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WHEN IS A GEODESIC FLOW OF ANOSOV TYPE? I

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Introduction

The geodesic flow on the unit tangent bundle of a compact surface of constant negative Gaussian curvature is one of the earliest known examples of an ergodic flow. The Anosov flows with an integral invariant on compact Riemannian manifolds form a more general class of ergodic flows which includes the geodesic flow on the unit tangent bundle of a compact Riemannian manifold of arbitrary dimension $n \geq 2$ and negative sectional curvature. Within the class of Anosov flows, the geodesic flows still retain a special importance; Anosov geodesic flows are K-systems, and as such, they satisfy stronger dynamical properties than ergodicity: mixing, for example. It is, therefore, of interest to find geometrical conditions on a compact Riemannian manifold M, which are equivalent to the condition that the geodesic flow on the unit tangent bundle of M be of Anosov type.

Under the hypothesis that M have no conjugate points, we restate the condition that the geodesic flow be of Anosov type in terms of various simple conditions on the Jacobi vector fields on unit speed geodesics of M. The restriction that M have no conjugate points is necessary, for Klingenberg [10] has proved that if M is a compact manifold with Anosov geodesic flow, then M has no conjugate points. Among other results, we prove that the geodesic flow of a compact Riemannian manifold without conjugate points is of Anosov type if and only if there exists no nonzero, perpendicular Jacobi vector field Y on a unit speed geodesic γ of M such that $\|Y(t)\|$ is bounded above for all $t \in R$. We also derive a formulation of the Anosov condition in terms of the growth rate of the function $t \to \|Y(t)\|$, where Y is a perpendicular Jacobi vector field on a unit speed geodesic γ, such that $Y(0) = 0$ and $\|Y'(0)\| = 1$. Our results are sharper if M has no focal points, or if M has nonpositive sectional curvature.

§ 1 contains basic facts. Let M denote any complete Riemannian manifold, and let TM and SM denote the tangent bundle and unit tangent bundle of M respectively. We describe the natural isomorphism between the tangent space $(TM)_v$ and the vector space of all Jacobi vector fields on γ_v, where v is an arbitrary vector in TM, and γ_v is the maximal geodesic of M with initial

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velocity v. This isomorphism provides an equivalent, if not simpler, formulation of the Anosov conditions for the geodesic flow in SM in terms of conditions on the Jacobi vector fields of M.

In § 2 we consider manifolds without conjugate points. If M now denotes a complete Riemannian manifold of dimension $n \geq 2$ without conjugate points, then for any unit vector $v \in SM$, there exists a naturally defined pair of $(n-1)$-dimensional subspaces $X_s(v)$ and $X_u(v)$ of $(SM)_v$. These subspaces have been studied by, among others, Anosov, in the case that the sectional curvature satisfies the condition $K \leq c < 0$, and by L. Green, in the general case of a manifold without conjugate points, but in a context not involving the geodesic flow. Our method of studying these subspaces is a synthesis of these two approaches. We prove in § 3 that if M is compact, then the geodesic flow in SM is of Anosov type if and only if $X_s(v) \cap X_u(v) = \{0\}$ for every vector $v \in SM$. In this case, the vector spaces $X_s(v)$ and $X_u(v)$ are those subspaces of $(SM)_v$ which according to the Anosov conditions are contracted and expanded exponentially by the differential maps of the geodesic flow transformations.

In § 3 we give the definition of Anosov flow, and we state and prove our equivalent formulations of the Anosov conditions for geodesic flows. Our results are valid for certain noncompact, as well as all compact, Riemannian manifolds without conjugate points. The crucial, although very elementary, results of this section are Lemmas 3.11 and 3.12. We conclude the paper with the construction of examples of compact Riemannian manifolds with an Anosov geodesic flow, nonpositive sectional curvature and large open sets on which the sectional curvature is identically zero. We are indebted to H. Karcher for the idea and the hardest part of the construction of these examples. We are also grateful to W. Klingenberg for many pertinent discussions, especially those regarding the relationship between the Anosov conditions for a geodesic flow and the growth rate of the norm of a nonzero, perpendicular Jacobi vector field which vanishes at time $t = 0$.

1. Preliminaries

In this section we sketch some basic material. Throughout let M denote a complete C^∞ Riemannian manifold of dimension $n \geq 2$. Let TM and SM denote respectively the full tangent bundle and unit tangent bundle of M, and π be the natural projection map onto M in either case. For any vector $v \in TM$, let γ_v be the unique maximal geodesic of M such that $\gamma_v'(0) = v$. Let $\langle , \rangle$ denote the inner product on M.

A *complete flow* on a C^∞ manifold N is a homeomorphism of (additive) R into the group of homeomorphisms of N; let $t \to T_t$ denote this homeomorphism. Then the flow is C^k differentiable, $0 \leq k \leq \infty$, if the map $(t, n) \to T_t n$: $R \times N \to N$ is C^k differentiable.

Definition 1.1. For any $t \in R$ we define a map $T_t\colon TM \to TM$ as follows: Given a vector $v \in TM$, let $T_t v = \gamma_v'(t)$, the velocity of γ_v at time t. The collection of maps T_t is called the *geodesic flow* in TM.

The geodesic flow is a complete C^∞ flow in TM, and also in SM, since T_t leaves SM invariant for every $t \in R$. If V denotes the vector field in TM defined by the geodesic flow, then the restriction of V to SM is a tangent vector field on SM.

Definition 1.2. A vector field Y on a maximal geodesic γ of M is a *Jacobi* vector field if

$$Y'' + R_{XY}X = 0$$

where the accent denotes covariant differentiation along γ, X is the velocity vector field of γ, and R is the curvature tensor of M.

A Jacobi field Y is uniquely determined by the values $Y(0)$ and $Y'(0)$. For any maximal geodesic γ, let $J(\gamma)$ be the $2n$-dimensional vector space of Jacobi vector fields on γ, and let $J_0(\gamma) \subseteq J(\gamma)$ be the $(2n-2)$-dimensional subspace of *perpendicular* Jacobi vector fields $Y(\langle Y(t), \gamma'(t)\rangle = 0$ for all $t \in R)$. Let $J = \bigcup_\gamma J(\gamma)$, γ a maximal geodesic of M, and let $J_0 = \bigcup_\gamma J_0(\gamma)$.

Definition 1.3. M is said to have no *conjugate points* if, for any maximal unit speed geodesic γ in M and any nonzero Jacobi vector field Y on γ, $\gamma(t) = 0$ for at most one number $t \in R$.

Let $N \subseteq M$ be a proper C^∞ Riemannian submanifold of M, and γ be a maximal unit speed geodesic of M such that $\gamma'(0)$ is perpendicular to $N_{\gamma(0)}$. A Jacobi vector field Y along γ is an *N-Jacobi* vector field if Y is perpendicular to γ, $Y(0) \in N_{\gamma(0)}$ and $SY(0) - Y'(0)$ is perpendicular to $N_{\gamma(0)}$, where S is the second fundamental form of N at $\gamma(0)$ determined by $\gamma'(0)$. Equivalently, Y is an N-Jacobi vector field if it can be written $Y(u) = dr(\partial/\partial v)(u, 0)$, where $r\colon (-\infty, \infty) \times (-\varepsilon, \varepsilon) \to M$ is a C^∞ variation of the form $r(u, v) = \exp(uZ(v)) = (\pi \circ T_u)Z(v)$, where $Z(v)$ is a C^∞ curve in the unit normal bundle of N such that $Z(0) = \gamma'(0)$. If N is a point, then a Jacobi vector field Y is an N-Jacobi vector field if and only if $Y(0) = 0$ and Y is perpendicular to γ. If N is a maximal geodesic σ and $Y(0) \neq 0$, then a Jacobi vector field Y is an N-Jacobi vector field if and only if $Y(0)$ is tangent to σ, Y is perpendicular to γ, and $\langle Y(0), Y'(0)\rangle = 0$. For an arbitrary proper submanifold $N \subseteq M$ and a perpendicular unit speed geodesic γ, the point $\gamma(a)$, $a \neq 0$, is a *focal point* of N along γ if there exists a nontrivial N-Jacobi vector field Y along γ such that $Y(a) = 0$. See [2] for a more complete discussion.

Definition 1.4. M is said to have no *focal points* if no maximal geodesic $N = \sigma$ has focal points along any unit speed geodesic perpendicular to σ.

The "no focal point" property is equivalent to the following: Let γ be a unit speed geodesic in M, and Y be a not necessarily perpendicular Jacobi vector field on γ such that $Y(0) = 0$ and $Y'(0) \neq 0$. Then for any $t > 0$,

$(\|Y\|^2)'(t) > 0$. Since an arbitrary Jacobi vector field on γ can be written as the sum of a perpendicular Jacobi vector field and a tangential Jacobi vector field, it suffices to prove this assertion for perpendicular Jacobi vector fields Y. This assertion, however, follows easily from the characterization of N-Jacobi vector fields Y, where N is a maximal geodesic and $Y(0) \neq 0$.

By the previous paragraph, M has no conjugate points if it has no focal points. If M has sectional curvature $K \leq 0$, then M has no focal points. Let Y be a Jacobi vector field on a unit speed geodesic γ such that $Y(0) = 0$ and $Y'(0) \neq 0$, and let $f(t) = \|Y(t)\|^2$. Then $f''(t) = 2[\|Y'(t)\|^2 - K(Y,\gamma')(t)\,\|Y \wedge \gamma'\|^2(t)]$. Since $f'(0) = 0$, $f''(0) > 0$ and $f''(t) \geq 0$ for all $t \in R$, it follows that $f'(t) > 0$ for $t > 0$.

