J. DIFFERENTIAL GEOMETRY
8 (1973) 555-564

THE MORSE INDEX THEOREM IN HILBERT SPACE

K. UHLENBECK

When does the critical point of a calculus of variations problem minimize the integral? The classical result is due to Jacobi, who proved that for a regular problem in one independent variable, the integral is minimized at a solution of the Euler-Lagrange equation up to the first conjugate point but not after. Morse extended the theorem to give a formula for the index of a critical curve in terms of the conjugate points along the curve. This result has since been generalized by Edwards [3], Simons [7] and Smale [8] to systems of higher order, minimal surfaces, and partial differential systems respectively. In this article we present an infinite dimensional proof of a general theorem on the index of a bilinear form in Hilbert space which can be applied to all these cases.

The first section contains the abstract formulation and proof of the main theorem (Theorem 1.11). The second section deals with single integral problems and the third with multiple integral problems. In the applications we assume less differentiability than the previous results.

1. The abstract theorem

Let $H_0 \subset H_t \subset H_1 = H$ be an increasing family of closed Hilbert spaces in H for $0 \leq t \leq 1$, and $A: H \to R$ be a C^2 function on H with 0 as a critical point. Clearly 0 is also a critical point of $A \mid H_t = A_t$. The Hessian of A at 0 is the bilinear form

$$B = d^2A(0): H \otimes H \to R .$$

Also the Hessian of A_t at 0 is $B_t = B \mid H_t \otimes H_t$.

We will be concerned with the properties of B and B_t only, so that we shall assume that $A(v) = \frac{1}{2}B(v, v)$. We recall that the *index* of 0 as a critical point of A is the dimension of any maximal subspace on which $B(v, v) < 0$ for $v \neq 0$. We define the two functions:

$i(t)$ = index of A_t = dimension of the maximal subspace of H_t on which B_t is negative,

$j(t)$ = dimension of the maximal subspace on which B_t is nonpositive =

Communicated by R. S. Palais, April 24, 1972.

codimension of the closure of a maximal subspace on which B_t is positive.

It is clear that $i(t) \leq j(t)$ from the definitions.

The bilinear form B_t induces a linear map $\underline{B}_t \in L(H_t, H_t^*)$ in the usual way. $B_t(u, v) = \underline{B}_t(u) \cdot v$. Let $N_t \in H_t$ denote the null space of this linear map. Then

$$\dim (N_t) = n(t) = j(t) - i(t) .$$

Lemma 1.1. *$i(t)$ and $j(t)$ are increasing functions of t.*

Proof. Let K_t be a maximal subspace on which B_t is negative definite. Since $K_t \subset H_t \subset H_{t+k}$, B_{t+k} is negative on K_t, and K_t can be enlarged to a maximal negative subspace for H_{t+k}. $i(t) = \dim K_t \leq i(t + k)$. A similar argument holds for $j(t)$.

Lemma 1.2. *If $j(1) < \infty$, then there exists a finite number of points $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $i(t)$ and $j(t)$ are constant on the open intervals (t_i, t_{i+1}).*

Proof. Both $i(t)$ and $j(t)$ are increasing integer-valued functions of t, and each can have at most $j(1)$ points of discontinuity since $j(1) < \infty$. At worst the points of discontinuity are separate, in which case $n = 2j(1)$.

For the rest of the section we assume that $j(1) < \infty$ and that the discontinuities of $i(t)$ and $j(t)$ occur at t_i, $0 \leq i \leq n$.

Definition 1.3. t is a *conjugate point* if $j(t) > i(t)$. The *degree of conjugacy of* t is $n(t) = j(t) - i(t) =$ dimension of the null space of $\underline{B}_t$. It follows that $n(t)$ is constant on the intervals (t_i, t_{i+1}).

