *all* f_i *nef for* $i \leq 9$, *and, for each* $i = 1,\dots,9$, *there is an effective decomposition* $H \sim 2f_i + 2g_i + h_i$, *where* $g_i > 0$ *and* $h_i > 0$ *are primitive and isotropic with* $f_i.g_i = g_i.h_i = 2$ *and* $f_i.h_i = 1$. *Furthermore,* $g_i + h_i$ *is not nef for at most one* $i \in \{1,\dots,9\}$.

Proof. Let $Q = E + E_1 + E_2$, so that $Q^2 = 10, \phi(Q) = 3$ and, by [**11**, Cor. 2.5.5], there is an isotropic effective 10-sequence $\{f_1, \dots, f_{10}\}$ such that $3Q \sim f_1 + \dots + f_{10}$. Since $E.Q = E_2.Q = 3$ we can assume that $f_1 = E, f_{10} = E_2$ and then $f_i.E_1 = 1$ for $i \in \{2, \dots, 9\}$. Suppose $i \leq 9$. By Lemma 4.5, f_i is nef and if $\phi(H - 2f_i) = 1$, then $H - 2f_i = 4F_1 + F_2$ for $F_k > 0$, $F_k^2 = 0$ and $F_1.F_2 = 1$, yielding $f_i.F_1 = 1$, whence $F_1.H = 3$, a contradiction. Therefore $\phi(H - 2f_i) = 2$, so that $H - 2f_i = 2g_i + h_i$ for isotropic $g_i > 0$ and $h_i > 0$ with $g_i.h_i = 2$. One easily sees that g_i and h_i are primitive, $f_i.g_i = 2, f_i.h_i = 1$ and g_i and h_i are quasi-nef by Lemma 4.5. By the ampleness of H and [**22**, Lemma 2.3], it follows that if $g_i + h_i$ is not nef for some $i \leq 9$, then there is a nodal curve R_i with $R_i.g_i = 0$, $R_i.f_i = 1$ and $h_i \equiv f_i + R_i$. Now if $g_i + h_i$ and $g_j + h_j$ are not nef for two distinct $i, j \leq 9$, then $H \equiv 3f_i + 2g_i + R_i \equiv 3f_j + 2g_j + R_j$. Since $f_j.H = 5$ and f_j is nef, we obtain $g_i.f_j = 1$ and $R_i.f_j = 0$. As $(R_i + R_j).H = 2 < \phi(H)$, we get $R_i.R_j \leq 1$. Hence $R_i.H = 1$ implies $R_i.g_j = 0$ and $R_i.R_j = 1$. Similarly $R_j.g_i = 0$, whence we get the absurdity $6 = g_i.H = 3g_i.f_j + 2g_i.g_j + g_i.R_j = 3 + 2g_i.g_j$. q.e.d.

By the claim we can assume that $H \sim 2E + 2E_1 + E_2$ with $E_1 + E_2$ nef. We have $(E_1 + E_2 - E)^2 = -2$. Since $1 = (E_1 + E_2).(E_1 + E_2 - E) < \phi(E_1 + E_2) = 2$, we have that $E_1 + E_2 - E$ is a nodal cycle, if effective. Hence, replacing E with $E + K_S$ if necessary, we can assume that $h^0(E_1+E_2-E) = 0$. As $h^2(E_1+E_2-E) = h^0(E-E_1-E_2+K_S) = 0$ by nefness of E, we get $h^1(E_1 + E_2 - E) = 0$.

