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ON THE CONTACT CLASS IN HEEGAARD FLOER HOMOLOGY

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Abstract

We present an alternate description of the Ozsváth-Szabó contact class in Heegaard Floer homology. Using our contact class, we prove that if a contact structure (M, ξ) has an adapted open book decomposition whose page S is a once-punctured torus, then the monodromy is right-veering if and only if the contact structure is tight.

1. Introduction

In the paper [**OS5**], Ozsváth and Szabó defined an invariant of a contact 3-manifold (M, ξ) which lives in the Heegaard Floer homology $\widehat{HF}(-M)$ of the manifold M with reversed orientation. It is defined via the work of Giroux [**Gi2**], who showed that there is a 1-1 correspondence between isomorphism classes of open book decompositions modulo positive stabilization and isomorphism classes of contact structures on closed 3-manifolds. Ozsváth and Szabó associated an element in Heegaard Floer homology to an open book decomposition and showed that its homology class is independent of the choice of the open book compatible with the given contact structure. They also showed that this invariant $c(\xi)$ is zero if the contact structure is overtwisted, and that it is nonzero if the contact structure is symplectically fillable. The *contact class* $c(\xi)$ has proven to be extremely powerful at (i) proving the tightness of various contact structures and (ii) distinguishing tight contact structures, especially in the hands of Lisca-Stipsicz [**LS1**, **LS2**] and Ghiggini [**Gh**].

The goal of this paper is to introduce an alternate, more hands-on, description of the contact class in Heegaard Floer homology and to use it in the context of our program of relating right-veering diffeomorphisms to tight contact structures.

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We briefly recall the notion of a *right-veering* diffeomorphism which was introduced in [**HKM2**]. Let S be a compact oriented surface with nonempty boundary (sometimes called a "bordered surface") and $h : S \xrightarrow{\sim} S$ be a diffeomorphism such that $h|_{\partial S} = id$. Given two properly embedded oriented arcs α and β with the same initial point $x \in \partial S$, we say α is *to the left* of β if the following holds: Isotop α and β, while fixing their endpoints, so that they intersect transversely (this include the endpoints) and with the fewest possible number of intersections. We then say α is to the left of β if either $\alpha = \beta$ or the tangent vectors $(\dot{\beta}(0), \dot{\alpha}(0))$ define the orientation on S at x. Then h is *right-veering* if for every choice of basepoint $x \in \partial S$ and every choice of properly embedded oriented arc α based at x, $h(\alpha)$ is to the right of α.

In [**HKM2**] we proved that if (S, h) is an open book decomposition compatible with a tight contact structure, then h is right-veering. In [**HKM3**] we continued the study of the monoid $Veer(S, \partial S)$ of right-veering diffeomorphisms and investigated its relationship with symplectic fillability in the pseudo-Anosov case. In particular we proved the following:

Theorem 1.1. *Let S be a bordered surface with connected boundary and h be pseudo-Anosov with fractional Dehn twist coefficient c. If $c \geq 1$, then the contact structure $\xi_{(S,h)}$ supported by (S, h) is isotopic to a perturbation of a taut foliation. Hence (S, h) is (weakly) symplectically fillable and universally tight if $c \geq 1$.*

Hence, when a contact structure is supported by an open book with "sufficiently" right-veering monodromy, it is symplectically fillable and therefore tight as a consequence of a theorem of Eliashberg and Gromov [**El**]. Unfortunately, a right-veering diffeomorphism with a small amount of rotation does not always correspond to a tight contact structure. In fact, any open book can be stabilized to a right-veering one (see Goodman [**Go**], as well as [**HKM2**]). However, we might optimistically conjecture that a minimal (i.e., not destabilizable) right-veering open book defines a tight contact structure. If we specialize to the case of a once-punctured torus, then we can use our description of the contact class to prove this conjecture.

Theorem 1.2. *Let (M, ξ) be a contact 3-manifold which is supported by an open book decomposition (S, h), where S is a once-punctured torus. Then ξ is tight if and only if h is right-veering.*

Very recently John Baldwin [**Ba**] also obtained results similar to Theorem 1.2.

The paper is organized as follows. In Section 2, we review the standard definition of $c(\xi)$. Then, in Section 3, we describe the class $EH(\xi) \in \widehat{HF}(-M)$, which arose in discussions between John Etnyre and the first author. We also prove that the class $EH(\xi)$ equals the Ozsváth-Szabó contact class $c(\xi)$, and hence $EH(\xi)$ is a contact invariant. In Section 4, the class $EH(\xi)$ is applied to contact structures with compatible genus one open book decompositions with connected boundary to prove Theorem 1.2.

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2. Open books and Ozsváth-Szabó contact invariants

In [**OS1**, **OS2**], Ozsváth and Szabó defined invariants of closed oriented 3-manifolds M which they called *Heegaard Floer homology*. Among the several versions of Heegaard Floer homology defined by Ozsváth and Szabó, we concentrate on the simplest one, namely $\widehat{HF}(M)$. It is defined as the homology associated to a chain complex determined by a Heegaard decomposition of M. Consider a Heegaard decomposition $(\Sigma, \alpha = \{\alpha_1, \dots, \alpha_g\}, \beta = \{\beta_1, \dots, \beta_g\})$ of genus g. Here Σ is the Heegaard surface, i.e., a closed oriented surface of genus g which splits M into two handlebodies H_1 and H_2, $\Sigma = \partial H_1 = -\partial H_2$, α_i are the boundaries of the compressing disks of H_1, and β_i are the boundaries of the compressing disks of H_2. Then consider two tori $\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \dots \times \beta_g$ in $Sym^g(\Sigma)$. Also pick a basepoint $z \in \Sigma - \cup_{i=1}^g \alpha_i - \cup_{i=1}^g \beta_i$. The complex $\widehat{CF}(M)$ is defined to be the free $\mathbb{Z}$-module generated by the points $\mathbf{x} = \{x_1, \dots, x_g\}$ of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The boundary map is defined by counting points in certain 0-dimensional moduli spaces of holomorphic maps of the unit disk into $Sym^g(\Sigma)$. It is, very roughly, defined as follows. Denote by $\mathcal{M}_{\mathbf{x},\mathbf{y}}$ the 0-dimensional (after quotienting by the natural $\mathbb{R}$-action) moduli space of holomorphic maps u from the unit disk $D^2 \subset \mathbb{C}$ (with complex coordinate ζ) to $Sym^g(\Sigma)$ that (i) send $1 \mapsto \mathbf{x}$, $-1 \mapsto \mathbf{y}$, $S^1 \cap \{\text{Im } \zeta \geq 0\}$ to $\mathbb{T}_\alpha$ and $S^1 \cap \{\text{Im } \zeta \leq 0\}$ to $\mathbb{T}_\beta$, and (ii) avoid $\{z\} \times Sym^{g-1}(\Sigma) \subset Sym^g(\Sigma)$. Then define

$$\partial \mathbf{x} = \sum_{\mu(\mathbf{x},\mathbf{y})=1} \#(\mathcal{M}_{\mathbf{x},\mathbf{y}})\ \mathbf{y},$$

where $\mu(\mathbf{x}, \mathbf{y})$ is the relative Maslov index of the pair and $\#(\mathcal{M}_{\mathbf{x},\mathbf{y}})$ is a signed count of points in $\mathcal{M}_{\mathbf{x},\mathbf{y}}$. The homology of this complex $\widehat{HF}(M)$ is shown to be independent of the various choices made in the definition. In particular, it is independent of the choice of a "weakly admissible" Heegaard decomposition.

Each intersection point $\mathbf{x}$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ defines a Spin^c structure $\mathbf{s}_\mathbf{x}$ on M. If there is a topological disk from $\mathbf{x}$ to $\mathbf{y}$ which satisfies (i) and (ii) in the previous paragraph, then the two Spin^c structures agree. Hence, the complex (as well as the homology of the complex) splits according to Spin^c structures. The Heegaard Floer homology decomposes as a direct sum

$$\widehat{HF}(M) = \oplus_{\mathbf{s}} \widehat{HF}(M, \mathbf{s}).$$

Given a contact structure ξ on M, let (S, h) be a compatible open book decomposition, where S is a surface of genus g (here genus means the genus of the surface capped off with disks) and h is a diffeomorphism of S which is the identity on ∂S. The manifold M is homeomorphic to $(S \times [0, 1])/\sim$ and the binding K is given by $(\partial S \times [0, 1])/\sim$. Here, the equivalence relation $\sim$ is generated by $(x, 1) \sim (h(x), 0)$ for $x \in S$ and $(y, t) \sim (y, t')$ for $y \in \partial S$, $t, t' \in [0, 1]$. From the above description of M we immediately see a natural Heegaard splitting of M by setting $H_1 = (S \times [0, \frac{1}{2}])/\sim$ and $H_2 = (S \times [\frac{1}{2}, 1])/\sim$. This gives a Heegaard decomposition of genus $2g$ with the splitting surface $\Sigma = S_{1/2} \cup -S_0$, where $S_t = S \times \{t\}$. A set of $2g$ properly embedded disjoint arcs $a_1, \dots, a_{2g}$ which cut S into a disk defines a set of compressing disks $a_i \times [0, \frac{1}{2}]$, $i = 1, \dots, 2g$, in H_1 and a set of compressing disks $a_i \times [\frac{1}{2}, 1]$, $i = 1, \dots, 2g$, in H_2. We then set $\alpha_i = \partial(a_i \times [0, \frac{1}{2}])$ and $\beta_i = \partial(a_i \times [\frac{1}{2}, 1])$, for $i = 1, \dots, 2g$. See Figure 1.