We shall describe a natural isomorphism between $(TM)_v$ and $J(\gamma_v)$ for any complete C^∞ Riemannian manifold M and any $v \in TM$. For each $v \in TM$, $d\pi\colon (TM)_v \to M_{\pi v}$ is linear and the kernel of $d\pi$ is the n-dimensional *vertical* subspace of $(TM)_v$. We define a *connection map* $K\colon T(TM) \to TM$ such that for each $v \in TM$, $K\colon (TM)_v \to M_{\pi v}$ is linear. The kernel of K is the n-dimensional *horizontal* subspace of $(TM)_v$, and the intersection of the horizontal and vertical subspaces is the zero vector.

Given a vector $\xi \in (TM)_v$, let $Z\colon (-\varepsilon, \varepsilon) \to TM$ be a C^∞ curve with initial velocity ξ, and let $\alpha = \pi \circ Z\colon (-\varepsilon, \varepsilon) \to M$. We define $K(\xi) = Z'(0) \in M_{\pi v}$, where $Z'(0)$ is the covariant derivative of Z along α evaluated at $t = 0$. $K(\xi)$ does not depend on the curve Z chosen. This definition of K is equivalent to that given in [7], where a more complete description of the map K may be found.

One may define a natural inner product on TM with respect to which the horizontal and vertical subspaces of $(TM)_v$ are orthogonal. Given vectors $\xi, \eta \in (TM)_v$ let $\langle \xi, \eta \rangle_v = \langle d\pi\xi, d\pi\eta \rangle_{\pi v} + \langle K\xi, K\eta \rangle_{\pi v}$. Relative to this inner product we have the following natural result.

Proposition 1.5. *Let $p\colon N \to M$ be a surjective local isometry of complete Riemannian manifolds. Then*

1) *$P = dp\colon TN \to TM$ is a surjective local isometry carrying SN onto SM,*

2) *$\|dT_t\xi\| = \|dT_t dP(\xi)\|$ for any $t \in R$ and any $\xi \in T(TN)$, where T_t denotes the geodesic flow in both TN and TM,*

3) *$dP\, V(v) = V(Pv)$ for any $v \in TN$, where V denotes the vector field defined by T_t in both TN and TM.*

Proof. Define the projection maps $\pi_1\colon TN \to N$, $\pi_2\colon TM \to M$, and the connection maps $K_1\colon T(TN) \to TN$ and $K_2\colon T(TM) \to TM$. The following relations are easily verified: a) $p \circ \pi_1 = \pi_2 \circ P$. b) $P \circ d\pi_1 = d\pi_2 \circ dP$. c) $P \circ K_1 = K_2 \circ dP$. d) $P \circ T_t = T_t \circ P$ for any $t \in R$. From relations b) and c) we see that P is a local isometry. P is surjective since p is surjective, and $P(SN) = SM$ since p is a local isometry. Assertion 2) follows from d) and 1), since $\|dT_t dP(\xi)\| = \|dP \circ dT_t(\xi)\| = \|dT_t\xi\|$. Assertion 3) follows immediately from d).

Definition 1.6. For any $v \in TM$ and any $\xi \in (TM)_v$, let Y_ξ be the unique Jacobi vector field on γ_v such that $Y_\xi(0) = d\pi\xi$ and $Y'_\xi(0) = K\xi$.

Let $r: (-\infty, \infty)\times(-\varepsilon, \varepsilon) \to M$ be the variation defined in the discussion of focal points above, with the difference that now Z is any curve in TM with initial velocity ξ. Then for any $u \in R$, $Y_\xi(u) = dr(\partial/\partial v)(u, 0)$. If $\xi \in (SM)_v$, then we may choose the curve Z to lie in SM; the u-parameter curves of the variation are then unit speed geodesics of M, and by Gauss's Lemma, $u \to \langle Y_\xi(u), \gamma'(u)\rangle$ is a constant function. The following result is now easy to prove; in 5) we also need the fact that $KV(v) = 0$ and $d\pi V(v) = v$ for any $v \in TM$.

Proposition 1.7. *Let $v \in TM$. Then*

1) *$\xi \to Y_\xi$ is a linear isomophism of $(TM)_v$ onto $J(\gamma_v)$,*

2) *$Y_\xi(t) = d\pi \circ dT_t(\xi)$ and $Y'_\xi(t) = K \circ dT_t(\xi)$ for every $t \in R$,*

3) *$\xi \in (TM)_v$ lies in $(SM)_v$ for $v \in SM$ if and only if $\langle K \circ dT_t(\xi), T_t v\rangle = \langle Y'_\xi(t), \gamma'_v(t)\rangle = 0$ for all $t \in R$, if and only if $t \to \langle Y_\xi(t), \gamma'_v(t)\rangle$ is a constant function,*

4) *$\langle \xi, V(v)\rangle = 0$ for $v \in SM$ and $\xi \in (SM)_v$ if and only if $\langle Y_\xi(t), \gamma'_v(t)\rangle = 0$ for all $t \in R$, where V is the flow vector field,*

5) *$\langle dT_t\xi, V(T_t v)\rangle = 0$ for all $t \in R$, if $v \in SM$ and $\xi \in (SM)_v$ satisfy the condition $\langle \xi, V(v)\rangle = 0$.*

Remark 1.8. It follows from 2) above that for any $t \in R$, any $v \in TM$ and any $\xi \in (TM)_v$, $\|dT_t\xi\|^2 = \|Y_\xi(t)\|^2 + \|Y'_\xi(t)\|^2$.

Definition 1.9. Let γ_n be a sequence of geodesics in M, and Y_n be a sequence of Jacobi vector fields such that Y_n is defined on γ_n for every integer n. If $v_n = \gamma'_n(0)$, choose $\xi_n \in (TM)_{v_n}$ so that $Y_n = Y_{\xi_n}$. We say that the Jacobi vecter fields Y_n converge to a Jacobi vector field Y on a geodesic γ if $\xi_n \to \xi$ in $T(TM)$, where $v = \gamma'(0)$, $\xi \in (TM)_v$ and $Y = Y_\xi$.

Note that $\xi_n \to \xi$ in $T(TM)$ if and only if $K\xi_n \to K\xi$ and $d\pi\xi_n \to d\pi\xi$. Therefore Y_n converges to Y if and only if $\gamma'_n(0) \to \gamma'(0)$, $Y_n(0) \to Y(0)$ and $Y'_n(0) \to Y'(0)$. If $Y_n \to Y$, and $u_n \subseteq R$ is a sequence converging to a finite number u, then by 2) of Proposition 1.7, $Y_n(u_n) \to Y(u)$ and $Y'_n(u_n) \to Y'(u)$.

Definition 1.10. M is said to be *compactly homogeneous* if there exists a compact set $B \subseteq M$ such that M is the union of all translates of B by the isometries of M.

If M is homogeneous or a Galois Riemannian covering of a compact Riemannian manifold, then M is compactly homogeneous. Since isometries preserve sectional curvature, all values of the sectional curvature of M are taken on at points of B, and therefore the sectional curvature of M is uniformly bounded above and below. The following result will be used often.

Proposition 1.11. *Let M be compactly homogeneous. For each integer $n > 0$, let Y_n be a Jacobi vector field on a geodesic γ_n with initial velocity v_n. If each of the sequences $\|v_n\|$, $\|Y_n(0)\|$, $\|Y'_n(0)\|$ is uniformly bounded above, then we can find a sequence ϕ_n of isometries of M and a Jacobi vector field*

Z on a geodesic σ, such that by passing to a subsequence, $Z_n = d\phi_n Y_n \to Z$ in the sense of Definition 1.9.

Proof. Let $p_n = \pi v_n$ and choose a sequence ϕ_n of isometries of M so that the sequence $q_n = \phi_n p_n$ is contained in a compact subset of M. $Z_n = d\phi_n Y_n$ is a Jacobi vector field on $\sigma_n = \phi_n \circ \gamma_n$, and $Z'_n(0) = d\phi_n Y'_n(0)$. If $w_n = \sigma'_n(0) = d\phi_n v_n$, then the sequences $\|w_n\|$, $\|Z_n(0)\|$ and $\|Z'_n(0)\|$ are uniformly bounded above. Passing to a subsequence, let $q_n \to q$ in M and let U be a local coordinate system around the point q. For each n, the three vectors w_n, $Z_n(0)$ and $Z'_n(0)$ lie in M_{q_n}, and relative to the induced coordinate system $\pi^{-1}(U)$ in TM, it is easy to see that the coordinates of each of the sequences w_n, $Z_n(0)$ and $Z'_n(0)$ are uniformly bounded in absolute value. Passing to a further subsequence, let $w_n \to w$, $Z_n(0) \to u$ and $Z'_n(0) \to v$, where u, v, w are vectors in M_q. Let $\sigma = \gamma_w$, and let Z be the Jacobi vector field on σ such that $Z(0) = u$ and $Z'(0) = v$. Then $Z_n \to Z$ by the discussion following Definition 1.9.

If ϕ is an isometry of M, then $T_\phi = d\phi$ is an isometry of TM, which leaves SM invariant. The following result is equivalent to that just proved.

