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NON-ALGEBRAIC HYPERKÄHLER MANIFOLDS

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Abstract

We study the algebraic dimension $a(X)$ of a compact hyperkähler manifold of dimension $2n$. We show that $a(X)$ is at most n unless X is projective. If a compact Kähler manifold with algebraic dimension 0 and Kodaira dimension 0 has a minimal model, then only the values 0, n and $2n$ are possible. In case of middle dimension, the algebraic reduction is holomorphic Lagrangian. If $n = 2$, then - *without any assumptions* - the algebraic dimension only takes the values 0, 2 and 4. The paper also gives structure results for "generalised hyperkähler" manifolds and studies nef lines bundles.

Contents

1. Introduction

In this paper, we are mainly interested in non-projective hyperkähler manifolds, especially their algebraic dimensions and algebraic reductions.

Let X be a compact Kähler manifold. The famous criterion of Kodaira [**Ko54**], Theorem 4, states that X is projective if and only if X admits an integral Kähler class. That is, X is projective if and only if

$$\mathcal{K}(X) \cap H^2(X, \mathbb{Q}) \neq \{0\}$$

in $H^2(X, \mathbb{R})$. Here $\mathcal{K}(X)$ is the Kähler cone of X, i.e., the cone consisting of the Kähler classes of X. As is well known, $\mathcal{K}(X)$ is an open convex

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cone in $H^{1,1}(X,\mathbb{R})$ (see e.g. [**GHJ03**], Page 84, Proposition 14.14). In particular, X is projective if $H^0(X,\Omega_X^2)=0$. In fact, if $H^0(X,\Omega_X^2)=0$, then $H^{1,1}(X,\mathbb{R})=H^2(X,\mathbb{R})$. Thus $\mathcal{K}(X)$ is an open convex cone of the *whole* $H^2(X,\mathbb{R})$, whence, meets a dense subset $H^2(X,\mathbb{Q})\setminus\{0\}$ of $H^2(X,\mathbb{R})$; one can then apply Kodaira's criterion above. So, if X is *not* projective, then X necessarily admits a non-zero global holomorphic 2-form, i.e., $H^0(X,\Omega_X^2)\neq 0$. In some sense, irreducible holomorphic symplectic manifolds (hyperkähler manifolds, for short) form the simplest class of manifolds having $H^0(X,\Omega_X^2)\neq 0$.

By definition, a *hyperkähler manifold* is a compact simply connected Kähler manifold X admitting a holomorphic $2-$form σ_X which is of maximal rank at every point such that $H^0(X,\Omega_X^2)=\mathbb{C}\sigma_X$. Note that dim X is then even, say $2n$, and $\wedge^n\sigma_X$ is a $2n-$form without zeroes. Though $H^0(X,\Omega_X^2)\neq 0$ is just a necessary condition for X to be non-projective, it is shown by Fujiki [**Fu83-2**], Theorem 4.8 (2) (see also [**Ca83**], Page 413, Théorème) that both projective and non-projective hyperkähler manifolds are dense in the Kuranishi space of X. This is based on Bogomolov's unobstructedness theorem [**Bo78**] (Theorem 1 and Corollary in Page 1464). We also note that a hyperkähler manifold is never rigid, as $H^1(X,T_X)\simeq H^1(X,\Omega_X^1)\neq 0$ by σ_X and the Kähler condition. So in the study of hyperkähler manifolds, it is natural and important to study not only projective ones but also non-projective ones and their interactions.

Let X be a *non-projective* compact Kähler manifold. The most basic numerical invariant of X is the algebraic dimension. The set $\mathbb{C}(X)$ of global meromorphic functions of X naturally form a field. It is a general fact, originally due to Siegel ([**Si55**], Satz 1, 2) that $\mathbb{C}(X)$ is a finitely generated field over $\mathbb{C}$, the field consisting of constant functions (see e.g. [**Ue75**], Page 24). The *algebraic dimension* $a(X)$ of X is the transcendental degree of $\mathbb{C}(X)$ over $\mathbb{C}$.

The roles of $\mathbb{C}(X)$ and $a(X)$ are, in many aspects, similar to the roles of the pluri-canonical ring and the Kodaira dimension in birational geometry of projective manifolds. As one often studies the pluri-canonical ring geometrically through the pluri-canonical map, one can also study the field $\mathbb{C}(X)$ and the algebraic dimension $a(X)$ more geometrically via the algebraic reduction. By definition, "the" *algebraic reduction*

$$f : X \dashrightarrow B$$

of X is a meromorphic map from X to a normal projective variety B such that

$$f^*(\mathbb{C}(B)) = \mathbb{C}(X).$$

We can naturally define the image and fibers of f through a resolution of indeterminacy of f. Then $f(X)=B$ and the fibers of f are connected. Note that $\mathbb{C}(B)$ is the same as the rational function field of

B, as B is *projective*. Thus, $a(X) = \dim B$ and $0 \le a(X) \le \dim X$. There are infinitely many ways to choose the base space B. However they are birational as their rational function fields are all isomorphic to $\mathbb{C}(X)$, whence "the" algebraic reduction f is unique up to birational modification of B (See [**Ue75**], section 3 in chapter I for more detail). When $a(X) = \dim X$, the algebraic reduction is bimeromorphic (and vice versa). In this case X is called *algebraic* or *Moishezon*. A famous theorem of Moishezon ([**Mo66**], Theorem 11) says that a compact *Kähler manifold* X is algebraic if and only if X is projective. In particular, non-projective and non-algebraic have the same meaning for hyperkähler manifolds.

In dimension 2, the algebraic reduction can always be taken as a *holomorphic* map from the *original* X (see e.g. [**Ue75**], Page 249). However, in dimension ≥ 3, this is no longer true, and makes the study of algebraic reduction for a higher dimensional manifold more difficult. Note that, as B is projective, the algebraic reduction is always given by a linear subsystem associated with a line bundle on X. Thus, to ask "*if one can take a holomorphic algebraic reduction from a hyperkähler manifold X or not*" is essentially the same as to ask the following:

Conjecture 1.1. *Let X be a non-projective hyperkähler manifold and L be a nef line bundle on X. Then L is semi-ample, i.e., the complete linear system $|mL|$ is base point free for some $m > 0$.*

Conjecture 1.1 is quite similar to one of the major problems in the minimal model theory; *if the pluri-canonical system $|mK_M|$ of a minimal model M is base point free for some $m > 0$* (abundance type problem). Even in dimension 4, this problem is very difficult and completely open both in non-projective hyperkähler manifolds and in minimal models. On the other hand, if Conjecture 1.1 holds true, then the following is an immediate consequence of Matsushita's result ([**Ma99**], Theorem 2 and Theorem 1 in Addendum; see also Proposition 3.3 in section 3):

Conjecture 1.2. *Let X be a non-projective hyperkähler manifold of dimension $2n$. Then its algebraic dimension takes only the values $0, n$. Moreover, if $a(X) = n$, then the algebraic reduction has a holomorphic model $f : X \longrightarrow B$ with B a normal projective variety of dimension n. Finally f is Lagrangian, that is $\sigma_X|F \equiv 0$ for a general fiber of f.*

Among others (see e.g. [**Og07**], Conjecture in section 3), Todorov ([**To03**], Conjectures 41 and 42) poses these two conjectures in connection with mirror-symmetry, especially the Strominger-Yau-Zaslow conjecture for hyperkähler manifolds. It would be really exciting if one could solve Conjectures 1.1 or 1.2 by using an idea from mirror-symmetry.

In this paper we give some significant results supporting these conjectures. Our main results are the following:

Theorem 1.3. *Conjecture 1.2 holds in dimension 4.*

We first reduce, via deformation, our proof of Theorem 1.3 to [**AC05**], Théorème 3.6, where "a version" of Conjecture 1.1 is solved for *projective* hyperkähler 4−folds (see ibid for the precise statement). As we explained above, algebraic reduction and pluricanonical map have some similarities. So, it might be worth noticing here that our proof is finally reduced to a certain universal estimate of the pluri-canonical system of a projective threefold ([**VZ07**], Corollary 0.4).

In higher dimensions, we can also solve Conjecture 1.2 up to the existence of minimal models of certain Kähler spaces:

Theorem 1.4. *Let X be a hyperkähler manifold of dimension $2n$. Then Conjecture 1.2 holds, provided that any compact Kähler manifold Y with $\dim Y \le 2n-1$, $a(Y) = \kappa(Y) = 0$ has a minimal model.*

A general conjecture from minimal model theory says that every compact Kähler manifold of non-negative Kodaira dimension should have a minimal model.

