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CODIMENSION ONE FOLIATIONS WITH BOTT-MORSE SINGULARITIES I

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Abstract

In this article we show how the classical theory of Reeb and others extends to the case of codimension one singular foliations on closed oriented manifolds, provided that at the singular set $\mathrm{sing}(\mathcal{F})$, the foliation is locally defined by Bott-Morse functions which are transversely centers. We prove, in this setting, the equivalent of the local and the complete stability theorems of Reeb. We show that if $\mathcal{F}$ has a compact leaf with finite fundamental group, or if a component of $\mathrm{sing}(\mathcal{F})$ has codimension ≥ 3 and finite fundamental group, then all leaves of $\mathcal{F}$ are compact and diffeomorphic, $\mathrm{sing}(\mathcal{F})$ consists of two connected components, and there is a Bott-Morse function $f : M \to [0,1]$ such that $f : M \setminus \mathrm{sing}(\mathcal{F}) \to (0,1)$ is a fiber bundle defining $\mathcal{F}$ and $\mathrm{sing}(\mathcal{F}) = f^{-1}(\{0,1\})$. This yields a topological description of the type of leaves that appear in these foliations, and also the type of manifolds admitting such foliations. These results unify and generalize well known results for cohomogeneity one isometric actions, and a theorem of Reeb for foliations with Morse singularities of center type.

Introduction

Cohomogeneity one isometric actions of Lie groups, *i.e.*, actions where the principal orbits have codimension 1, play an important role in Differential Geometry, particularly in the Theory of Minimal Submanifolds (see for instance [**13**]). A basic well-known fact about these actions is that whenever the group and the manifold are compact, if all orbits are principal then the space of orbits is S^1, and if there are special orbits then there are exactly two of them and the space of orbits is the interval $[0,1]$. Notice that such an action defines a codimension one foliation with compact leaves and singular set the special orbits. Since the action is isometric, the intersection of the orbits with a slice Σ transverse to a special orbit corresponds to a Morse singularity of center type.

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From the Foliation Theory viewpoint this reminds us of two important results of Reeb. The first of them is the Complete Stability Theorem, which states that a transversely oriented non-singular codimension one foliation having a compact leaf with finite fundamental group on a closed manifold, is a fibration over the circle. The second result concerns foliations with non-empty singular set. It states that if a codimension one transversely oriented foliation on a closed manifold has only Morse (isolated) singularities of center type then there are exactly two such singularities and the manifold is homeomorphic to a sphere.

In this article we unify these two situations by introducing the concept of foliations with Bott-Morse singularities. This means that the singular set of such a foliation is a disjoint union of "nondegenerate critical manifolds", as used by Bott in his classical proof of the periodicity theorem (a concept which is already present in the landmark work of Morse in his Colloquium Publication [**18**]). More precisely, in a neighborhood of each singular point, the foliation is defined by a Bott-Morse function; so it is a usual Morse function restricted to each transversal slice.

Given such a foliation, the transverse type of each connected component of the singular set $\operatorname{sing}(\mathcal{F})$ is well-defined, and we can speak of components of center type, of saddle type, etc., according to the Morse index of the foliation on a transversal slice.

Throughout this paper and in particular in the statements below, the manifold M is connected and the foliation $\mathcal{F}$ is smooth of codimension one. We prove the following Complete Stability Theorem:

Theorem A. *Let $\mathcal{F}$ be a foliation with Bott-Morse singularities on a closed oriented manifold M of dimension $m \geq 3$ having only center type components in $\operatorname{sing}(\mathcal{F})$. Assume that $\mathcal{F}$ has some compact leaf L_o with finite fundamental group, or there is a codimension ≥ 3 component N of $\operatorname{sing}(\mathcal{F})$ with finite fundamental group. Then all leaves of $\mathcal{F}$ are compact, stable, with finite fundamental group. If, moreover, $\mathcal{F}$ is transversely orientable, then $\operatorname{sing}(\mathcal{F})$ has exactly two components and there is a differentiable Bott-Morse function $f\colon M \to [0,1]$ whose critical values are $\{0,1\}$ and such that $f\big|_{M\setminus\operatorname{sing}(\mathcal{F})}\colon M\setminus\operatorname{sing}(\mathcal{F}) \to (0,1)$ is a fiber bundle with fibers the leaves of $\mathcal{F}$.*

The proof of Theorem A actually shows that every compact transversely oriented foliation with non-empty singular set, all of Bott-Morse type, has exactly two components in its singular set and is given by a Bott-Morse function $f\colon M \to [0,1]$ as is in the statement.

The first step for proving Theorem A is the following Local Stability Theorem:

Theorem B. *Let $\mathcal{F}$ be a foliation on a manifold M^m having Bott-Morse singularities and let $N^n \subset \operatorname{sing}(\mathcal{F})$ be a (compact) component* *with finite holonomy group (e.g., if N has finite fundamental group). Then there exists a neighborhood W of N in M where $\mathcal{F}$ is given by a Bott-Morse function $f\colon W \to \mathbb{R}$. If, moreover, the transverse type of $\mathcal{F}$ along N is a center, then N is stable and the leaves of $\mathcal{F}$ in W, for a suitable choice of W, are fiber bundles over N with fiber S^{m-n-1}.*

Theorem A and its proof lead to the following generalization of Theorem 1.5 in [**15**]:

Theorem C. *Let $\mathcal{F}$ be a transversely oriented, compact foliation with Bott-Morse singularities on a closed, oriented, connected manifold M^m, $m \geq 3$, with non-empty singular set $\operatorname{sing}(\mathcal{F})$. Let L be a leaf of $\mathcal{F}$. Then $\operatorname{sing}(\mathcal{F})$ has two connected components N_1, N_2, both of center type, and one has:*

(i) *$M \setminus (N_1 \cup N_2)$ is diffeomorphic to the cylinder $L \times (0,1)$.*
(ii) *L is a sphere fiber bundle over both manifolds N_1, N_2 and M is diffeomorphic to the union of the corresponding disc bundles over N_1, N_2, glued together along their common boundary L by some diffeomorphism $L \to L$.*
(iii) *In fact one has a double-fibration*

$$N_1 \xleftarrow{\pi_1} L \xrightarrow{\pi_1} N_2 \ ,$$

and M is homeomorphic to the corresponding mapping cylinder, i.e., to the quotient space of $(L \times [0,1]) \bigcup (N_1 \cup N_2)$ by the identifications $(x, 0) \sim \pi_1(x)$ and $(x, 1) \sim \pi_2(x)$.

This yields a description of this type of foliations on manifolds of dimensions 3 and 4 (see Section 4).

Unless it is stated otherwise, in this work all manifolds, bundles, foliations and maps are assumed to be of class C^∞. This is just for simplicity, because essentially everything we say holds in class C^r, for all $r \geq 1$.

In Section 1 we give the precise definition of foliations with Bott-Morse singularities and discuss key-examples of such foliations. In Section 2 we extend the concept of holonomy to the case of components of the singular set of a foliation with Bott-Morse singularities, and we prove the Local Stability Theorem. In Section 3 we prove Theorem A (the Complete Stability Theorem). For this we first explain the way to adapt to the setting of foliations with center-type Bott-Morse singularities, the theory of Dippolito [**7**] (see also [**6**, **9**]) about saturated open sets in compact codimension one foliated manifolds. In Section 4 we focus on topological implications of Theorem A, thus arriving to theorems C and D, giving a classification of the 3-manifolds, and the corresponding leaves, admitting this type of foliations.

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1. Definitions and examples

Let $\mathcal{F}$ be a codimension one smooth foliation on a manifold M of dimension $m \geq 2$. We denote by $\operatorname{sing}(\mathcal{F})$ the singular set of $\mathcal{F}$. We say that the singularities of $\mathcal{F}$ are of *Bott-Morse type* if $\operatorname{sing}(\mathcal{F})$ is a disjoint union of a finite number of disjoint closed connected submanifolds, $\operatorname{sing}(\mathcal{F}) = \bigcup\limits_{j=1}^{t} N_j$, each of codimension ≥ 2, which are non-degenerate in the following sense: For each $p \in N_j$ there exists a neighborhood V of p in M where $\mathcal{F}$ is defined by a Bott-Morse function. That is, there exist a disc $P \subset \mathbb{R}^n$, a disc D in $\mathbb{R}^{m-n}$ centered at the origin, equipped with a foliation $\mathcal{G}$ given by the fibers of a Morse function on D, and a diffeomorphism $\varphi\colon V \to P \times D$, taking $\mathcal{F}|_V$ into the product foliation $P \times \mathcal{G}$. In other words, we can find local coordinates

$$(x, y) = (x_1, \dots, x_n, y_1, \dots, y_{m-n}) \in V\,,$$

such that $N_j \cap V = \big\{y_1 = \dots = y_{m-n} = 0\big\}$ and $\mathcal{F}|_V$ is given by the levels of a function $J_{N_j}(x, y) = \sum\limits_{j=1}^{m-n} \lambda_j\, y_j^2$ where $\lambda_j \in \{\pm 1\}$.

