

**EXPONENTIAL ESTIMATES  
FOR PLURISUBHARMONIC FUNCTIONS  
AND STOCHASTIC DYNAMICS**

TIEN-CUONG DINH, VIÊT-ANH NGUYÊN & NESSIM SIBONY

**Abstract**

We prove exponential estimates for plurisubharmonic functions with respect to Monge-Ampère measures with Hölder continuous potential. As an application, we obtain several stochastic properties for the equilibrium measures associated to holomorphic maps on projective spaces. More precisely, we prove the exponential decay of correlations, the central limit theorem for general d.s.h. observables, and the large deviations theorem for bounded d.s.h. observables and Hölder continuous observables.

**1. Introduction**

In this paper we prove exponential estimates for plurisubharmonic functions with respect to a class of probability measures which contains the measures of maximal entropy for many dynamical systems in several complex variables. This permits us to prove the large deviations theorem for these dynamical systems and also sharp decay of correlation estimates. The results seem to be new even in dimension 1. This type of exponential estimates should play a role in the study of stochastic properties of dynamical systems in the complex domain.

Let  $X$  be a complex manifold of dimension  $k$  and  $K$  a compact subset of  $X$ . Let  $\mu$  be a positive measure on  $X$ . If  $\psi$  is a plurisubharmonic function and if  $\mu$  is given by a differential form with coefficients in  $L^p_{loc}$ ,  $p > 1$ , then  $e^{-\alpha\psi}$  restricted to  $K$  is integrable with respect to  $\mu$  for some constant  $\alpha > 0$ . The case where  $X$  is an open set in  $\mathbb{C}^k$  and  $\mu$  is the Lebesgue measure is a classical result (see Hörmander [22] and Skoda [29]). The general case is a direct consequence. These estimates are very useful in complex geometry; see e.g., Demailly's book [5] and the references therein. They are also very useful in Kähler-Einstein geometry and have been developed by Tian-Yau [30, 31, 32].

In this paper, we consider a class of measures satisfying an analogous property. We first recall some notions (see [8]). The measure  $\mu$  is said to be *locally moderate* if for any open set  $U \subset X$ , any compact set  $K \subset U$ ,

---

Received 3/5/2008.

and any compact family  $\mathcal{F}$  of plurisubharmonic functions (p.s.h. for short) on  $U$ , there are constants  $\alpha > 0$  and  $c > 0$  such that

$$\int_K e^{-\alpha\psi} d\mu \leq c \quad \text{for } \psi \in \mathcal{F}.$$

This inequality implies that  $\mathcal{F}$  is bounded in  $L^p_{loc}(\mu)$  for  $1 \leq p < \infty$ . In particular,  $\mu$  has no mass on pluripolar sets. The existence of  $c$  and  $\alpha$  is equivalent to the existence of  $c' > 0$  and  $\alpha' > 0$  satisfying

$$\mu\{z \in K, \psi(z) < -M\} \leq c'e^{-\alpha'M}$$

for  $M \geq 0$  and  $\psi \in \mathcal{F}$ . Note that the functions on  $\mathcal{F}$  are uniformly bounded from above on  $K$  (see e.g., [5]). Applying the above estimates to  $\log \|z - a\|$ , we obtain that the  $\mu$ -measure of a ball of center  $a \in K$  and of small radius  $r$  is bounded by  $r^{\alpha''}$  for some  $\alpha'' > 0$ . In the one variable case, this property is equivalent to the fact that  $\mu$  is locally moderate.

Fix a hermitian form  $\omega$ , i.e., a smooth strictly positive  $(1, 1)$ -form, on  $X$ . Let  $S$  be a positive closed current of bidegree  $(p, p)$  on  $X$ . Define the *trace measure* of  $S$  by  $\sigma_S := S \wedge \omega^{k-p}$ . We say that  $S$  is *locally moderate* if its trace measure is locally moderate. So, if  $S$  is given by a continuous differential form, then it is locally moderate. Observe that the notion of locally moderate current does not depend on the choice of  $\omega$ .

Consider a continuous real-valued function  $u$  on the support  $\text{supp}(S)$  of  $S$ . The multiplication  $uS$  defines a current on  $X$ , so the current  $dd^c(uS)$  is also well-defined. The function  $u$  is *S-p.s.h.* if  $dd^c(uS)$  is a positive current. If  $R$  is a positive closed  $(1, 1)$ -current on  $X$ , then we can locally write  $R = dd^c u$  where  $u$  is a p.s.h. function. We call  $u$  a *local potential* of  $R$ . If  $R$  has local continuous potentials, then the wedge product  $R \wedge S$  is well-defined and is locally given by  $R \wedge S := dd^c(uS)$ . Indeed, it is enough to have that  $u$  is locally integrable with respect to the trace measure of  $S$ . The wedge product is a positive closed  $(p+1, p+1)$ -current which does not depend on the choice of  $u$ . We refer the reader to [1, 5, 18, 2, 13] for the intersection theory of currents. Here is one of our main results.

**Theorem 1.1.** *Let  $S$  be a locally moderate positive closed  $(p, p)$ -current on a complex manifold  $X$ . If  $u$  is a Hölder continuous  $S$ -p.s.h. function, then  $dd^c(uS)$  is locally moderate. In particular, if  $R$  is a positive closed  $(1, 1)$ -current with Hölder continuous local potentials, then  $R \wedge S$  is locally moderate.*

If  $u$  is a continuous p.s.h. function on  $X$ , then the Monge-Ampère  $(p, p)$ -currents,  $1 \leq p \leq k$ , associated to  $u$  are defined by induction

$$(dd^c u)^p := dd^c u \wedge \dots \wedge dd^c u \quad (p \text{ times}).$$

These currents are very useful in complex analysis and geometry. We have the following corollary.

**Corollary 1.2.** *Let  $u$  be a Hölder continuous p.s.h. function on  $X$ . Then the Monge-Ampère currents  $(dd^c u)^p$  are locally moderate.*

We give now an application to dynamics. Consider a non-invertible holomorphic endomorphism  $f$  of the projective space  $\mathbb{P}^k$ . Let  $d \geq 2$  denote the algebraic degree of  $f$ . That is,  $f$  is induced by a homogeneous polynomial endomorphism of degree  $d$  on  $\mathbb{C}^{k+1}$ . If  $V$  is a subvariety of pure codimension  $p$  on  $\mathbb{P}^k$ , then  $f^{-1}(V)$  is a subvariety of pure codimension  $p$  and of degree (counted with multiplicity)  $d^p \deg(V)$ . More generally, if  $S$  is a positive closed  $(p, p)$ -current on  $\mathbb{P}^k$ , then  $f^*(S)$  is a well-defined positive closed  $(p, p)$ -current of mass  $d^p \|S\|$ . Here, we consider the metric on  $\mathbb{P}^k$  induced by the Fubini-Study form  $\omega_{\text{FS}}$  that we normalize by  $\int_{\mathbb{P}^k} \omega_{\text{FS}}^k = 1$ . The mass of  $S$  is given by  $\|S\| := \langle S, \omega_{\text{FS}}^{k-p} \rangle$ . We refer the reader to [26, 12] for the definition of the pull-back operator  $f^*$  on positive closed currents.

Recall some dynamical properties of  $f$  and its iterates  $f^n := f \circ \dots \circ f$ ,  $n$  times; see e.g., the survey article [28]. One can associate to  $f$  some canonical invariant currents. Indeed,  $d^{-n}(f^n)^*(\omega_{\text{FS}})$  converge to a positive closed  $(1, 1)$ -current  $T$  of mass 1 on  $\mathbb{P}^k$ . The current  $T$  has locally Hölder continuous potentials. So, one can define  $T^p := T \wedge \dots \wedge T$ ,  $p$  times. The currents  $T^p$  are the Green currents associated to  $f$ , and  $\mu := T^k$  is the Green measure of  $f$ . They are totally invariant by  $f$ :  $d^{-p} f^*(T^p) = d^{-k+p} f_*(T^p) = T^p$ .

**Corollary 1.3.** *Let  $f$  be a non-invertible holomorphic endomorphism of  $\mathbb{P}^k$ . Then the Green currents and the Green measure associated to  $f$  are locally moderate.*

This property of the Green measure  $\mu$  allows us to prove the central limit theorem and the large deviations theorem for a large class of observables. Recall that a quasi-p.s.h. function on  $\mathbb{P}^k$  is locally the difference of a p.s.h. function and a smooth function. A function is called d.s.h. if it is equal outside a pluripolar set to the difference of two quasi-p.s.h. functions. We identify two d.s.h. functions if they are equal out of a pluripolar set. Moreover, d.s.h. functions are integrable with respect to  $\mu$  (see e.g., [5, 10] and Section 3). Recall that  $\mu$  has no mass on pluripolar sets.

**Corollary 1.4.** *Let  $f$  be a non-invertible holomorphic endomorphism of  $\mathbb{P}^k$ . Let  $\mu$  denote its Green measure. If a d.s.h. function  $\psi$  on  $\mathbb{P}^k$  satisfies  $\langle \mu, \psi \rangle = 0$  and is not a coboundary, then it satisfies the central limit theorem with respect to  $\mu$ .*

The reader will find more details in Sections 3 and 4. Corollary 1.4 was known for  $\psi$  bounded d.s.h., and for  $\psi$  Hölder continuous, see

[17, 8, 10, 11]. The result was recently extended by Dupont to unbounded functions  $\psi$  with analytic singularities such that  $e^\psi$  is Hölder continuous [15]. He uses Ibragimov's approach and gives an application of the central limit theorem (see also [16]). Our result relies on the verification of Gordin's condition in Theorem 4.1. Some finer stochastic properties of  $\mu$  (the almost-sure invariance principle, the Donsker and Strassen principles, and the law of the iterated logarithm) can be deduced from the so-called Philipp-Stout's condition proved by Dupont [15] or Gordin's condition that we obtain here (see [20]). We refer to [6, 21, 25, 7, 27, 36] and the references therein for some results in the case of dimension 1. We will prove in Section 5 that bounded d.s.h. functions and Hölder continuous functions satisfy the large deviations theorem.

Note that Corollary 1.3 can be extended to several situations, in particular to Hénon maps, to regular polynomial automorphisms and also to automorphisms of compact Kähler manifolds [28, 9]. For a simple proof of the Hölder continuity of Green functions, see [13, Lemma 5.4.2].

**Acknowledgment.** We would like to thank the referee for his remarks.

## 2. Locally moderate currents

In this section, we give the proof of Theorem 1.1. Assume that  $S$  is moderate and  $u$  is a Hölder continuous function on  $\text{supp}(S)$  with Hölder exponent  $0 < \nu \leq 1$ .

The problem is local. So, we can assume that  $U = X$  is the ball  $B_2$  of center 0 and of radius 2 in  $\mathbb{C}^k$ ,  $K$  is the closed ball  $\overline{B}_{1/2}$  of radius  $1/2$ , and  $\omega$  is the canonical Kähler form  $dd^c\|z\|^2$ . Here,  $z = (z_1, \dots, z_k)$  is a coordinate system of  $\mathbb{C}^k$ . We replace  $S$  by  $S \wedge \omega^{k-p-1}$  in order to assume that  $S$  is of bidegree  $(k-1, k-1)$ . We have the following lemma.

