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THE SPACE OF CAUCHY-RIEMANN STRUCTURES ON 3-D COMPACT CONTACT MANIFOLDS

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Abstract

We study the action of the group of contact diffeomorphisms on CR deformations of compact three dimensional CR manifolds. Using anisotropic function spaces and an anisotropic structure on the space of contact diffeomorphisms, we establish the existence of local transverse slices to the action of the contact diffeomorphism group in the neighborhood of a fixed embeddable strongly pseudoconvex CR structure.

1. Introduction

Cauchy-Riemann manifolds arise naturally as the boundary of a bounded domain $D \subset \mathbb{C}^{n+1}$. In this case, the Cauchy-Riemann structure is simply that *residual* complex structure which is inherited from the complex structure on $\mathbb{C}^n$. Local coordinates for ∂D are said to be CR (for Cauchy-Riemann) if they are the restriction of holomorphic coordinates in $\mathbb{C}^{n+1}$, and they define a conjugate CR tangent space for ∂D in the same manner that the holomorphic coordinates on $\mathbb{C}^n$ define a conjugate holomorphic tangent space for $\mathbb{C}^{n+1}$. Intrinsically, one can define the *Cauchy-Riemann* structure on ∂D by specifying the space of conjugate CR tangent vectors in the same manner as one defines the complex structure on $\mathbb{C}^{n+1}$ by specifying the conjugate holomorphic tangent space. All questions which arise for abstract complex structures on a smooth manifold are equally valid for Cauchy-Riemann manifolds: for example, the embeddability and local embeddability, or the existence of holomorphic (CR) coordinates, or how many structures exist up to equivalence.

The significance of generalizing from complex structures on manifolds to studying Cauchy-Riemann structures can easily be seen from the following considerations. When D is a bounded domain in $\mathbb{C}^{n+1}, n \geq 1$, then holomorphic functions on D which extend smoothly to ∂D restrict to ∂D as CR functions; on the other hand, a slight generalization of Hartog's phenomenon in several complex variables states that CR functions on ∂D extend uniquely to D as holomorphic functions; that is, ∂D

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with its Cauchy-Riemann structure completely determines D with its complex structure. On the other hand, if we generalize to Σ a complex analytic space with an isolated singularity at $p \in \Sigma$, then the boundary of a small neighborhood of Σ inherits a smooth Cauchy-Riemann structure, whereas the space Σ is singular. On the basis of this observation, Kuranishi proposed [**Ku**] to study the deformation space for isolated singularities by studying the deformation space for Cauchy-Riemann structures on the boundary of the neighborhood, a smooth compact manifold.

A case of particular interest is that in which the domain D is strongly convex (more generally, strongly pseudoconvex). In this case, the boundary admits a natural family of positive definite metrics which are adapted to the CR structure, and play much the same role that Kähler metrics play in complex geometry. One consequence of particular importance is that when M is compact, strongly pseudoconvex, and $n \geq 2$ (so $\dim M \geq 5$), then M is embeddable. This is definitely not the case when $n = 1$, and this case has many deep and interesting features which have yet to be fully understood.

In this paper, we fix a smooth compact underlying manifold, and study the space of CR structures on the manifold up to equivalence. In particular, we study the local deformation theory for the space of CR structures, and the local action of the contact diffeomorphism group on the space of such structures. Although for much of the paper we set up the machinery to work in arbitrary dimensions, our main interest is in the three dimensional case, and we restrict our attention to this case in the latter sections of this paper. This was largely a matter of expedience, since in higher dimensions integrability factors play a role, and require the introduction of new operators and significantly different treatment than in the three dimensional case.

Most of the results in this paper rely heavily on [**BD1**] in which we developed the machinery to do analysis on contact manifolds using intrinsically defined anisotropic function spaces.

The outline of the paper is as follows. In Section 2, we give a quick review of strongly pseudoconvex Cauchy-Riemann structures and the relevant deformation theory. In Section 3, we define the weighted or anisotropic function spaces in which we will work, and recall the results from [**BD1**] on the space of weighted contact diffeomorphisms which we will need throughout the remainder of the paper. The inclusion of these two sections is to fix notation and to help make the paper self-contained. In Section 4, we study the action of contact diffeomorphisms on CR structures, computing both the linear and the fully nonlinear action; it is also in this section that we introduce the notion of complex contact vector fields, and explain their relation to the symmetry group. In Section 5, we collect results on homotopy operators for the $\bar{\partial}_b$-complex on compact CR manifolds and adapt them to our particular situation; we also indicate how to split complex contact vector fields into real contact vector fields and a transverse vector field. Section 6 contains the main results of the paper. In this section, we obtain normal forms for CR structures under the action of the group of contact diffeomorphisms with sharp regularity results. This is accomplished in two steps: first we obtain a weak normal form with a loss of regularity (Proposition 6.1.2, Corollary 6.1.4), and then using *a priori* estimates (Theorem 6.2.7) we recover the lost regularity (Theorem 6.1.5). It is believed that this approach to studying the action of infinite dimensional symmetry groups on underlying structures is new, and may have applications in other situations.

Leaving rigourous definitions and careful statements until Section 6, we formulate the main results roughly as follows. Let $(M, \bar{\partial}_b)$ be a compact embeddable strongly pseudoconvex CR manifold of dimension 3. Then Theorem 6.1.5 states that there is an open neighbourhoood in the space of CR structures in which all deformation tensors can be placed in the normal form $F^*_{\Psi X}\phi = i\bar{\partial}_b Y + \psi$, where $i\bar{\partial}_b Y$ corresponds to a "Kuranishi wiggle," that is, a deformation arising from deforming the embedding of $(M, \bar{\partial}_b)$ in its ambient surface through a diffeomorphism; ψ represents the obstruction to embeddability; and ΨX is a contact diffeomorphism corresponding to the contact vector field X. (Here, we assume that the CR structures have already been normalized to have the same underlying contact structure.) Moreover, the map $\phi \mapsto (X_\phi, Y_\phi, \psi_\phi)$ is continuous in the Γ^s norms, $s \geq 6$, where Γ^s refers to the Folland Stein anisotropic Sobolev spaces used throughout.

The second result is a regularity result. With $M, \bar{\partial}_b, \phi, X, Y, \psi$ as above, the linearization of the normalizing map at $\phi = 0$ is $\phi = \bar{\partial}_b(X - iY) + \psi$, and using homotopy operators, it easily follows that if $\phi \in \Gamma^s$, then $X, Y \in \Gamma^{s+1}, \psi \in \Gamma^s$; that is, the homotopy operators induce a linear decomposition of ϕ into $\bar{\partial}_b-$exact forms and their complement, where ψ is the image of ϕ under projection onto the complement. Theorem 6.2.7 states that this same regularity holds for the local nonlinear decomposition of ϕ: if $F^*_{\Psi X}\phi = -i\bar{\partial}_b Y + \psi$ where $\phi \in \Gamma^s$, then $X, Y \in \Gamma^{s+1}, \psi \in \Gamma^s$ with corresponding estimates.

Earlier results on normal forms were obtained in [**CL**] and [**B**]. The main idea in both papers was to study the linearized action, and to construct appropriate function spaces in which one can solve the linearized equation with good estimates. Since the $\bar{\partial}_b$–operator appears in the linearized equation, the anisotropic function spaces appear naturally. In [**CL**], they avoided using the anisotropic spaces by working in the Nash Moser category; they obtained a transverse slice for smooth CR structures. In [**B**], we restricted our attention to the case of the standard $S^3 \subset \mathbb{C}^2$, and used explicit information to construct an anisotropic Hilbert space structure on contact diffeomorphisms near the identity; the description of transverse slices follows easily from the linearized analysis. However, in [**B**], the action described for the contact diffeomorphism group was incorrectly asserted to be C^1, a necessary condition to apply the inverse function theorem in Banach spaces and obtain the transverse slices; a modified action is used in Section 6 of the current paper to correct this error, combined with a new regularity result to recover the lost regularity. With this modification and the generalization of the weighted function space structure for contact diffeomorphisms to arbitrary compact contact manifolds (see [**BD1**]), we are now able to obtain local transverse slices to the action of the contact diffeomorphism group on the space of CR structures for an arbitrary compact embeddable strongly pseudoconvex three dimensional CR structure.

1.1. Notation. Throughout the paper, M will denote a smooth compact $2n+1$ dimensional manifold equipped with a fixed contact distribution $H \subset TM$ and a fixed contact one-form η. As usual, TM and T^*M denote the tangent and cotangent bundles of M, respectively; $\Lambda^p M$ denotes the p-th exterior power of T^*M, $\Omega^p(M)$ the space of smooth p-forms on M, $\mathcal{L}_X\beta$ the Lie derivative of the form β with respect to the vector field X, and $X \lrcorner\, \beta$ interior evaluation.

We give M a fixed Riemannian metric g compatible with η (see Equation (2.1.3) for details), and let $|X|$ denote the norm of the tangent vector X with respect to g, and we let $\exp : TM \to M$ denote the exponential map of the g.

We let

$$\pi_H : T^*M \to H^*$$

denote the projection map. The *characteristic* (or *Reeb*) vector field T is the unique vector field satisfying the conditions $T \lrcorner\, \eta = 1$ and $T \lrcorner\, d\eta = 0$. We can then identify the dual contact distribution with the annihilator of T, i.e.

$$H^* = \{\beta \in T^*M \ : \ T \lrcorner\, \beta = 0\} \subset T^*M;$$

more generally,

$$\Lambda^p H^* = \{\beta \in \Lambda^p(M) \ : \ T \lrcorner\, \beta = 0\},$$

and we have the identity

$$\pi_H(\beta) = T \lrcorner\, (\eta \wedge \beta). \tag{1.1.1}$$

We endow $\mathbb{R}^{2n+1}$ with the contact structure defined by the one-form

$$\eta_0 = dx^{2n+1} - \sum_{j=1}^{n} x^{n+j} dx^j,$$

where $(x^1, \dots, x^n, x^{n+1}, \dots, x^{2n}, x^{2n+1})$ are the standard coordinates on $\mathbb{R}^{2n+1}$, and we let dV_0 denote the standard volume form:

$$dV_0 = \frac{1}{n!}\eta_0 \wedge (d\eta_0)^n.$$

We denote the contact distribution of η_0 by $H_0 \subset T\mathbb{R}^{2n+1}$ and we set

$$T_0 = \frac{\partial}{\partial x^{2n+1}}\,, X_j = \frac{\partial}{\partial x^j} + x^{n+j}\,\frac{\partial}{\partial x^{2n+1}}, \text{ and } X_{n+j} = \frac{\partial}{\partial x^{n+j}}\,, 1 \le j \le n.$$

Observe that the collection $\{X_j, 1 \le j \le 2n\}$ is a global framing for H_0. Note also that the 1-forms

$$\eta_0, dx^j,\, dx^{n+j},\, 1 \le j \le n,$$

are the dual coframe to T_0, X_j, X_{n+j}, $1 \le j \le n$.

Let $f = (f^1, \dots, f^m)$ be a smooth, $\mathbb{R}^m$-valued function defined on the closure of a domain $D \Subset \mathbb{R}^{2n+1}$. We define

$$X_I f = \begin{cases} X_{i_1} X_{i_2} \dots X_{i_t} f & \text{for } t > 0 \\ f & \text{for } t = 0\,, \end{cases}$$

where we have introduced the multi-index notation $I = (i_1, \dots, i_t)$, $1 \le i_j \le 2n$, and $X_I f = (X_I f^1, \dots, X_I f^m)$. (For $t = 0$, I denotes the *empty index* $I = ()$.) The integer t is called the *order* of I and written $|I|$.

Remark 1.1.2. We will often have to work in local coordinates adapted to the contact structure on M. An *adapted coordinate chart* for M is a chart $\phi : U \to \mathbb{R}^{2n+1}$ for which $\eta = \phi^*\eta_0$. It follows that $\phi_* T = T_0$ and $\phi_* H = H_0$. An *adapted atlas* consists of the following data: a fixed finite open cover $V_\ell \Subset U_\ell$, $\ell = 1, 2, \dots, m$ and an atlas $\{\phi_\ell : U_\ell \to \mathbb{R}^{2n+1}\}$, consisting of adapted coordinate charts. We set $D_\ell = \phi_\ell(V_\ell)$. By compactness of M and Darboux's Theorem for contact structures [**Arn**, page 362], M has an adapted atlas. We shall fix once and for all an adapted atlas and a partition of unity ρ_ℓ subordinate to $\{V_\ell\}$.

If $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a map between Banach spaces, with norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, respectively, then the expression

$$\|\mathcal{F}(f)\|_{\mathcal{B}} \prec \|f\|_{\mathcal{A}}$$

means that there is a constant $C > 0$ such that $\|\mathcal{F}(f)\|_{\mathcal{B}} \le C\|f\|_{\mathcal{A}}$ for all $f \in \mathcal{A}$.

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2. CR Structures

2.1. Deformation theory of CR structures. We begin with a quick review of the deformation theory of CR structures as presented in the paper of Akahori, Garfield, and Lee [**AGL**]. See also [**BD2**, section 16], where the special case of the deformation theory of S^{2n+1} is studied using a similar framework.

Definition 2.1.1. Let M be a $2n+1$ dimensional manifold. A (rank n) *Cauchy-Riemann structure (CR structure)* on M is a rank n complex subbundle $H_{(1,0)} \subset T_{\mathbb{C}}M$ of the complexified tangent bundle of M such that

(i) $H_{(1,0)} \cap \overline{H_{(1,0)}} = \{0\}$,
(ii) the integrability condition is satisfied:

$$[\Gamma^\infty(H_{(1,0)}), \Gamma^\infty(H_{(1,0)})] \subset \Gamma^\infty(H_{(1,0)})\,.$$

The bundle $H_{(1,0)}$ is called the *holomorphic tangent bundle* of the CR structure. As usual, we let $H_{(0,1)}$ denote the conjugate bundle $\overline{H_{(1,0)}}$. The transversality condition (i) implies that $H_{\mathbb{C}} = H_{(1,0)} \bigoplus H_{(0,1)} \subset T_{\mathbb{C}}M$ has complex codimension one.

Remark 2.1.2. We recall that when $n = 1$, the bundle $H_{(1,0)}$ is a complex line bundle, and condition (ii) is automatic. To see this, let Z be a section of $H_{(1,0)}$ that does not vanish on an open set $U \subset M$. Then since $[Z, Z] = 0$,

$$[fZ, gZ] = (fZ(g) - gZ(f))\,Z$$

for any two sections, say $X = fZ$ and $Y = gZ$ of $H_{(1,0)}$.

Two CR structures $H_{(1,0)}$ and $\hat{H}_{(1,0)}$ are said to be *equivalent* if there is a diffeomorphism $F : M \to M$ such that $F_* H_{(1,0)} = \hat{H}_{(1,0)}$. We are only interested in CR structures up to equivalence.

Observe that $H_{\mathbb{C}}$ is the complexification of a real codimension one subbundle $H \subset TM$ consisting of vectors of the form $X + \overline{X}$, $X \in H_{(1,0)}$. Let η be a real one-form dual to H. The CR structure $H_{(1,0)}$ is said to be *strongly pseudoconvex* if $-i\, d\eta(X, \overline{X}) > 0$ for all non-zero $X \in H_{(1,0)}$. In this case, $\eta \wedge (d\eta)^n$ is a nowhere vanishing $(2n+1)$-form. In other words, (M, H) is a contact manifold and η is a contact one-form.

The most common examples of CR structures are those arising from domains in $\mathbb{C}^{n+1}$. Let $D = \{z \in \mathbb{C}^{n+1} \ : \ \rho(z) < 1\}$ be a smoothly bounded domain in $\mathbb{C}^{n+1}$ with connected boundary, where ρ is a smooth nonnegative function defined on a neighborhood of $\overline{D}$, and $d\rho \neq 0$ on ∂D. The boundary ∂D is a CR-manifold for which the holomorphic tangent bundle is the intersection of the complexified tangent bundle of ∂D with the holomorphic tangent bundle of $\mathbb{C}^{n+1}$, and if the pull-back to ∂D of the one-form $i\bar{\partial}\rho$ is a contact form, then it is strongly pseudoconvex.[1]

We will assume, henceforth, that M is a contact manifold equipped with a fixed strongly pseudoconvex CR structure $H_{(1,0)} \subset T_{\mathbb{C}}M$ such that $H_{\mathbb{C}}$ is the complexification of the contact distribution of M. We shall refer to this CR structure as the *reference CR structure* on M.

The reference CR structure determines an endomorphism $J : H \to H$ satisfying the condition $J^2 = -Id$, which in turn defines a Riemannian metric g by the formula

$$g(X,Y) = \eta(X)\eta(Y) + d\eta(X, JY)\,. \tag{2.1.3}$$

The metric g is said to be *adapted to the CR structure*. We let $\exp : TM \to M$ denote the exponential map of g. Objects associated to any other CR structure on M will be decorated with hats. Two strongly pseudoconvex CR structures on M are said to be *isotopic* if they can be connected by a smooth 1-parameter family of strongly pseudoconvex CR structures. We consider only strongly pseudoconvex CR structures which are isotopic to the reference CR structure.

2.2. Representation by deformation tensors. Every CR structure that is isotopic to the reference one can be represented by a *deformation tensor* that takes values in $H_{(1,0)}$. The proof of this fact relies on a theorem of John Gray [**G**] which states that isotopic contact structures on a compact manifold are equivalent.

Theorem 2.2.1 (Gray). *Let η_t be a differentiable family of contact forms on a compact $2n+1$ dimensional manifold M. Then there is a differentiable family of diffeomorphisms $F_t : M \to M$ and a family of non-vanishing functions p_t such that*

$$F_t{}^*(\eta_t) = p_t\eta_0\,.$$

Corollary 2.2.2. *Every strongly pseudoconvex CR structure on M that is isotopic to the reference one is CR-equivalent to one of the form $\hat{H}_{(1,0)}$ where*

$$\hat{H}_{(0,1)} = \{X - \phi(X) \;:\; X \in H_{(0,1)}\} \tag{2.2.3}$$

and $\phi : H_{(0,1)} \to H_{(1,0)}$ is a map of complex vector bundles, called the deformation tensor *for $\hat{H}_{(1,0)}$.*

Proof. The fact that the CR structure is equivalent to one satisfying the inclusion relation $\hat{H}_{(0,1)} \subset H_{\mathbb{C}}$ follows immediately from Gray's theorem. Thus, there is a family $\hat{H}_{(0,1)}(t)$, $t \in [0,1]$, joining $H_{(0,1)}$ to

[1] The fact that D is bounded forces the Levi form to be positive at some point on ∂D, and hence by the non-degeneracy of $d\eta$ everywhere on the connected manifold ∂D.

