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AXIAL MINIMAL SURFACES IN $\mathbf{S}^2 \times \mathbf{R}$ ARE HELICOIDAL

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Abstract

We prove that if a complete, properly embedded, finite-topology minimal surface in $\mathbf{S}^2 \times \mathbf{R}$ contains a line, then its ends are asymptotic to helicoids, and that if the surface is an annulus, it must be a helicoid.

1. Introduction

There is a rich theory of complete properly embedded minimal surfaces of finite topology in $\mathbf{R}^3$. In particular, we now have a good understanding of the ends of such surfaces: aside from the plane, every such surface either has one end, in which case it is asymptotic to a helicoid [**BB08**], or it has more than one end, in which case each end is asymptotic to a plane or to a catenoid [**Col97**], [**HM89**], [**MR93**]. For the rest of this introduction, let us use "minimal surface" to mean "complete, properly embedded minimal surface with finite topology". (Colding and Minicozzi [**CM08**] have proved that every complete embedded minimal surface with finite topology in $\mathbf{R}^3$ is properly embedded, so the assumption of properness is not necessary.)

It is interesting to try to classify the ends of minimal surfaces in homogeneous 3-manifolds other than $\mathbf{R}^3$. This paper deals with the ambient manifold $\mathbf{S}^2 \times \mathbf{R}$. (The fundamental paper on minimal surfaces in $\mathbf{S}^2 \times \mathbf{R}$ is Rosenberg [**Ros02**]. The survey [**Ros03**] is a good introduction to this paper as well as to the papers of [**MR05**], [**Hau06**], and [**PR99**] mentioned below.) In that case, the only compact minimal surfaces are horizontal 2-spheres. Any noncompact example has exactly two ends, both annular, one going up and one going down. Therefore the genus-zero, noncompact minimal surfaces in $\mathbf{S}^2 \times \mathbf{R}$ are all annuli. The minimal annuli that are foliated by horizontal circles have been classified by Hauswirth [**Hau06**]. They form a two-parameter family that contains on its boundary the helicoids (defined in Section 1.2) and the unduloids constructed by Pedrosa and Ritore [**PR99**]. There are no other known minimal annuli.

These facts suggest the following two questions posed by Rosenberg:

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1) Is every minimal annulus in $\mathbf{S}^2 \times \mathbf{R}$ one of the known examples? That is, is every minimal annulus fibered by horizontal circles?
2) If so, must each end of any minimal surface in $\mathbf{S}^2 \times \mathbf{R}$ be asymptotic to one of the known minimal annuli?

In this paper, we show that the answer to both questions is "yes" in case the surface is an *axial surface*, i.e., in case the surface contains a vertical line. In particular, the axial minimal annuli in $\mathbf{S}^2 \times \mathbf{R}$ are precisely the helicoids.

The assumption that the surface contains a line is a very strong one, but there are many minimal surfaces that have that property. Indeed, in [**HW08**] we outlined the proof of the existence of axial examples of every genus g and every vertical flux. (See equation (5) in Remark 3.1 for a definition of vertical flux. The forthcoming [**HW11**] contains the full existence proof.) By Theorem 1.3 below, those examples are all asymptotic to helicoids, so we call them "genus-g helicoids".

Combining these results with Theorem 1.3, we have:

1.1. Theorem. *For every helicoid H of finite pitch in $\mathbf{S}^2 \times \mathbf{R}$ and for every genus $g > 0$, there are at least two genus-g properly embedded, axial minimal surfaces whose ends are, after suitable rotations, asymptotic to H. The two surfaces are not congruent to each other by any orientation-preserving isometry of $\mathbf{S}^2 \times \mathbf{R}$. If g is even, they are not congruent to each other by any isometry of $\mathbf{S}^2 \times \mathbf{R}$.*

The totally geodesic cylinder $\mathbf{S}^1 \times \mathbf{R}$ may be thought of as a helicoid of infinite pitch. In this case, the methods of [**HW11**, **HW08**] still produce two examples for each genus, but the proof that the two examples are not congruent breaks down. Earlier, by a different method, Rosenberg [**Ros02**] explicitly constructed, for each g, an axial, genus-g minimal surface asymptotic to a cylinder.