**Lemma 2.1.** *Let  $\mathcal{G}$  be a compact family of p.s.h. functions on  $B_2$ . Then  $\mathcal{G}$  is bounded in  $L_{loc}^1(\sigma_S)$ . Moreover, the mass of the measure  $dd^c\varphi \wedge S$  is locally bounded on  $B_2$ , uniformly on  $\varphi \in \mathcal{G}$ .*

*Proof.* Observe that on any compact set  $H$  of  $B_2$ , the functions of  $\mathcal{G}$  are bounded from above by the same constant. Subtracting from these functions a constant allows us to assume that they are negative on  $H$ . We deduce from the fact that  $S$  is locally moderate that  $\int_H \varphi d\sigma_S$  is bounded uniformly on  $\varphi \in \mathcal{G}$ . This proves the first assertion.

For the second assertion, consider a compact set  $H \subset B_2$ . Let  $0 \leq \chi \leq 1$  be a cut-off function, smooth, supported on a compact set  $L \subset B_2$  and equal to 1 on  $H$ . The mass of  $dd^c\varphi \wedge S$  is bounded by the following integral:

$$\int \chi dd^c(\varphi S) = \int dd^c\chi \wedge \varphi S \leq \|\chi\|_{\mathcal{C}^2} \int_L |\varphi| d\sigma_S.$$

We have seen that the last term is bounded uniformly on  $\varphi$ . q.e.d.

We will use the following classical lemma.

**Lemma 2.2.** *Let  $u$  be a  $\nu$ -Hölder continuous function on a closed subset  $F$  of  $B_2$ . Then  $u$  can be extended to a  $\nu$ -Hölder continuous function on  $B_1$ .*

*Proof.* We can assume that  $|u| \leq 1$ . Define for  $x \in B_1$ ,

$$\tilde{u}(x) := \min\{u(y) + A\|x - y\|^\nu, y \in F \cap \overline{B_1}\}$$

where  $A > 0$  is a constant large enough so that  $|u(a) - u(b)| \leq A\|a - b\|^\nu$  on  $F \cap \overline{B_1}$ . It follows that  $\tilde{u}(x) = u(x)$  for  $x \in F \cap B_1$ . We only have to check that  $\tilde{u}$  is  $\nu$ -Hölder continuous.

Consider two points  $x$  and  $x'$  in  $B_1$  such that  $\tilde{u}(x) \leq \tilde{u}(x')$ . The aim is to bound  $\tilde{u}(x') - \tilde{u}(x)$ . Let  $y$  be a point in  $F \cap \overline{B_1}$  such that  $\tilde{u}(x) = u(y) + A\|x - y\|^\nu$ . By definition of  $\tilde{u}(x')$ , we have

$$\begin{aligned} \tilde{u}(x') - \tilde{u}(x) &\leq u(y) + A\|x' - y\|^\nu - \tilde{u}(x) \\ &\leq A\|x' - y\|^\nu - A\|x - y\|^\nu \\ &\leq A(\|x' - x\| + \|x - y\|)^\nu - A\|x - y\|^\nu \\ &\leq A\|x' - x\|^\nu. \end{aligned}$$

This completes the proof. q.e.d.

We continue the proof of Theorem 1.1. For simplicity, let  $u$  denote the extension of  $u$  to  $B_1$  as above for  $F = \text{supp}(S)$ . Subtracting from  $u$  a constant allows us to assume that  $u \leq -1$  on  $B_1$ . Define  $v(z) := \max(u(z), A \log \|z\|)$  for a constant  $A$  large enough. Observe that since  $A$  is large,  $v$  is equal to  $u$  on  $B_{2/3}$  and to  $A \log \|z\|$  near the boundary of  $B_1$ . Moreover,  $v$  is  $\nu$ -Hölder continuous. We are interested in an estimate on  $\overline{B_{1/2}}$ . So, replacing  $u$  by  $v$  allows us to assume that  $u = A \log \|z\|$  on  $B_1 \setminus B_{1-4r}$  for some constant  $0 < r < 1/16$ . Fix a smooth function  $\chi$  with compact support in  $B_{1-r}$ , equal to 1 on  $B_{1-2r}$ , and such that  $0 \leq \chi \leq 1$ .

**Lemma 2.3.** *If  $\varphi$  is a p.s.h. function on  $B_2$ , then*

$$\begin{aligned} \int_{B_1} \chi \varphi dd^c(uS) &= - \int_{B_{1-r} \setminus B_{1-3r}} dd^c \chi \wedge \varphi u S - \int_{B_{1-r} \setminus B_{1-3r}} d\chi \wedge \varphi d^c u \wedge S \\ &\quad + \int_{B_{1-r} \setminus B_{1-3r}} d^c \chi \wedge \varphi du \wedge S + \int_{B_{1-r}} \chi u dd^c \varphi \wedge S. \end{aligned}$$

*Proof.* Observe that  $u$  is smooth on  $B_1 \setminus B_{1-4r}$ . So, all the previous integrals make sense (see also Lemma 2.1). On the other hand, one can approximate  $\varphi$  by a decreasing sequence of smooth p.s.h. functions (one reduces slightly  $B_2$  if necessary). Therefore, it is enough to prove the

lemma for  $\varphi$  smooth. A direct computation using integration by parts gives

$$\begin{aligned} \int_{B_1} \chi \varphi dd^c(uS) &= \int_{B_1} dd^c(\chi\varphi) \wedge uS = \int_{B_1} dd^c\chi \wedge \varphi uS + \int_{B_1} d\chi \wedge d^c\varphi \wedge uS \\ &\quad - \int_{B_1} d^c\chi \wedge d\varphi \wedge uS + \int_{B_1} \chi dd^c\varphi \wedge uS. \end{aligned}$$

We also have

$$\begin{aligned} \int_{B_1} d\chi \wedge \varphi d^c(uS) &= \int_{B_1} d^c(d\chi\varphi) \wedge uS = - \int_{B_1} dd^c\chi \wedge \varphi uS \\ &\quad - \int_{B_1} d\chi \wedge d^c\varphi \wedge uS \end{aligned}$$

and

$$\begin{aligned} \int_{B_1} d^c\chi \wedge \varphi d(uS) &= \int_{B_1} d(d^c\chi\varphi) \wedge uS = \int_{B_1} dd^c\chi \wedge \varphi uS \\ &\quad - \int_{B_1} d^c\chi \wedge d\varphi \wedge uS. \end{aligned}$$

It follows from the above computation that

$$\begin{aligned} \int_{B_1} \chi \varphi dd^c(uS) &= - \int_{B_1} dd^c\chi \wedge \varphi uS - \int_{B_1} d\chi \wedge \varphi d^c(uS) \\ &\quad + \int_{B_1} d^c\chi \wedge \varphi d(uS) + \int_{B_1} \chi dd^c\varphi \wedge uS. \end{aligned}$$

The fact that  $dd^c\chi$ ,  $d\chi$ ,  $d^c\chi$  are supported in  $B_{1-r} \setminus B_{1-2r}$  and  $\chi$  is supported in  $B_{1-r}$  implies the result. q.e.d.

*End of the proof of Theorem 1.1.* Since  $\mathcal{F}$  is compact, it is locally bounded from above. Subtracting from each function  $\varphi \in \mathcal{F}$  a constant allows us to assume that  $\varphi \leq 0$  on  $B_1$ . Define  $\varphi_M := \max(\varphi, -M)$  and  $\psi_M := \varphi_{M-1} - \varphi_M$  for  $\varphi \in \mathcal{F}$  and  $M \geq 0$ . Let  $\mathcal{G}$  denote the family of all these functions  $\varphi_M$ . This family is compact in  $L^1_{loc}$ . Lemma 2.1 implies that in  $B_1$  the masses of  $dd^c\varphi_M$  and of  $dd^c\varphi_M \wedge S$  are locally bounded independently of  $\varphi \in \mathcal{F}$  and of  $M$ . The function  $\psi_M$  is positive, bounded by 1, supported in  $\{\varphi < -M + 1\}$ , and equal to 1 on  $\{\varphi < -M\}$ . The mass of  $dd^c(uS)$  on  $\{\varphi < -M\}$  is bounded by

$$\int \chi \psi_M dd^c(uS),$$

with  $\chi$  as in Lemma 2.3 above. We will show that this integral is  $\lesssim e^{-\alpha M/3}$  for some  $\alpha > 0$ . This implies the result.

Since  $S$  is locally moderate, we have the following estimate for some  $\alpha > 0$ :

$$\sigma_S\{z \in B_{1-r}, \varphi(z) < -M + 1\} \lesssim e^{-\alpha M}.$$

Lemma 2.3 implies that

$$\begin{aligned} \int_{B_1} \chi \psi_M dd^c(uS) &= - \int_{B_{1-r} \setminus B_{1-3r}} dd^c \chi \wedge \psi_M u S - \int_{B_{1-r} \setminus B_{1-3r}} d\chi \wedge \psi_M d^c u \wedge S \\ &\quad + \int_{B_{1-r} \setminus B_{1-3r}} d^c \chi \wedge \psi_M du \wedge S + \int_{B_{1-r}} \chi u dd^c \psi_M \wedge S. \end{aligned}$$

The first three integrals on the right-hand side are  $\lesssim e^{-\alpha M}$ . This is a consequence of the above estimate on  $\sigma_S$  and the smoothness of  $u$  on  $B_1 \setminus B_{1-4r}$ . It remains to estimate the last integral.

We now use the  $\nu$ -Hölder continuity of  $u$ . Define  $\epsilon := e^{-\alpha M/3}$ . This is a small constant since we only have to consider  $M$  big. Write  $u = u_\epsilon + (u - u_\epsilon)$ , where  $u_\epsilon$  is defined on  $B_{1-r}$  and is obtained from  $u$  by convolution with a smooth approximation of identity. The convolution can be chosen so that  $\|u_\epsilon\|_{\mathcal{C}^2} \lesssim \epsilon^{-2}$  and  $\|u - u_\epsilon\|_\infty \lesssim \epsilon^\nu = e^{-\alpha\nu M/3}$ . Moreover, the  $\mathcal{C}^2$ -norm of  $u_\epsilon$  on  $B_{1-r} \setminus B_{1-3r}$  is bounded independently of  $\epsilon$  since  $u = \log \|z\|$  on  $B_1 \setminus B_{1-4r}$ . We have

$$\begin{aligned} \int \chi u dd^c \psi_M \wedge S &= \int \chi dd^c \psi_M \wedge S u_\epsilon + \int \chi dd^c \psi_M \wedge S (u - u_\epsilon) \\ &= \int \chi dd^c \psi_M \wedge S u_\epsilon + \int \chi (dd^c \varphi_{M-1} - dd^c \varphi_M) \wedge S (u - u_\epsilon). \end{aligned}$$

By Lemma 2.1, the last integral is  $\lesssim \|u - u_\epsilon\|_\infty \lesssim e^{-\alpha\nu M/3}$ .