We wish to give conditions on the family H_t and the functionals B_t such that $n(t) = 0$ on (t_i, t_{i+1}), $i(t)$ is lower semi-continuous and $j(t)$ is upper semi-continuous. It then follows that

$$i(1) - i(0) = \sum_{t \in [0,1)} n(t) .$$

Definition 1.4. B satisfies the *unique continuation property* with respect to the family H_t if $N_t \cap N_k = 0$ for $t \neq k$. (Recall that N_t is the null space of $\underline{B}_t$.)

Proposition 1.5. *If B has the unique continuation property with respect to the family H_t, then $n(t) = 0$ for $t \in (t_i, t_{i+1})$.*

Proof. Suppose the conclusion of the theorem is false. Choose an element $e \in N_t$. Let E^- be a maximal negative subspace for B_t, and E^+ be the perpendicular subspace under B in H_k, $t_i < t < k < t_{i+1}$. Then the minimum of B_k on the subspace E^+ is zero. $B(e, e) = 0$ and B takes on its minimum at e. It follows that $B(e, v) = 0$ for all $v \in E^- \oplus E^+ = H_k$. $e \in N_k$ violates the unique continuation hypothesis unless $e = 0$.

Definition 1.6. A bilinear form B on H is *Fredholm* if the associated linear transformation $\underline{B} : H \to H^*$ is a Fredholm map. Recall that a linear transformation is Fredholm if it has finite dimensional kernel and finite dimensional cokernel. Note that the index of $\underline{B}$ (as a Fredholm map) is 0, and should not be confused with the index of the bilinear form B.

If a bilinear form is Fredholm, a canonical form similar to the form for finite dimensional spaces exists. There exist an inner product $\langle\, ,\rangle$ on H and orthogonal projections P_- and P_0 with $P_-P_0 = P_0P_- = 0$, $P_-(H)$ a maximal negative subspace of $B, P_0(H)$ the null space of B, and [6]

$$B(e,f) = \langle e,f\rangle - 2\langle P_-e,f\rangle - \langle P_0e,f\rangle \,. \tag{1.7}$$

Lemma 1.8. *If B is Fredholm of finite index, then B_t is Fredholm.*

Proof. We use the existence of the canonical form (1.7). If we identify H and H^* by means of the inner product, we have the map $\underline{B}\colon H \to H^* \simeq H$ given by

$$\underline{B} = I - (2P_- + P_0) \,.$$

Let P_t be the orthogonal projection on the closed space H_t. Since $H_t \approx H_t^*$ by the same inner product,

$$\underline{B}_t = P_t(I - 2P_- + P_0)\,|\,H_t = I - P_t(2P_- + P_0)\,|\,H_t \,.$$

P_- and P_0 are projections on finite dimensional spaces, so $K = P_t(2P_- - P_0)$ has finite dimensional range. Therefore $B_t = I - K$ is Fredholm.

Lemma 1.9. *If $\overline{\bigcup_{t<k} H_t} = H_k$ and B has finite index, then $i(t)$ is upper semi-continuous.*

Proof. Let $\{e_l\}, l = 1, 2, \cdots, i(k)$, be a basis for a maximal negative subspace of H_k. Choose $f_{l,t} \in H_t$ with $\lim_{t\to k} f_{l,t} = e_l$. Since B is a continuous map, $\{f_{l,t}\}$ are linearly independent and lie in a negative subspace of B_t if t is close enough to k. So $i(t) \geq i(k)$ if $k - t > 0$ is sufficiently small. This argument does not apply if the index is not finite.

Lemma 1.10. *If B is Fredholm of finite index and $H_k = \bigcap_{t>k} H_t$, then $j(t)$ is upper semi-continuous.*

Proof. Let E be a maximal nonpositive subspace for H_k. We suppose that $j(t)$ is not upper semi-continuous at k, so there exists $e_t \in H_t$, $t > k$ such that $\{E, e_t\}$ span a subspace larger than E on which B_t is nonpositive. We may assume $B(f, e_t) = 0$ for $f \in E$.

