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WHEN IS A GEODESIC FLOW OF ANOSOV TYPE? II

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Introduction

A complete Riemannian manifold M has no *focal points* if no maximal geodesic σ of M has focal points along any perpendicular geodesic, where σ is regarded as a 1-dimensional submanifold of M. The simply connected Riemannian covering H of M also has no focal points and satisfies the following property: Let σ be a maximal geodesic of H, and p a point of H not lying on the point set σ. Then there exists a unique perpendicular geodesic from p to σ. If q is a point on σ closest to p, then the unique geodesic from p to q is perpendicular to σ. Let $P: H \to \sigma$ denote the map which sends each point p in H to the unique point q on σ which is closest to p. The map P is C^∞.

A complete Riemannian manifold M is *compactly homogeneous* if there exists a compact set $B \subseteq M$ such that the union of the images $\varphi(B)$, where φ is an isometry of M, is the entire manifold M. If M is homogeneous or admits a compact Riemannian quotient, then M is compactly homogeneous.

In this paper we prove the following result. For a definition of Anosov flow see [1] or [4].

Theorem. *Let M be a complete Riemannian manifold of dimension $m \geq 2$, without focal points, whose simply connected Riemannian covering H is compactly homogeneous. The following conditions are equivalent and imply that the geodesic flow in the unit tangent bundle of M is of Anosov type. Furthermore, if M has nonpositive sectional curvature, then the following conditions are equivalent to the condition that the geodesic flow be of Anosov type.*

1) *There exists a positive constant t_0 with the following property: Let σ be a maximal geodesic in H, let $P: H \to \sigma$ denote the projection map, and let v be a nonzero vector tangent to H at a point p such that $d(p, \sigma) \geq t_0$. Then $\|dP(v)\| < \|v\|$.*

2) *There exist positive constants a and c with the following property: Let σ be a maximal geodesic in H, let $P: H \to \sigma$ denote the projection map, and let v be a vector tangent to H at a point p. Then $\|dP(v)\| \leq ae^{-ct}\|v\|$, where $t = d(p, \sigma)$.*

3) *There exists a positive constant t_0 with the following property: Let Y*

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be a perpendicular Jacobi vector field on a unit speed geodesic γ of M such that $Y(0) \neq 0$ and $\langle Y(0), Y'(0)\rangle = 0$. Then $\|Y(t)\| > \|Y(0)\|$ for $t > t_0$.

4) *There exist a point p and positive constants c and t_0 with the following property: Let Y be a perpendicular Jacobi vector field on a unit speed geodesic γ such that $\gamma(0) = p$, $Y(0) = 0$ and $Y'(0) \neq 0$. Then $\{\log\|Y\|^2\}'(t) \geq c$ for $t \geq t_0$.*

5) *There exist positive constants c and t_0 with the following property: Let Y be a perpendicular Jacobi vector field on a unit speed geodesic γ of M such that $Y(0) = 0$ and $Y'(0) \neq 0$. Then $\{\log\|Y\|^2\}'(t) \geq c$ for $t \geq t_0$.*

6) *There exist positive constants c and t_0 with the following property: Let Z be a nontrivial perpendicular Jacobi vector field on a unit speed geodesic γ of M such that $\langle Z(0), Z'(0)\rangle \geq 0$. Then $\{\log\|Z\|^2\}'(t) \geq c$ for $t \geq t_0$.*

The constants c and t_0 are not necessarily the same in each of the six conditions.

A few remarks on the above conditions will be appropriate. Conditions 4), 5) and 6) represent a sharpened form of Corollary 3.3 of [4]. The conditions 1), 2) and 3) represent a geometric expression, not considered in [4], of the Anosov conditions for a geodesic flow. The idea of dropping perpendiculars from a point to a (geodesic) line or to a more general set is discussed in [3, pp. 6–8]. It is proved there that if M is a complete Riemannian manifold with nonpositive sectional curvature, and $A \subseteq M$ is a closed subset with the property (*total convexity*) that it contains all geodesic segments joining each pair of its points, then for any point p not in A there exists a unique perpendicular geodesic from p to A. One may therefore define a continuous projection map $P: M \to A$ exactly as above. Condition 1) of the theorem may now be compared with the following result, proved below in § 1, which is an infinitesimal version of Lemma 3.2 of [3]: Let H be a complete simply connected Riemannian manifold with nonpositive sectional curvature and dimension $m \geq 2$. Let σ be a maximal geodesic of H, and let $P: H \to \sigma$ denote the projection map. Then $\|dP(v)\| \leq \|v\|$ for any vector v tangent to H.

It will follow from the discussion in § 1 that M has no focal points if and only if it satisfies the following condition, which one may compare with condition 5): Let Y be a not necessarily perpendicular Jacobi vector field on a unit speed geodesic γ of M such that $Y(0) = 0$ and $Y'(0) \neq 0$. Then $\{\log\|Y\|^2\}'(0) > 0$ for $t > 0$. E. Hopf in [6] proved that the geodesic flow on the unit tangent bundle of a compact surface satisfying condition 5) is ergodic. Anosov [1] has remarked that the condition 5) for compact surfaces implies that the geodesic flow in the unit tangent bundle is of Anosov type and ergodic as a consequence.

For the definitions of geodesic flow, Jacobi vector field, focal points and other basic concepts not defined here, see, for example, § 1 of [4].

1. Basic facts

Before proceeding to the proof of the theorem we establish notation and list some essential facts. Let M always denote a complete Riemannian manifold of arbitrary dimension $m \geq 2$. For any point p in M, let M_p denote the tangent space to M at p. For any vector v tangent to M, let $\pi(v)$ denote the point of tangency of v. Let $\langle , \rangle$ denote the inner product, and d the Riemannian metric in M.