Proposition 1.12. *Let M be compactly homogeneous. Let $v_n \subseteq TM$ and $\xi_n \subseteq T(TM)$ be sequences such that $\xi_n \in (TM)_{v_n}$ for every n, and that the sequences $\|v_n\|$, $\|\xi_n\|$ are each uniformly bounded above. Then there exists a sequence ϕ_n of isometries of M such that by passing to a subsequence, $\xi_n^* = dT_{\phi_n}\xi_n$ converges to a vector $\xi^* \in T(TM)$.*

2. Manifolds without conjugate points

In this section we assume that M has no conjugate points. We consider mainly the unit tangent bundle SM rather than TM; all geodesics of M are assumed to have unit speed. For every vector $v \in SM$ we define a pair of $(n-1)$-dimensional subspaces $X_s(v)$, $X_u(v)$ of $(SM)_v$. Relative to the geodesic flow in SM, these subspaces are the candidates for the stable and unstable vector spaces $X_s^*(v)$, $X_u^*(v)$ which appear in the definition of an Anosov flow in § 3. If M is compact with sectional curvature $K < 0$, then the geodesic flow in SM is of Anosov type, where one lets $X_s^*(v) = X_s(v)$ and $X_u^*(v) = X_u(v)$ [1, pp. 182–189]. Conversely, as we shall see, if M is a compact manifold without conjugate points whose geodesic flow in SM is of Anosov type, then $X_s^*(v) = X_s(v)$ and $X_u^*(v) = X_u(v)$. In their dual formulations as spaces of Jacobi vector fields on the geodesic γ_v, the subspaces $X_s(v)$ and $X_u(v)$ have undoubtedly been known for a long time, at least in the case of surfaces of negative curvature.

We first describe a useful method of L. Green [6]. Let γ be a geodesic in M, and let $E_1(s), \cdots, E_n(s)$ be a system of parallel orthonormal vector fields along γ such that $E_n(s) = \gamma'(s)$ for every $s \in R$. (We call such a system an *adapted frame field*). If $Y(s) = \sum_{i=1}^{n-1} y_i(s)E_i(s)$ is a perpendicular vector field on γ, then we identify Y with the curve $s \to (y_1(s), \cdots, y_{n-1}(s)) \in R^{n-1}$. The covariant derivative $Y'(s) = \sum_{i=1}^{n-1} y_i'(s)E_i(s)$ is then identified with the curve $s \to (y_1'(s), \cdots, y_{n-1}'(s))$. Conversely, any curve $s \to (y_1(s), \cdots, y_{n-1}(s)) \in R^{n-1}$ defines a perpendicular vector field on γ. For each $s \in R$ we define a symmetric $(n-1)\times(n-1)$ matrix $R(s) = (R_{ij}(s))$, where $1 \leq i, j \leq n-1$, $R_{ij}(s) = \langle R_{E_n(s)E_i(s)}E_n(s), E_j(s)\rangle$ and R denotes the curvature tensor in M. Consider the $(n-1)\times(n-1)$ matrix Jacobi equation

$$\text{(J)} \qquad Y''(s) + R(s)Y(s) = 0 ,$$

where derivatives are taken componentwise. (For a discussion of second order linear matrix differential equations see [8]). If $Y(s)$ is a solution of (J), then for any $x \in R^{n-1}$ the curve $s \to Y(s)x$ corresponds to a perpendicular Jacobi vector field on γ. If $A(s)$ is the solution of (J) such that $A(0) = 0$ and $A'(0) = I$, then the perpendicular Jacobi vector fields on γ such that $Y(0) = 0$ and $\|Y'(0)\| = 1$ correspond to the curves $s \to A(s)x$, where $x \in R^{n-1}$ is a unit vector. Since M has no conjugate points, $A(s)$ is nonsingular for $s \neq 0$.

For any two solutions X, Y of (J), the Wronskian $W(X, Y)(s) = (X^*)'(s)Y(s) - X^*(s)Y'(s)$ is a constant matrix, where * denotes the transpose operation. If X is a solution of (J) which is nonsingular on an interval (a, b), then $U(s) = X'(s)X^{-1}(s)$ is a solution on (a, b) of the $(n-1)\times(n-1)$ matrix equation

$$\text{(R)} \qquad U'(s) + U(s)^2 + R(s) = 0 .$$

$U(s)$ is symmetric on (a, b) if and only if $W(X, X)(s) \equiv 0$ on (a, b).

For each number $t > 0$ there exists a unique solution D_t of (J), defined for all real numbers s, such that $D_t(0) = I$ and $D_t(t) = 0$. For $s > 0$, $D_t(s) = A(s)\int_s^t A^{-1}(u)A^{-1}(u)^*du$, where the integration is performed componentwise and $A(s)$ is the solution of (J) defined above. To see this, let $B_t(s)$ be the integral expression given above, for every $s > 0$. $B_t(s)$ is a solution of (J) defined for $s > 0$. If D_t is the solution of (J) such that $D_t(t) = B_t(t) = 0$ and $D_t'(t) = B_t'(t) = -A^{-1}(t)^*$, then $D_t(s) = B_t(s)$ for $s > 0$, and a Wronskian argument involving $W(A, D_t)$ shows that $D_t(0) = I$. For any numbers $0 < d < t$ and all $s > 0$, we have $D_t(s) - D_d(s) = A(s)\int_d^t A^{-1}(u)A^{-1}(u)^*du$, and the matrices $D_t'(0) - D_d'(0) = \int_d^t A^{-1}(u)A^{-1}(u)^*du$ are positive definite, symmetric and "monotone increasing" in t. Green [6] shows that $\lim_{t\to+\infty} D_t'(0) - D_d'(0)$ exists, and therefore that $\lim_{t\to+\infty} D_t'(0)$ exists. If $D(s)$ is the solution of (J) such that $D(0) = I$ and $D'(0) = \lim_{t\to+\infty} D_t'(0)$, then for any real number s, $D_t(s) \to D(s)$ as $t \to +\infty$, since solutions of (J) depend continuously on the initial conditions. Therefore, for $s > 0$, $D(s) = A(s)\int_s^\infty A^{-1}(u)A^{-1}(u)^* du$ (improper Riemann integral). $D(s)$ is clearly nonsingular for all $s > 0$ (in fact, for all s, as is shown in [6]).

Next, for any number $t \neq 0$, we define a linear map $\xi \to \xi_t : (TM)_v \to (TM)_v$ for every $v \in TM$. Given a vector $v \in TM$ and a vector $\xi \in (TM)_v$, let $\xi_t \in (TM)_v$ be the unique vector such that $d\pi(\xi_t) = d\pi(\xi)$ and $d\pi \circ dT_t(\xi_t) = 0$. If $Y = Y_\xi$ is the corresponding Jacobi vector field on γ_v, let Y_t be the unique Jacobi vector field on γ_v such that $Y_t(0) = Y(0)$ and $Y_t(t) = 0$. Then ξ_t corresponds to Y_t with respect to the isomorphism of Proposition 1.7. The kernel of the map $\xi \to \xi_t$ is the vertical subspace of $(TM)_v$. For $v \in SM$ it is not true in general that this map leaves $(SM)_v$ invariant; for $\xi \in (SM)_v$, $\xi_t \in (SM)_v$ if and only if $\langle \xi, V(v) \rangle = 0$. This assertion follows easily from Proposition 1.7 (3) and the fact that a Jacobi vector field which is perpendicular at two points is perpendicular everywhere.

Definition 2.1. For every $v \in SM$ let $X_s(v) = \{\xi \in (SM)_v$ such that $\langle \xi, V(v) \rangle = 0$ and $\xi_t \to \xi$ as $t \to +\infty\}$. Let $X_u(v) = \{\xi \in (SM)_v$ such that $\langle \xi, V(v) \rangle = 0$ and $\xi_t \to \xi$ as $t \to -\infty\}$.

Definition 2.2. Let γ be a unit speed geodesic in M with initial velocity v, and $J_s(\gamma)$, $J_u(\gamma)$ be respectively the images in $J(\gamma)$ of the sets $X_s(v)$, $X_u(v)$ under the isomorphism of Proposition 1.7.

$J_s(\gamma)(J_u(\gamma))$ may also be characterized as the set of all perpendicular Jacobi vector fields Y on γ such that $Y_t \to Y$ as $t \to +\infty (t \to -\infty)$, where Y_t is defined in the discussion above. $X_s(v)$, $X_u(v)$ are called the *stable* and *unstable subspaces* determined by v, and $J_s(\gamma)$, $J_u(\gamma)$ are called the *stable* and *unstable subspaces* of perpendicular Jacobi vector fields along γ. Propositions 2.4 and 2.6 show that these sets are vector spaces of dimension $n - 1$.

Remark 2.3. 1) Let γ be a unit speed geodesic in M. If M has sectional curvature $K \equiv 0$, then $J_s(\gamma) = J_u(\gamma) =$ the vector space of all perpendicular parallel vector fields on γ. At the other extreme, if $K \equiv -1$, then $J_s(\gamma) \cap J_u(\gamma) = \{0\}$. It will follow by Theorem 3.2 that for a compact manifold M without conjugate points the geodesic flow in SM is of Anosov type if and only if $J_s(\gamma) \cap J_u(\gamma) = \{0\}$ for every unit speed geodesic γ in M.