Let $\langle\, ,\rangle$ be an inner product with the properties defined in (1.7), and normalize it so that $\langle e_t, e_t\rangle = 1$. P_- and P_0 are projections on finite dimensional subspaces. Thus we may select a subsequence $e_{t(i)}$, $\lim_{i\to\infty} t(i) = k$, such that $e_{t(i)}$ converges weakly to $e \in H_k$, $P_-e_{t(i)}$ converges to P_-e and $P_0e_{t(i)}$ converges to P_0e. From (1.7) it follows that

$$0 \geq B(e_t, e_t) = \langle e_t, e_t \rangle - 2\langle P_- e_t, e_t \rangle - \langle P_0 e_t, e_t \rangle ,$$

and therefore that

$$2\langle P_- e, e \rangle + \langle P_0 e, e \rangle \geq 1 .$$

We find that $e \neq 0$ and $\langle e, e \rangle \leq 1$, so $B(e, e) \leq 0$. $\{E, e\}$ now span a larger subspace on which B_k is nonpositive, which is a contradiction. Therefore $j(t)$ must be upper semi-continuous.

The following main theorem follows directly from (1.5),(1.9) and (1.10).

Theorem 1.11. *Let B be a bilinear form on a Hilbert space H, and $H_0 \subset H_t \subset H_1 = H$, $0 \leq t \leq 1$, an increasing family of closed Hilbert spaces. If*

(i) *B satisfies the unique continuation property,*

(ii) *B is Fredholm of finite index,*

(iii) $\overline{\bigcup_{t<k} H_t} = H_k = \bigcap_{t>k} H_t$,

then there is only a finite number of conjugate points where $n(t) \neq 0$ and index B − index $B_0 = \sum_{0 \leq t < 1} n(t)$.

2. Applications to single integrals

In this section we consider the bilinear form

$$B(f, g) = \sum_{i=1}^{k} \sum_{j=1}^{k} \int_0^1 f^{(i)}(x) \cdot A_{ij}(x) g^{(j)}(x)\, dx . \tag{2.1}$$

If the $A_{ij}: [0, 1] \to L(R^m, R^m)$, $A_{ij}(x)$ are matrices with boudded measurable entries, and $A_{ij}(x) = A_{ji}(x)^*$, then B is defined and symmetric for $f, g \in H_{k,0}([0, 1], R^m)$, the Sobolev space of vector-valued functions on the interval $[0, 1]$ with k square-integrable derivatives and $k - 1$ derivatives which are zero at 0 and 1. We will make use of the inner product and norms

$$\langle f, g \rangle = \int_0 f(x) \cdot g(x) dx , \qquad \|f\|_k^2 = \sum_{j=0}^{k} \int_0^1 |f^{(j)}(x)|^2\, dx .$$

Thus

$$B(f, g) = \langle f, Lg \rangle, \quad \text{where} \quad L = \sum_{j=1}^{k} \sum_{i=1}^{k} (-1)^j \left(\frac{d}{dx}\right)^j A_{ji}(x) \left(\frac{d}{dx}\right)^i . \tag{2.2}$$

In applying the result of § 1 to the index of the form (2.1) we let $H_t = H_{k,0}([0, t], R^m) \subset H_{k,0}([0, 1], R^m) = H_1$. Here we are considering a function in H_t, which is naturally defined on the interval $[0, t]$, to be a function on $[0, 1]$ by extending it to be identically 0 for $x \geq t$. It is easy to see then that $\overline{\bigcup_{t<k} H_t} = H_k = \bigcap_{t>k} H_t$.

The form B_t on $H_{k,0}([0, t], R^n)$ is associated with the operator

$$L_t : H_{k,0}([0, t], R^m) \to H_{-k}([0, t], R^m) ,$$

where L_t has the same formal definition but different domain from L given in (2.2). The null space of B_t and that of the differential operator L_t are the same, so that the following definition of conjugate point agrees with (1.3).

