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BLOW-UP PHENOMENA FOR THE YAMABE EQUATION II

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Abstract

Let n be an integer such that $25 \leq n \leq 51$. We construct a smooth metric g on S^n with the property that the set of constant scalar curvature metrics in the conformal class of g is not compact.

1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. The Yamabe problem is concerned with finding metrics of constant scalar curvature in the conformal class of g. This problem leads to a semi-linear elliptic PDE for the conformal factor. More precisely, a conformal metric of the form $u^{\frac{4}{n-2}}\,g$ has constant scalar curvature c if and only if

$$\frac{4(n-1)}{n-2}\,\Delta_g u - R_g\,u + c\,u^{\frac{n+2}{n-2}} = 0, \tag{1}$$

where Δ_g is the Laplace operator with respect to g and R_g denotes the scalar curvature of g. Every solution of (1) is a critical point of the functional

$$E_g(u) = \frac{\int_M \big(\frac{4(n-1)}{n-2}\,|du|_g^2 + R_g\,u^2\big)\,dvol_g}{\big(\int_M u^{\frac{2n}{n-2}}\,dvol_g\big)^{\frac{n-2}{n}}}. \tag{2}$$

In this paper, we address the question whether the set of all solutions to the Yamabe PDE is compact in the C^2-topology. It has been conjectured that this should be true unless (M, g) is conformally equivalent to the round sphere (see [**15**], [**16**], [**17**]). The case of the round sphere S^n is special in that (1) is invariant under the action of the conformal group on S^n, which is non-compact. It follows from a theorem of Obata [**14**] that every solution of the Yamabe PDE on S^n is minimizing, and the space of all solutions to the Yamabe PDE on S^n can be identified with the unit ball B^{n+1}. Note that the round sphere is the only compact manifold for which the set of minimizing solutions is non-compact.

The Compactness Conjecture has been verified in low dimensions and in the locally conformally flat case. If (M, g) is locally conformally

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flat, compactness follows from work of R. Schoen [**15**], [**16**]. Moreover, Schoen proposed a strategy, based on the Pohozaev identity, for proving the conjecture in the non-locally conformally flat case. In [**12**], Y.Y. Li and M. Zhu followed this strategy to prove compactness in dimension 3. O. Druet [**6**] proved the conjecture in dimensions 4 and 5.

The case $n \geq 6$ is more subtle, and requires a careful analysis of the local properties of the background metric g near a blow-up point. The Compactness Conjecture is closely related to the Weyl Vanishing Conjecture, which asserts that the Weyl tensor should vanish to an order greater than $\frac{n-6}{2}$ at a blow-up point (see [**17**]). The Weyl Vanishing Conjecture has been verified in dimensions 6 and 7 by F. Marques [**13**] and, independently, by Y.Y. Li and L. Zhang [**10**]. Using these results and the positive mass theorem, these authors were able to prove compactness for $n \leq 7$. Moreover, Li and Zhang showed that compactness holds in all dimensions provided that $|W_g(p)| + |\nabla W_g(p)| > 0$ for all $p \in M$. In dimensions 10 and 11, it is sufficient to assume that $|W_g(p)| + |\nabla W_g(p)| + |\nabla^2 W_g(p)| > 0$ for all $p \in M$ (see [**11**]).

Very recently, M. Khuri, F. Marques, and R. Schoen [**9**] proved the Weyl Vanishing Conjecture up to dimension 24. This result, combined with the positive mass theorem, implies the Compactness Conjecture for those dimensions. After proving sharp pointwise estimates, they reduce these questions to showing a certain quadratic form is positive definite. It turns out the quadratic form has negative eigenvalues if $n \geq 25$.

In a recent paper [**4**], it was shown that the Compactness Conjecture fails for $n \geq 52$. More precisely, given any integer $n \geq 52$, there exists a smooth Riemannian metric g on S^n such that set of constant scalar curvature metrics in the conformal class of g is non-compact. Moreover, the blowing-up sequences obtained in [**4**] form exactly one bubble. The construction relies on a gluing procedure based on some local model metric. These local models are directions in which the quadratic form of [**9**] is negative definite. We refer to [**5**] for a survey of this and related results.

In the present paper, we extend these counterexamples to the dimensions $25 \leq n \leq 51$. Our main theorem is:

Theorem. *Assume that* $25 \leq n \leq 51$. *Then there exists a Riemannian metric* g *on* S^n *(of class* C^∞*) and a sequence of positive functions* $v_\nu \in C^\infty(S^n)$ $(\nu \in \mathbb{N})$ *with the following properties:*

(i) g *is not conformally flat,*
(ii) v_ν *is a solution of the Yamabe PDE (1) for all* $\nu \in \mathbb{N}$,
(iii) $E_g(v_\nu) < Y(S^n)$ *for all* $\nu \in \mathbb{N}$, *and* $E_g(v_\nu) \to Y(S^n)$ *as* $\nu \to \infty$,
(iv) $\sup_{S^n} v_\nu \to \infty$ *as* $\nu \to \infty$.

(Here, $Y(S^n)$ *denotes the Yamabe energy of the round metric on* S^n.*)*

We note that O. Druet and E. Hebey [**7**] have constructed blow-up examples for perturbations of (1) (see also [**8**]).

In Section 2, we describe how the problem can be reduced to finding critical points of a certain function $\mathcal{F}_g(\xi, \varepsilon)$, where ξ is a vector in $\mathbb{R}^n$ and ε is a positive real number. This idea has been used by many authors (see, e.g., [**1**], [**2**], [**3**], [**4**]). In Section 3, we show that the function $\mathcal{F}_g(\xi, \varepsilon)$ can be approximated by an auxiliary function $F(\xi, \varepsilon)$. In Section 4, we prove that the function $F(\xi, \varepsilon)$ has a critical point, which is a strict local minimum. Finally, in Section 5, we use a perturbation argument to construct critical points of the function $\mathcal{F}_g(\xi, \varepsilon)$. From this the non-compactness result follows.

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2. Lyapunov-Schmidt reduction

In this section, we collect some basic results established in [**4**]. Let

$$\mathcal{E} = \Big\{ w \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \cap W^{1,2}_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |dw|^2 < \infty \Big\}.$$

By Sobolev's inequality, there exists a constant K, depending only on n, such that

$$\Big(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} \Big)^{\frac{n-2}{n}} \leq K \int_{\mathbb{R}^n} |dw|^2$$

for all $w \in \mathcal{E}$. We define a norm on $\mathcal{E}$ by $\|w\|_{\mathcal{E}}^2 = \int_{\mathbb{R}^n} |dw|^2$. It is easy to see that $\mathcal{E}$, equipped with this norm, is complete.

Given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$, we define a function $u_{(\xi,\varepsilon)} : \mathbb{R}^n \to \mathbb{R}$ by

$$u_{(\xi,\varepsilon)}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \Big)^{\frac{n-2}{2}}.$$

The function $u_{(\xi,\varepsilon)}$ satisfies the elliptic PDE

$$\Delta u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} = 0.$$

It is well known that

$$\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}} = \Big(\frac{Y(S^n)}{4n(n-1)} \Big)^{\frac{n}{2}}$$

for all $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$. We next define

$$\varphi_{(\xi,\varepsilon,0)}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \Big)^{\frac{n+2}{2}} \frac{\varepsilon^2 - |x - \xi|^2}{\varepsilon^2 + |x - \xi|^2}$$

and

$$\varphi_{(\xi,\varepsilon,k)}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x-\xi|^2}\Big)^{\frac{n+2}{2}} \, \frac{2\varepsilon \, (x_k - \xi_k)}{\varepsilon^2 + |x-\xi|^2}$$

for $k = 1, \dots, n$. Finally, we define a closed subspace $\mathcal{E}_{(\xi,\varepsilon)} \subset \mathcal{E}$ by

$$\mathcal{E}_{(\xi,\varepsilon)} = \Big\{ w \in \mathcal{E} : \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)} \, w = 0 \quad \text{for } k = 0, 1, \dots, n \Big\}.$$

Clearly, $u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proposition 1. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$. There exists a positive constant $\alpha_0 \leq 1$, depending only on n, with the following significance: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}^n$, then, given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$ and any function $f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, there exists a unique function $w = G_{(\xi,\varepsilon)}(f) \in \mathcal{E}_{(\xi,\varepsilon)}$ such that*

$$\int_{\mathbb{R}^n} \Big(\langle dw, d\psi \rangle_g + \frac{n-2}{4(n-1)} R_g w \psi - n(n+2) u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} w \psi \Big) = \int_{\mathbb{R}^n} f \psi$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$.