2) If $p: N \to M$ is a surjective local isometry of complete Riemannian manifolds, and M has no conjugate points, then N has no conjugate points, and $dPX_s(v) = X_s(Pv)$, $dPX_u(v) = X_u(Pv)$, for every $v \in SN$, where $P = dp: TN \to TM$. N has no conjugate points since for any Jacobi vector field Y on a geodesic γ in N, PY is a Jacobi vector field on the geodesic $p \circ \gamma$ in M. If $\pi_1 : TN \to N$ and $\pi_2 : TM \to M$ are the projection maps, then $p \circ \pi_1 = \pi_2 \circ P$. Also, $P \circ T_t = T_t \circ P$ for every $t \in R$, where T_t is the geodesic flow in both TN and TM. From these two relations if follows that $(dP\xi)_t = dP(\xi_t)$ for any $t \neq 0$, any $v \in TN$, and any $\xi \in (TN)_v$. If $\langle \xi, V(v) \rangle = 0$, then $\langle dP\xi, V(Pv) \rangle$

$= 0$ by Proposition 1.5. Therefore $dPX_s(v) = X_s(Pv)$ and $dPX_u(v) = X_u(Pv)$.

Proposition 2.4. 1) *For every $v \in SM, X_s(v)$ and $X_u(v)$ are vector subspaces of $(SM)_v$.*

2) *If $S: SM \to SM$ is the map which takes a vector v into $-v$, then $X_u(-v) = dSX_s(v)$ and $X_s(-v) = dSX_u(v)$.*

3) *For any $t \in R$ and any $v \in SM$, $dT_tX_s(v) = X_s(T_tv)$ and $dT_tX_u(v) = X_u(T_tv)$.*

Proof. 1) is a direct consequence of the linearity of the map $\xi \to \xi_t: (TM)_v \to (TM)_v$ for any $v \in TM$ and any $t \neq 0$.

2) For any $\xi \in (TM)_v$ and any $t \neq 0$, $dS(\xi_t) = (dS\xi)_{-t}$; this fact is an easy consequence of the definitions and the relations $\pi \circ S = \pi$, $S \circ T_a = T_{-a} \circ S$ for every $a \in R$. The desired conclusion follows.

3) To prove this assertion we need the following:

Lemma 2.5. *Let $v \in SM$ be given. Then there exist numbers $b = b(v) > 0$ and $t_0 = t_0(v) > 0$ such that if $\xi \in (TM)_v$ satisfies $d\pi \circ dT_t(\xi) = 0$ for some $t \geq t_0$, then $\|K\xi\| \leq b\,\|d\pi\xi\|$.*

Proof. We may assume that $d\pi\xi \neq 0$, or otherwise the Jacobi vector field $Y_\xi \equiv 0$ (since it vanishes at 0 and $t \geq t_0 > 0$), and $K\xi = Y_\xi'(0) = 0$. We first consider the case where Y_ξ is perpendicular to γ_v. Relative to the matrix equation (J) defined by an adapted frame field along γ_v, let D_t be the unique solution of (J) such that $D_t(0) = I$ and $D_t(t) = 0$, where $t > 0$. If $d\pi \circ dT_t\xi = 0$ for some $t > 0$, then $Y = Y_\xi$ corresponds to the curve $s \to D_t(s)x$ for some vector $x \in R^{n-1}$, and $\|K\xi\| = \|Y_\xi'(0)\| = \|D_t'(0)x\| \leq \|D_t'(0)\|_\infty \|x\| = \|D_t'(0)\|_\infty \|d\pi\,\xi\|$, where $\|\cdot\|_\infty = \sup\{\|\cdot(x)\|: \|x\| = 1\}$. Since $D_t'(0) \to D'(0)$ as $t \to +\infty$ by the discussion at the beginning of § 2, we may choose $t_0 = t_0(v) > 1$ so that $\|D_t'(0)\|_\infty \leq 1 + \|D'(0)\|_\infty$ for $t \geq t_0$. If $b = 1 + \|D'(0)\|_\infty$, and $d\pi \circ dT_t(\xi) = 0$ for some $t \geq t_0$, then $\|K\xi\| \leq b\,\|d\pi\xi\|$.

If Y is an arbitrary Jacobi vector field on γ, we may write $Y(s) = Y_1(s) + Y_2(s)$, where Y_1 is a perpendicular Jacobi vector field, and Y_2 is a tangential Jacobi vector field of the from $Y_2(s) = (\alpha s + \beta)\gamma'(s)$ for suitable constants α and β. If $Y(t) = 0$ for some $t \geq t_0$, then $Y_1(t) = 0$, $Y_2(t) = 0$, and hence $\alpha = -\beta/t$. $\|Y_2'(0)\| = |\alpha| \leq |\beta| = \|Y_2(0)\|$ and $\|Y_1'(0)\| \leq b\,\|Y_1(0)\|$. Since $b \geq 1$ it follows that $\|K\xi\| = \|Y'(0)\| \leq b\,\|Y(0)\| = b\,\|d\pi\xi\|$.

We now complete the proof of 3). Let $v \in SM$ and $\xi \in (SM)_v$ be given, and fix a number $a \in R$. For any $t \neq 0$, $\|dT_a\xi - (dT_a\xi)_t\| \leq \|dT_a(\xi - \xi_{t+a})\| + \|dT_a\xi_{t+a} - (dT_a\xi)_t\|$. If $\psi_t = dT_a(\xi_{t+a}) - (dT_a\xi)_t$, then $d\pi \circ dT_t\psi_t = 0$ and $d\pi\psi_t = d\pi \circ dT_a(\xi_{t+a} - \xi)$. If $t \geq t_0 = t_0(T_av) > 0$, then the previous Lemma implies that $\|K\psi_t\| \leq b\,\|d\pi\psi_t\|$, where $b = b(T_av) > 0$ is independent of t and ψ_t. Therefore $\|\psi_t\| \leq (1 + b^2)^{1/2}\|d\pi \circ dT_a(\xi_{t+a} - \xi)\| \leq (1+b^2)^{1/2}\|dT_a(\xi_{t+a}-\xi)\|$, and $\|dT_a\xi-(dT_a\xi)_t\| \leq [1+(1+b^2)^{1/2}]\|dT_a(\xi_{t+a}-\xi)\|$ for $t \geq t_0$. If $\langle \xi, V(v) \rangle = 0$, then $\langle dT_a\xi, V(T_av) \rangle = 0$ by Proposition 1.7 (5). Since $a \in R$ is fixed, $\xi \in X_s(v)$ if and only if $dT_a\xi \in X_s(T_av)$. This fact and 2) of this Proposition imply the similar invariance relation for X_u.

Proposition 2.6. *Let there be given a point $p \in M$ and a unit vector $v \in M_p$, the tangent space to M at p. For each vector w in M_p, which is orthogonal to v, there exists a unique vector $\xi_w \in X_s(v)$ (respectively $X_u(v)$) such that $d\pi\xi_w = w$, and the map $w \to \xi_w$ is a linear isomorphism of the orthogonal complement of v in M_p onto $X_s(v)$ (respectively $X_u(v)$).*

Proof. We first prove the assertion for $X_s(v)$. Since $X_s(v)$ is a vector space, the uniqueness part of the assertion will follow when we show that $\xi = 0$ if $\xi \in X_s(v)$ and $d\pi\xi = 0$. Let $\xi \in X_s(v)$ satisfy the condition $d\pi\xi = 0$. For any $t > 0$, $d\pi(\xi_t) = d\pi\xi = 0$ and $d\pi \circ dT_t(\xi_t) = 0$. Therefore $\xi_t = 0$ since Y_{ξ_t} vanishes at 0 and t, and $\xi = 0$ since $\xi_t \to \xi$ as $t \to +\infty$.

To prove the existence of ξ_w, where $w \in M_p$ is orthogonal to v, we construct an adapted frame field on the geodesic γ_v. Relative to equation (J), let D, D_t be the solutions of (J) defined earlier, where $t > 0$. If w corresponds to $x \in R^{n-1}$ relative to the frame field, let $Y(s) = D(s)x$. Then $Y_t(s) = D_t(s)x$. Since $D_t'(0) \to D'(0)$ as $t \to +\infty$, $Y_t \to Y$ as $t \to +\infty$ by the criteria following Definition 1.9. Therefore $Y \in J_s(\gamma)$, and if $\xi \in (TM)_v$ corresponds to Y, then $\xi \in X_s(v) \subseteq (SM)_v$ since Y is perpendicular. Finally we see that $d\pi\xi = Y(0) = w$, identifying the tangent vector w with the corresponding vector $x \in R^{n-1}$. To prove the assertion for $X_u(v)$, it suffices to note that by 2) of Proposition 2.4, $\eta \in X_u(v)$ if and only if $\xi = dS(\eta) \in X_s(-v)$, and $d\pi\eta = d\pi\xi$ since $\pi \circ S = \pi$.

It follows from this Proposition that $X_s(v)$ and $X_u(v)$ both have dimension $n-1$. An interesting and basic question is whether the subspaces $X_s(v)$, $X_u(v)$ depend continuously on the vector v, that is, whether the sets $A_s = \bigcup_{v \in SM} X_s(v)$ and $A_u = \bigcup_{v \in SM} X_u(v)$ are closed subsets of $T(SM)$. If M has no focal points, we shall see that these sets are closed, but it is not clear that this is true with only the nonconjugacy hypothesis. In Proposition 2.13 we derive a sufficient condition for the sets A_s, A_u to be closed.

We now derive some results which will be useful in the next section.