For Theorems 1.3 and 1.4, the following a bit more technical result will be used:

Theorem 1.5. *Let X be a hyperkähler manifold of dimension $2n$. Then:*

1) *If the Néron-Severi group $NS(X)$ is elliptic, i.e., negative definite with respect to Beauville-Bogomolov-Fujiki's form, then $a(X) = 0$.*
2) *If $NS(X)$ is parabolic, i.e., negative semi-definite but not negative definite with respect to Beauville-Bogomolov-Fujiki's form, then*
$$0 \le a(X) \le n = \dim X/2\,.$$
3) *Assume that $NS(X)$ is parabolic and $a(X) > 0$. Then one can choose an algebraic reduction of one of the following two forms:*
   (i) $f : X \longrightarrow B$ is holomorphic Lagrangian, in particular, $a(X) = n$, or
   (ii) $f : X \dashrightarrow B$ is not almost holomorphic and the general fiber X_b ($b \in B$) is isotypically semi-simple, in particular, $a(X_b) = 0$ (see section 2 for the definition of the term "isotypically semisimple").
4) *Assume that any compact isotypically semi-simple Kähler manifold Y of $\dim Y \le 2n-1$, of algebraic dimension $a(Y) = 0$ and of Kodaira dimension $\kappa(Y) = 0$ and with effective canonical divisor K_Y, has a minimal model. Then Conjecture 1.2 holds.*

Here we recall that a meromorphic map between compact varieties is *almost holomorphic* if it is proper holomorphic over some Zariski dense open subset of the base space. The second cohomology group $H^2(X, \mathbb{Z})$

of a hyperkähler manifold admits a natural *miraculous*, integral symmetric bilinear form, called *Beauville-Bogomolov-Fujiki form* ([**Be83**], Théorème 5, see also [**GHJ03**], Definition 22.8, Corollary 23.11, Proposition 23.14)

$$q_X : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}.$$

This naturally induces a bilinear form on the Néron-Severi group $NS(X)$. According to the signature, $NS(X)$ falls into the three cases, *elliptic, parabolic* (the cases in Theorem 1.5) and *hyperbolic*, which is the case where $NS(X)$ has an element H such that $(H^2) > 0$ with respect to the Beauville-Bogomolov-Fujiki form. However, $NS(X)$ is never hyperbolic if X is non-projective, by the following deep result due to Huybrechts ([**Hu99**], Theorem 2 in Erratum, see also [**GHJ03**], Proposition 26.13):

Theorem 1.6. *$NS(X)$ is hyperbolic if and only if X is projective.*

It is worth noticing that, contrary to our proof of Theorem 1.3, Theorem 1.6 was proved by deforming X to "highly" non-algebraic hyperkähler manifolds (see ibid).

According to the Beauville-Bogomolov-Kobayashi decomposition theorem ([**Be83**], Théorème 1), hyperkähler manifolds, Calabi-Yau manifolds (in the strict sense) and complex tori form the building blocks of compact Kähler manifolds with vanishing first Chern class. In sharp contrast, Calabi-Yau manifolds X of dimension $n \geq 3$ are projective, as $H^0(X, \Omega_X^2) = 0$, hence always $a(X) = \dim X$, and for complex tori X of dimension $n \geq 2$, the algebraic dimension takes all values between 0 and n ([**We58**], Page 139, Proposition 10).

Parts (1) and (2) of Theorem 1.5 will be proved in section 3; parts (3) and (4) in section 4 and Theorem 1.3 finally in section 5. Theorem 1.4 is a special case of Theorem 1.5. All these sections make essential use of section 2, which contains structure results on meromorphic fibrations on compact Kähler manifolds, in particular, on those manifolds admitting a unique holomorphic 2-form which additionally is generically non-degenerate (Theorem 2.4 and Corollary 2.5). The final section gives some results on nef line bundles on hyperkähler manifolds. Theorem 6.1 there is closely related to Conjecture 1.1. In fact, we show that a non-trivial line bundle with a smooth metric of semi-positive curvature (a stronger assumption than nefness) has Kodaira dimension $\kappa(L) = a(X)$. In particular, $\kappa(L) \geq 0$, so some multiple of L must have a section.

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2. Fibrations on generalized hyperkähler manifolds

In this section we prove some general structure theorems on generalised hyperkähler manifolds.

Conventions and Notations 2.1. (1) By $X, X', X'', \ldots$, we denote n-dimensional compact irreducible complex spaces which are bimeromorphic to compact Kähler manifolds. The algebraic dimension ([**Ue75**], Page 24) is denoted by $a(X)$. We have the algebraic reduction (only defined up to obvious bimeromorphic equivalence, see [**Ue75**], Page 25)

$$f : X \dashrightarrow B\,.$$

We always take B to be normal projective and often we will choose B smooth.

(2) A *fibration* $f : X \dashrightarrow B$ is a dominant meromorphic map with connected fibers. A fiber of f is a *closed* analytic subset of X defined naturally through a resolution of indeterminacy of f. The fibration f is said to be *almost holomorphic* if f is proper holomorphic over some Zariski dense open subset of B, or equivalently, generic fibers (see (3) below) do not meet the indeterminacy locus of f. If B is not uniruled, then any fibration $f : X \dashrightarrow B$ is automatically almost holomorphic. Indeed, as each irreducible component E of the exceptional divisor of a resolution of indeterminacy of f is uniruled, E can not dominate B if B is not uniruled.
The fibration f is said to be *trivial* if $\dim B = 0$ or $\dim B = n$.

(3) A point b in B is said to be *generic* or *general* if it lies outside of a countable union of (suitable) proper closed analytic subsets of B. We denote by X_b the fiber of f over a generic $b \in B$.

(4) Recall ([**Fu82**], section 2, the third Definition, or [**Fu83**], Page 237) that a compact Kähler manifold X is said to be *simple* if X is not covered by positive-dimensional irreducible compact proper analytic subsets. By definition, if X is simple and dim $X \geq 2$, then necessarily $a(X) = 0$.

(5) Two complex spaces X and X' are *commensurable* if there exist a complex space X'' and generically finite surjective holomorphic maps $X'' \to X$ and $X'' \to X'$. This is easily seen to be an equivalence relation.

Notice that two *projective* varieties are commensurable if and only if they have the same dimension. But for non-algebraic X, this equivalence relation is very restrictive. For instance, if X is simple and if X' is commensurable to X, then X' is simple, too.

(6) We say (after [**Fu82**], section 2, the forth Definition) that X is *semi-simple* if it is commensurable to the product of simple manifolds, and that X is *isotypically semi-simple* if it is commensurable to the product S^k for some simple S and some $k > 0$.

Definition 2.2. Let $f : X \dashrightarrow B$ be a fibration from a compact (connected) Kähler manifold X.
We say that $f = h \circ g$ is a factorisation of f if $g : X \dashrightarrow S$ and $h : S \dashrightarrow B$ are fibrations with $f = h \circ g$.
This factorisation is said to be trivial if $\dim S = \dim X$ or $\dim S = \dim B$. The fibration f is said to be *minimal* if any factorisation of f is trivial. The variety itself X is minimal if the constant fibration $X \to \{pt\}$ is minimal.

The following easy observation is essential:

Theorem 2.3. *Let X be a compact Kähler manifold of dimension $2n$ and suppose that $h^{2,0}(X) = 1$, and the corresponding holomorphic $2-$form σ (which is unique up to a scalar) satisfies $\sigma^n \neq 0$. Then the following assertions hold.*

1) *If $f : X \dashrightarrow Y$ is a fibration with $\dim Y < \dim X$, then Y is Moishezon.*
2) *The algebraic reduction $f : X \dashrightarrow B$ is minimal.*
3) *If $a(X) = 0$, then X is minimal.*
4) *In particular, hyperkähler manifolds X with $a(X) = 0$ are minimal.*

Proof. Only (1) needs to be proved; the other statements are trivial consequences of (1). We show (1) by argue by contradiction. Suppose that $f : X \dashrightarrow Y$ would be a nontrivial fibration over *non-algebraic* Y. Then by a resolution of singularities and a resolution of indeterminacy, we may assume without loss of generality, that Y is smooth and f is holomorphic, i.e., $f : X \longrightarrow Y$ is a surjective holomorphic map from a compact Kähler manifold X to a compact non-algebraic manifold Y. By definition, Y is then of class $\mathcal{C}$ in the sense of Fujiki (see, e.g., [**Fu83**], p.235). Thus Y is bimeromorphic to a compact Kähler manifold by [**Va89**], p. 51, Theorem 5. Hence, again by a resolution of indeterminacy (twice), we may assume without loss of generality, that Y is Kähler and f is holomorphic. As Y is now Kähler but non-algebraic, it follows that $h^{2,0}(Y) > 0$ by [**Ko54**], Theorem 4 (see also the second paragraph of the introduction). Any non-zero holomorphic $2-$form on Y lifts to a non-zero holomorphic $2-$form on X by f. On the other hand, by our assumption, any non-zero holomorphic 2−form on X is generically of maximal rank. Thus, we would have $\dim Y = \dim X$, a contraction to $\dim Y < \dim X$, the assumption made in (1). This completes the proof of (1). q.e.d.

The main result of this section is:

Theorem 2.4. *Let X be a non-projective, compact Kähler manifold, $f : X \dashrightarrow B$ be a minimal fibration. Suppose $\dim X > \dim B$. Let X_b be a general fiber of f. Then*

1) *X_b is either Moishezon or isotypically semi-simple, in which case $a(X_b) = 0$.*
2) *Furthermore, if X_b are Moishezon, then f is almost holomorphic and X_b is an abelian variety.*

Theorem 2.4 will be proved at the end of this section.

By combining Theorem 2.3 (2) and Theorem 2.4, we get:

Corollary 2.5. *Let X be a non-projective, compact Kähler manifold of dimension $2n$ with $h^{2,0}(X) = 1$, carrying a holomorphic two-form σ such that $\sigma^n \neq 0$. Assume X is non-projective. Then the following assertions hold.*

1) *Let $f : X \dashrightarrow B$ be the algebraic reduction and X_b be a general fiber of f. Then either*
   2a. X_b is isotypically semi-simple or
   2b. f is almost holomorphic, and X_b is an abelian variety.
2) *In particular, if $a(X) = 0$, then X is isotypically semi-simple.*

We can obtain the following interesting result originally due to Fujiki ([**Fu87**], Proposition 5.16) also as an application of Theorem 2.4:

Corollary 2.6. *Let X be a simply connected compact Kähler manifold of dimension $2n$ with $h^{2,0}(X) = 1$, carrying a holomorphic two-form σ such that $\sigma^n \neq 0$. Assume moreover that X does not contain any effective divisor. Then X is simple.*

In particular, any hyperkähler manifold without effective divisors is simple and so does the generic member of the Kuransihi family of (any) hyperkähler manifold.