Given the contact manifold (M, ξ), we denote the associated Spin^c structure by $\mathbf{s}_\xi$. Ozsváth and Szabó define in [**OS5**] an element $c(\xi) \in \widehat{HF}(-M, \mathbf{s}_\xi)/(\pm 1)$ by using a Heegaard splitting associated to an open book decomposition (S, h) compatible with ξ. (At the time of the writing of the paper, the ± 1 ambiguity still exists. It is possible, however, that a careful study of orientations would remove this ambiguity. The ± 1 issue does not arise in Seiberg-Witten Floer homology. To avoid writing ± 1 everywhere, we either work with $\mathbb{Z}/2\mathbb{Z}$-coefficients or tacitly assume that $c(\xi)$ is well-defined up to a sign when $\mathbb{Z}$-coefficients are used.) The Heegaard splitting given in the previous paragraph is not quite the Heegaard splitting that Ozsváth and Szabó consider when defining $c(\xi)$. Instead, they use a Heegaard surface that can be viewed simultaneously as a Heegaard surface for M and for $M_0(K)$, the zero surgery along the binding K. (They also assume that ∂S is connected and hence K is a knot.) The contact element in $\widehat{HF}(-M)$ can be seen on this Heegaard surface as the image of a class in $\widehat{HF}(-M_0(K))$ (provided $g(K) > 1$), or, equivalently, as the image of a class in $\widehat{HFK}(-M, K, F, -g)$. To construct such a splitting, take a disk $D \subset int(S)$ which is contained in a small neighborhood of ∂S, dig $D \times [0, \frac{1}{2}]$ out of H_1, and then attach it to H_2. This produces two new handlebodies H_1' and H_2'. On H_2' we keep the same set of β-curves $\beta_1, \dots, \beta_{2g}$ as H_2 and add $\beta_0 = \partial D \times \{\frac{1}{4}\}$. Next, let d be a short arc connecting between the two boundary components of $S - D$, and let $\{b_1, \dots, b_{2g}\}$ be a set of arcs with endpoints on ∂D which are

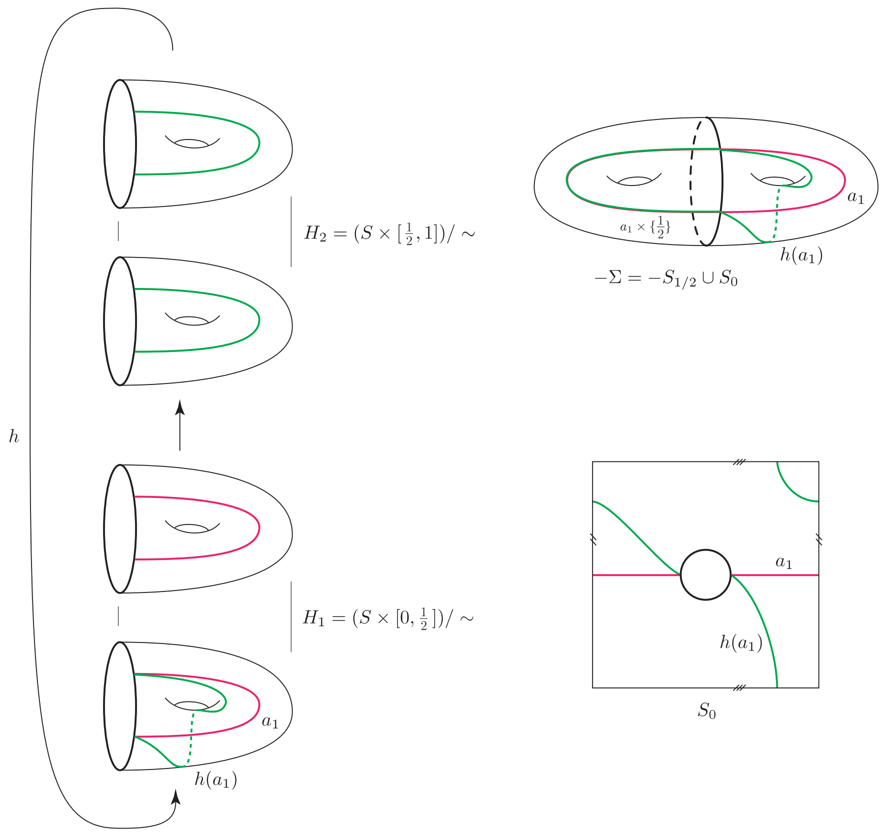

Figure 1. The left-hand portion of the figure shows the decomposition into the two handlebodies H_1 and H_2 and a compressing disk on each corresponding to a_1. The upper right portion shows $-\Sigma = -S_{1/2} \cup S_0$ and the boundaries of two compressing disks. We draw just the lower right portion to indicate the Heegaard decomposition and the effect of the monodromy on arcs.

"dual" to $\{a_1, \dots, a_{2g}\}$. (By this we mean $a_{2i+1} \cap b_j = \emptyset$ if $j \neq 2i$ and $a_{2i+1} \cap b_{2i} = \{x_{2i+1}\}$; also $a_{2i} \cap b_j = \emptyset$ if $j \neq 2i+1$ and $a_{2i} \cap b_{2i+1} = \{x_{2i}\}$.) Then on H_1', we let $\alpha_0 = \partial(d \times [0, \frac{1}{2}])$ and $\alpha_i = \partial(b_i \times [0, \frac{1}{2}])$. Also let $\alpha_0 \cap \beta_0 = \{x_0\}$.

The above choices determine a special point $\mathbf{x} = \{x_0, x_1, \dots, x_{2g}\}$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset Sym^{2g+1}(\Sigma)$. (Here, x_i means $(x_i, \frac{1}{2})$, for $i > 0$.) This point (after modifying the Heegaard diagram by winding in a region that does not affect $\mathbf{x}$ to adjust for admissibility) defines the special cycle in Heegaard Floer homology. The homology class of $\mathbf{x}$ is defined as the contact class $c(\xi)$ by Ozsváth-Szabó. They show that $\widehat{HFK}(-M, K, F, -g)$, the knot Floer homology for $(-M, K)$ at the lowest possible filtration level

$-g$, is isomorphic to $\mathbb{Z}$ and is generated by $\mathbf{x}$. Then $c(\xi)$ is defined to be the image of this generator in $\widehat{HF}(-M)$. For details, including the figures describing this decomposition and the corresponding generator of $c(\xi)$, see [**OS5**].

3. An alternate description of the contact element

3.1. Definition and main theorem. Let S be a bordered surface whose boundary is not necessarily connected. Let $\{a_1, \dots, a_r\}$ be a collection of disjoint, properly embedded arcs of S so that $S - \bigcup_{i=1}^r a_i$ is a single polygon. We will call such a collection a *basis for* S. Observe that every arc a_i of a basis is a nonseparating arc of S. Next let b_i be an arc which is isotopic to a_i by a small isotopy so that the following hold:

1) The endpoints of a_i are isotoped along ∂S, in the direction given by the boundary orientation of S.
2) a_i and b_i intersect transversely in one point in the interior of S.
3) If we orient a_i, and b_i is given the induced orientation from the isotopy, then the sign of the intersection $a_i \cap b_i$ is $+1$.

See Figure 2.

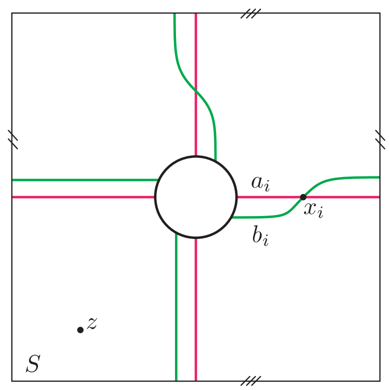

Figure 2. The arcs a_i and b_i for a once-punctured torus S.