Let γ_n be a sequence of (constant speed) geodesics in M, and Y_n a Jacobi vector field on γ_n for each integer n. We say that the sequence Y_n *converges* to a Jacobi vector field Y on a geodesic γ if $\gamma_n'(0) \to \gamma'(0)$, $Y_n(0) \to Y(0)$ and $Y_n'(0) \to Y'(0)$, where Y_n' and Y' denote the covariant derivative of Y_n and Y along γ_n and γ respectively. Let u_n be a sequence of numbers converging to a finite number u, and Y_n a sequence of Jacobi vector fields converging to a Jacobi vector field Y. Then $Y_n'(u_n) \to Y'(u)$ and $Y_n(u_n) \to Y(u)$; this assertion is a consequence, for example, of Proposition 1.7, 2) of [4]. The following fact, which is Proposition 1.11 of [4], will be used often:

Proposition 1. *Let M be compactly homogeneous. For each positive integer n, let Y_n be a Jacobi vector field on a geodesic γ_n with initial velocity v_n. If each of the sequences $\|v_n\|, \|Y_n(0)\|, \|Y_n'(0)\|$ is uniformly bounded above, then we can find a sequence φ_n of isometries of M and a Jacobi vector field Z on a geodesic γ such that $Z_n = d\varphi_n Y_n \to Z$ by passing to a subsequence.*

A manifold M has no *conjugate points* if no nontrivial Jacobi vector field Y along a geodesic γ of M vanishes at two distinct points. If M has no conjugate points, then the simply connected Riemannian covering H has the property that each exponential map $\exp_p\colon H_p \to H$ is a diffeomorphism; therefore there exists a unique geodesic joining any two distinct points of H. It is shown in § 1 of [4] that M has no focal points if and only if it satisfies the following property: Let Y be a not necessarily perpendicular Jacobi vector field on a unit speed geodesic γ of M such that $Y(0) = 0$ and $Y'(0) \neq 0$. Then $\{\|Y\|^2\}'(t) > 0$ for $t > 0$. Clearly, the condition that γ have unit speed can be dropped. This characterization shows that manifolds without focal points have no conjugate points. If M has no focal (conjugate) points, then any Riemannian covering of M has no focal (conjugate) points. A complete Riemannian manifold with nonpositive sectional curvature has no focal points.

If γ is a unit speed geodesic in a manifold M without conjugate points, then we may define an $(m-1)$-dimensional vector space $J_s(\gamma)$ of *stable* perpendicular Jacobi vector fields along γ. Let v be a nonzero vector in $M_{\gamma(0)}$ which is orthogonal to $\gamma'(0)$. For each positive number t, let Y_t be the unique Jacobi vector field on γ such that $Y_t(0) = v$ and $Y_t(t) = 0$. Then it is shown in [4] or [5] that there is a perpendicular Jacobi vector field Y on γ such that $Y_t \to Y$ as $t \to +\infty$, in the sense defined above. The set of perpendicular Jacobi vector fields Y constructed in this manner forms an $(m-1)$-dimensional vector space $J_s(\gamma)$; for every vector v orthogonal to $\gamma'(0)$ there is a unique $Y \in J_s(\gamma)$ such that $Y(0) = v$. If Y is a perpendicular Jacobi vector field along γ such that $\|Y(t)\|$ is bounded above for $t \geq 0$, then $Y \in J_s(\gamma)$. The converse assertion is true if M has no focal points. A complete discussion, with proofs, of the assertions in this paragraph may be found in § 2 of [4].

We next elaborate on some of the assertions concerning manifolds without focal points, which were made in the introduction. The following result is known, but does not seem to be in the literature.

Proposition 2. *Let H be a complete simply connected Riemannian manifold of arbitrary dimension $m \geq 2$ without conjugate points. Then H has no focal points if and only if for every maximal geodesic σ of H and every point p of H not lying on σ, there exists a unique geodesic from p to σ, which is perpendicular to σ.*

Proof. Suppose that the unique perpendicular property holds in H. Let a maximal geodesic σ and a point p not on σ be given. If q is a point on σ nearest to p, then a first variation argument shows that the unique geodesic from p to q is perpendicular to σ. Now let γ be any unit speed geodesic such that $\gamma(0)$ lies on σ and γ is perpendicular to σ. If $\gamma(a)$ is a focal point of σ along γ for some number $a > 0$, then $d(\gamma t, \sigma) < t$ for any number $t > a$. Therefore for any $t > a$ there exist two perpendiculars from $\gamma(t)$ to σ. This contradiction shows that σ has no focal points along γ, and since σ and γ are arbitrary, H has no focal points. To prove that H satisfies the unique perpendicular property if it has no focal points, we shall need the following result.

Lemma 1. *Let H be a complete simply connected Riemannian manifold without focal points. Then for any point p in H the function $q \to d^2(p, q)$ is a strictly convex C^∞ function in H, where d denotes the Riemannian metric in H.*

A real valued C^∞ function f on a manifold M is *strictly convex* if for every maximal geodesic γ of M, $(f \circ \gamma)''(t) > 0$ for every $t \in R$.

Proof of Lemma 1. For any point p in H, $f(q) = \|\exp_p^{-1}(q)\|^2$ is a C^∞ function since $\exp_p \colon H_p \to H$ is a diffeomorphism. To verify the strict convexity of f it suffices to consider unit speed geodesics in the definition above. Let a point p in H and a maximal unit speed geodesic γ of H be given. We need to show that $g''(t) > 0$ for every $t \in R$, where $g(t) = (f \circ \gamma)(t) = d^2(p, \gamma t)$. If p lies on γ, then $g''(t) \equiv 2$ so we may assume that p does not lie on γ. Define a curve $\varphi \colon R \to H_p$ by the formula $\varphi(t) = (\exp_p)^{-1}\gamma(t)$, and define a C^∞ variation $r \colon R \times R \to H$ by setting $r(u, v) = \exp_p(u\varphi(v))$. Define the vector functions $r_u(u, v) = dr(\partial/\partial u)(u, v)$ and $r_v(u, v) = dr(\partial/\partial v)(u, v)$. We observe that $g(v) = \int_0^1 \langle r_u, r_u\rangle(u, v)du$. A computation shows that $g'(v) = 2\langle r_u, r_v\rangle(1, v)$, and therefore $g'' = 2\langle \nabla_{r_v} r_u, r_v\rangle(1, v) = 2\langle \nabla_{r_u} r_v, r_v\rangle(1, v)$ since the curve $v \to r(1, v) = \gamma(v)$ is a geodesic. If we define $Y_v(u) = r_v(u, v)$ for every v, then Y_v is a Jacobi vector field on the nonconstant geodesic $u \to r(u, v)$ such that $Y_v(0) = 0$ and $Y_v'(0) \neq 0$. Then $g''(v) = 2\langle Y_v'(1), Y_v(1)\rangle > 0$ by the equivalent formulation of the no focal point property which appears above.