The discs $\Sigma_p = \varphi^{-1}(x(p) \times D)$ are transverse to $\mathcal{F}$ outside $\operatorname{sing}(\mathcal{F})$ and the restriction $\mathcal{F}|_{\sum_p}$ is an ordinary Morse singularity, whose Morse index does not depend on the point p in the component N_j. We shall refer to $\mathcal{G}(N_j) = \mathcal{F}|_{\sum_p}$ as the *transverse type* of $\mathcal{F}$ along N_j. This is a codimension one foliation in the disc Σ_p with an ordinary Morse singularity at $\{p\} = N_j \cap \Sigma_p$.

If N_j has dimension zero (or if we look at a transversal slice), then $\mathcal{F}$ has an ordinary Morse singularity at p and for suitable local coordinates, $\mathcal{F}$ is given by the level sets of a quadratic form $f = f(p) - (y_1^2 + \dots + y_r^2) + y_{r+1}^2 + \dots + y_m^2$, where $r \in \{0, \dots, m\}$ is the *Morse index* of f at p. The Morse singularity p is a *center* if r is 0 or m, otherwise p is called a *saddle*. In a neighborhood of a center, the leaves of $\mathcal{F}$ are diffeomorphic to $(m-1)$-spheres. In a neighborhood of a saddle q, we have conical leaves called *separatrices* of $\mathcal{F}$ through q, which are given by expressions $y_1^2 + \dots + y_r^2 = y_{r+1}^2 + \dots + y_m^2 \neq 0$. Each such leaf contains p in its closure.

Definition 1. A component $N \subset \operatorname{sing}(\mathcal{F})$ is of *center type* (or just *a center*) if the transverse type $\mathcal{G}(N) = \mathcal{F}|_{\sum_q}$ of $\mathcal{F}$ along N is a center. Similarly, the component $N \subset \operatorname{sing}(\mathcal{F})$ is of *saddle type* if its transverse type is a saddle.

As in the case of isolated singularities, these concepts do not depend on the choice of orientations. We denote by $C(\mathcal{F}) \subset \operatorname{sing}(\mathcal{F})$ the union of center type components, and by $S(\mathcal{F})$ the corresponding union of saddle components. Of course saddles can have different transversal Morse indices; this will be relevant for Part II of this article [**24**].

Definition 2. We say that $\mathcal{F}$ is *compact* if every leaf of $\mathcal{F}$ is compact (and consequently $S(\mathcal{F}) = \emptyset$). The foliation $\mathcal{F}$ is *closed* if every leaf of $\mathcal{F}$ is closed off $\operatorname{sing}(\mathcal{F})$.

If $\mathcal{F}$ is closed and M is compact, then all leaves are compact except for those containing separatrices of saddles in $S(\mathcal{F})$ and such a leaf is contained in a compact singular variety $\overline{L} = L \cup [\overline{L} \cap \operatorname{sing}(\mathcal{F})] \subset L \cup S(\mathcal{F})$. A closed foliation on a compact manifold is compact if and only if $S(\mathcal{F}) = \emptyset$.

Let $N \subset C(\mathcal{F})$ be a component of dimension k. Suppose that the nearby leaves of $\mathcal{F}$ are compact. We define $\Omega(N, \mathcal{F}) = \Omega(N) \subset M$ as the union of N and all the leaves $L \in \mathcal{F}$ which are compact and bound a compact invariant region $R(L, N)$ which is a neighborhood of N in M. The region $R(L, N)$ is equivalent to a fibre bundle with fibre the closed disc $\overline{D}^{m-k}$ over N, the fibers being transversal to the leaves of $\mathcal{F}$. As we will see, the notion of holonomy of the singular set, to be introduced in section 2.1, assures that if N is of center type and has finite holonomy group (e.g., if $\pi_1(N)$ is finite) then $\Omega(N, \mathcal{F})$ is an open subset of M.

Definition 3 (orientability and transverse orientability)**.** Let $\mathcal{F}$ be a codimension one foliation with Bott-Morse singularities on M^m, $m \geq 2$. The foliation $\mathcal{F}$ is *orientable* if there exists an $(m-1)$-form Ω on M^m, nonsingular on $M \setminus \operatorname{sing}(\mathcal{F})$, such that $\Omega\big|_L$ is a volume form on each leaf $L \in \mathcal{F}$. The choice of such an $(m-1)$-form Ω is called an *orientation* for $\mathcal{F}$. We shall say that $\mathcal{F}$ *is transversely orientable* if there exists a vector field X on M, possibly with singularities at $\operatorname{sing}(\mathcal{F})$, such that X is transverse to $\mathcal{F}$ at every point outside $\operatorname{sing}(\mathcal{F})$.

The following basic result is easily proved using the fact that we can always choose local orientations for $\mathcal{F}$, and also orientations along paths which are null-homotopic.

Proposition 1. *Let $\mathcal{F}$ be a codimension one foliation with Bott-Morse singularities on M^m, $m \geq 2$. Suppose M is orientable. Then:*

(i) *The foliation $\mathcal{F}$ is orientable if and only if it is transversely orientable.*

(ii) *If M is simply-connected, then $\mathcal{F}$ is transversely orientable.*

1.1. Examples. Basic examples of foliations with Bott-Morse singularities are given by Bott-Morse functions and by products of Morse foliations by closed manifolds. Next we give four types of examples of foliations with Bott-Morse singularities. We remark that foliations with Bott-Morse singularities are all examples of "generalized foliations" in the sense of [**25**].

Example 1 (Fiber bundles)**.** Let $\widetilde{M}^{m+k}$ and M^m be connected oriented manifolds. Let $\mathcal{F}$ be a foliation with Bott-Morse singularities on M and let $\pi\colon \widetilde{M} \to M$ be a proper submersion. Then the pull-back foliation $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ has only Bott-Morse singularities; hence $\widetilde{\mathcal{F}}$ is a foliation with Bott-Morse singularities and its transverse type at each component is that of $\mathcal{F}$ at the corresponding point.

For instance, take a vector field on an oriented closed surface S with non-degenerate singularities, and consider the corresponding foliation $\mathcal{L}$. Given any S^1-bundle $\pi\colon M \to S$, the pull-back foliation $\mathcal{F} = \pi^*(\mathcal{L})$ has Bott-Morse singularities on M; $\operatorname{sing}(\mathcal{F})$ is a union of circles.

In particular, the Hopf fibration $\pi\colon S^3 \to S^2$ gives rise, in this way, to Bott-Morse foliations on S^3. We can consider also $SO(3)$, regarded as the unit tangent bundle of S^2, to get examples on $SO(3) \cong \mathbb{R}P^3$.

Example 2 (Mapping cylinders and lens spaces)**.** Consider now a closed oriented manifold L that fibers as a sphere fiber bundle over two other manifolds N_1 and N_2, of possibly different dimensions, so that the corresponding disc bundles E_1, E_2 are compact manifolds with boundary L. Then each E_i can be foliated by copies of L by taking concentric spheres in the corresponding fibers. We may now glue E_1 and E_2 by some diffeomorphism of the common boundary L to get a closed oriented manifold M with a foliation with Bott-Morse singularities at N_1 and N_2, both of center type.

For instance, take two solid tori $S^1 \times D^2$, equipped with the same foliation, and glue their boundaries by a diffeomorphism that carries a meridian of the first torus into a curve on the second which is homologous to q-meridians and p-longitudes, with $p, q \geq 1$ coprime. We obtain foliations with Bott-Morse singularities on the so-called *lens spaces* $L(p, q)$ (see [**11**]).

Example 3 (Cohomogeneity one actions)**.** As mentioned before, a cohomogeneity one isometric action leads naturally to compact foliations with Bott-Morse singularities of center type.

For instance [**15**], consider $SO(n+1, \mathbb{R})$ as a subgroup of $SO(n+1, \mathbb{C})$. The standard action of this group on $\mathbb{C}^{n+1}$ defines an action of $SO(n+1, \mathbb{R})$ on $\mathbb{C}P(n)$, which is by isometries with respect to the Fubini-Study metric.

The special orbits are the complex quadric $Q_{n-1} \subset \mathbb{C}P(n)$, of points with homogeneous coordinates satisfying $\sum_{j=0}^{n} z_j^2 = 0$, and the real projective space $\mathbb{R}P(n) \subset \mathbb{C}P(n)$, consisting of the points which are fixed by the involution in $\mathbb{C}P(n)$ given by complex conjugation. The principal orbits are copies of the flag manifold

$$F_+^{n+1}(2,1) \cong SO(n+1,\mathbb{R})/(SO(n-1,\mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z}))\,,$$

of oriented 2-planes in $\mathbb{R}^{n+1}$ and (unoriented) lines in these planes. Each such orbit splits $\mathbb{C}P(n)$ in two pieces, each being a tubular neighborhood of a special orbit.

The case $n = 2$ is specially interesting because this provides an equivariant version of the Arnold-Kuiper-Massey theorem that $\mathbb{C}P(2)$ modulo conjugation is the 4-sphere, see for instance [**15**]. This is also proved in [**2**] and [**1**], where there are interesting generalizations of these constructions and theorem to the quaternionic and the octonian projective planes.