Proof. Since X has no effective divisors, we have $a(X) = 0$. Thus X is isotypically semi-simple by Theorem 2.4. Specifically there exist generically finite meromorphic maps $u : Z \to X$ and $v : Z \to S^k$, with Z smooth, and S simple. Our claim comes down to prove that u is bimeromorphic and that $k = 1$. Since X has no divisor, u is unramified, hence bimeromorphic, X being simply connected. Thus $h^{2,0}(X) = h^{2,0}(Z) = 1$. Since S is non-algebraic, for the same reason as in the proof of Theorem 2.3 (1), one has $h^{2,0}(S) \geq 1$, hence necessarily

$k = 1$. For the last statement, recall that generic hyperkähler manifolds in the Kuranishi family are of Picard number 0 ([**Ca83**], Page 413, Théorème, see also [**Og03**], Corollary 1.3). In particular, they are of algebraic dimension zero and do not contain any effective divisors.

q.e.d.

The assumption that X does not contain any divisors cannot be removed in (2.6). In fact, the hyperkähler 4-fold $S^{[2]}$, with S a K3 surface, $a(S) = 0$, is not simple but $a(S^{[2]}) = 0$.

The rest of the section is devoted to the proof of Theorem 2.4. We shall need the following two elementary lemmas, which are relative versions of results similar to [**Fu82**] (section 2, Theorem 1 and its proof) in a simplified form. We first recall some notions needed in the proof.

A *covering family* of X will be a compact irreducible analytic subset $S \subset \mathcal{C}(X)$ of the Chow variety (or Barlet-space) $\mathcal{C}(X)$ of X, such that if $Z \subset S \times X$ is its incidence graph, with natural projections $p : Z \to X$ and $q : Z \to S$, then p is surjective, and the generic fiber of q is irreducible. In other words, X is covered by the generically irreducible cycles Z_s, $s \in S$. We call $m = \dim Z_s$ the dimension of the family S.
If $f : X \dashrightarrow B$ is a fibration, we denote by $\mathcal{C}(X/B)$ the closed analytic subset of $\mathcal{C}(X)$ consisting of those points $s \in \mathcal{C}(X)$ such that the corresponding analytic compact pure-dimensional cycle Z_s of X has support contained in one fiber X_b of f. If $S \subset \mathcal{C}(X/B)$ is a covering family of X, the map f_* sending s to $b = f(Z_s)$ is a meromorphic dominant map $f_* : S \dashrightarrow B$.

Lemma 2.7. *Let X be a compact Kähler manifold and $f : X \dashrightarrow B$ be any fibration with $a(X_b) = 0$. Let $Z \longrightarrow S \subset \mathcal{C}(X/B)$ be a nontrivial covering family of X over B. Assume that* $\dim Z_s = m$ *($s \in S$) is maximal among the dimensions of nontrivial covering families of X over B. Then*

1) $\dim Z = \dim X$. *In particular, only finitely many of the Z_s's pass through the generic point of X.*
2) *S_b is simple, and no proper closed analytic subset of S_b is a covering family of X_b.*
3) *S_b is the union of finitely many irreducible components of $\mathcal{C}(X_b)$.*

Proof. (1) By definition of Z, we have $\dim Z \geq \dim X$. We shall show $\dim Z = \dim X$ by argue by contradiction. Assume that $\dim Z > \dim X$. Then also $\dim Z_b > \dim X_b$ so that we may assume $\dim B = 0$. The fibers of $p : Z \to X$ are Moishezon by [**Ca80**], Page 8, Théorème. (Note that X and Z here are denoted by Z and X there.) Thus we can find a covering family of S by curves $(C_v)_{v \in V}$. For general $v \in V$ we define

$$W_v := p(q^{-1}(C_v)).$$

This is an irreducible compact analytic subset of X and defines a covering family $(W_v)_{v\in V}$ of X with $\dim W_v = m+1$. As $W_v \subset X$, we have $m+1 \leq \dim X$. If $m+1 < \dim X$, then this would contradicts the maximality of m, and we are done. If $m+1 = \dim X$, then the Z_s $(s \in S)$ form *family* of divisors in X. This contradicts $a(X) = 0$, and we are done. This completes the proof of (1).
(2) The same argument shows that S_b is simple. In fact, if S_b were not simple, then in every fiber we find a covering family of proper subvarieties and in total we obtain a nontrivial covering family $(C_v)_{v\in V}$ of S and define W_v as above. By the maximality of m, we must have $W_v = X$ for all $v \in V$. But then $(Z_s)_{s\in C_v}$ is a nontrivial covering family of X. Let $Z' := q^{-1}(C_v)$ be the graph of this covering family. Then $\dim Z' = \dim X = \dim Z$. Thus by irreducibility we obtain $Z' = Z$ and $C_v = S$, a contradiction.
(3) The third assertion is an obvious consequence of the second.

q.e.d.

Lemma 2.8. *Let $k \in \mathbb{N}$ and let S_j be simple manifolds for $1 \leq j \leq k$. Put*

$$S = S_1 \times \cdots \times S_k$$

with projections $p_j : S \to S_j$. More generally, for a given subset $J = \{j_1, \dots, j_h\} \subset \{1, 2, \dots, k\}$, let

$$p_J : S \to S_J = S_{j_1} \times \ldots \times S_{j_h}$$

be the projection. Let $Y \subset S$ be an irreducible compact analytic subset such that $p_j(Y) = S_j$ for all j.
There exists J such that $p_J : Y \to S_J$ is surjective and generically finite. In particular, Y is commensurable to S_J and is therefore semi-simple. In particular, if $S_j \simeq S_k$ for all j, k, then Y is isotypically semi-simple.

Proof. Let $K = \{1, \dots, k-1\}$. If $p_K(Y) \neq S_K$, we proceed by induction on k. Thus we may assume that $p_K(Y) = S_K$ for $K = \{1, \dots, k-1\}$. If $p_K : Y \to S_K$ is not generically finite, let S'_K be its Stein factorisation with map $p'_K : Y \to S'_K$, and define a meromorphic map

$$\varphi : S'_K \dashrightarrow \mathcal{C}(S_k)$$

by sending a general $s \in S'_K$ to $p_k({p'_K}^{-1}(s))$. The image of φ gives a covering family of S_k. Because S_k is simple, we must have $\varphi(s) = S_k$ for all s. Thus $Y = S$ (in which case we take $J = \{1, \dots, k\}$).

q.e.d.

Proof of Theorem 2.4. Let $a_f : X \dashrightarrow Y$, $h : Y \dashrightarrow B$ with $f = h \circ a_f$ be the relative algebraic reduction of f (see [**Ca81**], Corollaire 2 or [**Fu83**], Page 238, second Definition and Proposition 2.1, for definition and existence). By definition, for general $b \in B$, the induced map $X_b \dashrightarrow Y_b$ is the algebraic reduction of X_b. Since f is minimal, either

$Y = X$ up to bimeromorphic equivalence and X_b is Moishezon, or $Y = B$ up to bimeromorphic equivalence and $a(X_b) = 0$.

(1) First, we consider the case where $a(X_b) = 0$. We need to show that X_b is isotypically semi-simple. If X_b is simple, we are already done. If X_b is not simple, let $S \subset \mathcal{C}(X/B)$ be a nontrivial covering family of X with $m = \dim Z_s$ maximal. By Lemma 2.7, S_b is simple and $p : Z \to X$ is generically finite onto X. Let δ be the degree of p and

$$\varphi : X \dashrightarrow Sym^{\delta}(S/B)$$

be the meromorphic map sending a general $x \in X$ to $q_*(p^{-1}(x))$. Here $Sym^{\delta}(S/B)$ denotes the subspace of $Sym^{\delta}(S)$ consisting of δ-tuples of S contained in some fiber of S over B. We adopt a similar convention for (S^{δ}/B).

Since f is minimal, this map is generically finite onto its image $Y_0 \subset Sym^{\delta}(S/B)$. Let $Y \subset (S^{\delta}/B)$ be a main component of the inverse image of Y_0 under the natural map $(S^{\delta}/B) \to Sym^{\delta}(S/B)$. Then Y maps surjectively onto S under all projections from (S^{δ}/B) to S; otherwise there would exist some irreducible proper compact analytic subset $S' \subset S$ parametrising a covering family of X, contradicting Lemma 2.7. From Lemma 2.8 we conclude that Y, and hence so X, is commensurable to (S^k/B) for some $k \leq \delta$. The first assertion of Theorem 2.4 is thus established.

(2) Next, we consider the case where X_b is Moishezon. We need to show that f is almost holomorphic and X_b ($b \in B$ general) is an abelian variety. By the minimality assumption of Theorem 2.4 and Theorem 2.3 (1), it follows that f is the algebraic reduction of X.

First we show that f is almost holomorphic (by argue by contradiction). In fact, if otherwise, one of the exceptional divisors, say E, of a resolution of indeterminacy of f, dominates Y. Note that E is birational to the blow up of some subvariety, say C, of (a bimeromorphic modification of) X_b. By definition, any bimeromorphic modification of X_b is algebraic. Thus, C is algebraic as well (See [**Ue75**], Corollary 3.9). Thus E is also algebraic. In particular, any general two points of E can be joined by a finite chain of algebraic curves. The same is true for X_b ($b \in B$) being general, as X_b are algebraic. As E dominates Y, any general two points of X is then joined by a finite chain of algebraic curves. However, X would then be algebraic by [**Ca81**], Page 212, Corollaire, a contradiction to our assumption that X is not algebraic. Hence f is almost holomorphic as claimed.