Set $D_0 := 2E + E_1$, so that D_0 is nef by Lemma 6.1 and $H - D_0 = E_1 + E_2$ is nef by assumption, whence base-point free. We have $(2D_0 - H).E = -1$, whence $h^0(2D_0 - H) = 0$, and by [**23**, Thm(iii)] we get that Φ_{H_D, ω_D} is surjective. The map μ_{V_D, ω_D} is surjective by Lemma 5.6, with $D_1 = E$ and $D_2 \in |E + E_1|$ a general smooth curve. Indeed, since $h^1(H - D_0 - D_1) = h^1(E_1 + E_2 - E) = 0$, the map $\mu_{V_{D_1}, \omega_{D_1}}$ is surjective by (15). Now $h^0(H - D_0 - D_2) = h^0(E_2 - E) = 0$, whence $h^0(E_2 - 2E) = 0$, so that $h^1(E_2 - 2E) = 1$ by Riemann-Roch. Therefore $h^0(\mathcal{O}_{D_2}(H - D_0 - D_1)) = h^0(\mathcal{O}_{D_2}(E_1 + E_2 - E)) \leq 1$ and $\mu_{V_{D_2}, \omega_{D_2}(D_1)}$ is surjective by (16).

10.2. The case $\beta = 3$. We can assume that $H \sim 3E + \gamma E_1 + M_2$, possibly after replacing E with $E + K_S$. Moreover let us see that

(26) $(\gamma - 1 - \varepsilon)E_1 + M_2$ is quasi-nef for $\varepsilon = 0, 1$.

Let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta.((\gamma - 1 - \varepsilon)E_1 + M_2) \leq -2$. If $\Delta.E_1 < 0$, then $\Delta.E \geq 2$ by the ampleness of H. By [**22**, Lemma 2.3] the divisor $A := E_1 + (E_1.\Delta)\Delta$ is primitive, effective and isotropic and $E.E_1 = 2$ yields the contradiction $E_1.\Delta = -1$, $E.\Delta = 2$ and $E \equiv A$. Hence $\Delta.E_1 \geq 0$, so that $M_2 > 0$ and $l := -\Delta.M_2 \geq 2$. Again we can write $M_2 \sim A_2 + l\Delta$ with $A_2 > 0$ primitive, $A_2^2 = M_2^2$ and

$\Delta.A_2 = l$. If $\Delta.E = 0$, then $\Delta.E_1 \geq 2$ by ampleness of H, whence $E_1.M_2 \geq 4$, so that $\gamma = 2$ by (24), which moreover implies $E_1.M_2 \leq 5$, so that $l = E_1.\Delta = 2$. As $(E_1 + \Delta)^2 = 2$, we must have $2\phi(L_1) \leq (E_1 + \Delta).L_1 = \phi(L_1) + \Delta.((3-\alpha)E + 2E_1 + M_2) = \phi(L_1) + 2$, and we get the contradiction $4 \leq E_1.M_2 \leq E_1.L_1 = \phi(L_1) \leq 2$. Therefore $\Delta.E > 0$, so that $E.M_2 \geq 3$. Thus $E.M_2 = 3$, $\gamma = 2$ and $\phi(H) = 7$ by (22), whence $M_2^2 \leq 4$ by (23). By (24) we must have $E_1.M_2 \leq 3$, but as $H^2 = 42 + 4E_1.M_2 + M_2^2 \geq 54$ by [**22**, Prop. 1.4], using (23), we get $E_1.M_2 = 3$. Since $E_1.(H - 2E) = 5 \leq \phi(H) = 7$ we have $\alpha = 2$, $L_1 \sim E + 2E_1 + M_2$ and $L_2 \sim E + M_2$. Since the latter is of small type and $M_2^2 \leq 4$, we must have $M_2^2 = 0$ or $M_2^2 = 4$. In the latter case we get $L_2^2 = 10$ and $\phi(L_2) = 3$. Now $(E + \Delta)^2 \geq 0$ and $(E + \Delta).M_2 \leq 1$, whence $\phi(M_2) = 1$ and we can write $M_2 \sim 2F_1 + F_2$ for some $F_i > 0$ with $F_i^2 = 0$ and $F_1.F_2 = 1$. Therefore $3 = \phi(L_2) \leq F_1.L_2 = F_1.E + 1$, so that $F_1.E \geq 2$, giving the contradiction $3 = E.M_2 = 2F_1.E + F_2.E \geq 4$. Hence $M_2^2 = 0$, $L_1^2 = 26$ and $\phi(L_1) = E_1.L_1 = 5$, contradicting [**22**, Prop. 1.4]. Therefore (26) is proved.