$\hat{H}_{(0,1)}$. For t small, it is clear that there are bundle maps $\phi(t)$ such that $\hat{H}_{(0,1)}(t)$ is the graph of $-\phi(t)$. The integrability conditions for CR structures imply that $\phi(t)$ satisfies certain symmetry properties, and when combined with the transversality condition they imply an *a priori* bound on the size of $\phi(t)$, from which the result follows.

We explain in brief. Choose a local basis Z_α for $H_{(1,0)}$, and let $i\eta[Z_\alpha, Z_{\bar{\beta}}] = -id\eta(Z_\alpha, Z_{\bar{\beta}}) = h_{\alpha\bar{\beta}}$ define the Levi form.[2] Then integrability implies in particular that $i\eta[Z_{\bar{\beta}} - \phi^\alpha_{\bar{\beta}} Z_\alpha, Z_{\bar{\delta}} - \phi^\gamma_{\bar{\delta}} Z_\gamma] = 0$. Since $\eta[Z_\alpha, Z_\gamma] = \eta[Z_{\bar{\beta}}, Z_{\bar{\delta}}] = 0$, it follows that $i\eta[-\phi^\alpha_{\bar{\beta}} Z_\alpha, Z_{\bar{\delta}}] + i\eta[Z_{\bar{\beta}}, -\phi^\gamma_{\bar{\delta}} Z_\gamma]$ $= \phi^\gamma_{\bar{\delta}} h_{\gamma\bar{\beta}} - \phi^\alpha_{\bar{\beta}} h_{\alpha\bar{\delta}} = 0$; this is the symmetry condition $\phi^\gamma_{\bar{\delta}} h_{\gamma\bar{\beta}} = \phi^\alpha_{\bar{\beta}} h_{\alpha\bar{\delta}}$. It follows that the operator $\phi \circ \bar{\phi} : H_{(1,0)} \to H_{(1,0)}$ has non-negative eigenvalues since

$$i\eta[(\phi \circ \bar{\phi}) Z_\alpha, Z_{\bar{\mu}}] = (\phi \circ \bar{\phi})^\delta_\alpha h_{\delta\bar{\mu}} = (\bar{\phi})^{\bar{\gamma}}_\alpha \phi^\delta_{\bar{\gamma}} h_{\delta\bar{\mu}} = h^{\bar{\gamma}\beta} (\bar{\phi})_{\beta\alpha} \phi_{\bar{\gamma}\bar{\mu}}$$

is hermitian positive semi-definite.

Next note that the transversality condition for CR structures (Definition 2.1.1(ii)) implies that none of the eigenvalues of $\phi \circ \bar{\phi}$ can be equal to one. Indeed, suppose to the contrary that $(\bar{\phi})^{\bar{\gamma}}_\alpha \phi^\delta_{\bar{\gamma}} - \delta^\delta_\alpha$ is a degenerate matrix. Then there exists v^α such that $v^\alpha (\bar{\phi})^{\bar{\gamma}}_\alpha \phi^\delta_{\bar{\gamma}} = v^\delta$, from which one obtains the relation

$$v^\alpha (Z_\alpha - (\bar{\phi})^{\bar{\beta}}_\alpha Z_{\bar{\beta}}) = v^\alpha (\bar{\phi})^{\bar{\beta}}_\alpha \phi^\delta_{\bar{\beta}} Z_\delta - v^\alpha (\bar{\phi})^{\bar{\beta}}_\alpha Z_{\bar{\beta}} = -v^\alpha (\bar{\phi})^{\bar{\beta}}_\alpha (Z_{\bar{\beta}} - \phi^\delta_{\bar{\beta}} Z_\delta);$$

that is, the transversality condition is violated for the subspace $\hat{H}_{(1,0)}(t)$ and its conjugate.

Since $\phi \circ \bar{\phi}$ is isotopic to the zero map by assumption, has non-negative eigenvalues, and $(\phi \circ \bar{\phi} - I)$ is nondegenerate, it follows that the eigenvalues of the operator $(\phi \circ \bar{\phi})$ are bounded between 0 and 1, which implies the norm condition. (See [**BD2**, page 83] where a similar argument is given.) q.e.d.

Remark 2.2.4. The choice to refer to the map $\phi : H_{(0,1)} \to H_{(1,0)}$ as the deformation tensor (rather than the conjugate map) is consistent with the deformation theory for complex structures, and has the advantage that ϕ may be thought of as a "vector-valued $(0,1)-$form," thus fitting naturally within a $\bar{\partial}-$complex (or in this case, a $\bar{\partial}_b-$complex).

In light of Corollary 2.2.2, we identify the space of CR structures with the subset of the space of $H_{(1,0)}$-valued $(0,1)$-forms. Specifically, if ω^α

[2]Here, and for the remainder of this section, we employ the Einstein summation conventions, with Greek indices ranging from 1 to n, and the conventions for raising and lowering indices by contraction with the hermitian form $h_{\alpha\bar{\delta}}$ and its inverse, with $\phi^\gamma_{\bar{\beta}} h_{\gamma\bar{\delta}} = \phi_{\bar{\beta}\bar{\delta}}$.

is a local coframe of $H^{(1,0)}$ with dual frame Z_α of $H_{(1,0)}$ such that

$$d\eta = \frac{i}{2}\delta_{\alpha\bar{\beta}}\omega^\alpha \wedge \omega^{\bar{\beta}},$$

then the *CR deformation tensor* can be written as

$$\phi = \phi^\alpha_{\bar{\beta}}\,\omega^{\bar{\beta}} \otimes Z_\alpha;$$

it uniquely determines the space of $(0,1)$-vectors for its corresponding CR structure as the subspace of $H_{\mathbb{C}}$ annihilated by the one-forms

$$\hat{\omega}^\alpha := \omega^\alpha + \phi^\alpha_{\bar{\beta}}\omega^{\bar{\beta}}.$$

The space of all smooth deformation tensors is given by

$$\mathcal{D}ef = \Omega^{(0,1)}(H_{(1,0)}) \simeq \Gamma^\infty\left(H_{(0,1)} \otimes H_{(1,0)}\right). \tag{2.2.5}$$

2.3. The deformation complex. Each deformation of a CR structure can be expressed as an $H_{(1,0)}$-valued (0,1)-form. In [**Aka**], Akahori studied CR deformations by developing the Hodge theory of a certain complex of vector-valued forms. A similar complex was studied in [**BD2**] and used to show that CR deformations of the standard CR structure on S^{2n+1} can be parameterized by *complex Hamiltonian vector fields*.

The space of smooth *forms of type* $(0,q)$, written $\Omega^{(0,q)}(M)$, is the space of sections of the bundle $\Lambda^q H^{(0,1)}$, where $H^{(0,1)}$ denotes the dual bundle of the complex vector bundle $H_{(0,1)}$. By the integrability condition for the CR structure, the exterior differential operator d naturally induces an operator

$$\bar{\partial}_b : \Omega^{(0,q)}(M) \to \Omega^{(0,q+1)}(M).$$

Set $T_{(1,0)}M = T_{\mathbb{C}}M/H_{(0,1)}$, where $T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ is the complexified tangent bundle of M, and let $\pi_{(1,0)} : T_{\mathbb{C}}M \to T_{(1,0)}M$ denote the quotient map. The space of *$T_{(1,0)}M$-valued forms of type* $(0,q)$ is the space of homomorphisms of complex vector bundles

$$\Omega^{(0,q)}(T_{(1,0)}M) = \Gamma^\infty\left(\mathrm{Hom}_{\mathbb{C}}(\Lambda^q H_{(0,1)}M, T_{(1,0)}M)\right).$$

By virtue of the integrability condition (Definition 2.1.1(ii)), the operator $\bar{\partial}_b$ extends to an operator on the space of $T_{(1,0)}M$-valued forms [**BlEp**, **BuMi**], which by abuse of notation we again denote by $\bar{\partial}_b$:

$$\bar{\partial}_b : \Omega^{(0,q)}(T_{(1,0)}M) \to \Omega^{(0,q+1)}(T_{(1,0)}M). \tag{2.3.1}$$

This operator is characterized by the following properties:

$$\bar{\partial}_b^2 = 0; \tag{2.3.2a}$$

$$\bar{\partial}_b(X)(\overline{Z}) = \pi_{(1,0)}[\overline{Z}, X], \tag{2.3.2b}$$

for $X \in \Omega^{(0,0)}(T_{(1,0)}M) = \Gamma(T_{(1,0)}M)$ and $\overline{Z} \in \Gamma(H_{(0,1)}M)$;

$$\bar{\partial}_b(\alpha \wedge \beta) = (\bar{\partial}_b\alpha) \wedge \beta + (-1)^{q_1}\alpha \wedge \bar{\partial}_b\beta, \tag{2.3.2c}$$

for $\alpha \in \Omega^{(0,q_1)}(M)$ and $\beta \in \Omega^{(0,q_2)}(T_{(1,0)}M)$.

Remark 2.3.3. The operator defined by Equation (2.3.2b) further lifts to an operator

$$\bar{\partial}_b : \Gamma(T_{\mathbb{C}}M) \to \Omega^{(0,1)}(T_{(1,0)}M)$$

via the formula

$$\bar{\partial}_b X := \bar{\partial}_b(\pi_{(1,0)}X)\,.$$

In particular, $\bar{\partial}_b X$ is well-defined in the special case where X is a real vector field. By abuse of notation, we again denote the lifted operator by $\bar{\partial}_b$.

Remark 2.3.4. When $(M, \eta, H_{(1,0)})$ is embedded, it bounds a strongly pseudoconvex complex space Σ, and there is a natural identification between $T_{(1,0)}M$ and the restriction of the holomorphic tangent bundle from Σ. In this case, $\bar{\partial}_b$ is naturally identified with the restriction of the $\bar{\partial}$ operator to the boundary.

3. The Group of Folland-Stein Contact Diffeomorphisms

In the previous section, we showed that every CR structure isotopic to a reference CR structure can be represented by a deformation tensor. In Section 4, we study the action of the group of contact diffeomorphisms on to space of CR structures. In this section, we recall the results from [**BD1**] that we need. Details can be found in [**BD1**].

3.1. Folland-Stein spaces. We begin by recalling the anisotropic function spaces $\Gamma^s(M)$ on M, introduced by Folland and Stein in [**FS**], and their generalizations. These spaces are the natural ones in which to work in order to obtain sharp estimates for the various operators which will arise.

Consider an open domain $D \Subset \mathbb{R}^{2n+1}$. The *Folland-Stein space* $\Gamma^s = \Gamma^s(D)$ is the Hilbert space completion of the set of smooth functions on $\overline{D}$ with respect to the inner product

$$(f, g)_{D,s} := \sum_{0 \le |I| \le s} \int_D |X_I f|\,|X_I g| dV,$$

with associated norm written $\|f\|_{D,s} = \sqrt{(f,f)}$, where X_I and dV are as in Section 1.1. Let $\Gamma^s(D, \mathbb{R}^m)$ denote the closure of the smooth $\mathbb{R}^m$ valued functions on $\overline{D}$ with inner product

$$(f, g)_{D,s} = \sum_{j=1}^{m} (f^j, g^j)_s$$

for smooth functions $f = (f^1, \ldots, f^m)$ and $g = (g^1, \ldots, g^m)$.

Let (M, η) be a smooth compact contact manifold, and let $\{\phi_\ell : \overline{U}_\ell \to \mathbb{R}^{2n+1}\}$ be an adapted atlas as in Section 1.1. A function $f : M \to \mathbb{R}$ is said to be a Γ^s *function* if the functions $f_\ell = f \circ \phi_\ell^{-1}$ lie in $\Gamma^s(D_\ell)$ for all ℓ. For functions $f, g \in \Gamma^s(M)$, we define the inner product

$$(f,g)_s := \sum_\ell (\rho_\ell f_\ell, \rho_\ell g_\ell)_{D_\ell, s}\,.$$

Similar definitions hold for $\Gamma^s(M, \mathbb{R}^m)$, $f, g \in \Gamma^s(M, \mathbb{R}^m)$. The definition of the function spaces is independent of the choice of adapted atlas and the local framings X_I and dV. Although the definition of the inner products depends upon the choices involved, different choices lead to equivalent norms.

Let $F : M \to \widetilde{M}$ be a C^1-map from M into a smooth $\widetilde{m}$-dimensional manifold $\widetilde{M}$. Choose an adapted atlas $\{(\phi_\ell, U_\ell, V_\ell)\}$ for M and a smooth atlas $\{\widetilde{\phi}_\ell : \widetilde{U}_\ell \to \mathbb{R}^{\widetilde{m}}\}$ for $\widetilde{M}$ such that

$$F(U_\ell) \subset \widetilde{U}_\ell, \text{ and } F_\ell(\bar{D}_\ell) \subset \widetilde{D}_\ell$$

for all ℓ, where $F_\ell = \widetilde{\phi}_\ell \circ F \circ \phi_\ell^{-1} : \phi_\ell\,(U_\ell) \to \mathbb{R}^{\widetilde{m}}$ and $\widetilde{D}_\ell \Subset \phi_\ell(\widetilde{U}_\ell)$ is a collection of open domains such that $\{\widetilde{\phi}_\ell^{-1}(\widetilde{D}_\ell)\}$ covers $\widetilde{M}$. The map F is said to be a Γ^s *map* if F_ℓ restricts to an element $F_\ell \in \Gamma^s(D_\ell, \mathbb{R}^{\widetilde{m}})$ for all ℓ. It is not difficult to show that the notion of a Γ^s map is independent of the choice of atlases and that F_ℓ restricts to an element in $\Gamma^s(D)$ for any open set $D \subset\subset \phi_\ell\,(U_\ell)$.

Let $\Gamma^s(M, \widetilde{M})$ for $s \geq n+4$ denote the topological space of Γ^s maps between M and $\widetilde{M}$. The restriction $s \geq n+4$ ensures that the maps are C^1. More generally, consider a smooth fiber bundle $\pi : P \to M$, with base a compact contact manifold. The space $\Gamma^s(P)$ of Γ^s sections of π is defined in the obvious way by choosing an adapted atlas for M such that $\pi^{-1}(U_\ell) \to U_\ell$ is trivial for all ℓ and requiring the local coordinate representations of sections to be Γ^s maps from U_ℓ into the fiber of π. (See [**BD1**] for details.)

3.2. The smooth manifold of Folland-Stein diffeomorphisms. Let $\mathcal{D}^s(M) \subset \Gamma^s(M, M)$ denote the space of Γ^s diffeomorphisms of M. We showed in [**BD1**] that $\mathcal{D}^s(M)$ is an open subset of $\Gamma^s(M, M)$ for all $s \geq 2n+4$. Let $\mathcal{D}^s_{cont}(M) \subset \mathcal{D}^s(M)$ denote the subspace of Γ^s contact diffeomorphisms of M. In [**BD1**], we obtained a local coordinate chart for contact diffeomorphisms in a neighborhood of the identity, and we showed that $\mathcal{D}^s_{cont}(M)$ is a topological group with respect to composition, provided that $s \geq 2n+4$.

More precisely, let g be a metric adapted to the contact structure such as the one constructed in Section 1.1. The exponential map induces various maps between Γ^s spaces that we need to parameterize contact diffeomorphisms. If X is a vector field, we use the notation F_X to denote the map

$$F_X := \exp \circ X : M \to M\,. \tag{3.2.1}$$

Recall that because M is compact, the map F_X is a diffeomorphism for X sufficiently small. The following proposition summarizes various smoothness properties of the maps that we need to construct our local coordinate charts for contact diffeomorphisms.

Proposition 3.2.2. *Let $\Gamma^s(TM)$ denote the space of Γ^s sections of TM. For $s \geq (2n+4)$, the map*

$$\exp : \Gamma^s(TM) \to \Gamma^s(M, M) \, : \, X \mapsto F_X = \exp \circ X$$

is smooth. Moreover, there is a neighborhood $\mathcal{U} \subset \Gamma^{2n+4}(TM)$ such that F_X is in $\mathcal{D}^s(M)$ for all $X \in \mathcal{U}^s$ and all $s \geq 2n+4$, where $\mathcal{U}^s := \mathcal{U} \cap \Gamma^s(TM)$; and the restriction

$$\exp : \mathcal{U}^s \to \mathcal{D}^s(M)$$

is a homeomorphism from $\mathcal{U}^s$ to a neighborhood of the identity diffeomorphism.

In general, the diffeomorphism F_X of Proposition 3.2.2 will not be a contact diffeomorphism. However, in [**BD1**] we showed that the subset of $\mathcal{U}^s$ for which it is a contact diffeomorphism is smoothly parameterized by the set of contact vector fields in a neighborhood of the zero section. As shown in [**BD1**], this implies that the space of Γ^s contact diffeomorphisms is a smooth Hilbert manifold.

We now introduce some notation that will be necessary to express the sharp estimates used later in the paper. Choose an adapted atlas $\phi_\ell : U_\ell \to \mathbb{R}^{2n+1}$ for M and a collection of open sets $V_\ell \Subset U_\ell$ covering M as in Section 1.1 and let ρ_ℓ be a partition of unity subordinate to $\{V_\ell\}$. By compactness of M, there is a constant $c > 0$ such that $\exp(x, X) \in U_\ell$ for all $x \in \overline{V_\ell}$, all $X \in TM_x$, with $|X| < c$, and all ℓ. Let X be a C^1 vector field with $|X| < c$.

Fix a chart, say ϕ_ℓ, and set $U = U_\ell$ and $V = V_\ell$. To simplify notation, we adopt the Einstein summation conventions, letting Roman indices range from 1 to $2n+1$. As explained in [**BD1**], by the second order Taylor's formula with integral remainder, there exist smooth functions $B^k_{ij}(x, X)$ (locally defined) on TM such that

$$F^k_X := \exp^k(x, X) = x^k + X^k + B^k_{ij}(x, X) X^i X^j \, . \tag{3.2.3}$$

A standard computation using Equation (3.2.3) then yields the following expansion for the pull-back of a q-form by F_X.

Lemma 3.2.4 ([**BD1**])**.** *Let ψ be a smooth q-form on M and choose a coordinate patch $U = U_\ell$, with $V = V_\ell \Subset U$. Let $c > 0$ be chosen so that $\exp(x, X) \in U$ for all $x \in \overline{V}$ and all $X \in T_x M$ with $|X| < c$. Then there are (locally defined) smooth fiber bundle maps*

$$Q^1_{ij} : BM|_V \to \Lambda^q M|_V \ \text{ and } Q^2_{ij} : BM|_V \to \Lambda^{q-1} M\big|_V \, ,$$

where $BM = \{X \in TM \;:\; |X| < c\}$, *such that for any* C^1 *vector field* $X : M \to BM \subset TM$ *the equation*

$$F_X^*\psi = \psi + \mathcal{L}_X\psi + Q^1_{ij}(X)\, X^i X^j + Q^2_{ij}(X)\, \wedge X^i dX^j$$

is satisfied on all of V.