Using an expansion as above, we obtain

$$\begin{aligned} \int \chi dd^c \psi_M \wedge S u_\epsilon &= \int_{B_{1-r} \setminus B_{1-3r}} dd^c \chi \wedge \psi_M S u_\epsilon + \int_{B_{1-r} \setminus B_{1-3r}} d\chi \wedge \psi_M S \wedge d^c u_\epsilon \\ &\quad - \int_{B_{1-r} \setminus B_{1-3r}} d^c \chi \wedge \psi_M S \wedge du_\epsilon + \int \chi \psi_M S \wedge dd^c u_\epsilon. \end{aligned}$$

As above, the first three integrals on the right-hand side are  $\lesssim e^{-\alpha M}$  because  $u_\epsilon$  has bounded  $\mathcal{C}^2$ -norm on  $B_{1-r} \setminus B_{1-3r}$ . Consider the last integral. Since  $\psi_M$  is supported in  $\{\varphi \leq -M + 1\}$ , the estimate on  $\sigma_S$  implies that the considered integral is  $\lesssim e^{-\alpha M} \|u_\epsilon\|_{\mathcal{C}^2} \lesssim e^{-\alpha M} \epsilon^{-2} = e^{-\alpha\nu M/3}$ . We deduce from all the previous estimates that

$$\int \chi u dd^c \psi_M \wedge S \lesssim e^{-\alpha\nu M/3}.$$

This completes the proof.

q.e.d.

**Remark 2.4.** On a compact Kähler manifold  $X$ , one can introduce the notion of (*globally*) *moderate current*. For this purpose, in the definition, one replaces local p.s.h. functions by (global) quasi-p.s.h. functions. In the case where  $X = \mathbb{P}^k$ , the first and third authors introduced in [13] a notion of super-potential for positive closed  $(p, p)$ -currents. One can prove that currents with Hölder continuous super-potentials are moderate and the intersection of currents with Hölder continuous

super-potentials admits Hölder continuous super-potentials. If a  $(p, p)$ -current admits a Hölder continuous potential, it has a Hölder continuous super-potential and then is moderate.

### 3. Decay of correlations

Let  $\mu$  be the Green measure of an endomorphism  $f$  of algebraic degree  $d \geq 2$  of  $\mathbb{P}^k$ . In this section, we will prove that  $\mu$  is *mixing* and *exponentially mixing* in different senses. If  $\phi$  is a d.s.h. function on  $\mathbb{P}^k$ , we can write  $dd^c\phi = R^+ - R^-$  where  $R^\pm$  are positive closed  $(1, 1)$ -currents. The *d.s.h. norm* of  $\phi$  is defined by

$$\|\phi\|_{\text{DSH}} := \|\phi\|_{L^1(\mathbb{P}^k)} + \inf \|R^\pm\|,$$

with  $R^\pm$  as above. Note that  $R^+$  and  $R^-$  have the same mass since they are cohomologous, and that  $\|\cdot\|_{\text{DSH}} \lesssim \|\cdot\|_{\mathcal{E}^2}$ . The following result was proved in [17, 8, 10] for  $p = +\infty$ .

**Theorem 3.1.** *Let  $f$  be a holomorphic endomorphism of algebraic degree  $d \geq 2$  and  $\mu$  its Green measure. Then for every  $1 < p \leq +\infty$  there is a constant  $c > 0$  such that*

$$|\langle \mu, (\varphi \circ f^n)\psi \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle| \leq cd^{-n} \|\varphi\|_{L^p(\mu)} \|\psi\|_{\text{DSH}}$$

for  $n \geq 0$ ,  $\varphi$  in  $L^p(\mu)$  and  $\psi$  d.s.h. Moreover, for  $0 \leq \nu \leq 2$  there is a constant  $c > 0$  such that

$$|\langle \mu, (\varphi \circ f^n)\psi \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle| \leq cd^{-n\nu/2} \|\varphi\|_{L^p(\mu)} \|\psi\|_{\mathcal{E}^\nu}$$

for  $n \geq 0$ ,  $\varphi$  in  $L^p(\mu)$  and  $\psi$  of class  $\mathcal{E}^\nu$ .

The expression  $\langle \mu, (\varphi \circ f^n)\psi \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle$  is called *the correlation of order  $n$*  between the observables  $\varphi$  and  $\psi$ . The measure  $\mu$  is said to be *mixing* if this correlation converges to 0 as  $n$  tends to infinity, for smooth observables (or equivalently, for continuous, bounded or  $L^2(\mu)$  observables).

Observe that the second assertion in Theorem 3.1 is a consequence of the first one. Indeed, on one hand, since  $\|\psi\|_{\text{DSH}} \lesssim \|\psi\|_{\mathcal{E}^2}$ , we obtain the second assertion for  $\nu = 2$ . On the other hand, we have since  $\mu$  is invariant

$$|\langle \mu, (\varphi \circ f^n)\psi \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle| \leq 2\|\varphi\|_{L^1(\mu)} \|\psi\|_{\mathcal{E}^0} \lesssim \|\varphi\|_{L^p(\mu)} \|\psi\|_{\mathcal{E}^0}.$$

So, the second assertion holds for  $\nu = 0$ . The theory of interpolation between the Banach spaces  $\mathcal{E}^0$  and  $\mathcal{E}^2$  [33] implies that

$$|\langle \mu, (\varphi \circ f^n)\psi \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle| \lesssim d^{-n\nu/2} \|\varphi\|_{L^p(\mu)} \|\psi\|_{\mathcal{E}^\nu}.$$

We prove now the first assertion in Theorem 3.1. Since  $\mu$  is invariant, the assertion is clear when  $\psi$  is constant. Indeed, in this case, the correlations vanish. So, subtracting from  $\psi$  a constant allows us to assume



that  $\langle \mu, \psi \rangle = 0$ . We have to bound  $|\langle \mu, (\varphi \circ f^n)\psi \rangle|$ . Consider the following *weak topology* on the space  $\text{DSH}(\mathbb{P}^k)$  of d.s.h. functions. We say that the sequence  $(\phi_n)$  converges to  $\phi$  in  $\text{DSH}(\mathbb{P}^k)$  if  $\phi_n \rightarrow \phi$  in the sense of currents and if  $\|\phi_n\|_{\text{DSH}}$  is bounded uniformly on  $n$ . Recall here some basic properties of d.s.h. functions (see [10]).

**Proposition 3.2.** *Let  $\phi$  be a d.s.h. function. There is a constant  $c > 0$  independent of  $\phi$  and two negative quasi-p.s.h. functions  $\phi^\pm$  such that  $\phi = \phi^+ - \phi^-$ ,  $\|\phi^\pm\|_{\text{DSH}} \leq c\|\phi\|_{\text{DSH}}$ , and  $dd^c\phi^\pm \geq -c\|\phi\|_{\text{DSH}}\omega_{\text{FS}}$ . Moreover,  $|\phi|$  is d.s.h. and  $\| |\phi| \|_{\text{DSH}} \leq c\|\phi\|_{\text{DSH}}$ . If  $\phi_n \rightarrow \phi$  in  $\text{DSH}(\mathbb{P}^k)$ , then  $\phi_n \rightarrow \phi$  in  $L^p$  for  $1 \leq p < +\infty$ .*

Since  $\mu$  is locally a Monge-Ampère measure with continuous potential,  $\psi \mapsto \langle \mu, \psi \rangle$  is continuous with respect to the considered topology on  $\text{DSH}(\mathbb{P}^k)$ . We say that  $\mu$  is PC. This allows us to prove that the DSH-norm of  $\phi$  is equivalent to the norm

$$\|\phi\|'_{\text{DSH}} := |\langle \mu, \phi \rangle| + \inf \|R^\pm\|$$

where we write as above  $dd^c\phi = R^+ - R^-$  (see [10]). In particular,  $\log|h|$  is d.s.h. for any rational function  $h$  on  $\mathbb{P}^k$ , and similarly for the potential  $u$  of any positive closed  $(1, 1)$ -current  $R$ , i.e., a quasi-p.s.h. function  $u$  such that  $dd^c u = R - c\omega_{\text{FS}}$  for some constant  $c$ .

Consider the codimension 1 subspace  $\text{DSH}_0(\mathbb{P}^k)$  of  $\text{DSH}(\mathbb{P}^k)$  defined by  $\langle \mu, \phi \rangle = 0$ . On this subspace, one has  $\|\phi\|'_{\text{DSH}} = \inf \|R^\pm\|$ . Recall that  $\mu$  is totally invariant :  $f^*\mu = d^k\mu$ . Then the space  $\text{DSH}_0(\mathbb{P}^k)$  is invariant under  $f_*$ . Recall that  $f_*\phi$  is defined by

$$f_*\phi(x) := \sum_{y \in f^{-1}(x)} \phi(y)$$

where the points in  $f^{-1}(x)$  are counted with multiplicities (there are exactly  $d^k$  points). The mass of a positive closed current on  $\mathbb{P}^k$  can be computed cohomologically. We have  $\|f_*R^\pm\| = d^{k-1}\|R^\pm\|$  and hence  $\|f_*\phi\|'_{\text{DSH}} \leq d^{k-1}\|\phi\|'_{\text{DSH}}$  on  $\text{DSH}_0(\mathbb{P}^k)$ . Define also the *Perron-Frobenius operator* by

$$\Lambda\phi := d^{-k}f_*\phi.$$

Since  $\mu$  is totally invariant, this is the adjoint operator of  $f^*$  on  $L^2(\mu)$ . Observe that  $\|\Lambda\phi\|'_{\text{DSH}} \leq d^{-1}\|\phi\|'_{\text{DSH}}$  on  $\text{DSH}_0(\mathbb{P}^k)$ . So,  $\Lambda$  has a spectral gap on  $\text{DSH}(\mathbb{P}^k)$ : the constant functions correspond to the eigenvalue 1 and the spectral radius on  $\text{DSH}_0(\mathbb{P}^k)$  is bounded by  $d^{-1} < 1$ .

**Proposition 3.3.** *There are constants  $c > 0$  and  $\alpha > 0$  such that for  $\psi \in \text{DSH}_0(\mathbb{P}^k)$  with  $\|\psi\|_{\text{DSH}} \leq 1$  and for every  $n \geq 0$  we have*

$$\langle \mu, e^{\alpha d^n |\Lambda^n \psi|} \rangle \leq c.$$

In particular, there is a constant  $c > 0$  independent of  $\psi \in \text{DSH}_0(\mathbb{P}^k)$  such that

$$\|\Lambda^n \psi\|_{L^q(\mu)} \leq cq d^{-n} \|\psi\|_{\text{DSH}}$$

for every  $n \geq 0$  and every  $1 \leq q < +\infty$ .