Now set $D_0 := 2E + E_1$, which is nef by Lemma 6.1. Moreover $H - D_0$ is easily seen to be nef by (26), whence base-point free. We have $h^0(2D_0 - H) = 0$ by nefness of E and (22), whence Φ_{H_D,ω_D} is surjective by [**23**, Thm(iii)].

If $M_2 > 0$ and $(\gamma, E.M_2, E_1.M_2) = (2, 1, 1)$, then $M_2^2 = 0$ by (23), $(H - 2D_0)^2 = -2$ and $(H - 2D_0).H = 0$, whence $h^1(H - 2D_0) = 0$, so that μ_{V_D,ω_D} is surjective by (12).

In the remaining cases, to show the surjectivity of μ_{V_D,ω_D} we apply Lemma 5.6 with $D_1 = E + K_S$ and D_2 general in $|E + E_1 + K_S|$. Since $h^1(H - D_0 - D_1) = h^1((\gamma - 1)E_1 + M_2 + K_S) = 0$ by (26) and [**21**, Cor. 2.5], we have that $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by (15). Similarly, $h^1(H - D_0 - D_2) = h^1((\gamma - 2)E_1 + M_2 + K_S) = 0$, whence $\mu_{V_{D_2},\omega_{D_2}(D_1)} = \mu_{\mathcal{O}_{D_2}(H-D_0),\omega_{D_2}(D_1)}$, which is surjective by [**15**, Cor. 4.e.4] if $M_2 > 0$, since we assume $(\gamma, E.M_2, E_1.M_2) \neq (2, 1, 1)$. If $M_2 = 0$, then $\gamma = 3$ by (22), whence E_1 is nef by Lemma 4.5. In particular, $h^1(H - 2D_0) = h^1(E_1 - E) = 1$ by Riemann-Roch. It is then easily checked that (16) is satisfied, so that $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective.

11. Case (S)

We have $H \sim \alpha E + L_1$ with $L_1^2 > 0$ by Lemma 4.4 and L_1 of small type by hypothesis. We also assume that H is not numerically 2-divisible in $\operatorname{Num} S$ and $H^2 \geq 32$ or $H^2 = 28$.

If $\alpha = 2$ we get $H^2 = 4E.L_1 + L_1^2 = 4\phi(H) + L_1^2$, whence $(\phi(H))^2 \leq 4\phi(H) + L_1^2$ and Lemma 4.3 yields $\phi(H) \leq 5$, incompatible with the hypotheses on H^2. Hence $\alpha \geq 3$. Write $L_1 \sim F_1 + \ldots + F_k$ as in Lemma 4.3 with $k = 2$ or 3 and $E.F_1 \geq \ldots \geq E.F_k$. If $E.F_k > 0$

then $\phi(H) + 1 \leq F_k.(L_1 + E) \leq F_k.L_1 + \frac{1}{k}E.L_1 = F_k.L_1 + \frac{1}{k}\phi(H)$ by definition of α, yielding $F_k.L_1 \geq 3$. As this also holds if $E.F_k = 0$, we get $L_1^2 = 10$, $k = 3$ and $\phi(H) = E.L_1 \leq 4$. Thus we can decompose $L_1 \sim E + E_1 + E_2$ to obtain the following cases

$$(27)\quad H \sim \beta E + E_1 + E_2,\ \beta := \alpha + 1 \geq 4,\ E.E_1 = 1,\ E.E_2 = E_1.E_2 = 2,$$

$$(28)\quad H \sim \beta E + E_1 + E_2,\ \beta := \alpha + 1 \geq 4,\ E.E_1 = E.E_2 = 2,\ E_1.E_2 = 1.$$

Claim 11.1. (i) *In the cases* (27) *and* (28) *we have that* $E + E_2$ *is nef and* E_2 *is quasi-nef.*