Henceforth, we will use the notation

$$Q_\psi(X) := F_X^*(\psi) - \psi - \mathcal{L}_X\psi \tag{3.2.5}$$

to denote the non-linear part of the pull-back $F_X^*\psi$. The lemma states that in local coordinates

$$Q_\psi(X) = Q^1_{ij}(X)\, X^i X^j + Q^2_{ij}(X) \wedge\, X^i dX^j \tag{3.2.6}$$

where Q^1_{ij} and Q^2_{ij} are smooth differential forms on $BM|_V \subset TM$, which depend on the smooth form ψ and on the coordinate chart ϕ_ℓ. Because the maps Q^a_{ij} are smooth differential forms for any smooth q-form ψ, and because M is compact, we have the following corollary to Lemma 3.2.4, which we prove in [**BD1**]:

Lemma 3.2.7. *Let* ψ *be a smooth* q *form. Then the following estimates are satisfied for all* $X \in \Gamma^s(TM)$, $s \geq 2n+6$, *such that* $|X| < c$:

(a) $$\|F_X^*\psi\|_{s-2} \prec \|\psi\|_{s-2} + \|\mathcal{L}_X\psi\|_{s-2} + \|X\|_{s-2}\, \|X\|_s\,,$$

(b) $$\|(F_X^*\psi) \wedge \eta\|_{s-1} \prec \|\psi \wedge \eta\|_{s-1} + \|\mathcal{L}_X\psi \wedge \eta\|_{s-1} + \|X\|_{s-1}\, \|X\|_s\,.$$

Moreover, the estimate

(c)
$$\begin{aligned}\|\,(Q_\psi(X_1) - Q_\psi(X_2)) \wedge \eta\|_{s-1} \prec \quad & \|X_1 - X_2\|_{s-1}\,(\|X_1\|_s + \|X_2\|_s) \\ & + \|X_1 - X_2\|_s\,(\|X_1\|_{s-1} + \|X_2\|_{s-1})\,,\end{aligned}$$

holds for any two vector fields X_i, $i = 1, 2$ *with* $|X_i| < c$.

Remark 3.2.9. As shown in [**BD1**], for ψ a smooth p-form, the maps $X \mapsto F_X^*\psi$ and $X \mapsto \eta \wedge F_X^*\psi$ define smooth maps

$$\Gamma^s(TM) \to \Gamma^{s-2}(\Lambda^p M)\,, \text{ for } s \geq 2n+6,$$

and

$$\Gamma^s(TM) \to \Gamma^{s-1}(\Lambda^{p+1} M)\,, \text{ for } s \geq 2n+5.$$

Recall that the condition for the diffeomorphism F_X to be a contact diffeomorphism is the vanishing of the one-form $F_X^*\eta$ mod η. Hence by Equation (3.2.5), F_X is a contact diffeomorphism if and only if it satisfies the condition

$$\mathcal{L}_X\eta + Q_\eta(X) = 0 \text{ mod } \eta\,. \tag{3.2.10}$$

Furthermore, by Equation (3.2.6), the linearization of this condition at the zero vector field is the condition

$$\mathcal{L}_X\eta = 0 \text{ mod } \eta\,,$$

i.e. X is a *contact vector field*.

Remark 3.2.11. Using the characteristic vector field T for the contact form η, we may express any vector field X as $X = X^0 T + X_H$, where X_H belongs to the contact distribution. Applying the Cartan identity

$$\mathcal{L}_X(\eta) = X \rfloor\, d\eta + d(X \rfloor\, \eta) = X_H \rfloor\, d\eta + dX^0$$

yields the well known facts (i) that the vector field X is a contact vector field if and only if

$$dX^0 = -X_H \rfloor\, d\eta \qquad \bmod \eta\,; \tag{3.2.12}$$

and (ii) that X is completely determined by the real-valued function $X^0 = X \rfloor\, \eta$. For this reason, $X \rfloor\, \eta$ is called the *generating function* for X and is denoted by g_X. In [**BD1**], we proved that there is an isomorphism

$$\Gamma^s_{cont}(TM) \to \Gamma^{s+1}(M) \,:\, X \mapsto g_X := X \rfloor\, \eta\,.$$

The main result of [**BD1**] is the construction of a smooth parametrization Ψ of the space of Γ^s-contact diffeomorphisms near the identity diffeomorphism by contact vector fields near the zero vector field. The parametrization Ψ in turn induces a smooth structure on the space $\mathcal{D}^s_{cont}(M)$ of all Γ^s-contact diffeomorphisms.

Theorem 3.2.13 ([**BD1**])**.** *For all $s \geq 2n+4$, and for $\mathcal{U} \subset \Gamma^{2n+4}(TM)$ sufficiently small, there is a smooth map*

$$\Psi : \Gamma^s_{cont}(TM) \cap \mathcal{U} \to \mathcal{U}^s \subset \Gamma^s(TM)$$

such that the following holds: for all $Y \in \mathcal{U} \cap \Gamma^s(TM)$, F_Y is a contact diffeomorphism if and only if $Y = \Psi(X)$ for some $X \in \Gamma^s_{cont}(TM) \cap \mathcal{U}$. Moreover, the map Ψ is of the form

$$\Psi(X) = X + B(X)(X, X)\,,$$

where $B : (\Gamma^s_{cont}(TM) \cap \mathcal{U}) \times \Gamma^s_{cont}(TM) \times \Gamma^s_{cont}(TM) \to \Gamma^s(TM)$ is smooth and bilinear in the last two factors.

This theorem implies the following global result:

Theorem 3.2.14 ([**BD1**])**.** *Let (M, η) be a compact contact manifold. For $s \geq (2n+4)$, the space of Γ^s contact diffeomorphisms is a smooth Hilbert manifold.*

We close this section with the *a priori* estimates for the nonlinear term $B(X)(X, X)$, which we proved in [**BD1**] and which we require in Section 6:

Proposition 3.2.15 ([**BD1**])**.** *For $X \in \mathcal{V}^s = \Gamma^s_{cont}(TM) \cap \mathcal{U}^s$,*

(a) $$\|\Psi(X) - X\|_s \prec \|X\|_s \|X\|_{s-1}\,.$$

Moreover, for all $X_1, X_2 \in \mathcal{V}^s$,

(b)

$$\|(\Psi(X_2) - X_2) - (\Psi(X_1) - X_1)\|_s \prec \|X_2 - X_1\|_{s-1}(\|X_2\|_s + \|X_1\|_s) \\ + \|X_2 - X_1\|_s(\|X_2\|_{s-1} + \|X_1\|_{s-1})\,.$$

4. The Action of the Contact Diffeomorphism Group

There is a natural action of contact diffeomorphisms on the space of CR deformations:

$$\begin{cases} \mathcal{D}^\infty_{cont}(M) \times \Omega^{(0,1)}(H_{(1,0)}) & \to \Omega^{(0,1)}(H_{(1,0)}) \\ \qquad (F, \phi) & \mapsto \mu = F^*\phi \end{cases}.$$

The main result of this section (Proposition 4.1.12) is a formula for $F^*\phi$ in the special case where $F = F_{\Psi(X)}$.

Let F be a contact diffeomorphism, and let ϕ be a deformation tensor. Let $\hat{H}_{(0,1)} \subset H_{\mathbb{C}} = H_{(0,1)} \oplus H_{(1,0)}$ denote the anti-holomorphic tangent bundle of the strongly pseudoconvex CR structure associated to ϕ, and define the *pull-back* CR structure $F^*\hat{H}_{(0,1)} \subset H_{\mathbb{C}}$ to be the CR structure with anti-holomorphic subbundle

$$F^*(\hat{H}_{(0,1)}) := \left\{ Z \in H_{\mathbb{C}} \ : \ F_* Z \in \hat{H}_{(0,1)} \right\}.$$

It is straightforward to check that if F_1 and F_2 are two contact diffeomorphisms, then the identity

$$(F_2 \circ F_1)^*(\hat{H}_{(0,1)}) = F_1^*(F_2^*(\hat{H}_{(0,1)}))$$

holds. By Corollary 2.2.2, if F is isotopic to the identity, then $F^*\hat{H}_{(0,1)}$ is represented by a deformation tensor, which we call the *pull-back CR deformation*, denoted by $F^*\phi$.

4.1. Local formulæ. We need a local formula for $F^*_{\Psi(X)}\phi$ that exhibits the non-linear dependence on the contact vector field X. It will also prove important to single out terms involving composition of the components of the tensor ϕ with $F_{\Psi(X)}$; we accomplish this by introducing an auxiliary contact vector field Y into some formulæ.

Choose an adapted atlas and subordinate partition of unity as in Remark 1.1.2. By smoothness of the map $X \mapsto F_{\Psi(X)}$ and compactness of M, for all sufficiently small X, the condition

$$F_{\Psi(X)}(V_\ell) \Subset U_\ell$$

holds for all ℓ. Next let $\eta, \omega^\alpha_\ell, \omega^{\bar\alpha}_\ell = \overline{\omega^\alpha_\ell}$ be a coframing for $T_{\mathbb{C}}M$ on U_ℓ, with $H_{(0,1)}$ the annihilator of η, ω^α.

For ease of notation, we temporarily suppress the index ℓ and set $F = F_{\Psi(X)}$. Then

(4.1.1)
$$F^*(\hat{H}_{(0,1)}) = \left\{ Z \in H_{\mathbb{C}} \;:\; Z\lrcorner\; F^*(\omega^\alpha + \phi^\alpha_{\bar\beta}\omega^{\bar\beta}) = 0,\ \text{for } \alpha = 1, 2, \dots, n \right\}.$$

One sees immediately that

$$F^*(\omega^\alpha + \phi^\alpha_{\bar\beta}\omega^{\bar\beta}) = A^\alpha_\beta \omega^\beta + B^\alpha_{\bar\beta}\omega^{\bar\beta} \mod \eta\,, \tag{4.1.2}$$

where

$$\begin{aligned} A^\alpha_\beta &= Z_\beta \lrcorner\; F^*(\omega^\alpha + \phi^\alpha_{\bar\gamma}\omega^{\bar\gamma}) \\ &= (Z_\beta\lrcorner\; F^*\omega^\alpha) + (\phi^\alpha_{\bar\gamma} \circ F)\left(Z_\beta \lrcorner\; F^*\omega^{\bar\gamma}\right) \end{aligned} \tag{4.1.3a}$$

$$\begin{aligned} B^\alpha_{\bar\beta} &= Z_{\bar\beta} \lrcorner\; F^*(\omega^\alpha + \phi^\alpha_{\bar\gamma}\omega^{\bar\gamma}) \\ &= \left(Z_{\bar\beta}\lrcorner\; F^*\omega^\alpha\right) + (\phi^\alpha_{\bar\gamma} \circ F)\left(Z_{\bar\beta} \lrcorner\; F^*\omega^{\bar\gamma}\right)\,. \end{aligned} \tag{4.1.3b}$$

By Lemma 3.2.4, one has the formulæ

$$F^*\omega^\alpha = \omega^\alpha + \mathcal{L}_X\omega^\alpha + \mathcal{L}_{(\Psi(X)-X)}w^\alpha + Q_{\omega^\alpha}(\Psi(X)) \tag{4.1.4a}$$

and

$$F^*\omega^{\bar\alpha} = \omega^{\bar\alpha} + \mathcal{L}_{\Psi(X)}\omega^{\bar\alpha} + Q_{\omega^{\bar\alpha}}(\Psi(X))\,, \tag{4.1.4b}$$

for one-forms Q_{ω^α} and $Q_{\omega^{\bar\alpha}}$ as in Equation (3.2.6). Consequently,

$$F^*\left(\omega^\alpha + \phi^\alpha_{\bar\gamma}\omega^{\bar\gamma}\right) = \omega^\alpha + \mathcal{L}_X\omega^\alpha + (\phi^\alpha_{\bar\gamma}\circ F_{\Psi(X)})\omega^{\bar\gamma} + Q^\alpha(X, X, \phi)\,, \tag{4.1.5}$$

where the expression $Q^\alpha(X, Y, \phi)$ is defined by the formula

$$\begin{aligned} Q^\alpha(X, Y, \phi) :=\, &\mathcal{L}_{\Psi(X)-X}\omega^\alpha + (\phi^\alpha_{\bar\gamma} \circ F_{\Psi(Y)})\mathcal{L}_{\Psi(X)}\omega^{\bar\gamma} \\ &+ \left\{ Q_{\omega^\alpha}(\Psi(X)) + (\phi^\alpha_{\bar\gamma}\circ F_{\Psi(Y)})\, Q_{\omega^{\bar\gamma}}(\Psi(X)) \right\} \end{aligned} \tag{4.1.6}$$

for Y a second, sufficiently small, contact vector field.

To single out the terms of the form $\phi^\alpha_{\bar\gamma} \circ F_{\Psi(X)}$, we replace the term $\phi^\alpha_{\bar\gamma} \circ F$ in Equations (4.1.3a) and (4.1.3b) by $\phi^\alpha_{\bar\gamma} \circ F_{\Psi(Y)}$ to get matrix-valued functions

$$A = A(X, Y, \phi) \text{ and } B = B(X, Y, \phi)\,. \tag{4.1.7}$$

Using the identity

$$Z_{\bar\beta}\lrcorner\; (\mathcal{L}_X\omega^\alpha) = -(\mathcal{L}_X Z_{\bar\beta})\lrcorner\; \omega^\alpha = (\mathcal{L}_{Z_{\bar\beta}}X)\lrcorner\; \omega^\alpha = (\bar\partial_b X)^\alpha_{\bar\beta}\,,$$

and the expression for $\bar\partial_b X$ in Remark 2.3.3, yields the following formulæ for the entries of $A(X, Y, \phi)$ and $B(X, Y, \phi)$:

$$A^\alpha_\beta = \delta^\alpha_\beta + Z_\beta\lrcorner\; \mathcal{L}_X\omega^\alpha + Z_\beta\lrcorner\; Q^\alpha(X, Y, \phi) \tag{4.1.8a}$$

and

$$B^\alpha_{\bar\beta} = (\bar\partial_b X)^\alpha_{\bar\beta} + (\phi^\alpha_{\bar\beta}\circ F_{\Psi(Y)}) + Z_{\bar\beta}\lrcorner\; Q^\alpha(X, Y, \phi)\,. \tag{4.1.8b}$$

Finally, expressing $A^{-1}B$ in the form $A^{-1}B = B + A^{-1}(I - A)B$ yields the identity

$$F^*_{\Psi(X)}\phi = \bar{\partial}_b X + \left(\phi^\alpha_{\bar{\beta}} \circ F_{\Psi(X)} + \mathcal{E}^\alpha_{\bar{\beta}}(X, X, \phi)\right) \omega^{\bar{\beta}} \otimes Z_\alpha$$

where

$$\mathcal{E}^\alpha_{\bar{\beta}}(X, Y, \phi) = Z_{\bar{\beta}} \lrcorner\, Q^\alpha(X, Y, \phi) + [A^{-1}(I - A)B]^\alpha_{\bar{\beta}} \tag{4.1.9}$$

and where $A = A(X, Y, \phi)$, $B = B(X, Y, \phi)$.

Using the partition of unity, we globalize these local formulæ to obtain the vector-valued one-forms

$$\phi \circ F_{\Psi(X)} := \sum_\ell \rho_\ell \cdot (\phi^\alpha_{\ell,\bar{\beta}} \circ F_{\Psi(X)})\, \omega^{\bar{\beta}}_\ell \otimes Z_{\ell,\alpha} \tag{4.1.10}$$

and

$$\mathcal{E}(X, Y, \phi) := \sum_\ell \rho_\ell \cdot \mathcal{E}^\alpha_{\ell,\bar{\beta}}(X, Y, \phi)\, \omega^{\bar{\beta}}_\ell \otimes Z_{\ell,\alpha}\,. \tag{4.1.11}$$

Noting that $\mu = \sum_\ell \rho_\ell \cdot \mu$ and $\bar{\partial}_b X = \sum_\ell \rho_\ell \bar{\partial}_b X$ then immediately gives the next proposition, which we need to prove the normal form theorem of Section 6.

Proposition 4.1.12. *Let* $(F_{\Psi(X)}, \phi) \in \mathcal{D}^{s+1}_{cont}(M) \times \Gamma^s \left(H^{(0,1)} \otimes H_{(1,0)}\right)$ *be near* $(id_M, 0)$*. Then* $F^*_{\Psi(X)}\phi$ *is given by*

$$F^*_{\Psi(X)}\phi = \bar{\partial}_b X + \phi \circ F_{\Psi(X)} + \mathcal{E}(X, X, \phi)\,. \tag{a}$$

The linearized action at the identity map and the zero deformation tensor is

$$(X, \dot{\phi}) \mapsto \bar{\partial}_b X + \dot{\phi}. \tag{b}$$

Remark 4.1.13. These equations require some care in interpretation. First, notice that the terms $F^*_{\Psi(X)}\phi$ and $\bar{\partial}_b X$ are in fact globally defined tensors, and make invariant sense. On the other hand, $\phi \circ F_{\Psi(Y)}$ and $\mathcal{E}(X, Y, \phi)$ have been defined using local coordinates and are coordinate dependent.

Remark 4.1.14. Observe that for $s \geq 2n + 4$ the map $(X, Y, \phi) \mapsto \mathcal{E}(X, Y, \phi)$ extends to the map

$$\mathcal{E} : \Gamma^{s+1}_{cont}(TM) \times \Gamma^{s+1}_{cont}(TM) \times \Gamma^s \left(H^{(0,1)} \otimes H_{(1,0)}\right) \to \Gamma^s \left(H^{(0,1)} \otimes H_{(1,0)}\right)$$

between Folland-Stein spaces. In Section 6.2 we obtain estimates for $\mathcal{E}$ that play a critical role in the proof of our normal form theorem.

4.2. Complex contact vector fields. By Equation (b), the action of the group of contact diffeomorphisms suggests normalizing deformation tensors by the image of $\bar{\partial}_b X$ where X is a real contact vector field. On the other hand, since $\bar{\partial}_b X = \bar{\partial}_b(\pi_{(1,0)}X)$, it is natural to normalize the deformation tensor by the image of $\bar{\partial}_b X$ for $X \in T_{(1,0)}M$. We accomplish this by introducing the notion of complex contact vector fields.[3]

Begin by recalling that $T_{(1,0)}M$ is defined as the quotient bundle

$$0 \to H_{(0,1)} \hookrightarrow T_{\mathbb{C}}M \xrightarrow{\pi_{(1,0)}} T_{(1,0)}M \to 0\,.$$

Noting that $T_{\mathbb{C}}M = H_{(1,0)} \oplus H_{(0,1)} \oplus \mathbb{C} \cdot T$, where T is the Reeb vector field, we see that the restriction of $\pi_{(1,0)}$ to $H_{(1,0)} \oplus \mathbb{C} \cdot T$ is an isomorphism of complex vector bundles. Thus, we shall identify $T_{(1,0)}M$ with $H_{(1,0)} \oplus \mathbb{C} \cdot T$ when it is convenient.