We also remark that the above foliations, given by a compact group action, are a special class of *Riemannian foliations*, introduced by P. Molino and studied by several authors (see [**17**]). Every singular Riemannian foliation has all its singularities of center-type.

Example 4 (Poisson manifolds)**.** A *Poisson structure* on a smooth manifold M consists of a Lie algebra structure on the ring of functions $C^\infty(M)$, generalizing the classical Poisson bracket on a symplectic manifold, which satisfies a Leibniz identity in such a way that $\{\,,h\}$ is a derivation. There is thus a vector bundle morphism $\psi\colon T^*M \to TM$ associated with $\{\,,\}$, satisfying an integrability condition, whose rank at each point is called the rank of the Poisson structure.

If the rank is constant, then the integrability condition implies one has a foliation on M, of dimension equal to the rank, and the tangent space of the foliation is, at each point $x \in M$, the image of $\psi(T_x^*M)$ in T_xM. If the rank is not constant, then one still has a generalized foliation in the sense of [**25**], *i.e.*, a foliation with singularities at the points where the rank drops, but at each such point one has a leaf of dimension the corresponding rank, whose tangent space is again given by $\psi(T_x^*M)$. The Dolbeault-Weinstein theorem implies that at such points the transversal structure plays a key role (see [**26**]).

It would be interesting to study Poisson structures for which the corresponding foliation has Bott-Morse singularities (*cf.* [**8**] for instance).

2. Holonomy and local stability

The notion of stability plays a fundamental role in the classical theory of (nonsingular) foliations. In what follows we bring this notion into our framework.

Definition 4. Let $\mathcal{F}$ be a (possibly singular) foliation on M. A subset $B \subset M$, invariant by $\mathcal{F}$, is *stable* (for $\mathcal{F}$) if for any given neighborhood W of B in M there exists a neighborhood $W' \subset W$ of B in M such that every leaf of $\mathcal{F}$ intersecting W' is contained in W.

The following technical result comes from the proof of the Complete Stability theorem of Reeb (cf. [**9**]):

Lemma 1. *Let $\mathcal{F}$ be a codimension one* (*nonsingular*) *foliation on M.*

(i) *Let L be a compact leaf of $\mathcal{F}$ and let L_n be a sequence of compact leaves of $\mathcal{F}$ accumulating on L. Then given a neighborhood W of L in M one has $L_n \subset W$ for all n sufficiently large.*

(ii) *Denote by L_x the leaf of $\mathcal{F}$ containing $x \in M$ and define M^* as the set of points $x \in M$ such that L_x is compact with finite fundamental group. Then every leaf contained in ∂M^* is closed in M.*

Remark 1 (Reeb's classical Complete Stability Theorem)**.** Let (M, $\mathcal{F}$) be a foliated manifold with M connected and compact, $\mathcal{F}$ smooth, transversely oriented of codimension one and $L_0 \in \mathcal{F}$ a compact leaf with finite fundamental group. Denote by $\Omega(\mathcal{F})$ the set of all compact leaves with finite fundamental group. Finally, let $\Omega(L_0) \subset \Omega(\mathcal{F})$ be the connected component containing L_0. By the Local Stability Theorem of Reeb, $\Omega(L_0)$ is an open subset of M. Put $U = \Omega(L_0)$ and suppose that $\partial U \neq \emptyset$. Let $V \subset M$ be a regular open set for $\mathcal{F}$ with $V \cap \partial U \neq \emptyset$ and denote by π the projection of V onto the space T of $\mathcal{F}$-plaques. The open set $\pi(V \cap U)$ of T is a countable union of disjoint open intervals which are bounded except for at most two of them. Given one of such bounded interval I, the saturation $\mathrm{Sat}_{\mathcal{F}\big|_V}(I) = \pi^{-1}(I)$ is at the same time open and closed in U and therefore each leaf of U crosses $\pi^{-1}(I)$. Hence $V \cap U$ has only finitely many connected components and therefore each leaf in the boundary of U is closed and compact. Such a leaf is also necessarily homeomorphic to the leaves in U, so it has finite fundamental group and cannot be a boundary leaf for ∂U, a contradiction. This proves in the classical framework that $U = M$, that is, $M = \Omega(L_0)$. This is the heart of the proof of the Complete Stability Theorem of Reeb. Later on, on Remark 4, we shall resume this discussion under another point of view.

2.1. Holonomy of the singular set. According to the proof of [**9**, Proposition 2.20], in case $\mathcal{F}$ is a compact foliation without singularities, stability of a leaf is equivalent to finiteness of its holonomy group. We will extend this result for compact codimension one foliations with Bott-Morse center singularities (see Proposition 2) using the following notion of holonomy.

Given a component $N \subset \operatorname{sing}(\mathcal{F})$, we consider a collection $\mathcal{U} = \{U_j\}_{j\in J}$ of open subsets $U_j \subset M$ and charts $\varphi_j \colon U_j \to \varphi_j(U_j) \subset \mathbb{R}^m$ with the following properties:

(1) Each $\varphi_j \colon U_j \to \varphi_j(U_j) \subset \mathbb{R}^m$ defines a local product trivialization of $\mathcal{F}$, $U_j \cap N$ is a disc and $\varphi_j(U_j)$ is a product of discs lying in plaques of $\varphi_j(\mathcal{F}\big|_{U_j})$.

(2) $\bigcup\limits_{j\in J} U_j$ is an open neighborhood of N in M.

(3) If $U_i \cap U_j \neq \emptyset$ then there exists an open subset $U_{ij} \subset M$ containing $U_i \cup U_j$ and a chart $\varphi_{ij} \colon U_{ij} \to \varphi_{ij}(U_{ij}) \subset \mathbb{R}^m$ of M, such that φ_{ij} defines a product structure for $\mathcal{F}$ in U_{ij} and $U_{ij} \cap N \supset (U_i \cup U_j \cap N) \neq \emptyset$.

Such a covering $\mathcal{U}$ will be called a *chain adapted* to $\mathcal{F}$ and N. When N is compact we can assume $\mathcal{U}$ to be finite, say $\mathcal{U} = \{U_1, \ldots, U_{\ell+1}\}$. Suppose now that $U_j \cap U_{j+1} \neq \emptyset$, for all $j \in \{1, \ldots, \ell\}$. In each U_j we choose a transverse disc Σ_j, $\Sigma_j \cap N = \{q_j\}$, such that $\Sigma_{j+1} \subset U_j \cap U_{j+1}$ if $j \in \{1, \ldots, \ell\}$. By the choice of $\mathcal{U}$, in each U_j the foliation is given by a smooth function $F_j \colon U_j \to \mathbb{R}$ which is the natural trivial extension of its restriction to any of the transverse discs Σ_j or Σ_{j+1}.

There is a C^∞ local diffeomorphism $\psi_j \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $F_{j+1}\big|_{\Sigma_{j+1}} = \psi_j \circ F_j\big|_{\Sigma_{j+1}}$. This implies that $F_{j+1} = \psi_j \circ F_j$ in $U_j \cap U_{j+1}$ (notice that by condition (3), if $U_i \cap U_k \neq \emptyset$ then every plaque of $\mathcal{F}$ in $U_i \backslash N$ intersects at most one plaque of $U_k \backslash N$).

Definition 5. The *holonomy map associated to the chain* $\mathcal{U} = \{U_1, \ldots, U_\ell\}$ is the local diffeomorphism $\psi \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ defined by the composition $\psi = \psi_\ell \circ \cdots \circ \psi_1$.

Given now a path $c \colon [0,1] \xrightarrow{C^0} N$, we can find a finite chain $\mathcal{U} = \{U_1, \ldots, U_{\ell+1}\}$ such that $\bigcup\limits_{j=1}^{\ell+1} U_j \supset c([0,1])$ and define *the holonomy map of* $c \colon [0,1] \to N$ as $\varphi = \varphi_\ell \circ \cdots \circ \varphi_1 \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. Clearly if $\widetilde{c} \colon [0,1] \to N$ is C^0-close to $c \colon [0,1] \to N$ and $\widetilde{c}(0) = c(0)$, $\widetilde{c}(1) = c(1)$ then c and $\widetilde{c}$ define the same holonomy map up to isotopy. This shows, by a standard argument, that the holonomy map of c is, up to isotopy, the same holonomy map of any curve $\widetilde{c}$ homotopic to c in N with $c(0) = \widetilde{c}(0)$, $c(1) = \widetilde{c}(1)$. If we now consider closed paths we obtain a map that associates to each homotopy class $[c] \in \pi_1(N, q_o)$ (where $q_o = c(0)$) the holonomy map of the path $c \colon [0,1] \to N$. This is indeed a group homomorphism $\operatorname{Hol} \colon \pi_1(N, q_o) \to \operatorname{Diff}^\infty(\mathbb{R}, 0)$ of the fundamental group of N based at q_o into the group of isotopy classes of germs of C^∞ diffeomorphisms fixing the origin $0 \in \mathbb{R}$. If we move either the base point or the discs Σ_j, or else if we change the coverings $\mathcal{U}$, then we obtain the same homomorphism up to conjugation in $\operatorname{Diff}^\infty(\mathbb{R}, 0)$.