Proposition 2.7. *Let the sectional curvature of M satisfy the relation $K > -k^2$ for some $k > 0$, and Y be a perpendicular Jacobi vector field on a unit speed geodesic γ in M such that $Y(0) = 0$. Then $\|Y'(s)\| \leq k \coth(ks) \|Y(s)\|$ for every $s > 0$.*

Proof. We shall first need the following:

Lemma 2.8. *For any integer $n \geq 2$ consider the $(n-1) \times (n-1)$ matrix Riccati equation*

$$\text{(R)} \qquad U'(s) + U(s)^2 + R(s) = 0\,,$$

where $R(s)$ is a symmetric matrix such that $\langle R(s)x, x\rangle > -k^2$ for some $k > 0$, all unit vectors $x \in R^{n-1}$ and all real numbers s. If $U(s)$ is a symmetric solution of (R), which is defined for all $s > 0$, then $|\langle U(s)x, x\rangle| \leq k \coth(ks)$ for all $s > 0$ and all unit vectors $x \in R^{n-1}$.

Proof. The proof is a slight modification of that of Lemma 3 in [6]. Let $U(s)$ be a symmetric solution of (R) defined for $s > 0$. For a fixed number $s_0 > 0$ and a fixed unit vector $x \in R^{n-1}$, choose a number $d > 0$ such that $\langle U(s_0)x, x\rangle < k \coth (ks_0 - d)$. We show that $\langle U(s)x, x\rangle < k \coth (ks - d)$ for $s \geq s_0$. The matrix $V(s) = k \coth (ks - d)I$ is a solution of the $(n-1)\times(n-1)$ matrix equation

$$V'(s) + V(s)^2 - k^2 I = 0$$

defined for $s > d/k > 0$. If $f(s) = \langle [U(s) - V(s)]x, x\rangle = \langle U(s)x, x\rangle - k \coth (ks - d)$, then $f(s_0) < 0$. If $f(s) = 0$ for some $s > s_0$, let s_1 be the first such value; one then shows as in the proof of Lemma 3 of [6] that $f'(s_1) < 0$, a contradiction.

Given a unit vector $x \in R^{n-1}$ and an integer $n > 0$, choose a number $d_n > 0$ such that $\langle U(1/n)x, x\rangle < k \coth (k/n - d_n)$. If $s > 0$ is given, then $s > 1/n$ for sufficiently large n, and the above argument shows that $\langle U(s)x, x\rangle < k \coth (ks - d_n)$. Since $d_n < k/n$, $d_n \to 0$ and we have $\langle U(s)x, x\rangle \leq k \coth (ks)$. One next proves as in Lemma 2.1 of [5] that $\langle U(s)x, x\rangle \geq -k$ for any $s > 0$ and any unit vector x. Since s and x are arbitrary, the Lemma is proved.

We conclude the proof of Proposition 2.7. Let γ be a unit speed geodesic in M. Relative to an adapted frame field along γ we consider the solution $A(s)$ to equation (J) discussed at the beginning of § 2. The curvature condition $K > -k^2$ is precisely the condition that $\langle R(s)x, x\rangle > -k^2$ for all unit vectors $x \in R^{n-1}$ and all real numbers s. Let Y be a perpendicular Jacobi vector field on γ such that $Y(0) = 0$. We may assume that $Y'(0) \neq 0$, since otherwise there is nothing to prove, and it suffices to consider the case where $\| Y'(0)\| = 1$. For some unit vector $x \in R^{n-1}$, Y may be identified with the curve $s \to A(s)x$ and Y' with the curve $s \to A'(s)x$. If $U(s) = A'(s)A^{-1}(s)$, then $U(s)$ is defined for $s > 0$ and is a solution to equation (R) of the previous Lemma. $U(s)$ is symmetric for $s > 0$ since $W(A, A)(s) \equiv W(A, A)(0) = 0$. For any $s > 0$, $\| Y'(s)\| = \|A'(s)x\| = \| U(s)A(s)x\| \leq \| U(s)\|_\infty \|A(s)x\| = \| U(s)\|_\infty \| Y(s)\|$. Since $U(s)$ is symmetric, $\| U(s)\|_\infty = \sup \{|\langle U(s)x, x\rangle| : \|x\| = 1\}$. The result now follows from Lemma 2.8.

Proposition 2.9. *Let $K > -k^2$ for some $k > 0$. Let γ be a unit speed geodesic in M. For any number $R > 0$ we can find a number $T = T(R, \gamma) > 0$ such that $\| Y(s)\| \geq R \| Y'(0)\|$ for $s \geq T$, where Y is any perpendicular Jacobi vector field on γ such that $Y(0) = 0$.*

Proof. It suffices to consider the case where $\| Y'(0)\| = 1$. Consider the matrix equation (J) relative to an adapted frame field along γ. For any $s > 0$ let $M(s) = D'(0) - D_s'(0) = \int_s^\infty A^{-1}(u)A^{-1}(u)^* du$, and let $U(s) = A'(s)A^{-1}(s)$ and $V(s) = D'(s)D^{-1}(s)$. $U(s)$ and $V(s)$ are solutions defined for $s > 0$ of equation (R) of Lemma 2.8. As remarked earlier, $U(s)$ is symmetric for $s > 0$.

$V(s)$ is symmetric for $s > 0$ since for every $t > 0$, $W(D_t, D_t)(t) = 0$, and therefore $W(D, D)(s) \equiv W(D, D)(0) = \lim_{t\to+\infty} W(D_t, D_t)(0) = 0$. Differentiating the expression $D(s) = A(s)M(s)$, which is valid for $s > 0$, or using a Wronskian argument involving $W(A, D)(s)$, we obtain the relation $U(s) - V(s) = A^{-1}(s)^*M^{-1}(s)A^{-1}(s)$. For any $s > 0$ and any unit vector $x \in R^{n-1}$, $|\langle M^{-1}(s)A^{-1}(s)x, A^{-1}(s)x\rangle| \leq |\langle U(s)x, x\rangle| + |\langle V(s)x, x\rangle| \leq 4k$ for $s \geq s_0 > 0$ by Lemma 2.8, where $s_0 > 0$ is chosen so that $k \coth(ks) \leq 2k$ for $s \geq s_0$. If $\lambda(s)$ is the largest eigenvalue of $M(s)$, then $\|M(s)\|_\infty = \lambda(s)$, and $1/\lambda(s)$ is the smallest eigenvalue of $M^{-1}(s)$. Since $M^{-1}(s)$ is positive definite and symmetric, we have $4k \geq |\langle M^{-1}(s)A^{-1}(s)x, A^{-1}(s)x\rangle| \geq (1/\lambda(s)) \|A^{-1}(s)x\|^2$ or $\|A^{-1}(s)x\|^2 \leq 4k \|M(s)\|_\infty$ for $s \geq s_0$. Since x is arbitrary, $\|A^{-1}(s)\|_\infty^2 \leq 4k \|M(s)\|_\infty$ for $s \geq s_0$. If $x \in R^{n-1}$ is any unit vector, then $\|A(s)x\| \geq 1/\|A^{-1}(s)\|_\infty \geq (4k \|M(s)\|_\infty)^{-1/2}$. Let $R > 0$ be given. Since $M(s) \to 0$ as $s \to +\infty$, we may choose $T > s_0 > 0$ so that $s \geq T$ implies $\|M(s)\|_\infty \leq 1/(4kR^2)$. If $s \geq T$, then $\|A(s)x\| \geq R$, and the result follows since x is an arbitrary unit vector.

Remark 2.10. We can strengthen the previous Proposition if the vector spaces $X_s(v)$, $X_u(v)$ depend continuously on v, that is, if the sets A_s, A_u defined earlier are closed in $T(SM)$. Relative to equation (J) on the unit speed geodesic γ_v, we see that for any $t > 0$, $\sup\{\|\xi - \xi_t\| : \xi \in X_s(v), \|d\pi\xi\| = 1\}$ $= \|D'(0) - D_t'(0)\|_\infty = \|M(t)\|_\infty = \left\|\int_t^\infty A^{-1}(u)A^{-1}(u)^* du\right\|_\infty$ is monotone decreasing in t. Using this observation and the fact that A_s is closed, one may show that for any compact set $C \subseteq M$ and any number $\varepsilon > 0$, we can find a number $s_0 = s_0(\varepsilon, C)$ such that if γ is a unit speed geodesic of M with $\gamma(0) \in C$, then, relative to any adapted frame field along γ, $\|M(s)\|_\infty \leq \varepsilon$ for $s \geq s_0$. The inequality in Proposition 2.9 which involves $\|A(s)x\|$ depends only on $\|M(s)\|_\infty$, and hence, in the statement of that proposition, we may choose $T = T(R, C)$ depending only on R and C so that the conclusion holds for any geodesic γ with the initial point in C. As a corollary of this strengthened result we would then obtain Theorem 1 of [4]. This strengthened result will not be needed here, and a more detailed proof is therefore omitted.

Proposition 2.11. *Let $K > -k^2$ for some $k > 0$. Then for any $v \in SM$ and any $\xi \in X_s(v)$ or $X_u(v)$ we have $\|K\xi\| \leq k \|d\pi\xi\|$.*

Proof. For any $t \neq 0$, $\|K(\xi_t)\| \leq k \coth(k|t|) \|d\pi(\xi_t)\| = k \coth(k|t|) \|d\pi\xi\|$ by Proposition 2.7. As $t \to +\infty$ (or as $t \to -\infty$) $\xi_t \to \xi$ and $k \coth(k|t|) \to k$. The result follows.