Next observe that the composite map

$$TM \hookrightarrow T_{\mathbb{C}}M \xrightarrow{\pi_{(1,0)}} T_{(1,0)}M$$

is injective with image the subbundle $\{Z \in T_{(1,0)}M \ : \ \eta(Z) \in \mathbb{R}\}$. Consequently, there are natural identifications

$$TM = H_{(1,0)} \oplus \mathbb{R} \cdot T := \{X \in T_{(1,0)}M \ : \ \eta(X) \in \mathbb{R}\}\,; \tag{4.2.1}$$

and it is easy to check that the inclusion $\Gamma^s(TM) \subset \Gamma^s(T_{(1,0)}M)$ is norm preserving.

Because $H_{(0,1)}$ is contained in the annihilator of η, the quantity $\eta(Z)$ is well-defined for all $Z \in T_{(1,0)}M$. In addition, the quantity $\pi^{(0,1)}(Z \lrcorner\, d\eta)$ is well-defined, where $\pi^{(0,1)} : T_{\mathbb{C}}M^* \to H^{(0,1)}$ denotes the natural projection map. More precisely, let[4]

$$\pi^{(0,1)}(Z \lrcorner\, d\eta) := \pi^{(0,1)}(\widetilde{Z} \lrcorner\, d\eta)\,,$$

for any $\widetilde{Z} \in T_{\mathbb{C}}M$ such that $\pi_{(1,0)}\widetilde{Z} = Z$.

Finally, recall from Remark 3.2.12 that a real vector field X is a contact vector field if and only if it satisfies the identity

$$dX^0 + X \lrcorner\, d\eta = 0 \qquad \mod \eta\,,$$

where $X^0 = X \lrcorner\, \eta$. This is equivalent to the two conditions

$W \lrcorner\, \left(dX^0 + X \lrcorner\, d\eta\right) = 0$ and $\overline{W} \lrcorner\, \left(dX^0 + X \lrcorner\, d\eta\right) = 0$
for all $W \in H_{(1,0)}$.

Since X is real, $W \lrcorner\, \left(dX^0 + X \lrcorner\, d\eta\right) = \overline{\overline{W} \lrcorner\, (dX^0 + X \lrcorner\, d\eta)}$. This leads us to the following definition.

[3] This corresponds to the notion of *Hamiltonian vector fields* as used in [**BD2**].

[4] Independence of the choice of $\widetilde{Z}$ is an immediate consequence of the identity $d\eta(W_1, W_2) = 0$ for all $W_1, W_2 \in H_{(0,1)}$.

Definition 4.2.2. We say that a $(1,0)$-vector field $Z \in \Gamma(T_{(1,0)}M)$ is a *complex contact vector field* if it satisfies the condition

$$\bar{\partial}_b\,(Z \lrcorner\ \eta) + \pi^{(0,1)}(Z \lrcorner\ d\eta) = 0\,.$$

We denote by $\Gamma^s_{cont}(T_{(1,0)}M)$ the Folland-Stein completion of the space of complex contact vector fields.

The following lemma places this definition in context.

Lemma 4.2.3. *Let $X \in \Gamma(T_{\mathbb{C}}M)$ be a real vector field. Then the following are equivalent:*

(a) *X is a contact vector field.*
(b) *X satisfies the identity*

$$\bar{\partial}_b(X \lrcorner\ \eta) + \pi^{(0,1)}(X \lrcorner\ d\eta) = 0$$

where $\pi^{(0,1)} : T^*_{\mathbb{C}}M \to H^*_{(0,1)}M$ is the restriction operator.
(c) *The vector-valued one-form $\bar{\partial}_b X$ takes values in $H_{(1,0)}$.*

Proof. By the observations above, a real vector field X is contact if and only if

$$\overline{W} \lrcorner\ \left(d(X \lrcorner\ \eta) + X \lrcorner\ d\eta\right) = 0 \qquad \forall\, \overline{W} \in H_{(0,1)}\ .$$

This is simply condition 4.2.3(b), thus establishing the equivalence of 4.2.3(a) and 4.2.3(b). The equivalence of 4.2.3(b) and 4.2.3(c) is a special case of Lemma 4.2.4 below. q.e.d.

The next lemma gives a useful characterization of complex contact vector fields. Before stating the lemma, we remark that the quotient bundle $T_{(1,0)}M$ has a naturally defined subbundle determined by the vanishing of η, that is

$$H_{\mathbb{C}}/H_{(0,1)} = \{Z \in T_{(1,0)}M\ :\ \eta(Z) = 0\} \subset T_{(1,0)}M\,.$$

A simple computation shows that the map $\pi_{(1,0)}$ defined above restricts to an isomorphism $H_{(1,0)} \simeq H_{\mathbb{C}}/H_{(0,1)}$. Hence, we may identify $H_{\mathbb{C}}/H_{(0,1)}$-valued forms with $H_{(1,0)}$-valued forms.

Lemma 4.2.4. *The vector field $Z \in \Gamma(T_{(1,0)}M)$ is a complex contact vector field if and only if $\bar{\partial}_b Z$ is an $H_{(1,0)}$-valued $(0,1)$-form.*

Proof. Suppose that $\bar{\partial}_b Z$ takes its values in $H_{(1,0)}$; that is, that

$$\eta(\bar{\partial}_b Z(\overline{W})) = 0 \qquad \forall\, \overline{W} \in H_{(0,1)}\ . \tag{4.2.5}$$

By Equation (2.3.2b),

$$\eta\left(\bar{\partial}_b Z(\overline{W})\right) = -\eta\left(\pi_{(1,0)}[Z,\overline{W}]\right) = -\eta\left([Z,\overline{W}]\right)\ ;$$

but

$$\begin{aligned}\eta\left([Z,\overline{W}]\right) &= -d\eta(Z,\overline{W}) + Z\eta(\overline{W}) - \overline{W}\eta(Z)\\ &= -d\eta(Z,\overline{W}) - \overline{W}(Z\lrcorner\ \eta)\\ &= -\overline{W}\lrcorner\ \left(Z\lrcorner\ d\eta + \bar{\partial}_b(Z\lrcorner\ \eta)\right)\ .\end{aligned}$$

Thus, $\bar{\partial}_b Z$ takes its values in $H_{(1,0)}$ if and only if $\overline{W}\lrcorner\ (Z\lrcorner\ d\eta+\bar{\partial}_b(Z\lrcorner\ \eta))$ $= 0$, for all $\overline{W}\in H_{(0,1)}$, which is equivalent to Z being complex contact.
q.e.d.

5. Homotopy Operators for CR Manifolds

In this section, we will collect various results concerning the existence and regularity of homotopy operators on compact, embedded strongly pseudoconvex CR manifolds. We restrict our statements to the special case of compact, embedded, three dimensional strongly pseudoconvex CR manifolds. More details of these constructions and their generalizations can be found in, e.g., [**BuMi**], [**Miy1**].

5.1. Miyajima's homotopy operators. First, we have the following result for the $\bar{\partial}_b$ complex. It follows immediately from the vector bundle valued version contained in [**Miy2**], where the vector bundle is the trivial line bundle, and $P = N\bar{\partial}_b^*$, but the result is essentially contained in [**BeGr**]. Roughly speaking, it says that there exists a partial inverse and a Szegö projector with good estimates.

Theorem 5.1.1. *There exist linear operators $H : C^\infty(M)\to C^\infty(M)$, $P:\Omega^{(0,1)}(M)\to C^\infty(M)$, and $S:\Omega^{(0,1)}(M)\to\Omega^{(0,1)}(M)$, such that the following identities and estimates are satisfied:*

(a) $\bar{\partial}_b\circ H = 0\,,\quad P\circ S = 0\,,\quad S\circ\bar{\partial}_b = 0$

(b) $u = P\bar{\partial}_b u + Hu$ *and* $\alpha = \bar{\partial}_b P\alpha + S\alpha$

(c) $\|H(u)\|_s \prec \|u\|_s,\quad \|P(\alpha)\|_s\prec\|\alpha\|_{s+1},\quad \|S(\alpha)\|_s\prec\|\alpha\|_s,$

for all $u\in C^\infty(M)$, $\alpha\in\Omega^{(0,1)}(M)$, and $s\geq 0$.

(d) *H extends to a self-adjoint, projection operator on $L^2(M,\mathbb{C})$.*

Similarly, homotopy operators for $T_{(1,0)}M$-valued $(0,1)$ forms also exist, with similar estimates [**Miy2**]. These estimates work in general for vector-valued forms, where the vector bundle is the restriction of a complex vector bundle which extends to the complex manifold bounded by M as a holomorphic bundle. (If the complex space X bounded by M is singular, we first resolve the singularities of X and then apply the above definition.)

Theorem 5.1.2 (Miyajima [**Miy1**], [**Miy2**]). *There exist linear operators*

(a) $\bar{\partial}_b \circ \rho = 0\,, \quad P \circ Q = 0, \qquad Q \circ \bar{\partial}_b = 0$

(b) $Z = P\bar{\partial}_b Z + \rho Z$ *and* $\phi = \bar{\partial}_b P\phi + Q\phi$

(c) $\|P\phi\|_{s+1} \prec \|\phi\|_s\,, \qquad \|Q\phi\|_s \prec \|\phi\|_s\,, \qquad \|\rho(Z)\|_s \prec \|Z\|_s\,,$

for all $Z \in \Gamma^\infty(T_{(1,0)}M)$, $\phi \in \Omega^{(0,1)}(T_{(1,0)}M)$, *and* $s \geq 0$.

Finally, there exist linear operators $L : \Omega^{(0,1)}(T_{(1,0)}M) \to \Gamma^\infty(T_{(1,0)}M)$ *and* $N : \Gamma^\infty(T_{(1,0)}M) \to \Gamma^\infty(T_{(1,0)}M)$, *with* L *a smooth horizontal linear first order differential operator such that*

(d) $P = N \circ L\,,$

and N *satisfies the estimate*

(e) $\|N(Z)\|_{s+2} \prec \|Z\|_s$ *for all* $Z \in \Gamma^\infty(T_{(1,0)}M)$, $s \geq 0$.

5.2. Homotopy operators for complex contact vector fields. These homotopy formulæ do not single out contact vector fields in any significant manner. We now show how to modify the homotopy operators in order to do so. We begin by introducing the *raising* and *lowering* operators induced by the nondegenerate two form $d\eta$:

Definition 5.2.1. The *lowering operator* is the vector bundle map

$$(\;)^\flat : TM \to H^* \;:\; X \mapsto X^\flat = X \lrcorner\; d\eta$$

whose restriction to $H \subset TM$ is an isomorphism between the contact distribution and its dual space. The *raising operator* is the inverse

$$(\;)^\sharp : H^* \to H \;:\; \phi \mapsto \phi^\sharp\,.$$

Remark 5.2.2. The maps $(\;)^\flat$ and $(\;)^\sharp$ of Definition 5.2.1 induce (complex) linear maps

$$(\;)^\flat : T_{(1,0)}M \to H^{(0,1)} \;:\; Z_{(1,0)} \mapsto Z_{(1,0)} \lrcorner\; d\eta$$

and

$$(\;)^\sharp : H^{(0,1)} \to H_{(1,0)}\,,$$

where the map $(\;)^\sharp$ is an isomorphism of complex vector bundles. Observe that by construction,

$$Z = \eta(Z)\,T + (Z^\flat)^\sharp \tag{5.2.3}$$

for all $Z \in \Gamma^\infty(T_{(1,0)}M)$. Notice also that by Definition 4.2.2, Z is an element of the space $\Gamma^\infty_{cont}(T_{(1,0)}M)$ of complex contact vector fields if and only if it satisfies the identity

$$\bar{\partial}_b\,(\eta(Z)) + Z^\flat = 0\,. \tag{5.2.4}$$

Thus, every complex contact vector field is of the form

$$Z_f = fT - (\bar{\partial}_b f)^\sharp \tag{5.2.5}$$

for f a smooth complex valued function. Moreover, the inclusion $TM \hookrightarrow T_{(1,0)}M$ induces the inclusion

$$\Gamma^\infty_{cont}(TM) \hookrightarrow \Gamma^\infty_{cont}(T_{(1,0)}M) \ : \ X \mapsto Z_{g_X} \,,$$

where $g_X = \eta(X)$. (See Remark 3.2.11.)

Proposition 5.2.6. *There exist smooth linear operators*

$$\hat{\mathcal{P}}, \hat{\mathcal{S}} \ : \ \Gamma^\infty(T_{(1,0)}M) \to \Gamma^\infty(T_{(1,0)}M)$$

satisfying the following:

(a) $Z = \hat{\mathcal{P}}Z + \hat{\mathcal{S}}Z$ *for all* $Z \in \Gamma^\infty(T_{(1,0)}M)$

(b) $\operatorname{range}(\hat{\mathcal{P}}) = \ker(\hat{\mathcal{S}}) = \Gamma^\infty_{cont}(T_{(1,0)}M)$

(c) $\hat{\mathcal{P}} \circ \hat{\mathcal{S}} = 0, \quad \hat{\mathcal{S}} \circ \hat{\mathcal{P}} = 0, \quad \hat{\mathcal{P}} \circ \hat{\mathcal{P}} = \hat{\mathcal{P}},$ *and* $\hat{\mathcal{S}} \circ \hat{\mathcal{S}} = \hat{\mathcal{S}}$

(d) $\|\hat{\mathcal{P}}Z\|_s \prec \|Z\|_s$ *and* $\|\hat{\mathcal{S}}Z\|_s \prec \|Z\|_s$ *for all* $Z \in \Gamma^\infty(T_{(1,0)}M)$.

Proof. Choose a vector field $Z \in \Gamma(T_{(1,0)}M)$, and compute as follows using the homotopy operators from Theorems 5.1.1 and 5.1.2:

$$\begin{aligned} Z &= \eta(Z)\,T + (Z^\flat)^\sharp \\ &= (H(\eta(Z)) + P(\bar{\partial}_b(\eta(Z))))\,T + \left\{\bar{\partial}_b P(Z^\flat) + S(Z^\flat)\right\}^\sharp . \end{aligned}$$

Add and subtract the term $P(Z^\flat)T$ and rearrange to get

$$\begin{aligned} Z = &\left\{(H(\eta(Z)) - P(Z^\flat))\,T + (\bar{\partial}_b P(Z^\flat))^\sharp\right\} \\ &+ \left\{\left(P(\bar{\partial}_b(\eta(Z))) + P(Z^\flat)\right)T + \left(S(Z^\flat)\right)^\sharp\right\} . \end{aligned} \tag{5.2.7}$$

Define $\hat{\mathcal{P}}, \hat{\mathcal{S}} : \Gamma^\infty(T_{(1,0)}M) \to \Gamma^\infty(T_{(1,0)}M)$ to be the linear operators given by the formulæ

$$\begin{aligned} \hat{\mathcal{P}}(Z) &= \left(H(\eta(Z)) - P(Z^\flat)\right)\,T + \left(\bar{\partial}_b P(Z^\flat)\right)^\sharp \\ \hat{\mathcal{S}}(Z) &= \left(P(\bar{\partial}_b(\eta(Z))) + P(Z^\flat)\right)\,T + \left(S(Z^\flat)\right)^\sharp . \end{aligned}$$

By construction, $Z = \hat{\mathcal{P}}Z + \hat{\mathcal{S}}Z$.

We claim that $\hat{\mathcal{P}}Z$ is a smooth complex contact vector field. This follows from Equation (5.2.4) and the computation

$$\bar{\partial}_b\left(\eta(\hat{\mathcal{P}}(Z))\right) + \hat{\mathcal{P}}(Z)^\flat = \bar{\partial}_b H(\eta(Z)) - \bar{\partial}_b P(Z^\flat) + \bar{\partial}_b P(Z^\flat) = \bar{\partial}_b H(\eta(Z)) = 0 \,.$$

Observe also that by (5.2.4),

$$\hat{\mathcal{S}}(Z) = \Big(P(\bar{\partial}_b(\eta(Z))) + P(Z^\flat)\Big)\, T + \Big(S(Z^\flat)\Big)^\sharp$$
$$= \Big(P(-Z^\flat) + P(Z^\flat)\Big)\, T + \big(S(-\bar{\partial}_b(\eta(Z)))\big)^\sharp = (-0)^\sharp = 0$$

for all $Z \in \Gamma^\infty_{cont}(T_{(1,0)}M)$.

We have shown that $\hat{\mathcal{P}}$ takes values in $\Gamma^\infty_{cont}(T_{(1,0)}M)$ and that $\hat{\mathcal{S}}$ vanishes on $\Gamma^\infty_{cont}(T_{(1,0)}M)$. These facts, combined with Equation (a), imply that $\hat{\mathcal{P}}$ and $\hat{\mathcal{S}}$ satisfy the identities:

$$\hat{\mathcal{P}} \circ \hat{\mathcal{S}} = 0\,, \quad \hat{\mathcal{S}} \circ \hat{\mathcal{P}} = 0\,, \quad \hat{\mathcal{P}} \circ \hat{\mathcal{P}} = \hat{\mathcal{P}}\,, \text{ and } \hat{\mathcal{S}} \circ \hat{\mathcal{S}} = \hat{\mathcal{S}}\,,$$

as well as the equalities

$$\Gamma^\infty_{cont}(T_{(1,0)}M) = \mathrm{range}(\hat{\mathcal{P}}) = \ker(\hat{\mathcal{S}})\,.$$

The estimates follow from the estimates in Theorems 5.1.1 and 5.1.2. q.e.d.

Remark 5.2.8. Because the projection operators $\hat{\mathcal{P}}, \hat{\mathcal{S}}$ in Proposition 5.2.6 preserve the Folland-Stein regularity, they extend to projection operators on the Folland-Stein space $\Gamma^s(T_{(1,0)}M)$ and they induce a direct sum decomposition

$$\Gamma^s(T_{(1,0)}M) = \Gamma^s_{cont}(T_{(1,0)}M) \oplus \ker(\hat{\mathcal{P}})$$

with $\Gamma^s_{cont}(T_{(1,0)}M) = \ker(\hat{\mathcal{S}}) = \mathrm{range}(\hat{\mathcal{P}}) \subset \Gamma^s(T_{(1,0)}M)$.

The following variant of Theorem 5.1.2 highlights the role of contact vector fields.