*Proof.* Since  $\|\cdot\|_{\text{DSH}}$  and  $\|\cdot\|'_{\text{DSH}}$  are equivalent, we assume for simplicity that  $\|\psi\|'_{\text{DSH}} = 1$ . Observe that  $d^n \Lambda^n \psi$  belongs to the family of functions in  $\text{DSH}_0(\mathbb{P}^k)$  with  $\|\cdot\|'_{\text{DSH}}$  norm bounded by 1. It follows from Proposition 3.2 that  $d^n |\Lambda^n \psi|$  belongs to a compact family of d.s.h. functions. By Theorem 1.1 and Proposition 3.2, there are positive constants  $\alpha$  and  $c$  such that

$$\langle \mu, e^{\alpha d^n |\Lambda^n \psi|} \rangle \leq c.$$

For the second assertion, it is enough to assume that  $q$  is a positive integer. Since  $e^x \geq x^q/q!$  for  $x \geq 0$ , we deduce, using the inequality  $q! \leq q^q$ , that  $\|d^n \Lambda^n \psi\|_{L^q(\mu)} \leq cq$  for some constant  $c > 0$  independent of  $\psi$ ,  $n$ , and  $q$ . q.e.d.

*End of the proof of Theorem 3.1.* Let  $1 \leq q < +\infty$  such that  $p^{-1} + q^{-1} = 1$ . Using a simple coordinate change, Proposition 3.3, and the Hölder inequality, we obtain that

$$\begin{aligned} |\langle \mu, (\varphi \circ f^n) \psi \rangle| &= d^{-kn} |\langle (f^n)^* \mu, (\varphi \circ f^n) \psi \rangle| = |\langle \mu, \varphi \Lambda^n \psi \rangle| \\ &\lesssim \|\varphi\|_{L^p(\mu)} \|\Lambda^n \psi\|_{L^q(\mu)} \lesssim d^{-n} \|\varphi\|_{L^p(\mu)} \|\psi\|_{\text{DSH}}. \end{aligned}$$

This completes the proof. q.e.d.

It is shown in [8] that  $\mu$  is mixing of any order and is K-mixing. More precisely, we have for every  $\psi$  in  $L^2(\mu)$ ,

$$\lim_{n \rightarrow \infty} \sup_{\|\varphi\|_{L^2(\mu)}=1} |\langle \mu, (\varphi \circ f^n) \psi \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle| = 0.$$

The reader can deduce the K-mixing from Theorem 3.1 and the fact that  $\Lambda$  has norm 1 when it acts on  $L^2(\mu)$ .

The following result gives the exponential mixing of any order. It can be extended to Hölder continuous observables using the interpolation of operators.

**Theorem 3.4.** *Let  $f$ ,  $d$ , and  $\mu$  be as in Theorem 3.1 and  $r \geq 1$  an integer. Then there is a constant  $c > 0$  such that*

$$\left| \langle \mu, \psi_0 (\psi_1 \circ f^{n_1}) \dots (\psi_r \circ f^{n_r}) \rangle - \prod_{i=0}^r \langle \mu, \psi_i \rangle \right| \leq cd^{-n} \prod_{i=0}^r \|\psi_i\|_{\text{DSH}}$$

for  $0 = n_0 \leq n_1 \leq \dots \leq n_r$ ,  $n := \min_{0 \leq i < r} (n_{i+1} - n_i)$ , and  $\psi_i$  d.s.h.

*Proof.* The proof is by induction on  $r$ . The case  $r = 1$  is a consequence of Theorem 3.1. Suppose the result is true for  $r - 1$ . We have to check it for  $r$ . Without loss of generality, assume that  $\|\psi_i\|_{\text{DSH}} \leq 1$ . This implies

that  $m := \langle \mu, \psi_0 \rangle$  is bounded. The invariance of  $\mu$  and the hypothesis of induction imply that

$$\begin{aligned} & \left| \langle \mu, m(\psi_1 \circ f^{n_1}) \dots (\psi_r \circ f^{n_r}) \rangle - \prod_{i=0}^r \langle \mu, \psi_i \rangle \right| \\ &= \left| \langle \mu, m\psi_1(\psi_2 \circ f^{n_2-n_1}) \dots (\psi_r \circ f^{n_r-n_1}) \rangle - m \prod_{i=1}^r \langle \mu, \psi_i \rangle \right| \leq cd^{-n} \end{aligned}$$

for some constant  $c > 0$ . In order to get the desired estimate, it is enough to show that

$$\left| \langle \mu, (\psi_0 - m)(\psi_1 \circ f^{n_1}) \dots (\psi_r \circ f^{n_r}) \rangle \right| \leq cd^{-n}.$$

Observe that the operator  $(f^n)^*$  acts on  $L^p(\mu)$  for  $p \geq 1$  and its norm is bounded by 1. Using the invariance of  $\mu$  and the Hölder inequality, we get for  $p := r + 1$ ,

$$\begin{aligned} & \left| \langle \mu, (\psi_0 - m)(\psi_1 \circ f^{n_1}) \dots (\psi_r \circ f^{n_r}) \rangle \right| \\ & \leq \left| \langle \mu, \Lambda^{n_1}(\psi_0 - m)\psi_1 \dots (\psi_r \circ f^{n_r-n_1}) \rangle \right| \\ & \leq \|\Lambda^{n_1}(\psi_0 - m)\|_{L^p(\mu)} \|\psi_1\|_{L^p(\mu)} \dots \|\psi_r \circ f^{n_r-n_1}\|_{L^p(\mu)} \\ & \leq cd^{-n_1} \|\psi_1\|_{L^p(\mu)} \dots \|\psi_r\|_{L^p(\mu)}, \end{aligned}$$

for some constant  $c > 0$ . Since  $\|\psi_i\|_{L^p(\mu)} \lesssim \|\psi_i\|_{\text{DSH}}$ , the previous estimates imply the result. Note that as in Theorem 3.1, it is enough to assume that  $\psi_i$  is d.s.h. for  $i \leq r - 1$  and  $\psi_r$  is in  $L^p(\mu)$  for some  $p > 1$ .  
q.e.d.

We obtain from Proposition 3.3 the following result.

**Proposition 3.5.** *Let  $0 < \nu \leq 2$  be a constant. There are constants  $c > 0$  and  $\alpha > 0$  such that if  $\psi$  is a  $\nu$ -Hölder continuous function with  $\|\psi\|_{\mathcal{E}^\nu} \leq 1$  and  $\langle \mu, \psi \rangle = 0$ , then*

$$\langle \mu, e^{\alpha d^{n\nu/2} |\Lambda^n \psi|} \rangle \leq c \quad \text{for every } n \geq 0.$$

Moreover, there is a constant  $c > 0$  independent of  $\psi$  such that

$$\|\Lambda^n \psi\|_{L^q(\mu)} \leq cq^{\nu/2} d^{-n\nu/2}$$

for every  $n \geq 0$  and every  $1 \leq q < +\infty$ .

*Proof.* We only consider the spaces of functions  $\psi$  such that  $\langle \mu, \psi \rangle = 0$ . By Proposition 3.3, since  $\|\cdot\|_{\text{DSH}} \lesssim \|\cdot\|_{\mathcal{E}^2}$ , we have

$$\|\Lambda^n \psi\|_{L^q(\mu)} \leq cq d^{-n} \|\psi\|_{\mathcal{E}^2},$$

with  $c > 0$  independent of  $q$  and of  $\psi$ . On the other hand, by definition of  $\Lambda$ , we have

$$\|\Lambda^n \psi\|_{L^q(\mu)} \leq \|\Lambda^n \psi\|_{L^\infty(\mu)} \leq \|\psi\|_{\mathcal{E}^0}.$$

The theory of interpolation between the Banach spaces  $\mathcal{C}^0$  and  $\mathcal{C}^2$  [33] (applied to the linear operator  $\psi \mapsto \Lambda^n \psi - \langle \mu, \psi \rangle$ ) implies that

$$\|\Lambda^n \psi\|_{L^q(\mu)} \leq A_\nu [cqd^{-n}]^{\nu/2} \|\psi\|_{\mathcal{C}^\nu},$$

for some constant  $A_\nu > 0$  depending only on  $\nu$  and on  $\mathbb{P}^k$ . This gives the second assertion in the proposition.

For the first assertion, assume that  $\|\psi\|_{\mathcal{C}^\nu} \leq 1$ . Fix a constant  $\alpha > 0$  small enough. We have (we can assume  $c \geq 1$ )

$$\langle \mu, e^{\alpha d^{n\nu/2} |\Lambda^n \psi|} \rangle = \sum_{q \geq 0} \frac{1}{q!} \langle \mu, |\alpha d^{n\nu/2} \Lambda^n \psi|^q \rangle \leq \sum_{q \geq 0} \frac{1}{q!} \alpha^q c^q q^q.$$

By Stirling's formula, the last sum converges. This implies the result. q.e.d.

#### 4. Central limit theorem

In this section, we give the proof of Corollary 1.4. We first recall some facts [23, 35]. Let  $(M, \mathcal{F}, m)$  be a probability space and  $g : M \rightarrow M$  a measurable map which preserves  $m$ , i.e.,  $m$  is  $g_*$ -invariant :  $g_* m = m$ . The measure  $m$  is *ergodic* if, for any measurable set  $A$  such that  $g^{-1}(A) = A$ , we have  $m(A) = 0$  or  $m(A) = 1$ . This is equivalent to the property that  $m$  is extremal in the convex set of invariant probability measures (if  $m$  is mixing, then it is ergodic). When  $m$  is ergodic, Birkhoff's theorem implies that if  $\psi$  is an observable in  $L^1(m)$ , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[ \psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x)) \right] = \langle m, \psi \rangle$$

for  $m$ -almost every  $x$ .

Assume now that  $\langle m, \psi \rangle = 0$ . Then the previous limit is equal to 0. The central limit theorem (CLT for short), when it holds, gives the speed of this convergence. We say that  $\psi$  *satisfies the CLT* if there is a constant  $\sigma > 0$  such that

$$\frac{1}{\sqrt{n}} \left[ \psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x)) \right]$$

converges in distribution to the Gaussian random variable  $\mathcal{N}(0, \sigma)$  of mean 0 and of variance  $\sigma$ . Recall that  $\psi$  is a *coboundary* if there is a function  $\psi'$  in  $L^2(\mu)$  such that  $\psi = \psi' - \psi' \circ g$ . In this case, one easily checks that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left[ \psi(x) + \psi(g(x)) + \cdots + \psi(g^{n-1}(x)) \right] = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left[ \psi'(x) - \psi'(g^n(x)) \right] = 0$$

in distribution. So,  $\psi$  does not satisfies the CLT (sometimes, one says that it satisfies the CLT for  $\sigma = 0$ ).

The CLT can be deduced from some strong mixing, (see [3, 19, 24, 34]). In the following result,  $\mathbf{E}(\psi | \mathcal{F}_n)$  denotes *the expectation* of  $\psi$  with

respect to  $\mathcal{F}_n$ , that is,  $\psi \mapsto \mathbf{E}(\psi|\mathcal{F}_n)$  is the orthogonal projection from  $L^2(m)$  onto the subspace generated by  $\mathcal{F}_n$ -measurable functions.