We complete the proof of Proposition 2. Let H have no focal points, and let there be given a maximal geodesic σ of H and a point p of H not lying on σ. Give σ a unit speed parametrization so that $\sigma(0)$ is a point on σ closest to p, and let $g(s) = d^2(p, \sigma s)$. The formula above for $g'(s)$ shows that the unique geodesic from p to $\sigma(s)$ is perpendicular to σ if and only if $g'(s) = 0$. By assumption, $g'(0) = 0$, and by Lemma 1, $g''(s) > 0$ for every $s \in R$. Therefore $g'(s) \neq 0$ for $s \neq 0$, and it follows that there is exactly one perpendicular from p to σ.

For the remainder of this section let H denote a complete simply connected Riemannian manifold of dimension $m \geq 2$ without focal points. Proposition 2 shows that for any maximal geodesic σ and any point p in H there is a unique point q on σ which is closest to p. We define the projection map $P\colon H \to \sigma$ by setting $P(p) = q$. Let $\sigma^\perp$ denote the normal bundle of σ, and $\exp^\perp\colon \sigma^\perp \to H$ the restriction of the exponential map to $\sigma^\perp$. Clearly $\exp^\perp$ is surjective, and Proposition 2 shows that $\exp^\perp$ is injective. The no focal point hypothesis implies that $(d\exp^\perp)_v$ is nonsingular for every vector $v \in \sigma^\perp$, and therefore $\exp^\perp\colon \sigma^\perp \to H$ is a diffeomorphism. If $k\colon \sigma^\perp \to \sigma$ denotes the natural projection map, then $P = k \circ (\exp^\perp)^{-1}$; P is therefore a C^∞ mapping. See also [3, pp. 6–8].

The following two lemmas describe the differential maps dP in terms of Jacobi vector fields.

Lemma 2. *Let σ be a maximal geodesic of H, and let $P\colon H \to \sigma$ denote the projection map. Let v be a vector tangent to H at a point p not on σ, and let $a = d(p, P(p))$. Let γ be the maximal unit speed geodesic such that $\gamma(0) = P(p)$ and $\gamma(a) = p$, and let Y be the unique Jacobi vector field on γ such that $Y(0) = dP(v)$ and $Y(a) = v$. Then $\langle Y(0), Y'(0)\rangle = 0$.*

Lemma 3. *Let Y be a perpendicular Jacobi vector field on a unit speed geodesic γ of H such that $Y(0) \neq 0$ and $\langle Y(0), Y'(0)\rangle = 0$. Let σ be the maximal geodesic tangent to $Y(0)$, and let $P\colon H \to \sigma$ denote the projection map. Then for any number $a \neq 0$, $dP\, Y(a) = Y(0)$.*

Proof of Lemma 2. Let σ, P, v, a, γ and Y be as defined in the statement of Lemma 2. Let σ be given a unit speed parametrization such that $\sigma(0) = P(p)$. Let $k\colon \sigma^\perp \to \sigma$ be the natural projection map. We may assume that $v \neq 0$, for otherwise $Y \equiv 0$ and the assertion is trivially true. Let α be the geodesic such that $\alpha'(0) = v$, and let $Z(t) = (\exp^\perp)^{-1}\alpha(t)$. Let $\beta(t) = (k \circ Z)(t) = (P \circ \alpha)(t) = \sigma(c(t))$, where $c\colon R \to R$ is a C^∞ function such that $c(0) = 0$. The equations $\beta'(t) = c'(t)\sigma'(ct)$ and $\beta''(t) = c''(t)\sigma'(ct)$ and the fact that Z is a curve in $\sigma^\perp$ imply that $\langle Z(t), \beta'(t)\rangle = \langle Z(t), \beta''(t)\rangle = 0$ for all t. Differentiating the expression $\langle Z(t), \beta'(t)\rangle \equiv 0$ we find that $\langle Z'(t), \beta'(t)\rangle \equiv 0$, where Z' denotes the covariant derivative of Z along β. Define a variation $r\colon R \times R \to H$ by the formula $r(u, t) = \exp^\perp((u/a)Z(t))$. It follows that $\gamma(u) = r(u, 0)$ and $Y(u) = dr(\partial/\partial t)(u, 0)$. One may then show that $\langle Y(0), Y'(0)\rangle = (1/a)\langle Z'(0), \beta'(0)\rangle = 0$.

Proof of Lemma 3. Let Y, γ, σ and P be defined as in the statement of

Lemma 3. Let $k\colon \sigma^{\perp} \to \sigma$ be the projection map. The conditions on Y imply that, in the terminology of [2], Y is an N-Jacobi field on γ, where $N = \sigma$. Therefore we may find a C^{∞} curve Z in the unit normal bundle of σ such that $Z(0) = \gamma'(0)$ and $Y(u) = dr(\partial/\partial v)(u, 0)$, where $r\colon R \times (-\varepsilon, \varepsilon) \to H$ is the variation given by the formula $r(u, v) = \exp^{\perp}(uZ(v))$. Thus for any number $a \neq 0$, $Y(a) = d\exp^{\perp}((d/dv)(aZ)|_{v=0})$. Since $P = k \circ (\exp^{\perp})^{-1}$, it follows that $dPY(a) = dk[(d/dv)(aZ)|_{v=0}] = dk[dZ/dv|_{v=0}] = Y(0)$.

If H has nonpositive sectional curvature, we may say even more about the differential maps dP.

Proposition 3. *Let H be a complete simply connected Riemannian manifold with nonpositive sectional curvature. Let σ be a maximal geodesic of H, and $P\colon H \to \sigma$ the projection map. Then for any vector v tangent to H, $\|dP(v)\| \leq \|v\|$.*

Proof. We first consider the case where $\pi(v)$ is a point of σ. Then $v = v_1 + v_2$, where v_1 is tangent to σ and v_2 is orthogonal to σ. Since P fixes every point of σ, $dP(v) = dP(v_1) + dP(v_2) = v_1$, and $\|dP(v)\|^2 = \|v_1\|^2 \leq \|v_1\|^2 + \|v_2\|^2 = \|v\|^2$. If $\pi(v)$ does not lie on σ, then let $a = d(\pi v, \sigma)$. Let γ be the unique unit speed geodesic such that $\gamma(0) = P(\pi v)$ and $\gamma(a) = \pi(v)$, and let Y be the unique Jacobi vector field on γ such that $Y(0) = dP(v)$ and $Y(a) = v$. By Lemma 2, $\langle Y(0), Y'(0)\rangle = 0$. If $f(t) = \|Y(t)\|^2$, then $f''(t) = 2\{\|Y'(t)\|^2 - K(Y, \gamma')(t)\|Y \wedge \gamma'\|^2(t)\} \geq 0$ for every t. Since $f'(0) = 0$, it follows that $f'(t) \geq 0$ for $t \geq 0$, and therefore $\|v\|^2 = f(a) \geq f(0) = \|dP(v)\|^2$.