Definition 2.3. If $L_t f = 0$ has a nonzero solution in $H_{k,0}([0, 1], R^m)$, then t is called a *conjugate point of multiplicity* $n(t)$ equal to the dimension of the solution space.

Theorem 2.4. *Let B be given as in* (2.1). *If* (i) $A_{ij}(x) = A_{ji}(x)^*$ *are matrices with bounded measurable coefficients,* (ii) $A_{kk}(x) \geq \varepsilon I$ *uniformly on* $0 \leq x \leq 1$, *then there is a finite number of conjugate points of B on the interval* $[0, 1]$, *and*

$$\operatorname{index} B = \sum_{0<t<1} n(t) .$$

Proof. The steps in the proof which we have not discussed are first that B is Fredholm of finite index and B_ε has zero index for small ε, and secondly that B satisfies unique continuation. Once we show that B satisfies these two properties we can apply Theorem 1.11 to get the result.

Because $A_{kk}(x) \geq \varepsilon I$ for $0 \leq x \leq 1$, we have the inequality that for some $N < \infty$ and all $f \in H_{k,0}([0, 1], R^m)$

$$B(f, f) \geq \tfrac{1}{2}\varepsilon \, \|f\|_k^2 - N \, \|f\|_{k-1}^2 .$$

Since the imbedding of H_k in H_{k-1} is completely continuous, it follows that B is Fredholm of finite index. A scale change shows that if f has support in $0 < x < t$, then $N = N(t)$ can be chosen as $N_0 - Ct^{-2k/(k-1)}$, so $B(f, f) > 0$ if $f \in H_{k,0}[0, \varepsilon]$.

The system L can be transformed into a first order system with bounded measurable coefficients. Let Z_i, $i = 1, \cdots, 2k$, be vector valued functions

$$Z_1 = Y , \quad \text{and} \quad Z_{i+1} = Z_i' \quad \text{for} \quad 1 \leq i \leq k-1 .$$

Then

$$(-1)^k Z_{i+1} = A_{kk} Z_k' + \sum_{j=1}^{k} A_{k,j-1} Z_j ,$$

$$Z_{k+i+1} = Z_{k+i}' + (-1)^{k-i} \left\{ \sum_{j=1}^{k} A_{k-1,j-1} Z_j + A_{k-i,k} A_{kk}^{-1} \left(Z_{k+1} - \sum_{j=1}^{k} A_{k-j-1} Z_j \right) \right\} ,$$

$$0 = Z'_{2k} + \sum_{j=1}^{k} A_{0,j-1}Z_j + A_{0,k}A_{kk}^{-1}\Big(Z_{k+1} - \sum_{i=1}^{k} A_{k,i-1}Z_i\Big) .$$

The last $2k$ equations can be made into a system. Since the uniqueness proof for given initial values applies to this system, it must apply to L itself. If $L_t f = 0$, then $f \in H_{k,0}([0, t], R^m)$ by definition of L_t. If $L_k f = 0$ also, then $f(x) = 0$ for $t \leq x \leq k$. Since f has zero initial data at t, f must be identically zero. So the null spaces of L_t and L_k, and therefore the null spaces of B_t and B_k, have zero intersection.

3. Applications to multiple integrals

Let Ω be a compact manifold (possibly with boundary) and L an $s \times s$ elliptic system of order k on Ω. We assume that in local coordinates L has the form

$$L = \sum_{|\alpha| \leq k} \sum_{|\beta| \leq k} D^\alpha A_{\alpha,\beta}(x) D^\beta ,$$

where the $A_{\alpha,\beta}: \Omega \to L(R^m, R^m)$, $A_{\alpha,\beta}(x)$ are matrices with bounded coefficients, $A_{\alpha,\beta}(x)$ is continuous if $|\alpha| = |\beta| = k$, and

$$\sum_{|\alpha|=k} \sum_{|\beta|=k} \eta^\alpha \cdot A_{\alpha,\beta}^{(x)} \eta^\beta > 0 \qquad \text{if } \eta \neq 0$$

