J. DIFFERENTIAL GEOMETRY
81 (2009) 197-224

DEFORMATIONS AND FOURIER-MUKAI TRANSFORMS

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Abstract

The aim of this paper is twofold. First we give an explicit construction of the infinitesimal deformations of the category $\mathrm{Coh}(X)$ of coherent sheaves on a smooth projective variety X. Secondly, we show that any Fourier-Mukai transform $\Phi\colon D^b(X) \to D^b(Y)$ extends to an equivalence between the derived categories of the deformed Abelian categories.

1. Introduction

Recent developments on derived categories, coming from Homological mirror symmetry [**11**] or birational geometry [**10**], motivate the necessity to establish a good deformation theory of derived categories. The general deformation theory of Abelian categories was previously studied in [**13**], and the A_∞-deformations of triangulated categories were studied in [**1**]. However, these analysis in these papers does not address the relationship between deformations and Fourier-Mukai transforms. So the following question arises:

"How do deformations interact with Fourier-Mukai transforms?"

In this paper we concentrate on the first order deformations of $\mathrm{Coh}(X)$, and answer the above question in this case. Here X is a smooth projective variety and $\mathrm{Coh}(X)$ is an Abelian category of coherent sheaves on X. By the philosophy of Kontsevich [**11**], the Hochschild cohomology $HH^*(X)$ should parameterize deformations of derived categories. The degree 2-part should consist of deformations of $\mathrm{Coh}(X)$, since $HH^2(X)$ contains $H^1(X, T_X)$ (deformations of complex structures) as a direct summand. The famous HKR-isomorphism says that N-th Hochschild cohomology is isomorphic to the direct sum $HT^N(X) := \oplus_{p+q=N} H^p(X, \wedge^q T_X)$. So there should be $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-linear Abelian category $\mathrm{Coh}(X, u)$ for $u \in HT^2(X)$. Roughly the goals of this paper can be summarized as follows.

- Give an explicit construction of $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-linear Abelian category $\mathrm{Coh}(X, u)$.

Received 05/15/2005.

- Understand the behavior of the deformed triangulated category

$$D^b(X,u) := D^b(\mathrm{Coh}(X,u))$$

under Fourier-Mukai transform $\Phi\colon D^b(X) \to D^b(Y)$.

Note that any $u \in HT^2(X)$ can be written as a sum $\alpha+\beta+\gamma$, with $\alpha \in H^2(X,\mathcal{O}_X)$, $\beta \in H^1(X,T_X)$, and $\gamma \in H^0(X,\wedge^2 T_X)$. Then β corresponds to a deformation of X as a scheme, γ is a non-commutative deformation. We will introduce "twisted" sheaves using α, and define $\mathrm{Coh}(X,u)$ as a combination of these components.

Next we make the second goal more precise. Let X and Y be smooth projective varieties such that there exists an equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then we have an induced isomorphism of Hochschild cohomologies $\phi\colon HH^*(X) \to HH^*(Y)$. By combining ϕ with HKR isomorphisms, we obtain the isomorphism $\phi_T\colon HT^2(X) \to HT^2(Y)$. Then the main theorem of this paper is the following:

Theorem 1.1. *For $u \in HT^2(X)$, let $v := \phi_T(u) \in HT^2(Y)$. Then there exists an equivalence*

$$\Phi^\dagger\colon D^b(X,u) \to D^b(Y,v),$$

such that the following diagram is 2-commutative:

$$\begin{array}{ccccc} D^b(X) & \xrightarrow{i_*} & D^b(X,u) & \xrightarrow{\mathbf{L}i^*} & D^-(X) \\ \Phi\downarrow & & \downarrow\Phi^\dagger & & \downarrow\Phi^- \\ D^b(Y) & \xrightarrow{i_*} & D^b(Y,v) & \xrightarrow{\mathbf{L}i^*} & D^-(Y). \end{array}$$

By the above theorem, we can compare deformation theories under Fourier-Mukai transforms. One of the interesting points of Theorem 1.1 is that ϕ_T does not necessary preserve direct summands of $HT^2(X)$. This indicates $\Phi^\dagger$ may produce new interesting Fourier-Mukai dualities, for example dualities between usual commutative schemes and non-commutative schemes. Recently in the paper [**3**], the equivalence $\Phi^\dagger$ of Theorem 1.1 has been extended to infinite order deformations, when X is an Abelian variety, Y is its dual, and Φ is given by the Poincare line bundle. This result is giving a new kind of dualities via deformations, and it seems we will be able to find more examples of Fourier-Mukai equivalences through deformation methods.

Acknowledgement. The author would like to express his profound gratitude to Professor Yujiro Kawamata, for many valuable comments, and warmful encouragement. The author also would like to thank T. Bridgeland for informing the author on the paper [**2**].

2. Hochschild cohomology and derived category

Let X be a smooth projective variety over $\mathbb{C}$ and $\Delta_X \subset X \times X$ be a diagonal. We write Δ_X as Δ if it causes no confusion. In this section we recall the definitions of Fourier-Mukai transform, Hochschild cohomology and their properties.

Definition 2.1. Let X and Y be smooth projective varieties and take $\mathcal{P} \in D^b(X \times Y)$. Let p_i be projections from $X \times Y$ onto the corresponding factors. We define $\Phi^{\mathcal{P}}_{X \to Y}$ as the following functor:

$$\Phi^{\mathcal{P}}_{X \to Y} := \mathbf{R}p_{2*}(p_1^*(*) \overset{\mathbf{L}}{\otimes} \mathcal{P}) \colon D^b(X) \to D^b(Y).$$

$\Phi^{\mathcal{P}}_{X \to Y}$ is called an integral transform with kernel $\mathcal{P}$. If $\Phi^{\mathcal{P}}_{X \to Y}$ gives an equivalence, then it is called a Fourier-Mukai transform.

The following theorem is fundamental in studying derived categories.

Theorem 2.2 (Orlov [**15**])**.** *Let $\Phi\colon D^b(X) \to D^b(Y)$ be an exact functor. Assume that Φ is fully faithful and has a right adjoint. Then there exists an object $\mathcal{P} \in D^b(X \times Y)$ such that Φ is isomorphic to the functor $\Phi^{\mathcal{P}}_{X \to Y}$. Moreover, $\mathcal{P}$ is uniquely determined up to isomorphism.*

Next we recall the Hochschild cohomology of the structure sheaf, given in [**11**].

Definition 2.3. We define $HH^N(X)$ and $HT^N(X)$ as follows:

$$HH^N(X) := \mathrm{Hom}_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[N]),$$
$$HT^N(X) := \bigoplus_{p+q=N} H^p(X, \bigwedge^q T_X).$$

Here Hom is a morphism in $D^b(X \times X)$. $HH^*(X)$ is called Hochschild cohomology.

Note that the object $\mathcal{F} \in D^b(X \times X)$ gives a functor $\Phi^{\mathcal{F}}_{X \to X}$, and the morphism $\mathcal{F} \to \mathcal{G}$ gives a natural transformation $\Phi^{\mathcal{F}}_{X \to X} \to \Phi^{\mathcal{G}}_{X \to X}$. In this sense, Hochschild cohomology is a natural transformation $\mathrm{id}_X \to [N]$. But as in [**6**], we can not consider $D^b(X \times X)$ as the category of functors precisely. (The map from the morphisms in $D^b(X \times X)$ to the natural transformations is not injective in general.) However, we can show the several properties of derived categories concerning $D^b(X \times X)$, for example categorical invariance of Hochschild cohomology, as if it is a category of functors. Since the natural transformations are categorical, Hochschild cohomology should be categorical invariant. In fact, we have the following theorem in [**6**].

Theorem 2.4 (Caldararu [**6**]). *Let X and Y be smooth projective varieties such that there exists an equivalence $\Phi\colon D^b(X) \to D^b(Y)$. Then Φ induces an isomorphism $\phi\colon HH^*(X) \to HH^*(Y)$.*

Outline of the proof. We will give the outline of the Caldararu's proof. Let $\mathcal{P} \in D^b(X \times Y)$ be a kernel of Φ, and $\mathcal{E} \in D^b(X \times Y)$ be a kernel of Φ^{-1}. Let $p_{ij}\colon X \times X \times Y \times Y \to X \times Y$ be projections onto corresponding factors. Caldararu [**6**] showed that the functor with kernel= $p_{13}^*\mathcal{P} \boxtimes p_{24}^*\mathcal{E} \in D^b(X \times X \times Y \times Y)$,

$$\Phi_{X\times X\to Y\times Y}^{p_{13}^*\mathcal{P}\boxtimes p_{24}^*\mathcal{E}}\colon D^b(X \times X) \to D^b(Y \times Y)$$

gives an equivalence which takes $\mathcal{O}_{\Delta_X}$ to $\mathcal{O}_{\Delta_Y}$. This equivalence implies the theorem immediately. q.e.d.

Next we can compare $HH^*(X)$ and $HT^*(X)$. Hochschild cohomology is useful since its definition is categorical. But it is difficult to write down Hochschild cohomology classes explicitly. In calculating Hochschild cohomology, we decompose it into direct sums of sheaf cohomologies of tangent bundles. The following theorem is due to Hochschild-Kostant-Rosenberg [**8**], Kontsevich [**11**], Swan [**17**], and Yekutieli [**19**].

Theorem 2.5. *There exists an isomorphism,*

$$I_{HKR}\colon HT^*(X) \to HH^*(X).$$

Outline of the proof. Note that $HH^N(X) \cong \mathrm{Hom}_X(\mathbf{L}\Delta^*\mathcal{O}_\Delta, \mathcal{O}_X[N])$ by adjunction. Let $\mathcal{O}_X^{\otimes i} \in \mathrm{Mod}(\mathcal{O}_X)$ be the sheaf associated to the following presheaf:

$$U \subset X \longmapsto \Gamma(U, \mathcal{O}_X)^{\otimes i}.$$

Here $\otimes$ is over $\mathbb{C}$, and $\mathcal{O}_X$-module structure on $\mathcal{O}_X^{\otimes i}$ is given by

$$a \cdot (x_0 \otimes x_1 \otimes \cdots \otimes x_i) := ax_0 \otimes x_1 \otimes \cdots \otimes x_i,$$

for $a, x_k \in \mathcal{O}_X$. Let $d^i\colon \mathcal{O}_X^{\otimes(i+1)} \to \mathcal{O}_X^{\otimes i}$ be

$$d^i(x_0 \otimes \cdots \otimes x_i) = \sum_{k=0}^{i-1}(-1)^k x_0 \otimes \cdots \otimes x_k x_{k+1} \otimes \cdots \otimes x_i + (-1)^i x_0 x_i \otimes x_1 \otimes \cdots \otimes x_{i-1}.$$

Then we have the complex of $\mathcal{O}_X$-modules:

$$\mathcal{C}_X := (\to \mathcal{O}_X^{\otimes(i+1)} \xrightarrow{d^i} \mathcal{O}_X^{\otimes i} \to \cdots \to \mathcal{O}_X \to 0).$$

By [**19**], we have an explicit quasi-isomorphism $\mathcal{C}_X \xrightarrow{\sim} \mathbf{L}\Delta^*\mathcal{O}_\Delta$ in $D(\mathrm{Mod}(\mathcal{O}_X))$. Yekutieli [**19**] describes this isomorphism by building a resolution using the formal neighborhood $X \subset X \times X \times \cdots \times X$. On the other hand, we have the following quasi-isomorphism $\mathcal{C}_X \to \oplus_{p\geq 0}\Omega_X^p[p]$:

$$\begin{array}{ccccccccccc} \longrightarrow & \mathcal{O}_X^{\otimes(i+1)} & \xrightarrow{d^i} & \cdots & \xrightarrow{d^2} & \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{0} & \mathcal{O}_X & \longrightarrow & 0 \\ & \downarrow{\scriptstyle I^i} & & & & \downarrow{\scriptstyle I^1} & & \downarrow{\scriptstyle I^0} & & \\ \longrightarrow & \Omega_X^i & \xrightarrow{0} & \cdots & \xrightarrow{0} & \Omega_X & \xrightarrow{0} & \mathcal{O}_X & \longrightarrow & 0. \end{array}$$

Here $I^i \colon \mathcal{O}_X^{\otimes(i+1)} \to \Omega_X^i$ is given by

$$I^i(x_0 \otimes \cdots \otimes x_i) = x_0 \cdot dx_1 \wedge \cdots \wedge dx_i.$$

One can consult [**12**] for the detail. Consequently, we get the quasi-isomorphism, $I \colon \mathbf{L}\Delta^*\mathcal{O}_\Delta \xrightarrow{\sim} \oplus_{p\geq 0}\Omega_X^p[p]$. Therefore we have the following isomorphism:

$$\mathrm{Hom}_X(\oplus_{p\geq 0}\Omega_X^p[p], \mathcal{O}_X[N]) \xrightarrow{I} \mathrm{Hom}_X(\mathbf{L}\Delta^*\mathcal{O}_\Delta, \mathcal{O}_X[N]).$$

The left hand side is $HT^N(X)$ and the right hand side is $HH^N(X)$.

q.e.d.