2. The main theorem

We shall prove the theorem stated in the introduction as a series of Propositions 4 through 9. The conditions referred to in the statements of these propositions are the conditions of the theorem. Since H is locally isometric to M, it suffices to prove that the conditions 1) through 6) are equivalent in H to conclude that they are equivalent in M.

Proposition 4. *Let M be a manifold without conjugate points, whose simply connected Riemannian covering H is compactly homogeneous. Then conditions* 5) *and* 6) *are equivalent in H, and imply that the geodesic flow in the unit tangent bundle of M is of Anosov type.*

Proof. Clearly condition 6) implies condition 5). Assume now that condition 5) is satisfied. We first establish the following fact: Let γ be a unit speed geodesic in M, and Y be a perpendicular Jacobi vector field on γ such that $\|Y(0)\| = 1$ and $Y \in J_s(\gamma)$ (defined in § 1). Then $\langle Y(0), Y'(0)\rangle \leq -\frac{1}{2}c < 0$, where c is the positive constant appearing in condition 5). To prove this assertion, let there be given a unit speed geodesic γ in H and a perpendicular Jacobi vector field $Y \in J_s(\gamma)$ such that $\|Y(0)\| = 1$. For any number $t > t_0$, the other constant of condition 5), let Y_t be the unique Jacobi vector field on γ such that $Y_t(0) = Y(0)$ and $Y_t(t) = 0$. Let $\sigma_t(u) = \gamma(t - u)$, and $Z_t(u) =$

$Y_t(t-u)$. Then $\langle Y_t(0), Y_t'(0)\rangle = -\langle Z_t(t), Z_t'(t)\rangle = -\frac{1}{2}\{\log\|Z_t\|^2\}'(t) \leq -\frac{1}{2}c$ by condition 5). By the definition of $J_s(\gamma)$ in § 1, $Y_t \to Y$ as $t \to +\infty$. It follows by continuity that $\langle Y(0), Y'(0)\rangle \leq -\frac{1}{2}c$.

Next we show that there exists a positive number t_1 with the following property: Let Y be a nontrivial perpendicular Jacobi vector field on a unit speed geodesic γ of H such that $\langle Y(0), Y'(0)\rangle \geq 0$. Then $\langle Y(t), Y'(t)\rangle > 0$ for $t > t_1$. If this assertion were false, then for each positive integer n we could find a unit speed geodesic γ_n, a perpendicular Jacobi vector field Y_n on γ_n, and a positive number t_n such that $\langle Y_n(0), Y_n'(0)\rangle \geq 0$, $\langle Y_n(t_n), Y_n'(t_n)\rangle \leq 0$, and $t_n \to +\infty$. For every n choose a number s_n such that $0 \leq s_n \leq t_n$ and $\|Y_n(t)\| \leq \|Y_n(s_n)\|$ for every number t such that $0 \leq t \leq t_n$. By passing to a subsequence we see that either $s_n \to \infty$ or $t_n - s_n \to \infty$. We consider only the first case since we obtain a contradiction to the second case in a similar way. For each n let $\sigma_n(t) = \gamma_n(s_n - t)$, and $Z_n(t) = Y_n(s_n - t)$. Then Z_n is a Jacobi vector field on the geodesic σ_n such that $\langle Z_n(0), Z_n'(0)\rangle = 0$ and $\|Z_n(t)\| \leq \|Z_n(0)\|$ for $0 \leq t \leq s_n$. Multiplying Z_n by a constant if necessary, we may assume that $\|Z_n(0)\|^2 + \|Z_n'(0)\|^2 = 1$. Since H is compactly homogenous, Proposition 1 asserts that there exist a sequence φ_n of isometries of H and a Jacobi vector field Z on a unit speed geodesic γ of H such that by passing to a subsequence, $Z_n^* = d\varphi_n Z_n \to Z$. For every integer n and every number t, we have $\|Z_n^*(t)\| = \|Z_n(t)\|$, $\|Z_n^{*\prime}(t)\| = \|Z_n'(t)\|$ and $\langle Z_n^*(0), Z_n^{*\prime}(0)\rangle = 0$. Therefore, by continuity, $\|Z(0)\|^2 + \|Z'(0)\|^2 = 1$, $\langle Z(0), Z'(0)\rangle = 0$ and $\|Z(t)\| \leq \|Z(0)\|$ for $t \geq 0$. Since Z is nontrivial it follows that $Z(0) \neq 0$, and moreover $Z \in J_s(\gamma)$ by the discussion in § 1. This contradicts the first paragraph of this proof since $\langle Z(0), Z'(0)\rangle = 0$.

The previous paragraph also shows that if Y is a Jacobi vector field of the type considered in condition 6), then $Y(t) \neq 0$ for $t > t_1$. It follows that $\{\log\|Y\|^2\}'(t) = 2\langle Y(t), Y'(t)\rangle/\|Y(t)\|^2 > 0$ for $t > t_1$. Suppose now that condition 6) does not hold. Then for each positive integer n we can find a unit speed geodesic γ_n, a nontrivial perpendicular Jacobi vector field Y_n on γ_n, and a positive number t_n such that $\langle Y_n(0), Y_n'(0)\rangle \geq 0$, $0 < \{\log\|Y_n\|^2\}'(t_n) < 1/n$, and $t_n \to +\infty$. For each n, choose a number s_n such that $0 \leq s_n \leq t_n$ and $\|Y_n(t)\| \leq \|Y_n(s_n)\|$ for every number t such that $0 \leq t \leq t_n$. If $s_n < t_n$ for all n, by passing to a subsequence if necessary, then $\langle Y_n(s_n), Y_n'(s_n)\rangle = 0$ for every n. Either $s_n \to +\infty$ or $t_n - s_n \to +\infty$ by passing to a further subsequence, and we obtain a contradiction exactly as in the previous paragraph. The remaining case to consider occurs when $s_n = t_n$ for sufficiently large n. In this case let $\sigma_n(t) = \gamma_n(t_n - t)$, and $Z_n(t) = Y_n(t_n - t)$ for each integer n. Then Z_n is a nontrivial perpendicular Jacobi vector field on the geodesic σ_n such that $\|Z_n(t)\| \leq \|Z_n(0)\|$ for $0 \leq t \leq t_n$. Multiply Z_n by a constant so that $\|Z_n(0)\|^2 + \|Z_n'(0)\|^2 = 1$; the scalar multiplication changes neither the inequality above nor the expression $\{\log\|Z_n\|^2\}'(t)$. Note that $2\|Z_n(0)\|^{-2}\langle Z_n(0), Z_n'(0)\rangle = \{\log\|Z_n\|^2\}'(0) = -\{\log\|Y_n\|^2\}'(t_n)$. Therefore $-\frac{1}{2}n^{-1}$