Let $M = M_{(S,h)}$ be the 3-manifold with open book decomposition (S, h). Recall the Heegaard decomposition for M described in the previous section, where $\Sigma = S_{1/2} \cup -S_0$. We choose the compressing disks to be $\alpha_i = \partial(a_i \times [0, \frac{1}{2}])$ and $\beta_i = \partial(b_i \times [\frac{1}{2}, 1])$. We will sometimes write $\alpha_i = (a_i, a_i)$ and $\beta_i = (b_i, h(b_i))$, where the first entry is the arc on $S_{1/2}$ and the second entry is the arc on S_0. Let x_i be the intersection point $(a_i \cap b_i) \times \{\frac{1}{2}\}$ lying in $S_{1/2} \subset \Sigma$, and let z be the basepoint which sits on $S_{1/2}$ and lies *outside* the thin strips of isotopy between the a_i's and the b_i's. Then $(\Sigma, \beta, \alpha, z)$ gives a weakly admissible Heegaard diagram, namely every periodic domain has positive and negative components. This is due to the fact that every periodic domain which involves α_i crosses x_i, at which point the sign of the connected component of $\Sigma - \bigcup_{i=1}^r \alpha_i - \bigcup_{i=1}^r \beta_i$ changes.

Let $\mathcal{D}_0, \dots, \mathcal{D}_m$ be the closures of the components of $\Sigma - \bigcup_{i=1}^r \alpha_i - \bigcup_{i=1}^r \beta_i$, and let z_i, $i = 0, \dots, m$, be a point in the interior of $\mathcal{D}_i$. Assume additionally that z_0 is the basepoint z. Given a Whitney disk $u : D \to Sym^r(\Sigma)$, the *domain associated to* u is the formal linear combination

$$\mathcal{D}(u) = \sum_{i=0}^{m} n_{z_i}(u)\mathcal{D}_i,$$

where $n_{z_i}(u)$ is the algebraic intersection number of u with $\{z_i\} \times Sym^{r-1}(\Sigma)$. To a holomorphic map $u : D \to Sym^r(\Sigma)$ there is the associated map $\tilde{u} : \widetilde{D} \to \Sigma$ of a branched cover of D to Σ. The domain $\mathcal{D}(u)$ can also be interpreted as the (homotopy class of the) image of $\tilde{u}$. In the definition of the boundary map in the $\widehat{HF}$ theory, we only count holomorphic disks $u : D \to Sym^r(\Sigma)$ that miss $\{z\} \times Sym^{r-1}(\Sigma)$, i.e., where the coefficient of $\mathcal{D}_0$ is zero. For such disks the intersection of the support of $\mathcal{D}(u)$ with $S_{1/2}$ is thus constrained to lie in the thin strips of isotopy of the a_i to b_i.

We claim that $\mathbf{x} = \{x_1, \dots, x_r\} \in \widehat{CF}(\Sigma, \beta, \alpha, z)$ is a cycle, thanks to the fortuitous placement of the basepoint z. (We write $\widehat{CF}(\Sigma, \beta, \alpha, z)$ instead of $\widehat{CF}(\Sigma, \alpha, \beta, z)$ to indicate homology on $-M$.) Suppose $u : D \to Sym^r(\Sigma)$ contributes nontrivially to $\partial\mathbf{x}$; in particular it is a holomorphic disk from $\mathbf{x}$ to $\mathbf{y}$ with $\mathbf{y} \neq \mathbf{x}$. Since $\mathbf{y} = (y_1, ..., y_r) \neq \mathbf{x} = (x_1, ..., x_r)$, at least one y_i lies on $S \times \{0\}$. There is an arc connecting y_i to some component x_i of $\mathbf{x}$ that is in the image of the boundary of the corresponding holomorphic map $\tilde{u}$. Because of the orientation of the holomorphic disk that starts at $\mathbf{x}$ (recall that $S^1 \cap \{\mathrm{Im}\ \zeta \geq 0\}$ maps to $\mathbb{T}_\beta$ and $S^1 \cap \{\mathrm{Im}\ \zeta \leq 0\}$ maps to $\mathbb{T}_\alpha$), the image of $\tilde{u}$ near x_i has to come out of x_i in the region containing z and hence the coefficient of $\mathcal{D}_0$ is nonzero, which contradicts the assumption that it contributes to the boundary operator.

Define $EH(S, h, \{a_1, \dots, a_r\})$ to be the homology class of the generator $\mathbf{x}$. The following is the main theorem of this section:

Theorem 3.1. *$EH(S, h, \{a_1, \dots, a_r\})$ is an invariant of the contact structure and equals $c(\xi_{(S,h)})$.*

In particular, $EH(S, h, \{a_1, \dots, a_r\})$ is independent of the choice of basis, and it will often be denoted by $EH(S, h)$.

In Theorem 3.1 we are not assuming that ∂S is connected.

Examples: To give some intuition for the class $EH(S, h)$, we give three examples when S is an annulus. Refer to Figure 3. The leftmost diagram gives a and b on $S_{1/2}$. The subsequent diagrams give S_0 for (1), (2), and (3) below (from left to right).

(1) If h is the identity, then (M, ξ) is the standard tight contact structure on $S^1 \times S^2$. Since there are two holomorphic disks from y to x, it follows that $EH(S, h) \neq 0$. One of the holomorphic disks from y to x has been shaded in Figure 3.

(2) If h is a positive Dehn twist about the core curve, then (M, ξ) is the standard tight contact structure on S^3. Since x is the unique intersection point on $\Sigma = T^2$, $EH(S, h) \neq 0$.

(3) If h is a negative Dehn twist about the core curve, then (M, ξ) is an overtwisted contact structure on S^3. We have $\partial y_1 = \partial y_2 = x$; hence $EH(S, h) = 0$.

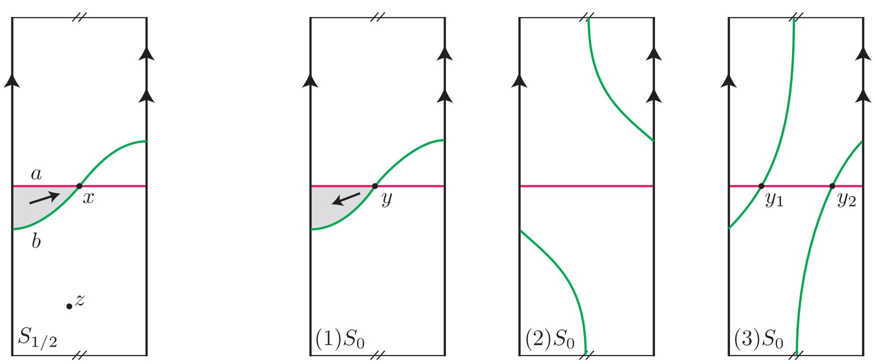

Figure 3. Examples when S is an annulus.

The following lemma echoes our result in [**HKM2**], which states that $\xi_{(S,h)}$ is overtwisted if h is not right-veering.

Lemma 3.2. *If h is not right-veering, then $EH(S, h) = 0$.*

Proof. If h is not right-veering, then there exists an arc a_1 on S so that $h(a_1)$ is to the left of a_1. If a_1 is nonseparating, then it can be completed to a basis $\{a_1, \dots, a_r\}$. There exists an intersection point $y_1 \in \alpha_1 \cap \beta_1$ and a unique (up to translation) holomorphic map from the unit disk $D \subset \mathbb{C}$ (with complex coordinate ζ) to Σ, where $1 \mapsto y_1$, $-1 \mapsto x_1$, $\partial D \cap \{\text{Im}\, \zeta \geq 0\}$ maps to β_i and $\partial D \cap \{\text{Im}\, \zeta \leq 0\}$ maps to α_i. Since z forces any holomorphic disk $u : D \to Sym^r(\Sigma)$ which contributes to $\partial\{y_1, x_2, \dots, x_n\}$ to be constant near x_i, $i = 2, \dots, r$, all the α_i and β_i, $i = 2, \dots, r$, are "used up", and the only holomorphic disk that remains is the unique one from y_1 to x_1. Hence $\partial\{y_1, x_2, \dots, x_n\} = \{x_1, x_2, \dots, x_n\}$.

If the arc a_1 is separating, then let us call its initial point p. Suppose the arcs $h(a_1)$ and a_1 intersect efficiently. Then $h(a_1)$ and a_1 must intersect at some point in the interior of $h(a_1)$; otherwise $h(a_1)$ will cut off a strictly smaller subsurface of S inside a subsurface of S cut off by a_1. Let q be the first point of intersection between $h(a_1)$ and a_1 in $int(h(a_1))$, c be the subarc of a_1 from p to q, and c' be the subarc of $h(a_1)$ from p to q. Then either $c(c')^{-1}$ is separating or it is not. If $c(c')^{-1}$ separates a region S' to the left of a_1, then there is a nonseparating arc $b \subset S'$ which begins and ends at p. On the other hand, if $c(c')^{-1}$ is nonseparating, then we let $b = c(c')^{-1}$. In either case, since b is strictly to the left of a_1 and strictly to the right of $h(a_1)$, it follows that $h(b)$ is strictly to the left of b. By separating the endpoints of b a little, we obtain a nonseparating, properly embedded, oriented arc which is moved to the left under h. q.e.d.

In view of Theorem 3.1 and the fact that every overtwisted contact structure admits an open book that is not right-veering, Lemma 3.2 immediately implies that $c(\xi) = 0$ for an overtwisted contact structure.