Let us show that X_b is an abelian variety. By [**Fu83**], Theorem 1, X_b is either a complex torus, whence an abelian variety (as X_b is algebraic), or a unirational manifold.

We have to exclude the second case. If $\kappa(X) \geq 0$ - and this is sufficient for all our applications - X_b cannot be uniruled and we conclude. In general, if X_b is unirational, then [**Fu83**], Proposition 2.5 implies that X is projective, which is excluded again by our assumption. q.e.d.

We now consider the restriction of holomorphic 2-forms to fibers.

Corollary 2.9. *Let X be a compact Kähler manifold and $f : X \dashrightarrow B$ be a fibration. Assume that the restriction of any holomorphic 2-form on X to the fiber X_b for generic $b \in B$ vanishes. Then*

1) *X_b is Moishezon for general b.*
2) *If f is the algebraic reduction of X and a minimal fibration, then f is almost holomorphic, and X_b is an abelian variety.*

Proof. The first statement is a lemma due to C. Voisin (see [Ca06], Proposition 2.1). The second follows from Theorem 2.4. q.e.d.

3. Basics on hyperkähler manifolds and first results

We begin by fixing some notations. For the rest of the paper we consider a hyperkähler manifold X of dimension $2n$, that is, a simply connected compact Kähler manifold admitting a holomorphic $2-$form σ which is of maximal rank at every point (hence σ^{2n} is a $2n-$form without zeroes), such that $H^0(X, \Omega^2_X) = \mathbb{C}\sigma$.
The non-degenerate symmetric bilinear form, constructed by Beauville ([**Be83**], Théorème 5, see also [**GHJ03**], Definition 22.8, Corollary 23.11, Proposition 23.14) will be denoted by

$$q = q_X : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}.$$

This form q_X has signature $(3, 0, b_2(X) - 3)$, more precisely, it is positive definite on $\mathbb{R}\langle \mathrm{Re}\,\sigma, \mathrm{Im}\,\sigma \rangle$ and of signature $(1, 0, h^{1,1}(X) - 1)$ on $H^{1,1}(X, \mathbb{R})$, and $q(\eta) > 0$ for each Kähler class η. We shall use the shorthand $q(a) = q(a, a)$. Let

$$\mathcal{P}(X) \subset H^{1,1}(X, \mathbb{R})$$

be the positive cone of X, that is, the connected component of

$$\{x \in H^{1,1}(X, \mathbb{R}) \,|\, q(x) > 0\}$$

containing the Kähler classes. As q_X is of signature $(1, 0, h^{1,1}(X))$ on $H^{1,1}(X, \mathbb{R})$, the signature of q_X on the Néron-Severi group

$$NS(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$

is one of the following.

- $(1, 0, \rho - 1)$ (hyperbolic case);
- $(0, 1, \rho - 1)$ (parabolic case);
- $(0, 0, \rho)$ (elliptic case).

Here ρ is the Picard number of X, i.e., the rank of $NS(X)$.
As we mentioned in the introduction, Huybrechts shows that X is projective if and only if $NS(X)$ is hyperbolic ([**Hu99**], Erratum, Theorem 2). So we are interested in the parabolic case and the elliptic case.

Theorem 3.1. *Let X be a hyperkähler manifold.*

1) *If $NS(X)$ is elliptic, then $a(X) = 0$.*
2) *If $0 < a(X) < 2n$, then $NS(X)$ is parabolic. Let $\ell \in NS(X)$ be the unique primitive isotropic vector of $NS(X)$ with $q_X(\ell, \eta) > 0$ for a Kähler class η. Then there is a line bundle L whose linear system defines the algebraic reduction, such that $c_1(L) \in \mathbb{Z}_{>0}\ell$. In particular $q_X(L) = 0$.*

Proof. Let X be a non-projective hyperkähler manifold and $f : X \dashrightarrow B$ be the algebraic reduction with B normal projective. If dim $B > 0$, i.e., $a(X) > 0$, then there is a line bundle L on X and a linear subsystem $\Lambda \subset |L|$ such that $f = \Phi_\Lambda$ (the meromorphic map associated with Λ) and such that any two general $D_1, D_2 \in |L|$ have no common component. In fact, L is given by a "pull-back" of a very ample line bundle on B (cf. setup (3.2) below). By the explicit description of q_X (see [**Be83**], Théorème 5), one has, up to positive constant multiple:

$$q_X(L, L) = \int_X c_1(D_1)c_1(D_2)(\sigma \wedge \overline{\sigma})^{n-1} = \int_{D_1 \cap D_2} (\sigma \wedge \overline{\sigma})^{n-1} \geq 0\,.$$

Thus $NS(X)$ is not elliptic if $a(X) > 0$. This proves (1). As X is not projective by our assumption, $NS(X)$ is not hyperbolic. Thus $NS(X)$ has to be parabolic (if $0 < a(X)$). Thus, by the inequality above, $q_X(L) = 0$. Hence

$$c_1(L) \in -\overline{\mathcal{P}(X)} \cup (\overline{\mathcal{P}(X)} \setminus \{0\})\,.$$

Here $\overline{\mathcal{P}(X)}$ is the closure of the positive cone. As L is non-zero effective, this implies $c_1(L) \in \overline{\mathcal{P}(X)} \setminus \{0\}$ and $q_X(L, \eta) > 0$, simply by reasons of signature (or the Cauchy-Schwarz inequality). This proves (2).

q.e.d.

Setup 3.2. (1) We shall assume that $0 < a(X) < 2n$. Thus X is not projective and $NS(X)$ is parabolic. We consider "the" algebraic reduction

$$f : X \dashrightarrow B.$$

From the previous section (Corollary 2.5) we recall that the general fiber is isotypically semi-simple or that f is almost holomorphic and the general fiber is abelian.
We always take B to be normal *projective* and often we will choose B smooth, too. Let

$$\pi : \tilde{X} \to X$$

be a resolution of indeterminacy of f so that the induced map

$$\tilde{f} : \tilde{X} \to B$$

is holomorphic.

(2) We fix a very ample line bundle A on B and set

$$L = \pi_*(\tilde{f}^*(A))^{**}.$$

Then L is an invertible sheaf, i.e., a holomorphic line bundle on X. For $D \in |f^*A|$, we have $\pi_* D \in |L|$. Here $\pi_* D$ is the pushforward of D as codimension one cycle, which is necessarily a Cartier divisor on X (as X is smooth). Thus, we find an effective divisor E on $\tilde{X}$ such that

$$\pi^*(L) = \tilde{f}^*(A) + E.$$

We set $\tilde{L} = \pi^*(L)$.

(3) In all what follows η will always denote a Kähler form on X. We set $\tilde{\eta} = \pi^*(\eta)$.

By the results of section 2, we may already state an "ideal" case (as we remarked in the introduction, it is however *highly non-trivial* if one can always take a holomorphic model as in this "ideal" case or not):

Proposition 3.3. *If the algebraic reduction $f : X \to B$ is holomorphic (with B normal, projective and $\dim B > 0$), then $a(X) = \dim B = n$ and f is Lagrangian, in particular, all smooth fibers are abelian.*

Proof. By [**Ma99**] (Theorem 2 and Theorem 1 in Addendum), f is Lagrangian and $\dim B = n$ - his argument works in the Kähler case as well. Then there is no holomorphic 2−form with non-zero restriction to the general fiber. There are several ways to conclude the last statement. For instance, we can conclude by Corollary 2.9. q.e.d.

Theorem 3.4. *Assume that $0 < a(X) < 2n$. Then $c_1(L) \in \overline{\mathcal{K}(X)}$, the closure of the Kähler cone, i.e., L is (analytically) nef. Moreover, $L.C = 0$ for all curves $C \subset X$.*

Proof. Let $\overline{\mathcal{P}(X)} \subset H^{1,1}(X, \mathbb{R})$ be the closure of the positive cone of X. In the proof of Theorem 3.1 (2), we already show that $c_1(L) \in \overline{\mathcal{P}(X)}$. Thus, by [**Hu03**], Proposition 3.2,

$$c_1(L) \in \overline{\mathcal{K}(X)},$$

i.e., L is nef, *if $L \cdot C \geq 0$ for all curves $C \subset X$*. So, it suffices to show the last statement.
As the form q_X is non-degenerate and defined over $H^2(X, \mathbb{Q})$, we have an isomorphism

$$\iota : H^2(X, \mathbb{Q}) \simeq H^2(X, \mathbb{Q})^* \simeq H^{4n-2}(X, \mathbb{Q}).$$

Here $H^2(X,\mathbb{Q})^*$ is the dual space of $H^2(X,\mathbb{Q})$, the first isomorphism is given by the map $x \mapsto q_X(*,x)$, and the second isomorphism is given by (the inverse of) the intersection pairing. Moreover, by the shape of q_X, the map ι is compatible with the Hodge decomposition. Therefore, ι induces an isomorphism

$$\iota : H^{1,1}(X,\mathbb{Q}) \simeq H^{2n-1,2n-1}(X,\mathbb{Q}).$$

Note here that $H^{1,1}(X,\mathbb{Q}) = NS(X) \otimes \mathbb{Q}$. Since $[C] \in H^{2n-1,2n-1}(X,\mathbb{Q})$, there is then an element $\alpha \in H^{1,1}(X,\mathbb{Q})$ such that $\iota(\alpha) = [C]$. Hence, by definition of ι, we have that

$$L \cdot C = q_X(L,\alpha) = 0\,.$$

Here the second equality follows from Theorem 3.1 (2), i.e., the fact that L is a multiple of the isotropic vector ℓ in parabolic $NS(X)$. This completes the proof. q.e.d.