1.2. Helicoids. Let O and O^* be a pair of antipodal points in $\mathbf{S}^2 \times \{0\}$ and let Z and Z^* be vertical lines passing through those points. Let $\sigma_{\alpha,v}$ denote the screw motion of $\mathbf{S}^2 \times \mathbf{R}$ consisting of rotation through angle α about the axes Z and Z^* followed by vertical translation by v. A *helicoid* with axes Z and Z^* is a surface of the form

$$\bigcup_{t \in \mathbf{R}} \sigma_{\kappa t, t} X \tag{1}$$

where X is a horizontal great circle that intersects Z and Z^*. The *pitch* of the helicoid is $2\pi/\kappa$, whose absolute value equals twice the vertical distance between successive sheets of the surface. Unlike the situation in $\mathbf{R}^3$, helicoids of different pitch do not differ by a homothety of $\mathbf{S}^2 \times \mathbf{R}$; there are no such homotheties. Note that a cylinder is a helicoid with infinite pitch ($\kappa = 0$), and that as $\kappa \to \infty$ the helicoids associated with κ converge to a minimal lamination of $\mathbf{S}^2 \times \mathbf{R}$ by level spheres with singular set of convergence equal to the axes $Z \cup Z^*$.

The main result of this paper is the following theorem:

1.3. Theorem. *Let M be a properly embedded, axial minimal surface in $\mathbf{S}^2 \times \mathbf{R}$ with bounded curvature and without boundary.*

(1) If E is an annular end of M, then E is asymptotic to a helicoid;
(2) If M is an annulus, then M is a helicoid;
(3) If M has finite topology, then each of its two ends is asymptotic to a helicoid, and the two helicoids are congruent to each other by a rotation.

1.4. Remarks. Meeks and Rosenberg [**MR05**] proved that a properly embedded minimal surface with finite topology in $\mathbf{S}^2 \times \mathbf{R}$ has bounded curvature. Thus our assumption that the surfaces we consider have bounded curvature is always satisfied.

In statement (1), it is not necessary that E be part of a complete surface without boundary. The statement is true (with essentially the same proof) for any properly embedded annulus $E \subset \mathbf{S}^2 \times [z_0, \infty)$ such that $\partial E \subset \partial \mathbf{S}^2 \times \{z_0\}$ and such that E contains a vertical ray.

We do not know whether the two helicoids referred to in statement (3) must be the same. See the discussion in Remark 3.2 below.

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2. A convexity lemma

2.1. Axial surfaces are symmetric and have two axes. Suppose that M is a properly embedded, axial minimal surface in $\mathbf{S}^2 \times \mathbf{R}$. Then M contains a vertical line Z. We claim that M must also contain the antipodal line Z^*, i.e., the line consisting of all points at distance πr from Z, where r is the radius of the $\mathbf{S}^2$. To see this, let $\rho_Z : \mathbf{S}^2 \times \mathbf{R} \to \mathbf{S}^2 \times \mathbf{R}$ denote rotation by π about Z. By Schwarz reflection, ρ_Z induces an orientation-reversing isometry of M. In particular, ρ_Z interchanges the two components of the complement of M. Thus no point in the complement of M is fixed by ρ_Z, so the fixed points of ρ_Z must all lie in M. The fixed point set of ρ_Z is precisely $Z \cup Z^*$, so Z^* must lie in M, as claimed.

2.2. The angle function θ. We will assume from now on that an axial surface in $\mathbf{S}^2 \times \mathbf{R}$ has axes Z and Z^* that pass through a fixed pair O and O^* of antipodal points in $\mathbf{S}^2 = \mathbf{S}^2 \times \{0\}$. Fix a stereographic projection from $(\mathbf{S}^2 \times \{0\}) \setminus \{O^*\}$ to $\mathbf{R}^2$, and let θ be the angle function on $(\mathbf{S}^2 \times \{0\}) \setminus \{O, O^*\}$ corresponding to the polar coordinate θ on $\mathbf{R}^2$. Extend θ to all of $(\mathbf{S}^2 \times \mathbf{R}) \setminus (Z \cup Z^*)$ by requiring that it be invariant under vertical translations. Of course, θ is only well-defined up to integer multiples of 2π.