(ii) *In case* (27) *both* $nE + E_2 - E_1$ *and* $nE + E_2 - E_1 + K_S$ *are effective and quasi-nef if* $n \geq 2$*, and moreover they are primitive and isotropic if* $n = 2$*.*

Proof. The proof of (i) is similar to many proofs above. As for (ii), note that $h^0(2E + E_2 - E_1) = h^0(2E + E_2 - E_1 + K_S) = 1$ by Lemma 4.5, whence also $h^1(2E + E_2 - E_1) = h^1(2E + E_2 - E_1 + K_S) = 0$ by Riemann-Roch. Since $E.(2E + E_2 - E_1) = 1$, the statement follows for $n = 2$ by [**21**, Cor. 2.5], and consequently for all $n \geq 2$ again by the same result. q.e.d.

Lemma 11.2. *Let* H *be as in* (27) *or* (28)*. Then* S *is nonextendable.*

Proof. We first treat case (27) with $\beta = 4$. Set $D_0 := 3E + E_2$, which is nef by Claim 11.1(i). Then $H - D_0$ is a base-component free pencil by Lemma 6.2. By Claim 11.1(ii) we have $h^0(2D_0 - H) = 1$ and $h^1(H - 2D_0) = 0$, so that Φ_{H_D,ω_D} surjects by [**23**, Thm(iv)] and so does μ_{V_D,ω_D} by (12).

In the general case, set $D_0 := \lfloor \frac{\beta}{2} \rfloor E + E_2$, which is nef by Claim 11.1(i), and $H - D_0$ is base-component free by Lemma 6.2. Since $2D_0 - H \leq E_2 - E_1$ we have $h^0(2D_0 - H) = 0$ as $(E + E_2).(E_2 - E_1) = -1$ in case (27) and $H.(E_2 - E_1) = 0$ in (28). Hence Φ_{H_D,ω_D} is surjective by [**23**, Thm(iii)]. Now if β is even and we are in case (28) we have $h^0(H - 2D_0) = h^2(H - 2D_0) = 0$ as $H.(H - 2D_0) = H.(E_2 - E_1) = 0$. It follows that $h^1(H - 2D_0) = 0$ and consequently μ_{V_D,ω_D} is surjective by (12). We can therefore assume that β is odd in case (28). In particular, $\beta \geq 5$, and we just need to prove the surjectivity of μ_{V_D,ω_D}, for which we will use Lemma 5.7.

We have $h^1(D_0 + K_S - 2E) = 0$ by Claim 11.1(i) and [**21**, Cor. 2.5]. Moreover $h^2(D_0 + K_S - 4E) = 0$ by the nefness of E. As $\beta \geq 5$, we have that $|H - D_0 - 2E|$ is base-component free by Lemma 6.2. Since $(E + E_2).(-E + E_1 - E_2) < 0$, we have that $h^0(H - 2D_0 - 2E) = h^0((\beta - 2\lfloor \frac{\beta}{2} \rfloor - 2)E + E_1 - E_2) \leq h^0(-E + E_1 - E_2) = 0$, whence (9) is equivalent to

$$(29)\qquad h^0(\mathcal{O}_D(H - D_0 - 4E)) \leq \left(\beta - \lfloor \frac{\beta}{2} \rfloor - 2\right) E.E_1 - 1.$$

In the case (28) with $\beta = 5$ we have $\deg \mathcal{O}_D(H - D_0 - 4E) = (-E + E_1).(2E + E_2) = 3$ and D is nontrigonal by [**22**, Cor. 1], therefore $h^0(\mathcal{O}_D(H - D_0 - 4E)) \leq 1$ and (29) is satisfied.