I_{HKR} is called the HKR (Hochschild–Kostant–Rosenberg)-isomorphism. In the rest of this paper we write I_{HKR} as I_X. Assume that X and Y are related by some Fourier-Mukai transform $\Phi \colon D^b(X) \to D^b(Y)$. By combining the isomorphisms I_X, I_Y and ϕ, we have the isomorphism:

$$\phi_T := I_Y^{-1} \circ \phi \circ I_X \colon HT^*(X) \xrightarrow{\sim} HT^*(Y).$$

In the following 2-sections, we will construct deformations of $\mathrm{Coh}(X)$ for $u \in HT^2(X)$.

3. Non-commutative deformations of affine schemes

Let R be a Noetherian commutative ring and $X = \operatorname{Spec} R$. In this section we will consider a sheaf $\mathcal{A}$ of (not necessary commutative) algebras on X. Let $\mathcal{U}_X$ be the category whose objects consist of Zariski open subset of X, and let $\mathcal{A}$ be a sheaf of algebra on X. Recall that a sheaf $\mathcal{M}$ of left $\mathcal{A}$-modules is quasi-coherent if for each $x \in X$, there exists an open neighborhood U of x and an exact sequence of left $\mathcal{A}_U$-modules,

$$(\mathcal{A}_U)^J \longrightarrow (\mathcal{A}_U)^I \longrightarrow \mathcal{M}_U \longrightarrow 0.$$

$\mathcal{M}$ is coherent if the following conditions are satisfied:

- $\mathcal{M}$ is finitely generated, i.e., for every $x \in X$, there exists an open neighborhood U of x and a surjection $(\mathcal{A}_U)^n \twoheadrightarrow M_U$.
- For every $U \in \mathcal{U}_X$ and every $n \in \mathbb{Z}_{>0}$, and an arbitrary morphism of left $\mathcal{A}_U$-modules $\phi \colon (\mathcal{A}_U)^n \to \mathcal{M}_U$, $\ker \phi$ is finitely generated.

We denote by $\mathrm{Mod}(\mathcal{A})$ the category of sheaves of left $\mathcal{A}$-modules, by $\mathrm{QCoh}(\mathcal{A})$ full-subcategory of quasi-coherent sheaves, and by $\mathrm{Coh}(\mathcal{A})$ coherent sheaves. Of course, it is well-known that if $\mathcal{A} = \mathcal{O}_X$, then a quasi-coherent sheaf is written as $\widetilde{M}$ for some R-module M, and a coherent sheaf is $\widetilde{M}$ for a finitely generated R-module M. We generalize these results to some non-commutative situations. Let γ be a bidifferential operator $\gamma\colon R \times R \longrightarrow R$. Using γ we define a (not necessary commutative) ring structure on $R[\varepsilon]/(\varepsilon^2)$ as follows:

$$(a + b\varepsilon) *_\gamma (c + d\varepsilon) := ac + (\gamma(a, c) + ad + bc)\varepsilon,$$

and denote it by $R^{(\gamma)}$. Let M be a left $R^{(\gamma)}$-module. Then the functor

$$\mathcal{U}_X \ni U \longmapsto \mathcal{O}_X(U)^{(\gamma)} \otimes_{R^{(\gamma)}} M \in (\text{left } \mathcal{O}_X(U)^{(\gamma)}\text{-modules})$$

determines a presheaf of sets on X. Let $\widetilde{M}$ be the associated sheaf. We have a sheaf of rings $\mathcal{O}_X^{(\gamma)} := \widetilde{R^{(\gamma)}}$ and $\widetilde{M}$ is a left $\mathcal{O}_X^{(\gamma)}$-module. Note that since $\mathcal{O}_X(U)^{(\gamma)}$ is right $R^{(\gamma)}$-left $\mathcal{O}_X(U)^{(\gamma)}$-module, $\mathcal{O}_X(U)^{(\gamma)} \otimes_{R^{(\gamma)}} M$ has a left $\mathcal{O}_X(U)^{(\gamma)}$-module structure.

As in the commutative case, we have the following lemma.

Lemma 3.1.

(1) *For $f \in R$, $\widetilde{M}(U_f) = R_f^{(\gamma)} \otimes_{R^{(\gamma)}} M$. In particular, $\widetilde{M}(X) = M$ and $\mathcal{O}_X^{(\gamma)}(X) = R^{(\gamma)}$.*

(2) *$\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X^{(\gamma)}$-module.*

(3) *The functor*

$$(\text{left } R^{(\gamma)}\text{-mod}) \ni M \longmapsto \widetilde{M} \in \mathrm{QCoh}(\mathcal{O}_X^{(\gamma)})$$

gives an equivalence of categories.

(4) *For $\mathcal{F} \in \mathrm{QCoh}(\mathcal{O}_X^{(\gamma)})$, $\mathcal{F}$ is coherent if and only if $M = \mathcal{F}(X)$ is a finitely generated left $R^{(\gamma)}$-module.*

Proof. Note that if we consider an R-module N as left $R^{(\gamma)}$-module by the surjection $R^{(\gamma)} \twoheadrightarrow R$, then the action of $\mathcal{O}_X^{(\gamma)}$ on $\widetilde{N}$ descends to $\mathcal{O}_X$, and $\widetilde{N}$ is a quasi-coherent $\mathcal{O}_X$-module.

(1) It suffices to show $\widetilde{M}(X) = M$. By the construction of $\widetilde{M}$, we have the natural morphism $M \to \widetilde{M}(X)$. Applying $\otimes_{R^{(\gamma)}} M$ to the surjection $R^{(\gamma)} \twoheadrightarrow R$, we obtain the exact sequence

$$0 \longrightarrow \ker(r) \longrightarrow M \xrightarrow{r} R \otimes_{R^{(\gamma)}} M \longrightarrow 0,$$

and the left action of $R^{(\gamma)}$ on $\ker(r)$ and $R \otimes_{R^{(\gamma)}} M$ descends to R. Therefore, $\widetilde{\ker(r)}$ and $\widetilde{R \otimes_{R^{(\gamma)}} M}$ are quasi-coherent $\mathcal{O}_X$-modules. By applying $M \mapsto \widetilde{M}$ and taking global sections, we obtain the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(r) & \longrightarrow & \widetilde{M}(X) & \longrightarrow & R \otimes_{R^{(\gamma)}} M & \longrightarrow & 0 \\ & & \| & & \uparrow & & \| & & \\ 0 & \longrightarrow & \ker(r) & \longrightarrow & M & \longrightarrow & R \otimes_{R^{(\gamma)}} M & \longrightarrow & 0. \end{array}$$

It is easy to check that the multiplicative set $S = \{f^{*_\gamma n}\}_{n \geq 0} \subset R^{(\gamma)}$ satisfies the right and left Ore localization conditions, and $R_f^{(\gamma)}$ is a localization $S^{-1}R^{(\gamma)}$. Therefore, the functor $M \mapsto \widetilde{M}$ is an exact functor. Moreover, since $H^1(X, \widetilde{\ker r}) = 0$, the top diagram is exact. By the 5-lemma, we have the isomorphism $M \to \widetilde{M}(X)$.

(2) Since $M \mapsto \widetilde{M}$ is an exact functor, we have an exact sequence

$$\widetilde{R^{(\gamma)J}} \longrightarrow \widetilde{R^{(\gamma)I}} \longrightarrow \widetilde{M} \longrightarrow 0.$$

(3) Take $\mathcal{F} \in \mathrm{QCoh}(\mathcal{O}_X^{(\gamma)})$. Applying $\otimes_{\mathcal{O}_X^{(\gamma)}} \mathcal{F}$ to the exact sequence

$$0 \longrightarrow \varepsilon\mathcal{O}_X \longrightarrow \mathcal{O}_X^{(\gamma)} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

we can easily see that $\mathcal{F}$ is given as an extension of quasi-coherent $\mathcal{O}_X$-modules. Therefore the problem is reduced to the following lemma:

Lemma 3.2. *Let $D^b(R^{(\gamma)})$ be the bounded derived category of left $R^{(\gamma)}$-modules, and $\mathrm{Mod}(X, \gamma) := \mathrm{Mod}(\mathcal{O}_X^{(\gamma)})$. The functor*

$$D^b(R^{(\gamma)}) \ni M \longmapsto \widetilde{M} \in D^b(\mathrm{Mod}(X, \gamma))$$

is fully faithful.

Proof. Take $M, N \in D^b(R^{(\gamma)}) \subset D(R^{(\gamma)})$ and we will show that

$$\mathrm{Hom}_{D(R^{(\gamma)})}(M, N) \longrightarrow \mathrm{Hom}_{D(\mathrm{Mod}(X,\gamma))}(\widetilde{M}, \widetilde{N})$$

is an isomorphism. By taking a free resolution, we may assume M is a bounded above complex of free $R^{(\gamma)}$-modules. Let $M_k := \sigma_{\geq -k} M$. Here $\sigma_{\geq -k}$ denotes the stupid truncation. Now we have a sequence of complexes $\to M_k \to M_{k+1} \to \cdots$ and if we take the homotopy colimit (cf. [**2**])

$$\oplus_k M_k \xrightarrow{s-\mathrm{id}} \oplus_k M_k \longrightarrow \mathrm{hocolim}(M_k) \longrightarrow \oplus_k M_k[1],$$

then there exists a quasi-isomorphism $\mathrm{hocolim}(M_k) \to M$. Here s is the shift map, whose coordinates are the natural maps $M_k \to M_{k+1}$. Therefore, we may assume M is a finite complex of free $R^{(\gamma)}$-modules. Again by taking stupid truncations, we may assume $M = R^{(\gamma)}$. Since

N is bounded, we may assume $N = N'[k]$ for some left $R^{(\gamma)}$-module N'. Now it suffices to show that the map

$$\mathrm{Hom}_{D(R^{(\gamma)})}(R^{(\gamma)}, N'[k]) \longrightarrow \mathrm{Hom}_{D(\mathrm{Mod}(X,\gamma))}(\mathcal{O}_X^{(\gamma)}, \widetilde{N}'[k])$$

is an isomorphism. If $k < 0$, then both sides are zero. If $k = 0$, then both sides are N'. If $k > 0$, then the left hand side is zero, so it suffices to show $H^k(X, \widetilde{N}') = 0$ for $k > 0$. But since $\widetilde{N}'$ is an extension of quasi-coherent $\mathcal{O}_X$-modules, $H^k(X, \widetilde{N}') = 0$ for $k > 0$. q.e.d.

(4) First we check that a submodule of a finitely generated $R^{(\gamma)}$-module is also finitely generated. In fact, let M be a finitely generated $R^{(\gamma)}$-module, and $N \subset M$ be a submodule. Then we have the natural morphism $g\colon N \to R \otimes_{R^{(\gamma)}} M$. It is enough to check that $\ker(g)$ and $\mathrm{im}(g)$ are finitely generated $R^{(\gamma)}$-modules. Note that we have $\ker(g) \subset \varepsilon M$ and $\mathrm{im}(g) \subset R \otimes_{R^{(\gamma)}} M$. Since R is Noetherian and εM, $R \otimes_{R^{(\gamma)}} M$ are both finitely generated R-modules, it follows that $\ker(g)$ and $\mathrm{im}(g)$ are both finitely generated R-modules. Thus, in particular, these are finitely generated $R^{(\gamma)}$-modules via the surjection $R^{(\gamma)} \twoheadrightarrow R$.

Using this fact, we can see that $\widetilde{M}$ for a finitely generated left $R^{(\gamma)}$-module M is coherent. On the other hand, take $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_X^{(\gamma)})$. Then by (3), $\mathcal{F}$ can be written as $\mathcal{F} = \widetilde{M}$ for some left $R^{(\gamma)}$-module M. Since $\mathcal{F}$ is given by an extension of coherent $\mathcal{O}_X$-modules, M is a finitely generated left $R^{(\gamma)}$-module. q.e.d.

For a full subcategory $\mathcal{C} \subset \mathrm{Mod}(X,\gamma)$, let $D^b_{\mathcal{C}}(\mathrm{Mod}(X,\gamma))$ denote the full subcategory of $D^b(\mathrm{Mod}(X,\gamma))$ whose objects have cohomologies contained in $\mathcal{C}$. As a corollary, we obtain the following:

Corollary 3.3. *There exist equivalences,*

$$D^b(R^{(\gamma)}) \xrightarrow{\sim} D^b_{\mathrm{QCoh}}(\mathrm{Mod}(X,\gamma)), \quad D^b_f(R^{(\gamma)}) \xrightarrow{\sim} D^b_{\mathrm{Coh}}(\mathrm{Mod}(X,\gamma)).$$

Here $D^b_f(R^{(\gamma)})$ is a derived category of finitely generated left $R^{(\gamma)}$-modules.

Proof. We have proved the full faithfulness in Lemma 3.2. Since an object of $\mathrm{QCoh}(\mathcal{O}_X^{(\gamma)})$ is written as $\widetilde{M}$ for a left $R^{(\gamma)}$-module M, the image from the left hand side generates the right hand side. q.e.d.

Remark 3.4. In general, we can show the unbounded case of the above corollary as in [**2**]. Here we gave a proof of the bounded case for the sake of simplicity. For details, the reader should refer to [**2**].