$< \langle Z_n(0), Z_n'(0)\rangle/\|Z_n(0)\|^2 < 0$ for every integer n. By Proposition 1 we may choose a sequence φ_n of isometries of H and a perpendicular Jacobi vector field Z on a unit speed geodesic γ such that $Z_n^* = d\varphi_n Z_n \to Z$, by passing to a subsequence if necessary. By continuity, $\|Z(0)\|^2 + \|Z'(0)\|^2 = 1$ and $\|Z(t)\| \leq \|Z(0)\|$ for $t \geq 0$. Since Z is nontrivial, $Z(0) \neq 0$, and furthermore $Z \in J_s(\gamma)$ by the discussion in § 1. However, $\langle Z(0), Z'(0)\rangle/\|Z(0)\|^2 = 0$ by continuity, and this contradicts the assertion in the first paragraph of the proof. Thus condition 5) implies condition 6).

We show that the geodesic flow in the unit tangent bundle of M is of Anosov type under condition 5). Let Y be a perpendicular Jacobi vector field on a unit speed geodesic γ such that $Y(0) = 0$ and $\|Y'(0)\| = 1$. An argument by contradiction similar to those used above shows that we can find a positive constant B, not depending on Y, such that $\|Y(t_0)\| \geq B$. Condition 5) then implies that for $t > t_0$, $\|Y(t)\| \geq (Be^{-ct_0/2})e^{ct/2}$. Furthermore, $\|Y(t)\| \geq \|Y(s)\|$ for any numbers $t \geq s \geq t_0$. These relations also hold for the Jacobi vector fields in M with the same initial conditions. By Theorem 3.2, 5) of [4], the geodesic flow in the unit tangent bundle of M is of Anosov type.

For the rest of this paper, unless otherwise specified, let M denote a complete Riemannian manifold without focal points, whose simply connected Riemannian covering H is compactly homogeneous.

Proposition 5. *Condition* 4) *implies condition* 6) *in* H.

Proof. We shall need a preliminary trigonometric result. If H has nonpositive sectional curvature, then the result is an immediate consequence of the law of cosines. The angle $\measuredangle(v, w)$ between two nonzero vectors tangent to a Riemannian manifold M at the same point is defined to be the unique number θ such that $\cos\theta = \langle v, w\rangle/(\|v\|\cdot\|w\|)$ and $0 \leq \theta \leq \pi$.

Lemma 4. *Let H be a complete simply connected Riemannian manifold without focal points. Let there be given a compact subset $C \subseteq H$ and positive numbers ε and A. There exists a positive number $t_0 = t_0(C, \varepsilon, A)$ with the following property: Let γ and σ be two unit speed geodesics such that $\gamma(0) = \sigma(0) \in C$ and $d(\gamma t, \sigma s) \leq A$ for some numbers $t \geq t_0$ and $s \geq t_0$. Then $\measuredangle(\gamma'(0), \sigma'(0)) < \varepsilon$.*

Proof. We first establish the following fact: Let there be given a compact set $C \subseteq H$ and a positive number R. Then there exists a positive number $t_0 = t_0(C, R)$ with the following property: Let γ be a unit speed geodesic such that $\gamma(0) \in C$, and Y a perpendicular Jacobi vector field on γ such that $Y(0) = 0$ and $\|Y'(0)\| = 1$. Then $\|Y(t)\| \geq R$ for $t \geq t_0$. We recall that because H has no focal points, the function $t \to \|Y(t)\|$ is monotone increasing for $t \geq 0$ and any Jacobi vector field Y such that $Y(0) = 0$ and $Y'(0) \neq 0$. Therefore it suffices to find a positive number t_0 such that $\|Y(t_0)\| \geq R$ for all perpendicular Jacobi vector fields Y on unit speed geodesics γ such that $\gamma(0) \in C$, $Y(0) = 0$, and $\|Y'(0)\| = 1$. If this were not the case, then for some compact set C and some positive number R we could find, for every positive integer n, a unit speed geodesic γ_n, a perpendicular Jacobi vector field Y_n on γ_n and a positive number t_n such that $\gamma_n(0) \in C$, $Y_n(0) = 0$, $\|Y_n'(0)\| = 1$, $\|Y_n(t_n)\| < R$, and $t_n \to +\infty$. Passing to a subsequence, we can find a unit speed geodesic γ and a perpendicular Jacobi vector field Y on γ such that $\gamma_n'(0) \to \gamma'(0)$, $Y_n'(0) \to Y'(0)$ and $Y_n(0) \to Y(0)$. Then $\gamma(0) \in C$, $Y(0) = 0$ and $\|Y'(0)\| = 1$ by continuity. By Proposition 2.9 of [4], $\|Y(t)\| \to \infty$ as $t \to \infty$, and therefore we can find a number $T > 0$ such that $\|Y(T)\| > 2R$. Since $Y_n(T) \to Y(T)$, it follows that $\|Y_n(T)\| > R$ for sufficiently large n. Then $t_n > T$ for large n, and we have $R > \|Y_n(t_n)\| > \|Y_n(T)\| > R$, which is a contradiction.