Proposition 2.12. *Let $K > -k^2$ for some $k > 0$. Let $v \in SM$ and let $\xi \in (SM)_v$ be such that $\langle \xi, V(v)\rangle = 0$ and $\|d\pi \circ dT_t\xi\|$ is bounded above for all $t \geq 0$ (respectively for all $t \leq 0$). Then $\xi \in X_s(v)$ (respectively $\xi \in X_u(v)$).*

Proof. We consider only the case where $\|d\pi \circ dT_t\xi\| \leq A$ for $t \geq 0$ and some $A > 0$, since the other case is proved similarly. For any $t > 0$, $\xi_t \in (SM)_v$ by Proposition 1.7 (3), since the Jacobi vector field Y_{ξ_t} is perpendicular at 0 and t and hence everywhere. Therefore $(\xi - \xi_t) \in (SM)_v$, and the Jacobi vector field which $\xi - \xi_t$ determines is perpendicular to γ_v since it vanishes at 0. Finally, $\|d\pi \circ dT_t(\xi-\xi_t)\| \leq A$, and by Proposition 2.9, $\|\xi-\xi_t\| = \|K(\xi-\xi_t)\| \to 0$ as $t \to +\infty$.

Proposition 2.13. *Let $K > -k^2$ for some $k > 0$. Suppose that there exist constants $A > 0$ and $s_0 \geq 0$ so that for any perpendicular Jacobi vector field Y such that $Y(0) = 0$ and for any numbers $t \geq s \geq s_0$ we have $\|Y(t)\| \geq A \|Y(s)\|$. Then each of the following is true:*

1) *For any $v \in SM$ and any $\xi \in (SM)_v$, $\xi \in X_s(v)$ (respectively $\xi \in X_u(v)$) if and only if $\langle \xi, V(v) \rangle = 0$ and $\|d\pi \circ dT_t\xi\|$ is bounded above for all $t \geq 0$ (respectively all $t \leq 0$). In either case, $(1/A)\|d\pi\xi\|$ is an upper bound.*

2) *$A_s = \bigcup_{v \in SM} X_s(v)$ and $A_u = \bigcup_{v \in SM} X_u(v)$ are closed subsets of $T(SM)$.*

Proof. 1) One half of the assertion follows from the preceding result. Now let $v \in SM$ and $\xi \in X_s(v)$, and fix a number $u \geq 0$. For $t \geq u + s_0$ it follows by the hypothesis that $\|d\pi \circ dT_u(\xi_t)\| \leq (1/A)\|d\pi(\xi_t)\| = (1/A)\|d\pi\xi\|$. Since $\xi_t \to \xi$ as $t \to +\infty$, $\|d\pi \circ dT_u\xi\| \leq (1/A)\|d\pi\xi\|$ by continuity. The proof for the case where $\xi \in X_u(v)$ is similar.

2) It suffices to prove that A_s is closed, since $A_u = dS(A_s)$ by Proposition 2.4. Let ξ_n be a sequence in A_s converging to $\xi \in T(SM)$. If $\xi \in (SM)_v$ and $\xi_n \in (SM)_{v_n}$, then $v_n \to v$ and $\langle \xi, V(v) \rangle = \lim_{n\to\infty} \langle \xi_n, V(v_n) \rangle = 0$. If $t \geq 0$ is given, then $\|d\pi \circ dT_t\xi\| = \lim_{n\to\infty} \|d\pi \circ dT_t(\xi_n)\| \leq \lim_{n\to\infty} (1/A)\|d\pi\xi_n\| = (1/A)\|d\pi\xi\|$. Therefore $\xi \in X_s(v)$ by 1).

Part 1) of this result says that for any unit speed geodesic γ in M, a perpendicular Jacobi vector field Y lies in $J_s(\gamma)$ (respectively $J_u(\gamma)$) if and only if $\|Y(t)\| \leq (1/A)\|Y(0)\|$ for $t \geq 0$ (respectively $t \leq 0$). It follows from the discussion in § 1 that if M has no focal points and $K > -k^2$ for some $k > 0$, then the hypothesis of Proposition 2.13 is satisfied for $A = 1$ and $s_0 = 0$.

Corollary 2.14. *Let M satisfy the hypotheses of Proposition 2.13, and let $B = [(1 + k^2)/A^2]^{1/2}$. Then for any $v \in SM$ and any $\xi \in (SM)_v$, $\xi \in X_s(v)$ (respectively $\xi \in X_u(v)$) if and only if $\langle \xi, V(v) \rangle = 0$ and $\|dT_t\xi\|$ is bounded above for all $t \geq 0$ (respectively $t \leq 0$). In either case, $B\|\xi\|$ is an upper bound.*

Proof. If $\|dT_t\xi\|$ is bounded above for all $t \geq 0$ (respectively $t \leq 0$), then $\|d\pi \circ dT_t\xi\| \leq \|dT_t\xi\|$ is bounded above, and $\xi \in X_s(v)$ (respectively $X_u(v)$) by the previous result. Conversely, if $\xi \in X_s(v)(X_u(v))$, then by Propositions 2.4, 3), 2.11 and 2.13 we have for $t \geq 0$ $(t \leq 0)$, $\|dT_t\xi\|^2 = \|d\pi \circ dT_t\xi\|^2 + \|K \circ dT_t\xi\|^2 \leq (1 + k^2)\|d\pi \circ dT_t\xi\|^2 \leq ((1 + k^2)/A^2)\|d\pi\,\xi\|^2 \leq B^2\|\xi\|^2$.

Let J_0^* be the set of all perpendicular Jacobi vector fields on unit speed geodesics of M.

Proposition 2.15. *Let the universal Riemannian covering H of M be compactly homogeneous. For each $s \geq 0$ let $g(s) = \inf\{\|Y(s)\| : Y \in J_0^*, Y(0)$*

$= 0$ *and* $\|Y'(0)\| = 1\}$. *Then* $g(s) > 0$ *for each* $s > 0$, *and* g *is semi-continuous in* $(0, \infty)$.

Proof. Since g is the infimum of a set of continuous functions, it follows that for any $\alpha > 0$, $\{s > 0: g(s) < \alpha\}$ is an open set in $(0, \infty)$. Thus g is semi-continuous. If $g^*(s)$ is the function defined in the same way relative to Jacobi vector fields in H, then $g^*(s) = g(s)$ since $p: H \to M$ is a surjective local isometry, and the Jacobi vector fields along geodesics in M are the projections of Jacobi vector fields along geodesics in H. To prove that $g(s) = g^*(s)$ is positive for $s > 0$, we need only to show that $g^*(s) = Y(s)$ for some $Y \in J_0^*$ such that $Y(0) = 0$ and $\|Y'(0)\| = 1$. Given $s > 0$, let $\xi_n \subseteq T(SH)$ be a sequence such that $d\pi\xi_n = 0$ and $\|\xi_n\| = 1$ for every n, and $\|d\pi \circ dT_s\xi_n\| \to g^*(s)$. By Proposition 1.12, there exist a sequence ϕ_n of isometries of H and a vector $\xi^* \in T(SH)$ such that $\xi_n^* = dT_{\phi_n}\xi_n \to \xi^*$ by passing to a subsequence, where $T_{\phi_n} = d\phi_n: TH \to TH$. Since T_{ϕ_n} is an isometry, $d\pi\xi_n^* = 0$, $\|\xi_n^*\| = 1$ and $\|d\pi \circ dT_s\xi_n^*\| = \|d\pi \circ dT_s\xi_n\| \to g^*(s)$ (see Proposition 1.5). By continuity, $d\pi\xi^* = 0$, $\|\xi^*\| = 1$ and $\|d\pi \circ dT_s\xi^*\| = g^*(s)$. If Y is the Jacobi vector field corresponding to ξ^*, then $g^*(s) = Y(s)$.

3. The Anosov equivalences

We first define an Anosov flow. One usually assumes that the underlying manifold is compact, but we shall not assume this. For convenience we also assume more differentiability than is necessary. For a discussion of the properties of Anosov flows on compact Riemannian manifolds see [1].

Definition 3.1. Let T_t be a complete C^∞ flow on a (complete) C^∞ Riemannian manifold N of dimension $n \geq 3$. The flow is said to be *of Anosov type* if the following conditions are satisfied:

1) The vector field V defined by the flow never vanishes on N.
2) For each $n \in N$ the tangent space N_n splits into a direct sum.

$$N_n = X_s^*(n) \oplus X_u^*(n) \oplus Z(n)$$

($\dim X_s^* = k > 0$, $\dim X_u^* = l > 0$, $\dim Z = 1$), where $Z(n)$ is generated by $V(n)$, and there exist positive numbers a, b, c, such that

i) for any $\xi \in X_s^*(n)$

$$\|dT_t\xi\| \leq a\,\|\xi\|\,e^{-ct} \quad \text{for } t \geq 0, \geq b\,\|\xi\|\,e^{-ct} \quad \text{for } t \leq 0\ ;$$

ii) for any $\eta \in X_u^*(n)$

$$\|dT_t\eta\| \leq a\,\|\eta\|\,e^{ct} \quad \text{for } t \leq 0, \geq b\,\|\eta\|\,e^{ct} \quad \text{for } t \geq 0\ .$$

We now state the main results. For the rest of this section we assume that M is a complete, C^∞ Riemannian manifold of dimension $n \geq 2$ without conjugate points such that the universal Riemannian covering H is compactly homogeneous. Let T_t denote the geodesic flow in TM, SM, TH and SH. Since M has the same values for the sectional curvature as H, there exists a number $k > 0$ such that $K > -k^2$ in M.