4. Infinitesimal deformations of $\mathrm{Coh}(X)$

From this section on, we will assume that X is a smooth projective variety over $\mathbb{C}$. The aim of this section is to construct the first order deformations of $\mathrm{Coh}(X)$. First we begin with the general situation. Let us take an affine open cover $X = \cup_{i=1}^{N} U_i$, and denote by $\mathfrak{U}$ this open cover. Let $U_{i_0\cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$, and $j_{i_0\cdots i_p} \colon U_{i_0\cdots i_p} \hookrightarrow X$ be open immersions. For a sheaf $\mathcal{F}$ on X, let $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})$, $\mathbf{C}^p(\mathfrak{U}, \mathcal{F})$ be

$$\mathcal{C}^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0\cdots i_p} j_{i_0\cdots i_p *} j^{*}_{i_0\cdots i_p} \mathcal{F}, \quad \mathbf{C}^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0\cdots i_p} \Gamma(U_{i_0\cdots i_p}, \mathcal{F}).$$

Let us consider a sheaf of algebras $\mathcal{A}$ on X and its center $Z(\mathcal{A})$. Take $\tau \in H^2(X, Z(\mathcal{A})^{\times})$. Then τ is represented by a Čech cocycle $\tau = \{\tau_{i_0 i_1 i_2}\} \in \mathbf{C}^2(\mathfrak{U}, Z(\mathcal{A})^{\times})$. We define the category $\mathrm{Mod}(\mathcal{A}, \tau)$ as follows:

Definition 4.1. We define $\mathrm{Mod}(\mathcal{A}, \tau)$ as an Abelian category of τ-twisted left $\mathcal{A}$-modules. Namely objects of $\mathrm{Mod}(\mathcal{A}, \tau)$ are collections

$$\mathcal{F} = (\{\mathcal{F}_i\}_{1 \le i \le N}, \phi_{i_0 i_1}),$$

where $\mathcal{F}_i \in \mathrm{Mod}(\mathcal{A}|_{U_i})$ and $\phi_{i_0 i_1}$ are isomorphisms

$$\phi_{i_0 i_1} \colon \mathcal{F}_{i_0}|_{U_{i_0 i_1}} \xrightarrow{\cong} \mathcal{F}_{i_1}|_{U_{i_0 i_1}}$$

as left $\mathcal{A}|_{U_i}$-modules. These data must satisfy the equality

$$\phi_{i_2 i_0} \circ \phi_{i_1 i_2} \circ \phi_{i_0 i_1} = \tau_{i_0 i_1 i_2} \cdot id_{\mathcal{F}_0}.$$

We say $\mathcal{F} \in \mathrm{Mod}(\mathcal{A}, \tau)$ is quasi-coherent if $\mathcal{F}_i \in \mathrm{QCoh}(\mathcal{A}|_{U_i})$, and coherent if $\mathcal{F}_i \in \mathrm{Coh}(\mathcal{A}|_{U_i})$. We denote by $\mathrm{QCoh}(\mathcal{A}, \tau)$ the category of quasi-coherent τ-twisted left $\mathcal{A}$-modules, and by $\mathrm{Coh}(\mathcal{A}, \tau)$ coherent twisted sheaves.

Lemma 4.2.
Up to equivalence, the categories $\mathrm{Mod}(\mathcal{A}, \tau)$, $\mathrm{QCoh}(\mathcal{A}, \tau)$, $\mathrm{Coh}(\mathcal{A}, \tau)$ *are independent of choices of* $\mathfrak{U}$ *and Čech representative of* α.

Proof. The proof is easy and left to the reader. q.e.d.

Fundamental properties and operations on $\mathrm{Mod}(\mathcal{A}, \tau)$**.**

- j^*, j_*, $j_!$ *for an open immersion* $j \colon U \hookrightarrow X$

Let $j \colon U \hookrightarrow X$ be an open immersion. We have the obvious functors:

$$j^* \colon \mathrm{Mod}(\mathcal{A}, \tau) \longrightarrow \mathrm{Mod}(\mathcal{A}|_U, \tau|_U),$$
$$j_*, j_! \colon \mathrm{Mod}(\mathcal{A}|_U, \tau|_U) \longrightarrow \mathrm{Mod}(\mathcal{A}, \tau).$$

j_* is right adjoint of j^*, and $j_!$ is left adjoint of j^*. For $\mathcal{F} = (\{\mathcal{F}_{i_0}\}, \phi_{i_0 i_1}) \in \mathrm{Mod}(\mathcal{A}|_U, \tau_U)$, with $\mathcal{F}_{i_0} \in \mathrm{Mod}(\mathcal{A}|_{U \cap U_{i_0}})$, $j_*\mathcal{F}$ and $j_!\mathcal{F}$ are given by

$$j_*(\mathcal{F})_{i_0} := (j|_{U \cap U_{i_0}})_* \mathcal{F}_{i_0}, \quad j_!(\mathcal{F})_{i_0} := (j|_{U \cap U_{i_0}})_! \mathcal{F}_{i_0}.$$

Here

$$(j|_{U \cap U_{i_0}})_! \colon \mathrm{Mod}(\mathcal{A}|_{U \cap U_{i_0}}) \longrightarrow \mathrm{Mod}(\mathcal{A}|_{U_{i_0}})$$

is extension by zero.

- *Tensor product*

Let us take $\mathcal{F} \in \mathrm{Mod}(\mathcal{A}^{op}, \tau)$. Assume that the right action of the subalgebra $\mathcal{B} \subset \mathcal{A}$ on $\mathcal{F}$ is centralized. Then we have the functor

$$\mathcal{F} \otimes * \colon \mathrm{Mod}(\mathcal{A}, \tau') \longrightarrow \mathrm{Mod}(\mathcal{B}, \tau \cdot \tau').$$

In particular, if $\mathcal{B}$ is contained in the center of $\mathcal{A}$, then we have the functor,

$$\otimes \colon \mathrm{Mod}(\mathcal{A}^{op}, \tau) \times \mathrm{Mod}(\mathcal{A}, \tau') \longrightarrow \mathrm{Mod}(\mathcal{B}, \tau \cdot \tau').$$

- *Pull-back*

Let $f \colon Y \to X$ be a morphism of varieties, and $\mathcal{A}$, $\mathcal{B}$ be sheaves of algebra on X and Y. If there exists a morphism of algebras $f^{-1}\mathcal{A} \to \mathcal{B}$ which preserves their centers, then we have the pullback

$$f^* \colon \mathrm{Mod}(\mathcal{A}, \tau) \longrightarrow \mathrm{Mod}(\mathcal{B}, f^*\tau),$$

which takes $(\{\mathcal{F}_i\}, \phi_{i_0 i_1})$ to $(\{\mathcal{B} \otimes_{\mathcal{A}} f^{-1}\mathcal{F}_i\}, 1 \otimes \phi_{i_0 i_1})$.

- *Push-forward*

In the same situation as above, we have a morphism of algebras $\mathcal{A} \to f_*\mathcal{B}$ which preserves their centers. We have the push-forward:

$$f_* \colon \mathrm{Mod}(\mathcal{B}, f^*\tau) \longrightarrow \mathrm{Mod}(\mathcal{A}, \tau).$$

Clearly f_* is a right adjoint of f^*.

- *Enough injectives and flats*

Lemma 4.3.

(i) $\mathrm{Mod}(\mathcal{A}, \tau)$ *has enough injectives.*

(ii) *For every $A \in \mathrm{Mod}(\mathcal{A}, \tau)$, there exists a flat object $P \in \mathrm{Mod}(\mathcal{A}, \tau)$ and a surjection $P \twoheadrightarrow A$. Here we say $\mathcal{F} = (\{\mathcal{F}_i\}, \phi_{i_0 i_1})$ is flat if each $\mathcal{F}_i$ is a flat $\mathcal{A}_{U_i}$-module.*

Proof.

(i) Take $A \in \mathrm{Mod}(\mathcal{A}, \tau)$. Since $\mathrm{Mod}(\mathcal{A}|_{U_i})$ has enough injective, there exists an injection $j^*A \hookrightarrow I_i$ for an injective object $I_i \in \mathrm{Mod}(\mathcal{A}|_{U_i})$. Let $\widetilde{I}_i := j_* I_i$. Then the composition

$$A \longrightarrow j_* j^* A \longrightarrow \prod_i \widetilde{I}_i$$

is an injection. Since j_* is a right adjoint of j^*, $\prod_i \widetilde{I}_i$ is an injective object of $\mathrm{Mod}(\mathcal{A}, \tau)$.

(ii) Take $A \in \mathrm{Mod}(\mathcal{A}, \tau)$. We can take a surjection $P_i \twoheadrightarrow j^*A$ for flat $\mathcal{O}_{U_i}^{(\gamma)}$-module P_i. Let $\bar{P}_i := j_! P_i$. Then the composition

$$\bigoplus_i \bar{P}_i \longrightarrow j_! j^* A \longrightarrow A$$

is surjective and $\bigoplus_i \bar{P}_i$ is flat. q.e.d.

Let us take an element

$$u = (\alpha, \beta, \gamma) \in HT^2(X) = H^2(\mathcal{O}_X) \oplus H^1(T_X) \oplus H^0(\wedge^2 T_X).$$

First we construct a sheaf $\mathcal{O}_X^{(\beta,\gamma)}$ of $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-algebras on X. Note that we can consider γ as a bidifferential operator $\mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$, and $\beta_{i_0 i_1}$ as a differential operator $\mathcal{O}_{U_{i_0 i_1}} \to \mathcal{O}_{U_{i_0 i_1}}$. As a sheaf $\mathcal{O}_X^{(\beta,\gamma)}$ is $\mathcal{O}_X^{(\beta_{i_0 i_1})}$, the kernel of the following morphism:

$$\mathcal{O}_X \oplus \mathcal{C}^0(\mathfrak{U}, \mathcal{O}_X) \ni (a, \{b_i\}) \overset{\delta^{(\beta_{i_0 i_1})}}{\longmapsto} -\beta_{i_0 i_1}(a) + \delta\{b_i\} \in \mathcal{C}^1(\mathfrak{U}, \mathcal{O}_X).$$

We define the product on $\mathcal{O}_X \oplus \mathcal{C}^0(\mathfrak{U}, \mathcal{O}_X)$ by the formula:

$$(a, \{b_i\}) *_\gamma (c, \{d_i\}) := (ac, \{ad_i + cb_i + \gamma(a, c)\}_i).$$

Then it is easy to see that $\mathcal{O}_X^{(\beta_{i_0 i_1})}$ is a subalgebra of $\mathcal{O}_X \oplus \mathcal{C}^0(\mathfrak{U}, \mathcal{O}_X)$, and denote by $\mathcal{O}_X^{(\beta,\gamma)}$ this sheaf of algebras. It is also easy to check that $\mathcal{O}_X^{(\beta,\gamma)}$ doesn't depend on the choices of $\mathfrak{U}$ and Čech representative of β. Note that $\mathcal{O}_X^{(\beta,\gamma)}|_{U_i} \cong \mathcal{O}_{U_i}^{(\gamma)}$ as a sheaf of algebra. Since $(1 - \alpha_{i_0 i_1 i_2}\varepsilon)$ is contained in the center of $\mathcal{O}_{U_{i_0 i_1 i_2}}^{(\gamma)}$, we have an element

$$\widetilde{\alpha} := \{(1 - \alpha_{i_0 i_1 i_2}\varepsilon)\}_{i_0 i_1 i_2} \in \mathbf{C}^2(X, Z(\mathcal{O}_X^{(\beta,\gamma)})),$$

which is a cocycle. Let $\mathrm{Mod}\,(X, u) := \mathrm{Mod}\,(\mathcal{O}_X^{(\beta,\gamma)}, \widetilde{\alpha})$, and define $\mathrm{QCoh}(X, u)$ and $\mathrm{Coh}(X, u)$ as above.

Now we can define $D^*(X, u)$ for $* = b, \pm, \emptyset$ as follows.

Definition 4.4. We define $\mathbb{C}\,[\varepsilon]\,/\,(\varepsilon^2)$–linear triangulated category $D^*(X, u)$ as

$$D^*(X, u) := D^*(\mathrm{Coh}(X, u)), \qquad (* = b, \pm, \emptyset).$$

As in [**2**], we have the following proposition:

Proposition 4.5. *There exist natural equivalences:*

$$D^*(\mathrm{QCoh}(X, u)) \xrightarrow{\sim} D^*_{\mathrm{QCoh}}(\mathrm{Mod}(X, u)),$$
$$D^*(X, u) \xrightarrow{\sim} D^*_{\mathrm{Coh}}(\mathrm{Mod}(X, u)),$$

for $* = b, \pm, \emptyset$.

Proof. The proof is the same as in [**2**]. Take an affine open cover $X = \cup_{i=1}^N U_i$. We use the induction on N to prove the proposition, and the case of $N = 1$ and $* = b$ has been proved in the previous section. q.e.d.