Moreover, we have $\|w\|_{\mathcal{E}} \leq C \, \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$, where C is a constant that depends only on n.

Proposition 2. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$. Moreover, let $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$. There exists a positive constant $\alpha_1 \leq \alpha_0$, depending only on n, with the following significance: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$ for all $x \in \mathbb{R}^n$, then there exists a function $v_{(\xi,\varepsilon)} \in \mathcal{E}$ such that $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ and*

$$\int_{\mathbb{R}^n} \Big(\langle dv_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)} \, R_g \, v_{(\xi,\varepsilon)} \, \psi - n(n-2) \, |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} \, v_{(\xi,\varepsilon)} \, \psi \Big) = 0$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Moreover, we have the estimate

$$\begin{aligned} &\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{\mathcal{E}} \\ &\leq C \, \Big\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n-2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}, \end{aligned}$$

where C is a constant that depends only on n.

We next define a function $\mathcal{F}_g : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{F}_g(\xi, \varepsilon) = &\int_{\mathbb{R}^n} \Big(|dv_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)} \, R_g \, v_{(\xi,\varepsilon)}^2 - (n-2)^2 \, |v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} \Big) \\ &- 2(n-2) \, \Big(\frac{Y(S^n)}{4n(n-1)} \Big)^{\frac{n}{2}}. \end{aligned}$$

If we choose α_1 small enough, then we obtain the following result:

Proposition 3. *The function $\mathcal{F}_g$ is continuously differentiable. Moreover, if $(\bar{\xi}, \bar{\varepsilon})$ is a critical point of the function $\mathcal{F}_g$, then the function $v_{(\bar{\xi},\bar{\varepsilon})}$ is a non-negative weak solution of the equation*

$$\Delta_g v_{(\bar{\xi},\bar{\varepsilon})} - \frac{n-2}{4(n-1)}\, R_g\, v_{(\bar{\xi},\bar{\varepsilon})} + n(n-2)\, v_{(\bar{\xi},\bar{\varepsilon})}^{\frac{n+2}{n-2}} = 0.$$

3. An estimate for the energy of a "bubble"

Throughout this paper, we fix a real number τ and a multi-linear form $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The number τ depends only on the dimension n. The exact choice of τ will be postponed until Section 4. We assume that W_{ijkl} satisfies all the algebraic properties of the Weyl tensor. Moreover, we assume that some components of W are non-zero, so that

$$\sum_{i,j,k,l=1}^{n} (W_{ijkl} + W_{ilkj})^2 > 0.$$

For abbreviation, we put

$$H_{ik}(x) = \sum_{p,q=1}^{n} W_{ipkq}\, x_p\, x_q$$

and

$$\overline{H}_{ik}(x) = f(|x|^2)\, H_{ik}(x),$$

where $f(s) = \tau + 5s - s^2 + \frac{1}{20}\, s^3$. It is easy to see that $H_{ik}(x)$ is trace-free, $\sum_{i=1}^{n} x_i\, H_{ik}(x) = 0$, and $\sum_{i=1}^{n} \partial_i H_{ik}(x) = 0$ for all $x \in \mathbb{R}^n$.

We consider a Riemannian metric of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$,

$$|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$$

for all $x \in \mathbb{R}^n$, and

$$h_{ik}(x) = \mu\, \lambda^6\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)$$

for $|x| \leq \rho$. We assume that the parameters λ, μ, and ρ are chosen such that $\mu \leq 1$ and $\lambda \leq \rho \leq 1$. Note that $\sum_{i=1}^{n} x_i\, h_{ik}(x) = 0$ and $\sum_{i=1}^{n} \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$.

Given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$, there exists a unique function $v_{(\xi,\varepsilon)}$ such that $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ and

$$\int_{\mathbb{R}^n} \Big(\langle dv_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)}\, R_g\, v_{(\xi,\varepsilon)}\, \psi - n(n-2)\, |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\, v_{(\xi,\varepsilon)}\, \psi \Big) = 0$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$ (see Proposition 2).

For abbreviation, let

$$\Omega = \Big\{ (\xi, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} : |\xi| < 1,\ \frac{1}{2} < \varepsilon < 2 \Big\}.$$

The following result is proved in the Appendix A of [**4**]. A similar formula is derived in [**2**].

Proposition 4. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $|h(x)| \leq 1$ for all $x \in \mathbb{R}^n$. Let R_g be the scalar curvature of g. There exists a constant C, depending only on n, such that*

$$\Big| R_g - \partial_i\partial_k h_{ik} + \partial_i(h_{il}\,\partial_k h_{kl}) - \frac{1}{2}\,\partial_i h_{il}\,\partial_k h_{kl} + \frac{1}{4}\,\partial_l h_{ik}\,\partial_l h_{ik} \Big| \leq C\,|h|^2\,|\partial^2 h| + C\,|h|\,|\partial h|^2.$$

Proposition 5. *Assume that $(\xi, \varepsilon) \in \lambda\,\Omega$. Then we have*

$$\Big\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)} + n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\,\lambda^8\,\mu + C\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}$$

and

$$\Big\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)} + n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} + \sum_{i,k=1}^n \mu\,\lambda^6\,f(\lambda^{-2}\,|x|^2)\,H_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\,\lambda^{\frac{8(n+2)}{n-2}}\,\mu^2 + C\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}.$$

Proof. Note that $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$. Hence, it follows from Proposition 4 that

$$|R_g(x)| \leq C\,|h(x)|^2\,|\partial^2 h(x)| + C\,|\partial h(x)|^2 \leq C\,\mu^2\,(\lambda + |x|)^{14}$$

for $|x| \leq \rho$. This implies

$$\begin{aligned} &\Big| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)} + n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big| \\ &= \Big| \sum_{i,k=1}^n \partial_i \Big[(g^{ik} - \delta_{ik})\,\partial_k u_{(\xi,\varepsilon)} \Big] - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)} \Big| \\ &\leq C\,\lambda^{\frac{n-2}{2}}\,\mu\,(\lambda + |x|)^{8-n} \end{aligned}$$

and

$$\begin{aligned}
&\Big|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} + \sum_{i,k=1}^n h_{ik}\, \partial_i \partial_k u_{(\xi,\varepsilon)}\Big| \\
&= \Big| \sum_{i,k=1}^n \partial_i \big[(g^{ik} - \delta_{ik} + h_{ik})\, \partial_k u_{(\xi,\varepsilon)}\big] - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)}\Big| \\
&\leq C\, \lambda^{\frac{n-2}{2}}\, \mu^2\, (\lambda + |x|)^{16-n} \\
&\leq C\, \lambda^{\frac{n-2}{2}}\, \mu^2\, (\lambda + |x|)^{\frac{8(n+2)}{n-2}-n}
\end{aligned}$$

for $|x| \leq \rho$. From this the assertion follows.

Corollary 6. *The function $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}$ satisfies the estimate*

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\, \lambda^8\, \mu + C\, \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}$$

whenever $(\xi,\varepsilon) \in \lambda\, \Omega$.

Proof. It follows from Proposition 2 that

$$\begin{aligned}
&\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\
&\leq C\, \Big\|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)},
\end{aligned}$$

where C is a constant that depends only on n. Hence, the assertion follows from Proposition 5.