Definition 6. We define *the holonomy group* of the component $N \subset \operatorname{sing}(\mathcal{F})$ as the image of the homomorphism $\operatorname{Hol}\colon \pi_1(N, q_o) \to \operatorname{Diff}^\infty(\mathbb{R}, 0)$ up to conjugacy in $\operatorname{Diff}^\infty(\mathbb{R}, 0)$.

In what follows, $N \subset \operatorname{sing}(\mathcal{F})$ is compact, connected and of Bott-Morse type. The following lemma proves the first statement in Theorem B.

Lemma 2. *If the holonomy group of the component $N \subset \operatorname{sing}(\mathcal{F})$ is finite, then there is a neighborhood W of N in M where $\mathcal{F}$ is given by a smooth function $f\colon W \to \mathbb{R}$.*

Proof. We recall (see for instance [**4**], Lemma 5, page 73) that a finite subgroup of $\operatorname{Diff}^\infty(\mathbb{R}, 0)$ is either trivial or has order two and therefore it is conjugate to the group generated by the involution $\varphi(x) = -x$ in $\operatorname{Diff}^\infty(\mathbb{R}, 0)$. Assume first that the holonomy is trivial. The proof is by a standard argument of extension by holonomy. We fix a point $q_o \in N$ and a transverse disc Σ_{q_o} such that $\mathcal{F}\big|_{\Sigma_{q_o}}$ is given by a Morse function $f_o\colon \Sigma_{q_o} \to \mathbb{R}$ singular only at $\{q_o\} = \Sigma_{q_o} \cap N$. Take a point $q \in N$ and consider a transverse disc Σ_q given by a transverse fibration as in the above definition of holonomy. Fix any curve $c_q\colon [0,1] \to N$ with $c_q(0) = q_o$ and $c_q(1) = q$. Given a point $y_o \in \Sigma_{q_o}$ we consider the lift $\tilde{c}_{y_o}\colon [0,1] \to L_y$ of the curve c_q to the leaf L_{y_o} of $\mathcal{F}$ through the point y_o. Put $y = \tilde{c}_{y_o}(1) \in \Sigma_q$. We define the value $f(y) = f_o(y_o)$. By triviality of the holonomy of N, the value $f(y)$ does not depend on the curve c_q. Thus we can define a function $f\colon W \to \mathbb{R}$ in an invariant tubular neighborhood W of N in M with the following properties:

(i) $f\big|_{\Sigma_{q_o}} = f_o$.

(ii) f is constant along the leaves of $\mathcal{F}$ in W.

(iii) The restriction $f\big|_{\Sigma_q}$ to a transverse disc Σ_q to N at q is conjugate to f_o by a holonomy map diffeomorphism $h_{c_q}\colon (\Sigma_{q_o}, q_o) \to (\Sigma_q, q)$.

And finally,

(iv) This extension f is a smooth first integral for $\mathcal{F}$ which is a submersion in $W \setminus N$.

Assume now that N has holonomy group generated by the real map $\varphi(x) = -x$. Then we can use the same proof of Lemma 2 above but replacing f_o by $(f_o)^2 = f_o.f_o$. This function $(f_o)^2(x) = \big(f_o(x)\big)^2$ is invariant by the holonomy $\varphi(x) = -x$ and therefore extends to a well-defined first integral for $\mathcal{F}$ in a neighborhood W of N in M. q.e.d.

Remark 2. If the holonomy has order 2, then we cannot assure that the first integral $f\colon W \to \mathbb{R}$ has connected fibers. Nevertheless, if $\mathcal{F}$ is transversely oriented then the holonomy of N consists of orientation preserving elements in $\operatorname{Diff}^\infty(\mathbb{R}, 0)$ and therefore it is finite if and only if it is trivial. This shows that the order 2 case in the proof of Lemma 2 does not occur if $\mathcal{F}$ is transversely oriented.

2.2. Proof of the Local stability. To prove Theorem B we use:

Proposition 2. *Let $\mathcal{F}$ be a transversely orientable foliation with Bott-Morse singularities on M. Given a compact component $N \subset$ $\operatorname{sing}(\mathcal{F})$ of center type we have:*

(i) *If all leaves near N are compact then the holonomy group of N is finite.*

(ii) *If the holonomy group of N is finite, then N is stable and the nearby leaves are all compact.*

Proof. First we prove (i). Suppose that N is a center and all leaves in a neighborhood of N are compact. Since the nearby leaves are compact, the holonomy group $\operatorname{Hol}(\mathcal{F}, N) \subset \operatorname{Diff}^\infty(\mathbb{R}, 0)$ is an orientation preserving group with finite orbits. This implies that this group is trivial, proving (i), and therefore $\mathcal{F}$ has a fibre bundle structure in a neighborhood of N in M, as noticed in the paragraph before Definition 3. In fact, as noticed there, one has a fundamental system of neighborhoods of N such that each neighborhood is equivalent to a fibre bundle with fibre the closed disc $\overline{D}^{m-k}$ over N, the fibers being transversal to the leaves of $\mathcal{F}$.

Thanks to this fibre bundle structure given any leaf L close enough to N, the leaf L bounds a region $R(L)$ in M, this region is invariant by $\mathcal{F}$ and such that $\lim_{L \to N} R(L) = N$. Because of the transverse orientation for $\mathcal{F}$ we can assume that the above limit is a decreasing limit so that N is stable.

Proof of (ii): As already mentioned, if $\operatorname{Hol}(\mathcal{F}, N)$ is finite then it is trivial and $\mathcal{F}$ has a fibre bundle structure in a neighborhood of N, which implies that N is stable with compact nearby leaves. q.e.d.

Remark 3. For codimension one transversely oriented *nonsingular* foliations, a *compact leaf* is stable if and only if it has trivial holonomy, this is due to Reeb [**23**]. This is not true for components of the singular set of foliations with Bott-Morse singularities and a counterexample for a one-dimensional singular component is given in Section 3.3. Also a compact leaf which is a limit of compact stable leaves is not necessarily stable, as shown by the following construction. Consider the sphere S^m as obtained by gluing $S^1 \times D^{m-1}$ and $D^2 \times S^{m-2}$ along their boundary. On $S^1 \times D^{m-1}$ we consider a non-compact foliation with leaves diffeomorphic to $\mathbb{R} \times S^{m-2}$, except for the boundary leaf which is diffeomorphic to $S^1 \times S^{m-2}$, and on $D^2 \times S^{m-2}$ we consider the trivial foliation with compact leaves $S^1(r) \times S^{m-2}$. These foliations can be glued together along the common boundary leaf $S^1 \times S^{m-2}$. The resulting foliation $\mathcal{F}$ is partially depicted in Figure 2 and has a non-stable compact leaf which is diffeomorphic to $S^1 \times S^{m-2}$, which is a limit of compact stable leaves $S^1(r) \times S^{m-2}$.

Proof of Theorem B. The first part of the theorem is exactly the content of Lemma 2. Assume now that the transverse type of $\mathcal{F}$ along N is a center. Then N is stable with compact nearby leaves (cf. Proposition 2 (ii)). It remains to prove that these leaves are fibre bundles over N with fiber S^{m-n-1}. The local product structure off N and the triviality of the holonomy group of N give a retraction of a suitable saturated neighborhood W of N onto N having as fibers transverse discs Σ to N. The restriction of this retraction to any leaf $L \subset W$ gives a proper smooth submersion of L onto N. The fibration theorem of Ehresmann [**9**] and the center type of N give the fibre bundle structure of L. q.e.d.

3. Complete Stability

In this section we prove the Complete Stability Theorem A. We begin with a brief discussion of the standard theory of saturated open sets, that we use in the sequel.

3.1. Dippolito's Semistability Theorem. Let us recall Dippolito's semistability theorem in [**7**] (see also [**6**, Sections 5.2, 5.3], [**9**, Section IV. 4] and [**12**, Chapter V, Sections 3, 4]).

We consider a codimension one, non-singular foliation $\mathcal{G}$ of class C^∞ in a closed C^∞ manifold M. Given such a pair $(M, \mathcal{G})$, we denote by $\mathcal{O}(\mathcal{G})$ the set of all open $\mathcal{G}$-saturated subsets of M. We assume that $\mathcal{G}$ is transversely oriented, and we let $\mathcal{L}$ be a one dimensional oriented foliation, defined by a smooth non-singular vector field transverse to $\mathcal{G}$.