Now we can construct transformations between derived categories. Take two smooth projective varieties X and Y, and $u = (\alpha, \beta, \gamma) \in$

$HT^2(X)$, $v = (\alpha', \beta', \gamma') \in HT^2(Y)$. For a perfect object (i.e., locally quasi-isomorphic to bounded complexes of free modules) $\mathcal{P}^\dagger \in D^b_{\mathrm{perf}}(X \times Y, -p_1^*\check{u} + p_2^* v)$, we will construct a functor,

$$\Phi^\dagger \colon D^b(X, u) \longrightarrow D^b(Y, v).$$

Here $\check{u} := (\alpha, -\beta, \gamma)$. First take $\mathcal{F} \in D^b(X, u)$. Since we have a morphism of algebras

$$p_1^{-1}\mathcal{O}_X^{(\beta,\gamma)} \longrightarrow \mathcal{O}_{X\times Y}^{(p_1^*\beta + p_2^*\beta', p_1^*\gamma - p_2^*\gamma')},$$

we obtain the object

$$\begin{aligned} p_1^*\mathcal{F} &\in D^b(\mathrm{Coh}(\mathcal{O}_{X\times Y}^{(p_1^*\beta + p_2^*\beta', p_1^*\gamma - p_2^*\gamma')}, p_1^*\widetilde{\alpha})) \\ &\simeq D^b(\mathrm{Coh}(\mathcal{O}_{X\times Y}^{(p_1^*\beta + p_2^*\beta', -p_1^*\gamma + p_2^*\gamma'), op}, p_1^*\widetilde{\alpha})). \end{aligned}$$

Now, by Lemma 4.3, we can define $\overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger$. Since the right action of $p_2^{-1}\mathcal{O}_Y^{(\beta',\gamma')}$ on each term of $p_1^*\mathcal{F}$ is centralized, we obtain the object,

$$p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger \in D^b(\mathrm{Mod}(p_2^{-1}\mathcal{O}_Y^{(\beta',\gamma')}, p_2^*\widetilde{\alpha}')).$$

(Since $\mathcal{P}^\dagger$ is perfect, $\overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger$ preserves boundedness.) Applying $\mathbf{R}p_{2*}$, we obtain the object,

$$\mathbf{R}p_{2*}(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger) \in D^b(\mathrm{Mod}(\mathcal{O}_Y^{(\beta',\gamma')}, \widetilde{\alpha}')).$$

If all the cohomologies $R^i p_{2*}(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger)$ are coherent, we can define $\Phi^\dagger$ as

$$\Phi^\dagger(\mathcal{F}) := \mathbf{R}p_{2*}(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger) \in D^b_{\mathrm{Coh}}(\mathrm{Mod}(\mathcal{O}_Y^{(\beta',\gamma')}, \alpha')) \simeq D^b(Y, v),$$

by Lemma 4.5. In fact we have the following:

Lemma 4.6. *For each i, the object $R^i p_{2*}(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger)$ is coherent.*

Proof. Since there exists a distinguished triangle,

$$p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times Y} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger \longrightarrow p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger \longrightarrow p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times Y} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger$$

in $D^b(\mathrm{Mod}(p_2^{-1}\mathcal{O}_Y^{(\beta',\gamma')}, p_2^*\alpha'))$, it suffices to show that $R^i p_{2*}(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times Y} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger)$ is coherent. But since $H^q(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times Y} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger)$ are coherent $\mathcal{O}_{X\times Y}$-modules, the existence of a first quadrant spectral sequence

$$E_2^{p,q} := R^p p_{2*} H^q(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times Y} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger) \Rightarrow R^{p+q} p_{2*}(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times Y} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger)$$

shows $R^i p_{2*}(p_1^*\mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times Y} \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger)$ is coherent. q.e.d.

Since we have a morphism of algebras $i\colon \mathcal{O}_X^{(\beta,\gamma)} \to \mathcal{O}_X$, we have functors:

$$i_*\colon \mathrm{Coh}(X) \to \mathrm{Coh}(X,u), \qquad i^*\colon \mathrm{Coh}(X,u) \to \mathrm{Coh}(X).$$

Passing to derived categories and using Proposition 4.5, we obtain the derived functors:

$$i_*\colon D^b(X) \to D^b(X,u), \qquad \mathbf{L}i^*\colon D^b(X,u) \to D^-(X).$$

Note that an equivalence $\Phi\colon D^b(X) \to D^b(Y)$ extends to an equivalence $\Phi^-\colon D^-(X) \to D^-(Y)$, using the same kernel with Φ. Now we can state our main theorem.

Theorem 4.7. *Let X and Y be smooth projective varieties such that there exists an equivalence of derived categories $\Phi\colon D^b(X) \to D^b(Y)$. Take $u \in HT^2(X)$ and $v := \phi_T(u) \in HT^2(Y)$. Then there exists an object $\mathcal{P}^\dagger \in D^b_{\mathrm{perf}}(X \times Y, -p_1^*\check{u} + p_2^*v)$ such that the associated functor*

$$\Phi^\dagger\colon D^b(X,u) \longrightarrow D^b(Y,v)$$

gives an equivalence. Moreover, the following diagram is 2-commutative:

$$\begin{array}{ccccc} D^b(X) & \xrightarrow{i_*} & D^b(X,u) & \xrightarrow{\mathbf{L}i^*} & D^-(X) \\ \Phi\downarrow & & \downarrow\Phi^\dagger & & \downarrow\Phi^- \\ D^b(Y) & \xrightarrow{i_*} & D^b(Y,v) & \xrightarrow{\mathbf{L}i^*} & D^-(Y). \end{array}$$

5. Atiyah classes and FM-transforms

In this section we will analyze Atiyah classes of kernels of Fourier-Mukai transforms, and give the preparation for the proof of the main theorem. Firstly, let us recall the universal Atiyah class. Let X be a smooth projective variety and Δ be a diagonal or diagonal embedding. We write Δ as Δ_X when needed. Let $I_\Delta \subset \mathcal{O}_{X\times X}$ be an ideal sheaf of Δ. Consider the exact sequence

$$(\star) \qquad 0 \longrightarrow I_\Delta/I_\Delta^2 \longrightarrow \mathcal{O}_{X\times X}/I_\Delta^2 \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

Definition 5.1. The universal Atiyah class

$$a_X\colon \mathcal{O}_\Delta \longrightarrow \Delta_*\Omega_X[1]$$

is the extension class of the exact sequence $(\star)$.

Consider the composition

$$\mathcal{O}_\Delta \xrightarrow{a_X} \Delta_*\Omega_X[1] \xrightarrow{a_X\otimes p_2^*\Omega_X} \Delta_*\Omega_X^{\otimes 2}[2] \longrightarrow \cdots \longrightarrow \Delta_*\Omega_X^{\otimes i}[i].$$

By composing anti-symmetrization $\epsilon\colon \Omega_X^{\otimes i} \to \Omega_X^i$, we get a morphism

$$a_{X,i}\colon \mathcal{O}_\Delta \longrightarrow \Delta_*\Omega_X^i[i].$$

Definition 5.2. The exponential universal Atiyah class is a morphism

$$\exp(a)_X := \bigoplus_{i\ge 0} a_{X,i} \colon \mathcal{O}_\Delta \longrightarrow \bigoplus_{i\ge 0} \Delta_* \Omega_X^i[i].$$

Here $a_{X,0} = id$.

Caldararu [**5**] showed the following:

Proposition 5.3 (Caldararu [**5**])**.** *$\exp(a)_X$ is equal to the composition*

$$\mathcal{O}_\Delta \longrightarrow \Delta_* \mathbf{L}\Delta^* \mathcal{O}_\Delta \xrightarrow{\Delta_* I} \bigoplus_{i\ge 0} \Delta_* \Omega_X^i[i].$$

Here $\mathcal{O}_\Delta \to \Delta_* \mathbf{L}\Delta^* \mathcal{O}_\Delta$ is an adjunction, and I is a morphism which appeared in the proof of Theorem 2.5

By the above proposition, HKR-isomorphism is nothing but the following morphism

$$HT^*(X) \ni u \longmapsto \Delta_* u \circ \exp(a)_X \in HH^*(X).$$

Next, let us recall the Atiyah class and exponential Atiyah class for an object $\mathcal{P} \in D^b(X)$. By applying $\mathbf{R}p_{2*}(p_1^*\mathcal{P} \overset{\mathbf{L}}{\otimes} *)$ to the exact sequence $(\star)$, we obtain the distinguished triangle,

$$(\star_{\mathcal{P}}) \quad \mathcal{P} \otimes \Omega_X \longrightarrow \mathbf{R}p_{2*}\left(p_1^*\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times X}/I_\Delta^2\right) \longrightarrow \mathcal{P} \longrightarrow \mathcal{P} \otimes \Omega_X[1].$$

Definition 5.4. The Atiyah class $a(\mathcal{P}) \in \operatorname{Ext}_X^1(\mathcal{P}, \mathcal{P} \otimes \Omega_X)$ is a morphism

$$a(\mathcal{P}) \colon \mathcal{P} \longrightarrow \mathcal{P} \otimes \Omega_X[1]$$

in the distinguished triangle $(\star_{\mathcal{P}})$.

As in the exponential universal Atiyah class, let us take the composition

$$a(\mathcal{P}) \circ \cdots a(\mathcal{P}) \colon \mathcal{P} \longrightarrow \mathcal{P} \otimes \Omega_X[1] \longrightarrow \cdots \longrightarrow \mathcal{P} \otimes \Omega_X^{\otimes i}[i].$$

By composing $\epsilon \colon \Omega_X^{\otimes i} \to \Omega_X^i$, we get the morphism

$$a(\mathcal{P})_i \colon \mathcal{P} \longrightarrow \mathcal{P} \otimes \Omega_X^i[i].$$

Definition 5.5. The exponential Atiyah class of $\mathcal{P}$ is a morphism

$$\exp a(\mathcal{P}) := \bigoplus_{i\ge 0} a(\mathcal{P})_i \colon \mathcal{P} \longrightarrow \bigoplus_{i\ge 0} \mathcal{P} \otimes \Omega_X^i[i].$$

Here $a(\mathcal{P})_0 = id$.

Now let us consider two smooth projective varieties X and Y, and an equivalence of derived categories $\Phi \colon D^b(X) \to D^b(Y)$. Let $\mathcal{P} \in D^b(X \times Y)$ be a kernel of Φ. By Theorem 2.4, Φ induces the isomorphism $\phi \colon HH^*(X) \to HH^*(Y)$. We have the following proposition:

Proposition 5.6. *ϕ factors into the isomorphisms:*

$$HH^*(X) \xrightarrow{\sim} \operatorname{Ext}^*_{X\times Y}(\mathcal{P},\mathcal{P}) \xrightarrow{\sim} HH^*(Y).$$

Proof. Let p_{ij} be projections from $X \times Y \times Z$ onto corresponding factors. For $a \in D^b(X\times Y)$ and $b \in D^b(Y \times Z)$, let $b \circ a \in D^b(X \times Z)$ be

$$b \circ a := \mathbf{R}p_{13*}(p_{12}^*(a) \overset{\mathbf{L}}{\otimes} p_{23}^*(b)).$$

It is easy to see $\Phi^b_{Y\to Z} \circ \Phi^a_{X\to Y} \cong \Phi^{b\circ a}_{X\to Z}$. We have the following functor:

$$\mathcal{P}\circ\colon D^b(X\times X) \ni a \longmapsto \mathcal{P}\circ a \in D^b(X\times Y).$$

The above functor is an equivalence, since the functor $D^b(X\times Y) \ni b \mapsto \mathcal{E}\circ b \in D^b(X\times X)$ gives a quasi-inverse. Here $\mathcal{E}$ is a kernel of Φ^{-1}. Similarly, we have an equivalence $\circ\mathcal{P}\colon D^b(Y\times Y) \ni a \mapsto a\circ\mathcal{P} \in D^b(X\times Y)$. Consider the following diagrams:

(♠)

$$\begin{array}{ccc} D^b(X\times X) & \xrightarrow{\mathcal{P}\circ} & D^b(X\times Y) \\ {\scriptstyle \Delta_{X*}}\uparrow & & \uparrow{\scriptstyle p_1^*(*)\overset{\mathbf{L}}{\otimes}\mathcal{P}} \\ D^b(X) & = & D^b(X), \end{array} \qquad \begin{array}{ccc} D^b(Y\times Y) & \xrightarrow{\circ\mathcal{P}} & D^b(X\times Y) \\ {\scriptstyle \Delta_{Y*}}\uparrow & & \uparrow{\scriptstyle p_2^*(*)\overset{\mathbf{L}}{\otimes}\mathcal{P}} \\ D^b(Y) & = & D^b(Y). \end{array}$$

The above diagrams are 2-commutative. Let us check that the left diagram commutes. Take $a \in D^b(X)$. Then

$$\begin{aligned} \mathcal{P}\circ(\Delta_{X*}a) &\cong \mathbf{R}p_{13*}\left(p_{12}^*\Delta_{X*}a \overset{\mathbf{L}}{\otimes} p_{23}^*\mathcal{P}\right) \\ &\cong \mathbf{R}p_{13*}\left((\Delta_X\times \mathrm{id})_*p_1^*a \overset{\mathbf{L}}{\otimes} p_{23}^*\mathcal{P}\right) \\ &\cong \mathbf{R}p_{13*}(\Delta_X\times\mathrm{id})_*\left(p_1^*a \overset{\mathbf{L}}{\otimes} (\Delta_X\times\mathrm{id})^*p_{23}^*\mathcal{P}\right) \\ &\cong p_1^*a \overset{\mathbf{L}}{\otimes} \mathcal{P}. \end{aligned}$$

The second isomorphism follows from flat base change of the diagram below

$$\begin{array}{ccc} X\times Y & \xrightarrow{\Delta_X\times\mathrm{id}_Y} & X\times X\times Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_{12}} \\ X & \xrightarrow{\Delta_X} & X\times X, \end{array}$$

and the third isomorphism is the projection formula. By the above commutative diagram, we have $\mathcal{P}\circ\mathcal{O}_{\Delta_X} \cong \mathcal{P}$, $\mathcal{O}_{\Delta_Y}\circ\mathcal{P} \cong \mathcal{P}$. Therefore, we have the isomorphisms:

$$HH^*(X) \xrightarrow{\sim} \operatorname{Ext}^*_{X\times Y}(\mathcal{P},\mathcal{P}) \xrightarrow{\sim} HH^*(Y).$$

Since the equivalence $\Phi^{p_{13}^*\mathcal{P}\boxtimes p_{24}^*\mathcal{E}}_{X\times X\to Y\times Y}$ given in Theorem 2.4 is nothing but the following functor:

$$\mathcal{P}\circ(*)\circ\mathcal{E}\colon D^b(X\times X)\longrightarrow D^b(Y\times Y),$$

the composition of the above isomorphisms is equal to ϕ. q.e.d.