Hence we can assume, for the rest of the proof, that $\beta \geq 5$ in case (27) and $\beta \geq 7$ (and odd) in case (28). This implies $\beta - \lfloor \frac{\beta}{2} \rfloor - 4 \geq -1$ in case (27) and ≥ 0 in case (28), so that we have $h^0((\beta - \lfloor \frac{\beta}{2} \rfloor - 4)E + E_1) = (\beta - \lfloor \frac{\beta}{2} \rfloor - 4)E.E_1 + 1$ by Lemma 6.2 and Riemann-Roch. Hence

$$h^0(\mathcal{O}_D(H - D_0 - 4E)) \leq h^0(H - D_0 - 4E) + h^1(H - 2D_0 - 4E))$$
$$\leq \left(\beta - \lfloor \tfrac{\beta}{2} \rfloor - 4\right) E.E_1 + 1 + h^1\Big(K_S + \left(2\lfloor \tfrac{\beta}{2} \rfloor + 4 - \beta\right) E + E_2 - E_1\Big),$$

and to prove (29) it remains to show

$$h^1\left(K_S + \left(2\lfloor \frac{\beta}{2} \rfloor + 4 - \beta\right) E + E_2 - E_1\right) \leq 2E.E_1 - 2. \tag{30}$$

In case (27) the inequality (30) follows from Claim 11.1(ii). In case (28), as $h^2(K_S + 3E + E_2 - E_1) = h^0(E_1 - 3E - E_2) = 0$ and β is odd, (30) is equivalent to $h^0(N) \leq 2$, where $N := K_S + 3E - E_1 + E_2$. If, by contradiction, $h^0(N) \geq 3$, then we can write $|N| = |M| + \Delta$ for Δ fixed and $h^0(M) \geq 3$. Since $E.N = 0$ and E is nef, we must have $E.M = E.\Delta = 0$, whence $M \sim 2lE$ for an integer $l \geq 2$ and $E_2.\Delta \geq 0$ by the nefness of $E + E_2$. Now $5 = E_2.N \geq 4l \geq 8$, a contradiction. Hence (30) is proved. q.e.d.

12. Proof of Theorem 1.5 and surfaces of genus 15 and 17

We have shown in Sections 5-11 that every Enriques surface $S \subset \mathbb{P}^r$ of genus $g \geq 18$ is nonextendable, thus proving Theorem 1.5. Moreover we have a more precise version if $g = 15$ or $g = 17$:

Proposition 12.1. *Let $S \subset \mathbb{P}^r$ be a smooth Enriques surface with hyperplane divisor H such that $H^2 = 32$ or $H^2 = 28$ and $E > 0$ such that $E.H = \phi(H)$. Then S is nonextendable if H satisfies:*

(a) *$H^2 = 32$ and either $\phi(H) \neq 4$ or $\phi(H) = 4$ and neither H nor $H - E$ are 2-divisible in* Pic S.

(b) *$H^2 = 28$ and either $\phi(H) = 5$ or $(\phi(H), \phi(H - 3E)) = (4, 2)$ or $(\phi(H), \phi(H - 4E)) = (3, 2)$.*

Proof. We have shown that S is nonextendable except for the following ladder decompositions:

(a1) $H \sim 4E + 4E_1$, $E.E_1 = 1$ (see 7.1);
(a2) $H \sim 4E + 2E_1$, $E.E_1 = 2$ (see 7.2);
(a3) $H \sim 3E + 2E_1 + 2E_2$, $E.E_1 = E.E_2 = E_1.E_2 = 1$ (see §9);
(b1) $H \sim 3E + 2E_1 + E_2$, $E.E_1 = E_1.E_2 = 1$, $E.E_2 = 2$ (see Prop. 8.5(i));

(b2) $H \sim 4E + 2E_1 + E_2$, $E.E_1 = E.E_2 = E_1.E_2 = 1$ (see Prop. 8.5(ii)).

One easily sees that cases (a1)-(a3) do not satisfy (a) and (b1)-(b2) do not satisfy (b). Moreover, $H^2 = 32$ in (a1)-(a3) and $H^2 = 28$ in (b1)-(b2). q.e.d.

13. A new Enriques-Fano threefold

We now prove a more precise version of Proposition 1.4.