If H is a helicoid with axes Z and Z^*, we will call the components of $H \setminus (Z \cup Z^*)$ *half-helicoids*. From (1) it follows that the half-helicoids are precisely the surfaces given by

$$\theta = \kappa z + b. \tag{2}$$

Here $2\pi/\kappa$ is the pitch of the helicoid. Note that at each level z, specifying $\theta = c$ specifies a great semicircle connecting Z to Z^*. Thus the half helicoid specified by $\theta = \kappa z + b$ is foliated by great semicircles turning at a constant rate. Rotating H by an angle γ changes the corresponding b to $b + \gamma$.

Note that the entire helicoid H consists of $Z \cup Z^*$ (where θ is not defined) together with all points not in $Z \cup Z^*$ such that

$$\theta \cong \kappa z + b \pmod{\pi}.$$

2.2. The restriction of θ to an annular slice. Let $I \subset \mathbf{R}$ be a closed interval (possibly infinite) and let

$$E = M \cap (\mathbf{S}^2 \times I)$$

be the portion of M in $\mathbf{S}^2 \times I$. Suppose that E is an annulus. Then $E \setminus (Z \cup Z^*)$ consists of two simply connected domains that are congruent by the involution ρ_Z. Denote by D one of these domains, and consider the restriction of θ to D. Because D is simply connected, we may choose a single-valued branch of this function, and we will also refer to it as θ when there is no ambiguity. Since I is closed, $\overline{D} \setminus D \subset Z \cup Z^*$. Because $\overline{D}$ has a well-defined tangent halfplane at each point of $\overline{D} \setminus D$, the function θ extends continuously to all of $\overline{D}$.

At each level z, the angle function θ has a maximum and a minimum on the compact set $\overline{D} \cap (\mathbf{S}^2 \times \{z\})$. This allows us to make the following definition:

2.3. Definition.

$$\begin{aligned}
\alpha(z) &= \max\{\theta(p,z) \,:\, (p,z) \in \overline{D} \cap (\mathbf{S}^2 \times \{z\})\}, \\
\beta(z) &= \min\{\theta(p,z) \,:\, (p,z) \in \overline{D} \cap (\mathbf{S}^2 \times \{z\})\}, \\
\phi(z) &= \alpha(z) - \beta(z).
\end{aligned}$$

Note that $\alpha(z) = \beta(z)$ if and only if $D \cap (\mathbf{S}^2 \times \{z\})$ is half of a great circle. Note also that E is a portion of a helicoid if and only if

$$\alpha(z) \equiv \beta(z) \equiv \kappa z + b$$

for some κ and b.

2.4. Lemma. *The functions α, $-\beta$, and $\phi = \alpha - \beta$ are convex, and they are strictly convex unless they are linear (in which case E is contained in a helicoid and $\phi \equiv 0$).*

Proof. Suppose that α is not strictly convex. Then there exists $z_1 < z_2$ and $0 < \lambda < 1$ such that

$$\alpha(z_1 + \lambda(z_2 - z_1)) \geq \alpha(z_1) + \kappa(z_1 + \lambda(z_2 - z_1)), \tag{3}$$

where

$$\kappa = \frac{\alpha(z_2) - \alpha(z_1)}{z_2 - z_1}$$

is the slope of the line segment connecting $(z_1, \alpha(z_1))$ and $(z_2, \alpha(z_2))$. It follows that there is a line of the form

$$y = \kappa z + \tilde{b}$$

that lies above the graph of α between z_1 and z_2 and touches that graph at a point $(\tilde{z}, \alpha(\tilde{z}))$ with $z_1 < \tilde{z} < z_2$. By the definition of α, there is a point $(\tilde{p}, \tilde{z})$ in the compact set $\overline{D} \cap (\mathbf{S}^2 \times \{\tilde{z}\})$ with