Parts of the following Lemma are certainly well-known; we include full proofs for the convenience of the reader.

Lemma 3.5. 1) *$L^n \neq 0$ in $H^{2n}(X,\mathbb{R})$ and $L^{n+1} = 0$ in $H^{2n+2}(X,\mathbb{R})$. In particular, the numerical dimension $\nu(L) = n$.*

2) *For all $a, b \geq 0$ with $a + b > n$ we have*

$$\tilde{f}^*(A)^a \cdot \tilde{L}^b \cdot \tilde{\eta}^{2n-(a+b)} = 0.$$

3) *For all $k \geq 0$ we have*

$$\tilde{f}^*(A)^k \cdot \tilde{L}^{2n-k} = 0.$$

Proof. By Verbitsky [**Ve96**], Theorem 1.5 (see also [**Bo96**] or [**GHJ03**], proof of Proposition 24.1, for a more geometric proof), we have a graded ring isomorphism

$$SH^2(X) = \mathrm{Sym}^* H^2(X,\mathbb{C})/I\,.$$

Here $SH^2(X)$ is the subalgebra of the total cohomology ring $H^*(X,\mathbb{C})$ generated by $H^2(X,\mathbb{C})$, and I is the ideal of the symmetric algebra $\mathrm{Sym}^* H^2(X,\mathbb{C})$ generated by all elements α^{n+1} such that $\alpha \in H^2(X,\mathbb{C})$ and $q_X(\alpha) = 0$. By definition of I, we have that $L^n \notin I$ and $L^{n+1} \in I$. Thus, by the isomorphism above, we have that $L^n \neq 0$ and $L^{n+1} = 0$ in $SH^2(X) \subset H^*(X,\mathbb{C})$. This implies (1).

Let $c > n$ and $a + b = c$. By (1) we have

$$\tilde{L}^c \cdot \tilde{\eta}^{2n-c} = \pi^*(L^c.\eta^{2n-c}) = 0.$$

Hence

$$(\tilde{f}^*(A) + E) \cdot \tilde{L}^{c-1} \cdot \tilde{\eta}^{2n-c} = 0.$$

Since $\tilde{L}$ is nef, this gives

$$\tilde{f}^*(A) \cdot \tilde{L}^{c-1} \cdot \tilde{\eta}^{2n-c} = E \cdot \tilde{L}^{c-1} \cdot \tilde{\eta}^{2n-c} = 0.$$

Continuing in this way, we obtain

$$\tilde{f}^*(A)^a \cdot \tilde{L}^{c-a} \cdot \tilde{\eta}^{2n-c} = 0$$

proving (2).
Claim (3) is the following special case of (2): $a = k$ and $b = 2n - k$.
q.e.d.

Theorem 3.6. *Let X be a non-algebraic hyperkähler manifold of dimension $2n$. Then $a(X) \leq n$.*

Proof. Recall that $a(X) = \dim B$. We shall argue by contradiction. Suppose that $k = \dim B > n$. Then we can take $a = k$ and $b = 0$ in Lemma 3.5 (3), i.e., $(\tilde{f}^*A)^k \tilde{\eta}^{2n-k} = 0$. The class $(\tilde{f}^*A)^k$ is represented by a positive multiple of general fiber $\tilde{F}$ of $\tilde{f}$. So, if we put $F = \pi_* \tilde{F}$, then F is a $(2n-k)$-dimensional non-zero effective cycle on X, but

$$0 < (\eta|F)^{2n-k} = F\eta^{2n-k} = \tilde{F}\tilde{\eta}^{2n-k} = 0\,,$$

a contradiction. This completes the proof. q.e.d.

Theorem 3.7. *Let X be a hyperkähler manifold of dimension $2n$. Suppose $a(X) = n$. Then any nef line bundle D on X is semi-ample. In particular its algebraic reduction can be taken holomorphic.*

Proof. Note that $NS(X) = \mathbb{Z}\ell \oplus V$ and q_X is negative definite on V. Thus, $D = L$ up to a non-negative multiple, as D is nef. By Lemma 3.5 (2), we know $\nu(L) = n$. On the other hand, $\kappa(L) = n$ by $a(X) = n$. We also note that the canonical line bundle K_X is trivial, as X is a hyperkähler manifold. Hence [**Na87**], Theorem 5.5 (see also [**Fn08**], Theorem 4.8 for a complete proof) applies and $D = L$ is semi-ample.
q.e.d.

4. Almost holomorphic algebraic reductions

We use the same notations as in section 3 (setup (3.2)) and first prove that an almost holomorphic algebraic reduction has in fact a holomorphic model.

Theorem 4.1. *Let X be a hyperkähler manifold of dimension $2n$ such that $0 < a(X) < 2n$. If the algebraic reduction f is almost holomorphic, then f has a Lagrangian holomorphic model.*

Proof. It suffices to show that L is semi-ample. By Hironaka's flattening theorem ([**Hi75**], Page 504, Corollary 1) applied to $\tilde{f} : \tilde{X} \longrightarrow B$ and the normalization for the resulting source space, we have an equidimensional modification $\hat{f} : \hat{X} \longrightarrow \hat{B}$ of $\tilde{f} : \tilde{X} \longrightarrow B$. More precisely, there are a normal space $\hat{X}$, a proper bimeromorphic morphism

$$\mu = \hat{\mu} \circ \pi : \hat{X} \longrightarrow \tilde{X} \longrightarrow X,$$

a smooth projective manifold $\hat{B}$, a birational morphism $\mu_B : \hat{B} \longrightarrow B$ and an equi-dimensional morphism $\hat{f} : \hat{X} \longrightarrow \hat{B}$ such that $\mu_B \circ \hat{f} = \tilde{f} \circ \hat{\mu}$ and $\mu_B \circ \hat{f} = f \circ \mu$. Note that we can make $\hat{B}$ smooth, as flatness is preserved under base change. Note also that we can make $\hat{X}$ normal as the normalization map is a finite map.

Put $\hat{A} = \mu_B^*(A)$ so that $\hat{A}$ is big, nef and semi-ample and set

$$\hat{L} = \mu^* L = \hat{f}^* \hat{A} + \sum a_i \hat{E}_i \, ,$$

where $\cup \hat{E}_i$ is the exceptional divisor of μ and a_i are non-negative integers. As $\hat{D}$ and $\hat{f}^* \hat{A}$ are Cartier, so is $\sum a_i \hat{E}_i$. Since f is almost holomorphic and $\hat{f}$ is equi-dimensional, $\hat{f}(E_i)$ is a divisor on $\hat{B}$. As $\hat{B}$ is smooth, it is not only a Weil divisor but also a Cartier divisor. Let C be a sufficiently general ample complete intersection curve on $\hat{B}$. Let

$$\hat{V} = \hat{X} \times_{\hat{B}} C \, ,$$

$\nu : Z \longrightarrow \hat{V} \subset \hat{X}$ be a resolution of $\hat{V}$ and $\varphi : Z \longrightarrow C$ be the induced morphism. Choose a Kähler class η_Z of Z. Then, as $\varphi^*(\hat{A}|C)$ and $\nu^*(\sum a_i \hat{E}_i))$ are supported in the fibers of φ, we have:

$$\int_Z \nu^* \hat{L} \wedge \varphi^*(\hat{A}|C) \wedge \eta_Z^{n-1}$$

$$= \int_Z (\varphi^*(\hat{A}|C) + \nu^*(\sum a_i \hat{E}_i)) \wedge \varphi^*(\hat{A}|C) \wedge \eta_Z^{n-1} = 0 \, .$$

As $\nu^* \hat{L}$, $\varphi^*(\hat{A}|C) \in \overline{\mathcal{K}(Z)}$, they are proportional in $NS(Z)$ by the Hodge index theorem. So are $\varphi^*(\hat{A}|C)$ and $\nu^*(\sum a_i \hat{E}_i)$. Consequently

$$\nu^*(N(\sum a_i \hat{E}_i)) = \varphi^*(\Theta)$$

for some positive integer N and an effective divisor Θ on C. As $\hat{f}$ is equi-dimensional and C is a general ample complete intersection curve, the Cartier divisor $N(\sum a_i \hat{E}_i)$ on $\hat{X}$ is then of the form $\hat{f}^* \Delta$ for some effective Cartier divisor Δ on $\hat{B}$. Thus, replacing L by some positive multiple, we have $\mu^* \hat{L} = \hat{f}^*(\hat{A} + \Delta)$ for some semi-ample big divisor $\hat{A}$ on $\hat{B}$ and an effective Cartier divisor Δ on $\hat{B}$. As $\mu^* \hat{L} \in \overline{\mathcal{K}(\hat{X})}$ (strictly speaking, this makes a sense after passing to a resolution of $\hat{X}$. But this would not matter, as we will not need equi-dimensionality any longer), it follows that $\hat{A} + \Delta \in \overline{\mathcal{K}(\hat{B})}$. As $\hat{B}$ is projective, this implies that the divisor $\hat{A} + \Delta$ is nef. As $\hat{A}$ is big and Δ is effective, the divisor $\hat{A} + \Delta$ is also big. Thus,

$$\kappa(\hat{L}) = \dim B = \nu(\hat{A} + \Delta) = \nu(\hat{L}) \, .$$

As $\hat{L} = \mu^* L$, we have $\kappa(L) = \nu(L) > 0$ as well. Thus L is semi-ample by [**Na87**], Theorem 5.5 (see also [**Fn08**], Theorem 4.8 for a complete proof). The morphism given by $|mL|$ is then Lagrangian fibration by a result of Matsushita [**Ma99**] (Addendum, Theorem 1). This completes the proof. q.e.d.