Theorem 5.2.9. *There exist linear operators $\mathcal{P} : \Omega^{(0,1)}(T_{(1,0)}M) \to \Gamma^\infty_{cont}(T_{(1,0)}M)$ and $\mathcal{H} : \Omega^{(0,1)}(T_{(1,0)}M) \to \Omega^{(0,1)}(T_{(1,0)}M)$ such that:*

(a) $Z = \mathcal{P}\bar{\partial}_b Z + \rho Z$ *for all* $Z \in \Gamma^\infty_{cont}(T_{(1,0)}M)$

(b) $\phi = \bar{\partial}_b \mathcal{P}\phi + \mathcal{H}\phi$ *for all* $\phi \in \Omega^{(0,1)}(T_{(1,0)}M)$

(c) $\bar{\partial}_b \mathcal{P} \circ \mathcal{H} = 0\,, \qquad \mathcal{H} \circ \bar{\partial}_b \mathcal{P} = 0$

(d) $\mathcal{H} \circ \bar{\partial}_b Z = 0$ *for all* $Z \in \Gamma^\infty_{cont}(T_{(1,0)}M)$

(e) $\|\mathcal{P}\phi\|_{s+1} \prec \|\phi\|_s\,, \quad \|\mathcal{H}\phi\|_s \prec \|\phi\|_s$ *for all* $\phi \in \Omega^{(0,1)}(T_{(1,0)}M)$.

Moreover,

(f) $\|\rho(Z)\|_s \prec \|Z\|_s$ *for all* $Z \in \Gamma^\infty_{cont}(T_{(1,0)}M)$, $s \geq 0$.

Finally, there exist smooth linear operators $\mathcal{L} : \Omega^{(0,1)}(T_{(1,0)}M) \to \Gamma^\infty(T_{(1,0)}M)$ *and* $\mathcal{N} : \Gamma^\infty(T_{(1,0)}M) \to \Gamma^\infty(T_{(1,0)}M)$ *with* $\mathcal{L}$ *a horizontal first order differential operator, such that*

(g) $\mathcal{P} = \mathcal{N} \circ \mathcal{L}$

(h) $\|\mathcal{N}(Z)\|_{s+2} \prec \|Z\|_s$ *for all* $Z \in \Gamma^\infty(T_{(1,0)}M)$, $s \geq 0$.

Proof. The key step in the proof is to express the homotopy operator P of Theorem 5.1.2 as the sum of two operators $\mathcal{P}$ and $\mathcal{S}$, defined by the formulas

$$\mathcal{P} = \hat{\mathcal{P}} \circ P \text{ and } \mathcal{S} = \hat{\mathcal{S}} \circ P\,.$$

By Proposition 5.2.6, $P = \mathcal{P} + \mathcal{S}$ and the image of $\mathcal{P}$ is contained in the space $\Gamma^\infty_{cont}(T_{(1,0)}M)$ of smooth complex contact vector fields. Next let $\mathcal{H} = \bar{\partial}_b \circ \mathcal{S} + Q$, where Q is as in Theorem 5.1.2.

To prove (a), let Z be a complex contact vector field and note that by 5.1.2(b),

$$Z = P\bar{\partial}_b Z + \rho Z = \mathcal{P}\bar{\partial}_b Z + \mathcal{S}\bar{\partial}_b Z + \rho Z\,.$$

We need only show that $\mathcal{S}\bar{\partial}_b Z = 0$, for Z complex contact. First observe that whenever Z is a complex contact vector field, then $P\bar{\partial}_b Z$ is also complex contact. This follows easily from Lemma 4.2.4, the formula $P\bar{\partial}_b Z = Z - \rho(Z)$, and $\bar{\partial}_b \rho(Z) = 0$. Consequently, $\mathcal{S}(\bar{\partial}_b Z) = \hat{\mathcal{S}}(P\bar{\partial}_b Z) = 0$, for all $Z \in \Gamma^\infty_{cont}(T_{(1,0)}M)$.

To prove the homotopy formula (b), notice that Proposition 5.2.6(a) implies the decomposition

$$P = \mathcal{P} + \mathcal{S}\,;$$

then use the homotopy formula 5.1.2(b) to compute as follows:

$$\phi = \bar{\partial}_b P\phi + Q\phi = \bar{\partial}_b \mathcal{P}\phi + \bar{\partial}_b \mathcal{S}\phi + Q\phi = \bar{\partial}_b \mathcal{P}\phi + \mathcal{H}\phi\,.$$

We now prove parts (c) and (d). First observe that $\mathcal{S} \circ \bar{\partial}_b \mathcal{P} = 0$. Since $\mathcal{P}\phi$ is complex contact, $P\bar{\partial}_b \mathcal{P}\phi$ is complex contact. Therefore, $\mathcal{S}\bar{\partial}_b \mathcal{P}\phi = \hat{\mathcal{S}}(P\bar{\partial}_b \mathcal{P}\phi) = 0$. Next observe that $\bar{\partial}_b(\mathcal{P} \circ \bar{\partial}_b \mathcal{S}) = 0$ as follows: For $\phi \in \Omega^{(0,1)}(T_{(1,0)}M)$, compute as follows:

$$\bar{\partial}_b \mathcal{P}\phi = \bar{\partial}_b \mathcal{P}\left(\bar{\partial}_b \mathcal{P}\phi + \bar{\partial}_b \mathcal{S}\phi + Q\phi\right) = \bar{\partial}_b \mathcal{P}\bar{\partial}_b \mathcal{P}\phi + \bar{\partial}_b \mathcal{P}\bar{\partial}_b \mathcal{S}\phi\,;$$

on the other hand,

$$\bar{\partial}_b \mathcal{P}\phi = \bar{\partial}_b \mathcal{P}(\bar{\partial}_b \mathcal{P}\phi) + \bar{\partial}_b \mathcal{S}(\bar{\partial}_b \mathcal{P}\phi) + Q(\bar{\partial}_b \mathcal{P}\phi) = \bar{\partial}_b \mathcal{P}\bar{\partial}_b \mathcal{P}\phi\,.$$

Thus, $\bar{\partial}_b(\mathcal{P} \circ \bar{\partial}_b \mathcal{S}) = 0$. Finally, the identities $Q \circ \bar{\partial}_b = \mathcal{P} \circ Q = \mathcal{S} \circ Q = 0$ follow immediately from Theorem 5.1.2. Then the identities $\bar{\partial}_b \mathcal{P} \circ \mathcal{H} = 0$ and $\mathcal{H} \circ \bar{\partial}_b \mathcal{P} = 0$ follow from the identities $\bar{\partial}_b(\mathcal{P} \circ \bar{\partial}_b \mathcal{S}) = 0$ and $\mathcal{S} \circ \bar{\partial}_b \mathcal{P} = 0$.

To prove part (g), set $\mathcal{L} = L$ and $\mathcal{N} = \hat{\mathcal{P}} \circ N$. Since $\mathcal{P} = \hat{\mathcal{P}} \circ P$ and by (5.1.2c) $P = N \circ L$, it follows that $\mathcal{P} = \mathcal{N} \circ \mathcal{L}$.

The estimates (e), (f), and (h) follow immediately from the estimates in Theorems 5.1.1 and 5.1.2. q.e.d.

Notice that in the last theorem, since $\Omega^{(0,1)}(H_{(1,0)}) \subset \Omega^{(0,1)}(T_{(1,0)}M)$, it follows that for $\phi \in \Omega^{(0,1)}(H_{(1,0)})$, we have $\phi = \bar{\partial}_b\mathcal{P}\phi + \mathcal{H}\phi$. Moreover, since the range of $\mathcal{P}$ is the space of complex contact vector fields, then $\bar{\partial}_b\mathcal{P}\phi \in \Omega^{(0,1)}(H_{(1,0)})$ (see Lemma 4.2.4). It follows that $\mathcal{H}$ restricts to an operator $\mathcal{H} : \Omega^{(0,1)}(H_{(1,0)}) \to \Omega^{(0,1)}(H_{(1,0)})$. Therefore, we can restrict the homotopy formula to the horizontal vector valued forms. We state this next, using the same symbols to denote the restricted operators without risk of confusion.

Corollary 5.2.10. *There exist homotopy operators* $\mathcal{P} : \Omega^{(0,1)}(H_{(1,0)}) \to \Gamma^{\infty}_{cont}(T_{(1,0)}M)$, $\mathcal{H} : \Omega^{(0,1)}(H_{(1,0)}) \to \Omega^{(0,1)}(H_{(1,0)})$ *such that:*

(a) $\phi = \bar{\partial}_b\mathcal{P}\phi + \mathcal{H}\phi$ *for all* $\phi \in \Omega^{(0,1)}(H_{(1,0)})$

(b) $\bar{\partial}_b\mathcal{P} \circ \mathcal{H} = 0 \qquad \mathcal{H} \circ \bar{\partial}_b\mathcal{P} = 0$

(c) $\mathcal{H} \circ \bar{\partial}_b Z = 0$ *for all* $Z \in \Gamma^{\infty}_{cont}(T_{(1,0)}M)$

(d) $\|\mathcal{P}\phi\|_{s+1} \prec \|\phi\|_s \qquad \|\mathcal{H}\phi\|_s \prec \|\phi\|_s$ *for all* $\phi \in \Omega^{(0,1)}(H_{(1,0)})$.

Moreover, noting that the harmonic projection ρ *restricts to a map* $\rho\colon \Gamma^{\infty}_{cont}(T_{(1,0)}M) \to \Gamma^{\infty}_{cont}(T_{(1,0)}M)$*:*

(e) $Z = \mathcal{P}\bar{\partial}_b Z + \rho Z$ *for all* $Z \in \Gamma^{\infty}_{cont}(T_{(1,0)}M)$

(f) $\|\rho(Z)\|_s \prec \|Z\|_s$ *for all* $Z \in \Gamma^{\infty}_{cont}(T_{(1,0)}M), s \geq 0$.

5.3. Harmonic decomposition of complex contact vector fields. In this section, we obtain a decomposition of complex contact vector fields into real contact vector fields and a complementary subspace. Recall from Equation (5.2.5) that the space of complex contact vector fields is parameterized by complex valued functions as follows:

$$f \mapsto Z_f = fT - (\bar{\partial}_b f)^{\sharp}\,.$$

The observation that this parametrization agrees with the parametrization of real contact vector fields as introduced in Remark 3.2.11 suggests constructing the decomposition using the naïve projection operator $\pi_{\mathrm{Re}} : Z_f \mapsto Z_{\mathrm{Re}(f)}$. Unfortunately, this projection map is not continuous in the Folland-Stein norm. We see this as follows. By virtue of the identification $T_{(1,0)}M = H_{(1,0)} \oplus \mathbb{C}\cdot T$, the Folland-Stein structure on the space of complex contact vector fields is

$$\begin{aligned}(Z_f, Z_g)_s &= \Big((fT - (\bar\partial_b f)^{\#}), (gT - (\bar\partial_b g)^{\#})\Big)_s \\ &= (fT, gT)_s + \Big((\bar\partial_b f)^{\#}, (\bar\partial_b g)^{\#}\Big)_s \\ &= (f,g)_s + \big(\bar\partial_b f, \bar\partial_b g\big)_s \\ \|Z_f\|_s^2 &= \|f\|_s^2 + \|\bar\partial_b f\|_s^2 \,.\end{aligned}$$

On the other hand, since $Z_{\mathrm{Re}(f)} = 1/2(Z_f + Z_{\overline{f}})$,

$$\|Z_{\mathrm{Re}(f)}\|_s^2 = 1/4\|Z_f + Z_{\overline{f}}\|_s^2 = \|\mathrm{Re}(f)\|_s^2 + 1/4\|\bar\partial_b f + \overline{\partial_b f}\|_s^2 \,.$$

Let f_k be a sequence of CR functions with $\|\partial_b f_k\|_s \to \infty$ and $\|f_k\|_s$ bounded. Then $\|Z_{f_k}\|_s$ is bounded, but

$$\|Z_{\mathrm{Re}(f_k)}\|_s^2 = \|\mathrm{Re}(f_k)\|_s^2 + \frac{1}{4}\|\bar\partial_b \overline{f_k}\|_s^2 = \|\mathrm{Re}(f_k)\|_s^2 + \frac{1}{4}\|\partial_b f_k\|_s^2 \to \infty \,.$$

Therefore, to obtain a bounded projection, we have to proceed differently. We need the following regularity lemma.

Lemma 5.3.1. *The estimate* $\|u\|_{s+2} \prec \|\mathrm{Re}(I + \Box_b)u\|_s$ *holds for any smooth, real-valued function* u*. In particular, if* $\mathrm{Re}(I + \Box_b)u$ *is smooth, then so is* u*.*

Proof. One easily verifies that for u real, the identity $\mathrm{Re}(u + \Box_b u) = u + \frac{1}{2n+2}\Delta_R u$ holds, where Δ_R is the Laplace operator in the Rumin complex. The estimate follows from the corresponding estimate for Δ_R, proved in [**R**, **BD3**]. q.e.d.

Next let f be a smooth, complex valued function f. Then $\mathrm{Re}(f + \Box_b f)$ is smooth, and Lemma 5.3.1 implies that there is a unique, smooth, real-valued function u, satisfying the equation

$$(I + \frac{1}{2n+2}\Delta_R)u = \mathrm{Re}(f + \Box_b f) \,.$$

Proposition 5.3.2. *For all* $s \geq 2n + 4$*, the map*

$$f \mapsto u := (I + \frac{1}{2n+2}\Delta_R)^{-1}\mathrm{Re}(f + \Box_b f)$$

induces a bounded projection operator

$$\pi_{\mathrm{Re}} : \Gamma^s_{cont}(T_{(1,0)}M) \to \Gamma^s_{cont}(T_{(1,0)}M) \ : \ Z_f \mapsto Z_u$$

with image $\Gamma^s_{cont}(TM)$*.*

Proof. By construction, $\pi_{\mathrm{Re}}(Z_u) = Z_u$ for u real. Consequently, π_{Re} is a projection operator, as claimed. To prove that π_{Re} is bounded, note that regularity for Δ_R justifies estimating as follows:

$$\begin{aligned}\|Z_u\|_s \prec \|u\|_s + \|\bar\partial_b u\|_s \prec \|u\|_{s+1} \prec \|(I + \frac{1}{2n+2}\Delta_R)u\|_{s-1} \\ \prec \|\mathrm{Re}(f + \Box_b f)\|_{s-1} \prec \|f\|_{s-1} + \|\Box_b f\|_{s-1} \,.\end{aligned}$$

But $\|\Box_b f\|_{s-1} = \|\bar{\partial}_b^* \bar{\partial}_b f\|_{s-1} \prec \|\bar{\partial}_b f\|_s$ implies the estimate

$$\|Z_u\|_s \prec \|f\|_{s-1} + \|\bar{\partial}_b f\|_s \prec \|Z_f\|_s .$$

q.e.d.

The projection map π_{Re} induces the decomposition

$$\Gamma^\infty_{cont}(T_{(1,0)}M) = \Gamma^\infty_{cont}(TM) \oplus iV ,$$

where

$$V := \{Y \in \Gamma^\infty_{cont}(T_{(1,0)}M) \;:\; \pi_{\mathrm{Re}}(iY) = 0\} . \tag{5.3.3}$$

Let V^s denote the closure of V in the Γ^s norm. It will prove convenient to adopt the notational convention

$$Z_f = X_f - i\,Y_f , \tag{5.3.4}$$

where $X_f := \pi_{\mathrm{Re}}(Z_f) \in \Gamma^\infty_{cont}(TM)$ and $Y_f := \pi_{\mathrm{Im}}(Z_f)$ is the projection

$$\pi_{\mathrm{Im}} := i(\mathrm{Id} - \pi_{\mathrm{Re}}) : Z_f \mapsto Y_f .$$

Moreover, the estimate

$$\|X_f\|_s + \|Y_f\|_s \prec \|Z_f\|_s$$

holds for all $f \in \Gamma^s(M, \mathbb{C})$, with $s \geq 2n+4$.

Remark 5.3.5. We caution the reader that although X_f is real, it is <u>not</u> the real part of Z_f.

Remark 5.3.6. We could at this point let iV be a rather arbitrary complement to $\Gamma^\infty_{cont}(TM)$. The only properties for V that are important in what follows are:

(a) $\Gamma^\infty_{cont}(T_{(1,0)}M) \cong \Gamma^\infty_{cont}(TM) \oplus iV$
(b) $\Gamma^s_{cont}(T_{(1,0)}M) \cong \Gamma^s_{cont}(TM) \oplus iV^s$
(c) $\|X\|_s + \|Y\|_s \prec \|X - iY\|_s$,

for all $s \geq 2n+4$.

6. Normal Form for CR Deformations

In this section, we study the action of the contact diffeomorphism group on the space of deformations of a fixed embeddable CR structure $(M, H_{(1,0)})$ on a compact three dimensional manifold M.

There are significant differences in the analysis between the three dimensional case and higher dimensions. These arise since first, there are no integrability conditions in dimension three, and second, the relevant operators are not subelliptic in three dimensions. While the analysis generalizes to higher dimensions, the details are numerous and everything requires a separate statement, including the introduction of new operators to take into account the integrability conditions. Since in dimensions at least five, it is well known that all compact, strongly pseudoconvex CR manifolds are embeddable, our main interest is in the three dimensional case where the situation is more subtle and less well understood. Henceforth, we will restrict our attention to this case.

Before beginning the statement and proof of the main results, we make some comments to motivate the definitions and statements. The contact diffeomorphism group acts on the space of deformation tensors, and the linearization of the action at the identity map and the zero deformation tensor is $(X, \dot{\phi}) \mapsto (\bar{\partial}_b X + \dot{\phi})$, where X is a contact vector field. On the other hand, the Hodge decomposition of Corollary 5.2.10 shows that a deformation tensor can be split as $\phi = \bar{\partial}_b \mathcal{P}\phi + \mathcal{H}\phi$, where $\mathcal{P}\phi$ is a complex contact vector field, and $\mathcal{H}\phi$ serves as the "harmonic part" of the deformation. If we split the complex contact vector fields as $\mathcal{P}\phi = X - iY$, where X is a real contact vector field, and Y lies in a transverse subspace (see Section 5.3), then Y can be heuristically thought of as infinitesimally arising from one of Kuranishi's "wiggles" of the embedded CR manifold within its ambient surface. The normal form should then be $i\bar{\partial}_b Y + \phi_{\mathcal{H}}$, that is, a harmonic form plus a wiggle.

This overview suggests that we should consider a map $\Gamma^{\infty}_{cont}(TM) \oplus iV \oplus \ker \mathcal{P} \to \mathcal{D}ef$ and show that for all $\phi \in \mathcal{D}ef$, there exist (X, Y, ψ) such that $F^*_{\Psi(X)}\phi = i\bar{\partial}_b Y + \psi$; here, $F_{\Psi(X)}$ is the contact diffeomorphism defined by $\Psi(X)$ as in Theorem 3.2.13. Unfortunately, this map loses regularity since the linearization involves differentiation of ϕ in the direction of X. To circumvent this difficulty, we carry along a copy of ϕ and consider the modified map $(\phi, X, Y, \psi) \mapsto (\phi, F^*_{\Psi(X)}\phi - (i\bar{\partial}_b Y + \psi))$. This map is now invertible (modulo a kernel—the CR vector fields–which is easily incorporated) giving a weak normal form:

for every ϕ, there is a triple (X, Y, ψ) such that $F^*_{\Psi(X)}\phi = i\bar{\partial}_b Y + \psi$.