**Theorem 4.1** (Gordin). *Consider the decreasing sequence  $\mathcal{F}_n := g^{-n}(\mathcal{F})$ ,  $n \geq 0$ , of algebras. Let  $\psi$  be a real-valued function in  $L^2(m)$  such that  $\langle m, \psi \rangle = 0$ . Assume that*

$$\sum_{n \geq 0} \|\mathbf{E}(\psi|\mathcal{F}_n)\|_{L^2(m)} < \infty.$$

Then the positive number  $\sigma$  defined by

$$\sigma^2 := \langle m, \psi^2 \rangle + 2 \sum_{n \geq 1} \langle m, \psi(\psi \circ g^n) \rangle$$

is finite. It vanishes if and only if  $\psi$  is a coboundary. Moreover, when  $\sigma \neq 0$ , then  $\psi$  satisfies the CLT with variance  $\sigma$ .

Note that  $\sigma$  is equal to the limit of  $n^{-1/2} \|\psi + \dots + \psi \circ g^{n-1}\|_{L^2(m)}$ . The last expression is equal to  $\|\psi\|_{L^2(m)}$  if the family  $(\psi \circ g^n)_{n \geq 0}$  is orthogonal in  $L^2(m)$ .

We now prove Corollary 1.4. Since  $\mu$  is mixing, it is ergodic. So, we can apply Gordin's theorem to the map  $f$  on  $(\mathbb{P}^k, \mathcal{B}, \mu)$  where  $\mathcal{B}$  is the canonical Borel algebra.

**Lemma 4.2.** *Let  $\mathcal{B}_n := f^{-n}(\mathcal{B})$  for  $n \geq 0$ . Then for  $\phi \in L^2(\mu)$  we have*

$$\mathbf{E}(\phi|\mathcal{B}_n) = (\Lambda^n \phi) \circ f^n \quad \text{and} \quad \|\mathbf{E}(\phi|\mathcal{B}_n)\|_{L^2(\mu)} = \|\Lambda^n \phi\|_{L^2(\mu)}.$$

*Proof.* Consider a function in  $L^2(\mu)$  which is measurable with respect to  $\mathcal{B}_n$ . It has the form  $\xi \circ f^n$ . We have  $\|\xi \circ f^n\|_{L^2(\mu)} = \|\xi\|_{L^2(\mu)}$  since  $\mu$  is invariant. Hence,  $\xi \in L^2(\mu)$ . We deduce from the identity  $d^{-kn}(f^n)^*\mu = \mu$  that

$$\begin{aligned} \|\mathbf{E}(\phi|\mathcal{B}_n)\|_{L^2(\mu)} &= \sup_{\|\xi\|_{L^2(\mu)}=1} |\langle \mu, (\xi \circ f^n)\phi \rangle| \\ &= \sup_{\|\xi\|_{L^2(\mu)}=1} d^{-kn} |\langle (f^n)^*\mu, (\xi \circ f^n)\phi \rangle| \\ &= \sup_{\|\xi\|_{L^2(\mu)}=1} |\langle \mu, \xi \Lambda^n \phi \rangle| \\ &= \|\Lambda^n \phi\|_{L^2(\mu)}. \end{aligned}$$

The computation also shows that the previous supremum is reached when  $\xi$  is proportional to  $\Lambda^n \phi$ . It follows that  $\mathbf{E}(\phi|\mathcal{B}_n) = (\Lambda^n \phi) \circ f^n$ .  
q.e.d.

*End of the proof of Corollary 1.4.* By Proposition 3.3 and Lemma 4.2, since  $\psi$  is d.s.h., we have  $\|\mathbf{E}(\psi|\mathcal{B}_n)\|_{L^2(\mu)} \lesssim d^{-n}$ . Hence,  $\sum_{n \geq 0} \|\mathbf{E}(\psi|\mathcal{B}_n)\|_{L^2(\mu)}$  converges. It is enough to apply Theorem 4.1 in order to get the result.  
q.e.d.

**Remark 4.3.** If  $\psi$  is an observable in  $L^\infty(\mu)$ , then  $\|\Lambda^n \psi\|_{L^\infty(\mu)} \leq \|\psi\|_{L^\infty(\mu)}$ . Hence, by Lemma 4.2, the Gordin’s condition in Theorem 4.1 is a consequence of the condition  $\sum_{n \geq 1} \|\Lambda^n \psi\|_{L^1(\mu)}^{1/2} < +\infty$ . In particular, Hölder continuous observables satisfy the CLT; see Proposition 3.5 and [10, 11] for meromorphic maps.

The following proposition gives us the next term in the expansion of the  $L^2$ -norm of Birkhoff’s sums.

**Proposition 4.4.** *Let  $\psi$  be a d.s.h. or an  $\nu$ -Hölder continuous function, with  $0 < \nu \leq 2$ , such that  $\langle \mu, \psi \rangle = 0$ . Let  $\sigma \geq 0$  and  $\gamma$  be the constants defined by*

$$\sigma^2 := \langle \mu, \psi^2 \rangle + 2 \sum_{n \geq 1} \langle \mu, \psi(\psi \circ f^n) \rangle \quad \text{and} \quad \gamma := 2 \sum_{n \geq 1} n \langle \mu, \psi(\psi \circ f^n) \rangle.$$

Then

$$\|\psi + \dots + \psi \circ f^{n-1}\|_{L^2(\mu)}^2 - n\sigma^2 + \gamma$$

is of order  $O(d^{-n})$  if  $\psi$  is d.s.h. and  $O(d^{-n\nu/2})$  if  $\psi$  is  $\nu$ -Hölder continuous.

*Proof.* Since  $\mu$  is invariant, we have  $\langle \mu, (\psi \circ f^l)(\psi \circ f^m) \rangle = \langle \mu, \psi(\psi \circ f^{m-l}) \rangle$  for  $m \geq l$ . It follows that

$$\begin{aligned} \|\psi + \dots + \psi \circ f^{n-1}\|_{L^2(\mu)}^2 &= \sum_{0 \leq l, m \leq n-1} \langle \mu, (\psi \circ f^l)(\psi \circ f^m) \rangle \\ &= n \langle \mu, \psi^2 \rangle + \sum_{1 \leq j \leq n-1} 2(n-j) \langle \mu, \psi(\psi \circ f^j) \rangle \\ &= n\sigma^2 - \gamma + \sum_{j \geq n} 2(j-n) \langle \mu, \psi(\psi \circ f^j) \rangle. \end{aligned}$$

Theorem 3.1 implies the result. Note that this theorem also implies that the series in the definition for  $\sigma^2$  and for  $\gamma$  are convergent. Moreover, the previous computation gives that  $\sigma^2$  is the limit of  $n^{-1} \|\psi + \dots + \psi \circ f^{n-1}\|_{L^2(\mu)}^2$ , which is a positive number. q.e.d.

### 5. Large deviations theorem

In this section, we prove the large deviations theorem (LDT for short) for the equilibrium measure of holomorphic endomorphisms of  $\mathbb{P}^k$ . We have the following result, which holds in particular for  $\mathcal{C}^2$  observables.

**Theorem 5.1.** *Let  $f$  be a holomorphic endomorphism of  $\mathbb{P}^k$  of algebraic degree  $d \geq 2$ . Then the equilibrium measure  $\mu$  of  $f$  satisfies the large deviations theorem (LDT) for bounded d.s.h. observables. More precisely, if  $\psi$  is a bounded d.s.h. function, then for every  $\epsilon > 0$  there is*

a constant  $h_\epsilon > 0$  such that

$$\mu\left\{z \in \mathbb{P}^k : \left|\frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j(z) - \langle \mu, \psi \rangle\right| > \epsilon\right\} \leq e^{-n(\log n)^{-2}h_\epsilon}$$

for all  $n$  large enough.

Note that in the classical large deviations theorem for independent random variables, there is no factor  $(\log n)^{-2}$  in the previous estimate.

We start with the following Bennett’s type inequality (see [4, Lemma 2.4.1]).

**Lemma 5.2.** *Let  $(M, \mathcal{F}, m)$  be a probability space and  $\mathcal{G}$  a  $\sigma$ -subalgebra of  $\mathcal{F}$ . Assume that there is a constant  $0 < \nu < 1$  and an element  $A \in \mathcal{F}$  such that  $m(A \cap B) = \nu m(B)$  for every  $B \in \mathcal{G}$ . Define  $s^- := \max\{1, \nu^{-1}(1 - \nu)\}$  and  $s^+ := \max\{1, \nu(1 - \nu)^{-1}\}$ . Let  $\psi$  be a real-valued function on  $M$  such that  $\|\psi\|_{L^\infty(m)} \leq b$  and  $\mathbf{E}(\psi|\mathcal{G}) = 0$ . Then*

$$\mathbf{E}(e^{\lambda\psi}|\mathcal{G}) \leq \nu e^{-s^-\lambda b} + (1 - \nu)e^{s^+\lambda b}$$

for every  $\lambda \geq 0$ .

*Proof.* Fix a strictly positive constant  $\lambda$ . Let  $\psi_0$  be the function which is equal to  $t^- := -s^-\lambda b$  on  $A$  and to  $t^+ := s^+\lambda b$  on  $M \setminus A$ . We have  $\psi_0^2 \geq (\lambda b)^2 \geq (\lambda\psi)^2$ . We deduce from the hypothesis on  $A$  and the relation  $-\nu s^- + (1 - \nu)s^+ = 0$  that  $\mathbf{E}(\psi_0|\mathcal{G}) = 0$ . Let  $g(t) = a_0 t^2 + a_1 t + a_2$  be the unique quadratic function such that  $h(t) := g(t) - e^t$  satisfies  $h(t^+) = 0$  and  $h(t^-) = h'(t^-) = 0$ . We have  $g(\psi_0) = e^{\psi_0}$ .

Since  $h''(t) = 2a_0 - e^t$  admits at most one zero,  $h'$  admits at most two zeros. The fact that  $h(t^-) = h(t^+) = 0$  implies that  $h'$  vanishes in  $]t^-, t^+[$ . Hence  $h'$  admits exactly one zero at  $t^-$  and another in  $]t^-, t^+[$ . We deduce that  $h''$  admits a zero. This implies that  $a_0 > 0$ . Moreover,  $h$  vanishes only at  $t^-, t^+$  and  $h'(t^+) \neq 0$ . It follows that  $h(t) \geq 0$  on  $[t^-, t^+]$  because  $h$  is negative near  $+\infty$ . Thus,  $e^t \leq g(t)$  on  $[t^-, t^+]$  and then  $\mathbf{E}(e^{\lambda\psi}|\mathcal{G}) \leq \mathbf{E}(g(\lambda\psi)|\mathcal{G})$ .