Suppose $0 < a(X) < 2n$. In order to prove that always $a(X) = n$, we are reduced to the case that for the general fiber F has algebraic dimension $a(F) = a(\tilde{F}) = 0$ and moreover that $\tilde{F}$ is isotypically semi-simple. Unfortunately not much is known about compact Kähler manifolds $\tilde{F}$ with $a(\tilde{F}) = 0$. If however $\tilde{F}$ has a minimal model, things work out:

Proposition 4.2. *If every isotypically semi-simple compact Kähler manifold Z of dimension at most $2n-1$ with $\kappa(Z) = 0$ and $h^0(K_Z) = 1$ has a minimal model with numerically trivial canonical bundle (i.e. a bimeromorphically equivalent normal Kähler variety which is $\mathbb{Q}-$ Gorenstein with numerically trivial canonical class), then every compact hyperkähler manifold X of dimension $2n$ has algebraic dimension $a(X) = 0, n, 2n$.*

Proof. We must rule out that $1 \leq a(X) \leq n-1$. We argue by contradiction, hence we are in situation of setup (3.2). We write the second equation in setup (3.2)(2) more precisely as

$$\tilde{L} = \tilde{f}^*(A) + \sum_I a_i E_i$$

with $a_i \geq 0$. Furthermore we have

$$K_{\tilde{X}} = \sum_I b_i E_i \tag{1}$$

with $b_i > 0$. Let $D_i = E_i \cap \tilde{F}$ and $I' \subset I, I' \neq \emptyset$, the set of all i such that $E_i \cap \tilde{F} \neq \emptyset$.
Then

$$\tilde{L}_{\tilde{F}} = \sum_{I'} a_i D_i$$

and by the adjunction formula,

$$K_{\tilde{F}} = \sum_{I'} b_i D_i. \tag{2}$$

Moreover

$$\tilde{L}_{\tilde{F}} = \sum_{I'} a_i D_i. \tag{3}$$

Let $h : \tilde{F} \dashrightarrow F'$ be a minimal model of $\tilde{F}$; then $K_{F'} \equiv 0$. Choose a modification $\tau : \hat{F} \to \tilde{F}$ from a compact Kähler manifold $\hat{F}$ such that the bimeromorphic map $h : \tilde{F} \dashrightarrow F'$ induces a holomorphic map $\hat{h} : \hat{F} \to F'$. Since $K_{F'} \equiv 0$, by (2) every $D_i, i \in I'$ is contracted by h. Moreover by (2) and (3)

$$mK_{\tilde{F}} = \tilde{L}_{\tilde{F}} + D'$$

with D' effective and supported in $\bigcup_{I'} D_i$ and $m \gg 0$. Therefore $\tau^*(\tilde{L}_{\tilde{F}})$ is on one hand nef, on the other hand effective with support necessarily in the exceptional locus of $\hat{h}$. This is only possible when all $a_i = 0$ for $i \in I'$ by the following proposition, a contradiction. q.e.d.

Proposition 4.3. *Let X be a (not necessarily compact) Kähler manifold, $\phi : X \to Y$ bimeromorphic with exceptional divisor $E = \bigcup E_i$. Let $L = \mathcal{O}_X(\sum a_i E_i)$ with $a_i \geq 0$. If L is $\phi-$nef, then all $a_i = 0$.*

Proof. This proposition is of course folklore. Since we were not able to trace a proof in the literature, here we give a proof. By taking local sections in Y, we reduce to the case that E is mapped to a point p, which is then an isolated singularity of Y. Since isolated singularities are algebraic by Artin's theorem, we next reduce to the case that X and Y are algebraic (X quasi-projective and Y affine). Then reduce to $\dim X = 2$ by taking hyperplane sections in X. Finally for surfaces the claim is obvious, the intersection matrix $(E_i \cdot E_j)$ being negative definite. q.e.d.

5. The 4-dimensional case

In this section we settle Conjecture 1.2 in dimension 4 completely. What still needs to be proved is

Theorem 5.1. *Let X be a 4-dimensional hyperkähler manifold. Then* $a(X) \neq 1$.

Proof. We shall argue by contradiction. So we assume to the contrary that $a(X) = 1$ and shall derive a contradiction.
By $a(X) = 1$, the base space B of the algebraic reduction has to be a smooth projective curve (as we always assume that the base space is normal and projective). Since $H^0(X, \Omega^1_X) = 0$ as X is simply connected, the base space B is then $\mathbb{P}_1$. Now let

$$f : X \dashrightarrow B \simeq \mathbb{P}_1$$

be the algebraic reduction with the setup (3.2); we set specifically $A = \mathcal{O}_B(1)$. By Theorem 2.4 and Theorem 4.1, we know that $a(F) = a(\tilde{F}) = 0$; moreover $\tilde{F}$ is isotypically semi-simple.
(In fact, otherwise the second case (2) in Theorem 2.4 would happen. In particular, f would be almost holomorphic. However then Theorem 4.1 applies to conclude dim $B = 2$, a contradiction.)
But since $\dim \tilde{F} = 3$, necessarily $\tilde{F}$ must be simple.
(In fact, otherwise $\tilde{F}$ would be commensurable to the self product of a curve, say C^3. Then however $a(\tilde{F}) = a(C^3) = 3$, a contradiction to $a(\tilde{F}) = 0$.)
We may also assume, without loss of generality, that $q(\tilde{F}) = 0$. For this statement, recall that any subvarietry V of a positive dimensional

complex torus T that generates T is of Kodaira dimension > 0 ([**Ue75**], Corollary 10.5), and hence $a(V) > 0$, and that if the Albanese map from a compact Kähler manifold U is surjective to the Albanse torus $\mathrm{Alb}(U)$, then U is of Kodaira dimension > 0 unless the Albanese map is bimeromorphic (by the ramification formula). Thus, if $q(\tilde{F}) > 0$, then the Albanese map of $\tilde{F}$ must be bimeromorphic onto $\mathrm{Alb}(\tilde{F})$, so that $\tilde{F}$ has a minimal model, and we conclude (i.e., get a contradiction) now by Proposition 4.2.

Since $A = \mathcal{O}_{\mathbb{P}_1}(1)$, we have $h^0(L) = 2$; we take $F_1, F_2 \in |L|$, both necessarily irreducible. Set

$$S = F_1 \cap F_2$$

as complex spaces. Hence S is a possibly non-reduced complete intersection. Let $\mathcal{I}_S \subset \mathcal{O}_X$ (resp. $\mathcal{I}_E \subset \mathcal{O}_{\tilde{X}}$) be the ideal sheaf of S (resp. of E). Notice that

$$\pi_*(\mathcal{I}_E) = \mathcal{I}_S. \qquad (*)$$

In fact, we have on the level of *analytic preimages* (complex subspaces)

$$\pi^*(S) = \pi^*(F_1) \cap \pi^*(F_2) = (\tilde{F}_1 + E) \cap (\tilde{F}_2 + E) = E.$$

In other words

$$\pi^*(\mathcal{I}_S) \cdot \mathcal{O}_{\tilde{X}} = \mathcal{I}_E,$$

where the left hand side denotes the image of $\pi^*(\mathcal{I}_S)$ in $\mathcal{O}_{\tilde{X}}$. Therefore the canonical monomorphism $\mathcal{I}_S \to \pi_*(\mathcal{I}_E)$ must be an isomorphism. We first show

5.2 Claim. $H^q(X, L) = 0$ *for* $q = 1, 3, 4$ *and* $\dim H^2(X, L) = 1$.

Proof. We proceed in several steps. (1) $H^1(X, L \otimes \mathcal{I}_S) = 0$. To verify this vanishing, we deduce from (*)

$$\pi_*(\tilde{f}^*(\mathcal{O}_B(1))) = \pi_*(\mathcal{I}_E \otimes \tilde{L}) = \mathcal{I}_S \otimes L.$$

Thus our claim (1) certainly holds if we can show

$$H^1(\tilde{X}, \tilde{f}^*(\mathcal{O}_B(1))) = 0.$$

By the Leray spectral sequence (and the projection formula for $\tilde{f}$), this in turn comes down to

$$R^1\tilde{f}_*(\mathcal{O}_{\tilde{X}}) = 0. \qquad (**)$$

Since $q(\tilde{F}) = 0$, the sheaf $R^1\tilde{f}_*(\mathcal{O}_{\tilde{X}})$ is torsion, supported on a finite set. Thus if the sheaf would not be 0, again the Leray spectral sequence would yield $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \neq 0$, which is absurd.

(2) $\chi(L_S) = 0$.
Now there is a constant K such that

$$a^2 \cdot c_2(X) = Kq(a)$$

for all $(1,1)-$classes a ([**GHJ03**], Proposition 23.17, see also [**Fu87**], Page 147, Statement (B)). Hence

$$L^2 \cdot c_2(X) = 0$$

and via Riemann-Roch we obtain

$$\chi(X, mL) = \chi(\mathcal{O}_X) = 3$$

for all $m \in \mathbb{Z}$. Here the last equality follows from [**Be83**], Proposition 3. The Koszul complex

$$0 \to L^* \to \mathcal{O}_X \oplus \mathcal{O}_X \to L \otimes \mathcal{I}_S \to 0$$

gives then $\chi(L \otimes \mathcal{I}_S) = 2\chi(\mathcal{O}_X) - \chi(L^*) = 3$, so that

$$\chi(L_S) = \chi(L) - \chi(L \otimes \mathcal{I}_S) = 0.$$

Here and hereafter, L^* is the dual bundle of L. This establishes (2).