Now let us take the exponential Atiyah class of $\mathcal{P}$

$$\exp a(\mathcal{P})\colon \mathcal{P}\longrightarrow \bigoplus_{i\ge 0}\mathcal{P}\otimes\Omega^i_{X\times Y}[i],$$

and take direct summands,

$$\exp a(\mathcal{P})_X\colon \mathcal{P}\longrightarrow \bigoplus_{i\ge 0}\mathcal{P}\otimes p_1^*\Omega^i_X[i],\quad \exp a(\mathcal{P})_Y\colon \mathcal{P}\longrightarrow \bigoplus_{i\ge 0}\mathcal{P}\otimes p_2^*\Omega^i_Y[i].$$

By the commutative diagram (♠) in the proof of Lemma 5.6, we have two morphisms

$$\exp(a)^+_X\colon \mathcal{O}_{\Delta_X}\longrightarrow \bigoplus_{i\ge 0}\Delta_*\Omega^i_X[i],\quad \exp(a)^+_Y\colon \mathcal{O}_{\Delta_Y}\longrightarrow \bigoplus_{i\ge 0}\Delta_*\Omega^i_Y[i],$$

such that $\mathcal{P}\circ\exp(a)^+_X=\exp a(\mathcal{P})_X$, $\exp(a)^+_Y\circ\mathcal{P}=\exp a(\mathcal{P})_Y$. We will investigate the relationship between $\exp(a)^+_X$, $\exp(a)^+_Y$, and the universal exponential Atiyah classes of X and Y. Let $\sigma\colon X\times X\to X\times X$ be the involution $\sigma(x,x')=(x',x)$.

Lemma 5.7. *We have the following equalities:*

$$\exp(a)^+_X=\sigma_*\circ\exp(a)_X,\qquad \exp(a)^+_Y=\exp(a)_Y.$$

Proof. We show $\exp(a)^+_X=\sigma_*\circ\exp(a)_X$. Let

$$a^+_{X,i}\colon \mathcal{O}_{\Delta_X}\longrightarrow \Delta_*\Omega^i_X[i],\quad a(\mathcal{P})_{X,i}\colon \mathcal{P}\longrightarrow \mathcal{P}\otimes p_1^*\Omega^i_X[i]$$

be direct summands of $\exp(a)^+_X$ and $\exp a(\mathcal{P})_X$ respectively. For $i=1$, we write $*_1=*$ for $*=a^+_X$ or $a(\mathcal{P})_X$. We will show $a^+_{X,i}=\sigma_*a_{X,i}$. This is equivalent to $a(\mathcal{P})_{X,i}=\mathcal{P}\circ(\sigma_*a_{X,i})$. First we treat the case of $i=1$.

Let p_{ij} and q_{ij} be projections from $X\times X\times Y$, $X\times Y\times X\times Y$ onto corresponding factors. Let

$$\Delta_X\times\mathrm{id}\colon X\times Y\hookrightarrow X\times X\times Y,$$
$$\mathrm{id}\times\Delta_Y\colon X\times X\times Y\hookrightarrow X\times Y\times X\times Y$$

be $(\Delta_X\times\mathrm{id})(x,y)=(x,x,y)$, $(\mathrm{id}\times\Delta_Y)(x,x',y)=(x,y,x',y)$. Let $I_{\Delta_{X^{(2)},Y}}$ be the kernel of the composition

$$\mathcal{O}_{X\times Y\times X\times Y}\longrightarrow(\mathrm{id}\times\Delta_Y)_*\mathcal{O}_{X\times X\times Y}\longrightarrow(\mathrm{id}\times\Delta_Y)_*\mathcal{O}_{X\times X\times Y}/p_{12}^*I^2_{\Delta_X}.$$

Then we have a morphism of distinguished triangles (in fact, morphism of exact sequence)

$$(\diamondsuit)\qquad \begin{array}{ccccc} \mathcal{O}_{X\times Y\times X\times Y}/I^2_{\Delta_{X\times Y}} & \longrightarrow & \mathcal{O}_{\Delta_{X\times Y}} & \longrightarrow & \Delta_{X\times Y*}\Omega_{X\times Y}[1] \\ \downarrow & & \| & & \downarrow \\ \mathcal{O}_{X\times Y\times X\times Y}/I_{\Delta_{X^{(2)},Y}} & \longrightarrow & \mathcal{O}_{\Delta_{X\times Y}} & \longrightarrow & \Delta_{X\times Y*}p_1^*\Omega_X[1]. \end{array}$$

Note that since

$$\begin{aligned} \Delta_{X\times Y*}p_1^*\Omega_X &\cong (\mathrm{id}\times\Delta_Y)_*(\Delta_X\times\mathrm{id})_*p_1^*\Omega_X \\ &\cong (\mathrm{id}\times\Delta_Y)_*p_{12}^*\Delta_{X*}\Omega_X, \end{aligned}$$

and

$$\mathcal{O}_{X\times Y\times X\times Y}/I_{\Delta_{X^{(2)},Y}} \cong (\mathrm{id}\times\Delta_Y)_*p_{12}^*\mathcal{O}_{X\times X}/I^2_{\Delta_X},$$

the bottom sequence of $(\diamondsuit)$ is obtained by applying $(\mathrm{id}\times\Delta_Y)_*p_{12}^*$ to the distinguished triangle,

$$\Delta_{X*}\Omega_X \longrightarrow \mathcal{O}_{X\times X}/I^2_{\Delta_X} \longrightarrow \mathcal{O}_{\Delta_X} \xrightarrow{a_X} \Delta_{X*}\Omega_X[1].$$

Let $\widetilde{\Phi}\colon D^b(X\times Y\times X\times Y)\to D^b(X\times Y)$ be the functor $\widetilde{\Phi}:=\mathbf{R}q_{34*}(*\overset{\mathbf{L}}{\otimes} q_{12}^*\mathcal{P})$. Then we have the isomorphisms of functors,

$$\begin{aligned} &\widetilde{\Phi}\circ(\mathrm{id}\times\Delta_Y)_*\circ p_{12}^*(*) \\ &= \mathbf{R}q_{34*}\left((\mathrm{id}\times\Delta_Y)_*p_{12}^*(*)\overset{\mathbf{L}}{\otimes} q_{12}^*\mathcal{P}\right) \\ &\cong \mathbf{R}q_{34*}(\mathrm{id}\times\Delta_Y)_*\left(p_{12}^*(*)\overset{\mathbf{L}}{\otimes}(\mathrm{id}\times\Delta_Y)^*q_{12}^*\mathcal{P}\right) \\ &\cong \mathbf{R}p_{23*}(p_{12}^*(*)\overset{\mathbf{L}}{\otimes} p_{13}^*\mathcal{P}) \\ &\cong \mathbf{R}p_{23*}(\sigma\times\mathrm{id})_*\left((\sigma\times\mathrm{id})^*p_{12}^*(*)\overset{\mathbf{L}}{\otimes}(\sigma\times\mathrm{id})^*p_{13}^*\mathcal{P}\right) \\ &\cong \mathbf{R}p_{13*}(p_{12}^*\sigma_*(*)\overset{\mathbf{L}}{\otimes} p_{13}^*\mathcal{P}) \\ &= \mathcal{P}\circ\sigma_*(*). \end{aligned}$$

Therefore, if we apply $\widetilde{\Phi}$ to the diagram $(\diamondsuit)$, we obtain the morphism of distinguished triangles,

$$\begin{array}{ccccc} \widetilde{\mathcal{P}} & \longrightarrow & \mathcal{P} & \xrightarrow{a(\mathcal{P})} & \mathcal{P}\otimes\Omega_{X\times Y}[1] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}\circ\left(\sigma_*\mathcal{O}_{X\times X}/I^2_{\Delta_X}\right) & \longrightarrow & \mathcal{P}\circ(\mathcal{O}_{\Delta_X}) & \xrightarrow{\mathcal{P}\circ(\sigma_*a_X)} & \mathcal{P}\circ(\Delta_{X*}\Omega_X[1]). \end{array}$$

Here $\widetilde{\mathcal{P}} := \widetilde{\Phi}\left(\mathcal{O}_{X\times Y\times X\times Y}/I^2_{\Delta_{X\times Y}}\right)$. Since the morphism $\mathcal{P} \to \mathcal{P} \circ (\mathcal{O}_{\Delta_X})$, $\mathcal{P} \otimes \Omega_{X\times Y} \to \mathcal{P} \circ (\Delta_{X*}\Omega_X)$ of the above diagram are equal to $\mathrm{id}_{\mathcal{P}}$, and direct summand $\mathcal{P}\otimes\Omega_{X\times Y} \to \mathcal{P}\otimes p_1^*\Omega_X$ under the isomorphism $\mathcal{P} \circ \Delta_* \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} p_1^*(*)$ of the diagram in Lemma 5.6, we can conclude $a(\mathcal{P})_X = \mathcal{P} \circ (\sigma_* a_X)$.

Secondly, we show $a(\mathcal{P})_{X,i} = \mathcal{P} \circ (\sigma_* a_{X,i})$ for all i. Since

$$\begin{aligned}
\mathcal{P} \circ \sigma_*(a_X \otimes p_2^*\Omega_X^{\otimes i}) &= \mathcal{P} \circ (\sigma_* a_X \otimes p_1^*\Omega_X^{\otimes i}) \\
&= \mathbf{R}p_{13*}(p_{12}^*\sigma_* a_X \otimes p_{12}^* p_1^*\Omega_X^{\otimes i} \overset{\mathbf{L}}{\otimes} p_{23}^*\mathcal{P}) \\
&= \mathbf{R}p_{13*}(p_{12}^*\sigma_* a_X \otimes p_{13}^* p_1^*\Omega_X^{\otimes i} \overset{\mathbf{L}}{\otimes} p_{23}^*\mathcal{P}) \\
&= \mathbf{R}p_{13*}(p_{12}^*\sigma_* a_X \overset{\mathbf{L}}{\otimes} p_{23}^*\mathcal{P}) \otimes p_1^*\Omega_X^{\otimes i} \\
&= (\mathcal{P} \circ \sigma_* a_X) \otimes p_1^*\Omega_X^{\otimes i} \\
&= a(\mathcal{P})_X \otimes p_1^*\Omega_X^{\otimes i},
\end{aligned}$$

we have $a(\mathcal{P})_{X,i} = \mathcal{P} \circ (\sigma_* a_{X,i})$. q.e.d.

Using the above proposition, we can find the relationship between HKR–isomorphism, the isomorphism $HH^*(X) \to \mathrm{Ext}^*_{X\times Y}(\mathcal{P}, \mathcal{P})$ of Lemma 5.6 and the exponential Atiyah-classes. In fact, we have the following lemma:

Lemma 5.8. *The following diagrams commute:*

$$\begin{array}{ccc}
HT^*(X\times Y) & \xrightarrow{\times \exp a(\mathcal{P})} & \mathrm{Ext}^*_{X\times Y}(\mathcal{P},\mathcal{P}) \\
{\scriptstyle p_1^*}\uparrow & & \uparrow{\scriptstyle \mathcal{P}\circ} \\
HT^*(X) & \xrightarrow[\sigma_* I_X]{} & HH^*(X),
\end{array}$$

$$\begin{array}{ccc}
HT^*(X\times Y) & \xrightarrow{\times \exp a(\mathcal{P})} & \mathrm{Ext}^*_{X\times Y}(\mathcal{P},\mathcal{P}) \\
{\scriptstyle p_2^*}\uparrow & & \uparrow{\scriptstyle \circ\mathcal{P}} \\
HT^*(Y) & \xrightarrow[I_Y]{} & HH^*(Y).
\end{array}$$

Here $\times \exp a(\mathcal{P})$ *means multiplying by* $\exp a(\mathcal{P})$ *and taking* $\mathrm{Ext}^*(\mathcal{P},\mathcal{P})$*-component.*

Proof.