Let $U \in \mathcal{O}(\mathcal{G})$ be connected; fix a riemannian metric on M and take its restriction to U. Let $d : U \times U \to [0, \infty)$ be the induced topological metric, and denote by $\widehat{U}$ its completion with respect to this metric. One has (see propositions 5.2.10 to 5.2.12 in [**6**]):

Proposition 3. i) *The space $\widehat{U}$ is a complete connected, C^∞ manifold with finitely many boundary components, and its interior $Int\widehat{U}$ is diffeomorphic to U.*

ii) *The manifold $\widehat{U}$ has a foliation $\widehat{\mathcal{G}}$ induced from that in U, and the inclusion $i : U \hookrightarrow M$ extends to a C^∞ immersion $\widehat{i} : \widehat{U} \hookrightarrow M$ that carries leaves of $\widehat{\mathcal{G}}$ diffeomorphically onto leaves of $\mathcal{G}$.*

iii) *If we let $\delta U = \widehat{i}(\partial\widehat{U})$ be the image of the boundary of $\widehat{U}$, then δU is a union of leaves of $\mathcal{G}$, and if L is a leaf in δU then $\widehat{i}^{-1}(L)$ consists of one or two leaves in $\partial\widehat{U}$.*

iv) *There is also an induced oriented foliation $\widehat{\mathcal{L}}$ on $\widehat{U}$, defined by a vector field transverse to $\widehat{\mathcal{G}}$, which is carried by $\widehat{i}$ into the foliation $\mathcal{L}$.*

Definition 7. The manifold $\widehat{U}$ is the *(abstract transverse) completion* of $U \in \mathcal{O}(\mathcal{G})$. The set $\delta(U)$ is the *border* of U; the leaves in δU are the *border leaves* of U.

That is, a border leaf $L \subset \delta U$ is the image under $\widehat{i}$ of a leaf in the boundary of the completion $\widehat{U}$, and there are at most two such leaves in $\partial\widehat{U}$ corresponding to L.

A *biregular* cover of $(M, \mathcal{G}, \mathcal{L})$ is a cover $M = \bigcup U_\alpha$ by open sets U_α equipped with coordinates $(x_\alpha, y_\alpha) \in U_\alpha$, such that $\mathcal{G}\big|_{U_\alpha}$ and $\mathcal{L}\big|_{U_\alpha}$ are trivial, the plaques of $\mathcal{G}$ in U_α are the level sets of y_α and the plaques of $\mathcal{L}$ in U_α are the level sets of x_α. Every foliated manifold $(M, \mathcal{G}, \mathcal{L})$ admits a biregular cover, which can be taken to be finite if M is compact.

We say that $U \in \mathcal{O}(\mathcal{G})$ is a *foliated product* (with respect to $\mathcal{L}$) if the restriction $\mathcal{L}\big|_U$ fibers U by open intervals over some $(m-1)$-manifold N. Since $\mathcal{L}$ is defined by a vector field, this bundle is trivial. Thus U is diffeomorphic to $N \times (0,1)$, but the restriction $\mathcal{G}\big|_U$ is not necessarily a product foliation.

We now recall that a leaf $L \in \mathcal{G}$ is *proper* if it is locally path-connected. This is equivalent to saying that L does not cluster on itself (see [**7**, p. 408] or [**6**, Def. 4.3.3]).

Definition 8. A leaf $L \in \mathcal{G}$ is *semiproper* if it is proper or does not cluster on itself from one side, which is called *positive* (see [**6**, p. 118] or [**9**, page 228]). The leaf $L \in \mathcal{G}$ is *semistable* if it is semiproper and on the proper side of L and in a transverse arc J on this side, that meets L only at a point $x_0 \in L$, the fixed points of the holonomy of L cluster at x_0 (by definition, a point $x \in J$ is a fixed point of the holonomy if, for every loop $\gamma \subset L$ based at the point x_0, the corresponding holonomy map h_γ either is not defined at the point $x \in J$ or it fixes the point $x \in J$, *i.e.*, $h_\gamma(x) = x$, see [**6**, p. 134]).

One has that a leaf $L \in \mathcal{G}$ is *semiproper* if and only if it is a border leaf of some $U \in \mathcal{O}(\mathcal{G})$ (see [**6**, Lemma 5.3.2]). Therefore each component of $\partial\widehat{U}$ is identified with a semiproper leaf of $\mathcal{G}$; some pairs of components may be identified with a same leaf.

Theorem 1 (Semistability theorem of Dippolito)**.** *Let L be a semiproper leaf which is semistable on the proper side defined by the transverse arc $J = [x_0, y_0)$. Then there is a point $y_1 \in J \setminus \{x_0\}$ such that the $\mathcal{G}$-saturation $U = \mathrm{Sat}_{\mathcal{G}}((x_0, y_1))$ is a foliated product having as border leaves the* (*distinct*) *leaves through x_0 and y_1. Also, there exists a sequence $\{y_k\}_{k=1}^{\infty} \subset (x_0, y_1]$ converging monotonically to x_0, such that the leaf $L_k \ni y_k$ is carried by the $\widehat{\mathcal{L}}$-fibration $\pi \colon \widehat{U} \to L$ homeomorphically onto L, for all $k \geq 1$.*

Remark 4 (Classical complete stability - revisited)**.** Let $(M, \mathcal{G})$ be as above with M compact and connected, and assume there is a compact leaf $L_0 \in \mathcal{G}$ with finite fundamental group. Denote by $\Omega(\mathcal{G})$ the set of all compact leaves with finite fundamental group, and let $\Omega(L_0) \subset \Omega(\mathcal{G})$ be the connected component containing L_0. Reeb's Complete Stability

Theorem claims that $\Omega(L_0)$ is all of M. To prove this, notice that by the Local Stability Theorem of Reeb, $\Omega(L_0)$ is an open subset of M; hence, by connectedness, we only need to prove that this set is also closed in M. Suppose that $\partial\Omega(L_0) \neq \emptyset$, so there is a border leaf $L \subset \delta\Omega(L_0)$. This leaf $L \subset \delta\Omega(L_0)$ is semiproper (see the paragraph just above Theorem 1), and since each leaf in $\Omega(L_0)$ is compact, the border leaf L must be semistable. By Dippolito's theorem 1, the leaf L is homeomorphic to the leaves inside $\Omega(L_0)$ and therefore it is compact with finite fundamental group. Thence $L \subset \Omega(L_0)$, a contradiction.

This sort of ideas apply to our situation, of foliations with singularities of center type, as we shall see below (see Lemma 3).

We now consider the case of a compact foliated manifold $(M, \mathcal{F}, \mathcal{L})$ with Bott-Morse singularities, all of center type and $\mathcal{F}$ transversely orientable. The basic concept of biregular cover can be extended as follows:

Definition 9. Choose $\mathcal{L}$ a one-dimensional foliation such that $\operatorname{sing}(\mathcal{L}) \subset \operatorname{sing}(\mathcal{F})$ and $\mathcal{L}$ is defined by a smooth vector field transverse to $\mathcal{F}$ off $\operatorname{sing}(\mathcal{F})$. Let $N \subset \operatorname{sing}(\mathcal{F})$ be a component of dimension $\ell < m$ and $p \in N$ a point. A *bi-distinguished* open neighborhood of p is an open set $V \subset M$, with $p \in V$, equipped with a diffeomorphism $\varphi\colon V \to \mathbb{R}^{m-\ell} \times \mathbb{R}^{\ell}$, such that φ conjugates the restriction $\mathcal{F}\big|_V$ to the foliation given by the quadratic form $\sum\limits_{j=\ell+1}^{\ell} y_m^2$ and conjugates the restriction $\mathcal{L}\big|_V$ to the foliation given by the vector field $X = \sum\limits_{j=\ell+1}^{m} y_j \frac{\partial}{\partial y_j}$. We define a *biregular cover* of $(M, \mathcal{F}, \mathcal{L})$ as an open cover of M which is a union of an ordinary biregular cover of $\Big(M \setminus \operatorname{sing}(\mathcal{F}), \mathcal{F}\big|_{\operatorname{sing}(\mathcal{F})}, \mathcal{L}\big|_{M\setminus\operatorname{sing}(\mathcal{F})}\Big)$ and an open cover of $\operatorname{sing}(\mathcal{F}) \subset M$, consisting of bi-distinguished neighborhoods as above.

It is now easy to prove:

Proposition 4. *Let M be a closed oriented manifold, $\mathcal{F}$ a foliation on M with Bott-Morse singularities, all of center type, having an oriented transverse foliation $\mathcal{L}$ off its singular set* $\operatorname{sing}(\mathcal{F})$. *Then $(M, \mathcal{F}, \mathcal{L})$ admits a finite biregular cover.*

Now, for a connected $\mathcal{F}$-invariant open subset $U \subset M \setminus \operatorname{sing}(\mathcal{F})$ we construct its (abstract transverse) completion $\widehat{U}$ in the same way as above. This is again a manifold with a singular foliation $\widehat{\mathcal{F}}$, and the inclusion of U in M extends to an immersion $\widehat{i} : \widehat{U} \to M$ carrying $\widehat{\mathcal{F}}$ onto $\mathcal{F}$. The border of U is $\widehat{i}(\partial\widehat{U})$ and it consists of finitely many components: some are semiproper leaves of $\mathcal{F}$ (recall that some pairs of leaves in $\partial\widehat{U}$ may be identified with the same leaf of $\mathcal{F}$) and others are connected components of $\operatorname{sing}(\mathcal{F})$.