Since  $a_0 > 0$ , if an observable  $\phi$  satisfies  $\mathbf{E}(\phi|\mathcal{G}) = 0$ , then  $\mathbf{E}(g(\phi)|\mathcal{G})$  is an increasing function of  $\mathbf{E}(\phi^2|\mathcal{G})$ . Now, using the properties of  $\psi$  and  $\psi_0$ , we obtain

$$\begin{aligned} \mathbf{E}(e^{\lambda\psi}|\mathcal{G}) &\leq \mathbf{E}(g(\lambda\psi)|\mathcal{G}) \leq \mathbf{E}(g(\psi_0)|\mathcal{G}) \\ &= \mathbf{E}(e^{\psi_0}|\mathcal{G}) \leq \nu e^{-s^-\lambda b} + (1 - \nu)e^{s^+\lambda b}, \end{aligned}$$

which completes the proof.

q.e.d.

We continue the proof of Theorem 5.1. Without loss of generality we may assume that  $\langle \mu, \psi \rangle = 0$ ,  $|\psi| \leq 1$  and  $\|\psi\|_{\text{DSH}} \leq 1$ . The general idea is to write  $\psi = \psi' + (\psi'' - \psi'' \circ f)$  for functions  $\psi'$  and  $\psi''$  in  $\text{DSH}_0(\mathbb{P}^k)$  such that

$$\mathbf{E}(\psi' \circ f^n | \mathcal{B}_{n+1}) = 0, \quad n \geq 0,$$

where  $\mathcal{B}_n := f^{-n}(\mathcal{B})$  as above with  $\mathcal{B}$  the canonical Borel algebra of  $\mathbb{P}^k$ . In the language of probability theory, these identities mean that  $(\psi' \circ f^n)_{n \geq 0}$  is a *reversed martingale difference* as in Gordin's approach (see also [34]). The strategy is to prove the LDT for  $\psi'$  and for the coboundary  $\psi'' - \psi'' \circ f$ . Theorem 5.1 is in fact a consequence of Lemmas 5.4 and 5.8 below.

Define

$$\psi'' := - \sum_{n=1}^{\infty} \Lambda^n \psi, \quad \psi' := \psi - (\psi'' - \psi'' \circ f).$$

Using the estimate  $\|\Lambda\phi\|'_{\text{DSH}} \leq d^{-1}\|\phi\|'_{\text{DSH}}$  on  $\text{DSH}_0(\mathbb{P}^k)$ , we see that  $\psi'$  and  $\psi''$  are in  $\text{DSH}_0(\mathbb{P}^k)$  with norms bounded by some constant. In particular, they belong to  $L^2(\mu)$ . However, we lose the boundedness: these functions are not necessarily in  $L^\infty(\mu)$ .

**Lemma 5.3.** *We have  $\Lambda^n \psi' = 0$  for  $n \geq 1$  and  $\mathbf{E}(\psi' \circ f^n | \mathcal{B}_m) = 0$  for  $m > n \geq 0$ .*

*Proof.* We deduce from the definition of  $\psi''$  that

$$\Lambda\psi' = \Lambda\psi - \Lambda\psi'' + \Lambda(\psi'' \circ f) = \Lambda\psi - \Lambda\psi'' + \psi'' = 0.$$

It follows that  $\Lambda^n \psi' = 0$  for  $n \geq 1$ . For every function  $\phi$  in  $L^2(\mu)$ , since  $\mu$  is invariant, we have

$$\langle \mu, (\psi' \circ f^n)(\phi \circ f^m) \rangle = \langle \mu, \psi'(\phi \circ f^{m-n}) \rangle = \langle \mu, (\Lambda^{m-n} \psi')\phi \rangle = 0,$$

which completes the proof. q.e.d.

Given a function  $h \in L^1(\mu)$ , define the *Birkhoff's sum*  $\mathbf{S}_n h$  by

$$\mathbf{S}_0 h := 0 \quad \text{and} \quad \mathbf{S}_n h := \sum_{j=0}^{n-1} h \circ f^j \quad \text{for } n \geq 1.$$

**Lemma 5.4.** *The coboundary  $\psi'' - \psi'' \circ f$  satisfies the LDT.*

*Proof.* By Proposition 3.3, up to multiplying  $\psi$  by a constant, we can assume that  $\langle \mu, e^{|\psi''|} \rangle \leq c$  for some constant  $c > 0$ . Observe that  $\mathbf{S}_n(\psi'' - \psi'' \circ f) = \psi'' - \psi'' \circ f^n$ . Consequently, for a given  $\epsilon > 0$ , we have, using the invariance of  $\mu$ ,

$$\begin{aligned} \mu\{|\mathbf{S}_n(\psi'' - \psi'' \circ f)| > n\epsilon\} &\leq \mu\left\{|\psi'' \circ f^n| > \frac{n\epsilon}{2}\right\} + \mu\left\{|\psi''| > \frac{n\epsilon}{2}\right\} \\ &= 2\mu\left\{|\psi''| > \frac{n\epsilon}{2}\right\} = 2\mu\left\{e^{|\psi''|} > e^{\frac{n\epsilon}{2}}\right\} \\ &\leq 2e^{-\frac{n\epsilon}{2}} \langle \mu, e^{|\psi''|} \rangle \leq 2ce^{-\frac{n\epsilon}{2}}. \end{aligned}$$

Hence,  $\psi'' \circ f - \psi''$  satisfies the LDT. q.e.d.

It remains to show that  $\psi'$  satisfies the LDT. Fix a number  $\delta$  such that  $1 < \delta^5 < d$ . We will use the following lemma for a positive constant  $b$  of order  $O(\log n)$ .



**Lemma 5.5.** *There are constants  $c > 0$  and  $\alpha > 0$  such that for every  $b \geq 1$  we have*

$$\mu\{|\psi'| > b\} \leq ce^{-\alpha\delta^b}.$$

*Proof.* Define  $\varphi := \sum_{n \geq 1} \delta^{5n} |\Lambda^n \psi|$ . Since  $\|\Lambda\phi\|'_{\text{DSH}} \leq d^{-1} \|\phi\|'_{\text{DSH}}$  on  $\text{DSH}_0(\mathbb{P}^k)$ ,  $\varphi$  is in  $\text{DSH}_0(\mathbb{P}^k)$  and has a bounded DSH-norm. So,  $\langle \mu, e^{\alpha\varphi} \rangle \leq c'$  for some constants  $c' > 0$  and  $\alpha > 0$ . It is enough to consider the case where  $b = 5l$  for some integer  $l$ . Since  $|\psi| \leq 1$ , we also have  $|\Lambda^n \psi| \leq 1$ . Hence

$$|\psi''| \leq \sum_{n \geq 1} |\Lambda^n \psi| \leq \delta^{-5l} \sum_{n \geq 1} \delta^{5n} |\Lambda^n \psi| + \sum_{1 \leq n \leq l} |\Lambda^n \psi| \leq \delta^{-5l} \varphi + l.$$

Consequently, we deduce that

$$\mu\{|\psi''| > 2l\} \leq \mu\{\varphi > \delta^{5l}\} \leq e^{-\alpha\delta^{5l}} \langle \mu, e^{\alpha\varphi} \rangle \leq c' e^{-\alpha\delta^{5l}}.$$

Therefore, by definition of  $\psi'$ , since  $|\psi| \leq 1 \leq l$  and  $\mu$  is invariant, we obtain

$$\begin{aligned} \mu\{|\psi'| > 5l\} &\leq \mu\{|\psi''| > 2l\} + \mu\{|\psi'' \circ f| > 2l\} \\ &= 2\mu\{|\psi''| > 2l\} \leq 2c' e^{-\alpha\delta^{5l}}. \end{aligned}$$

This implies the lemma. q.e.d.

In order to apply Lemma 5.2, we will need the following property.

**Lemma 5.6.** *There is a Borel set  $A$  such that  $\mu(A \cap B) = (1 - d^{-1})\mu(B)$  for every  $B$  in  $\mathcal{B}_1$ .*

*Proof.* Recall that  $f$  defines a ramified covering of degree  $d^k$ . Since  $\mu$  has no mass on analytic sets, it does not charge the critical values of  $f$ . So, there is a Borel set  $Z$  of total  $\mu$  measure such that  $f^{-1}(Z)$  is the union of  $d^k$  disjoint Borel sets  $Z_i$ ,  $1 \leq i \leq d^k$ . Moreover, one can choose  $Z_i$  so that  $f : Z_i \rightarrow Z$  is bijective. Since  $\mu$  is totally invariant, we have  $\mu(Z_i) = d^{-k}$  for every  $i$ . Define  $A := \cup_{i > d^{k-1}} Z_i$ .

Since  $B$  is an element of  $\mathcal{B}_1$ , we have  $B = f^{-1}(B')$  with  $B' := f(B)$ . We also have  $\mu(Z_i \cap f^{-1}(B')) = d^{-k} \mu(B') = d^{-k} \mu(B)$ . Therefore,

$$\mu(A \cap B) = \sum_{i > d^{k-1}} \mu(Z_i \cap f^{-1}(B')) = \sum_{i > d^{k-1}} d^{-k} \mu(B) = (1 - d^{-1})\mu(B).$$

This gives the lemma. q.e.d.

**Lemma 5.7.** *For every  $b \geq 1$ , there are Borel sets  $W_n$  such that  $\mu(W_n) \leq cne^{-\alpha\delta^b}$  and*

$$\int_{\mathbb{P}^k \setminus W_n} e^{\lambda S_n \psi'} d\mu \leq d \left[ \frac{(d-1)e^{-\lambda b} + e^{(d-1)\lambda b}}{d} \right]^n,$$

where  $c > 0$  and  $\alpha > 0$  are constants independent of  $b$ .

*Proof.* For  $n = 1$ , define  $W := \{|\psi'| > b\}$ ,  $W' := f(W)$ , and  $W_1 := f^{-1}(W')$ . Since  $\mu$  is totally invariant and  $f$  has topological degree  $d^k$ , we have  $\mu(f(W)) \leq d^k \mu(W)$ . This and Lemma 5.5 imply that

$$\mu(W_1) = \mu(W') \leq d^k \mu(W) \leq ce^{-\alpha \delta^b}$$

for some constant  $c > 0$ . We also have

$$\int_{\mathbb{P}^k \setminus W_1} e^{\lambda S_1 \psi'} d\mu = \int_{\mathbb{P}^k \setminus W_1} e^{\lambda \psi'} d\mu \leq e^{\lambda b} \leq d \left[ \frac{(d-1)e^{-\lambda b} + e^{\lambda b}}{d} \right].$$

So, the lemma holds for  $n = 1$ .

Suppose the lemma for  $n \geq 1$ ; we need to prove it for  $n + 1$ . Define  $W_{n+1} := f^{-1}(W_n) \cup W_1 = f^{-1}(W_n \cup W')$ . We have

$$\mu(W_{n+1}) \leq \mu(f^{-1}(W_n)) + \mu(W_1) = \mu(W_n) + \mu(W_1) \leq c(n+1)e^{-\alpha \delta^b}.$$

We will apply Lemma 5.2 to  $M := \mathbb{P}^k$ ,  $\mathcal{F} := \mathcal{B}$ ,  $\mathcal{G} := \mathcal{B}_1 = f^{-1}(\mathcal{B})$ ,  $m := \mu$ ,  $\nu := 1 - d^{-1}$  (see Lemma 5.6), and to the function  $\psi^*$  such that  $\psi^* = \psi'$  on  $\mathbb{P}^k \setminus W_1$  and  $\psi^* = 0$  on  $W_1$ . By Lemma 5.3, we have  $\mathbf{E}(\psi^* | \mathcal{G}) = \mathbf{E}(\psi^* | \mathcal{B}_1) = 0$  since  $W_1$  is an element of  $\mathcal{B}_1$ .