Theorem 3.2. *The following properties are equivalent:*

1) *The geodesic flow in SM is of Anosov type.*

2) *For every $v \in SM$, $X_s(v) \cap X_u(v) = \{0\}$, where X_s, X_u are the subspaces defined in* § 2.

3) *For every $v \in SM$, $(SM)_v = X_s(v) \oplus X_u(v) \oplus Z(v)$, where $Z(v)$ is the one-dimensional subspace generated by $V(v)$.*

4) *There exists no nonzero perpendicular Jacobi vector field Y on a unit speed geodesic γ of M such that $\|Y(t)\|$ is bounded above for all $t \in R$.*

5) *The following two conditions hold:*

i) *There exist numbers $A > 0$, $s_0 \geq 0$, such that for any perpendicular Jacobi vector field Y on a unit speed geodesic γ satisfying the condition* $Y(0) = 0$, *we have* $\|Y(t)\| \geq A\,\|Y(s)\|$ *for any numbers* $t \geq s \geq s_0$.

ii) $\int_1^\infty [1/g(t)]dt < \infty$, *where for* $t > 0$, $g(t) = \inf\{\|Y(t)\| : Y \in J_0^*, Y(0) = 0, \|Y'(0)\| = 1\}$.

(*This is the Lebesgue integral of a positive measurable function.*)

As a consequence of the proof of this result, we also show that if the geodesic flow in SM is of Anosov type, then $X_s(v) = X_s^*(v)$, and $X_u(v) = X_u^*(v)$ for every $v \in SM$.

A vector field Z on a differentiable curve $\sigma: R \to M$ is *parallel* along σ if the covariant derivative $Z'(t) = 0$ for every real number t.

Corollary 3.3. *If M has no focal points, then the following properties are equivalent:*

1) *The geodesic flow in SM is of Anosov type.*

2) *There exists no nonzero perpendicular parallel Jacobi vector field Y on a unit speed geodesic γ of M.*

3) $\int_1^\infty [1/g(t)]dt < \infty$.

Corollary 3.4. *If the geodesic flow in SM is of Anosov type, then the following property holds: Let γ be any unit speed geodesic of M, and $E(t)$ be any nonzero perpendicular parallel vector field on γ. Then the sectional curvature $K(E, \gamma')(t) < 0$ for some real number t.*

Corollary 3.5. *If M has no focal points, and M satisfies the property of Corollary* 3.4, *then the geodesic flow in SM is of Anosov type.*

Corollary 3.6. *If M is a two-dimensional manifold without focal points, then the geodesic flow in SM is of Anosov type if and only if every geodesic of M passes through a point of negative Gaussian curvature.*

It follows from the proof of Theorem 3.2 and Corollary 3.3 that if

$\int_1^\infty [1/g(t)]dt < \infty$, then there exist positive constants a, c, and t_0 such that $g(t) \geq ae^{ct}$ for $t \geq t_0$.

A consequence of Corollary 3.5 is the theorem that if M is a compact Riemannian manifold with sectional curvature $K < 0$, then the geodesic flow in SM is of Anosov type. We also use Corollary 3.5 to construct examples of compact manifolds with curvature $K \leq 0$, large patches of zero curvature, and Anosov geodesic flow in SM.

We omit the proof of Corollary 3.6, which is an immediate consequence of Corollaries 3.4 and 3.5.

Theorem 3.2 will be established by means of the series of Propositions 3.7 through 3.16. Before beginning the proof, we remark that if $p: N \to M$ is a surjective local isometry of complete Riemannian manifolds, and the geodesic flow in SM is of Anosov type, then the geodesic flow in SN is of Anosov type. $P = dp: SN \to SM$ is a local isometry by Proposition 1.5. If we are given $v \in SN$, then we define $X_s^*(v)$, $X_u^*(v)$ to be those subspaces in $(SN)_v$ which are mapped isomorphically onto $X_s^*(Pv)$, $X_u^*(Pv)$ by dP. It follows by Proposition 1.5 that $(SN)_v = X_s^*(v) \oplus X_u^*(v) \oplus Z(v)$, and that the spaces $X_s^*(v)$, $X_u^*(v)$ satisfy the Anosov conditions.

Proposition 3.7. *Let M admit no nonzero perpendicular Jacobi vector field Y on a unit speed geodesic γ such that $\|Y(t)\|$ is bounded above for all $t \in R$. There exists a constant $A > 0$ such that if Y is a nonzero perpendicular Jacobi vector field on a unit speed geodesic γ such that $Y(0) = 0$, then $\|Y(t)\| \geq A\|Y(s)\|$ for any numbers $t \geq s \geq 1$.*

Proof. By reasoning similar to that used in the proof of Proposition 2.15, it suffices to prove this assertion for Jacobi vector fields of the same kind in the universal Riemannian covering H of M. If the assertion were false, then there would exist nonzero perpendicular Jacobi vector fields Y_n on unit speed geodesics γ_n in H, and sequences $1 \leq s_n \leq t_n$ such that for every integer $n > 0$, $Y_n(0) = 0$ and $\|Y_n(t_n)\| \leq (1/n)\|Y_n(s_n)\|$. Multiplying Y_n by a scalar if necessary, we may assume that $\|Y_n'(0)\| = 1$. Choose a sequence $u_n \subseteq R$ such that $0 \leq u_n \leq t_n$ and $\|Y_n(s)\| \leq \|Y_n(u_n)\|$ for every $0 \leq s \leq t_n$.

We assert that $u_n \geq \delta > 0$ for some number $\delta > 0$ and every integer $n > 0$. If this were false, then $u_n \to 0$, by passing to a subsequence. By Proposition 1.11, we may choose a sequence ϕ_n of isometries of H and a Jacobi vector field Z on a unit speed geodesic γ in H such that $Z_n = d\phi_n Y_n \to Z$ by passing to a subsequence. By continuity, $0 = \|Z(0)\| = \lim_{n\to\infty}\|Z_n(u_n)\| = \lim_{n\to\infty}\|Y_n(u_n)\|$. However, since $t_n \geq 1$ for every n, we have $\|Y_n(u_n)\| \geq \|Y_n(1)\| \geq g(1) > 0$ by Proposition 2.15. This is a contradiction.

For each n, let $\sigma_n(t) = \gamma_n(u_n + t)$ and $Z_n(t) = Y_n(u_n + t)/\|Y_n(u_n)\|$. Then Z_n is a Jacobi vector field on the unit speed geodesic σ_n such that $\|Z_n(0)\| = 1$, $\|Z_n(s)\| \leq 1$ for $-u_n \leq s \leq t_n - u_n$, $Z_n(-u_n) = 0$ and $\|Z_n(t_n - u_n)\| \leq 1/n$. Since H is compactly homogeneous, $K > -k^2$ for some $k > 0$, and by Proposition 2.7, $\|Z_n'(0)\| = \|Y_n'(u_n)\|/\|Y_n(u_n)\| \leq k \coth(ku_n) \leq k \coth(k\delta)$. By Proposition 1.11, there exist a sequence ϕ_n of isometries of H and a Jacobi vector field Z^* on a unit speed geodesic γ such that $Z_n^* = d\phi_n Z_n \to Z^*$, by passing to a subsequence. $\|Z^*(0)\| = 1$ by continuity, so $Z^* \not\equiv 0$.

There are four cases to consider: 1) $t_n - u_n$ and u_n both contain bounded subsequences; 2) $u_n \to +\infty$ and $t_n - u_n$ contains a bounded subsequence; 3) $t_n - u_n \to +\infty$, and u_n contains a bounded subsequence; 4) $t_n - u_n \to +\infty$ and $u_n \to +\infty$. We derive contradictions in all cases. If case 1) occurs, then we can find numbers $t \geq 0$ and $u \geq \delta > 0$ such that $t_n - u_n \to t$ and $u_n \to u$, by passing to a subsequence. By continuity, $Z^*(-u) = 0$ and $Z^*(t) = 0$. Since $t \geq 0$ and $-u \leq -\delta$, this contradicts the fact that $Z^* \not\equiv 0$. If case 2) occurs, then let $t_n - u_n \to t < \infty$ by passing to a subsequence. By continuity, $Z(t) = 0$ and $\|Z(u)\| \leq 1$ for all $u \leq t$. By Proposion 2.9, however, $\|Z(t)\| \to \infty$ as $t \to -\infty$. The contradiction to case 3) is similar to that of case 2). If case 4) occurs, then $\|Z(t)\| \leq 1$ for all $t \in R$ by continuity, but this contradicts the hypothesis of the proposition.

Proposition 3.8. *Conditions* 2), 3) *and* 4) *of Theorem* 3.2 *are equivalent.*

Proof. For any $v \in SM$, $(SM)_v = V(v)^\perp \oplus Z(v)$, where $V(v)^\perp$ is the orthogonal complement to $V(v)$, and $Z(v)$ is the one-dimensional subspace generated by $V(v)$. Since $V(v)^\perp$ has dimension $2n - 2$ and $X_s(v)$, $X_u(v)$ are $(n-1)$-dimensional subspaces of $V(v)^\perp$, the conditions 2) and 3) are clearly equivalent. Suppose that M admits a nonzero perpendicular Jacobi vector field Y on a unit speed geodesic γ such that $\|Y(t)\|$ is bounded above for all $t \in R$. If $v = \gamma'(0)$, and $\xi \in (SM)_v$ corresponds to Y, then it follows by Proposition 2.12 that $\xi \in X_s(v) \cap X_u(v)$. Hence 2) implies 4). That assertion 4) implies 2) follows from Propositions 2.13 and 3.7.