3.2. Proof of Theorem A. From now on, $\mathcal{F}$ is a smooth foliation with non-empty singular set which is Bott-Morse of center type, in a compact, oriented, smooth manifold M of dimension ≥ 3. Such a foliation will be called *stable* if every leaf is compact and stable, and each component of the singular set is stable. Our first step is the following proposition:

Proposition 5. *Suppose there exists a compact leaf $L_o \in \mathcal{F}$ with finite fundamental group. Then every leaf of $\mathcal{F}$ is compact with finite fundamental group and the foliation is stable. If one also has that $\mathcal{F}$ is transversely orientable, then all leaves are diffeomorphic to L_o.*

Proof. Using the 2-fold transversely orientable covering of $\mathcal{F}$ we can assume in what follows that $\mathcal{F}$ is transversely orientable. To prove Proposition 5 denote by $\Omega(\mathcal{F})$ the union of leaves $L \in \mathcal{F}$ which are compact with finite fundamental group and by $\Omega(L_o)$ the connected component of $\Omega(\mathcal{F})$ that contains the leaf L_o. By the Reeb local stability theorem $\Omega(L_o)$ is open in $M \setminus \operatorname{sing}(\mathcal{F})$. Since $\Omega(L_o)$ is connected and $\mathcal{F}$ is transversely oriented, all the leaves in $\Omega(L_o)$ are diffeomorphic.

The following lemma implies that all the leaves are compact and diffeomorphic. It is here that we use the previous results about the structure of saturated open sets.

Lemma 3. *We have $\Omega(L_o) = M \setminus \operatorname{sing}(\mathcal{F})$.*

Proof. Put $U = \Omega(L_o)$ and let $\widehat{U}$ be its abstract transverse completion. We prove first that $\partial\widehat{U}$ has no nonsingular component, *i.e.*, it contains no leaf. Assume by contradiction that there is a leaf $\widehat{L} \subset \partial\widehat{U}$. By the above theory of saturated open sets, its image in M by the map $\widehat{i}$ in 3 is a leaf L contained in the border δU; this leaf is semiproper (because it is a border leaf of a saturated open subset; see the comment after Definition 8) and indeed semistable, due to the compactness of the leaves in U. Thence, as in the classical non-singular case, by Dippolito's Semistability Theorem 1, L is homeomorphic to the leaves in $U = \Omega(L_o)$. Thus L is compact with a finite fundamental group, and one has $L \subset U$, giving a contradiction.

Now assume there is a leaf $L \subset \partial U$ in the boundary of the original open set $U \subset M$ (which is not a border leaf, $L \not\subset \delta U$). Since U is a union of compact leaves of the foliation, this leaf L is compact by a well-known theorem of Haefliger. On the other hand, the leaves in U are compact. This implies that L is semistable on whatever side is approached by points of U. Hence Dippolito's semistability theorem implies that L is indeed homeomorphic to the leaves inside U, contradicting the fact that L is a leaf in the boundary of U and proving 3. q.e.d.

In order to conclude that $\mathcal{F}$ is stable it is enough to apply the Local stability theorem of Reeb for the stability of the leaves, and Proposition 2 for the stability of the singular set. q.e.d.

The second step in the proof of Theorem A is:

Proposition 6. *Assume $\mathcal{F}$ has a codimension ≥ 3 component $N_o \subset \operatorname{sing}(\mathcal{F})$ with finite fundamental group. Then all leaves are compact with finite fundamental group.*

Proof. Again, we can assume that $\mathcal{F}$ is transversely orientable. Let $N_o^{n_o}$ be as above, with codimension $m - n_o \geq 3$. By the Local Stability (Theorem B) or by Proposition 2, N_o is stable with compact nearby leaves, and it has trivial holonomy (see Remark 2). Hence each leaf of $\mathcal{F}$, in a suitable neighborhood W of N, is diffeomorphic to an S^{m-n_o-1}-fibre bundle over $N_o^{n_o}$. Since $m - n_o \geq 3$, the homotopy sequence of the fibration $S^{m-n_o-1} \hookrightarrow L \to N_o^{n_o}$ implies that L has finite fundamental group, so we can apply Proposition 5 to conclude the statement in Proposition 6. q.e.d.

Remark 5. The condition on the codimension of N_o in Proposition 6 is indeed necessary. For instance, consider a foliation with Bott-Morse singularities on $S^2 \times S^2$ given by the product of a non-periodic flow with exactly two center type singularities on S^2 by the sphere S^2, which has non-compact leaves.

The existence of the function $f\colon M \to [0,1]$ describing $\mathcal{F}$ in Theorem A is a consequence of the following lemma.

Lemma 4. *Assume $\mathcal{F}$ is transversely orientable. Then $\operatorname{sing}(\mathcal{F})$ has exactly two connected components, say N_1, N_2, and there exists an arc $\gamma\colon [0,1] \to M$ transverse to $\mathcal{F}$ such that $\gamma(0) \in N_1$, $\gamma(1) \in N_2$, whose image meets every leaf of $\mathcal{F}$ at a single point.*

Proof. Let us prove first that $\operatorname{sing}(\mathcal{F})$ has at most two connected components. Take a component $N \subset \operatorname{sing}(\mathcal{F})$ and denote by $\mathcal{A}(N)$ the subset of M which is the union of leaves $L \in \mathcal{F}$ such that L bounds a region $R(L) \subset M$ containing N and such that $\mathcal{F}\big|_{R(L)\setminus N}$ is a fibre bundle over N. Clearly $N \subset \partial\mathcal{A}(N)$ and $\partial R(L) \setminus L \subset \operatorname{sing}(\mathcal{F})$. Suppose there is a singular component $N' \subset \partial\mathcal{A}(N) \setminus N$ and let us prove one has $M = N \cup \mathcal{A}(N) \cup N'$. First we claim that $S = N \cup \mathcal{A}(N) \cup N'$ is an open subset of M. To see this, take an invariant neighborhood W of N' given by the local stability theorem for N'. Since $N' \subset \partial\mathcal{A}(N)$, there is a leaf $L \subset \mathcal{A}(N)$ which intersects W and therefore is entirely contained in W. By definition of $\mathcal{A}(N)$ the leaf L bounds a region $R(L) \subset M$ such that $\mathcal{F}\big|_{R(L)\setminus N}$ is a fiber bundle over N and by the choice of W, L bounds a region $R'(L) \subset W$ such that $\mathcal{F}\big|_{R(L')\setminus N'}$ is a fiber bundle over N'. Finally, since $\mathcal{F}$ has a local product structure in a neighborhood of L we conclude that $W \setminus N' \subset \mathcal{A}(N)$. Thus S is open in M. The above arguments also show that S contains the union of two compact submanifolds with boundary which are glued along their common boundary (L above). Hence S equals M and therefore $\operatorname{sing}(\mathcal{F})$ has exactly two connected components, as stated in 4.

To construct the arc γ in the statement we first need:

Claim 1. *Let $\gamma_o \colon S^1 \to M$ be a closed curve transverse to $\mathcal{F}$ and to* $\operatorname{sing}(\mathcal{F})$*. Then γ_o intersects all leaves of $\mathcal{F}$ and all components of* $\operatorname{sing}(\mathcal{F})$*.*

Proof of Claim 1. Denote by Ω the set of all leaves $L \in \mathcal{F}$ such that $\gamma_o \cap L \neq \emptyset$. By transversality this is an open set. To see this set is also closed in $M \setminus \operatorname{sing}(\mathcal{F})$ take a nonsingular point $p \in \partial\Omega$ and choose an invariant neighborhood W of the leaf L_p, given by the local stability, where $\mathcal{F}$ is trivial. Because of this triviality, every curve transverse to $\mathcal{F}$ intersects all leaves in W. Since $p \in \partial\Omega$ we have $\gamma_o \cap W \neq \emptyset$ and therefore $\gamma_o \cap L_p \neq \emptyset$. Thus Ω is closed in $M \setminus \operatorname{sing}(\mathcal{F})$ and $\Omega = M \setminus \operatorname{sing}(\mathcal{F})$. Similar arguments prove that γ_o intersects each component of $\operatorname{sing}(\mathcal{F})$.

q.e.d.

As a consequence we have:

Claim 2. *Let $\gamma \colon [0,1] \to M$ be a curve such that $\gamma(0) \in N_1$ and $\gamma(1) \in N_2$ and $\gamma(0,1)$ is everywhere transverse to $\mathcal{F}$. Then $\#(\gamma \cap L) \leq 1$, for each leaf L of $\mathcal{F}$.*

Proof of Claim 2. Suppose by contradiction that γ intersects twice some leaf L of $\mathcal{F}$. Let us choose two such points $p_1 = \gamma(t_1) \in L$ and $p_2 = \gamma(t_2) \in L$, $t_2 > t_1$, and a path $\beta \colon [0,1] \to L$ joining p_1 to p_2. By a classical argument, there exists $\delta > 0$ and a smooth closed curve γ_o transverse to $\mathcal{F}$, such that γ_o contains the arc $C = \alpha([t_1 + \delta, t_2 - \delta])$ and the complement $\gamma_o([0,1]) \setminus C$ projects onto β via a transverse fibration with basis β. Thus we can construct a closed curve γ_o transverse to $\mathcal{F}$ and which avoids a neighborhood of N. This contradicts Claim 1.

q.e.d.