Proof of Theorem 3.1. Let us denote a positive Dehn twist about a closed curve γ by ϕ_γ. Assume ∂S is connected. We first prove the theorem for a special case, namely when $h = \phi^n_{\partial S}$ with $n > 0$, in Section 3.2. Next, in Section 3.3 we prove that $EH(S, h, \{a_1, \dots, a_r\})$ only depends on the isotopy class of h (relative to the boundary), and in Section 3.4 we show that $EH(S, h, \{a_1, \dots, a_r\})$ is independent of the choice of basis by using handleslides. Then in Section 3.5 we prove that the class $EH(S, h)$ is mapped to the class $EH(S, \phi_\gamma^{-1} \circ h)$ under the natural map

$$\widehat{HF}\left(-M_{(S,h)}\right) \to \widehat{HF}\left(-M_{(S,\phi_\gamma^{-1}\circ h)}\right),$$

which corresponds to a Legendrian (+1)-surgery. We then start with $\phi^n_{\partial S}$ with $n \gg 0$ and apply a sequence of negative Dehn twists until we reach the desired monodromy map h. In Section 3.6 we reduce the case of multiple boundary components to the case when ∂S is connected.

q.e.d.

3.2. Primordial Example. Let S be a once-punctured torus and $h = \phi_{\partial S}$, i.e., a positive Dehn twist about ∂S. The same argument works if S is a genus g surface with one boundary component and $h = \phi^n_{\partial S}$, $n > 0$.

The subarcs of α_i and β_i that live in S_0 are given in Figure 4. We change notation and the constituent points of $\mathbf{x}$ representing $EH(S, h)$ will be denoted $x_0 = x_0'$ and $y_0 = y_0'$ as in Figure 4. Although $x_0 = x_0'$ and $y_0 = y_0'$, strictly speaking, live on $S_{1/2}$, we view them as sitting on ∂S_0. (Also, the points x_0 and x_0', as well as y_0 and y_0', are drawn as distinct points on ∂S_0, but we hope this will not cause any confusion for the reader.)

We then place the basepoint w on S_0 as indicated in Figure 4. Observe that z and w together represent the binding K. The binding K is isotopic to the dotted curve γ_0 which consists of two subarcs c_1 and c_2 between z and w, where c_1 intersects only the α-curves and c_2 intersects only the β-curves. Then $(\Sigma, \beta, \alpha, z, w)$ is a doubly-pointed Heegaard diagram for the knot Floer homology of K.

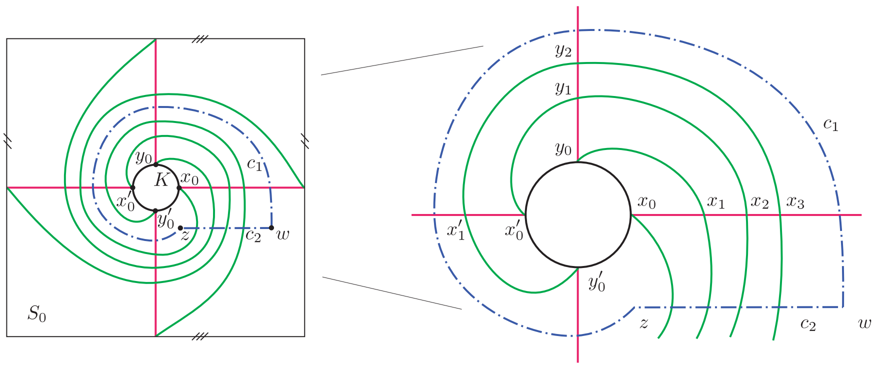

Figure 4. S_0 when $h = \phi_{\partial S}$, and a zoomed-in ammonite-like region.

If we stabilize this Heegaard splitting by digging a handle in $S \times [0, \frac{1}{2}]$ which is parallel to the arc c_2, then we obtain a Heegaard surface Σ' on which we can see both $-M$ and $-M_0(K)$. See Figure 5. Here $-M$ is given by $\{\beta_0\} \cup \beta$ and $\{\alpha_0\} \cup \alpha$, whereas $-M_0(K)$ is given by $\gamma = \{\gamma_0\} \cup \beta$ and $\{\alpha_0\} \cup \alpha$. (Here γ_0 is viewed as a curve that passes through the handle once.) The stabilization sends $\mathbf{x} = \{x_0, y_0\}$ to $\mathbf{x}' = \{z_0, x_0, y_0\}$, where z_0 is the intersection of the two new compressing curves α_0 and β_0.

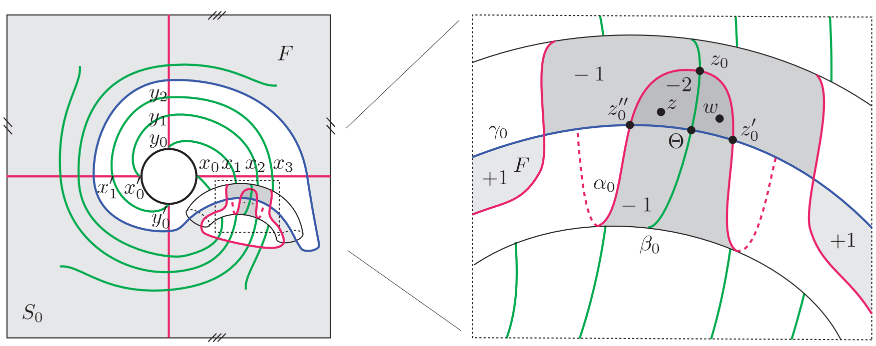

Figure 5. Part of the stabilized Heegaard surface $-\Sigma'$. The domain F has been shaded.

As a first step in exploiting the Ozsváth-Szabó characterization of $c(\xi)$, we show that the lowest filtration level is generated by $\mathbf{x}' = \{z_0, x_0, y_0\}$ as well as the other intersection points $\mathbf{y} = \{z_0, x, y\}$, where x and y live inside the dotted lines of Figure 4. The filtration level is computed by first letting $F \subset \Sigma'$ be the domain bounded by α_0 and γ_0 which does not intersect $S_{1/2}$ (and hence lives mostly on S_0). We additionally assume that F is oriented so that the surface $\hat{F}$, obtained from F by capping off ∂F, is an oriented fiber of the fibration of $M_0(K)$. In order to find generators $\mathbf{y}$ which are at the lowest filtration level, we minimize $\langle c_1(\mathbf{s}_{\mathbf{y}'}), [\hat{F}]\rangle$. Here $\mathbf{y}' = \{z_0', x, y\}$ and z_0' is the intersection point on $\alpha_0 \cap \gamma_0$ which forms a small triangle with $\Theta \in \beta_0 \cap \gamma_0$ and z_0 as in Figure 5. (Keep in mind that since we are dealing with $\widehat{HF}$ of $-M$ and $-M_0(K)$, the Heegaard surface is $-\Sigma'$; otherwise our calculations will be off by a negative sign.)

To this end, we recall the formula for the first Chern class (Section 7.1 of [**OS2**]; for some details, see Rasmussen [**Ra**]):

$$\langle c_1(\mathbf{s}_{\mathbf{y}'}), [A]\rangle = \chi(\mathcal{P}) - 2\overline{n}_z(\mathcal{P}) + 2\sum_{p\in\mathbf{y}'} \overline{n}_p(\mathcal{P}).$$

Here $[A] \in H_2(M_0(K), \mathbb{Z})$, $\mathbf{s}_{\mathbf{y}'}$ is a Spin^c structure corresponding to $\mathbf{y}'$, $\mathcal{P}$ is the periodic domain for $[A]$ (where we do not require that $\mathcal{P}$ avoid z) and χ is the *Euler measure*. Let $\mathcal{D}$ be a component of $(-\Sigma') - \bigcup_i \alpha_i - \bigcup_i \gamma_i$. Then $\overline{n}_p(\mathcal{D})$ equals (i) 1 if p is in the interior of $\mathcal{D}$, (ii) 0 if p does not intersect $\mathcal{D}$, (iii) $\frac{1}{2}$ if p is on an edge of $\mathcal{D}$ (but not a corner), and (iv) $\frac{1}{4}$ if p is on a corner of $\mathcal{D}$. We then extend $\overline{n}_p$ linearly to $\mathcal{P}$.

In the case at hand, the possible x's and y's are either in the interior of F or not in F, and therefore they either contribute 1 or 0. On the other hand, $\overline{n}_z(\mathcal{P}) = -2$, $\chi(\mathcal{P}) = -2g(S)$, and $\overline{n}_{z_0'}(\mathcal{P}) = -1$ are constant, and it follows that $\langle c_1(\mathbf{s}_{\mathbf{y}'}), [\hat{F}]\rangle = 2 - 2g(S)$ is the minimal value and it is attained when both x and y are not in F. (In fact, $(\{\beta_0\} \cup \beta, \{\alpha_0\} \cup \alpha, z, w)$ is a "sutured Heegaard diagram" in the sense of [**Ni**].)