for all $x \in \Omega$. We would like L to be self-adjoint, so we assume that there exists a measure μ on Ω such that $\langle Lf, g \rangle = \langle f, Lg \rangle$ for all smooth f and g with support in the interior of M. Here $\langle\ ,\ \rangle$ indicates the L_2 inner product

$$\langle f, g \rangle = \int_\Omega f(x) \cdot g(x) d\mu .$$

The Hilbert space we will use for the bilinear form is the Sobolev space $H = H_{k,0}(\Omega, R^m)$ of vector-valued functions with partial derivatives up to order k in $L_2(\Omega)$ and $k - 1$ derivatives which are zero on the boundary of Ω. The symbol $\|\ \|_k$ will be a norm for this space. $B(f, g) = \langle Lf, g \rangle$ is defined for all $f, g \in H$. The following lemma is similiar to Lemma 7 of Smale's paper [8].

Lemma 3.1. *B is a symmetric bilinear form on $H = H_{k,0}(\Omega, R^m)$. If L has the properties described above, then there exist constants ε and N such that*

$$B(f, f) \geq \varepsilon \|f\|_k^2 - N \|f\|_{k-1}^2 , \qquad \text{for all } f \in H .$$

Further, there exists a constant δ such that if the support of f lies in a set of measure less than δ, then N may be taken to be zero.

In this lemma the inequality is Gårding's inequality [1], and the fact that N may be taken to be zero follows from the Sobolev inequality [9]

$$\|f\|^2_{k-1,p} \leq C_1 \|f\|^2_k ,$$

where $p = 2n/(n-2)$ for $n = \dim \Omega$ or any $p < \infty$ for $\dim \Omega = 2$, and $\| \|_{k-1,p}$ is a norm for the Sobolev space of functions with $k-1$ derivatives which are p integrable. Hölder's inequality shows that

$$\|f\|^2_{k-1} \leq C_2 \quad \text{meas}^{2-2\alpha}(\text{support } f) \quad \|f\|^2_{k-1,2/\alpha} ,$$

and the three inequalities can be put together to get

$$B(f, f) \geq \tfrac{1}{2}\varepsilon \|f\|^2_k ,$$

when measure (support $f \leq (\varepsilon/(2NC_1C_2))^{(n-1)/2}$.

In order to apply § 1 to the bilinear form B, we define $H_t = H_{k,0}(\Omega_t, R^m)$ which are those functions in H with support in $\Omega_t \subset \Omega$; the family Ω_t may be constructed as follows:

Let h be a smooth real-valued function on Ω with the properties:

$$0 \leq h(x) \leq 1 , \qquad h(\partial\Omega) \subseteq \{0, 1\} .$$

The critical points of h are nondegenerate and occur in the interior of Ω, and the local maxima and minima occur only at 1 and 0 respectively. We choose $\Omega_t = h^{-1}[0, t]$, and $H_t = H_{k,0}(\Omega_t, R^m)$.

Lemma 3.2. *If h has no strict local maxima, then*

$$\overline{\bigcap_{t<k} H_t} = H_k = \bigcap_{t>k} H_t .$$

This identity is true for $2k \leq \dim \Omega$ even if h has local maxima.

Proof. By definition H_t is the closure in H of smooth functions with support in the interior of Ω_t, so when $\bigcup_{t<k} \Omega_t = \text{interior } \Omega_k \subset \Omega_k = \bigcap_{t>k} \Omega_t$, the identity is immediate. However, if h has local maxima at points $\{x_1, \cdots, x_n\}$ in the interior of Ω for which $h(x_i) = k$, then

$$\bigcup_{t<1} \Omega_t = \text{interior } \Omega_k - \{x_1, \cdots, x_n\} .$$

If $2k > \dim \Omega$, then $H \subset C^0(\Omega)$; for $t < 1$, H_t contains only functions which are zero at $\{x_1, \cdots, x_n\}$, and so the functions in the limit must be zero at $\{x_1, \cdots, x_n\}$. Thus it is clear that the restriction $2k \leq \dim \Omega$ is necessary.