However, in the proof, the normal form $i\bar{\partial}_b Y + \psi$ has less regularity than ϕ. This can be viewed as a weak Hodge decomposition for the nonlinear theory. But, taking our lead from the proof of regularity for the standard linear Hodge theory, we obtain *a priori* estimates in Section 6.2 to improve the regularity and establish a strong normal form:

if $F^*_{\Psi(X)}\phi = i\bar{\partial}_b Y + \psi$ with $\phi \in \Gamma^s$, then $X, Y \in \Gamma^{s+1}, \psi \in \Gamma^s$.

Remark. We expect that this approach of first using linear analysis to obtain a weak normal form and then *a priori* estimates to obtain the strong normal form will find a wide range of use in other applications.

6.1. Statement of the Normal Form Theorem. Throughout the remainder of the paper, $(M, H_{(1,0)})$ is a fixed embeddable compact three dimensional CR manifold.

We first establish notation. Let $\mathrm{H}^1 = \ker \mathcal{P} \subset \Omega^{(0,1)}(H_{(1,0)})$ represent the "harmonic deformation tensors," where $\mathcal{P}$ is as in Corollary 5.2.10, and denote the CR vector fields by $\Gamma^{s+1}_{CR}(TM) = \ker \bar{\partial}_b \cap \Gamma^{s+1}(T_{(1,0)}M)$. Let $\Gamma^s(\mathrm{H}^1)$ denote the Folland-Stein completion of H^1 in

$\Gamma^s(\Omega^{(0,1)}(H_{(1,0)}))$. Notice that $\Gamma^s(\mathrm{H}^1)$ is closed in the space of deformation tensors $\Gamma^s(\mathcal{D}ef) = \Gamma^s(\Omega^{(0,1)}(H_{(1,0)}))$ and that by Corollary 5.2.10,

$$\Gamma^s(\Omega^{(0,1)}(H_{(1,0)})) = \mathrm{range}(\bar{\partial}_b) \oplus \Gamma^s(\mathrm{H}^1)\,.$$

We define the map:[5]

$$\text{(6.1.1a)}\quad \begin{aligned}\Phi : \Gamma^{s+2}(\mathcal{D}ef) \oplus \Gamma^{s+1}_{cont}(TM) \oplus V^{s+1} \oplus \Gamma^s(\mathrm{H}^1)&\\ \longrightarrow \Gamma^{s+2}(\mathcal{D}ef) \oplus \Gamma^s(\mathcal{D}ef) \oplus \Gamma^{s+1}_{CR}(TM)&\end{aligned}$$

by the formula

$$\text{(6.1.1b)}\qquad (\phi, X, Y, \psi) \mapsto \Big(\phi, F^*_{\Psi(X)}\phi - i\bar{\partial}_b Y - \psi, \rho(X - iY)\Big)\,.$$

Proposition 6.1.2. *The map Φ is a local diffeomorphism in a neighborhood of the origin.*

Proof. By the inverse function theorem for Banach spaces, it is sufficient to establish that:

(1) Φ is locally C^1;
(2) $d\Phi|_{(0,0,0,0)}$ is invertible.

To establish (1), notice that all terms in the map Φ are linear, and smooth (see Theorem 5.1.2), except $F^*_{\Psi(X)}\phi$, so it suffices to check the regularity of this term. By Remark 3.2.9, $X \mapsto (Z_\beta \lrcorner\, F^*_{\Psi(X)}\omega)$ and $X \mapsto (Z_\beta \lrcorner\, F^*_{\Psi(X)}\bar{\omega})$ are smooth maps from Γ^{s+1} contact vector fields to Γ^s functions. We proved in [**BD1**] that the map[6]

$$\Gamma^{s+2}(M) \oplus \mathcal{D}^{s+1}_{cont}(M) \to \Gamma^s(M) \;:\; (u, F) \mapsto u \circ F$$

is C^1. From the local expressions in formulæ (4.1.3a) and (4.1.3b) and the fact that the matrix A in these formulæ is invertible, it follows that the term $(\phi, X) \mapsto F^*_{\Psi(X)}\phi$ is C^1, completing the proof that the map Φ is C^1.

We next check that $d\Phi$ is invertible at the origin. Let $(\dot{\phi}, \dot{X}, \dot{Y}, \dot{\psi})$ be a tangent vector at the origin. Then

$$d\Phi(\dot{\phi}, \dot{X}, \dot{Y}, \dot{\psi}) = \Big(\dot{\phi}, \bar{\partial}_b\dot{X} - i\bar{\partial}_b\dot{Y} + \dot{\phi} - \dot{\psi}, \rho(\dot{X} - i\dot{Y})\Big)\,.$$

[5]This corrects an error in [**B**] when we mistakenly asserted the map Φ to be C^1 if we take the first factor on each side to be in $\Gamma^s(\mathcal{D}ef)$. In Section 6.2, we obtain *a priori* estimates to establish a local nonlinear Hodge theory and recover the lost regularity.

[6]Notice that the differential of this map is the Lie derivative of u, which explains the loss in regularity on u.

It is clear that this map has trivial kernel and that it is surjective. In fact, using the homotopy operators $\mathcal{P}, \mathcal{H}$, we can verify that the inverse map

$$(d\Phi)^{-1} : \Gamma^{s+2}(\mathcal{D}ef) \oplus \Gamma^{s}(\mathcal{D}ef) \oplus \Gamma^{s+1}_{CR}(TM) \longrightarrow \Gamma^{s+2}(\mathcal{D}ef) \oplus \Gamma^{s+1}_{cont}(TM) \oplus V^{s+1} \oplus \Gamma^{s}(\mathrm{H}^1)$$

is given by

$$(d\Phi)^{-1}(\dot\phi, \chi, \xi) = (\dot\phi, \pi_{\mathrm{Re}}(\mathcal{P}(\chi - \dot\phi) + \xi), \pi_{\mathrm{Im}}(\mathcal{P}(\chi - \dot\phi) + \xi), -\mathcal{H}(\chi - \dot\phi))\,.$$

To verify that this is the inverse of $d\Phi_{(0,0,0)}$, compute as follows:

$$\begin{aligned}
(d\Phi)^{-1}(\dot\phi, \bar\partial_b \dot X - i\bar\partial_b \dot Y + \dot\phi - \dot\psi, \rho(\dot X - i\dot Y))& \\
= (\dot\phi, \pi_{\mathrm{Re}}(\mathcal{P}(\bar\partial_b \dot X - i\bar\partial_b \dot Y - \dot\psi) + \rho(\dot X - i\dot Y)),& \\
\pi_{\mathrm{Im}}(\mathcal{P}(\bar\partial_b \dot X - i\bar\partial_b \dot Y - \dot\psi) + \rho(\dot X - i\dot Y)),& \\
- \mathcal{H}(\bar\partial_b \dot X - i\bar\partial_b \dot Y - \dot\psi)) = (\dot\phi, \dot X, \dot Y, \dot\psi)\,.&
\end{aligned}$$

q.e.d.

By the implicit function theorem, inverting Φ gives rise to the C^1 map

$$\text{(6.1.3a)} \quad \Gamma^{s+2}(\mathcal{D}ef) \to \mathcal{D}^{s+1}_{cont}(M) \oplus V^{s+1} \oplus \Gamma^{s}(\mathrm{H}^1) \,:\, \phi \mapsto (F_\phi, Y_\phi, \psi_\phi)$$

defined by the constraint

$$\text{(6.1.3b)} \qquad (\phi, X_\phi, Y_\phi, \psi_\phi) = \Phi^{-1}(\phi, 0, 0),$$

with $F_\phi = F_{\Psi X_\phi}$ and ϕ in a sufficiently small neighborhood of the origin.

Corollary 6.1.4. *There exist neighborhoods $0 \in U \subset \Gamma^{s+2}(\mathcal{D}ef)$ and $id_M \in \tilde U \subset \mathcal{D}^{s+1}_{cont}(M)$ such that for any $\phi \in U$, there is a contact diffeomorphism $F_\phi \in \widetilde U$ such that $F_\phi^*\phi$ is contained in the subspace $\bar\partial_b\left(iV^{s+1}\right) \oplus \Gamma^s(\mathrm{H}^1) \subset \Gamma^s(\mathcal{D}ef)$. The equation*

$$F_\phi^*\phi = i\bar\partial_b Y_\phi + \psi_\phi \in \Gamma^s(\mathcal{D}ef)$$

determines F_ϕ, Y_ϕ, and ψ_ϕ up to the CR-vector field $\rho(X_\phi - iY_\phi)$, which is in turn determined by the additional constraint $\rho(X_\phi - iY_\phi) = 0$.

We call the deformation tensor

$$F_\phi {*} \phi = i\bar\partial_b Y_\phi + \psi_\phi \in \Gamma^s(\mathcal{D}ef)$$

the *normal form* of ϕ. The following theorem, which is proved using *a priori* estimates, gives increased regularity for the normal form. It is an immediate corollary to Theorem 6.2.7 below.

Theorem 6.1.5. *The map $\phi \mapsto (F_\phi, Y_\phi, \psi_\phi)$ defines a C^0 map of the form*

$$\Gamma^{s+2}(\mathcal{D}ef) \to \mathcal{D}^{s+3}_{cont}(M) \oplus V^{s+3} \oplus \Gamma^{s+2}(\mathrm{H}^1)\,,$$

for sufficiently small $\phi \in \Gamma^{s+2}(\mathcal{D}ef)$. In particular, the normal form

$$F^*_\phi \phi = \left(i\bar{\partial}_b Y_\phi + \psi_\phi\right)$$

is contained in $\bar{\partial}_b\left(iV^{s+3}\right) \oplus \Gamma^{s+2}(\mathrm{H}^1) \subset \Gamma^{s+2}(\mathcal{D}ef)$.

Remark 6.1.6. As noted in Remark 5.3.6, we have some freedom in the choice of V^{s+3}, the complementary subspace to $\Gamma^{s+3}_{cont}(TM)$ in $\Gamma^{s+3}_{cont}(T_{(1,0)}M)$. If the original CR manifold admits a free S^1 action as a symmetry, we can choose all homotopy operators to be S^1 equivariant. Complex contact vector fields then have Fourier expansions, and we can choose our complement V to consist of complex vector fields of the form Z_f, where f has only positive (respectively negative) Fourier coefficients. In [**B**], we made these choices to obtain the interior (respectively exterior) normal form.

In general, since M is embeddable it follows that $M \hookrightarrow \Sigma$ for some compact complex surface Σ as a separating hypersurface (see [**Le**]). The elements of V correspond on the infinitesimal level to Kuranishi's "wiggles," that is, CR structures which are induced on M through infinitesimal isotopies of M within Σ. In this regard, one expects the factor ψ to correspond to deformations of the singularities of the "fill-in" of M (that is, the pseudoconvex side of Σ bounded by M) or to non-embeddable structures on M.

6.2. A priori estimates for the action on CR structures. We now proceed to establish the *a priori* regularity estimates for the action of the contact diffeomorphism group on the space of deformation tensors that we need to establish Theorem 6.1.5.

Let X be a contact vector field and let ϕ be a CR deformation, expressed relative to a local frame Z_α and dual coframe $\omega^{\bar\beta}$ as $\phi = \phi^\alpha_{\bar\beta}\omega^{\bar\beta} \otimes Z_\alpha$. For X and ϕ sufficiently small, we will obtain estimates for the deformation tensor for the pull-back CR structure $\mu = F^*\phi$.[7]

Remark 6.2.1. Since we are restricting ourselves to a small neighborhood of the embeddable structure, we may choose the neighborhood small enough to have the following uniform estimates:

$$\|\phi\|_{s+2} < C\,, \quad \|\mu\|_{s+2} < C\,, \text{ and } \|X\|_{s+1} < C$$

where C is a fixed (sufficiently small) constant. Since when $\|X\|_{s+1} < C$ one has $\|X\|_{s+1} \sim \|F_{\Psi(X)}\|_{s+1}$, and one can choose C such that in addition

$$\|F_{\Psi(X)}\|_{s+1} < C,$$

[7]Although we have restricted to the three dimensional case $n = 1$, we continue to use index notation to help distinguish between functions and coefficients of tensors.

here and in what follows we use the norm on contact diffeomorphisms $\|F_{\Psi(X)}\|_{s+1} := \|\Psi(X)\|_{s+1}$, where $F_{\Psi(X)} = \exp \circ \Psi(X)$.

Remark 6.2.2. We will repeatedly use the estimates

$$\|fg\|_s \prec \|f\|_s \|g\|_{s-1} + \|f\|_{s-1}\|g\|_s$$

and

$$\begin{aligned} \|g \circ F\|_s &\prec (\|g\|_s + \|g\|_s\|F\|_{s-1} + \|g\|_{s-1}\|F\|_s) \cdot (1 + \|F\|_{s-1})^{s-1} \\ &\prec \|g\|_s + \|g\|_s\|F\|_{s-1} + \|g\|_{s-1}\|F\|_s \end{aligned}$$

for all $s > 2n+4 = 6$ (since $n = 1$), $f, g \in \Gamma^s(M)$, and $F \in \mathcal{D}^s_{cont}(M)$, $\|F\| \prec 1$, without comment. The first estimate was proved in [**BD1**] (for general n). The second estimate follows easily by writing $g \circ F$ in local coordinates and computing $\|g \circ F\|_{U,s}$ in a coordinate neighborhood $U \subset M$ using the chain rule. In the last estimate, we used the fixed bound on X to conclude that $(1 + \|F\|_{s-1})^{s-1} \prec 1$.

Our next goal is to obtain estimates for the deformation tensor for the pull-back $\mu = F^*_{\Psi(X)}\phi$.

Lemma 6.2.3. *Let $F = F_{\Psi(X)}$ and $s > 6$. Then*

$$\|\phi \circ F\|_s \prec \|\phi\|_s + \|\phi\|_s \cdot \|X\|_{s-1} + \|\phi\|_{s-1} \cdot \|X\|_s \, .$$

Proof. Observe that on each coordinate patch U_ℓ,

$$\begin{aligned} \|\rho_\ell(\phi \circ F)_\ell\|_{U_\ell,s} &\prec \|\phi\|_{U_\ell,s} + \|\phi\|_{U_\ell,s}\|F_\ell\|_{U_\ell,s-1} + \|\phi\|_{U_\ell,s-1}\|F_\ell\|_{U_\ell,s} \\ &\prec \|\phi\|_s + \|\phi\|_s\|F\|_{s-1} + \|\phi\|_{s-1}\|F\|_s \\ &\prec \|\phi\|_s + \|\phi\|_s\|\Psi(X)\|_{s-1} + \|\phi\|_{s-1}\|\Psi(X)\|_s \\ &\prec \|\phi\|_s + \|\phi\|_s\|X\|_{s-1} + \|\phi\|_{s-1}\|X\|_s. \end{aligned}$$

The result follows from finiteness of the cover U_ℓ. q.e.d.

Next let $\mathcal{E}(X, Y, \phi)$ be the vector-valued one-form defined by Equation (4.1.11). Then we have the following estimates:

Lemma 6.2.4. *For $s > 6$, let $\phi \in \Gamma^s(\mathcal{D}ef)$ be a deformation tensor with $\|\phi\|_s < C$ and let $X, Y \in \Gamma^{s+1}_{cont}(TM)$ be vector fields with $\|X\|_{s+1} < C$, $\|Y\|_{s+1} < C$, for C chosen as in Remark (6.2.1). Then*

$$\|\mathcal{E}(X, Y, \phi)\|_s \prec (\|X\|_s + \|\phi \circ F_{\Psi(Y)}\|_s) \cdot \|X\|_{s+1} \, .$$

Let $\phi_j \in \Gamma^s(\mathcal{D}ef)$, $j = 1, 2$ be two deformation tensors with $\|\phi_j\|_s < C$, and let $X_j, Y_j \in \Gamma^{s+1}_{cont}(TM)$, $j = 1, 2$ be contact vector fields with

$\|X_j\|_{s+1} < C$, $\|Y_j\|_{s+1} < C$. *Then*

$$
\begin{aligned}
&\|\mathcal{E}(X_1, Y_1, \phi_1) - \mathcal{E}(X_2, Y_2, \phi_2)\|_s \\
&\prec (\|X_1\|_{s+1} + \|X_2\|_{s+1}) \cdot \|X_1 - X_2\|_s + (\|X_1\|_s + \|X_2\|_s) \cdot \|X_1 - X_2\|_{s+1} \\
&\qquad + (\|X_1\|_{s+1} + \|X_2\|_{s+1}) \cdot \|\phi_1 \circ F_1 - \phi_2 \circ F_2\|_s \\
&\qquad\qquad + (\|\phi_1 \circ F_1\|_s + \|\phi_2 \circ F_2\|_s) \cdot \|X_1 - X_2\|_{s+1},
\end{aligned}
$$

where $F_j = F_{\Psi(Y_j)}$, $j = 1, 2$.

Proof. By Equation (4.1.11), our proof amounts to obtaining sufficiently good estimates on the entries of the matrices $A(X, Y, \phi)$ and $B(X, Y, \phi)$ defined in Equations (4.1.7). Recall the local formulæ for A and B:

$$
\begin{aligned}
A^\alpha_\beta &= \delta^\alpha_\beta + Z_\beta \lrcorner\ \mathcal{L}_X \omega^\alpha + Z_\beta \lrcorner\ Q^\alpha(X, Y, \phi) \\
B^\alpha_{\bar\beta} &= (\bar\partial_b X)^\alpha_{\bar\beta} + (\phi^\alpha_{\bar\beta} \circ F_{\Psi(Y)}) + Z_{\bar\beta} \lrcorner\ Q^\alpha(X, Y, \phi)\,.
\end{aligned}
$$

The estimate

$$
\|\mathcal{E}^\alpha_{\bar\beta}\|_s \prec \|Z_{\bar\beta} \lrcorner\ Q^\alpha(X, Y, \phi)\|_s + \|A^{-1}\|_s \|[(I - A)B]^\alpha_{\bar\beta}\|_s
$$

follows immediately from the formula for $\mathcal{E}^\alpha_{\bar\beta}$.