The graded complex for calculating $\widehat{HFK}(-M, K, -1)$ is generated by:

$$\{z_0, x_0, y_0\}, \{z_0, x_0, y_2\}, \{z_0, x_1', y_1\}, \{z_0, x_1, y_1\},$$
$$\{z_0, x_2, y_0\}, \{z_0, x_2, y_2\}, \{z_0, x_3, y_1\}.$$

Our task is to identify $\mathbf{x}' = \{z_0, x_0, y_0\}$ as a generator of $\widehat{HFK}(-M, K, -1) \simeq \mathbb{Z}$. We will show that all the generators besides $\mathbf{x}'$ correspond to Spin^c structures which are different from that of the contact structure ξ. An easy computation shows that $H_2(M; \mathbb{Z}) \simeq \mathbb{Z}^2$ and is generated by tori T_δ of the form $(\delta \times [0,1])/\sim$, where δ is any nonseparating curve on S and $(x, 1) \sim (h(x), 0)$ as before. Since ξ is close to the foliation $S \times \{t\}$ on $(S \times [0,1])/\sim$, it follows that $\langle c(\xi), [T_\delta]\rangle = 0$. Now, let δ_1 be a $(0,1)$-curve on S and δ_2 be a $(1,0)$-curve. Then $[T_{\delta_1}]$ is given by the periodic domain $\mathcal{P}_{\delta_1}$, which consists of two rectangles $y_0y_2y_4'y_2'$ and $y_0'y_2'y_4y_2$ with opposite signs, shown in Figure 6. Similarly, $[T_{\delta_2}]$ is represented by $\mathcal{P}_{\delta_2}$, consisting of $x_0x_2x_4'x_2'$ and $x_0'x_2'x_4x_2$ with opposite signs.

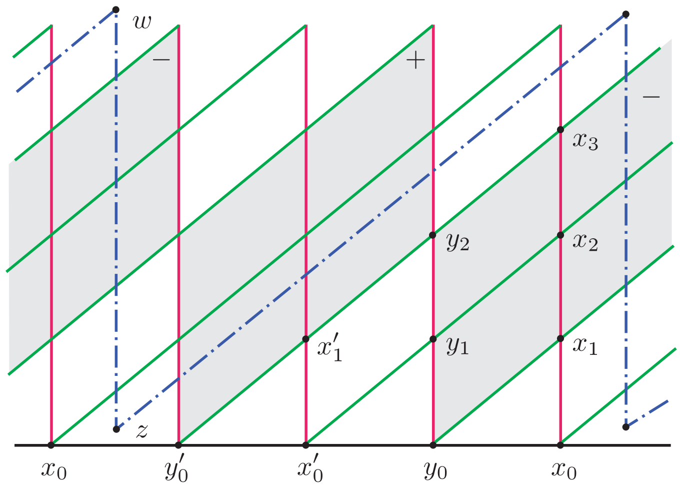

Figure 6. Cover of a neighborhood of ∂S.

Now refer to Figure 6, which is a cover of an annular neighborhood of $\partial S \subset S$. The dotted curve is (a lift of) c_1c_2. Points below the dotted curve are not in F, so only they have the proper filtration level to represent generators.

We will show $\mathbf{s}_{\{z_0,x,y\}} \neq \mathbf{s}_{\{z_0,x_0,y_0\}}$ if $\{x,y\} \neq \{x_0,y_0\}$, by showing that $\langle c(\mathbf{s}_{\{z_0,x,y\}}), [T_{\delta_i}]\rangle \neq 0$ for $i=1$ and 2 if $\{x,y\} \neq \{x_0,y_0\}$.

First consider the intersection points on the vertical lines starting at x_0 and at x_0'. Suppose that $\langle c(\mathbf{s}_{\{z_0,x,y\}}), [T_{\delta_2}]\rangle = 0$. The rectangle $x_0'x_2'x_4x_2$ of the periodic domain $\mathcal{P}_{\delta_2}$ contributes $\frac{1}{2}$ if $x = x_3$ or $x = x_1'$. Since there is no value of y below the dotted curve with a contribution of $-\frac{1}{2}$ from the rectangle $x_0x_2x_4'x_2'$ to cancel the $\frac{1}{2}$, the possibilities $x = x_3, x_1'$ are eliminated. Since $x_0x_2x_4'x_2'$ gives a contribution of $-\frac{1}{2}$ to x_1, and $x_0'x_2'x_4x_2$ contributes 0 to y_0, $\frac{1}{2}$ to y_1 and 1 to y_1, the only generator containing x_1 that is allowed is $\{z_0,x_1,y_1\}$. Any generator containing y_2 is also disallowed since $x_0'x_2'x_4x_2$ contributes 1 to y_2, and there is no x value that will offset it from the $x_0x_2x_4'x_2'$ rectangle. The only generator allowed to contain y_1 is again $\{z_0,x_1,y_1\}$. The same rectangle gives x_1' a contribution of $-\frac{1}{2}$ that cannot be offset.

It therefore remains to consider the generator $\{z_0,x_1,y_1\}$, as well as pairs with $x = x_0$ or x_2. Moreover, the only possible y-coordinates are y_0 and y_1, and $\{z_0,x_1,y_1\}$ is the only option allowed for $y = y_1$. Now use the periodic domain $\mathcal{P}_{\delta_1}$. The rectangle $y_0y_2y_4'y_2'$ contributes -1 to $\{z_0,x_1,y_1\}$, thus eliminating it as a possibility. The only other option different from $\{z_0, x_0, y_0\}$ is $\{z_0, x_2, y_0\}$ (since y_2 was banned) which gets a nonzero contribution from $y_0 y_2 y_4' y_2'$.

To show how this argument generalizes to higher genus surfaces, let us examine the genus two case. The generators will have the form $\{z_0, x, y, u, v\}$, and there will be 8 intersection points on each vertical segment in a picture analogous to Figure 6. Denote the points on the boundary by u_0, v_0, u_0', v_0', x_0, y_0, x_0', y_0' going from right to left. Start by considering the rectangles $u_0 u_2 u_8' u_6'$ and $u_0' u_6' u_8 u_2$. We eliminate $u_3, \dots, u_7$ and all the u' values besides u_0', by noticing that there is no allowable v value to offset the $\frac{1}{2}$ contribution from $u_0' u_6' u_8 u_2$. The contribution of 1 from the same rectangle eliminates all values of v other than v_0 and v_1 (though no v_i' are yet disallowed). If $v = v_1$, only generators of the form $\{z_0, x, y, u_1, v_1\}$ are allowed.

Now use the periodic domain represented by the rectangles $v_0 v_2 v_8' v_6'$ and $v_0' v_6' v_8 v_2$. The generators of the form $\{z_0, x, y, u_1, v_1\}$ get a contribution of -1 from $v_0 v_2 v_8' v_6'$ and there is no positive contribution from the allowable x, y coordinates that can be gained from $v_0' v_6' v_8 v_2$; therefore all the $\{z_0, x, y, u_1, v_1\}$ are eliminated. Next, u_2 gets a contribution of -1 from $v_0 v_2 v_8' v_6'$ that cannot be canceled since there is no v value that gets a contribution of 1 needed from $v_0' v_6' v_8 v_2$. It follows that u_0 is the only allowable u-coordinate. Generators $\{z_0, x, y, u_0, v_i'\}$, $i \neq 0$, are eliminated since v_i' gets a contribution of $\frac{1}{2}$ from $v_0' v_6' v_8 v_2$ that cannot be canceled. Therefore we are left with $\{z_0, x, y, u_0, v_0\}$. The argument is now reduced to eliminating the possible x, y coordinates, and this follows just as in the genus one argument given above.

This shows how the proof works for arbitrary genus. The inductive step is done in the same way by eliminating all extra options in the two new coordinates, thus reducing to the case of lower genus.

Since the contact invariant is the image of a generator of $\widehat{HFK}(-M, K, -g)$ in $\widehat{HF}(-M)$, it follows that $c(\xi_{(S,h)}) = EH(S, h)$. It is not hard to see how the above argument generalizes to the $h = \phi_{\partial S}^n$, $n > 0$ case.