(3) The vanishing (1) and the isomorphism $H^0(\mathcal{I}_S \otimes L) \to H^0(L)$ give

$$H^0(L_S) = 0.$$

Finally we obtain

$$H^3(L) = H^4(L) = 0 \ ; \ h^2(L) = 1$$

as follows. Concerning H^2 we calculate using the adjunction formula $K_S = 2L_S$:

$$H^2(L_S) = H^0(L_S^* \otimes 2L_S) = H^0(L_S) = 0.$$

Hence by (2):

$$H^1(L_S) = 0.$$

Therefore

$$H^1(X, L) = 0.$$

Next

$$H^q(X, L) = 0$$

for $q = 4, 3$ by Serre duality resp. by a Kodaira vanishing theorem in the Kähler case [**DP03**], Theorem 0.1 (observe $L^2 \neq 0$). Hence by Riemann-Roch

$$\dim H^2(X, L) = 1.$$

Thus we completely determined the cohomology of L and Claim (5.2) is established.

We continue with the proof of Theorem 5.1. Notice that $h^{1,1}(X) \geq 2$,; otherwise X would be projective. Thus

$$h^1(T_X) = h^{1,1}(X) \geq 2.$$

Let us consider the small deformation of the pair (X, L) of X and its line bundle L. Such a deformation is realized as a smooth hypersurface, say V, in the Kuranishi space of X (see e.g. [**Hu99**], Pages 74, 75, Paragraph 1.14). As $h^1(T_X) \geq 2$, we have dim $V \geq 1$. Thus, we can choose a small disk Δ in V centered at 0 (the point corresponding to (X, L)) and obtain one dimensional deformation

$$p : \mathcal{X} \to \Delta$$

of (X, L). By definition and by [**Fu83-2**], Theorem 4.8 (2) (see also [**Ca83**], Page 413, Théorème), this deformation has the following properties

- $X \simeq X_0$;
- there is a line bundle $\mathcal{L}$ over $\mathcal{X}$ such that $\mathcal{L}|X_0 \simeq L$;
- there is a sequence (t_k) in T converging to 0 such that X_{t_k} is projective;
- the set Δ_1 of all t such that X_t is not projective is dense in Δ with countable complement.

Here $X_t = p^{-1}(t)$, the fiber of p over $t \in \Delta$, is again a hyperkähler manifold (see, e.g., [**GHJ03**], Proposition 22.7). Let $L_t = \mathcal{L}|X_t$. From the knowledge of the cohomology of L_0, semi-continuity theorem and the constancy of $\chi(L_t)$, we obtain immediately (possibly after shrinking Δ) for all t:

$$h^0(L_t) = 2\ , h^2(L_t) = 1$$

and

$$h^q(L_t) = 0$$

for $q = 1, 3, 4$. Therefore

$$a(X_t) \geq 1$$

for all t. Notice that by Theorem 3.6, $a(X_t)$ takes the values 1, 2 and 4; the set

$$\Delta_1 = \{t \in \Delta\, |a(X_t) = 1\}$$

is moreover dense in Δ with countable complement. In fact, if otherwise, we conclude $a(X_0) \geq 2$ by the upper-semicontinuity of algebraic dimensions (see e.g. [**FP09**], Proposition 4.1 for an explicit statement and proof). We should remark that the Kähler condition of fibers is essential, as there is an explicit counter-example in non-Kähler case ([**FP09**], Corollary 1.2).

We consider the meromorphic map

$$f_t : X_t \dashrightarrow B_t \simeq \mathbb{P}_1$$

defined by $|L_t|$. Our plan is to apply [**AC05**] to X_{t_k}; [**AC05**], Théorème 3.6 gives a composition of flops $X_{t_k} \dashrightarrow X'$ to some other projective hyperkähler manifold X' such that the induced rational map $X' \dashrightarrow B$ is actually a morphism. But then by [**Ma99**], Theorem 2, B_t cannot have dimension 1 (a projective hyperkähler 4-fold does not admit a surjective morphism to a curve), a contradiction. This contradiction would then complete the proof of Theorem 5.1. *However, in order to* *be able to apply* [**AC05**]*, Théorème 3.6, we need to check one condition that*

$$\kappa(F_{t_k}) \leq 0.$$

Here $\kappa(F_t)$ is the Kodaira dimension of a desingularisation of a general fiber of f_t. Let us complete the proof by checking this condition. The maps f_t fits together in a family

$$\tilde{f} : \mathcal{X} \dashrightarrow \mathbb{P}_1 \times \Delta.$$

We introduce a resolution of indeterminacy of $\tilde{f}$:

$$\tilde{\varphi} : \tilde{\mathcal{X}} \to \mathbb{P}_1 \times \Delta.$$

Choose a general point $b \in \mathbb{P}_1$ and set $\tilde{F}_t = \tilde{\varphi}^{-1}((b,t))$, the fiber of $\tilde{\varphi}$ over $(b,t) \in \mathbb{P}_1 \times \Delta$. Then we can consider a family $(\tilde{F}_t)$ of general fibers of f_t, i.e.,

$$\varphi = \tilde{\varphi}|W : W = \tilde{\varphi}^{-1}(\{b\} \times \Delta) \longrightarrow \{b\} \times \Delta = \Delta\,.$$

After possibly shrinking Δ, we may assume that all $\tilde{F}_t$ are smooth except for $t = 0$. Here we have an abuse of language and $\tilde{F}_0$ may split in a component which was called $\tilde{F}_0$ formerly, and possibly some other components. However, this possible imbellicity is avoided by considering instead of X_0 some X_s with $s \in \Delta_1 (s \neq 0)$ (defined above) and by treating this X_s as our new X_0. Thus we may assume that $p : \tilde{\mathcal{X}} \to \Delta$, where $p = \mathrm{pr}_2 \circ \tilde{f}$, is a submersion and that $\tilde{F}_0$ is smooth.

Now choose a universal number M such that $|MK_Z|$ defines the Iitaka fibration for all smooth projective threefolds Z. This number exists by [**FM00**], Corollary 6.2 and [**VZ07**], Corollary 0.4 (including references for the general type case and the case of Kodaira dimension 0, which actually are not needed here). Note that $K_{\tilde{F}_t} = K_W|\tilde{F}_t$ by the adjunction formula. Thus, by the semi-continuity theorem, there is a neighborhood $U \subset \Delta$ of 0 such that

$$h^0(MK_{\tilde{F}_0}) \geq h^0(MK_{\tilde{F}_t})$$

for all $t \in U$. Recall that $\tilde{F}_0$ is a resolution of singularities of X_0. So, $a(\tilde{F}_0) = a(F_0) = 0$ (see at the beginning of the proof of Theorem 5.1). In particular, $h^0(MK_{\tilde{F}_0}) \leq 1$. Thus,

$$h^0(MK_{\tilde{F}_{t_k}}) \leq 1$$

for all $t_k \in U$. Therefore, by the choice of M, we conclude $\kappa(\tilde{F}_{t_k}) = \kappa(F_{t_k}) \leq 0$ for all $t_k \in U$. Now we are done. q.e.d.

6. Nef line bundles on hyperkähler manifolds

If X is a non-algebraic hyperkähler manifold, then $NS(X)$ is parabolic if and only if X carries a nef non-trivial line bundle L, which is then unique up to a multiple (see the proof of Theorem 3.7). We expect that L is actually semi-ample (cf. Conjecture 1.1). In this section we give some results pointing in this direction.
A line bundle L on a compact complex manifold is *hermitian semi-positive* if there exists a (smooth) hermitian metric on L whose curvature form is semi-positive. Equivalently, there exists a semi-positive $(1,1)-$form ω such that

$$c_1(L) = [\omega].$$

A hermitian semi-positive line bundle is nef, but the converse is not true even in dimension 2, see [**DPS01**], Corollary 2.9.

Theorem 6.1. *Let X be a non-projective hyperkähler manifold of dimension $2n$. Let L be a non-trivial hermitian semi-positive line bundle on X. Then $a(X) = \kappa(L)$; in particular $\kappa(L) \geq 0$.*

Proof. We use Riemann-Roch in the following form (see e.g. [**GHJ03**], Corollary 23.18)

$$\chi(mL) = \sum_{i=0}^{n} b_i m^i q_X(L)^i,$$

where b_i are some numbers which do not depend on L. Since X is assumed to be non-algebraic, we have $q_X(L) = 0$ and Riemann-Roch reads

$$\chi(mL) = b_0 = \chi(\mathcal{O}_X) = n + 1.$$

Here the last equality follows from [**Be83**], Proposition 3:

$$H^q(X, \mathcal{O}_X) = 0$$

for q odd and

$$\dim H^q(X, \mathcal{O}_X) = 1$$

for q even, $0 \leq q \leq 2n$.
Note that X is a hyperkähler manifold of dimension $2n$. If $h^0(mL) \geq n + 1$ for all $m >> 0$, then $\kappa(L) \geq 1$, in particular $a(X) \geq 1$. Since L defines the algebraic reduction in the sense of the setup 3.2 (recall that $NS(X)$ must be parabolic and that we have only one nef line bundle up to scalars), we obtain $\kappa(L) = a(X)$.
So we may assume that there is a sequence (m_k) converging to ∞ and some number $q > 0$ (actually even) such that

$$H^q(X, m_k L) \neq 0$$

for all m_k. Fix a Kähler form ω. By the Hard Lefschetz Theorem in the semi-positive case [**DPS01**], Corollary 2.2 (see also [**Mou99**], Théorème

2.6 and [**Ta97**], Theorem 1), the canonical morphism

$$\wedge\omega^q : H^0(X, \Omega_X^{2n-q} \otimes m_k L) \to H^q(X, m_k L) \qquad (*)$$

is surjective. Thus

$$H^0(X, \Omega_X^{2n-q} \otimes m_k L) \neq 0$$

for all k. Now we apply [**DPS01**], Proposition 2.15: one has $a(X) \geq 1$ or $\kappa(L) \geq 0$. In both cases we argue as above and conclude $\kappa(L) = a(X)$. q.e.d.