Let X be a vector field transverse to $\mathcal{F}$ on M. Let $N \subset \operatorname{sing}(\mathcal{F})$ be given, we can assume that X is radial pointing outwards in a neighborhood of N. Consider a point $p \in N \subset \operatorname{sing}(\mathcal{F})$ and the orbit γ of X whose α-limit is p. We consider the ω-limit $\omega(\gamma)$. Then $\omega(\gamma)$ avoids a neighborhood of N. In fact

Claim 3. *We have $\omega(\gamma) = \{q\}$ where $q \in \operatorname{sing}(\mathcal{F}) \setminus N$.*

Proof of Claim 3. Suppose $\omega(\gamma)$ contains some non-singular point q. Then γ cuts the leaf L_q infinitely many times. Let us choose two such points $p_1 = \gamma(t_1)$ and $p_2 = \gamma(t_2)$, $t_2 > t_1$ close enough to q so that they avoid a neighborhood of N and a path $\beta \colon [0,1] \to L_q$ joining p_1

to p_2. By a classical argument $\exists\,\delta > 0$ and a smooth closed curve γ_o, transverse to $\mathcal{F}$ such that γ contains the arc $C = \alpha([t_1+\delta, t_2-\delta])$ and the complement $\gamma_o([0,1])\backslash C$ projects onto β via a transverse fibration with basis β. Thus we can construct a closed curve γ_o transverse to $\mathcal{F}$ and which avoids a neighborhood of N. This contradicts Claim 1. Therefore we must have $\omega(\gamma) \subset \operatorname{sing}(\mathcal{F})$. Since X is radial in a neighborhood of $\operatorname{sing}(\mathcal{F})$ we have that $\omega(\gamma)$ has to be a single point, say $\omega(\gamma) = q \in \operatorname{sing}(\mathcal{F})$. Because X points outwards in a neighborhood of N we have that q cannot belong to N, proving the claim. q.e.d.

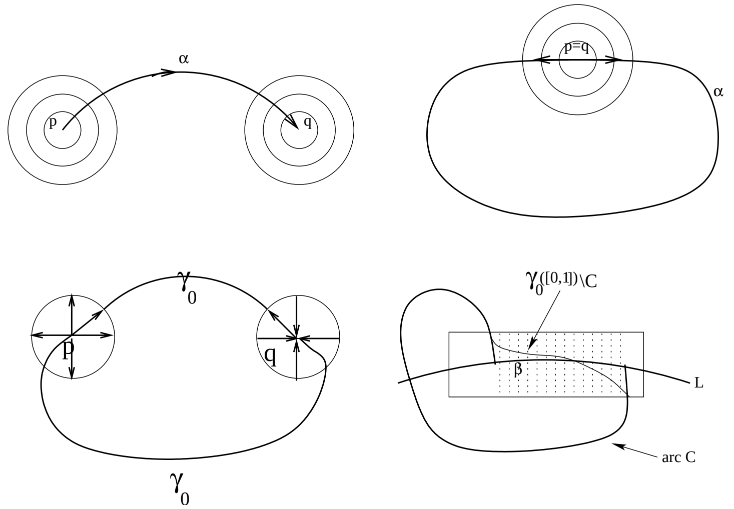

Figure 1

By Claim 3 we have an arc $\gamma_o\colon [0,1] \to M$ such that $\gamma_o(0) \in N_1$ and $\gamma(1) \in N_2$ and $\gamma(0,1)$ is everywhere transverse to $\mathcal{F}$. By Claim 2 γ_o intersects each leaf of $\mathcal{F}$ at most once. Thus, there is a leaf $L_o \in \mathcal{F}$ such that γ_o cuts L_o exactly one time.

Claim 4. $\#(\gamma_o \cap L) = 1$, *for each leaf* L *of* $\mathcal{F}$.

Proof of Claim 4. Let $\mathcal{O}$ be the set of points $x \in M\backslash \operatorname{sing}(\mathcal{F})$ such that $\#(\gamma_o \cap L_x) = 1$. By Local Stability we have a local product structure for $\mathcal{F}$ around each compact leaf L and therefore $\mathcal{O}$ is open in $M\backslash \operatorname{sing}(\mathcal{F})$. We claim that $\partial\mathcal{O} = \operatorname{sing}(\mathcal{F})$. Assume by contradiction that there is a leaf $L \subset \partial\mathcal{O}$. Then by the local product structure we have $\#(\gamma_o \cap L_x) = \#(\gamma_o \cap L)$, for all x close enough to L and therefore we get a contradiction. This shows that $M = \mathcal{O} \cup \operatorname{sing}(\mathcal{F})$ and proves the claim. q.e.d.

Now, by the local structure of $\mathcal{F}$ around the singularities we obtain that also $\#(\gamma_o \cap N) = 1$ for each component $N \subset \operatorname{sing}(\mathcal{F})$. This ends the proof of the lemma. q.e.d.

Theorem A is now an immediate consequence of Propositions 5 and 6 and Lemma 4.

3.3. Examples and remarks on stability. The condition on the codimension of the singular set in Theorem A is necessary, for otherwise one can construct examples of foliations with Bott-Morse singularities with only center type singularities and non-compact leaves, as for instance the example in Remark 3. A modification of that construction gives a foliation with center type singularities which are limits of compact leaves and also of noncompact leaves. For this, let A^m be a compact annulus (*i.e.*, an m-disc minus a smaller m-disc in its interior), and consider a foliation $\mathcal{F}_A$ in $S^1 \times A^m$ tangent to the boundary $\partial(S^1 \times A^m) = (S^1 \times S_1^{m-1}) \uplus (S^1 \times S_2^{m-1})$, transverse to the annuli $\{z\} \times A^m, \;\; z \in S^1$ and such that each restriction $\mathcal{F}_A\big|_{\{z\}\times A^m}$ is equivalent to the trivial foliation by $(m-1)$-spheres concentric and tangent to the boundary of A^m. We may also choose $\mathcal{F}_A$ so that each leaf on $S^1 \times A^m$, outside the boundary, is non-compact and accumulates on both components of $\partial(S^1 \times A^m)$ as $\mathcal{F}_o$ above. Now we consider a sequence of positive numbers $1 = r_1 > r_2 > \cdots > r_j > r_{j+1} > \cdots$ converging to zero. Let A_j be the annulus of internal radius r_{j+1} and external radius r_j. On each solid annulus we put a copy $\mathcal{F}_{A_j}$ of $\mathcal{F}_A$. Glue all these foliations in a foliation $\mathcal{F}'_o$ of the product $S^1 \times D^m$ to get a foliation there, with singular set $S^1 \times \{pt\}$ of center type. Finally glue two copies of $\mathcal{F}'_o$ into a foliation $\mathcal{F}$ of $S^1 \times S^m$ with two circles N_1, N_2 as singular set, both with center types. Each component N_j is a limit of compact leaves (diffeomorphic to $S^1 \times S^{m-2}$) and also of noncompact leaves (diffeomorphic to $\mathbb{R} \times S^{m-2}$) as well. In particular, N_j *is stable without trivial holonomy* and $\mathcal{F}$ is not compact, although N_j is of center type and is a limit of compact leaves.

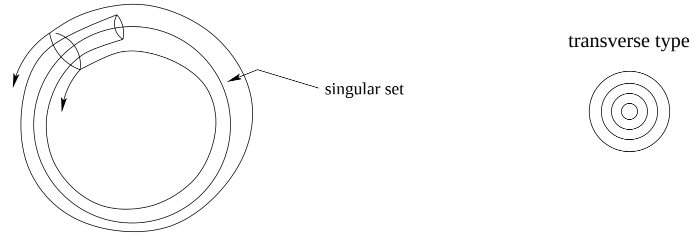

Figure 2

Example 5. We decompose S^3 as the union of two solid tori $S^3 = (S_1^1 \times D_1^2) \cup (D_2^2 \times S_2^1)$ with common boundary $S_1^1 \times S_2^1$. In the solid torus $S_1^1 \times D_1^2$ we consider the product foliation $S^1 \times \mathcal{C}$ where $\mathcal{C}$ is the foliation of the 2-disc by concentric circles. We decompose the second solid torus as the union of a solid torus and a solid annulus with common boundary $S_3^1 \times S_2^1$, *i.e.*, $D_2^2 \times S_2^1 = (A^2 \times S_2^1) \cup (D_3^2 \times S_2^1)$. In the solid torus $D_3^2 \times S_2^1$ we put another trivial foliation $\mathcal{C} \times S^1$. Finally, in the solid annulus $A^2 \times S_2^1$ we consider a product foliation $\mathcal{F} \times S_2^1$ where $\mathcal{F}$ is a one-dimensional foliation in A^2 as follows:

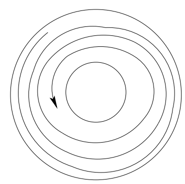

Figure 3

$\mathcal{F}$ has a noncompact leaf accumulating the two circles in the boundary of A^2. Gluing all together we obtain a foliation $\mathcal{F}$ on S^3 with singular set $\operatorname{sing}(\mathcal{F})$ = union of two circles which are stable with respect to $\mathcal{F}$, however $\mathcal{F}$ is not a foliation by compact leaves due to the noncompact leaves in $A^2 \times S_2^1$. This construction cannot be performed for dimension $m \geq 4$ as is implied by Proposition 6.