We next establish a more precise estimate for the function $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}$. Applying Proposition 1 with $h = 0$, we conclude that there exists a unique function $w_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\langle dw_{(\xi,\varepsilon)}, d\psi\rangle - n(n+2)\, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\, w_{(\xi,\varepsilon)}\, \psi\Big) \\
&= -\int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, \lambda^6\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, \psi
\end{aligned}$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proposition 7. *The function $w_{(\xi,\varepsilon)}$ is smooth. Moreover, if $(\xi,\varepsilon) \in \lambda\, \Omega$, then the function $w_{(\xi,\varepsilon)}$ satisfies the estimates*

$$\begin{aligned}
|w_{(\xi,\varepsilon)}(x)| &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{10-n} \\
|\partial w_{(\xi,\varepsilon)}(x)| &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{9-n} \\
|\partial^2 w_{(\xi,\varepsilon)}(x)| &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{8-n}
\end{aligned}$$

for all $x \in \mathbb{R}^n$.

Proof. There exist real numbers $b_k(\xi, \varepsilon)$ such that

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big(\langle dw_{(\xi,\varepsilon)}, d\psi \rangle - n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w_{(\xi,\varepsilon)} \, \psi \Big) \\ &= - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu \, \lambda^6 \, f(\lambda^{-2} \, |x|^2) \, H_{ik}(x) \, \partial_i \partial_k u_{(\xi,\varepsilon)} \, \psi \\ &\quad + \sum_{k=0}^n b_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)} \, \psi \end{aligned}$$

for all test functions $\psi \in \mathcal{E}$. Hence, standard elliptic regularity theory implies that $w_{(\xi,\varepsilon)}$ is smooth.

It remains to prove quantitative estimates for $w_{(\xi,\varepsilon)}$. To that end, we consider a pair $(\xi, \varepsilon) \in \lambda \, \Omega$. One readily verifies that

$$\Big\| \sum_{i,k=1}^n \mu \, \lambda^6 \, f(\lambda^{-2} \, |x|^2) \, H_{ik}(x) \, \partial_i \partial_k u_{(\xi,\varepsilon)} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \, \lambda^8 \, \mu.$$

As a consequence, the function $w_{(\xi,\varepsilon)}$ satisfies $\|w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \, \lambda^8 \, \mu$. Moreover, we have $\sum_{k=0}^n |b_k(\xi, \varepsilon)| \leq C \, \lambda^8 \, \mu$. This implies

$$\begin{aligned} &\big| \Delta w_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w_{(\xi,\varepsilon)} \big| \\ &= \Big| \sum_{i,k=1}^n \mu \, \lambda^6 \, f(\lambda^{-2} \, |x|^2) \, H_{ik}(x) \, \partial_i \partial_k u_{(\xi,\varepsilon)} - \sum_{k=0}^n b_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)} \Big| \\ &\leq C \, \lambda^{\frac{n-2}{2}} \, \mu \, (\lambda + |x|)^{8-n} \end{aligned}$$

for all $x \in \mathbb{R}^n$. We claim that

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\frac{n-2}{2}} \, |w_{(\xi,\varepsilon)}(x)| \leq C \, \lambda^8 \, \mu.$$

To show this, we fix a point $x_0 \in \mathbb{R}^n$. Let $r = \frac{1}{2} \, (\lambda + |x_0|)$. Then

$$u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}} \leq C \, r^{-2}$$

and

$$\big| \Delta w_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w_{(\xi,\varepsilon)} \big| \leq C \, \lambda^{\frac{n-2}{2}} \, \mu \, r^{8-n}$$

for all $x \in B_r(x_0)$. Hence, it follows from standard interior estimates that

$$\begin{aligned} r^{\frac{n-2}{2}} \, |w_{(\xi,\varepsilon)}(x_0)| &\leq C \, \|w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(B_r(x_0))} \\ &\quad + C \, r^{\frac{n+2}{2}} \, \big\| \Delta w_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w_{(\xi,\varepsilon)} \big\|_{L^\infty(B_r(x_0))} \\ &\leq C \, \lambda^8 \, \mu + C \, \lambda^{\frac{n-2}{2}} \, \mu \, r^{-\frac{n-18}{2}} \\ &\leq C \, \lambda^8 \, \mu. \end{aligned}$$

Therefore, we have

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\frac{n-2}{2}} \, |w_{(\xi,\varepsilon)}(x)| \leq C \, \lambda^8 \, \mu,$$

as claimed. Since $\sup_{x \in \mathbb{R}^n} |x|^{\frac{n-2}{2}} \, |w_{(\xi,\varepsilon)}(x)| < \infty$, we can express the function $w_{(\xi,\varepsilon)}$ in the form

$$w_{(\xi,\varepsilon)}(x) = -\frac{1}{(n-2)\,|S^{n-1}|} \int_{\mathbb{R}^n} |x-y|^{2-n} \, \Delta w_{(\xi,\varepsilon)}(y) \, dy \tag{3}$$

for all $x \in \mathbb{R}^n$.

We are now able to use a bootstrap argument to prove the desired estimate for $w_{(\xi,\varepsilon)}$. It follows from (3) that

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta} \, |w_{(\xi,\varepsilon)}(x)| \leq C \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta+2} \, |\Delta w_{(\xi,\varepsilon)}(x)|$$

for all $0 < \beta < n-2$. Since

$$\begin{aligned} |\Delta w_{(\xi,\varepsilon)}(x)| &\leq n(n+2) \, u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}} \, |w_{(\xi,\varepsilon)}(x)| \\ &\quad + C \, \lambda^{\frac{n-2}{2}} \, \mu \, (\lambda + |x|)^{8-n} \end{aligned}$$

for all $x \in \mathbb{R}^n$, we conclude that

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta} \, |w_{(\xi,\varepsilon)}(x)| &\leq C \, \lambda^2 \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta-2} \, |w_{(\xi,\varepsilon)}(x)| \\ &\quad + C \, \lambda^{\beta - \frac{n-18}{2}} \, \mu \end{aligned}$$

for all $0 < \beta \leq n-10$. Iterating this inequality, we obtain

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{n-10} \, |w_{(\xi,\varepsilon)}(x)| \leq C \, \lambda^{\frac{n-2}{2}} \, \mu.$$

The estimates for the first and second derivatives of $w_{(\xi,\varepsilon)}$ follow now from standard interior estimates.

Corollary 8. *The function $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}$ satisfies the estimate*

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \, \lambda^{\frac{8(n+2)}{n-2}} \, \mu^{\frac{n+2}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}$$

whenever $(\xi, \varepsilon) \in \lambda \, \Omega$.

Proof. Consider the functions

$$B_1 = \sum_{i,k=1}^n \partial_i \big[(g^{ik} - \delta_{ik}) \, \partial_k w_{(\xi,\varepsilon)} \big] - \frac{n-2}{4(n-1)} \, R_g \, w_{(\xi,\varepsilon)}$$

and

$$B_2 = \sum_{i,k=1}^n \mu \, \lambda^6 \, f(\lambda^{-2} \, |x|^2) \, H_{ik}(x) \, \partial_i \partial_k u_{(\xi,\varepsilon)}.$$

By definition of $w_{(\xi,\varepsilon)}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \Big(\langle dw_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)}\, R_g\, w_{(\xi,\varepsilon)}\, \psi - n(n+2)\, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\, w_{(\xi,\varepsilon)}\, \psi \Big) \\ &= -\int_{\mathbb{R}^n} (B_1 + B_2)\, \psi\end{aligned}$$

for all functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Since $w_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$, it follows that

$$w_{(\xi,\varepsilon)} = -G_{(\xi,\varepsilon)}(B_1 + B_2).$$

Moreover, we have

$$v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)}\big(B_3 + n(n-2)\, B_4\big),$$

where

$$B_3 = \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}$$

and

$$B_4 = |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\, v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}\, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}).$$

Thus, we conclude that

$$v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)}\big(B_1 + B_2 + B_3 + n(n-2)\, B_4\big),$$

where $G_{(\xi,\varepsilon)}$ denotes the solution operator constructed in Proposition 1. As a consequence, we obtain

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\, \big\|B_1 + B_2 + B_3 + n(n-2)\, B_4\big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}.$$

It follows from Proposition 7 that

$$|B_1(x)| \leq C\, \lambda^{\frac{n-2}{2}}\, \mu^2\, (\lambda + |x|)^{16-n} \leq C\, \lambda^{\frac{n-2}{2}}\, \mu^2\, (\lambda + |x|)^{\frac{8(n+2)}{n-2} - n}$$

for $|x| \leq \rho$ and

$$|B_1(x)| \leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, |x|^{8-n}$$

for $|x| \geq \rho$. This implies

$$\|B_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\, \lambda^{\frac{8(n+2)}{n-2}}\, \mu^2 + C\, \rho^8\, \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}.$$

Moreover, observe that

$$\|B_2 + B_3\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\, \lambda^{\frac{8(n+2)}{n-2}}\, \mu^2 + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}$$

by Proposition 5. Finally, Corollary 6 implies that

$$\begin{aligned}\|B_4\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} &\leq C\, \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{n+2}{n-2}} \\ &\leq C\, \lambda^{\frac{8(n+2)}{n-2}}\, \mu^{\frac{n+2}{n-2}} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n+2}{2}}.\end{aligned}$$

Putting these facts together, we obtain

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\,\lambda^{\frac{8(n+2)}{n-2}}\,\mu^{\frac{n+2}{n-2}} + C\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}.$$

This completes the proof.