Now, let a compact subset $C \subseteq H$ and positive numbers ε and A be given. Choose a positive number R so that $A/R < \varepsilon$, and let $t_0 = A + t_0(C, R)$, where $t_0(C, R)$ is the constant appearing in the assertion of the previous paragraph. We claim that t_0 has the property stated in the lemma. Let γ and σ be unit speed geodesics in H such that $\gamma(0) = \sigma(0) = p \in C$, and let $d(\gamma t, \sigma s) \leq A$ for some numbers $t \geq t_0$ and $s \geq t_0$. Let $\alpha\colon [0, 1] \to H$ be the unique geodesic segment such that $\alpha(0) = \gamma(t)$ and $\alpha(1) = \sigma(s)$. Then $d(p, \alpha v) \geq t_0(C, R)$ for every $v \in [0, 1]$. Let $\beta\colon [0, 1] \to H_p$ be defined by $\beta(v) = (\exp_p)^{-1}\alpha(v)$. Define a C^∞ variation $r\colon R \times [0, 1] \to H$ by the formula $r(u, v) = \exp_p(u[\|\beta(v)\|\beta(v)])$. Let $r_u(u, v)$ and $r_v(u, v)$ denote the vector functions $dr(\partial/\partial u)(u, v)$ and $dr(\partial/\partial v)(u, v)$. Since $\alpha(v) = r(\|\beta(v)\|, v)$, we see that $\alpha'(v) = \|\beta\|'(v) r_u(\|\beta(v)\|, v) + r_v(\|\beta(v)\|, v)$. By Gauss's Lemma, $r_u(u, v)$ and $r_v(u, v)$ are always orthogonal since the curves $u \to r(u, v)$ are unit speed geodesics emanating from the point p. Therefore $\|\alpha'(v)\| \geq \|r_v(\|\beta(v)\|, v)\|$. For each v the vector field $Y_v(u) = r_v(u, v)$ is a perpendicular Jacobi vector field on the geodesic $u \to r(u, v)$ such that $Y_v(0) = 0$. Using the fact proved in the previous paragraph we see that $\|\alpha'(v)\| \geq \|r_v(\|\beta(v)\|, v)\| = \|Y_v(\|\beta(v)\|)\| \geq R\|Y_v'(0)\|$, since $\|\beta(v)\| = d(p, \alpha v) \geq t_0(R, C)$. $Y_v'(0)$ denotes the covariant derivative at time zero of Y_v along the geodesic $u \to r(u, v)$.

Let $E_1, \cdots, E_n$ be an orthogonal frame field in a neighborhood of p, the initial point of γ and σ. If $\varphi(v) = \beta(v)/\|\beta(v)\| = r_u(0, v)$, then we may write $\varphi(v) = \sum_{i=1}^{n} \varphi_i(v) E_i(p)$. We may regard φ either as a vector field on the point curve $\{p\} = r(0, v)$ or as a curve with coordinates $\{\varphi_i(v)\}$ in the unit sphere of H_p. In these two senses, the covariant derivative of φ along $\{p\}$ and the velocity vector field $\varphi'(v)$ have the same coordinates $\{\varphi_i'(v)\}$. Since the covariant derivative of φ along $\{p\}$ is the vector field $v \to Y_v'(0)$, we find that $\|\varphi'(v)\| = \|Y_v'(0)\|$ for all $v \in [0, 1]$. If $\theta = \measuredangle(\gamma'(0), \sigma'(0))$, then $\theta \leq \int_0^1 \|\varphi'(v)\| dv = \int_0^1 \|Y_v'(0)\| dv$ since θ is the spherical distance between $\gamma'(0)$ and $\sigma'(0)$, and φ joins $\gamma'(0)$ to $\sigma'(0)$. Finally, $A \geq d(\gamma t, \sigma s) = \int_0^1 \|\alpha'(v)\| dv \geq R \int_0^1 \|Y_v'(0)\| dv \geq R\theta$, or $\theta \leq A/R < \varepsilon$.

Now let condition 4) hold in H. It will follow (by the identical argument used in Proposition 4) that condition 6) holds in H, once we have established the following assertion: Let Y be a perpendicular Jacobi vector field on a unit speed geodesic γ in H such that $Y \in J_s(\gamma)$ and $\|Y(0)\| = 1$. Then $\langle Y(0), Y'(0)\rangle \leq -\frac{1}{2}c$, where c is the positive constant appearing in condition 4).

Let there be given a point p in H and positive constants c and t_0 which satisfy the hypotheses of condition 4). If φ is an isometry of H, then $\varphi(p)$ and the constants c and t_0 also satisfy the hypotheses of condition 4). Let there be given a unit speed geodesic γ in H and a perpendicular Jacobi vector field Y on γ such that $Y \in J_s(\gamma)$ and $\|Y(0)\| = 1$. Since H is compactly homogeneous, we can find a positive number A and a sequence φ_n of isometries of H such that $d(\varphi_n(p), \gamma(n)) < A$ for every positive integer n. Let γ_n be the unique unit speed geodesic joining $\gamma(0)$ to $\varphi_n p$ such that $\gamma_n(0) = \gamma(0)$. Then Lemma 4 implies that $\gamma_n'(0) \to \gamma'(0)$. Let X_n be the unique Jacobi vector field on γ_n such that $X_n(0) = Y(0)$ and $X_n(t_n) = 0$, where $t_n = d(\gamma(0), \varphi_n p)$. We may write $X_n = Y_n + Z_n$, where Y_n is a perpendicular Jacobi vector field on γ_n, and $Z_n(t) = (a_n t + b_n)\gamma_n'(t)$ for all t and suitable constants a_n and b_n.

We show that $Y_n \to Y$. First, we note that $\|Y(0) - Y_n(0)\| = \|X_n(0) - Y_n(0)\| = \|Z_n(0)\| = |\langle X_n(0), \gamma_n'(0)\rangle| \to |\langle Y(0), \gamma'(0)\rangle| = 0$. Hence $Y_n(0) \to Y(0)$. Next, since H is compactly homogeneous, there exists a positive constant k such that the sectional curvature satisfies the condition $K > -k^2$. Since $Y_n(t_n) = 0$, Proposition 2.7 of [4] implies that $\|Y_n'(0)\| \leq k \coth(kt_n)\|Y_n(0)\| \leq k \coth(kt_n) \leq 2k$ for sufficiently large n. We may choose a vector $w \in H_{\gamma(0)}$ and a subsequence n_r of integers such that $Y_{n_r}'(0) \to w$. Let Y^* be the Jacobi vector field on γ such that $Y^*(0) = Y(0)$ and $Y^{*\prime}(0) = w$. By the no focal point property, $\|Y_{n_r}(t)\| \leq \|Y_{n_r}(0)\|$ for $0 \leq t \leq t_n$. We have seen that $\gamma_{n_r}'(0) \to \gamma'(0)$, $Y_{n_r}(0) \to Y(0) = Y^*(0)$, and $Y_{n_r}'(0) \to Y^{*\prime}(0)$. Hence $Y_{n_r} \to Y^*$, and it follows that Y^* is a perpendicular Jacobi vector field on γ such that $\|Y^*(t)\| \leq \|Y^*(0)\|$ for all $t \geq 0$. Therefore $Y^* \in J_s(\gamma)$, and since a vector field in $J_s(\gamma)$ is uniquely determined by its value at zero, we see that $Y^* = Y$. Since $Y_{n_r}'(0)$ was an arbitrary convergent subsequence of $Y_n'(0)$, it follows that $Y_n'(0) \to Y'(0)$. Therefore $Y_n \to Y$.