Proposition 13.1. *There exists an Enriques-Fano threefold $X \subseteq \mathbb{P}^9$ of genus 9 satisfying:*

(a) *X does not have a $\mathbb{Q}$-smoothing. In particular, it does not lie in the closure of the component of the Hilbert scheme made of the examples of Fano-Conte-Murre-Bayle-Sano.*

(b) *Let $\mu : \widetilde{X} \to X$ be the normalization. Then $\widetilde{X}$ has canonical but not terminal singularities, it does not have a $\mathbb{Q}$-smoothing and $(\widetilde{X}, \mu^*\mathcal{O}_X(1))$ does not belong to the list of Fano-Conte-Murre-Bayle-Sano.*

(c) *On the general smooth Enriques surface $S \in |\mathcal{O}_X(1)|$, we have $\mathcal{O}_S(1) \cong \mathcal{O}_S(2E_1 + 2E_2 + E_3)$, where E_1, E_2 and E_3 are smooth irreducible elliptic curves with $E_1.E_2 = E_1.E_3 = E_2.E_3 = 1$.*

Proof. Let $Y \subset \mathbb{P}^{13}$ be the well-known Enriques-Fano threefold of genus 13. By [**13**, **9**] we have that Y is the image of the blow-up of $\mathbb{P}^3$ along the edges of a tetrahedron, via the linear system of sextics double along the edges. This description of Y allows to identify the linear system embedding its general hyperplane section $T \subset \mathbb{P}^{12}$. Let $P_1, \dots, P_4$ be four independent points in $\mathbb{P}^3$, let l_{ij} be the line joining P_i and P_j and denote by $\widetilde{\mathbb{P}}^3$ the blow-up of $\mathbb{P}^3$ along the l_{ij}'s with exceptional divisors E_{ij} and by $\widetilde{H}$ the pull-back of a plane in $\mathbb{P}^3$. Let $\widetilde{L} = 6\widetilde{H} - 2\sum E_{ij}$. Therefore T is just a general element $\widetilde{S} \in |\widetilde{L}|$, embedded with $\widetilde{L}_{|\widetilde{S}}$. Now let $\widetilde{l}_{ij}$ be the inverse image of l_{ij} on $\widetilde{S}$. Then by [**16**, Ch.4, §6], for each pair of disjoint lines l_{ij}, l_{kl} on $\widetilde{S}$ there is a genus one pencil $|2\widetilde{H}_{|\widetilde{S}} - \widetilde{l}_{ik} - \widetilde{l}_{il} - \widetilde{l}_{jk} - \widetilde{l}_{jl}| = |2\widetilde{l}_{ij}|$. Therefore $\widetilde{L}_{|\widetilde{S}} \sim 2\widetilde{l}_{12} + 2\widetilde{l}_{13} + 2\widetilde{l}_{14}$ and the hyperplane bundle of T is as in (c) with $E_i := \widetilde{l}_{1,i+1}$.

Consider the linear span $M \cong \mathbb{P}^3$ of E_3, the projection $\pi_M : \mathbb{P}^{13} - - \to \mathbb{P}^9$ and set $X = \pi_M(Y) \subset \mathbb{P}^9$. Let $\psi : \widetilde{Y} \to Y$ be the blow up of Y along E_3 with exceptional divisor F, set $\mathcal{H} = (\psi^*\mathcal{O}_Y(1))(-F)$ and let $\widetilde{T} \in |\mathcal{H}| \cong |\mathcal{J}_{E_3/Y}(1)|$ be the smooth Enriques surface isomorphic to T. Then one can easily check that $|\mathcal{H}|$ is base-point free and defines a morphism $\varphi_{\mathcal{H}}$ such that $X = \varphi_{\mathcal{H}}(\widetilde{Y}) \subseteq \mathbb{P}^9$. Also $\mathcal{H}^3 = (2E_1 + 2E_2 + E_3)^2 = 16$, whence X is a threefold.