Proposition 9. *We have*

$$\begin{aligned} \bigg| \int_{\mathbb{R}^n} \Big(&|dv_{(\xi,\varepsilon)}|_g^2 - |du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)}\,R_g\,(v_{(\xi,\varepsilon)}^2 - u_{(\xi,\varepsilon)}^2)\Big) \\ &+ \int_{\mathbb{R}^n} n(n-2)\,(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\,u_{(\xi,\varepsilon)}\,v_{(\xi,\varepsilon)} \\ &- \int_{\mathbb{R}^n} n(n-2)\,(|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}) \\ &- \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\,\lambda^6\,f(\lambda^{-2}\,|x|^2)\,H_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)}\,w_{(\xi,\varepsilon)}\bigg| \\ \leq C\,\lambda^{\frac{16n}{n-2}}\,\mu^{\frac{2n}{n-2}} &+ C\,\lambda^8\,\mu\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} + C\,\Big(\frac{\lambda}{\rho}\Big)^{n-2} \end{aligned}$$

whenever $(\xi,\varepsilon) \in \lambda\,\Omega$*.*

Proof. By definition of $v_{(\xi,\varepsilon)}$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(&|dv_{(\xi,\varepsilon)}|_g^2 - \langle du_{(\xi,\varepsilon)}, dv_{(\xi,\varepsilon)}\rangle_g + \frac{n-2}{4(n-1)}\,R_g\,v_{(\xi,\varepsilon)}\,(v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)})\Big) \\ &- \int_{\mathbb{R}^n} n(n-2)\,|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\,v_{(\xi,\varepsilon)}\,(v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) = 0. \end{aligned}$$

Using Proposition 5 and Corollary 6, we obtain

$$\begin{aligned} &\bigg| \int_{\mathbb{R}^n} \Big(\langle du_{(\xi,\varepsilon)}, dv_{(\xi,\varepsilon)}\rangle_g - |du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)}\,(v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)})\Big) \\ &\quad - \int_{\mathbb{R}^n} n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\,(v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \\ &\quad - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\,\lambda^6\,f(\lambda^{-2}\,|x|^2)\,H_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)}\,(v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)})\bigg| \\ &\leq \bigg\|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)} + n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \\ &\quad + \sum_{i,k=1}^n \mu\,\lambda^6\,f(\lambda^{-2}\,|x|^2)\,H_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)}\bigg\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ &\quad \cdot \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ &\leq C\,\lambda^{\frac{16n}{n-2}}\,\mu^3 + C\,\lambda^8\,\mu\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} + C\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}. \end{aligned}$$

Moreover, we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\sum_{i,k=1}^n \mu\,\lambda^6\,f(\lambda^{-2}\,|x|^2)\,H_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)}\,(v_{(\xi,\varepsilon)}-u_{(\xi,\varepsilon)}-w_{(\xi,\varepsilon)})\Big|\\&\leq C\,\lambda^8\,\mu\,\|v_{(\xi,\varepsilon)}-u_{(\xi,\varepsilon)}-w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}\\&\leq C\,\lambda^{\frac{16n}{n-2}}\,\mu^{\frac{2n}{n-2}}+C\,\lambda^8\,\mu\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}\end{aligned}$$

by Corollary 8. Putting these facts together, the assertion follows.

Proposition 10. *We have*

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}-u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\,u_{(\xi,\varepsilon)}\,v_{(\xi,\varepsilon)}-\frac{2}{n}\int_{\mathbb{R}^n}(|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}}-u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}})\Big|\\&\leq C\,\lambda^{\frac{16n}{n-2}}\,\mu^{\frac{2n}{n-2}}+C\,\Big(\frac{\lambda}{\rho}\Big)^n\end{aligned}$$

whenever $(\xi,\varepsilon)\in\lambda\,\Omega$.

Proof. We have the pointwise estimate

$$\begin{aligned}&\Big|(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}-u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\,u_{(\xi,\varepsilon)}\,v_{(\xi,\varepsilon)}-\frac{2}{n}\,(|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}}-u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}})\Big|\\&\leq C\,|v_{(\xi,\varepsilon)}-u_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}},\end{aligned}$$

where C is a constant that depends only on n. This implies

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}-u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\,u_{(\xi,\varepsilon)}\,v_{(\xi,\varepsilon)}-\frac{2}{n}\int_{\mathbb{R}^n}(|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}}-u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}})\Big|\\&\leq C\,\|v_{(\xi,\varepsilon)}-u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{2n}{n-2}}\\&\leq C\,\lambda^{\frac{16n}{n-2}}\,\mu^{\frac{2n}{n-2}}+C\,\Big(\frac{\lambda}{\rho}\Big)^n\end{aligned}$$

by Corollary 6.

Proposition 11. *We have*

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\Big(|du_{(\xi,\varepsilon)}|_g^2+\frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)}^2-n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}\Big)\\&\qquad-\int_{B_\rho(0)}\frac{1}{2}\sum_{i,k,l=1}^n h_{il}\,h_{kl}\,\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)}\\&\qquad+\int_{B_\rho(0)}\frac{n-2}{16(n-1)}\sum_{i,k,l=1}^n(\partial_l h_{ik})^2\,u_{(\xi,\varepsilon)}^2\Big|\\&\leq C\,\lambda^{\frac{16n}{n-2}}\,\mu^3+C\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}\end{aligned}$$

whenever $(\xi,\varepsilon)\in\lambda\,\Omega$.

Proof. Note that

$$\begin{aligned}&\Big|g^{ik}(x) - \delta_{ik} + h_{ik}(x) - \frac{1}{2}\sum_{l=1}^n h_{il}(x)\,h_{kl}(x)\Big| \\ &\leq C\,|h(x)|^3 \leq C\,\mu^3\,(\lambda + |x|)^{24} \leq C\,\mu^3\,(\lambda + |x|)^{\frac{16n}{n-2}}\end{aligned}$$

for $|x| \leq \rho$. This implies

$$\begin{aligned}&\bigg|\int_{\mathbb{R}^n} \big(|du_{(\xi,\varepsilon)}|_g^2 - |du_{(\xi,\varepsilon)}|^2\big) + \int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik}\,\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)} \\ &\qquad - \int_{B_\rho(0)} \frac{1}{2}\sum_{i,k,l=1}^n h_{il}\,h_{kl}\,\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)}\bigg| \\ &\leq C\,\lambda^{n-2}\,\mu^3 \int_{B_\rho(0)} (\lambda + |x|)^{\frac{16n}{n-2}+2-2n} + C\,\lambda^{n-2} \int_{\mathbb{R}^n \setminus B_\rho(0)} (\lambda + |x|)^{2-2n} \\ &\leq C\,\lambda^{\frac{16n}{n-2}}\,\mu^3 + C\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}.\end{aligned}$$