$$\theta(\tilde{p}, \tilde{z}) = \alpha(\tilde{z}).$$

Then in a neighborhood of $(\tilde{p}, \tilde{z})$, the surface D lies on one side of the half-helicoid H given by $\theta = \kappa z + \tilde{b}$ as in (2), and the two surfaces touch at $(\tilde{p}, \tilde{z})$. By the maximum principle (or the boundary maximum principle if $(\tilde{p}, \tilde{z})$ is a boundary point of D), a neighborhood in $\overline{D}$ of $(\tilde{p}, \tilde{z})$ lies in H. By analytic continuation, all of D lies in H. Thus $\theta = \kappa z + \tilde{b}$ on D, so $\alpha(z) \equiv \beta(z) \equiv \kappa z + \tilde{b}$ and $\phi(z) \equiv 0$.

The statements about convexity and strict convexity of $-\beta$ (or, equivalently, about concavity and strict concavity of β) are proved in exactly the same way. The assertions about $\alpha - \beta$ follow, since the sum of two convex functions is convex, and the sum is strictly convex if either summand is strictly convex. q.e.d.

3. The proof of Theorem 1.3

Consider first an annular end E. We may suppose that E is properly embedded in $\mathbf{S}^2 \times [a, \infty)$. Let

$$c := \limsup_{z \to \infty} \phi(z)$$

where $\phi = \alpha - \beta$ is as in Definition 2.3. Choose $z_n \to \infty$ such that

$$\lim_{n \to \infty} \phi(z_n) = c.$$

(A priori, c might be infinite, but we will show below that it is equal to zero.)

Let E_n and D_n be the result of translating E and D downward by z_n. Since we are assuming that the curvature of E is bounded, we may assume by passing to a subsequence that the E_n converge smoothly to a properly embedded minimal annulus $\hat{E}$ (see [**MR05**]), and that the D_n converge smoothly to $\hat{D}$, one of the connected components of

$\hat{E} \setminus (Z \cup Z^*)$. The smooth convergence $D_n \to \hat{D}$ implies that the angle-difference functions

$$\phi_n(z) := \phi(z + z_n)$$

for the disks D_n converge smoothly to the angle-difference function $\hat{\phi}$ for the disk $\hat{D}$. Note that because $\hat{\phi}$ is the angle-difference function for the simply connected surface $\hat{D}$, it has a well defined and finite value for all $z \in \mathbf{R}$. In particular, its value at $z = 0$ is finite. Also,

$$\hat{\phi}(z) = \lim_{n\to\infty} \phi_n(z) = \lim_{n\to\infty} \phi(z + z_n) \leq c$$

by the definition of c, with equality if $z = 0$. Thus $\hat{\phi}(z)$ attains its maximum value of c at $z = 0$. Consequently, $\hat{\phi}$ is not strictly convex, so by Lemma 2.4, $\hat{D}$ is contained in helicoid, and therefore $\hat{\phi} \equiv 0$. In particular, $c = 0$.

Returning our attention to the original surface D, we will now prove statement (1) of the theorem. Since $\alpha \geq \beta$, the sets

$$\{(z, y) : y > \alpha(z), z > a\},$$
$$\{(z, y) : y < \beta(z), z > a\}$$

are disjoint, convex subsets of $\mathbf{R}^2$, so they are separated by a line $y = az + d$. Thus

$$\beta(z) \leq az + d \leq \alpha(z).$$

Since $\alpha(z) - \beta(z) \to 0$ as $z \to \infty$, the graphs of α and of β are asymptotic to the line $y = az + d$. Thus D is C^0-asymptotic to the half-helicoid whose equation is $\theta = az + d$. Since the curvature of D is bounded, the surface D is smoothly asymptotic to that half-helicoid. It follows immediately that the end

$$E = \overline{D} \cup \rho_Z \overline{D}$$

is asymptotic to the corresponding helicoid. This proves statement (1) of the theorem.

To prove statement (2), suppose that M is a properly embedded, axial minimal annulus. Let D be one of the simply connected components of $M \setminus (Z \cup Z^*)$. We know from Lemma 2.4 that ϕ is convex on all of $\mathbf{R}$, and from the proof above of the first statement of Theorem 1.3 (applied to the ends of M) that

$$\lim_{z\to\pm\infty} \phi(z) = 0.$$

Thus $\phi(z) \equiv 0$, so by Lemma 2.4, M is a helicoid.