3.3. Isotopy. In this subsection we prove the following:

Lemma 3.3. *If $h_t : S \xrightarrow{\sim} S$, $t \in [0,1]$, is a 1-parameter family of diffeomorphisms which restrict to the identity on ∂S, then $EH(S, h_0, \{a_1, \dots, a_r\}) = EH(S, h_1, \{a_1, \dots, a_r\})$.*

Proof. Let $\alpha_i = (a_i, a_i)$ and $\beta_i^t = (b_i, h_t(b_i))$. In other words, we fix the α_i and isotop the β_i. Observe that the β_i^t remain constant on $S \times \{1\}$. According to Theorem 7.3 of [**OS1**], we can reduce to the case where h_t is a Hamiltonian isotopy. Let $\Psi_t : \Sigma \xrightarrow{\sim} \Sigma$ be the Hamiltonian isotopy which restricts to the identity on $S \times \{1\}$ and restricts to h_t on $S \times \{0\}$. We use the same notation for the induced isotopy on $Sym^r(\Sigma)$. Then the chain map $\Phi : \widehat{CF}(\beta^0, \alpha) \to \widehat{CF}(\beta^1, \alpha)$ is obtained by counting holomorphic disks $u : [0,1] \times \mathbb{R} \to Sym^r(\Sigma)$ which satisfy $\lim_{t \to +\infty} u(s+$

$it) = \mathbf{x}$, $\lim_{t\to-\infty} u(s+it) = \mathbf{x}'$, $u(0+it) \in \Psi_t(\mathbb{T}_\beta)$, and $u(1+it) \in \mathbb{T}_\alpha$, and avoid $\{z\} \times Sym^{r-1}(\Sigma)$. Here $\mathbf{x} \in \widehat{CF}(\beta^0,\alpha)$ and $\mathbf{x}' \in \widehat{CF}(\beta^1,\alpha)$. Now, if $\mathbf{x}$ is the unique r-tuple of points on $S \times \{1\}$ representing the generator of $EH(S, h_0, \{a_1, \dots, a_r\}$, then the only holomorphic disks of the above type are constant holomorphic disks, due to the placement of the basepoint z. This implies that $EH(S, h_0, \{a_1, \dots, a_r\})$ is mapped to $EH(S, h_1, \{a_1, \dots, a_r\})$ under the isomorphism $\Phi : \widehat{HF}(\beta^0, \alpha) \xrightarrow{\sim} \widehat{HF}(\beta^1, \alpha)$. q.e.d.

3.4. Change of basis. In this subsection we prove the following proposition:

Proposition 3.4. *$EH(S, h, \{a_1, \dots, a_r\})$ is independent of the choice of basis $\{a_1, \dots, a_r\}$.*

Let $\{a_1, a_2, \dots, a_r\}$ be a basis for S. After possibly reordering the a_i's, suppose a_1 and a_2 are adjacent arcs on ∂S, i.e., there is an arc $\tau \subset \partial S$ with endpoints on a_1 and a_2 such that τ does not intersect any a_i in $\mathrm{int}(\tau)$. Define $a_1 + a_2$ to be the isotopy class of $a_1 \cup \tau \cup a_2$, relative to the endpoints. Then the modification $\{a_1, a_2, \dots, a_r\} \mapsto \{a_1 + a_2, a_2, \dots, a_r\}$ is called an *arc slide.*

Proposition 3.4 is immediate from the following two lemmas.

Lemma 3.5. *$EH(S, h)$ is invariant under an arc slide $\{a_1, a_2, \dots, a_r\} \mapsto \{a_1 + a_2, a_2, \dots, a_r\}$.*

Proof. Without loss of generality, consider the case where S is a once-punctured torus. We show that the chain map which corresponds to an arc slide takes the representative of $EH(S, h, \{a_1, a_2\})$ determined by $\mathbf{x} = \{x_1, x_2\}$ to the representative of $EH(S, h, \{a_1 + a_2, a_2\})$ determined by the intersection point $\mathbf{w} = \{w_1, w_2\}$. Observe that an arc slide corresponds to a sequence of two handleslides for the corresponding Heegaard splitting.

Let $(\Sigma, \beta, \alpha, z)$ be the pointed Heegaard diagram corresponding to a_i, b_i as described above, with z a point in $S_{1/2}$ lying outside the thin strips of isotopy between the a_i's and the b_i's. If we slide α_2 over α_1 along a path parallel to ∂S, then we obtain a new pair $\gamma = \{\gamma_1, \gamma_2\}$, where $\gamma_1 = (a_1 + a_2, a_1 + a_2)$ and γ_2 is a suitable pushoff of (a_2, a_2) as in the proof of the invariance of Heegaard Floer homology under handleslides in [**OS1**]. Figure 7 depicts the case where a_1 is to the right of a_2 with respect to τ; the case where a_2 is to the right of a_1 is treated similarly.

We claim that $(\Sigma, \gamma, \beta, \alpha, z)$ is a weakly admissible Heegaard triple-diagram. Recall that a triple-diagram is *weakly admissible* if each non-trivial triply-periodic domain which can be written as a sum of doubly-periodic domains has both positive and negative coefficients. First let us restrict to a neighborhood $\mathcal{R}$ of the labeled regions of $\Sigma - \cup_i\alpha_i - \cup_i\beta_i - \cup_i\gamma_i$ on the right-hand side of Figure 7. Due to the placement of z, the only potential doubly-periodic region involving β, α on $\mathcal{R}$ is $D_2 + D_3 - D_5 - D_6$. (Here D_i is the domain labeled i.) Similarly, for γ, β we have $D_1 + D_2 - D_4 - D_5$ and for α, γ we have $D_1 + D_6 - D_3 - D_4$. Taking linear combinations, we have

$$\begin{aligned} &a(D_2 + D_3 - D_5 - D_6) + b(D_1 + D_2 - D_4 - D_5) \\ &\qquad + c(D_1 + D_6 - D_3 - D_4) \\ &= (b + c)D_1 + (a + b)D_2 + (a - c)D_3 \\ &\qquad - (b + c)D_4 - (a + b)D_5 + (-a + c)D_6. \end{aligned}$$

Since the coefficients come in pairs, e.g., $a + b$ and $-(a + b)$, if any of $a + b$, $b + c$, $a - c$ does not vanish, then the triply-periodic domain has both positive and negative coefficients. Hence, if any of α_1, β_1 and γ_1 is used, then we are done. Otherwise, we may assume that none of α_1, β_1 and γ_1 is used in the periodic domain. This allows us to erase all three, and apply the above considerations to α_2, β_2, and γ_2. The verifications of weak admissibility of all other triple-diagrams in this paper are identical, and are omitted.

Let $\Theta = \{\Theta_1, \Theta_2\}$ be the top generator of $\widehat{HF}(\#(S^1 \times S^2)) = \widehat{HF}(\alpha, \gamma)$. Define the map

$$\psi : \widehat{HF}(\beta, \alpha) \otimes \widehat{HF}(\alpha, \gamma) \to \widehat{HF}(\beta, \gamma),$$

where $\psi(\mathbf{y} \otimes \mathbf{y}')$ counts holomorphic triangles, two of whose vertices are $\mathbf{y} \in \widehat{CF}(\beta, \alpha)$ and $\mathbf{y}' \in \widehat{CF}(\alpha, \gamma)$. Then the isomorphism $g : \widehat{HF}(\beta, \alpha) \xrightarrow{\sim} \widehat{HF}(\beta, \gamma)$ is given by $g(\mathbf{y}) = \psi(\mathbf{y} \otimes \Theta)$.

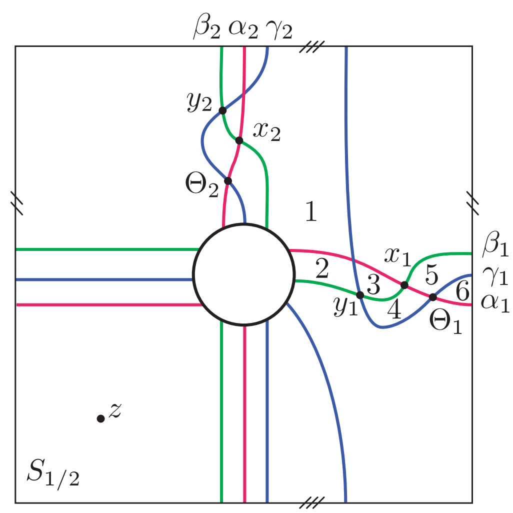

Figure 7. The first handleslide.

We claim that the representative $\mathbf{x} = \{x_1, x_2\}$ of $EH(S, h, \{a_1, a_2\})$ gets mapped to $\mathbf{y} = \{y_1, y_2\} \in \widehat{CF}(\beta, \gamma)$ given in Figure 7. By the placement of z, we see that the unique domain which has x_1 and some Θ_i as corners and avoids z must be a triangle with vertices x_1, Θ_1, y_1. Now that α_1, β_1, and γ_1 are used up, it easily follows that the unique domain which involves x_2 and Θ_2 and avoids z is a triangle with vertices x_2, Θ_2, y_2. This proves the claim.

Let us now consider the effect of the second handleslide, depicted in Figure 8. Let $\delta = \{\delta_1, \delta_2\}$, where δ_1 and δ_2 are suitable pushoffs of $(a_1 + a_2, h(a_1 + a_2))$ and $(a_2, h(a_2))$, respectively. A similar argument as above shows that, under the map

$$\widehat{HF}(\delta, \beta) \otimes \widehat{HF}(\beta, \gamma) \to \widehat{HF}(\delta, \gamma),$$

$\Theta \otimes \mathbf{y}$ gets mapped to $\mathbf{w}$. This shows that $\mathbf{x}$ and $\mathbf{w}$ determine the same element in Heegaard Floer homology, and consequently $EH(S, h, \{a_1, a_2\}) = EH(S, h, \{a_1 + a_2, a_2\})$. q.e.d.