By Proposition 4, the scalar curvature of g satisfies the estimate

$$\begin{aligned}&\Big|R_g(x) + \frac{1}{4}\sum_{i,k,l=1}^n (\partial_l h_{ik}(x))^2\Big| \\ &\leq C\,|h(x)|^2\,|\partial^2 h(x)| + C\,|h(x)|\,|\partial h(x)|^2 \\ &\leq C\,\mu^3\,(\lambda + |x|)^{22} \leq C\,\mu^3\,(\lambda + |x|)^{\frac{16n}{n-2}-2}\end{aligned}$$

for $|x| \leq \rho$. This implies

$$\begin{aligned}&\bigg|\int_{\mathbb{R}^n} R_g\,u_{(\xi,\varepsilon)}^2 + \int_{B_\rho(0)} \frac{1}{4}\sum_{i,k,l=1}^n (\partial_l h_{ik})^2\,u_{(\xi,\varepsilon)}^2\bigg| \\ &\leq C\,\lambda^{n-2}\,\mu^3 \int_{B_\rho(0)} (\lambda + |x|)^{\frac{16n}{n-2}+2-2n} + C\,\lambda^{n-2} \int_{\mathbb{R}^n \setminus B_\rho(0)} (\lambda + |x|)^{4-2n} \\ &\leq C\,\lambda^{\frac{16n}{n-2}}\,\mu^3 + C\,\rho^2\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}.\end{aligned}$$

At this point, we use the formula

$$\begin{aligned}&\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,\partial_i\partial_k(u_{(\xi,\varepsilon)}^2) \\ &= \frac{1}{n}\Big(|du_{(\xi,\varepsilon)}|^2 - \frac{n-2}{4(n-1)}\,\Delta(u_{(\xi,\varepsilon)}^2)\Big)\,\delta_{ik}.\end{aligned}$$

Since h_{ik} is trace-free, we obtain

$$\sum_{i,k=1}^n h_{ik}\,\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)} = \frac{n-2}{4(n-1)}\sum_{i,k=1}^n h_{ik}\,\partial_i\partial_k(u_{(\xi,\varepsilon)}^2),$$

hence

$$\int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} = \int_{\mathbb{R}^n} \frac{n-2}{4(n-1)} \sum_{i,k=1}^n \partial_i \partial_k h_{ik}\, u_{(\xi,\varepsilon)}^2.$$

Since $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$, it follows that

$$\bigg| \int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} \bigg| \leq C \int_{\mathbb{R}^n \setminus B_\rho(0)} u_{(\xi,\varepsilon)}^2 \leq C\, \rho^2 \Big(\frac{\lambda}{\rho}\Big)^{n-2}.$$

Putting these facts together, the assertion follows.

Corollary 12. *The function $\mathcal{F}_g(\xi,\varepsilon)$ satisfies the estimate*

$$\begin{aligned} &\bigg| \mathcal{F}_g(\xi,\varepsilon) - \int_{B_\rho(0)} \frac{1}{2} \sum_{i,k,l=1}^n h_{il}\, h_{kl}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} \\ &\qquad + \int_{B_\rho(0)} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l h_{ik})^2\, u_{(\xi,\varepsilon)}^2 \\ &\qquad - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, \lambda^6\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, w_{(\xi,\varepsilon)} \bigg| \\ &\leq C\, \lambda^{\frac{16n}{n-2}}\, \mu^{\frac{2n}{n-2}} + C\, \lambda^8\, \mu \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} + C \Big(\frac{\lambda}{\rho}\Big)^{n-2} \end{aligned}$$

whenever $(\xi,\varepsilon) \in \lambda\, \Omega$.

4. Finding a critical point of an auxiliary function

We define a function $F : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ as follows: given any pair $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$, we define

$$\begin{aligned} F(\xi,\varepsilon) &= \int_{\mathbb{R}^n} \frac{1}{2} \sum_{i,k,l=1}^n \overline{H}_{il}(x)\, \overline{H}_{kl}(x)\, \partial_i u_{(\xi,\varepsilon)}(x)\, \partial_k u_{(\xi,\varepsilon)}(x) \\ &\quad - \int_{\mathbb{R}^n} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\, u_{(\xi,\varepsilon)}(x)^2 \\ &\quad + \int_{\mathbb{R}^n} \sum_{i,k=1}^n \overline{H}_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}(x)\, z_{(\xi,\varepsilon)}(x), \end{aligned}$$

where $z_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ satisfies the relation

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big(\langle dz_{(\xi,\varepsilon)}, d\psi \rangle - n(n+2)\, u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}}\, z_{(\xi,\varepsilon)}\, \psi \Big) \\ &= - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \overline{H}_{ik}\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, \psi \end{aligned}$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Our goal in this section is to show that the function $F(\xi,\varepsilon)$ has a critical point.

Proposition 13. *The function $F(\xi,\varepsilon)$ satisfies $F(\xi,\varepsilon) = F(-\xi,\varepsilon)$ for all $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$. Consequently, we have $\frac{\partial}{\partial \xi_p} F(0,\varepsilon) = 0$ and $\frac{\partial^2}{\partial\varepsilon\,\partial\xi_p} F(0,\varepsilon) = 0$ for all $\varepsilon > 0$ and $p = 1,\ldots,n$.*

Proof. This follows immediately from the relation $\overline{H}_{ik}(-x) = \overline{H}_{ik}(x)$.

Proposition 14. *We have*

$$\begin{aligned}&\int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 \, x_p \, x_q \\ &= \frac{2}{n(n+2)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \, r^{n+3} \\ &\quad + \frac{1}{n(n+2)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \, r^{n+3}\end{aligned}$$

and

$$\begin{aligned}&\int_{\partial B_r(0)} \sum_{i,k=1}^n H_{ik}(x)^2 \, x_p \, x_q \\ &= \frac{2}{n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \, r^{n+5} \\ &\quad + \frac{1}{2n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \, r^{n+5}.\end{aligned}$$

Proof. See [**4**], Proposition 16.

Proposition 15. *We have*

$$\begin{aligned}&\int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \, x_p \, x_q \\ &= \frac{2}{n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \\ &\qquad \cdot r^{n+3} \Big[(n+4) \, f(r^2)^2 + 8r^2 \, f(r^2) \, f'(r^2) + 4r^4 \, f'(r^2)^2 \Big] \\ &\quad + \frac{1}{n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \\ &\qquad \cdot r^{n+3} \Big[(n+4) \, f(r^2)^2 + 4r^2 \, f(r^2) \, f'(r^2) + 2r^4 \, f'(r^2)^2 \Big].\end{aligned}$$

Proof. Using the identity

$$\partial_l \overline{H}_{ik}(x) = f(|x|^2)\, \partial_l H_{ik}(x) + 2\, f'(|x|^2)\, H_{ik}(x)\, x_l$$

and Euler's theorem, we obtain

$$\begin{aligned}
&\sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \\
&= f(|x|^2)^2 \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 \\
&\quad + 4\, f(|x|^2)\, f'(|x|^2) \sum_{i,k,l=1}^n H_{ik}(x)\, x_l\, \partial_l H_{ik}(x) \\
&\quad + 4\, |x|^2\, f'(|x|^2)^2 \sum_{i,k=1}^n H_{ik}(x)^2 \\
&= f(|x|^2)^2 \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 \\
&\quad + \big[8\, f(|x|^2)\, f'(|x|^2) + 4\, |x|^2\, f'(|x|^2)^2\big] \sum_{i,k=1}^n H_{ik}(x)^2.
\end{aligned}$$

Hence, the assertion follows from the previous proposition.