Let now $\mathcal{F}$ and f be as in Theorem A and assume $m \geq 4$. If $\operatorname{sing}(\mathcal{F})$ has some isolated singularity then by Reeb complete stability theorem all leaves are diffeomorphic to S^{m-1}. Nevertheless, $\operatorname{sing}(\mathcal{F})$ is not necessarily of dimension zero. For instance, take the classical Hopf fibration of S^3 over S^2 with fiber S^1. Now consider the corresponding disc bundle over S^2. Its total space E^4 is a four dimensional manifold with boundary S^3. Using the discs D^2 bounded by the fibers we can construct a foliation with Bott-Morse singularities $\mathcal{F}_1$ of E^4 having compact leaves diffeomorphic to S^3 and singular set $S^2 \cong \mathbb{C}P(1)$. Now glue to E^4 a four dimensional disc in the obvious way to obtain the complex projective plane $\mathbb{C}P(2)$ and a foliation with Bott-Morse singularities $\mathcal{F}$ of $\mathbb{C}P(2)$ with leaves S^3 and singular set S^2 union a point. This same construction generalizes to $\mathbb{C}P(n)$ regarded as the union of a $2n$-disc and $\mathbb{C}P(n-1)$.

4. Compact foliations with Bott-Morse singularities

Let $\mathcal{F}$ be a transversely oriented, compact foliation with Bott-Morse singularities on the closed, oriented, connected manifold M^m, $m \geq 3$. Notice that Proposition 2 implies that (besides each leaf L of $\mathcal{F}$) each component N of the singular set is stable with finite holonomy. By Lemma 4 one has Theorem C as an immediate consequence. This obviously imposes stringent conditions on both, the topology of M and L. Let us see what this says when M has dimensions 3 and 4. If $m = 3$, then L must be a two-dimensional closed oriented manifold that fibers over another manifold of dimension 0 or 1, with fiber a sphere. The only possibilities for L are to be S^2, fibered over a point, or the 2-torus $T = S^1 \times S^1$, since the are no other S^1-bundles over S^1, except for the Klein bottle which is not orientable. Hence the possibilities for the double-fibration in Theorem C are:

(i) If N_1 is a point, then L must be a 2-sphere S^2, and this surface does not fiber over S^1, hence N_2 must be also a point. This is the classical case envisaged by Reeb and others, the leaves are copies of S^2 and M is the 3-sphere, regarded as the suspension over S^2.

(ii) If N_1 is a circle, then L is the torus $T = S^1 \times S^1$ and M is the result of gluing together two solid tori along their common boundary. The manifolds one gets in this way are either orientable S^1-bundles over S^2 (and there is one such bundle for each integer, being classified by their Euler class), or a lens space $L(p, q)$, obtained by identifying two solid tori by a diffeomorphism of their boundaries that carries a meridian into a curve of type (p, q) in T.

Notice that Theorem C implies:

Theorem D. *Let M be a closed oriented connected 3-manifold equipped with a transversely oriented compact foliation $\mathcal{F}$ with Bott-Morse singularities. Then either $\operatorname{sing}(\mathcal{F})$ consists of two points, the leaves are 2-spheres and M is S^3, or $\operatorname{sing}(\mathcal{F})$ consists of two circles, the leaves are tori and M is homeomorphic to a lens space or to an S^1-bundle over S^2.*

We remark that the hypothesis of having a compact foliation is necessary, otherwise the conclusion of Theorem D does not hold. For instance, decompose S^3 as a union of two solid tori T_1, T_2, as usual. Foliate $T_1 = S^1 \times D^2$ by concentric tori $S^1 \times S^1$, and put Reeb's foliation on T_2. We get a foliation on S^3 with singular set a circle of center type.

Examples 1 and 2 show that all S^1-bundles over S^2 and all lens spaces admit compact foliations as in Theorem D.

When $m = 4$ the list of possibilities for L and M is larger. For instance, we can foliate S^4 in various ways:

- By 3-spheres with two isolated centers.
- By copies of $S^1 \times S^2$ with two circles as singular set.
- Think of S^4 as being the space of real 3×3 symmetric matrices A of trace zero and $\operatorname{tr}(A^2) = 1$. The group $SO(3, \mathbb{R})$ acts on S^4 by

$A \mapsto O^t AO$, for a given $O \in SO(3,\mathbb{R})$ and $A \in S^4$. As noticed in [**13**] this gives an isometric action of $SO(3,\mathbb{R})$ on the sphere S^4 with two copies of $\mathbb{R}P(2)$ as singular set. The leaves are copies of the flag manifold

$$F^3(2,1) \cong SO(3,\mathbb{R})/(\mathbb{Z}/2\mathbb{Z} \times Z/2\mathbb{Z}) \cong L(4,1)/(\mathbb{Z}/2\mathbb{Z}),$$

of (unoriented) planes in $\mathbb{R}^3$ and lines in these planes.

- Now consider the complex projective plane $\mathbb{C}P(2)$. Thinking of it as being $\mathbb{C}^2$ union the line at infinity, one gets a foliation by copies of S^3 with an isolated singularity at the origin and a copy of $S^2 \cong \mathbb{C}P(1)$ at infinity.
- Notice that, as in Example 3, the group $SO(3,\mathbb{R})$ is a subgroup of $SO(3,\mathbb{C})$ and therefore acts on $\mathbb{C}P(2)$ in the usual way. The orbits of this action are copies of the Flag manifold $F^3_+(2,1) \cong SO(3,\mathbb{R})/(\mathbb{Z}/2\mathbb{Z})$, which is a double cover of $F^3(2,1)$. The singular set now consists of the quadric $\sum\limits_{j=0}^{2} z_j^2 = 0$, which is diffeomorphic to S^2, and a copy of $\mathbb{R}P(2)$. As in Example 3, this foliation is mapped to the above foliation of S^4 by the projection $\mathbb{C}P(2) \to \mathbb{C}P(2)/j \cong S^4$, where $j\colon \mathbb{C}P(2) \to \mathbb{C}P(2)$ is complex conjugation (by [**2**], [**1**] or [**15**]).

Let us discuss the various possibilities for L and M. Let N_1 and N_2 be the connected components of $\operatorname{sing}(\mathcal{F})$.

(i) If N_1 is a point then each leaf L must be S^3.

We claim that there are three possibilities for N_2: it can be either a point, the 2-sphere or the projective plane $\mathbb{R}P(2)$. Indeed, L fibers over N_2 with fiber a sphere, and S^3 does not fiber over S^1. This implies that N_2 has cannot have dimension one. If N_2 has dimension two then necessarily is diffeomorphic to S^2 or to $\mathbb{R}P(2)$. Thus the possibilities are the following:

(i.a) If N_2 is also a point, then M is S^4 by Reeb's theorem.

(i.b) If N_2 is the 2-sphere then one has a fiber bundle

$$S^1 \hookrightarrow S^3 \longrightarrow N_2\,;$$

such a bundle necessarily corresponds to a free S^1-action on S^3. The effective actions of S^1 on 3-manifolds are classified in [**19**], and the only free action on S^3 is the usual one, which gives the Hopf fibration $S^1 \hookrightarrow S^3 \longrightarrow S^2$, and M is the complex projective plane $\mathbb{C}P^2$. Of course the projection $S^2 \to \mathbb{R}P(2)$ gives a fibre bundle $S^1 \hookrightarrow S^3 \longrightarrow \mathbb{R}P(2)$.

(ii) If N_1 is a circle, then L fibers over $S^1 \cong N_1$ with fiber a 2-sphere, so L is $S^1 \times S^2$, and N_2 can be either a circle S^1, S^2 or $\mathbb{R}P(2)$. If $N_2 \cong S^1$ then both fibrations $L \xrightarrow{\pi_i} N_i$, $i = 1,2$, necessarily coincide. Then M is the result of taking two copies of the corresponding disc bundle, and glued them along their common boundary L by some diffeomorphism. If N_2 is S^2 or $\mathbb{R}P(2)$ then L is a product $S^1 \times S^2$.

(iii) If N_1 and N_2 are both surfaces, then they can be oriented or not, and L is a closed, oriented Seifert manifold. The manifolds N_1 and N_2 cannot be arbitrary, since L must fiber over both of them simultaneously, but there is a lot of freedom. For instance, notice that we can use the procedure in Example 2 to construct compact foliations with Bott-Morse singularities whenever we have a double-fibration as in Theorem C, regardless of whether or not the hypothesis of Theorem A are satisfied.

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