We show that the top diagram commutes. Take $u \in H^p(X, \wedge^q T_X)$. By Lemma 5.7, we have $\sigma_* a_{X,q} = a^+_{X,q}$. So by Proposition 5.3, $\sigma_* I_X(u)$

is the composition:

$$\mathcal{O}_\Delta \xrightarrow{a^{\dagger}_{X,q}} \Delta_*\Omega^q_X[q] \xrightarrow{\Delta_* u} \Delta_*\mathcal{O}_X[p+q].$$

Therefore $\mathcal{P}\circ\sigma_* I_X(u)$ is the composition

$$\mathcal{P} \xrightarrow{a(\mathcal{P})_{X,q}} \mathcal{P}\otimes p_1^*\Omega^q_X[q] \xrightarrow{\times p_1^* u} \mathcal{P}[p+q].$$

But this is equal to the composition

$$\mathcal{P} \xrightarrow{a(\mathcal{P})_q} \mathcal{P}\otimes \Omega^q_{X\times Y}[q] \xrightarrow{\times p_1^* u} \mathcal{P}[p+q].$$

Therefore the diagram commutes. q.e.d.

6. Proof of the main theorem

In this section we will prove Theorem 4.7. Let X, Y and Φ, $\mathcal{P}$ be as in the previous sections. We want to extend $\mathcal{P}$ to $\mathcal{P}^\dagger \in D^b_{\mathrm{perf}}(X\times Y, -p_1^*\check{u}+p_2^*v)$. For this purpose we have to investigate the relationship between u, v, and the exponential Atiyah-class of $\mathcal{P}$. For $u\in H^p(X,\wedge^q T_X)$, let $\check{u} := (-1)^q u$, and extend the operation to $HT^*(X)$ linearly. Then it is clear that $\sigma_* I_X(u) = I_X(\check{u})$. Take $u\in HT^*(X)$ and $v=\phi_T(u)$. By Lemma 5.8 and the above remark, we have

$$\begin{aligned}(-p_1^*\check{u}+p_2^*v)\cdot \exp a(\mathcal{P}) &= -\mathcal{P}\circ\sigma_* I_X(\check{u}) + I_Y(v)\circ\mathcal{P}\\ &= -\mathcal{P}\circ I_X(u) + (\phi\circ I_X(u))\circ\mathcal{P}\\ &= -\mathcal{P}\circ I_X(u) + \mathcal{P}\circ I_X(u)\circ\mathcal{E}\circ\mathcal{P}\\ &= 0,\end{aligned}$$

in $\mathrm{Ext}^2_{X\times Y}(\mathcal{P},\mathcal{P})$. Therefore, to extend $\mathcal{P}$ to $\mathcal{P}^\dagger$, it suffices to show the following proposition.

Proposition 6.1. *Take $\mathcal{P}\in D^b(X)$ and $u\in HT^2(X)$. Assume that $u\cdot\exp a(\mathcal{P}) = 0$ in $\mathrm{Ext}^2_X(\mathcal{P},\mathcal{P})$. Then there exists an object $\mathcal{P}^\dagger \in D^b_{\mathrm{perf}}(X,u)$ such that $\mathbf{L}i^*\mathcal{P}^\dagger \cong \mathcal{P}$.*

Proof. Let $\mathcal{P}^\bullet$ be a complex of locally free sheaves on X, which represents $\mathcal{P}$. Since $\mathcal{P}^n$ is locally free, we have

$$\widetilde{\mathcal{P}}^i := \mathbf{R}p_{2*}\left(p_1^*\mathcal{P}^n \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X\times X}/I^2_{\Delta_X}\right) = p_{2*}\left(p_1^*\mathcal{P}^n\otimes\mathcal{O}_{X\times X}/I^2_{\Delta}\right),$$

and the distinguished triangle

$$\mathcal{P}\otimes\Omega_X \longrightarrow \mathbf{R}p_{2*}\left(p_1^*\mathcal{P}\overset{\mathbf{L}}{\otimes}\mathcal{O}_{X\times X}/I^2_{\Delta_X}\right) \longrightarrow \mathcal{P}\longrightarrow \mathcal{P}\otimes\Omega_X[1]$$

is represented by the exact sequence of complexes,

$$0\longrightarrow \mathcal{P}^\bullet\otimes\Omega_X \longrightarrow \widetilde{\mathcal{P}}^\bullet \xrightarrow{\psi^\bullet} \mathcal{P}^\bullet \longrightarrow 0.$$

But $\psi^n \colon \widetilde{\mathcal{P}}^n \to \mathcal{P}^n$ has a $\mathbb{C}$-linear section $\lambda^n \colon \mathcal{P}^n \to \widetilde{\mathcal{P}}^n$,

$$\lambda^n(U_i) \colon \mathcal{P}^n(U_i) \ni x \longmapsto x \otimes 1 \in \widetilde{\mathcal{P}}^n(U_i) = \mathcal{P}^n(U_i) \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i \times U_i}/I_\Delta^2$$

and $\lambda^\bullet \colon \mathcal{P}^\bullet \to \widetilde{\mathcal{P}}^\bullet$ gives a $\mathbb{C}$-linear splitting of $\psi^\bullet \colon \widetilde{\mathcal{P}}^\bullet \to \mathcal{P}^\bullet$. Therefore the Atiyah class $a(\mathcal{P})$ becomes the zero map after applying the forgetful functor $D^b(X) \to D^b(\mathrm{Mod}(X, \mathbb{C}))$. Here $\mathrm{Mod}(X, \mathbb{C})$ is a category of sheaves of $\mathbb{C}$-vector spaces on X.

On the other hand, in the derived category of quasi-coherent sheaves, the Atiyah class is represented by some morphism of complexes of quasi-coherent sheaves, denoted by the same symbol $a(\mathcal{P})$:

$$a(\mathcal{P}) \colon \mathcal{P}^\bullet \longrightarrow T\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{P}^\bullet \otimes \Omega_X).$$

Here T is a translation functor $T(X^\bullet) = X^{\bullet+1}$, $T(d_X) = -d_X$. By the above remark, $a(\mathcal{P})$ is homotopic to zero as complexes of $\mathbb{C}$-vector spaces, and we are now going to construct a homotopy. Let us choose connections

$$\nabla_i^{(n)} \colon \mathcal{P}^n|_{U_i} \longrightarrow \mathcal{P}^n|_{U_i} \otimes \Omega_X$$

on U_i for all i. Then it is easy to check that a homotopy between $a(\mathcal{P})$ and zero as morphisms of complexes of $\mathbb{C}$-vector spaces is given by $\nabla \colon \mathcal{P}^\bullet \to \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{P}^\bullet \otimes \Omega_X)$, defined as follows:

$$\nabla^n \colon \mathcal{P}^n \ni x \longmapsto \{\nabla_i^{(n)}(x)\}_i \in \mathcal{C}^0(\mathfrak{U}, \mathcal{P}^n \otimes \Omega_X) \subset \mathcal{C}^n(\mathfrak{U}, \mathcal{P}^\bullet \otimes \Omega_X).$$

Namely $a(\mathcal{P}) = \nabla \circ d_{\mathcal{P}} + T(d_{\mathcal{C}}) \circ \nabla$. (cf. [**9**]). Also $a(\mathcal{P})_2$ is represented by a morphism of complexes of quasi-coherent sheaves,

$$a(\mathcal{P})_2 \colon \mathcal{P}^\bullet \longrightarrow T^2\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{P}^\bullet \otimes \Omega_X^2) = \mathcal{C}^{\bullet+2}(\mathfrak{U}, \mathcal{P}^\bullet \otimes \Omega_X^2),$$

which is homotopic to zero as complexes of $\mathbb{C}$-vector spaces. In fact, we can calculate $a(\mathcal{P})_2$ as follows:

$$\begin{aligned} a(\mathcal{P})_2 &= \epsilon \circ T(a(\mathcal{P}) \otimes 1) \circ a(\mathcal{P}) \\ &= -\epsilon(\nabla \circ d_{\mathcal{C}} \circ \nabla) \circ d_{\mathcal{P}} - d_{\mathcal{C}} \circ \epsilon(\nabla \circ d_{\mathcal{C}} \circ \nabla). \end{aligned}$$

Hence, the homotopy is given by

$$-\epsilon(\nabla \circ d_{\mathcal{C}} \circ \nabla) \colon \mathcal{P}^\bullet \longrightarrow T\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{P}^\bullet \otimes \Omega_X^2).$$

Here $d_{\mathcal{P}}$ and $d_{\mathcal{C}}$ are differentials of $\mathcal{P}^\bullet$ and $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{P}^\bullet)$ respectively. Therefore, if we take a Čech representative of β and consider the morphism of complexes of quasi-coherent sheaves

$$\beta \cdot a(\mathcal{P}) + \gamma \cdot a(\mathcal{P})_2 \colon \mathcal{P}^\bullet \longrightarrow \mathcal{C}^{\bullet+2}(\mathfrak{U}, \mathcal{P}^\bullet),$$

then this is homotopic to zero as morphisms of complexes of $\mathbb{C}$-vector spaces. The homotopy is given by

$$\nabla^\dagger := \beta \circ \nabla - \gamma \circ \epsilon(\nabla \circ d_{\mathcal{C}} \circ \nabla) \colon \mathcal{P}^\bullet \longrightarrow T\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{P}^\bullet).$$

By the assumption,

$$\alpha \otimes \mathrm{id}_{\mathcal{P}} + \beta \cdot a(\mathcal{P}) + \gamma \cdot a(\mathcal{P})_2 \colon \mathcal{P}^\bullet \longrightarrow \mathcal{C}^{\bullet+2}(\mathfrak{U}, \mathcal{P}^\bullet)$$

is homotopic to zero as a map of complexes of quasi-coherent sheaves, and let $h^\bullet\colon \mathcal{P}^\bullet \longrightarrow T\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{P}^\bullet)$ be such a homotopy. Note that h^n is a $\mathcal{O}_X$-module homomorphism. Combining these, we can conclude $\alpha \otimes \mathrm{id}_{\mathcal{P}}$ is homotopic to zero as complexes of $\mathbb{C}$-vector spaces and the homotopy is given by $h^\dagger := h - \nabla^\dagger$.

Now we are going to construct the complex $(\mathcal{P}^\dagger)^\bullet$ whose terms are objects in $\mathrm{QCoh}(X, u)$ by using $h^\dagger$. First define $(\mathcal{P}^\dagger)^n_i$ to be

$$(\mathcal{P}^\dagger)^n_i := \mathcal{P}^n|_{U_i} \oplus \mathcal{C}^n(\mathfrak{U}, \mathcal{P}^\bullet)|_{U_i}.$$

We introduce a left $\mathcal{O}^{(\beta,\gamma)}_{U_i}$-module structure on $(\mathcal{P}^\dagger)^n_i$. For $a \in \mathcal{O}_{U_i}$, let $\gamma_a \in T_{U_i}$ be a differential operator $\gamma_a := \gamma(a, *)$. Then for

$$(a, b) \in \mathcal{O}_{U_i} \oplus \mathcal{C}^0(\mathfrak{U}, \mathcal{O}_X)|_{U_i}, \quad (x, y) \in \mathcal{P}^n|_{U_i} \oplus \mathcal{C}^n(\mathfrak{U}, \mathcal{P}^\bullet)|_{U_i},$$

define $(a, b) *_\gamma (x, y)$ to be

$$(a, b) *_\gamma (x, y) := (ax, bx + ay + \{\gamma_a \circ \nabla^{(n)}_{i_0}(x|_{U_{i_0}})\}_{i_0}) \in (\mathcal{P}^\dagger)^n_i.$$

Here $\{\gamma_a \circ \nabla^{(n)}_{i_0}(x|_{U_{i_0}})\}_{i_0} \in \mathcal{C}^0(\mathfrak{U}, \mathcal{P}^n)|_{U_i}$. We have to check the following:

Lemma 6.2.
$*_\gamma$ defines the left action of $\mathcal{O}^{(\beta,\gamma)}_{U_i} \subset \mathcal{O}_X|_{U_i} \oplus \mathcal{C}^0(\mathfrak{U}, \mathcal{O}_X)|_{U_i}$ on $(\mathcal{P}^\dagger)^n_i$.

Proof. Take $(a, b), (c, d) \in \mathcal{O}^{(\beta,\gamma)}_X(U_i)$ and $(x, y) \in (\mathcal{P}^\dagger)^n_i$. Then

$$\begin{aligned}
&(a, b) *_\gamma \{(c, d) *_\gamma (x, y)\} \\
&= (a, b) *_\gamma (cx, cy + dx + \{\gamma_c \circ \nabla^{(n)}_{i_0}(x|_{U_{i_0}})\}_{i_0}) \\
&= (acx, acy + adx + bcx + \{a\gamma_c \circ \nabla^{(n)}_{i_0}(x|_{U_{i_0}}) + \gamma_a \circ \nabla^{(n)}_{i_0}(cx|_{U_{i_0}})\}_{i_0}),
\end{aligned}$$

and

$$\begin{aligned}
&\{(a, b) *_\gamma (c, d)\} *_\gamma (x, y) \\
&= (ac, \gamma(a, c) + ad + bc) *_\gamma (x, y) \\
&= (acx, acy + adx + bcx + \gamma(a, c) \cdot x + \{\gamma_{ac} \circ \nabla^{(n)}_{i_0}(x|_{U_{i_0}})\}_{i_0}).
\end{aligned}$$

Since $\nabla^{(n)}_{i_0}$ is a connection, we have

$$\begin{aligned}
&a\gamma_c \circ \nabla^{(n)}_{i_0}(x|_{U_{i_0}}) + \gamma_a \circ \nabla^{(n)}_{i_0}(cx|_{U_{i_0}}) \\
&= a\gamma_c \circ \nabla^{(n)}_{i_0}(x|_{U_{i_0}}) + \gamma_a \circ \{dc \otimes (x|_{U_{i_0}}) + c \cdot \nabla^{(n)}_{i_0}(x|_{U_{i_0}})\} \\
&= \gamma(a, c) \cdot (x|_{U_{i_0}}) + (c\gamma_a + a\gamma_c) \circ \nabla^{(n)}_{i_0}(x|_{U_{i_0}}) \\
&= \gamma(a, c) \cdot (x|_{U_{i_0}}) + \gamma_{ac} \circ \nabla^{(n)}_{i_0}(x|_{U_{i_0}}).
\end{aligned}$$

Therefore, the lemma follows. q.e.d.