Corollary 16. *We have*

$$\begin{aligned}
&\int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \\
&= \frac{1}{n(n+2)}\, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\
&\qquad \cdot r^{n+1} \Big[(n+2)\, f(r^2)^2 + 4\, r^2\, f(r^2)\, f'(r^2) + 2\, r^4\, f'(r^2)^2\Big].
\end{aligned}$$

Proposition 17. *We have*

$$\begin{aligned}
F(0,\varepsilon) = &-\frac{n-2}{16n(n-1)(n+2)}\, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\
&\cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{2-n}\, r^{n+1} \\
&\qquad \cdot \Big[(n+2)\, f(r^2)^2 + 4\, r^2\, f(r^2)\, f'(r^2) + 2\, r^4\, f'(r^2)^2\Big]\, dr.
\end{aligned}$$

Proof. Note that $z_{(0,\varepsilon)}(x) = 0$ for all $x \in \mathbb{R}^n$. This implies

$$F(0,\varepsilon) = -\int_{\mathbb{R}^n} \frac{n-2}{16(n-1)}\, \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{2-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2.$$

Using Corollary 16, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varepsilon^{n-2} \, (\varepsilon^2 + |x|^2)^{2-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \\
&= \frac{1}{n(n+2)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\
&\quad \cdot \int_0^\infty \varepsilon^{n-2} \, (\varepsilon^2 + r^2)^{2-n} \, r^{n+1} \\
&\qquad \cdot \Big[(n+2) \, f(r^2)^2 + 4 \, r^2 \, f(r^2) \, f'(r^2) + 2 \, r^4 \, f'(r^2)^2 \Big].
\end{aligned}$$

This proves the assertion.

Proposition 18. *The function $F(0, \varepsilon)$ can be written in the form*

$$\begin{aligned}
F(0, \varepsilon) = -\frac{n-2}{16n(n-1)(n+2)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\
\cdot I(\varepsilon^2) \int_0^\infty (1 + r^2)^{2-n} \, r^{n+7} \, dr,
\end{aligned}$$

where

$$\begin{aligned}
I(s) = \frac{n-12}{n+6} \, \frac{n-10}{n+4} \, (n-8) \, \tau^2 \, s^2 + 10 \, \frac{n-12}{n+6} \, (n-10) \, \tau \, s^3 \\
+ \Big(25 \, \frac{n-12}{n+6} \, (n+8) - 2(n-12) \, \tau \Big) \, s^4 \\
+ \Big(\frac{n+8}{10} \, \tau - 10(n+12) \Big) \, s^5 \\
+ \frac{n+8}{n-14} \, \frac{3n+52}{2} \, s^6 - \frac{n+8}{n-14} \, \frac{n+10}{n-16} \, \frac{n+24}{10} \, s^7 \\
+ \frac{n+8}{n-14} \, \frac{n+10}{n-16} \, \frac{n+12}{n-18} \, \frac{n+32}{400} \, s^8.
\end{aligned}$$

Proof. It is straightforward to check that

$$\begin{aligned}
&(n+2) \, f(s)^2 + 4s \, f(s) \, f'(s) + 2s^2 \, f'(s)^2 \\
&= (n+2)\tau^2 + 10(n+4)\tau \, s + \Big(25(n+8) - 2(n+6)\tau \Big) \, s^2 \\
&\quad + \Big(\frac{n+8}{10} \, \tau - 10(n+12) \Big) \, s^3 + \frac{3n+52}{2} \, s^4 - \frac{n+24}{10} \, s^5 + \frac{n+32}{400} \, s^6.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_0^\infty \varepsilon^{n-2}\,(\varepsilon^2+r^2)^{2-n}\,r^{n+1} \\
&\qquad\cdot \Big[(n+2)\,f(r^2)^2 + 4\,r^2\,f(r^2)\,f'(r^2) + 2r^4\,f'(r^2)^2\Big]\,dr \\
&= (n+2)\tau^2\,\varepsilon^4 \int_0^\infty (1+r^2)^{2-n}\,r^{n+1}\,dr \\
&\quad + 10(n+4)\tau\,\varepsilon^6 \int_0^\infty (1+r^2)^{2-n}\,r^{n+3}\,dr \\
&\quad + \Big(25(n+8) - 2(n+6)\tau\Big)\,\varepsilon^8 \int_0^\infty (1+r^2)^{2-n}\,r^{n+5}\,dr \\
&\quad + \Big(\frac{n+8}{10}\,\tau - 10(n+12)\Big)\,\varepsilon^{10} \int_0^\infty (1+r^2)^{2-n}\,r^{n+7}\,dr \\
&\quad + \frac{3n+52}{2}\,\varepsilon^{12} \int_0^\infty (1+r^2)^{2-n}\,r^{n+9}\,dr \\
&\quad - \frac{n+24}{10}\,\varepsilon^{14} \int_0^\infty (1+r^2)^{2-n}\,r^{n+11}\,dr \\
&\quad + \frac{n+32}{400}\,\varepsilon^{16} \int_0^\infty (1+r^2)^{2-n}\,r^{n+13}\,dr.
\end{aligned}$$

Using the identity

$$\int_0^\infty (1+r^2)^{2-n}\,r^{\beta+2}\,dr = \frac{\beta+1}{2n-\beta-7} \int_0^\infty (1+r^2)^{2-n}\,r^{\beta}\,dr,$$

we obtain

$$\begin{aligned}
&\int_0^\infty \varepsilon^{n-2}\,(\varepsilon^2+r^2)^{2-n}\,r^{n+1} \\
&\qquad\cdot \Big[(n+2)\,f(r^2)^2 + 4\,r^2\,f(r^2)\,f'(r^2) + 2r^4\,f'(r^2)^2\Big]\,dr \\
&= I(\varepsilon^2) \int_0^\infty (1+r^2)^{2-n}\,r^{n+7}\,dr.
\end{aligned}$$

This completes the proof.

In the next step, we compute the Hessian of F at $(0,\varepsilon)$.

Proposition 19. *The second order partial derivatives of the function $F(\xi, \varepsilon)$ are given by*

$$\begin{aligned}\frac{\partial^2}{\partial\xi_p\,\partial\xi_q} F(0,\varepsilon) &= \int_{\mathbb{R}^n} (n-2)^2\, \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{-n} \sum_{l=1}^n \overline{H}_{pl}(x)\, \overline{H}_{ql}(x) \\ &\quad - \int_{\mathbb{R}^n} \frac{(n-2)^2}{4} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 x_p\, x_q \\ &\quad + \int_{\mathbb{R}^n} \frac{(n-2)^2}{8(n-1)} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \delta_{pq}.\end{aligned}$$

Proof. See [**4**], Proposition 21.

Proposition 20. *The second order partial derivatives of the function $F(\xi, \varepsilon)$ are given by*

$$\begin{aligned}&\frac{\partial^2}{\partial\xi_p\,\partial\xi_q} F(0,\varepsilon) \\ &= -\frac{2(n-2)^2}{n(n+2)(n+4)}\, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp})\, (W_{iqkl} + W_{ilkq}) \\ &\qquad \cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{-n}\, r^{n+5} \Big[2\, f(r^2)\, f'(r^2) + r^2\, f'(r^2)^2\Big]\, dr \\ &\quad - \frac{(n-2)^2}{2n(n+2)(n+4)}\, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2\, \delta_{pq} \\ &\qquad \cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{-n}\, r^{n+5} \Big[2\, f(r^2)\, f'(r^2) + r^2\, f'(r^2)^2\Big]\, dr \\ &\quad + \frac{(n-2)^2}{4n(n-1)(n+2)}\, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2\, \delta_{pq} \\ &\qquad \cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{1-n}\, r^{n+5}\, f'(r^2)^2\, dr.\end{aligned}$$

Proof. Using the identity

$$\begin{aligned}&\int_{\partial B_r(0)} \sum_{l=1}^n \overline{H}_{pl}(x)\, \overline{H}_{ql}(x) \\ &= \int_{\partial B_r(0)} \sum_{i,j,k,l,m=1}^n W_{ipkl}\, W_{jqml}\, x_i\, x_j\, x_k\, x_m\, f(|x|^2)^2 \\ &= \frac{1}{n(n+2)}\, |S^{n-1}| \\ &\qquad \cdot \sum_{i,j,k,l,m=1}^n W_{ipkl}\, W_{jqml}\, (\delta_{ij}\, \delta_{km} + \delta_{ik}\, \delta_{jm} + \delta_{im}\, \delta_{jk})\, r^{n+3}\, f(r^2)^2\end{aligned}$$

$$= \frac{1}{2n(n+2)}\, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp})\, (W_{iqkl} + W_{ilkq})\, r^{n+3}\, f(r^2)^2,$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n \overline{H}_{pl}(x)\, \overline{H}_{ql}(x) \\
&= \frac{1}{2n(n+2)}\, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp})\, (W_{iqkl} + W_{ilkq}) \\
&\quad \cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{-n}\, r^{n+3}\, f(r^2)^2\, dr.
\end{aligned}$$