Statement (3) of the theorem follows from a standard flux argument as follows. Let $s < t$ and let

$$\Sigma = \Sigma(s, t) = M \cap (\mathbf{S}^2 \times (s, t)).$$

Let $\nu(p)$ be the outward unit co-normal at $p \in \partial\Sigma \subset M$ (a vector field along $\partial\Sigma$ that is tangent to M). Since $\partial/\partial\theta$ is a Killing field on $\mathbf{S}^2 \times \mathbf{R}$,

$$\int_{\partial\Sigma} \left(\nu \cdot \frac{\partial}{\partial\theta}\right) ds = 0$$

by the first variation formula. Equivalently, if we let $M_a = M \cap \{z \leq a\}$, then the flux

$$\int_{\partial M_a} \left(\nu \cdot \frac{\partial}{\partial\theta}\right) ds \tag{4}$$

is independent of a. We call (4) the *rotational flux* of M (with respect to the axes Z and Z^*).

If M is asymptotic (as $z \to \infty$ or as $z \to -\infty$) to a helicoid H, then M and H clearly have the same rotational flux. Thus to prove statement (3) of the theorem, it suffices to check that if two helicoids with axes $Z \cup Z^*$ have the same rotational flux, then they are congruent by rotation. If we let $F(\kappa)$ denote the rotational flux of the helicoid $H(\kappa)$ given by

$$\theta \cong \kappa\, z \ (\mathrm{mod}\ \pi),$$

then it suffices to show that $F(\kappa)$ depends strictly monotonically on κ. To see it does, consider expression (4) for $F(\kappa)$:

$$F(\kappa) = \int_{\partial(H(\kappa)\cap\{z\leq 0\})} \left(\nu \cdot \frac{\partial}{\partial\theta}\right) ds.$$

Note that the domain $\partial(H(\kappa) \cap \{z \leq 0\})$ is a fixed great circle C, and that for each point in $C \setminus \{O, O^*\}$, the integrand is a strictly increasing function of κ (because the larger κ is, the smaller the angle between the vectors $\partial/\partial\theta$ and ν). Thus $F(\kappa)$ is a strictly increasing function of κ. q.e.d.

3.1. Remark. The reader may wonder why we used rotational flux rather than the vertical flux

$$\int_{\partial M_a} \left(\nu \cdot \frac{\partial}{\partial z}\right) ds. \tag{5}$$

The problem with vertical flux is that the helicoid $H(\kappa)$ and its mirror image $H(-\kappa)$ have the same vertical flux. Thus vertical flux alone does not rule out the possibility that the two ends of M might be asymptotic to helicoids that are mirror images of each other.

3.2. Remark. We have not proved that the constant terms b in the equations

$$\theta \cong \kappa\, z + b \quad (\mathrm{mod}\ \pi)$$

for the helicoids asymptotic to the ends of M are the same. There is some reason to expect that b can change from end to end.

A change in b corresponds to a rotation, and when $\kappa \neq 0$ (i.e., when the helicoid is not a cylinder), a rotation by β is equivalent to a vertical translation by β/κ. In the Introduction, we discussed known examples of properly embedded axial minimal surfaces of finite genus. Those examples may be regarded as desingularizing the intersection of a helicoid H with the totally geodesic sphere $\mathbf{S}^2 \times \{0\}$. Such desingularization might well cause a slight vertical separation of the top and bottom ends of the helicoid, in order to "make room" for the sphere. A similar situation exists in $\mathbf{R}^3$ when considering the Costa-Hoffman-Meeks surfaces as desingularizations of the intersection of a vertical catenoid with a horizontal plane passing through the waist of the helicoid [**HM90**], [**HK97**]. While the top and bottom catenoidal ends have the same logarithmic growth rate, corresponding to the vertical flux, numerical evidence from computer simulation of these surfaces indicates a vertical separation of those ends. (In other words, the top end is asymptotic to the top of a catenoid, the bottom end is asymptotic to the bottom of a catenoid, and numerical evidence indicates that the two catenoids are related by a nonzero vertical translation.)

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