To see that X is not a cone over its general hyperplane section $S := \psi(\widetilde{T})$, consider the four planes $H_1, ..., H_4$ in $\mathbb{P}^3$ defined by the faces of the tetrahedron. As any sextic hypersurface in $\mathbb{P}^3$ that is double on the edges of the tetrahedron and goes through another point of H_i must contain H_i, we see that these four planes are contracted to four singular points $Q_1, \dots, Q_4 \in Y$. Moreover their linear span $\langle Q_1, \dots, Q_4 \rangle$ in $\mathbb{P}^{13}$ has dimension 3, since the hyperplanes containing $Q_1, \dots, Q_4$ correspond to sextics in $\mathbb{P}^3$ containing $H_1, \dots, H_4$. Now suppose that X is a cone with vertex V. Then $Q_1, \dots, Q_4$ project to V, whence $\dim \langle M, Q_1, \dots, Q_4 \rangle \leq 4$ and $\dim M \cap \langle Q_1, \dots, Q_4 \rangle \geq 2$. On the other hand we know that $M = \langle E_3 \rangle \subset \overline{H}$, where $\overline{H}$ is a general hyperplane. Therefore we have that $Q_i \notin \overline{H}, 1 \leq i \leq 4$, whence $\dim \overline{H} \cap \langle Q_1, \dots, Q_4 \rangle = \dim M \cap \langle Q_1, \dots, Q_4 \rangle = 2$, so that $\overline{H} \cap \langle Q_1, \dots, Q_4 \rangle = M \cap \langle Q_1, \dots, Q_4 \rangle$. Now choose the projection from $M' = \langle E_2 \rangle \subset \overline{H}$. If also $\pi_{M'}(Y)$ is a cone then, aguing as above, we get $\overline{H} \cap \langle Q_1, \dots, Q_4 \rangle = M' \cap \langle Q_1, \dots, Q_4 \rangle$, whence $\dim M \cap M' \geq 2$. But this is absurd since $\dim M \cap M' = 6 - \dim \langle E_2 \cup E_3 \rangle = -6 + h^0(\mathcal{O}_T(2E_1 + E_2 + E_3)) = 0$. Hence X is an Enriques-Fano threefold satisfying (c).

Now let X' be the only threefold in $\mathbb{P}^9$ appearing in Bayle-Sano's list, namely an embedding, by a line bundle L', of a quotient by an involution of a smooth complete intersection Z of two quadrics in $\mathbb{P}^5$. Let S' be a general hyperplane section of X'. We claim that the hyperplane bundle $L'_{|S'}$ is 2-divisible in $\operatorname{Num} S'$. As $2E_1 + 2E_2 + E_3$ is not 2-divisible in $\operatorname{Num} S$, this shows in particular that X does not belong to the list of Bayle-Sano.

By [**2**, §3, p. 11], if we let $\pi : Z \to X'$ be the quotient map, we have that $-K_Z = \pi^*(L')$ and the K3 cover $\pi_{|S''} : S'' \to S'$ is an anticanonical surface in Z, that is a smooth complete intersection S'' of three quadrics in $\mathbb{P}^5$. Therefore, if H_Z is the line bundle giving the embedding of Z in $\mathbb{P}^5$, we have $-K_Z = 2H_Z$. Hence, setting $p = \pi_{|S''}$ and $H_{S''} = (H_Z)_{|S''}$, we deduce that $p^*(L'_{|S'}) \cong (\pi^* L')_{|S''} = 2H_{S''}$. Suppose now that $L'_{|S'}$ is not 2-divisible in $\operatorname{Num} S'$. Then $(L'_{|S'})^2 = 16$ and by [**22**, Prop. 1.4] we have that $\phi(L'_{|S'}) = 3$ and it is easily seen that there are three isotropic effective divisors E_1, E_2, E_3 such that either (i) $L'_{|S'} \sim 2E_1 + 2E_2 + E_3$ with $E_1.E_2 = E_1.E_3 = E_2.E_3 = 1$ or (ii) $L'_{|S'} \sim 2E_1 + E_2 + E_3$ with $E_1.E_2 = 1$, $E_1.E_3 = E_2.E_3 = 2$. In case (i) we get that $p^*(E_3) \sim 2D$, for some $D \in \operatorname{Pic} S''$. Since $(p^*(E_3))^2 = 0$, we have $D^2 = 0$ and, as we are on a K3 surface, either D or $-D$ is effective. As $4H_{S''}.D = p^*(L'_{|S'}).p^*(E_3) = 8$, we have $H_{S''}.D = 2$ and D is a conic of arithmetic genus 1, a contradiction. In case (ii) we get that $p^*(E_2 + E_3) \sim 2D'$, for some $D' \in \operatorname{Pic} S''$ with $(D')^2 = 2$ and $H_{S''}.D' = 5$. But now $|D'|$ cuts out a g^2_5 on the general element $C \in |H_{S''}|$ and this is a contradiction since