Similarly, it follows from Proposition 15 that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\, x_p\, x_q \\
&= \frac{2}{n(n+2)(n+4)}\, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp})\, (W_{iqkl} + W_{ilkq}) \\
&\quad \cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{-n}\, r^{n+3} \\
&\quad \cdot \Big[(n+4)\, f(r^2)^2 + 8r^2\, f(r^2)\, f'(r^2) + 4r^4\, f'(r^2)^2\Big]\, dr \\
&\quad + \frac{1}{n(n+2)(n+4)}\, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2\, \delta_{pq} \\
&\quad \cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{-n}\, r^{n+3} \\
&\qquad \cdot \Big[(n+4)\, f(r^2)^2 + 4r^2\, f(r^2)\, f'(r^2) + 2r^4\, f'(r^2)^2\Big]\, dr.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\, \delta_{pq} \\
&= \frac{1}{n(n+2)}\, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2\, \delta_{pq} \\
&\quad \cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{1-n}\, r^{n+1} \\
&\quad \cdot \Big[(n+2)\, f(r^2)^2 + 4\, r^2\, f(r^2)\, f'(r^2) + 2\, r^4\, f'(r^2)^2\Big]\, dr
\end{aligned}$$

by Corollary 16. A straightforward calculation yields

$$\begin{aligned}&(\varepsilon^2 + r^2)^{1-n}\, r^{n+1} \big[(n+2)\, f(r^2)^2 + 4r^2\, f(r^2)\, f'(r^2)\big] \\ &= 2(n-1)\, (\varepsilon^2 + r^2)^{-n}\, r^{n+3}\, f(r^2)^2 + \frac{d}{dr}\big[(\varepsilon^2 + r^2)^{1-n}\, r^{n+2}\, f(r^2)^2\big].\end{aligned}$$

This implies

$$\begin{aligned}&\int_{\mathbb{R}^n} \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\, \delta_{pq} \\ &= \frac{2(n-1)}{n(n+2)}\, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2\, \delta_{pq} \\ &\quad \cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{-n}\, r^{n+3}\, f(r^2)^2\, dr \\ &\quad + \frac{2}{n(n+2)}\, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2\, \delta_{pq} \\ &\quad \cdot \int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{1-n}\, r^{n+5}\, f'(r^2)^2\, dr.\end{aligned}$$

Putting these facts together, the assertion follows.

Proposition 21. *We have*

$$\begin{aligned}&\int_0^\infty \varepsilon^{n-2}\, (\varepsilon^2 + r^2)^{-n}\, r^{n+5} \Big[2\, f(r^2)\, f'(r^2) + r^2\, f'(r^2)^2\Big]\, dr \\ &= J(\varepsilon^2) \int_0^\infty (1 + r^2)^{-n}\, r^{n+9}\, dr,\end{aligned}$$

where

$$\begin{aligned}J(s) &= 10\, \frac{n-10}{n+8}\, \frac{n-8}{n+6}\, \tau\, s^2 + \frac{n-10}{n+8}\, (75 - 4\tau)\, s^3 \\ &\quad + \Big(\frac{3}{10}\, \tau - 50\Big)\, s^4 + \frac{23}{2}\, \frac{n+10}{n-12}\, s^5 - \frac{11}{10}\, \frac{n+10}{n-12}\, \frac{n+12}{n-14}\, s^6 \\ &\quad + \frac{3}{80}\, \frac{n+10}{n-12}\, \frac{n+12}{n-14}\, \frac{n+14}{n-16}\, s^7.\end{aligned}$$

Proof. Note that

$$\begin{aligned}&2\, f(s)\, f'(s) + s\, f'(s)^2 \\ &= 10\tau + (75 - 4\tau)\, s + \Big(\frac{3}{10}\, \tau - 50\Big)\, s^2 + \frac{23}{2}\, s^3 - \frac{11}{10}\, s^4 + \frac{3}{80}\, s^5.\end{aligned}$$

This implies

$$\begin{aligned}
&\int_0^\infty \varepsilon^{n-2}\,(\varepsilon^2+r^2)^{-n}\,r^{n+5}\Big[2\,f(r^2)\,f'(r^2)+r^2\,f'(r^2)^2\Big]\,dr \\
&= 10\tau\,\varepsilon^4 \int_0^\infty (1+r^2)^{-n}\,r^{n+5}\,dr \\
&\quad + (75-4\tau)\,\varepsilon^6 \int_0^\infty (1+r^2)^{-n}\,r^{n+7}\,dr \\
&\quad + \Big(\frac{3}{10}\,\tau-50\Big)\,\varepsilon^8 \int_0^\infty (1+r^2)^{-n}\,r^{n+9}\,dr \\
&\quad + \frac{23}{2}\,\varepsilon^{10} \int_0^\infty (1+r^2)^{-n}\,r^{n+11}\,dr \\
&\quad - \frac{11}{10}\,\varepsilon^{12} \int_0^\infty (1+r^2)^{-n}\,r^{n+13}\,dr \\
&\quad + \frac{3}{80}\,\varepsilon^{14} \int_0^\infty (1+r^2)^{-n}\,r^{n+15}\,dr.
\end{aligned}$$

Hence, the assertion follows from the identity

$$\int_0^\infty (1+r^2)^{-n}\,r^{\beta+2}\,dr = \frac{\beta+1}{2n-\beta-3}\int_0^\infty (1+r^2)^{-n}\,r^{\beta}\,dr.$$

Proposition 22. *Assume that $25 \leq n \leq 51$. Then we can choose $\tau \in \mathbb{R}$ such that $I'(1) = 0$, $I''(1) < 0$, and $J(1) < 0$.*

Proof. The condition $I'(1) = 0$ is equivalent to

$$a_n\,\tau^2 + b_n\,\tau + c_n = 0,$$

where

$$\begin{aligned}
a_n &= 2\,\frac{n-12}{n+6}\,\frac{n-10}{n+4}\,(n-8) \\
b_n &= 30\,\frac{n-12}{n+6}\,(n-10) - 8(n-12) + \frac{n+8}{2} \\
c_n &= 100\,\frac{n-12}{n+6}\,(n+8) - 50(n+12) + 3\,\frac{n+8}{n-14}\,(3n+52) \\
&\quad - 7\,\frac{n+8}{n-14}\,\frac{n+10}{n-16}\,\frac{n+24}{10} + \frac{n+8}{n-14}\,\frac{n+10}{n-16}\,\frac{n+12}{n-18}\,\frac{n+32}{50}.
\end{aligned}$$

By inspection, one verifies that $49\,a_n - 7\,b_n + c_n < 0$ for $25 \leq n \leq 51$. Since a_n is positive, there exists a unique real number $\tau < -7$ such that $a_n\,\tau^2 + b_n\,\tau + c_n = 0$. Moreover, we have

$$I''(1) - I'(1) = \alpha_n\,\tau + \beta_n$$

and

$$J(1) = \gamma_n\,\tau + \delta_n,$$

where

$$\alpha_n = 30\,\frac{n-12}{n+6}\,(n-10) - 16(n-12) + \frac{3(n+8)}{2}$$
$$\beta_n = 200\,\frac{n-12}{n+6}\,(n+8) - 150(n+12) + 12\,\frac{n+8}{n-14}\,(3n+52)$$
$$- 35\,\frac{n+8}{n-14}\,\frac{n+10}{n-16}\,\frac{n+24}{10} + 3\,\frac{n+8}{n-14}\,\frac{n+10}{n-16}\,\frac{n+12}{n-18}\,\frac{n+32}{25}$$
$$\gamma_n = 10\,\frac{n-10}{n+8}\,\frac{n-8}{n+6} - \frac{4(n-10)}{n+8} + \frac{3}{10}$$
$$\delta_n = 75\,\frac{n-10}{n+8} - 50 + \frac{23}{2}\,\frac{n+10}{n-12} - \frac{11}{10}\,\frac{n+10}{n-12}\,\frac{n+12}{n-14}$$
$$+ \frac{3}{80}\,\frac{n+10}{n-12}\,\frac{n+12}{n-14}\,\frac{n+14}{n-16}.$$

By inspection, one verifies that $7\alpha_n > \beta_n > 0$ and $7\gamma_n > \delta_n > 0$ for $25 \leq n \leq 51$. This implies $I''(1) = \alpha_n\,\tau + \beta_n < -7\alpha_n + \beta_n < 0$ and $J(1) = \gamma_n\,\tau + \delta_n < -7\gamma_n + \delta_n < 0$. This completes the proof.