By Lemma 6.2, we have obtained the object,

$$(\mathcal{P}^{\dagger})_i^n \in \mathrm{Mod}(\mathcal{O}_{U_i}^{(\gamma)}).$$

If we regard $\mathcal{P}^n|_{U_i}$ and $\mathcal{C}^n(\mathfrak{U}, \mathcal{P}^{\bullet})|_{U_i}$ as $\mathcal{O}_{U_i}^{(\gamma)}$-modules by the surjection, $\mathcal{O}_{U_i}^{(\gamma)} \twoheadrightarrow \mathcal{O}_{U_i}$, then we have the exact sequence in $\mathrm{Mod}(\mathcal{O}_{U_i}^{(\gamma)})$,

$$0 \longrightarrow \mathcal{C}^n(\mathfrak{U}, \mathcal{P}^{\bullet})|_{U_i} \longrightarrow (\mathcal{P}^{\dagger})_i^n \longrightarrow \mathcal{P}^n|_{U_i} \longrightarrow 0.$$

Since $\mathcal{P}^n|_{U_i}$, $\mathcal{C}^n(\mathfrak{U}, \mathcal{P}^{\bullet})|_{U_i}$ are objects in $\mathrm{QCoh}(\mathcal{O}_{U_i}^{(\gamma)})$, we have

$$(\mathcal{P}^{\dagger})_i^n \in \mathrm{QCoh}(\mathcal{O}_{U_i}^{(\gamma)}),$$

by Lemma 3.1(3) and Corollary 3.3. Next define $\phi_{i_0 i_1}^n \colon (\mathcal{P}^{\dagger})_{i_0}^n|_{U_{i_0 i_1}} \to (\mathcal{P}^{\dagger})_{i_1}^n|_{U_{i_0 i_1}}$ to be

$$\phi_{i_0 i_1}^n(x, y) := (x, -\{\alpha_{i_0 i_1 j} \cdot x\}_j + y).$$

Here $-\{\alpha_{i_0 i_1 j} \cdot x\}_j \in \mathcal{C}^0(\mathfrak{U}, \mathcal{P}^n)|_{U_{i_0 i_1}}$. Then $\phi_{i_0 i_1}^n$ is clearly $\mathcal{O}_{U_{i_0 i_1}}^{(\beta,\gamma)}$-module homomorphism, and the cocycle condition of α implies the following:

$$(\mathcal{P}^{\dagger})^n := ((\mathcal{P}^{\dagger})_i^n, \phi_{i_0 i_1}^n) \in \mathrm{QCoh}(X, u).$$

Now we will construct a differential $d^n \colon (\mathcal{P}^{\dagger})^n \to (\mathcal{P}^{\dagger})^{n+1}$. On U_i, we define d_i^n as

$$d_i^n(x, y) := (d_{\mathcal{P}} x, d_{\mathcal{C}} y + h^{\dagger}(x) - \{\alpha_{ikl} \cdot x\}_{kl}) \in \mathcal{P}^{n+1}|_{U_i} \oplus \mathcal{C}^{n+1}(\mathfrak{U}, \mathcal{P}^{\bullet})|_{U_i},$$

for $(x, y) \in \mathcal{P}^n|_{U_i} \oplus \mathcal{C}^n(\mathfrak{U}, \mathcal{P}^{\bullet})|_{U_i}$. Here $\{\alpha_{ikl} \cdot x\}_{kl} \in \mathcal{C}^1(\mathfrak{U}, \mathcal{P}^n)|_{U_i}$. Then

$$\begin{aligned}
& d_i^{n+1} \circ d_i^n(x, y) \\
&= (0, d_{\mathcal{C}}(d_{\mathcal{C}} y + h^{\dagger}(x) - \{\alpha_{ikl} \cdot x\}_{kl}) + h^{\dagger}(d_{\mathcal{P}} x) - \{\alpha_{ikl} \cdot d_{\mathcal{P}} x\}_{kl}) \\
&= (0, d_{\mathcal{C}} h^{\dagger}(x) + h^{\dagger} d_{\mathcal{P}}(x) - \{\alpha_{i_0 i_1 i_2} \cdot x\}_{i_0 i_1 i_2}) \\
&= (0, 0).
\end{aligned}$$

The second equality comes from the cocycle condition of α. We can check $\phi_{i_0 i_1}^{n+1} \circ d_{i_0}^n = d_{i_1}^n \circ \phi_{i_0 i_1}^n$ similarly. We have to check the following:

Lemma 6.3. *d_i^n is $\mathcal{O}_{U_i}^{(\beta,\gamma)}$-module homomorphism.*

Proof.

Take $(a, b) \in \mathcal{O}_{U_i}^{(\beta,\gamma)} \subset \mathcal{O}_{U_i} \oplus \mathcal{C}^0(\mathfrak{U}, \mathcal{O}_X)|_{U_i}$, i.e., $\delta b = \{\beta_{i_0 i_1}(a)\}_{i_0 i_1}$, and $(x, y) \in \mathcal{P}^n|_{U_i} \oplus \mathcal{C}^n(\mathfrak{U}, \mathcal{P}^{\bullet})|_{U_i}$. Then

$$\begin{aligned}
& (a, b) *_{\gamma} d_i^n(x, y) \\
&= (a, b) *_{\gamma} (d_{\mathcal{P}} x, d_{\mathcal{C}} y + h^{\dagger}(x) - \{\alpha_{ijk} \cdot x\}_{jk}) \\
&= (a d_{\mathcal{P}} x, a d_{\mathcal{C}} y + a h^{\dagger}(x) - a\{\alpha_{ijk} \cdot x\}_{jk} + b d_{\mathcal{P}} x + \{\gamma_a \circ \nabla_{i_0}^{(n+1)}(d_{\mathcal{P}} x)\}_{i_0}),
\end{aligned}$$

and

$$\begin{aligned} & d_i^n\{(a,b) *_\gamma (x,y)\} \\ & = d_i^n(ax, ay+bx+\{\gamma_a \circ \nabla_{i_0}^{(n)}(x)\}_{i_0}) \\ & = (d_{\mathcal{P}}(ax), d_{\mathcal{C}}(ay+bx+\{\gamma_a \circ \nabla_{i_0}^{(n)}(x)\}_{i_0}) + h^\dagger(ax) - \{\alpha_{ijk}\cdot ax\}_{jk}). \end{aligned}$$

Therefore it suffices to check the following:

$$-a\nabla^\dagger(x) + \{\gamma_a \circ \nabla_{i_0}^{(n+1)}(d_{\mathcal{P}}x)\}_{i_0} = \delta(bx) + d_{\mathcal{C}}\{\gamma_a \circ \nabla_{i_0}^{(n)}(x)\}_{i_0} - \nabla^\dagger(ax).$$

We calculate $\nabla^\dagger(ax) - a\nabla^\dagger(x)$. Since

$$\nabla(ax) - a\nabla(x) = da \otimes x$$

and

$$\begin{aligned} & \nabla \circ d_{\mathcal{C}} \circ \nabla(ax) - a\nabla \circ d_{\mathcal{C}} \circ \nabla(x) \\ & = \nabla \circ d_{\mathcal{C}} \circ (da \otimes x + a\nabla(x)) - a\nabla \circ d_{\mathcal{C}} \circ \nabla(x) \\ & = \nabla \circ (da \otimes d_{\mathcal{P}}x + ad_{\mathcal{C}} \circ \nabla(x)) - a\nabla \circ d_{\mathcal{C}} \circ \nabla(x) \\ & = da \otimes \nabla \circ d_{\mathcal{P}}x + d_{\mathcal{C}} \circ \nabla(x) \otimes da, \end{aligned}$$

we have

$$\begin{aligned} & \nabla^\dagger(ax) - a\nabla^\dagger(x) \\ & = \{\beta_{i_0 i_1}(a)\}_{i_0 i_1} \cdot x - \gamma_a \circ \nabla \circ d_{\mathcal{P}}(x) + \gamma_a \circ d_{\mathcal{C}} \circ \nabla(x) \\ & = \delta(b)x - \gamma_a \circ \nabla \circ d_{\mathcal{P}}(x) + \gamma_a \circ d_{\mathcal{C}} \circ \nabla(x). \end{aligned}$$

So the lemma follows. q.e.d.

We have constructed an unbounded complex of $\mathrm{QCoh}(X,u)$:

$$\mathcal{P}^\dagger := \cdots \longrightarrow (\mathcal{P}^\dagger)^n \xrightarrow{d^n} (\mathcal{P}^\dagger)^{n+1} \longrightarrow \cdots,$$

with $d^n|_{U_i} = d_i^n$. The next lemma finishes the proof of Proposition 6.1. q.e.d.

Lemma 6.4. *$\mathcal{P}^\dagger$ is locally quasi-isomorphic to a bounded complex of free $\mathcal{O}_X^{(\beta,\gamma)}$ -modules of finite rank, and $\mathbf{L}i^*\mathcal{P}^\dagger \cong \mathcal{P}$.*

Proof. Let

$$p_i^n \colon \mathcal{C}^n(\mathfrak{U}, \mathcal{P}^\bullet)|_{U_i} \longrightarrow \mathcal{C}^0(\mathfrak{U}, \mathcal{P}^n)|_{U_i} \longrightarrow \mathcal{P}^n|_{U_i}$$

be a projection and $\widetilde{h}_i^n$ be the composition,

$$\widetilde{h}_i^n := p_i^{n+1} \circ (h^\dagger)_i^n \colon \mathcal{P}^n|_{U_i} \longrightarrow \mathcal{C}^{n+1}(\mathfrak{U}, \mathcal{P}^\bullet)|_{U_i} \longrightarrow \mathcal{P}^{n+1}|_{U_i}.$$

Let $\widetilde{\mathcal{P}}_i^n := \mathcal{P}^n|_{U_i}[\varepsilon]/(\varepsilon^2)$ be a free left $\mathcal{O}_{U_i}^{(\gamma)}$-module, the left action given by for $a + b\varepsilon \in \mathcal{O}_{U_i}^{(\gamma)}$, $x + y\varepsilon \in \widetilde{\mathcal{P}}_i^n$,

$$(a + b\varepsilon) *_\gamma (x + y\varepsilon) := ax + (ay + bx + \gamma_a \circ \nabla_i^{(n)}(x))\varepsilon.$$

Then define the complex $\widetilde{\mathcal{P}}_i^{\bullet}$ whose differential is given by

$$\widetilde{\mathcal{P}}_i^n \ni x + y\varepsilon \longmapsto d_{\mathcal{P}}(x) + (d_{\mathcal{C}}y + \widetilde{h}_i^n(x))\varepsilon \in \widetilde{\mathcal{P}}_i^{n+1}.$$

We will show that the natural map

$$(\mathcal{P}^{\dagger})_i^n \ni (x, y) \longmapsto x + p_i^n(y)\varepsilon \in \widetilde{\mathcal{P}}_i^n$$

gives a quasi-isomorphism between $\mathcal{P}^{\dagger}|_{U_i}$ and $\widetilde{\mathcal{P}}_i^{\bullet}$. It is clear that the above map is a morphism of complexes of $\mathcal{O}_{U_i}^{(\beta,\gamma)}$-modules. Note that $p_i^{\bullet}$ gives a splitting of the Čech resolution on U_i, so we have the decomposition,

$$\mathcal{C}^{\bullet}(\mathfrak{U}, \mathcal{P}^{\bullet})|_{U_i} \cong \mathcal{P}^{\bullet}|_{U_i} \oplus \mathcal{Q}_i^{\bullet},$$

for some complex $\mathcal{Q}_i^{\bullet}$ with $H^{\bullet}(\mathcal{Q}_i^{\bullet}) = 0$. We have the following diagram:

$$\begin{array}{ccccc} & & T\mathcal{Q}_i^{\bullet} & & \\ & & \big\downarrow & & \\ \mathcal{P}^{\bullet}|_{U_i} & \xrightarrow{h^{\dagger}-\{\alpha_{ijk}\}_{jk}} & T\mathcal{C}^{\bullet}(\mathfrak{U}, \mathcal{P}^{\bullet})|_{U_i} & \longrightarrow & T(\mathcal{P}^{\dagger})_i^{\bullet} \\ \Big\| & & \big\downarrow {\scriptstyle Tp_i^{\bullet}} & & \\ \mathcal{P}^{\bullet}|_{U_i} & \xrightarrow{\widetilde{h}} & T\mathcal{P}^{\bullet}|_{U_i} & \longrightarrow & T\widetilde{\mathcal{P}}_i^{\bullet}. \end{array}$$

Therefore we obtain the distinguished triangle

$$\mathcal{Q}_i^{\bullet} \longrightarrow (\mathcal{P}^{\dagger})_i^{\bullet} \longrightarrow \widetilde{\mathcal{P}}_i^{\bullet} \longrightarrow T\mathcal{Q}_i^{\bullet}.$$

Since $\mathcal{Q}_i^{\bullet}$ is acyclic, the first part of the lemma follows. For the second part, we have a morphism of complexes $(\mathcal{P}^{\dagger})^{\bullet} \to i_*\mathcal{P}^{\bullet}$ by construction. By taking adjoint, we have a morphism $\mathbf{L}i^*\mathcal{P}^{\dagger} \to \mathcal{P}$ in $D^b(X)$. This morphism is quasi-isomorphic on U_i, and hence quasi-isomorphic. q.e.d.