C is a smooth complete intersection of three quadrics in $\mathbb{P}^4$. Therefore $L'_{|S'}$ is 2-divisible in $\operatorname{Num} S'$.

Now assume that X has a $\mathbb{Q}$-smoothing, that is a small deformation $\mathcal{X} \longrightarrow \Delta$ over the 1-parameter unit disk, such that, if we denote a fiber by X_t, we have that $X_0 = X$ and X_t has only cyclic quotient terminal singularities. Let $L = \mathcal{O}_X(1)$. We have that $H^1(N_{S/X_0}) = H^1(\mathcal{O}_S(1)) = 0$, whence the Enriques surface S deforms with any deformation of X_0. Therefore we can assume, after restricting Δ if necessary, that there is an $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ such that $h^0(\mathcal{L}) > 0$ and $\mathcal{L}_{|X} = L$ (this also follows from the proof of [**17**, Thm. 5], since $H^1(T_{\mathbb{P}^9}{}_{|X}) = 0$). Taking a general element of $|\mathcal{L}|$ we therefore obtain a family $\mathcal{S} \longrightarrow \Delta$ of surfaces whose fibers S_t belong to $|L_t|$, where $L_t := \mathcal{L}_{|X_t}$ and $S_0 = S \in |L|$ is general, whence a smooth Enriques surface with hyperplane bundle $H_0 := L_{|S_0} \sim 2E_1 + 2E_2 + E_3$ of type (i) above. Therefore, after restricting Δ if necessary, we can also assume that the general fiber S_t is a smooth Enriques surface ample in X_t, so that (X_t, S_t) belongs to the list of Bayle [**2**, Thm. B] and is therefore a threefold like $X' \subset \mathbb{P}^9$.

Let $H_t = (L_t)_{|S_t}$. As above $H_t \equiv 2A_t$, for some $A_t \in \operatorname{Pic} S_t$. Taking the limit, we get $H_0 \sim 2E_1 + 2E_2 + E_3 \equiv 2A_0$ for some $A_0 \in \operatorname{Pic} S_0$, yielding that E_3 is 2-divisible in $\operatorname{Num} S_0$, a contradiction.

We have therefore shown that X does not have a $\mathbb{Q}$-smoothing. In particular it does not lie in the closure of the component of the Hilbert scheme consisting of Enriques-Fano threefolds with only cyclic quotient terminal singularities. Hence (a) is proved.

To see (b) note that $\widetilde{Y}$ is terminal (because Y is), whence the morphism $\varphi_{\mathcal{H}}$ factorizes through $\widetilde{X}$. Since $\widetilde{X}$ is $\mathbb{Q}$-Gorenstein by [**6**], an easy calculation, using a common resolution of singularities of $\widetilde{Y}$ and $\widetilde{X}$ and the facts that $-K_{\widetilde{X}} \equiv \mu^*\mathcal{O}_X(1)$ and $-K_Y \equiv \mathcal{O}_Y(1)$, shows that $\widetilde{X}$ is canonical.

Finally, the same proof as above shows that $(\widetilde{X}, \mu^*\mathcal{O}_X(1))$ does not belong to the list of Fano-Conte-Murre-Bayle-Sano and that $\widetilde{X}$ has no $\mathbb{Q}$-smoothing, whence is nonterminal by [**25**, MainThm. 2]. This proves (b). q.e.d.

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