For each integer n let $\sigma_n(t) = \gamma_n(t_n - t)$ and $W_n(t) = Y_n(t_n - t)$. Then W_n is a perpendicular Jacobi vector field on the geodesic σ_n, and $\sigma_n(0) = \varphi_n p$. By condition 4), $\langle Y_n(0), Y_n'(0)\rangle/\|Y_n(0)\|^2 = -\langle W_n(t_n), W_n'(t_n)\rangle/\|W_n(t_n)\|^2 = -\frac{1}{2}\{\log\|W_n\|^2\}'(t_n) \leq -\frac{1}{2}c$ for sufficiently large n. By continuity, $\langle Y(0), Y'(0)\rangle = \langle Y(0), Y'(0)\rangle/\|Y(0)\|^2 \leq -\frac{1}{2}c$ since $Y_n \to Y$. We have proved the assertion in the paragraph immediately following the proof of Lemma 4, and by the remarks in that paragraph this completes the proof of Proposition 5.

Proposition 6. *Condition* 1) *implies condition* 3) *in* H.

Proof. Let condition 1) hold in H, and let Y be a perpendicular Jacobi vector field on a unit speed geodesic γ of H such that $Y(0) \neq 0$ and $\langle Y(0), Y'(0)\rangle = 0$. Let σ be the maximal geodesic of H tangent to $Y(0)$, and let $P\colon H \to \sigma$ denote the projection map. Since γ and σ are orthogonal, $d(\gamma t, \sigma) = |t|$ for every number t. If t_0 is the positive constant of condition 1), then we use Lemma 3 to conclude that $\|Y(0)\| = \|dPY(t)\| < \|Y(t)\|$ for $t > t_0$.

Proposition 7. *Condition* 3) *implies condition* 5) *in* H.

Proof. Let condition 3) hold in H. If Y is a perpendicular Jacobi vector field on a unit speed geodesic γ such that $Y(0) = 0$ and $Y'(0) \neq 0$, then $\{\log\|Y\|^2\}'(t) > 0$ for each $t > 0$. If condition 5) does not hold in H, then for each positive integer n we can find a unit speed geodesic γ_n, a perpendicular Jacobi vector field Y_n on γ_n, and a positive number t_n such that $Y_n(0) = 0$, $Y_n'(0) \neq 0$, $0 < \{\log\|Y\|^2\}'(t_n) < 1/n$, and $t_n \to +\infty$. For each n let $\sigma_n(t) = \gamma_n(t_n - t)$, and $Z_n(t) = Y_n(t_n - t)$. Then Z_n is a perpendicular Jacobi vector field on the geodesic σ_n, and since the function $t \to \|Y_n(t)\|$ is monotone increasing for $t > 0$ and each integer n, it follows that $\|Z_n(t) \leq \|Z_n(0)\|$ for $0 \leq t \leq t_n$. We now proceed exactly as in the latter part of the proof of Proposition 4 to show that there exists a perpendicular Jacobi vector field Z on a unit speed geodesic γ in H such that $Z(0) \neq 0$, $\langle Z(0), Z'(0)\rangle = 0$ and $\|Z(t)\| \leq \|Z(0)\|$ for $t > 0$. This is a contradiction to condition 3).

Proposition 8. *Condition* 6) *implies condition* 2) *in* H.

Proof. There exists a positive number ε with this property: Let Y be a perpendicular Jacobi vector field on a unit speed geodesic γ such that $\|Y(0)\| = 1$ and $\langle Y(0), Y'(0)\rangle = 0$. Then $\|Y(t)\| \geq \frac{1}{2}$ for $0 \leq t \leq \varepsilon$. First, a familiar argument by contradiction and an application of Proposition 1 show that there exists a positive number A, not depending on Y, such that $\|Y(t)\| \leq A$ for $0 \leq t \leq t_0$, where t_0 is the positive constant of condition 6). Next, since H is compactly homogeneous, we can find a positive number R such that the sectional curvature of H satisfies the inequality $K < R$. We obtain the desired constant $\varepsilon > 0$ by choosing ε to be so small that $\varepsilon < t_0$ and $1 - 2RA^2\varepsilon^2 > \frac{1}{4}$. Let $f(t) = \|Y(t)\|^2$. Then $f''(t) = 2\{\|Y'(t)\|^2 - K(Y, \gamma')(t)\|Y(t)\|^2\} \geq -2Rf(t) \geq -2RA^2$ for $0 \leq t \leq \varepsilon$. Since $f'(0) = 0$, it follows that $f'(t) = \int_0^t f''(u)du \geq -2RA^2\varepsilon$ for $0 \leq t \leq \varepsilon$. Therefore $f(t) = 1 + \int_0^t f'(u)du \geq 1 - 2RA^2\varepsilon^2 > \dfrac{1}{4}$ for $0 \leq t \leq \varepsilon$.