Assume for convenience that the only local maxima occurs at x_1, and choose a coordinate patch with $x_1 = 0$. Let ϕ be a smooth function which is identically 1 outside the coordinate patch and which is zero in a neighborhood of 0. Define

$$f_N(x) = f(x)\phi(Nx) \in H_t \quad \text{for some} \quad t < 1 .$$

If f is a smooth function with support in the interior of Ω, then

$$\|f_N(x) - f(x)\|_k \leq C(N)^{2k-n} ,$$

where C depends on the function f as well as ϕ. If $2k < n$, then $\text{limit}_{N\to\infty} f_N(x) = f(x) \in \overline{\bigcup_{t<1} H_t}$. If $2k = n$, then weak $\text{limit}_{N\to\infty} f_N(x) = f(x)$. Since the subspace $\overline{\bigcup_{t<1} H_t}$ is closed under weak limits, $f \in \overline{\bigcup_{t<1} H_t}$.

Definition 3.3. A differential operator L has the *unique continuation property* on a domain Ω if there are no solutions of $Lu = 0$, on Ω, $u \neq 0$, such that u has support on a domain with closure properly contained in Ω.

There are examples of elliptic operators and systems of operators which violate this condition. $\Delta^3 + B$, where B is of order less than six, can have a solution with support in a compact region of R^n. Much work has been done on this subject, and we refer the reader to [4] for a general discussion. However there are at least two tractibe cases.

Proposition 3.4. *If L is an elliptic system with analytic coefficients, then L has the unique continuation property. If L is any elliptic second order operator with C^2 coefficients, then L has the unique continuation property.*

The first part of this theorem is a result of Holmgren's uniqueness theorem, and a proof of the second can be found in Hörmander [4]. In fact, Hörmander's proof applies to any elliptic system of second order with a symbol which is a scalar. This fact will be useful in dealing with minimal surfaces.

Theorem 3.5. *Let L be a self-adjoint system as described above. If L has the unique continuation on property on Ω and $2k \leq \dim \Omega$, then the index of the form $\langle f, Lg\rangle$ on $H_{k,0}(\Omega, R^m)$ is equal to the number of linearly independent solutions $Lu = 0$ on Ω_t for $u \in H_{k,0}(\Omega_t, R^m)$ on the interval $0 < t < 1$.*

Proof. The assumption of unique continuation assures that condition (i) of Theorem 1.11 holds. Gårding's inequality, which is stated in (3.1), is sufficient to prove that $B(f, g) = \langle f, Lg\rangle$ is Fredholm of finite index, since the inclusion of $H_{k,0}(\Omega, R^m)$ in $H_{k-1,0}(\Omega, R^m)$ is completely continuous. $\lim_{t\to 0} \text{meas}\,(\Omega_t) = \lim_{t\to 0} \text{meas}\,(h^{-1}[0, t]) = 0$, and $B_t = B \mid H_t$ has finite index for small t according to (3.1). The last condition in the hypotheses of (1.11) has been varified in (3.2), so the proof is complete. L may be taken to be an operator on a vector bundle with no change in the proof.

We can make a direct application to minimal surfaces. We assume that $S\colon \Omega \to E$ is a smooth immersed minimal surface in a Riemannian manifold E. S is then a critical point of the area integral $A(S) = \int_\Omega dS^*\mu$, where μ is the volume element of E. There is no difference between this case and the previous case except that Diff (Ω) leaves the integral invariant. However we can still define the index to be the dimension of a maximal subspace in $C_0^1(\Omega, S^*TE)$ on which the second variation of $A(U)$ is negative definite. The null space of the second variation will always have the reparametrizations in it, however it may contain certain elements which are not infinitesimal reparametrizations (for example, S may be contained in a family of minimal surfaces with common boundary).