Now let us return to the situation of the first part of this section. By Proposition 6.1, we obtain the object $\mathcal{P}^{\dagger} \in D^b_{\mathrm{perf}}(X \times Y, -p_1^*\check{u} + p_2^*v)$. Therefore we can construct a functor $\Phi^{\dagger} \colon D^b(X, u) \to D^b(Y, v)$. Next we will show $\Phi^{\dagger}$ fits some commutative diagram.

Lemma 6.5. *The following diagram is 2-commutative,*

$$\begin{array}{ccccc} D^b(X) & \xrightarrow{i_*} & D^b(X, u) & \xrightarrow{\mathbf{L}i^*} & D^-(X) \\ {\scriptstyle \Phi}\big\downarrow & & \big\downarrow{\scriptstyle \Phi^{\dagger}} & & \big\downarrow{\scriptstyle \Phi^-} \\ D^b(Y) & \xrightarrow{i_*} & D^b(Y, v) & \xrightarrow{\mathbf{L}i^*} & D^-(Y). \end{array}$$

Proof. To distinguish the notation, let

$$\mathbf{R}p_{2*}^{\dagger} \colon D^b(\mathrm{Mod}(p_2^{-1}\mathcal{O}_Y^{(\beta',\gamma')}, p_2^*\widetilde{\alpha}')) \longrightarrow D^b(\mathrm{Mod}(\mathcal{O}_Y^{(\beta',\gamma')}), \widetilde{\alpha}'),$$

$$p_1^{\dagger *} \colon D^b(X, u) \longrightarrow D^b(X \times Y, p_1^*u + p_2^*(0, \beta', \gamma')),$$

be derived push-forward and pull-back. Let us take $a \in D^b(X)$. Then

$$\begin{aligned}\Phi^\dagger \circ i_*(a) &= \mathbf{R}p^\dagger_{2*}(p_1^{\dagger *} i_* a \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger) \\ &\cong \mathbf{R}p^\dagger_{2*}(i_* p_1^* a \overset{\mathbf{L}}{\otimes} \mathcal{P}^\dagger) \\ &\cong \mathbf{R}p^\dagger_{2*} i_*(p_1^* a \overset{\mathbf{L}}{\otimes} \mathbf{L}i^*\mathcal{P}^\dagger) \\ &\cong i_*\mathbf{R}p_{2*}(p_1^* a \overset{\mathbf{L}}{\otimes} \mathcal{P}) \\ &\cong i_* \circ \Phi(a).\end{aligned}$$

The second isomorphism follows from flat base change, and the third from projection formula. These properties are verified in our case as in the commutative case. We have proved that the left diagram commutes. The right diagram commutes similarly. q.e.d.

Proof of Theorem 4.7. It remains to show $\Phi^\dagger$ gives an equivalence. Take $a \in D^b(X,u)$ and $b \in D^-(X)$. Then we have

$$\begin{aligned}\mathrm{Hom}(\Phi^\dagger(a), \Phi^\dagger i_*(b)) &\cong \mathrm{Hom}(\Phi^\dagger(a), i_*\Phi(b)) \\ &\cong \mathrm{Hom}(\mathbf{L}i^*\Phi^\dagger(a), \Phi(b)) \\ &\cong \mathrm{Hom}(\Phi^-\mathbf{L}i^* a, \Phi(b)) \\ &\cong \mathrm{Hom}(\mathbf{L}i^* a, b) \\ &\cong \mathrm{Hom}(a, i_* b).\end{aligned}$$

Therefore the map $\mathrm{Hom}(a, i_* b) \xrightarrow{\Phi^\dagger} \mathrm{Hom}(\Phi^\dagger(a), \Phi^\dagger(i_* b))$ is an isomorphism. Next take $a, b \in D^b(X,u)$. Since we have the distinguished triangle,

$$i_*\mathbf{L}i^* b \longrightarrow b \longrightarrow i_*\mathbf{L}i^* b \longrightarrow i_*\mathbf{L}i^* b[1],$$

we have the following morphism of exact sequences, $(b' := \mathbf{L}i^* b)$

$$\begin{array}{ccccc}\mathrm{Hom}(a, i_* b') & \longrightarrow & \mathrm{Hom}(a,b) & \longrightarrow & \mathrm{Hom}(a, i_* b') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(\Phi^\dagger(a),\Phi^\dagger(i_* b')) & \longrightarrow & \mathrm{Hom}(\Phi^\dagger(a),\Phi^\dagger(b)) & \longrightarrow & \mathrm{Hom}(\Phi^\dagger(a),\Phi^\dagger(i_* b')).\end{array}$$

Therefore, the morphism $\mathrm{Hom}(a,b) \longrightarrow \mathrm{Hom}(\Phi^\dagger(a), \Phi^\dagger(b))$ is an isomorphism by 5-lemma. Now we have proved $\Phi^\dagger$ is fully-faithful. Finally, we show $\Phi^\dagger$ is essentially surjective. Take $\mathcal{F} \in D^b(Y,v)$. Again we have the distinguished triangle,

$$i_*\mathbf{L}i^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_*\mathbf{L}i^*\mathcal{F} \xrightarrow{t_{\mathcal{F}}} i_*\mathbf{L}i_*\mathcal{F}[1].$$

Let $\mathcal{F}' := \mathbf{L}i^*\mathcal{F}$. Since we have

$$\begin{aligned}i_*\mathcal{F}' &\cong i_*\Phi \circ \Psi(\mathcal{F}') \\ &\cong \Phi^\dagger \circ i_*\Psi(\mathcal{F}'),\end{aligned}$$

the morphism $t_{\mathcal{F}}\colon i_*\mathcal{F}' \to i_*\mathcal{F}'[1]$ is obtained by applying $\Phi^\dagger$ to some morphism, $s_{\mathcal{F}}\colon i_*\Psi(\mathcal{F}') \to i_*\Psi(\mathcal{F}')[1]$. Let $\mathcal{G} := \mathrm{Cone}(s_{\mathcal{F}})$. Then $\mathcal{F}$ is isomorphic to $\Phi^\dagger(\mathcal{G})$. It remains to show $\mathcal{G}$ is bounded. Note that by the definition of $\Phi^\dagger$, there exists $N > 0$ such that if $H^i(A) = 0$ for $i \ge l$ and some l, then $H^i(\Phi^\dagger(A)) = 0$ for $i \ge l+N$. Let us take an intelligent truncation of $\mathcal{G}$:

$$\tau_{\le l-1}\mathcal{G} \longrightarrow \mathcal{G} \longrightarrow \tau_{\ge l}\mathcal{G}.$$

Then by the above remark, $H^i(\Phi^\dagger(\tau_{\le l-1}\mathcal{G})) = 0$ for $i \ge l+N$. Therefore $\Phi^\dagger(\tau_{\le l-1}\mathcal{G}) \to \Phi^\dagger(\mathcal{G}) = \mathcal{F}$ is zero-map for sufficiently small l. Since $\Phi^\dagger$ is fully-faithful, this implies $\tau_{\le l-1}\mathcal{G} \to \mathcal{G}$ is zero-map. Therefore $\tau_{\le l-1}\mathcal{G} = 0$. q.e.d.

7. Examples

Abelian varieties. We give an example in which ϕ_T does not preserve direct summands of $HT^2(X)$. Let A be an Abelian variety, and $\hat{A}$ be its dual Abelian variety. Let $\mathcal{U} \in \mathrm{Pic}(A \times \hat{A})$ be the Poincare line bundle. Then the functor

$$\Phi^{\mathcal{U}}_{\hat{A}\to A}\colon D(\hat{A}) \longrightarrow D(A)$$

gives an equivalence (cf. [**14**]). In this particular example, ϕ_T takes some $\alpha \in H^2(\mathcal{O}_{\hat{A}})$ to $\gamma \in H^0(\wedge^2 T_A)$. Hence $\Phi^\dagger$ give equivalences between gerby deformations and non-commutative deformations of Abelian varieties first orderly. This phenomenon has been extended to infinite order deformations in [**3**].

Birational geometry. In this example, we discuss the situation in which ϕ_T preserves some direct summands of $HT^2(X)$. This example comes from the equivalences under some birational transforms, e.g., flops. Recently the relationship between derived categories and birational geometry has been developed. For example, see [**4**], [**7**], and [**10**]. Two smooth projective varieties X, Y are called K-equivalent if and only if there is a common resolution $p\colon Z \to X$, $q\colon Z \to Y$ such that $p^*K_X = q^*K_Y$. Kawamata [**10**] conjectured that derived categories are equivalent under K-equivalence. On the other hand, Wang [**18**] conjectured that the deformation theories of complex structures are invariant under K-equivalence. Since derived category contains much information, it is reasonable to guess that Kawamata's conjecture is stronger than Wang's conjecture. We will see the relationship between two conjectures using Theorem 4.7. Recall that $X \xrightarrow{f} W \xleftarrow{g} Y$ is called a flop if

- f and g are isomorphisms in codimension one.
- Relative Picard numbers of f, g are one.
- $K_X = f^*K_W$, $K_Y = g^*K_W$.
- Birational map $g^{-1} \circ f\colon X \dashrightarrow Y$ is not an isomorphism.

If X and Y are connected by flops, then X and Y are K-equivalent. We denote by $\mathrm{Def}(X)$ the Kuranishi deformation spaces, and by $T_0\,\mathrm{Def}(X)$ its tangent space at the origin . Let $\mathcal{X} \to \mathrm{Def}(X)$, $\mathcal{Y} \to \mathrm{Def}(Y)$ be Kuranishi families. For $\beta \in T_0\,\mathrm{Def}(X)$, let $\mathcal{X}_\beta$ be a scheme over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, the infinitesimal deformation of X corresponding to β.

Theorem 7.1. *Let X and Y be smooth projective varieties, which are connected by a flop, $X \xrightarrow{f} W \xleftarrow{g} Y$. Assume that there exists an object $\mathcal{P} \in D^b(X \times Y)$, which is supported on $X \times_W Y$, such that the functor $\Phi^{\mathcal{P}}_{X \to Y} \colon D^b(X) \to D^b(Y)$ gives an equivalence. Then there exists an isomorphism $\phi_D \colon T_0\,\mathrm{Def}(X) \to T_0\,\mathrm{Def}(Y)$ such that Φ extends to an equivalence,*

$$\Phi^\dagger \colon D^b(\mathrm{Coh}(\mathcal{X}_\beta)) \longrightarrow D^b(\mathrm{Coh}(\mathcal{Y}_{\phi_D(\beta)})).$$

Proof. Let $\phi_T \colon HT^2(X) \to HT^2(Y)$ be the isomorphism induced by Φ. It suffices to show ϕ_T takes $(0, \beta, 0)$ to $(0, \beta', 0)$. Let $U \subset W$ be the maximum open subset on which f and g are isomorphic. Then, since $\mathrm{codim}(X \setminus U) \geq 2$, $\mathrm{codim}(Y \setminus U) \geq 2$, and $f|_{X \setminus U}$, $g|_{Y \setminus U}$ has positive dimensional fibers, it follows that $\mathrm{codim}(W \setminus U) \geq 3$. On the other hand, since $\mathcal{P}$ is supported on $X \times_W Y$, the following diagram commutes:

$$\begin{array}{ccc} HT^2(X) & \xrightarrow{\phi_T} & HT^2(Y) \\ \downarrow & & \downarrow \\ HT^2(U) & = & HT^2(U). \end{array}$$

Here the vertical arrows are restrictions. Let $(\alpha', \beta', \gamma') := \phi_T(0, \beta, 0)$. By the above diagram, we have $\alpha'|_U = 0$, $\gamma'|_U = 0$. It is clear that $\gamma' = 0$. On the other hand, since $\mathbf{R}g_*\mathcal{O}_Y = \mathcal{O}_W$, we have $H^2(Y, \mathcal{O}_Y) \cong H^2(W, \mathcal{O}_W)$. Since $\mathrm{codim}(W \setminus U) \geq 3$, the restriction $H^2(W, \mathcal{O}_W) \to H^2(U, \mathcal{O}_U)$ is injective by [**16**]. Therefore, $\alpha' = 0$. q.e.d.

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