Observe that  $|\psi'| \leq b$  on  $\mathbb{P}^k \setminus W_1$ . Hence,  $|\psi^*| \leq b$ . By Lemma 5.2, we have

$$\mathbf{E}(e^{\lambda \psi^*} | \mathcal{B}_1) \leq \frac{(d-1)e^{-\lambda b} + e^{(d-1)\lambda b}}{d} \quad \text{on } \mathbb{P}^k \quad \text{for } \lambda \geq 0.$$

It follows that

$$\mathbf{E}(e^{\lambda \psi'} | \mathcal{B}_1) \leq \frac{(d-1)e^{-\lambda b} + e^{(d-1)\lambda b}}{d} \quad \text{on } \mathbb{P}^k \setminus W_1 \quad \text{for } \lambda \geq 0.$$

Now, using the fact that  $W_{n+1}$  and  $e^{\lambda S_n(\psi' \circ f)}$  are  $\mathcal{B}_1$ -measurable, we can write

$$\begin{aligned} \int_{\mathbb{P}^k \setminus W_{n+1}} e^{\lambda S_{n+1} \psi'} d\mu &= \int_{\mathbb{P}^k \setminus W_{n+1}} e^{\lambda \psi'} e^{\lambda S_n(\psi' \circ f)} d\mu \\ &= \int_{\mathbb{P}^k \setminus W_{n+1}} \mathbf{E}(e^{\lambda \psi'} | \mathcal{B}_1) e^{\lambda S_n(\psi' \circ f)} d\mu. \end{aligned}$$

Since  $W_{n+1} = f^{-1}(W_n) \cup W_1$ , the last integral is bounded by

$$\begin{aligned} &\sup_{\mathbb{P}^k \setminus W_1} \mathbf{E}(e^{\lambda \psi'} | \mathcal{B}_1) \int_{\mathbb{P}^k \setminus f^{-1}(W_n)} e^{\lambda S_n(\psi' \circ f)} d\mu \\ &\leq \left[ \frac{(d-1)e^{-\lambda b} + e^{(d-1)\lambda b}}{d} \right] \int_{\mathbb{P}^k \setminus W_n} e^{\lambda S_n \psi'} d\mu \\ &\leq d \left[ \frac{(d-1)e^{-\lambda b} + e^{(d-1)\lambda b}}{d} \right]^{n+1}, \end{aligned}$$

where the last inequality follows from the hypothesis of induction for  $n$ . So, the lemma holds for  $n + 1$ . q.e.d.

The following lemma, together with Lemma 5.4, implies Theorem 5.1.

**Lemma 5.8.** *The function  $\psi'$  satisfies the LDT.*

*Proof.* Fix an  $\epsilon > 0$ . By Lemma 5.7, we have, for every  $\lambda \geq 0$ ,

$$\begin{aligned} \mu\{|\mathcal{S}_n\psi'| \geq n\epsilon\} &\leq \mu(W_n) + e^{-\lambda n\epsilon} \int_{\mathbb{P}^k \setminus W_n} e^{\lambda \mathcal{S}_n\psi'} d\mu \\ &\leq cne^{-\alpha\delta^b} + de^{-\lambda n\epsilon} \left[ \frac{(d-1)e^{-\lambda b} + e^{(d-1)\lambda b}}{d} \right]^n. \end{aligned}$$

Take  $b := \log n(\log \delta)^{-1}$  and  $\lambda := u\epsilon b^{-2}$  with a fixed  $u > 0$  small enough. We have

$$cne^{-\alpha\delta^b} = cne^{-\alpha n} \leq e^{-\alpha n/2}$$

for  $n$  large. Since  $u$  is small,  $\lambda b$  is small. It follows that

$$\begin{aligned} &\frac{(d-1)e^{-\lambda b} + e^{(d-1)\lambda b}}{d} \\ &\leq \frac{(d-1)(1 - \lambda b + \lambda^2 b^2) + (1 + (d-1)\lambda b + (d-1)^2 \lambda^2 b^2)}{d} \\ &\leq 1 + d^2 \lambda^2 b^2 \leq e^{d^2 \lambda^2 b^2} = e^{d^2 u^2 \epsilon^2 b^{-2}}. \end{aligned}$$

Therefore

$$de^{-\lambda n\epsilon} \left[ \frac{(d-1)e^{-\lambda b} + e^{(d-1)\lambda b}}{d} \right]^n \leq de^{-nu\epsilon^2 b^{-2}(1-d^2u)} = de^{-n(\log n)^{-2}h_\epsilon}$$

for some constant  $h_\epsilon > 0$ . We deduce from the previous estimates that

$$\mu\{|\mathcal{S}_n\psi'| \geq n\epsilon\} \leq e^{-n(\log n)^{-2}h_\epsilon}$$

for some constant  $h_\epsilon > 0$  and for  $n$  large. So,  $\psi'$  satisfies the LDT. q.e.d.

Now, using Proposition 3.5 we can prove the LDT for Hölder continuous observables.

**Theorem 5.9.** *Let  $f$  be a holomorphic endomorphism of  $\mathbb{P}^k$  of algebraic degree  $d \geq 2$ . Then the equilibrium measure  $\mu$  of  $f$  satisfies the large deviations theorem for Hölder continuous observables. More precisely, if  $\psi$  is a Hölder continuous function, then for every  $\epsilon > 0$  there is a constant  $h_\epsilon > 0$  such that*

$$\mu\left\{z \in \mathbb{P}^k : \left| \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j(z) - \langle \mu, \psi \rangle \right| > \epsilon\right\} \leq e^{-n(\log n)^{-2}h_\epsilon}$$

for all  $n$  large enough.

The proof follows along the same lines as Theorem 5.1. Fix a  $\nu$ -Hölder continuous function  $\psi$  with  $0 < \nu \leq 2$  and a constant  $1 < \delta < d^{\nu/10}$ . We define as above the functions  $\psi', \psi''$  and  $\varphi := \sum_{n \geq 0} \delta^{5n} |\Lambda^n \psi|$ . We only have to check that  $\langle \mu, e^{\alpha\varphi} \rangle \leq c$  for some constants  $\alpha > 0$  and  $c > 0$ . In fact, this implies the inequality  $\langle \mu, e^{\alpha|\psi''|} \rangle \leq c$  and the crucial estimate in Lemma 5.5. We deduce the estimate  $\langle \mu, e^{\alpha\varphi} \rangle \leq c$  from Proposition 3.5 and the following lemma for  $\theta := \delta^5 d^{-\nu/2}$  and  $\eta_n := \alpha d^{n\nu/2} |\Lambda^n \psi|$ .

**Lemma 5.10.** *Let  $\eta_n$  be positive measurable functions and  $0 < \theta < 1$  be a constant. Assume there is a constant  $c > 0$  such that  $\langle \mu, e^{\eta_n} \rangle \leq c$  for every  $n \geq 0$ . If  $\xi := (1 - \theta) \sum_{n \geq 0} \theta^n \eta_n$ , then  $e^\xi$  is  $\mu$ -integrable.*

*Proof.* Define  $\xi_m := (1 - \theta) \sum_{n \geq m} \theta^{n-m} \eta_n$ . We have  $\xi_0 = \xi$  and  $\xi_m = (1 - \theta)\eta_m + \theta\xi_{m+1}$ . The Hölder inequality implies that  $\langle \mu, e^{\xi_m} \rangle = \langle \mu, e^{(1-\theta)\eta_m} e^{\theta\xi_{m+1}} \rangle \leq \langle \mu, e^{\eta_m} \rangle^{1-\theta} \langle \mu, e^{\xi_{m+1}} \rangle^\theta \leq c^{1-\theta} \langle \mu, e^{\xi_{m+1}} \rangle^\theta$ . By induction, this implies that

$$\langle \mu, e^{\xi_0} \rangle \leq c^{(1-\theta)(1+\theta+\theta^2+\dots)},$$

which implies the result. q.e.d.

**Remark 5.11.** In the main estimate of Theorem 5.9, we can remove the factor  $(\log n)^{-2}$  if  $\|\Lambda^n \psi\|_{L^\infty(\mu)}$  tends to 0 exponentially fast when  $n \rightarrow \infty$ . This is the case in dimension 1 (see Drasin-Okuyama [14]; and when  $f$  is a **generic** map in higher dimension, see [13]). LDT was recently proved for Lipschitz observables in dimension 1 by Xia-Fu [36]. It seems there is a slip in their paper: they state the main result for Hölder continuous observables.

## 6. Abstract version of large deviations theorem

In this section, we give a version of the large deviations theorem in an abstract setting. Let  $(M, \mathcal{F}, m)$  be a probability space and  $f : M \rightarrow M$  a measurable map which preserves  $m$ , i.e.,  $f_*(m) = m$ . Define  $\mathcal{F}_1 := f^{-1}(\mathcal{F})$ . We say that  $f$  has *bounded jacobian* if there is a constant  $\kappa > 0$  such that  $m(f(A)) \leq \kappa m(A)$  for every  $A \in \mathcal{F}$ . Observe that  $f^*$  defines a linear operator of norm 1 from  $L^2(m)$  into itself.

**Theorem 6.1.** *Let  $f : (M, \mathcal{F}, m) \rightarrow (M, \mathcal{F}, m)$  be a map with bounded jacobian which preserves  $m$  as above. Let  $\Lambda$  denote the adjoint of  $f^*$  on  $L^2(m)$ . Let  $\psi$  be a bounded real-valued measurable function. Assume there are constants  $\delta > 1$  and  $c > 0$  such that*

$$\langle m, e^{\delta^n |\Lambda^n \psi - \langle m, \psi \rangle|} \rangle \leq c \quad \text{for every } n \geq 0.$$

*Then  $\psi$  satisfies the large deviations theorem, that is, for every  $\epsilon > 0$ , there exists a constant  $h_\epsilon > 0$  such that*

$$m \left\{ z \in M : \left| \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j(z) - \langle m, \psi \rangle \right| > \epsilon \right\} \leq e^{-n(\log n)^{-2} h_\epsilon}$$

*for all  $n$  large enough.*

The proof follows the same steps as in Section 5. The details are left to the reader. We only note two important points. The property that  $f$  is of bounded jacobian allows us to prove an analog of Lemma 5.7. Indeed, in the proof of that lemma, the inequality  $\mu(W') \leq d^k \mu(W)$

should be replaced by  $m(W') \leq \kappa m(W)$ . The following version of the Bennett’s inequality replaces Lemma 5.2.