Corollary 23. *Assume that τ is chosen such that $I'(1) = 0$, $I''(1) < 0$, and $J(1) < 0$. Then the function $F(\xi, \varepsilon)$ has a strict local minimum at $(0, 1)$.*

Proof. Since $I'(1) = 0$, we have $\frac{\partial}{\partial\varepsilon} F(0, 1) = 0$. Therefore, $(0, 1)$ is a critical point of the function $F(\xi, \varepsilon)$. Since $J(1) < 0$, we have

$$\int_0^\infty (1+r^2)^{-n}\, r^{n+5}\, \Big[2\, f(r^2)\, f'(r^2) + r^2\, f'(r^2)^2\Big]\, dr < 0$$

by Proposition 21. Hence, it follows from Proposition 20 that the matrix $\frac{\partial^2}{\partial\xi_p\,\partial\xi_q} F(0, 1)$ is positive definite. Using Proposition 18 and the inequality $I''(0) < 0$, we obtain $\frac{\partial^2}{\partial\varepsilon^2} F(0, 1) > 0$. Consequently, the function $F(\xi, \varepsilon)$ has a strict local minimum at $(0, 1)$.

5. Proof of the main theorem

Proposition 24. *Assume that $25 \leq n \leq 51$. Moreover, let g be a smooth metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ such that $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq \alpha_1$ for all $x \in \mathbb{R}^n$, $h(x) = 0$ for $|x| \geq 1$, and*

$$h_{ik}(x) = \mu\, \lambda^6\, f(\lambda^{-2}\, |x|^2)\, H_{ik}(x)$$

for $|x| \leq \rho$. If α and $\rho^{2-n}\, \mu^{-2}\, \lambda^{n-18}$ are sufficiently small, then there exists a positive function v such that

$$\Delta_g v - \frac{n-2}{4(n-1)}\, R_g\, v + n(n-2)\, v^{\frac{n+2}{n-2}} = 0,$$

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} < \Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}},$$

and $\sup_{|x| \leq \lambda} v(x) \geq c\,\lambda^{\frac{2-n}{2}}$*. Here, c is a positive constant that depends only on n.*

Proof. By Corollary 23, the function $F(\xi,\varepsilon)$ has a strict local minimum at $(0,1)$. It follows from Proposition 17 that $F(0,1) < 0$. Hence, we can find an open set $\Omega' \subset \Omega$ such that $(0,1) \in \Omega'$ and

$$F(0,1) < \inf_{(\xi,\varepsilon) \in \partial\Omega'} F(\xi,\varepsilon) < 0.$$

Using Corollary 12, we obtain

$$\begin{aligned}&|\mathcal{F}_g(\lambda\xi,\lambda\varepsilon) - \lambda^{16}\,\mu^2\,F(\xi,\varepsilon)|\\ &\leq C\,\lambda^{\frac{16n}{n-2}}\,\mu^{\frac{2n}{n-2}} + C\,\lambda^8\,\mu\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} + C\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}\end{aligned}$$

for all $(\xi,\varepsilon) \in \Omega$. This implies

$$\begin{aligned}&|\lambda^{-16}\,\mu^{-2}\,\mathcal{F}_g(\lambda\xi,\lambda\varepsilon) - F(\xi,\varepsilon)|\\ &\leq C\,\lambda^{\frac{32}{n-2}}\,\mu^{\frac{4}{n-2}} + C\,\rho^{\frac{2-n}{2}}\,\mu^{-1}\,\lambda^{\frac{n-18}{2}} + C\,\rho^{2-n}\,\mu^{-2}\,\lambda^{n-18}\end{aligned}$$

for all $(\xi,\varepsilon) \in \Omega$. Hence, if $\rho^{2-n}\,\mu^{-2}\,\lambda^{n-18}$ is sufficiently small, then we have

$$\mathcal{F}_g(0,\lambda) < \inf_{(\xi,\varepsilon) \in \partial\Omega'} \mathcal{F}_g(\lambda\xi,\lambda\varepsilon) < 0.$$

Consequently, there exists a point $(\bar{\xi},\bar{\varepsilon}) \in \Omega'$ such that

$$\mathcal{F}_g(\lambda\bar{\xi},\lambda\bar{\varepsilon}) = \inf_{(\xi,\varepsilon) \in \Omega'} \mathcal{F}_g(\lambda\xi,\lambda\varepsilon) < 0.$$

By Proposition 3, the function $v = v_{(\lambda\bar{\xi},\lambda\bar{\varepsilon})}$ is a non-negative weak solution of the partial differential equation

$$\Delta_g v - \frac{n-2}{4(n-1)}\,R_g\,v + n(n-2)\,v^{\frac{n+2}{n-2}} = 0.$$

Using a result of N. Trudinger, we conclude that v is smooth (see [**18**], Theorem 3 on p. 271). Moreover, we have

$$\begin{aligned}2(n-2)\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} &= 2(n-2)\,\Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}} + \mathcal{F}_g(\lambda\bar{\xi},\lambda\bar{\varepsilon})\\ &< 2(n-2)\,\Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}}.\end{aligned}$$

Finally, it follows from Proposition 2 that $\|v - u_{(\lambda\bar{\xi},\lambda\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\,\alpha$. This implies

$$|B_\lambda(0)|^{\frac{n-2}{2n}}\,\sup_{|x| \leq \lambda} v(x) \geq \|v\|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} \geq \|u_{(\lambda\bar{\xi},\lambda\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} - C\,\alpha.$$

Hence, if α is sufficiently small, then we obtain $\lambda^{\frac{n-2}{2}}\,\sup_{|x| \leq \lambda} v(x) \geq c$.

Proposition 25. *Let $25 \leq n \leq 51$. Then there exists a smooth metric g on $\mathbb{R}^n$ with the following properties:*

(i) $g_{ik}(x) = \delta_{ik}$ *for* $|x| \geq \frac{1}{2}$.

(ii) g *is not conformally flat.*

(iii) *There exists a sequence of non-negative smooth functions v_ν ($\nu \in \mathbb{N}$) such that*

$$\Delta_g v_\nu - \frac{n-2}{4(n-1)} R_g v_\nu + n(n-2) v_\nu^{\frac{n+2}{n-2}} = 0$$

for all $\nu \in \mathbb{N}$,

$$\int_{\mathbb{R}^n} v_\nu^{\frac{2n}{n-2}} < \Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}}$$

for all $\nu \in \mathbb{N}$, *and* $\sup_{|x| \leq 1} v_\nu(x) \to \infty$ *as* $\nu \to \infty$.

Proof. Choose a smooth cutoff function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$. We define a trace-free symmetric two-tensor on $\mathbb{R}^n$ by

$$h_{ik}(x) = \sum_{N=N_0}^{\infty} \eta(4N^2 \, |x - y_N|) \, 2^{-4N} \, f(2^N \, |x - y_N|^2) \, H_{ik}(x - y_N),$$

where $y_N = (\frac{1}{N}, 0, \ldots, 0) \in \mathbb{R}^n$. It is straightforward to verify that $h(x)$ is C^∞ smooth. Moreover, if N_0 is sufficiently large, then we have $h(x) = 0$ for $|x| \geq \frac{1}{2}$ and $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha$ for all $x \in \mathbb{R}^n$. (Here, α is the constant that appears in Proposition 24.) We now define a Riemannian metric g by $g(x) = \exp(h(x))$. The assertion is then a consequence of Proposition 24.

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