We estimate each term on the right-hand side. First, using Proposition 3.2.15 to estimate $\Psi(X) - X$ and observing that the estimate $\|Z_\beta \lrcorner\ Q_{\omega^\alpha}(\Psi(X))\|_s \prec \|\Psi(X)\|_s \|\Psi(X)\|_{s+1}$ follows immediately from the local formula (3.2.6), we obtain

$$
\begin{aligned}
&\|Z_\beta \lrcorner\ Q^\alpha(X, Y, \phi)\|_s \\
&\quad \prec \|Z_\beta \lrcorner\ \mathcal{L}_{\Psi(X) - X} \omega^\alpha\|_s + \|(\phi^\alpha_{\bar\gamma} \circ F_{\Psi(Y)}) Z_\beta \lrcorner\ \mathcal{L}_{\Psi(X)} \omega^{\bar\gamma}\|_s \\
&\qquad + \|Z_\beta \lrcorner\ Q_{\omega^\alpha}(\Psi(X))\|_s + \|(\phi^\alpha_{\bar\gamma} \circ F_{\Psi(Y)}) Z_\beta \lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X))\|_s \\
&\quad \prec \|\Psi(X) - X\|_{s+1} + \|\phi \circ F_{\Psi(Y)}\|_s \, \|Z_\beta \lrcorner\ \mathcal{L}_{\Psi(X)} \omega^{\bar\gamma}\|_{s-1} \\
&\qquad + \|\phi \circ F_{\Psi(Y)}\|_{s-1} \, \|Z_\beta \lrcorner\ \mathcal{L}_{\Psi(X)} \omega^{\bar\gamma}\|_s + \|Z_\beta \lrcorner\ Q_{\omega^\alpha}(\Psi(X))\|_s \\
&\qquad + \|\phi \circ F_{\Psi(Y)}\|_{s-1} \|Z_\beta \lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X))\|_s \\
&\qquad + \|\phi \circ F_{\Psi(Y)}\|_s \|Z_\beta \lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X))\|_{s-1} \\
&\quad \prec \|X\|_s \, \|X\|_{s+1} + \|\phi \circ F_{\Psi(Y)}\|_s \, \|\Psi(X)\|_s \\
&\qquad + \|\phi \circ F_{\Psi(Y)}\|_{s-1} \, \|\Psi(X)\|_{s+1} + \|\Psi(X)\|_s \, \|\Psi(X)\|_{s+1} \\
&\qquad + \|\phi \circ F_{\Psi(Y)}\|_{s-1} \|\Psi(X)\|_s \, \|\Psi(X)\|_{s+1} \\
&\qquad + \|\phi \circ F_{\Psi(Y)}\|_s \|\Psi(X)\|_{s-1} \, \|\Psi(X)\|_s \\
&\quad \prec \|X\|_s \, \|X\|_{s+1} + \|\phi \circ F_{\Psi(Y)}\|_s \, \|X\|_s + \|\phi \circ F_{\Psi(Y)}\|_{s-1} \, \|X\|_{s+1} \\
&\qquad + \|X\|_s \, \|X\|_{s+1} + \|\phi \circ F_{\Psi(Y)}\|_{s-1} \|X\|_s \, \|X\|_{s+1} \\
&\qquad + \|\phi \circ F_{\Psi(Y)}\|_s \|X\|_{s-1} \, \|X\|_s \\
&\quad \prec (\|X\|_s + \|\phi \circ F_{\Psi(Y)}\|_s) \cdot \|X\|_{s+1}
\end{aligned}
$$

with a similar estimate for $\|Z_{\bar\beta}\lrcorner\ Q^\alpha(X,Y,\phi)\|_s$. Next

$$\begin{aligned}\|[(I-A)]^\alpha_\gamma\|_s &= \|(Z_\gamma\lrcorner\ \mathcal{L}_X\omega^\alpha + Z_\gamma\lrcorner\ Q^\alpha(X,Y,\phi))\|_s\\ &\prec \|X\|_{s+1} + (\|X\|_s + \|\phi\circ F_{\Psi(Y)}\|_s)\cdot\|X\|_{s+1}\,,\end{aligned}$$

which implies in particular that $A = I-(I-A)$ is invertible. More precisely, because the matrix $A = \left[A^\alpha_{\bar\beta}\right]$ is of the form $I+$(small matrix), a series expansion for A^{-1} yields the estimate $\|A^{-1}-I\|_s \prec \|X\|_{s+1}+\|\phi\circ F_{\Psi(Y)}\|_s\cdot\|X\|_{s+1}$ which is uniformly bounded by a constant depending only on the constant C in Remark 6.2.1. Also

$$\begin{aligned}\|B^\gamma_{\bar\beta}\|_s &= \|((\bar\partial_b X)^\gamma_{\bar\beta} + (\phi^\gamma_{\bar\beta}\circ F_{\Psi(Y)}) + Z_{\bar\beta}\lrcorner\ Q^\gamma(X,Y,\phi))\|_s\\ &\prec \|X\|_{s+1} + \|\phi\circ F_{\Psi(Y)}\|_s + (\|X\|_s + \|\phi\circ F_{\Psi(Y)}\|_s)\cdot\|X\|_{s+1}\,,\end{aligned}$$

so

$$\begin{aligned}&\|[(I-A)B]^\alpha_{\bar\beta}\|_s\\ &\quad\prec \|[(I-A)]^\alpha_\gamma\|_s\|[B]^\gamma_{\bar\beta}\|_{s-1} + \|[(I-A)]^\alpha_\gamma\|_{s-1}\|[B]^\gamma_{\bar\beta}\|_s\\ &\quad\prec \left\{\|X\|_{s+1} + (\|X\|_s + \|\phi\circ F_{\Psi(Y)}\|_s)\cdot\|X\|_{s+1}\right\}\\ &\qquad\cdot\left\{\|X\|_s + \|\phi\circ F_{\Psi(Y)}\|_{s-1} + (\|X\|_{s-1} + \|\phi\circ F_{\Psi(Y)}\|_{s-1})\cdot\|X\|_s\right\}\\ &\qquad+\left\{\|X\|_s + (\|X\|_{s-1} + \|\phi\circ F_{\Psi(Y)}\|_{s-1})\cdot\|X\|_s\right\}\\ &\qquad\cdot\left\{\|X\|_{s+1} + \|\phi\circ F_{\Psi(Y)}\|_s + (\|X\|_s + \|\phi\circ F_{\Psi(Y)}\|_s)\cdot\|X\|_{s+1}\right\}\\ &\quad\prec \|X\|_{s+1}\cdot(\|X\|_s + \|(\phi\circ F_{\Psi(Y)})\|_{s-1})\\ &\qquad\quad+\|X\|_s\cdot(\|X\|_{s+1} + \|(\phi\circ F_{\Psi(Y)})\|_s)\\ &\quad\prec (\|X\|_s + \|\phi\circ F_{\Psi(Y)}\|_s)\cdot\|X\|_{s+1}\,.\end{aligned}$$

This completes the proof of the first estimate.

To prove the second estimate, let $A_j = A(X_j,Y_j,\phi_j)$, $B_j = B(X_j,Y_j,\phi_j)$, $\mathcal{E}_j = \mathcal{E}(X_j,Y_j,\phi_j)$, $j=1,2$. Then

$$\begin{aligned}A_1^{-1}B_1 - A_2^{-1}B_2 &= A_1^{-1}(B_1-B_2) - (A_2^{-1}-A_1^{-1})B_2\\ &= [(B_1-B_2) + A_1^{-1}(I-A_1)(B_1-B_2)]\\ &\quad -[A_2^{-1}(A_1-A_2)A_1^{-1}B_2]\,.\end{aligned}$$

Using this in Equation (4.1.9), we obtain the equality

$$\begin{aligned}[\mathcal{E}_1-\mathcal{E}_2]^\alpha_{\bar\beta} &= Z_{\bar\beta}\lrcorner\ (Q^\alpha(X_1,Y_1,\phi_1) - Q^\alpha(X_2,Y_2,\phi_2))\\ &\quad + [A_1^{-1}(I-A_1)(B_1-B_2)]^\alpha_{\bar\beta} - [A_2^{-1}(A_1-A_2)A_1^{-1}B_2]^\alpha_{\bar\beta}\,.\end{aligned}$$

Choose the constant C in Remark 6.2.1 sufficiently small to ensure that $\|A_j^{-1}\|_s < C'$ for some fixed constant C'. The triangle inequality then gives

$$\text{(6.2.5)}\quad \begin{aligned}\|[\mathcal{E}_1-\mathcal{E}_2]^\alpha_{\bar\beta}\|_s \prec \|Z_{\bar\beta}\lrcorner\ Q^\alpha(X_1,Y_1,\phi_1) - Z_{\bar\beta}\lrcorner\ Q^\alpha(X_2,Y_2,\phi_2)\|_s \\ + \|(I-A_1)(B_1-B_2)\|_s + \|(A_1-A_2)\|_s\|B_2\|_{s-1} \\ + \|(A_1-A_2)\|_{s-1}\|B_2\|_s\,.\end{aligned}$$

We estimate all four terms on the right-hand side of (6.2.5) in a similar manner. We present the estimate of the first term in detail and leave the verification of the estimates of the remaining two terms to the reader. Rearranging terms and simplifying gives

$$\begin{aligned}
&Z_{\bar\beta}\lrcorner\ Q^\alpha(X_1,Y_1,\phi_1) - Z_{\bar\beta}\lrcorner\ Q^\alpha(X_2,Y_2,\phi_2)\\
&\quad= \big\{Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_1)-X_1}\omega^\alpha + (\phi^\alpha_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_1)}\omega^{\bar\gamma}\\
&\qquad + Z_{\bar\beta}\lrcorner\ Q_{\omega^\alpha}(\Psi(X_1)) + (\phi^\alpha_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_1))\big\}\\
&\quad- \big\{Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_2)-X_2}\omega^\alpha + (\phi^\alpha_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_2)}\omega^{\bar\gamma}\\
&\qquad + Z_{\bar\beta}\lrcorner\ Q_{\omega^\alpha}(\Psi(X_2)) + (\phi^\alpha_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_2))\big\}\\
&\quad= Z_{\bar\beta}\lrcorner\ \mathcal{L}_{(\Psi(X_1)-X_1)-(\Psi(X_2)-X_2)}(\omega^\alpha)\\
&\qquad + (Z_{\bar\beta}\lrcorner\ Q_{\omega^\alpha}(\Psi(X_1)) - Z_{\bar\beta}\lrcorner\ Q_{\omega^\alpha}(\Psi(X_2))\\
&\quad+ \big\{(\phi^\alpha_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_1)}\omega^{\bar\gamma} - (\phi^\alpha_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_2)}\omega^{\bar\gamma}\big\}\\
&\quad+ \big\{(\phi^\alpha_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_1)) - (\phi^\alpha_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_2))\big\}
\end{aligned}$$

By our previous estimates, we may estimate as follows:

$$\begin{aligned}
&\|Z_{\bar\beta}\lrcorner\ Q^\alpha(X_1,Y_1,\phi_1) - Z_{\bar\beta}\lrcorner\ Q^\alpha(X_2,Y_2,\phi_2)\|_s\\
&\quad\prec\ \|Z_{\bar\beta}\lrcorner\ \mathcal{L}_{(\Psi(X_1)-X_1)-(\Psi(X_2)-X_2)}\omega^\alpha\|_s\\
&\qquad+\ \|Z_{\bar\beta}\lrcorner\ Q_{\omega^\alpha}(\Psi(X_1)) - Z_{\bar\beta}\lrcorner\ Q_{\omega^\alpha}(\Psi(X_2))\|_s\\
&\qquad+\ \|(\phi^\alpha_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_1)}\omega^{\bar\gamma} - (\phi^\alpha_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_2)}\omega^{\bar\gamma}\|_s\\
&\qquad+\ \|(\phi^\alpha_{1,\bar\gamma}\circ F_1)\,Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_1)) - (\phi^\alpha_{2,\bar\gamma}\circ F_2)\,Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_2))\|_s\\
&\quad\prec\ \|(\Psi(X_1)-X_1)-(\Psi(X_2)-X_2)\|_{s+1}\\
&\qquad+\ \|X_1-X_2\|_s\cdot(\|X_1\|_{s+1}+\|X_2\|_{s+1})\\
&\qquad+\ \|X_1-X_2\|_{s+1}\cdot(\|X_1\|_s+\|X_2\|_s)\\
&\qquad+\ \|(\phi^\alpha_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_1)}\omega^{\bar\gamma} - (\phi^\alpha_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_2)}\omega^{\bar\gamma}\|_s\\
&\qquad+\ \|(\phi^\alpha_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_1)) - (\phi^\alpha_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_2))\|_s
\end{aligned}$$

where we have used Lemma 3.2.7(c),

$$\begin{aligned}
&\quad\prec\ \|X_1-X_2\|_s\cdot(\|X_1\|_{s+1}+\|X_2\|_{s+1})\\
&\qquad+\ \|X_1-X_2\|_{s+1}\cdot(\|X_1\|_s+\|X_2\|_s),\\
&\qquad+\ \|(\phi^\alpha_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_1)}\omega^{\bar\gamma} - (\phi^\alpha_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_2)}\omega^{\bar\gamma}\|_s\\
&\qquad+\ \|(\phi^\alpha_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_1)) - (\phi^\alpha_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ Q_{\omega^{\bar\gamma}}(\Psi(X_2))\|_s,
\end{aligned}$$

where we have used Proposition 3.2.15(b).

Observe that

$$
\begin{aligned}
&\|(\phi^{\alpha}_{1,\bar\gamma}\circ F_1)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_1)}\omega^{\bar\gamma} - (\phi^{\alpha}_{2,\bar\gamma}\circ F_2)Z_{\bar\beta}\lrcorner\ \mathcal{L}_{\Psi(X_2)}\omega^{\bar\gamma}\|_s \\
&\qquad \prec \quad (\|\phi_1\circ F_1\|_{s-1} + \|\phi_2\circ F_2\|_{s-1})\cdot\|X_1 - X_2\|_{s+1} \\
&\qquad\qquad + (\|\phi_1\circ F_1\|_s + \|\phi_2\circ F_2\|_s)\cdot\|X_1 - X_2\|_s \\
&\qquad\qquad + \ (\|X_1\|_s + \|X_2\|_s)\cdot\|\phi_1\circ F_1 - \phi_2\circ F_2\|_s \\
&\qquad\qquad + (\|X_1\|_{s+1} + \|X_2\|_{s+1})\cdot\|\phi_1\circ F_1 - \phi_2\circ F_2\|_{s-1} \\
&\qquad \prec \quad (\|\phi_1\circ F_1\|_s + \|\phi_2\circ F_2\|_s)\cdot\|X_1 - X_2\|_{s+1} \\
&\qquad\qquad + \ (\|X_1\|_{s+1} + \|X_2\|_{s+1})\cdot\|\phi_1\circ F_1 - \phi_2\circ F_2\|_s\,,
\end{aligned}
$$

where we have used the identity $f_1\,g_1 - f_2\,g_2 = f_1\,(g_1 - g_2) + (f_1 - f_2)\,g_2$ and the corresponding estimate

$$
\begin{aligned}
&\|f_1\,g_1 - f_2\,g_2\|_s \\
&\quad \prec (\|f_1\|_{s-1} + \|f_2\|_{s-1})\cdot\|g_1 - g_2\|_s + (\|f_1\|_s + \|f_2\|_s)\cdot\|g_1 - g_2\|_{s-1} \\
&\quad + (\|g_1\|_{s-1} + \|g_2\|_{s-1})\cdot\|f_1 - f_2\|_s + (\|g_1\|_s + \|g_2\|_s)\cdot\|f_1 - f_2\|_{s-1}\,.
\end{aligned}
$$

A similar argument yields the estimate

$$
\begin{aligned}
&\|(\phi_1\circ F_1)Z_{\bar\beta}\lrcorner\ Q_{\overline{\varpi}}(\Psi(X_1)) - (\phi_2\circ F_2)Z_{\bar\beta}\lrcorner\ Q_{\overline{\varpi}}(\Psi(X_2))\|_s \\
&\qquad \prec \quad (\|\phi_1\circ F_1\|_s + \|\phi_2\circ F_2\|_s)\cdot\|X_1 - X_2\|_{s+1} \\
&\qquad\qquad + (\|X_1\|_{s+1} + \|X_2\|_{s+1})\cdot\|\phi_1\circ F_1 - \phi_2\circ F_2\|_s\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|Z_{\bar\beta}\lrcorner\ Q^{\alpha}(X_1, Y_1, \phi_1) - Z_{\bar\beta}\lrcorner\ Q^{\alpha}(X_2, Y_2, \phi_2)\|_s \\
&\qquad \prec \quad (\|X_1\|_{s+1} + \|X_2\|_{s+1})\cdot\|X_1 - X_2\|_s \\
&\qquad\qquad + (\|X_1\|_s + \|X_2\|_s)\cdot\|X_1 - X_2\|_{s+1} \\
&\qquad\qquad + (\|X_1\|_{s+1} + \|X_2\|_{s+1})\cdot\|\phi_1\circ F_1 - \phi_2\circ F_2\|_s \\
&\qquad\qquad + (\|\phi_1\circ F_1\|_s + \|\phi_2\circ F_2\|_s)\cdot\|X_1 - X_2\|_{s+1}\,.
\end{aligned}
$$

q.e.d.

Our proof of the *a priori* estimates from which Theorem 6.1.5 follows requires one more technical lemma. For $k > 6$ and $\epsilon > 0$ small, let $\phi \in \Gamma^k(\mathcal{D}ef)$ and $X_0 \in \Gamma^k_{cont}(TM)$ with $\|\phi\|_k < \epsilon$ and $\|X_0\| < \epsilon$. Then the map

$$T^{k+1}_{\phi,X_0}\ :\ \Gamma^{k+1}_{cont}(T_{(1,0)}M) \to \Gamma^{k+1}_{cont}(T_{(1,0)}M)\ :\ Z \mapsto Z + \mathcal{P}\left(\mathcal{E}(\pi_{\mathrm{Re}}(Z), X_0, \phi)\right)$$

is defined for all Z in a sufficiently small ball about the origin.

Lemma 6.2.6. *There exists a sufficiently small $\epsilon > 0$ such that the following holds. For all $\phi \in \Gamma^k(\mathcal{D}ef)$ and $X_0 \in \Gamma^k_{cont}(TM)$ such that $\|X_0\|_k < \epsilon$ and $\|\phi\|_k < \epsilon$, the equation $T^{k+1}_{\phi,X_0}(Z) = W$ has a unique solution $Z \in \Gamma^{k+1}_{cont}(T_{(1,0)}M)$ such that $\|Z\|_{k+1} \leq 2\,\epsilon$ for all*

$W \in \Gamma^{k+1}_{cont}(T_{(1,0)}M)$ with $\|W\|_{k+1} < \epsilon$. Moreover, the solution satisfies the estimate $\|Z\|_{k+1} \le 2\,\|W\|_{k+1}$.