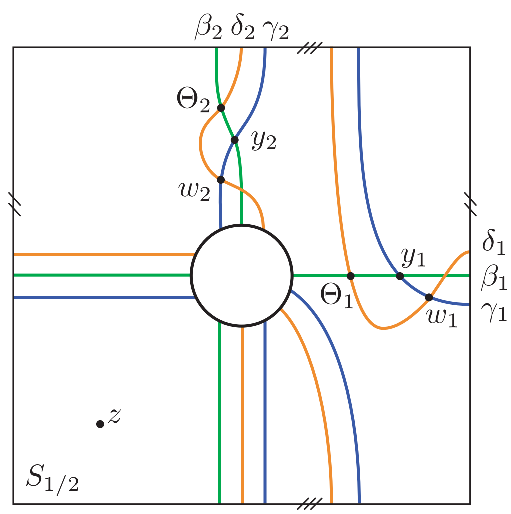

Figure 8. The second handleslide.

Lemma 3.6. *Let $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ be two bases for S. Then there is a sequence of arc slides that takes $\{a_1, \dots, a_r\}$ to $\{b_1, \dots, b_r\}$.*

We do not need to assume that ∂S is connected.

Proof. We argue that we can reduce the total number of intersections of $\bigcup_i a_i$ and $\bigcup_i b_i$ by replacing $\{a_1, \dots, a_r\}$ with $\{a'_1, \dots, a'_r\}$, which is obtained from $\{a_1, \dots, a_r\}$ by a sequence of arc slides. By inducting on the number of intersection points, this shows that we can perform a sequence of arc slides until $\bigcup_i a_i$ and $\bigcup_i b_i$ become disjoint. We then show that two disjoint bases can be brought one into another by a sequence of arc slides.

Let $P = S - \bigcup_i a_i$. Then P is a polygon whose boundary ∂P consists of $4r$ arcs, $2r$ of which are a_i or a_i^{-1} and $2r$ of which are arcs $\tau_1, \dots, \tau_{2r}$ of ∂S.

Suppose $(\bigcup_i a_i) \cap (\bigcup_i b_i) \neq \emptyset$, where we are assuming efficient intersections. After possibly reordering the arcs, there is a subarc $b_1^0 \subset b_1$ which starts on $\tau_1 \subset \partial S$ and ends on a_1, and whose interior $int(b_1^0)$ does not intersect $\bigcup_i a_i$. (In other words, b_1^0 is a properly embedded arc of P.) We may assume that a_1 is not adjacent to τ_1; otherwise, isotop the relevant endpoint of b_1 along τ_1. The subarc b_1^0 separates the polygon P into two regions P_1 and P_2, only one of which contains a boundary arc that is labeled a_1^{-1} (say P_2). We can then slide a_1 over all the arcs of type a_i or a_i^{-1} in the other region P_1, and obtain the new curve a_1' as in Figure 9 so that the new basis $\{a_1', a_2, \dots, a_r\}$ has fewer intersections with $\bigcup_i b_i$. (Note that trying to slide over a_1^{-1} presents a problem, so we

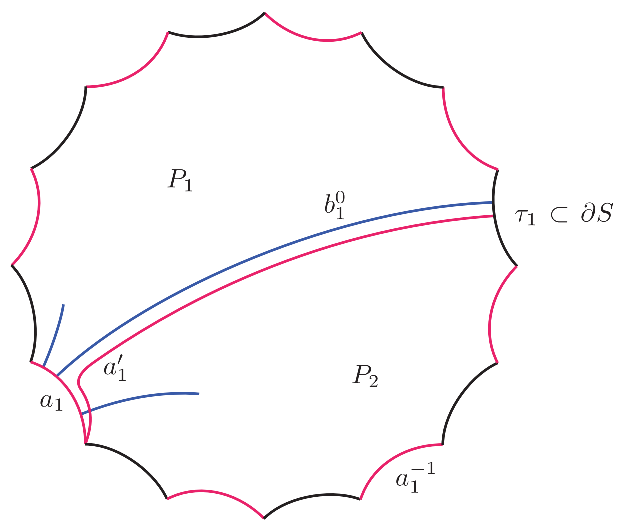

Figure 9. Simplifying the intersections of $\bigcup a_i$ and $\bigcup b_i$.

must go the other way around.) There is one situation when the above strategy needs a little more thought, namely when ∂P_2 only intersects a_1 and a_1^{-1} (among all the a_i and a_i^{-1}). In this case, b_1 exits the polygon P along a_1 and reenters through a_1^{-1}. Eventually we find a subarc of b_1 which starts on some τ_2 and ends on an adjacent a_1^{-1}, a contradiction. We now apply the same procedure to $\{a_1', a_2, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ until they become disjoint.

Now suppose that the two bases $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ are disjoint. We consider the polygon $P = S - \bigcup_i a_i$. Some of the b_i arcs may be parallel to a_j or a_j^{-1}. An arc b_1 that is not parallel to any of the a_i will cut P into two components P_1 and P_2, each containing more than one of a_i, a_i^{-1}, $i = 1, \dots, r$. Recall that b_1 is nonseparating. One can easily verify that b_1 being nonseparating is equivalent to the existence of some a_i such that $a_i \in P_1$ and $a_i^{-1} \in P_2$ (or vice versa). (If there is some a_i, then take an arc c in P from $a_i \subset P_1$ to $a_i^{-1} \subset P_2$. The closed curve in S obtained by gluing up c is dual to b_1.) If each such a_i is parallel to some b_j, then $S - \bigcup_i b_i$ would be disconnected. Hence we may additionally assume that there is some a_i which is not parallel to any b_j. Now we slide a_i across all the arcs of type a_j, a_j^{-1} in P_1 until it becomes parallel to b_1. q.e.d.

3.5. Legendrian surgery. Let δ be a nonseparating curve and ϕ_δ^{-1} be a negative Dehn twist about δ. We now transfer EH from $M = M_{(S,h)}$ to $M' = M_{(S,\phi_\delta^{-1}\circ h)}$. Recall that there is a natural map

$$f : \widehat{HF}(-M) \to \widehat{HF}(-M'),$$

which arises from tensoring with the top generator Θ of $\widehat{HF}(\#(S^1 \times S^2))$.

Proposition 3.7. $f(EH(S,h)) = EH(S, \phi_\delta^{-1} \circ h)$.

Proof. By Proposition 3.4 we may take a basis $\{a_1, \dots, a_r\}$ for S so that δ is disjoint from $h(b_2), \dots, h(b_r)$, intersects $h(b_1)$ exactly once, and is parallel to $h(b_2)$. Then the result of performing $(+1)$-surgery along δ (or, equivalently, a negative Dehn twist along δ) is given by Figure 10.

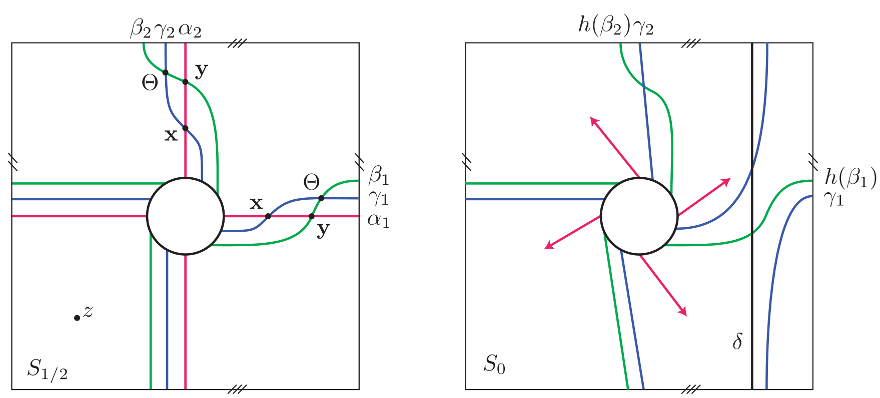

Figure 10. Legendrian (+1)-surgery. The second figure shows curves and their *relative* position correctly, but they are positioned on S_0 as if h^{-1} had been applied to each of them.

The α-curves and β-curves are as before, and we define the γ-curves as follows: Let $\gamma_1 = (b_1, \phi_\delta^{-1} \circ h(b_1))$ and $\gamma_i = (b_i, h(b_i))$ for $i > 1$. Let $\Theta \in \widehat{HF}(\gamma, \beta)$ be the top generator of $\#(S^1 \times S^2)$, given in Figure 10. Define the map

$$\phi : \widehat{HF}(\gamma, \beta) \otimes \widehat{HF}(\beta, \alpha) \to \widehat{HF}(\gamma, \alpha),$$

where $\phi(\mathbf{y} \otimes \mathbf{y}')$ counts holomorphic triangles, two of whose vertices are $\mathbf{y}$ and $\mathbf{y}'$. Then the map $f : \widehat{HF}(\beta, \alpha) \to \widehat{HF}(\gamma, \alpha)$ is given by $f(\mathbf{y}) = \phi(\Theta \otimes \mathbf{y})$. By the convenient placement of z, it follows that we only have small triangles in the Heegaard diagram. Hence if $[\mathbf{x}] = EH(S, h, \{a_1, a_2\})$, then $\phi([\Theta \otimes \mathbf{x}]) = EH(S, \phi_\delta^{-1} \circ h, \{a_1, a_2\})$. q.e.d.