**Lemma 6.2.** *Let  $\psi$  be a real-valued function on  $M$  such that  $\|\psi\|_{L^\infty(m)} \leq b$  and  $\mathbf{E}(\psi|\mathcal{F}_1) = 0$ . Then*

$$\mathbf{E}(e^{\lambda\psi}|\mathcal{F}_1) \leq \frac{e^{-\lambda b} + e^{\lambda b}}{2}$$

for every  $\lambda \geq 0$ .

*Proof.* We decompose the measure  $m$  using the fibers of  $f$ . For  $m$ -almost every  $x \in M$ , there is a positive measure  $m_x$  on  $M_x := f^{-1}(x)$  such that if  $\varphi$  is a function in  $L^1(m)$ , then

$$\langle m, \varphi \rangle = \int_M \langle m_x, \varphi \rangle dm(x).$$

Since  $m$  is invariant by  $f$ , we have

$$\langle m, \varphi \rangle = \langle m, \varphi \circ f \rangle = \int_M \langle m_x, \varphi \circ f \rangle dm(x) = \int_M \|m_x\| \varphi(x) dm(x).$$

Therefore,  $m_x$  is a probability measure for  $m$ -almost every  $x$ . Using also the invariance of  $m$ , we obtain for  $\varphi$  and  $\phi$  in  $L^2(m)$  that

$$\begin{aligned} \langle m, \varphi(\phi \circ f) \rangle &= \int_M \langle m_x, \varphi(\phi \circ f) \rangle dm(x) = \int_M \langle m_x, \varphi \rangle \phi(x) dm(x) \\ &= \int_M \langle m_{f(x)}, \varphi \rangle \phi(f(x)) dm(x). \end{aligned}$$

We deduce that

$$\mathbf{E}(\varphi|\mathcal{F}_1)(x) = \langle m_{f(x)}, \varphi \rangle.$$

By hypothesis, we have  $\langle m_x, \psi \rangle = 0$  for  $m$ -almost every  $x$ . It suffices to check that

$$\langle m_x, e^{\lambda\psi} \rangle \leq \frac{e^{-\lambda b} + e^{\lambda b}}{2}.$$

Consider first the particular case where there is an element  $A \subset \mathcal{F}$  such that  $A \subset M_x$  and  $m_x(A) = 1/2$ . Applying Lemma 5.2 to  $M_x := f^{-1}(x)$ ,  $m_x$ ,  $A$ ,  $\nu := 1/2$  and for  $\mathcal{G} := \{\emptyset, M_x\}$  the trivial  $\sigma$ -algebra of  $M_x$  yields the result. The general case is deduced from the previous particular case. Indeed, it is enough to apply this case to the disjoint union of  $(M, \mathcal{F}, m)$  with a copy  $(M', \mathcal{F}', m')$  of this space, i.e., to the space  $(M \cup M', \mathcal{F} \cup \mathcal{F}', \frac{m}{2} + \frac{m'}{2})$  and to the function equal to  $\psi$  on  $M$  and on  $M'$ . q.e.d.

### References

[1] E. Bedford & B.A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math., **149** (1982), no. 1-2, 1–40, MR 0674165, Zbl 0547.32012.  
 [2] B. Berndtsson & N. Sibony, *The  $\bar{\partial}$ -equation on a positive current*, Invent. Math., **147** (2002), no. 2, 371–428, MR 1881924, Zbl 1031.32005.

- [3] C. Bonatti, L.J. Díaz & M. Viana, *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective*, Encyclopaedia of Mathematical Sciences, **102**, Mathematical Physics, III. Springer-Verlag, Berlin, 2005, MR 2105774, Zbl 1060.37020.
- [4] A. Dembo & O. Zeitouni, *Large deviations techniques and applications*, second edition, Springer-Verlag, 1998, MR 1619036, Zbl 1177.60035.
- [5] J.-P. Demailly, *Complex analytic geometry*, available at [www.fourier.ujf-grenoble.fr/~demailly](http://www.fourier.ujf-grenoble.fr/~demailly).
- [6] M. Denker, F. Przytycki & M. Urbański, *On the transfer operator for rational functions on the Riemann sphere*, Ergodic Theory Dynam. Systems, **16** (1996), no. 2, 255–266, MR 1389624, Zbl 0852.46024.
- [7] T.-C. Dinh, V.-A. Nguyễn & N. Sibony, *On thermodynamics of rational maps on the Riemann sphere*, Ergodic Theory Dynam. Systems, **27** (2007), no. 4, 1095–1109, MR 2342967.
- [8] T.-C. Dinh & N. Sibony, *Dynamique des applications d'allure polynomiale*, J. Math. Pures Appl., **82** (2003), 367–423, MR 1992375, Zbl 1033.37023.
- [9] ———, *Dynamics of regular birational maps in  $\mathbb{P}^k$* , J. Funct. Anal., **222** (2005), no. 1, 202–216, MR 2129771, Zbl 1067.37055.
- [10] ———, *Distribution des valeurs de transformations méromorphes et applications*, Comment. Math. Helv., **81** (2006), no. 1, 221–258, MR 2208805, Zbl 1094.32005.
- [11] ———, *Decay of correlations and the central limit theorem for meromorphic maps*, Comm. Pure Appl. Math., **59** (2006), no. 5, 754–768, MR 2172806, Zbl 1137.37023.
- [12] ———, *Pull-back of currents by holomorphic maps*, Manuscripta Math., **123** (2007), no. 3, 357–371, MR 2314090, Zbl 1128.32020.
- [13] ———, *Super-potentials of positive closed currents, intersection theory and dynamics*, Acta Math., **203** (2009), no. 1, 1–82, MR 2545825.
- [14] D. Drasin & Y. Okuyama, *Equidistribution and Nevanlinna theory*, Bull. Lond. Math. Soc., **39** (2007), no. 4, 603–613, MR 2346941, Zbl 1123.37018.
- [15] C. Dupont, *Bernoulli coding map and singular almost-sure invariance principle for endomorphisms of  $\mathbb{P}^k$* , Probab. Theory Related Fields **146** (2010), no. 3-4, 337–359.
- [16] ———, *Formule de Pesin et applications méromorphes*, Bull. Braz. Math. Soc. (N.S.), **37** (2006), no. 3, 393–418, MR 2267785, Zbl 1130.37340.
- [17] J.-E. Fornæss & N. Sibony, *Complex dynamics in higher dimension*, in *Complex potential theory*, (Montréal, PQ, 1993), Nato ASI series Math. and Phys. Sci., vol. **C439**, Kluwer (1994), 131–186, MR 1332961.
- [18] ———, *Oka's inequality for currents and applications*, Math. Ann., **301** (1995), no. 3, 399–419, MR 1324517, Zbl 0832.32010.
- [19] M.I. Gordin, *The Central Limit Theorem for stationary processes*, Soviet. Math. Dokl., **10** n.5 (1969), 1174–1176, MR 0251785, Zbl 0212.50005.
- [20] P. Hall & C.C. Heyde, *Martingale limit theory and its application*, Probability and Mathematical Statistics, Academic Press, New York-London, 1980, MR 0624435, Zbl 0462.60045.
- [21] N. Haydn, *Convergence of the transfer operator for rational maps*, Ergodic Theory Dynam. Systems, **19** (1999), no. 3, 657–669, MR 1695914, Zbl 0953.37006.

- [22] L. Hörmander, *An introduction to complex analysis in several variables*, third edition, North-Holland Mathematical Library, **7**, North-Holland Publishing Co., Amsterdam, 1990, MR 1045639, Zbl 0685.32001.
- [23] A. Katok & B. Hasselblatt, *Introduction to the modern theory of dynamical systems. Encyclopedia of Mathematics and its Applications*, **54**, Cambridge University Press, Cambridge, 1995, MR 1326374, Zbl 0878.58020.
- [24] C. Liverani, *Central limit theorem for deterministic systems*, International Conference on Dynamical Systems (Montevideo, 1995), 56–75, Pitman Res. Notes Math. Ser., **362**, Longman, Harlow, 1996, MR 1460797, Zbl 0871.58055.
- [25] N. Makarov & S. Smirnov, *On “thermodynamics” of rational maps, I, Negative spectrum*, Comm. Math. Phys., **211** (2000), no. 3, 705–743, MR 1773815, Zbl 0983.37033.
- [26] M. Méo, *Image inverse d’un courant positif fermé par une application analytique surjective*, C. R. Acad. Sci. Paris Sér. I Math., **322** (1996), no. 12, 1141–1144, MR 1396655, Zbl 0858.32012.
- [27] F. Przytycki & J. Rivera-Letelier, *Statistical properties of topological Collet-Eckmann maps*, Ann. Sci. École Norm. Sup. (4), **40** (2007), no. 1, 135–178, MR 2332354, Zbl 1115.37048.
- [28] N. Sibony, *Dynamique des applications rationnelles de  $\mathbb{P}^k$* , Panoramas et Synthèses, **8** (1999), 97–185, MR 1760844, Zbl 1020.37026.
- [29] H. Skoda, *Sous-ensembles analytiques d’ordre fini ou infini dans  $\mathbb{C}^n$* , Bull. Soc. Math. France, **100** (1972), 353–408, MR 0352517, Zbl 0246.32009.
- [30] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with  $C_1(M) > 0$* , Invent. Math., **89** (1987), no. 2, 225–246, MR 0894378, Zbl 0599.53046.
- [31] G. Tian & S.-T. Yau, *Kähler-Einstein metrics on complex surfaces with  $C_1 > 0$* , Comm. Math. Phys., **112** (1987), no. 1, 175–203, MR 0904143, Zbl 0631.53052.
- [32] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure Appl. Math., **31** (1978), no. 3, 339–411, MR 0480350, Zbl 0369.53059.
- [33] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, 1978, MR 0503903, Zbl 0387.46032.
- [34] M. Viana, *Stochastic dynamics of deterministic systems*, volume **21**, IMPA, 1997.
- [35] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, **79**, Springer-Verlag, New York-Berlin, 1982, MR 0648108, Zbl 0958.28011.
- [36] H. Xia & X. Fu, *Remarks on large deviation for rational maps on the Riemann sphere*, Stochastics and Dynamics, **7**, No. 3 (2007), 357–363, MR 2351042, Zbl 1143.37037.

UPMC UNIV PARIS 06, UMR 7586  
 INSTITUT DE MATHÉMATIQUES DE JUSSIEU  
 4 PLACE JUSSIEU  
 F-75005 PARIS, FRANCE

*E-mail address:* dinh@math.jussieu.fr, www.math.jussieu.fr/~dinh

UNIVERSITÉ PARIS-SUD  
MATHÉMATIQUE—BÂTIMENT 425  
91405 ORSAY, FRANCE

*E-mail address:* VietAnh.Nguyen@math.u-psud.fr

UNIVERSITÉ PARIS-SUD  
MATHÉMATIQUE—BÂTIMENT 425  
91405 ORSAY, FRANCE

*E-mail address:* nessim.sibony@math.u-psud.fr