There exists a positive constant B with the following property: Let Y be a perpendicular Jacobi vector field on a unit speed geodesic γ in H such that $Y(0) \neq 0$ and $\langle Y(0), Y'(0)\rangle = 0$, and let t_0 be the positive constant of condition 6). Then for any number s such that $0 \leq s \leq t_0$, $\|Y(s)\| \geq B\|Y(0)\|$. If this assertion were false, then for each positive integer n we could find a unit speed geodesic γ_n, a perpendicular Jacobi vector field Y_n on γ_n, and a number t_n such that $\|Y_n(0)\| = 1$, $\langle Y_n(0), Y_n'(0)\rangle = 0$, $\|Y_n(t_n)\| < 1/n$, and $0 \leq t_n \leq t_0$. If ε is the positive constant determined in the previous paragraph, then $t_n > \varepsilon$ for $n \geq 2$. Passing to a subsequence we let t_n converge to a number t^*, where $0 < \varepsilon \leq t^* \leq t_0$. If the sequence $\|Y_n'(0)\|$ contains a bounded subsequence, then an application of Proposition 1 shows that there exists a perpendicular

Jacobi vector field Y on a unit speed geodesic γ such that $\|Y(0)\| = 1$, $\langle Y(0), Y'(0)\rangle = 0$ and $Y(t^*) = 0$. If σ is the maximal geodesic of H tangent to $Y(0)$, then $\gamma(t^*)$ is a focal point of σ along γ since $t^* > 0$. This contradicts the assumption that H has no focal points. Suppose now that $\|Y_n'(0)\| \to \infty$, and define Jacobi vector fields $Z_n(t) = Y_n(t)/\|Y_n'(0)\|$. Then $\|Z_n'(0)\| = 1$, $\langle Z_n(0), Z_n'(0)\rangle = 0$, $\|Z_n(0)\| \to 0$ and $\|Z_n(t_n)\| \to 0$. Another application of Proposition 1 shows that there exists a Jacobi vector field Z on a unit speed geodesic γ such that $\|Z'(0)\| = 1$, $Z(0) = 0$ and $Z(t^*) = 0$. Since $t^* > 0$ and Z is nontrivial, this contradicts the fact that H has no conjugate points. This proves the existence of the constant B.

Let Y be a perpendicular Jacobi vector field on a unit speed geodesic γ in H such that $Y(0) \neq 0$ and $\langle Y(0), Y'(0)\rangle = 0$. By condition 6) and the result of the previous paragraph, $\|Y(t)\| \geq e^{-ct_0/2}e^{ct/2}\|Y(t_0)\| \geq (Be^{-ct_0/2})e^{ct/2}\|Y(0)\|$ for every $t \geq t_0$. Let σ be a maximal geodesic of H, and let $P\colon H \to \sigma$ denote the projection map. Let v be a vector tangent to H. If $\pi(v)$ lies on σ, then $\|dP(v)\| \leq \|v\|$ by the argument of Proposition 3. If $\pi(v)$ does not lie on σ, then let $a = d(\pi v, \sigma)$, and let γ be the unit speed geodesic such that $\gamma(0) = P(\pi v)$ and $\gamma(a) = \pi(v)$. Write $v = v_1 + v_2$, where v_1 is a vector at $\pi(v)$ which is orthogonal to $\gamma'(a)$, and v_2 is a vector at $\pi(v)$ which is a scalar multiple of $\gamma'(a)$. If Y is the perpendicular Jacobi vector field on γ such that $Y(0) = dP(v) = dP(v_1)$ and $Y(a) = v_1$, then $\langle Y'(0), Y(0)\rangle = 0$ by Lemma 2. If $a \leq t_0$, then $\|dP(v)\| = \|Y(0)\| \leq (1/B)\|Y(a)\| = (1/B)\|v_1\| \leq (1/B)\|v\|$. If $a \geq t_0$, then $\|dP(v)\| \leq [(1/B)e^{ct_0/2}]e^{-ca/2}\|v_1\| \leq [(1/B)e^{ct_0/2}]e^{-ca/2}\|v\|$. It follows that condition 2) is satisfied.

In the Propositions 4 through 8 we proved the assertion 5) implies 6), 6) implies 5), 4) implies 6), 1) implies 3), 3) implies 5), and 6) implies 2). That the assertion 2) implies 1) and that 6) implies 4) are immediate. Therefore the six conditions are equivalent in manifolds without focal points. The assertion that these conditions imply that the geodesic flow in the unit tangent bundle of M is of Anosov type was proved in Proposition 4.

Proposition 9. *Let M have nonpositive sectional curvature, the simply connected Riemannian covering H of M be compactly homogeneous, and the geodesic flow in the unit tangent bundle of M be of Anosov type. Then the conditions* 1) *through* 6) *are satisfied.*

Proof. We shall prove that conditon 3) is satisfied in H. If condition 3) were not satisfied, then for every positive integer n we could find a unit speed geodesic γ_n in H, a perpendicular Jacobi vector field Y_n on γ_n, and a positive number t_n such that $Y_n(0) \neq 0$, $\langle Y_n(0), Y_n'(0)\rangle = 0$, $\|Y_n(t_n)\| \leq \|Y_n(0)\|$ and $t_n \to +\infty$. Let $f_n(t) = \|Y_n(t)\|^2$. Then $f_n''(t) \geq 0$ for every $t \in R$ since the sectional curvature of H is nonpositive. Since $f_n'(t) = 0$, it follows that $f_n'(t) \geq 0$ for every $t \geq 0$ and every integer n. By assumption $f_n(t_n) \leq f_n(0)$, and therefore we see that $\|Y_n(t)\| = \|Y_n(0)\|$ for $0 \leq t \leq t_n$. Let $\sigma_n(t) = \gamma_n(t + \frac{1}{2}t_n)$, and $Z_n(t) = Y_n(t + \frac{1}{2}t_n)$. Then $\|Z_n(t)\| = \|Z_n(0)\|$ for each number t such that $-\frac{1}{2}t_n \leq t \leq \frac{1}{2}t_n$. Multiplying Z_n by a constant, we may assme that $\|Z_n(0)\|^2 + \|Z_n'(0)\|^2 = 1$ for every n. Applying Proposition 1 we can find a perpendicular Jacobi vector field Z on a unit speed geodesic γ such that $\|Z(0)\|^2 + \|Z'(0)\|^2 = 1$ and $\|Z(t)\| = \|Z(0)\|$ for all $t \in R$. Since Z is nontrivial, $Z(0) \neq 0$. If $f(t) = \|Z(t)\|^2$, then $0 = f''(t) = 2\{\|Z'(t)\|^2 - K(Z, \gamma')(t)\|Z(t)\|^2\} \geq 0$. Therefore Z is a nonzero perpendicular Jacobi vector field on γ such that $Z'(t) = 0$ for every t. This is a contradiction to Corollary 3.3 of [4], and concludes the proof of the theorem.

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