3.6. Multiple boundary components. Consider (S, h) where S has disconnected boundary. For simplicity, assume S has two boundary components. Pick a basis $\{a_1, \dots, a_r\}$ for S. Next consider $(S', h\#id)$, where S' is obtained from S by attaching a 1-handle between the two boundary components and we are extending h by the identity. If a_0 is the cocore of the 1-handle, then $\{a_0, \dots, a_r\}$ is a basis for S'. Our argument is similar to that of Lemma 4.4 of [**OS3**]. The natural map

$$F_U : \widehat{HF}\left(\left(-M_{(S,h)}\right) \# \left(S^1 \times S^2\right)\right) \to \widehat{HF}\left(-M_{(S,h)}\right),$$

which corresponds to the cobordism U attaching a 3-handle as in Section 4.3 of [**OS4**], sends

$$EH\left(S', h\#id, \{a_0, a_1, \dots, a_r\}\right) \mapsto EH\left(S, h, \{a_1, \dots, a_r\}\right).$$

Since S' has only one boundary component, we already know that

$$c(S', h\#id) = EH\left(S', h\#id, \{a_0, a_1, \dots, a_r\}\right).$$

Moreover, if δ is a closed curve on S' which is "dual" to a_0, then there is a natural map

$$F_W : \widehat{HF}\left(-M_{(S,h)}\right) \to \widehat{HF}\left(\left(-M_{(S,h)}\right) \# \left(S^1 \times S^2\right)\right)$$

which maps $c(S, h)$ to $c(S', h\#id)$. Here (S, h) and $(S', \phi_\delta \circ (h\#id))$ represent the same 3-manifold, and W is the cobordism from $-M_{(S', \phi_\delta \circ (h\#id))}$ to $-M_{(S', h\#id)}$, obtained by attaching a 2-handle along the curve δ. Finally, $U \circ W \simeq [0,1] \times M_{(S,h)}$, so

$$c(S,h) = F_U \circ F_W(c(S,h)) = F_U(c(S', h\#id)) = EH(S, h, \{a_1, \dots, a_r\}).$$

4. Right-veering and holomorphic disks

In this section we prove Theorem 1.2.

Proof of Theorem 1.2. Let S be a once-punctured torus.

Suppose first that h has pseudo-Anosov monodromy. If the fractional Dehn twist coefficient $c \geq 1$, then the contact structure is already symplectically fillable and universally tight. It also follows that $c(\xi_{(S,h)}) \neq 0$. If $c = \frac{1}{2}$, then $c(\xi_{(S,h)}) \neq 0$ follows from Theorem 4.1 below. If $c \leq 0$, then ξ is overtwisted since S is not right-veering. (See [**HKM2**].)

If h is periodic, then ξ is right-veering if and only if h is a product of positive Dehn twists by [**HKM3**].

If h is reducible, then $c(\xi_{(S,h)}) \neq 0$ follows from Theorem 4.3 below.

q.e.d.

Theorem 4.1. *Let (S,h) be an open book decomposition for M, where S is a once-punctured torus and h is pseudo-Anosov with fractional Dehn twist coefficient $c = \frac{1}{2}$. Then $c(\xi_{(S,h)}) = EH(S,h) \neq 0$, and hence the contact structure $\xi_{(S,h)}$ is tight.*

Proof. We show that $EH(S,h) \neq 0$ by choosing a basis for S for which there are no holomorphic disks in the corresponding Heegaard diagram that map to the generator $\mathbf{x} = \{x_0, y_0\}$ defining $EH(S,h)$.

The following lemma furnishes us with a convenient basis:

Lemma 4.2. *Let $A \in SL(2,\mathbb{Z})$ be a matrix with $tr(A) < -2$. Then A is conjugate in $SL(2,\mathbb{Z})$ to a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where (a,c) and (b,d) are in the third quadrant.*

Proof. Let Λ^s and Λ^u be the stable and unstable laminations for A. The slopes of Λ^s and Λ^u will be written slope(Λ^s) and slope(Λ^u). (Recall that these slopes are irrational.) Let us consider the Farey tessellation on the hyperbolic unit disk D^2. Pick a vertex s_1 on the clockwise edge along ∂D^2 from slope(Λ^s) to slope(Λ^u), and pick a vertex s_2 on the counterclockwise edge from slope(Λ^s) to slope(Λ^u), so that there is an edge of the Farey tessellation between s_1 and s_2. (The existence of such a pair s_1, s_2 is an exercise.) Then $A(s_1)$ (resp. $A(s_2)$) is closer to slope(Λ^s) than s_1 (resp. s_2) is. An oriented basis corresponding to (s_1, s_2) will have the desired property. q.e.d.

With the choice of basis as above, we can represent $M = M_{(S,h)}$ by the Heegaard diagram below. We have drawn a picture of the diagram corresponding to $A = \left(\begin{smallmatrix} -1 & -1 \\ -1 & -2 \end{smallmatrix}\right)$, but the same argument works for any such A as described in the previous lemma. We prove that there is no holomorphic disk from any $\mathbf{y}$ to $\mathbf{x} = \{x_0, y_0\}$. Suppose on the contrary that there is such a holomorphic disk u. Let $\partial\mathcal{D}(u)$ be the boundary of the support of $\mathcal{D}(u)$. Assuming $\partial\mathcal{D}(u)$ is connected, it is given by a subarc of a_1 from some $x_i \in a_1 \cap h(a_2)$ to x_0, followed by a subarc of $h(a_1)$ from x_0 to some $y_j \in a_2 \cap h(a_1)$, followed by a subarc of a_2 from y_j to y_0 (you either turn left or turn right at y_j), and then by a subarc of $h(a_2)$ from y_0 to x_i. If we lift $\partial\mathcal{D}(u)$ to the universal cover of the capped off surface $T^2 = S \cup D^2$, then in all cases we see that $\partial\mathcal{D}(u)$ is not contractible. This implies that $\partial\mathcal{D}(u)$ cannot bound a surface in S. We argue similarly when $\partial\mathcal{D}(u)$ has two components. It follows that the class $EH(S,h)$ of $\mathbf{x} = \{x_0, y_0\}$ is nonzero. q.e.d.

Theorem 4.3. *$EH(S,h) \neq 0$ if h is reducible and right-veering.*

Proof. Suppose h is reducible. Let g be an element of $Aut(S, \partial S)$ which is the minimally right-veering representative for the matrix $A = -id$. (In terms of positive Dehn twists, $g = (A_1A_2A_1)^2$, where $A_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$.) After changing bases if necessary, $h = g^n \phi_\gamma^m$,

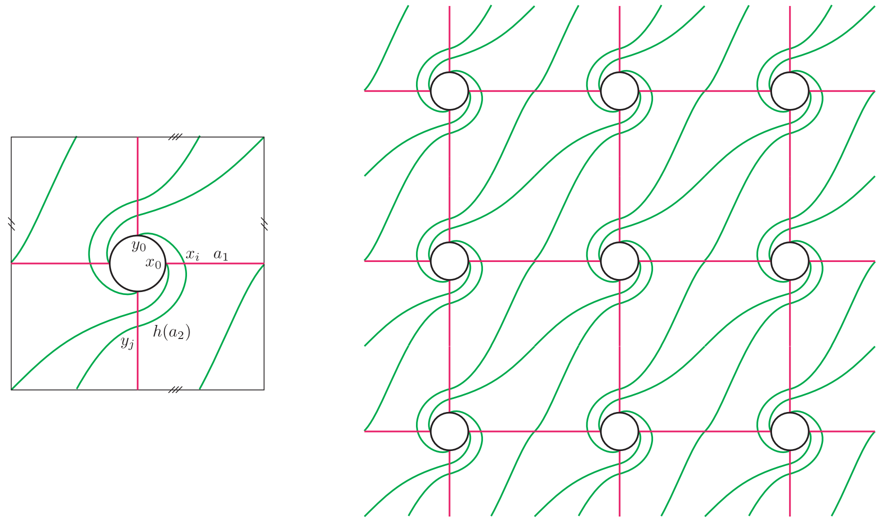

Figure 11

where n is a positive integer, m is an integer, and ϕ_γ is a positive Dehn twist about a $(0,1)$-curve γ. If m is nonnegative, then h is a product of positive Dehn twists, and $EH(S,h) \neq 0$.

Suppose $m < 0$. It suffices to prove the theorem for $n = 1$, since the contact structures corresponding to larger n are obtained from the $n = 1$ case by Legendrian surgery. Take a basis corresponding to slopes $0, \infty$ and matrix $A = \left(\begin{smallmatrix} -1 & 0 \\ -m & -1 \end{smallmatrix}\right)$. Then $EH(S,h)$ is nonzero by the same argument as in Theorem 4.1. q.e.d.

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