The arguments of Theorem 6.1 actually sometimes work also in the nef case, namely when the zero locus of a suitable multiplier ideal is not too large. This leads to the following

Theorem 6.2. *Let X be a parabolic hyperkähler manifold of dimension $2n \geq 4$. Then X contains a positive dimensional compact subvariety of dimension at least 2.*

Proof. Assume to the contrary that all compact subvarieties of X have dimension at most 1, in particular $a(X) = 0$. Since $NS(X)$ is parabolic, there is, as already mentioned at the beginning of this section, a non-trivial nef line bundle L, unique up to a scalar . On $L^{\otimes m}$ we introduce a singular metric h_m with multiplier ideal $\mathcal{I}_m$ with zero locus V_m. We argue similarly as in Theorem 6.1. From Riemann-Roch we deduce the existence of a positive even number $q \geq 2$ such that $H^q(X, m_k L) \neq 0$ for a sequence (m_k) converging to ∞. Since $\dim V_{m_k} \leq 1$ by our assumption, we conclude

$$H^q(X, m_k L \otimes \mathcal{I}_{m_k}) \neq 0$$

for all k. By the Hard Lefschetz Theorem for nef line bundles [**DPS01**], Theorem 2.1 (see also [**Ta97**], Theorem 1), we obtain the non-vanishing

$$H^0(X, \Omega_X^{2n-q} \otimes m_k L \otimes \mathcal{I}_{m_k}) \neq 0.$$

Now [**DPS01**], Proposition 2.15 implies $a(X) \geq 1$ or $\kappa(L) \geq 0$. Since the only positive-dimensional subvarieties in X are curves, the first alternative is only possible when $a(X) = 2n - 1$, contradicting Theorem 3.6. In the second alternative X contains a divisor, since L cannot be trivial, again a contradiction. q.e.d.

References

[AC05] E. Amerik & F. Campana, *Fibrations méromorphes sur certaines variétés de classe canonique triviale*, Pure Appl. Math. Q. **4** (2008), no. 2, part 1, 509–545. MR 2400885, Zbl 1143.14035.

[Be83] A. Beauville, *Variétés kählériennes dont la première classe de Chern est nulle*, J. Diff. Geom. **18** (1983) 755–782. MR 0730926, Zbl 0537.53056.

[Bo78] F. Bogomolov, *Hamiltonian Kählerian manifolds*, Dokl. Akad. Nauk SSSR **243** (1978) 1462–1465. MR 0522939, Zbl 0439.14002.

[Bo96] F. Bogomolov, *On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky)*, Geom. Funct. Anal. **6** (1996) 612–618. MR 1406665, Zbl 0862.53050.

[Ca80] F. Campana, *Algébricité et compacité dans l'espace des cycles d'un espace analytique complexe*, Math. Ann. **251** (1980) 7–18. MR 0583821, Zbl 0445.32021.

[Ca81] F. Campana, *Coréduction algébrique d'un espace analytique faiblement kählérien compact*, Invent. Math. **63** (1981) 187–223. MR 0610537, Zbl 0436.32024.

[Ca83] F. Campana, *Densité des variétés hamiltoniennes primitives projectives*, C. R. Acad. Sci. Paris Série 1. **297** (1983) 413–416. MR 0732847, Zbl 0537.32004.

[Ca06] F. Campana, *Isotrivialité de certaines familles kählériennes de variétés non projectives*, Math. Z. **252** (2006) 147–156. MR 2209156, Zbl 1104.32008.

[DPS01] J.P. Demailly, Th. Peternell & M. Schneider, *Pseudo-effective line bundles on compact Kähler manifolds*, Intl. J. Math. **12** (2001) 689–741 MR 1875649, Zbl 1111.32302.

[DP03] J.P. Demailly & Th. Peternell, *A Kodaira vanishing theorem on compact Kähler manifolds*, J.Diff.Geom. **63** (2003) 231–277. MR 2039988, Zbl 1077.32504.

[Fu81] A. Fujiki, *A theorem on bimeromorphic maps of Kähler manifolds and its applications*, Publ. Res. Inst. Math. Sci. **17** (1981) 735–754. MR 0642659, Zbl 0515.53049.

[Fu82] A. Fujiki, *Semisimple reductions of compact complex varieties*, Conference on complex analysis (Nancy, 1982), Inst. Élie Cartan **8**, Univ. Nancy, Nancy (1983) 79–133. MR 0748315, Zbl 0562.32014.

[Fu83] A. Fujiki, *On the structure of compact complex manifolds in $\mathcal{C}$*, Adv. Stud. Pure Math. **1**, North-Holland, Amsterdam (1983) 231–302. MR 0715653, Zbl 0513.32027.

[Fu83-2] A. Fujiki, *On primitively symplectic compact Kähler V-manifolds of dimension four*, Classification of algebraic and analytic manifolds (Katata, 1982), 71–250, Progr. Math., 39, Birkhäuser Boston, Boston, MA, 1983. MR 0728609, Zbl 0549.32018.

[Fu87] A. Fujiki, *On the de Rham cohomology group of a compact Kähler symplectic manifold*, Adv. Stud. Pure Math. **10** North-Holland, Amsterdam (1987) 105–165. MR 0946237, Zbl 0654.53065.

[FP09] A. Fujiki & M. Pontecorovo, *Non-upper-semicontinuity of algebraic dimension for families of compact complex manifolds*, arXiv:0903.4232.

[FM00] O. Fujino & S. Mori, *A canonical bundle formula.* J. Diff. Geom. **56** (2000) 176–188 MR 1863025, Zbl 1032.14014.

[Fn08] O. Fujino, *On Kawamata's theorem.* arXiv:0910.1156

[GHJ03] M. Gross, D. Huybrechts & D. Joyce, *Calabi-Yau manifolds and related geometries*, Universitext. Springer-Verlag, Berlin (2003). MR 1963559, Zbl 1001.00028.

[Hu99] D. Huybrechts, *Compact hyperkähler manifolds:basic results*, Invent. Math. **135** (1999) 63–113; Erratum: "Compact hyper-Kähler manifolds: basic results" Invent. Math. **152** (2003) 209–212. MR 1664696, Zbl 0953.53031, MR 1965365, Zbl 1029.53058

[Hu03] D. Huybrechts, *The Kähler cone of a compact hyperkähler manifold*, Math. Ann. **326** (2003) 499–513. MR 1992275, Zbl 1023.14015.

[Hi75] H. Hironaka, *Flattening theorem in complex-analytic geometry*, Amer. J. Math. **97** (1975) 503–547. MR 0393556, Zbl 0307.32011.

[Ko54] K. Kodaira, *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)*, Ann. of Math. **60** (1954) 28–48. MR 0068871, Zbl 0057.14102.

[Ma99] D. Matsushita, *On fiber space structures of a projective irreducible symplectic manifold*, Topology **38** (1999) 79–83; Addendum ibid. **40** (2001) 431–432. MR 1644091, MR 1808227, Zbl 0932.32027.

[Mo66] B. G. Moishezon, *On n-dimensional compact complex manifolds having n algebraically independent meromorphic functions. I.*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966) 133–174. Zbl 0186.26204.

[Mou99] C. Mourougane, *Théorèmes d'annulation génériques pour les fibrés vectoriels seminégatifs*, Bull. Soc. Math. France **127** (1999) 115–133 MR 1700471, Zbl 0939.32020.

[Na87] N. Nakayama, *The lower semicontinuity of the plurigenera of complex varieties*, Adv. Stud. Pure Math. **10** North-Holland, Amsterdam (1987) 551–590. MR 0946250, Zbl 0649.14003.

[Og03] K. Oguiso, *Local families of K3 surfaces and applications*, J. Alg. Geom. **12** (2003) 405–433. MR 1966023, Zbl 1085.14510.

[Og07] K. Oguiso, *Salem polynomials and birational transformation groups for hyperkähler manifolds*, Sugaku **59** (2007) 1–23. MR 2301428.

[Si55] C.L. Siegel, *Meromorphe Funktionen auf kompakten analytischen Mannigfaltigkeiten*, Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. (1955) 71–77. MR 0074061, Zbl 0064.08201.

[Ta97] K. Takegoshi, *On cohomology groups of nef line bundles tensorized with multiplier ideal sheaves on compact Kähler manifolds*, Osaka J. Math. **34** (1997) 783–802. MR 1618661, Zbl 0895.32008.

[To03] A. Todorov, *Large Radius Limit and SYZ Fibrations of Hyper-Kähler Manifold*, arXiv:math/0308210.

[Ue75] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics **439** Springer-Verlag, Berlin-New York (1975). MR 0506253, Zbl 0299.14007.

[Va89] J. Varouchas, *Kähler Spaces and Proper Open Morphisms*, Math. Ann. **283** (1989) 13–52. MR 0973802, Zbl 0632.53059.

[Ve96] M. Verbitsky, *Cohomology of compact hyper-Kähler manifolds and its applications*, Geom. Funct. Anal. **6** (1996) 601–611. MR 1406664, Zbl 0861.53069.

[VZ07] E. Viehweg, De-Qi Zhang, *Effective Iitaka fibrations*, J. Algebraic Geom. **18** (2009), no. 4, 711–730. MR 2524596, Zbl 1177.14039.

[We58] A. Weil, *Introduction à L'étude de Variétés Kählériennes*, Paris Hermann (1958). MR 0111056, Zbl 0137.41103.

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