Definition 3.6. The *multiplicity* of a minimal surface S is defined as the maximal dimension of a subspace $N_0 \subseteq C_0^1(\Omega, S^*TE)$ such that N_0 does not contain any reparametrizations and N_0 lies in the null space of the second variation of the area integral. A minimal surface is *conjugate* if its multiplicity is nonzero.

This is of course the infinitesimal version of saying that S is contained in a smooth n dimensional family of minimal surfaces with common boundary. To procede further we must use the parametrization which is given to us by the fact that S is an immersion.

Theorem 3.7. *Let $S\colon \Omega \to E$ be a smoothly immersed minimal surface, and Ω_t as before. Then the index of S is finite and equal to the number of points $t \in (0, 1)$ counted with multiplicity such that $S_t\colon \Omega_t \to E$ is a conjugate surface.*

Proof. This theorem is proved by showing that the situation is really the situation in Theorem 3.5 in disguise. Let N be the normal bundle to S, so $S(\Omega) \subseteq N \subseteq E$. Every nearby surface to S can be given as the section of the normal bundle N, which incidentally fixes its parametrization. In these coordinates, S is the zero section. We compute in local coordinates:

$$A(U) = \int_\Omega G(dU, U, x) dS^*\mu ,$$

$$\begin{aligned} G(dU, U, x)dS^*\mu &= g(U, x) \text{ Jacobian } (\delta_{ij}, dU^k/dx_j)dx_1 \cdots dx_n \\ &= g(U, x)\Big[1 + \sum_{i,k} (dU^k/dx_j)^2 + 0(|dU|)^4\Big]^{1/2} dx_1 \cdots dx_n \\ &= \sum_{i,k} [g(x, 0) + G_i(x)U^i + G_{ik}(x)U^iU^k \\ &\qquad + \tfrac{1}{2}g(0, x)(dU^k/dx_i)^2 + 0(|u| + |dU|)^3]dx_1 \cdots dx_n \\ &= [\tilde{G}(U, dU) + 0(|u| + |dU|)^3]dS^*\mu . \end{aligned}$$

Here we have brought out the part of the integrand, which is quadratic and is involved in the computation of the second variation. Now, if we apply the results of Theorem 3.5 to the system L on the normal bundle N, which is given in local coordinates by

$$(LU)_k = \sum_i -\frac{\partial}{\partial x_i} g(0, x) \frac{\partial}{\partial x_i} U^k + G_{ik}(x)U^i ,$$

we find that the results apply also to the parametric integral involved in the computation of the minimal surface. In particular, due to the regularity theory for systems of equations, it is irrelevant which space of functions is used to determine the index.

It would be very interesting, although not at all straightforward, to try to apply this theorem to the case of minimal surfaces with singularities in the imbedding.

References

[1] S. Agmon, *Lectures on elliptic boundary value problems,* Van Nostrand, Princeton, 1965.

[2] P. J. Cohen, *The Non-uniqueness of the Cauchy problem,* ONR Technical Report 93, Stanford, 1960.

[3] H. M. Edwards, *A generalized Sturm theorem,* Ann. of Math. **80** (1964) 22–57.

[4] L. Hörmander, *Linear partial differential operators,* Springer, Berlin, 1963, Chapter VIII.

[5] M. Morse, *Introduction to analysis in the large,* Mimeographed notes, Institute for Advanced Study, Princeton, 1947.

[6] J. T. Schwartz, *Nonlinear functional analysis,* Gordon and Beach, New York, 1969, p. 136.

[7] J. Simons, *Minimal varieties in riemannian manifolds,* Ann. of Math. 88 (1969) 62–105.

[8] S. Smale, *On the Morse index theorem,* J. Math. Mech. **14** (1965) 1049–1056.

[9] S. L. Sobolev, *Applications of functional analysis in mathematical physics,* Transl. Math. Monographs, Vol. 7, Amer. Math. Soc., 1963.

UNIVERSITY OF ILLINOIS