Proof. We first show that we can choose $\delta > 0$ so that the map $Z \mapsto \mathcal{P}(\mathcal{E}(\pi_{\mathrm{Re}}(Z), X_0, \phi))$ is a contraction mapping in $\Gamma^{k+1}_{cont}(T_{(1,0)}M)$ for $\|Z\|_{k+1} < \delta$. To see this, first note that by Lemma 6.2.4, for $\phi \in \Gamma^k(\mathcal{D}ef)$ with $\|\phi\|_k < C$, for C sufficiently small, the estimate

$$\|\mathcal{E}(X_1, X_0, \phi) - \mathcal{E}(X_2, X_0, \phi)\|_k \prec (\|X_1\|_k + \|X_2\|_{k+1} + \|\phi \circ F_0\|_k) \cdot \|X_1 - X_2\|_{k+1}$$

holds for all $X_1, X_2 \in \Gamma^{k+1}_{cont}(TM)$, with $\|X_j\|_{k+1} < C$, $j = 1, 2$. Thus,

$$\|\mathcal{P}(\mathcal{E}(X_1, X_0, \phi)) - \mathcal{P}(\mathcal{E}(X_2, X_0, \phi))\|_{k+1} \prec (\|X_1\|_{k+1} + \|X_2\|_{k+1} + \|\phi \circ F_0\|_k) \cdot \|X_1 - X_2\|_{k+1}\,.$$

Consequently, for $\delta' > 0$ sufficiently small,

$$\|\mathcal{P}(\mathcal{E}(X_1, X_0, \phi)) - \mathcal{P}(\mathcal{E}(X_2, X_0, \phi))\|_{k+1} < \frac{1}{2}\|X_1 - X_2\|_{k+1}\,,$$

provided $\|\phi\|_k < \delta'$, $\|X_j\|_{k+1} < \delta'$, $j = 1, 2$.

Now choose $\delta < \delta'$ so that $\|\pi_{\mathrm{Re}}(Z)\|_{k+1} < \delta'$ for $\|Z\|_{k+1} < \delta$. Then for $X_j = \pi_{\mathrm{Re}}(Z_j)$, $j = 1, 2$,

$$\|\mathcal{P}(\mathcal{E}(\pi_{\mathrm{Re}} Z_1, X_0, \phi)) - \mathcal{P}(\mathcal{E}(\pi_{\mathrm{Re}} Z_2, X_0, \phi))\|_{k+1} < \frac{1}{2}\|Z_1 - Z_2\|_{k+1}\,,$$

provided $\|Z_j\|_{k+1} < \delta$.

Finally, set $\epsilon = \delta/2$. Choose any $W \in \Gamma^{k+1}_{cont}(T_{(1,0)}M)$ and define the sequence Z_n, $n = 0, 1, 2, \ldots$ inductively by $Z_0 = 0$, $Z_{n+1} = W - \mathcal{P}(\mathcal{E}(\pi_{\mathrm{Re}}(Z_n), X_0, \phi))$. Since $\mathcal{E}(0, X_0, \phi) = 0$, $Z_1 = W$. Consequently, $\{Z_n\}$ is Cauchy with $\|Z_{n+1} - Z_n\|_{k+1} < \frac{1}{2}\|Z_n - Z_{n-1}\|_{k+1}$. Therefore, $\|Z_n\|_{k+1} < 2\|W\|_{k+1}$. Thus, the sequence converges to a solution Z of the equation $T_{\phi, X_0}(Z) = W$ satisfying the estimate $\|Z\|_{k+1} \le 2\|W\|_{k+1}$. Uniqueness of the solution follows from the contraction mapping property. q.e.d.

We are now able to obtain the *a priori* estimates that we promised and from which Theorem 6.1.5 follows.

Theorem 6.2.7. *Fix a smooth background embeddable CR structure on M as above, and let*

$$\mathcal{P} : \Omega^{(0,1)}(H_{(1,0)}) \to \Gamma^{\infty}_{cont}(T_{(1,0)}M) \text{ and } \mathcal{H} : \Omega^{(0,1)}(H_{(1,0)}) \to \Omega^{(0,1)}(H_{(1,0)})$$

be the linear operators of Corollary 5.2.10. Then for $s > 6$, there exists $\epsilon > 0$ such that the following holds:
Suppose that $\phi \in \Gamma^{s+2}(\mathcal{D}ef)$, $X \in \Gamma^{s+1}_{cont}(TM)$ (so $F_{\Psi(X)} \in \mathcal{D}^{s+1}_{cont}(M)$), $Y \in V^{s+1}$, and $\psi \in \Gamma^s(\mathcal{D}ef)$ satisfy the conditions

$$\rho(X - iY) = 0\,, \quad \|F_{\Psi(X)}\|_{s+1} < \epsilon\,, \quad \|\phi\|_{s+2} < \epsilon\,, \text{ and } \psi \in \ker \mathcal{P}\,.$$

If the deformation tensor $\mu = F^*_{\Psi(X)}\phi - i\bar\partial_b Y - \psi$ *is contained in* $\Gamma^{s+2}(\mathcal{D}ef)$ *and* $\|\mu\|_{s+2} < \epsilon$*, then*

$$F_{\Psi(X)} \in \mathcal{D}^{s+3}_{cont}(M)\,, \quad Y \in V^{s+3}, \text{ and } \psi \in \Gamma^{s+2}(\mathcal{D}ef)\,.$$

Moreover, the following estimates are satisfied:

$$\begin{aligned}\|F_{\Psi(X)}\|_{s+3} &\prec \|\phi\|_{s+2} + \|\mu\|_{s+2}\,,\\ \|Y\|_{s+3} &\prec \|\phi\|_{s+2} + \|\mu\|_{s+2}\,,\\ \|\psi\|_{s+2} &\prec \|\phi\|_{s+2} + \|\mu\|_{s+2}\,.\end{aligned}$$

Proof. Substitution of the expression for $F^*_{\Psi(X)}\phi$ given in Proposition 4.1.12 in the formula for μ gives

$$\mu = \bar\partial_b X + \phi \circ F_{\Psi(X)} - i\bar\partial_b Y - \psi + \mathcal{E}(X, X, \phi)$$

where $\phi \circ F$ and $\mathcal{E}$ are defined as in (4.1.10) and (4.1.11).

We first prove that $\|X\|_{s+3}$, $\|Y\|_{s+3}$, and $\|\psi\|_{s+2}$ are finite. Applying the operator $\mathcal{P}$ and using the hypothesis $\mathcal{P}(\psi) = 0$ gives

$$\mathcal{P}(\mu) = \mathcal{P}(\bar\partial_b X - i\bar\partial_b Y) + \mathcal{P}(\phi \circ F) + \mathcal{P}(\mathcal{E}(X, X, \phi))\,. \tag{6.2.8}$$

Since $\rho(X - iY) = 0$, it follows that $X - iY = \mathcal{P}(\bar\partial_b(X - iY))$, and solving for $X - iY$ in the last equation, we have:

$$X - iY + \mathcal{P}(\mathcal{E}(X, X, \phi)) = \mathcal{P}(\mu - \phi \circ F_{\Psi(X)})\,. \tag{6.2.9}$$

Next we "freeze coefficients" in (6.2.9). Let $X_0 - iY_0 = X - iY$ and set $W = \mathcal{P}(\mu - \phi \circ F_{\Psi(X_0)}) \in \Gamma^{s+3}_{cont}(T_{(1,0)}M)$. Choose ϵ such that Lemma 6.2.6 holds for $k = s+1, s+2, s+3$. Then $X_0 - iY_0$ is the unique solution in $\Gamma^{s+1}_{cont}(T_{(1,0)}M)$ of the equation

$$T^k_{\phi, X_0}(Z) = W \text{ for } k = s+1. \tag{6.2.10}$$

We now perform the first of two bootstrapping steps. Notice that ϕ and μ are small in Γ^{s+2} and, hence, small in Γ^{s+1}, and that X_0 is also small in Γ^{s+1}. Consequently, the map T^k_{ϕ,X_0} is defined for $k = s+2$. Lemma 6.2.6 then shows that Equation (6.2.10) with $k = s+2$ has a unique solution in $\Gamma^{s+2}_{cont}(T_{(1,0)}M)$. It follows that $X_0 - iY_0$ is in $\Gamma^{s+2}_{cont}(T_{(1,0)}M)$. Lemma 6.2.3 then gives the *a priori* estimate

$$\|X_0 - iY_0\|_{s+2} \prec \|W\|_{s+2} \prec \|\mu - \phi \circ F_{\Psi(X_0)}\|_{s+1} \prec \|\mu\|_{s+1} + \|\phi\|_{s+1}\,.$$

The second bootstrap proceeds as follows. We now know that X_0 and ϕ are both in Γ^{s+2} and that $W = \mathcal{P}(\mu - \phi \circ F_{\Psi(X_0)})$ is in Γ^{s+3}. By our choice of ϵ, we can solve Equation (6.2.10) with $k = s+3$ to conclude that $X_0 - iY_0$ is in $\Gamma^{s+3}_{cont}(T_{(1,0)}M)$. Moreover, we have $X - iY = X_0 - iY_0$ with the *a priori* estimate

$$\|X - iY\|_{s+3} \prec \|\mu\|_{s+2} + \|\phi\|_{s+2}\,.$$

Finally, since $\psi = F^*\phi - i\bar\partial_b Y - \mu$, it follows that ψ is in Γ^{s+2} and satisfies the *a priori* estimate

$$\|\psi\|_{s+2} \prec \|\mu\|_{s+2} + \|\phi\|_{s+2} + \|X\|_{s+3} + \|Y\|_{s+3} \prec \|\mu\|_{s+2} + \|\phi\|_{s+2}\,.$$

This establishes the *a priori* bounds, and hence the *a priori* estimates for $F_{\Psi(X)}$, Y, and ψ. q.e.d.

Proof of Theorem 6.1.5. That F_ϕ, Y_ϕ, and ψ_ϕ are in the appropriate spaces is an immediate corollary of Theorem 6.2.7. It remains only to show that the map is continuous.

Choose smooth $\phi_j \in \Gamma^{s+2}(\mathcal{D}ef)$ and $\mu_j \in \Gamma^{s+2}(\mathcal{D}ef)$ such that $\phi_j \underset{\|\cdot\|_{s+2}}{\longrightarrow} \phi$ and $\mu_j \underset{\|\cdot\|_{s+2}}{\longrightarrow} \mu$. By the analysis above, there exist $X_j \in \Gamma^{s+3}_{cont}(TM)$, $Y_j \in V^{s+3}$, and $\psi_j \in \Gamma^{s+2}(\mathrm{H}^1)$ such that the contact diffeomorphisms $F_j = F_{\Psi(X_j)} \in \mathcal{D}^{s+3}_{cont}(M)$ satisfy the conditions

$$\mu_j = F_j^*\phi_j - i\bar\partial_b Y_j - \psi_j\,, \quad \rho(X_j - iY_j) = 0\,,$$

with $F_j \underset{\|\cdot\|_{s+1}}{\longrightarrow} F$, $Y_j \underset{\|\cdot\|_{s+1}}{\longrightarrow} Y$, and $\psi_j \underset{\|\cdot\|_{s}}{\longrightarrow} \psi$. By the *a priori* estimates $\|F_j\|_{s+2} \prec \|\phi_j\|_{s+1} + \|\mu_j\|_{s+1}$, $\|Y_j\|_{s+2} \prec \|\phi_j\|_{s+1} + \|\mu_j\|_{s+1}$, and $\|\psi_j\|_{s+1} \prec \|\phi_j\|_{s+1} + \|\mu_j\|_{s+1}$ established above, we note, in particular, that F_j, Y_j, ψ_j are bounded sequences in Γ^{s+2}, Γ^{s+2}, and Γ^{s+1}, respectively. Also note that, by continuity of composition, $F_j \underset{\|\cdot\|_{s+1}}{\longrightarrow} F$ and $\phi_j \underset{\|\cdot\|_{s+1}}{\longrightarrow} \phi$ together imply $\phi_j \circ F_j \underset{\|\cdot\|_{s+1}}{\longrightarrow} \phi \circ F$.

We now show that the sequences X_j and Y_j are Cauchy in $\Gamma^{s+2}_{cont}(TM)$. We estimate $\|X_j - X_i\|_{s+2}$ as follows. Writing

$$\mu_j = \bar\partial_b(X_j - i\,Y_j) + \phi_j \circ F_j - \psi_j + \mathcal{E}_j \qquad \text{(see (4.1.10) and (4.1.11))}$$

with $F_j = F_{\Psi(X_j)}$, $\mathcal{E}_j = \mathcal{E}(X_j, Xj, \phi_j)$ yields the formula

$$\mu_j - \mu_i = \bar\partial_b(X_j - X_i) - i\bar\partial_b(Y_j - Y_i) - (\psi_j - \psi_i) + (\phi_j \circ F_j - \phi_i \circ F_i) + (\mathcal{E}_j - \mathcal{E}_i)\,.$$

Applying the operator $\mathcal{P}$ and using the facts $\mathcal{P}(\psi_j) = 0$, $\mathcal{P}\bar\partial_b(X_j - iY_j) = X_j - iY_j$ as above, gives:

$$\begin{aligned}\mathcal{P}(\mu_j - \mu_i) =& \mathcal{P}\left(\bar\partial_b(X_j - X_i) - i\bar\partial_b(Y_j - Y_i)\right)\\ &+ \mathcal{P}(\phi_j \circ F_j - \phi_i \circ F_i) + \mathcal{P}(\mathcal{E}_j - \mathcal{E}_i)\\ =&(X_j - X_i) - i(Y_j - Y_i) + \mathcal{P}(\phi_j \circ F_j - \phi_i \circ F_i) + \mathcal{P}(\mathcal{E}_j - \mathcal{E}_i)\,.\end{aligned}$$

Solving for $(X_j - X_i) - i(Y_j - Y_i)$, we have:

$$(X_j - X_i) - i(Y_j - Y_i) = \mathcal{P}(\mu_j - \mu_i) - \mathcal{P}(\phi_j \circ F_j - \phi_i \circ F_i) - \mathcal{P}(\mathcal{E}_j - \mathcal{E}_i)\,.$$

We can estimate the $s+2$ norm for $(X_j - X_i)$ as follows, using our *a priori* bound $\|X_j\|_{s+2} \le K$ on the sequence:

$$\begin{aligned}
&\|X_j - X_i\|_{s+2} + \|Y_j - Y_i\|_{s+2} \\
&\quad \prec \|(X_j - X_i) - i(Y_j - Y_i)\|_{s+2} \\
&\quad \prec \|\mathcal{P}(\mu_j - \mu_i)\|_{s+2} + \|\mathcal{P}(\phi_j \circ F_j - \phi_i \circ F_i)\|_{s+2} + \|\mathcal{P}(\mathcal{E}_j - \mathcal{E}_i)\|_{s+2} \\
&\quad \prec \|\mu_j - \mu_i\|_{s+1} + \|\phi_j \circ F_j - \phi_i \circ F_i\|_{s+1} + \|\mathcal{E}_j - \mathcal{E}_i\|_{s+1} \\
&\quad \prec \|\mu_j - \mu_i\|_{s+1} + \|\left(\phi_j \circ F_j - \phi_i \circ F_i\right)\|_{s+1} \\
&\qquad\qquad + \left(\|X_i\|_{s+2} + \|X_j\|_{s+2}\right) \cdot \|(\phi_j \circ F_j - \phi_i \circ F_i)\|_{s+1} \\
&\qquad\qquad + \left(\|\phi_j \circ F_j\|_{s+1} + \|\phi_i \circ F_i\|_{s+1}\right) \cdot \|X_j - X_i\|_{s+2} \\
&\qquad\qquad + \left(\|X_j\|_{s+2} + \|X_i\|_{s+2}\right) \cdot \|X_j - X_i\|_{s+1} \\
&\qquad\qquad + \left(\|X_j\|_{s+1} + \|X_i\|_{s+1}\right) \cdot \|X_j - X_i\|_{s+2} \\
&\quad \prec \|\mu_j - \mu_i\|_{s+1} + \|\left(\phi_j \circ F_j - \phi_i \circ F_i\right)\|_{s+1} + K\,\|(\phi_j \circ F_j - \phi_i \circ F_i)\|_{s+1} \\
&\qquad\qquad + \left(\|\phi_j\|_{s+1} + \|\phi_j\|_{s+1}\|F_j\|_s + \|\phi_j\|_s\|F_j\|_{s+1}\right) \cdot \|X_j - X_i\|_{s+2} \\
&\qquad\qquad + \left(\|\phi_i\|_{s+1} + \|\phi_i\|_{s+1}\|F_i\|_s + \|\phi_i\|_s\|F_i\|_{s+1}\right) \cdot \|X_j - X_i\|_{s+2} \\
&\qquad\qquad + K\,\|X_j - X_i\|_{s+1} + C\,\|X_j - X_i\|_{s+2} \\
&\quad \prec \|\mu_j - \mu_i\|_{s+1} + \|\phi_j \circ F_j - \phi_i \circ F_i\|_{s+1} \\
&\qquad\quad + K\,\left(\|(\phi_j \circ F_j - \phi_i \circ F_i)\|_{s+1} + \|X_j - X_i\|_{s+1}\right) + C\|X_j - X_i\|_{s+2}
\end{aligned}$$

For $\|\phi\|_{s+1}, \|\mu\|_{s+1}$ sufficiently small (that is, for C sufficiently small), we can absorb the last term on the right-hand side to obtain an *a priori* estimate on the sequence:

$$\begin{aligned}
\|X_j - X_i\|_{s+2} &\prec \|\mu_j - \mu_i\|_{s+1} + \|\left(\phi_j \circ F_j - \phi_i \circ F_i\right)\|_{s+1} \\
&\qquad\qquad + \|X_j - X_i\|_{s+1}\,; \\
\|\Psi(X_j) - \Psi(X_i)\|_{s+2} &\prec \|X_j - X_i\|_{s+2} \\
&\prec \|\mu_j - \mu_i\|_{s+1} + \|\left(\phi_j \circ F_j - \phi_i \circ F_i\right)\|_{s+1} \\
&\qquad\qquad + \|X_j - X_i\|_{s+1}\,; \\
\|Y_j - Y_i\|_{s+2} &\prec \|\mu_j - \mu_i\|_{s+1} + \|\left(\phi_j \circ F_j - \phi_i \circ F_i\right)\|_{s+1} \\
&\qquad\qquad + \|X_j - X_i\|_{s+1}\,.
\end{aligned}$$

Using the facts that $\phi_j \circ F_j$, μ_j, and X_j are Cauchy in Γ^{s+1}, we have that X_j and Y_j are Cauchy in Γ^{s+2} and that $X_j \to X$ and $Y_j \to Y$ in Γ^{s+2}.

Bootstrapping one more time, using the facts that $\phi_j \circ F_j$, μ_j, and X_j are Cauchy in Γ^{s+2}, we have that X_j and Y_j are Cauchy in Γ^{s+3} and that $X_j \to X$ and $Y_j \to Y$ in Γ^{s+3}. This establishes continuity of the map in Theorem 6.1.5 and completes the proof of the theorem. q.e.d.

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