Proposition 3.9. *Let M admit no nonzero perpendicular Jacobi vector field Y on a unit speed geodesic γ such that $\|Y(t)\|$ is bounded above for all $t \in R$. Then for any number $\varepsilon \in 0$, there exists a number $T > 0$ such that:*

1) *for any $v \in SM$ and any $\xi \in X_s(v)$, $\|dT_t\xi\| \leq \varepsilon\|\xi\|$ for $t \geq T$;*

2) *for any $v \in SM$ and any $\xi \in X_u(v)$, $\|dT_t\eta\| \leq \varepsilon\|\eta\|$ for $t \leq -T$.*

Proof. 1) By Proposition 1.5 and Remark 2.3, 2), it suffices to prove this assertion for all $v \in SH$ and $\xi \in X_s(v)$, where H is the universal Riemannian covering of M. If the assertion were false, then for some $\varepsilon > 0$ there would exist sequences $t_n \subseteq R$, $v_n \subseteq SH$ and $\xi_n \subseteq T(SH)$ such that $\xi_n \in X_s(v_n)$, $t_n \to +\infty$ and $\|dT_t\xi_n\| > \varepsilon\|\xi_n\|$ for every integer n. We may assume that $\|\xi_n\| = 1$ for every n. If $\psi_n = dT_{t_n}\xi_n$, then $\psi_n \in A_s = \bigcup_{v \in SH} X_s(v)$ and $\|\psi_n\| > \varepsilon$ for every n. It follows from Corollary 2.14 and Proposition 3.7 that there exists a number $B > 0$ such that for every n, $\|dT_t\psi_n\| \leq B$ for $-t_n \leq t < \infty$. By Proposition 1.12, there exist a sequence ϕ_n of isometries of H and a vector $\psi^* \in T(SH)$ such that $\psi_n^* = dT_{\phi_n}\psi_n \to \psi^*$, by passing to a subsequence, where $T_{\phi_n} = d\phi_n$. The vector ψ_n^* lies in A_s for every n by Remark 2.3, 2). Since $T_t \circ d\phi_n = d\phi_n \circ T_t$ for every $t \in R$, we have $\|dT_t\psi_n^*\| = \|dT_t\psi_n\| \leq B$

for $-t_n \leq t < \infty$. The vector $\psi^* \neq 0$ since $\|\psi_n^*\| = \|\psi_n\| > \varepsilon$ for every n, and by continuity $\|dT_t\psi^*\| \leq B$ for all $t \in R$. If $\psi^* \in (SH)_{v^*}$ and $\psi_n^* \in (SH)_{v_n^*}$, then $\langle \psi^*, V(v^*) \rangle = 0$ since $\langle \psi_n^*, V(v_n^*) \rangle = 0$ for every n. If $Y^* = Y_{\psi^*}$, then Y^* is a nonzero perpendicular Jacobi vector field on γ_{v^*} such that $\|Y^*(t)\| \leq B$ for all $t \in R$. $Y(t) = dpY^*(t)$ is a nonzero perpendicular Jacobi vector field on the geodesic $p \circ \gamma_{v^*}$ such that $\|Y(t)\| = \|Y^*(t)\| \leq B$ for all $t \in R$, but this contradicts the hypothesis of the proposition.

2) $\eta \in X_u(v)$ if and only if $\xi = dS(\eta) \in X_s(-v)$, where $S\colon SM \to SM$ takes v into $-v$. Since S is an isometry of SM satisfying the relation $S \circ T_t = T_{-t} \circ S$ for all $t \in R$, we have $\|dT_{-t}\eta\| = \|dS \circ dT_t\xi\| = \|dT_t\xi\| \leq \varepsilon \|\xi\| = \varepsilon \|\eta\|$ for $t \geq T$.

Proposition 3.10. *Let M admit no nonzero perpendicular Jacobi vector field Y on a unit speed geodesic γ such that $\|Y(t)\|$ is bounded above for all $t \in R$. Then the geodesic flow in SM is of Anosov type.*

This is the result that 4) implies 1) in Theorem 3.2. We first need some preliminary results.

Lemma 3.11. *Let M be a complete manifold without conjugate points, which satisfies the hypothesis of Proposition* 2.13. *For each $s \geq 0$, let $\phi(s) =$* $\sup \{\|dT_s\xi\| \colon \xi \in A_s, \|\xi\| = 1\}$. *Then*

1) *there exists a constant $B > 0$ such that $0 \leq \phi(s) \leq B$ for every $s \geq 0$,*

2) *$\phi(s + t) \leq \phi(s) \cdot \phi(t)$ for all numbers $s \geq 0$, $t \geq 0$.*

Furthermore, if the universal Riemannian covering H is compactly homogeneous, and M also satisfies the hypothesis of Proposition 3.10, *then*

3) *$\phi(s) \to 0$ as $s \to +\infty$.*

Proof. 1) If $B > 0$ is the constant of Corollary 2.14, then $0 \leq \phi(s) \leq B$ for every $s \geq 0$.

2) For any $\xi \in A_s$ and any $s \geq 0$, $\|dT_s\xi\| \leq \phi(s)\|\xi\|$ by definition. Let $\xi \in A_s$, $\|\xi\| = 1$ be given, and let $s \geq 0$, $t \geq 0$ be arbitrary numbers. Then $\|dT_{t+s}\xi\| = \|dT_t(dT_s\xi)\| \leq \phi(t)\|dT_s\xi\|$ (since A_s is invariant under dT_s) $\leq \phi(t) \cdot \phi(s)$. Since ξ was arbitrary, 2) follows from the definition of $\phi(s + t)$.

3) This is a consequence of Proposition 3.9.

Lemma 3.12. *Let $\phi\colon (0, \infty) \to (0, \infty)$ be a function satisfying properties* 1), 2) *and* 3) *of Lemma* 3.11. *Then there exist numbers $a > 0$ and $c > 0$ such that $\phi(s) \leq ae^{-cs}$ for all $s \geq 0$.*

Proof. It follows from property 2) that if $s \geq 0$ is any number and $n > 0$ is any integer, then $\phi(ns) \leq \phi(s)^n$. By 3) we may choose $s_0 > 0$ so that $\phi(s) \leq \frac{1}{2}$ for $s \geq s_0$. The lemma will be proved when we show that $\phi(s) \leq e^{-cs}$ for $s \geq s_0$, where $c = \frac{1}{2}(\log 2)/s_0$. Given any number $s \geq s_0$ we may choose an integer $n \geq 1$ such that $s_0 \leq s/n \leq 2s_0$. Let $s^* = s/n$. Then $[\log \phi(s)]/s = [\log \phi(ns^*)]/(ns^*) \leq [\log \phi(s^*)^n]/(ns^*) = [\log \phi(s^*)]/s^* \leq -\frac{1}{2}(\log 2)/s_0 = -c$.

Proof of Proposition 3.10. It follows from Proposition 3.7, Lemma 3.11 and Lemma 3.12 that for every $v \in SM$ and every $\xi \in X_s(v)$ we have $\|dT_t\xi\| \leq a\|\xi\|e^{-ct}$ for $t \geq 0$, where $a > 0$ and $c > 0$ do not depend on ξ or v.

Therefore $\|dT_t\xi\| \geq (1/a)\,\|\xi\|\, e^{-ct}$ for $t \leq 0$, $v \in SM$ and $\xi \in X_s(v)$. If $v \in SM$ and $\xi \in X_u(v)$ are given, then $\xi = dS(\eta) \in X_s(-v)$ by Proposition 2.4. Hence for $t \geq 0$, $\|dT_t\eta\| = \|dT_t dS(\xi)\| = \|dS \circ dT_{-t}\xi\| = \|dT_{-t}\xi\| \geq (1/a)\,\|\xi\|\, e^{ct} = (1/a)\,\|\eta\|\, e^{ct}$. For $t \leq 0$, $v \in SM$ and $\eta \in X_u(v)$ we have $\|dT_t\eta\| \leq a\,\|\eta\|\, e^{ct}$. Therefore, for any $v \in SM$, it follows that $X_s(v) \cap X_u(v) = \{0\}$, and by Proposition 3.8 we have $(SM)_v = X_s(v) \oplus X_u(v) \oplus Z(v)$. If we set $X_s^*(v) = X_s(v)$ and $X_u^*(v) = X_u(v)$, then the conditions for an Anosov flow are satisfied.

Proposition 3.13. *Let the geodesic flow in SM be of Anosov type. Then M admits no nonzero perpendicular Jacobi vector field Y on a unit speed geodesic γ such that $\|Y(t)\|$ is bounded above for all $t \in R$.*

This is the result that 1) implies 4) in Theorem 3.2.

Proof. Suppose that there exists a nonzero perpendicular Jacobi vector field Y on a unit speed geodesic such that $\|Y(t)\| \leq c$ for some number $c > 0$ and all $t \in R$. If $v = \gamma'(0)$, let $\xi \in (SM)_v$ correspond to Y. Choose $k > 0$ so that the sectional curvature satisfies the condition $K > -k^2$. Since Y is perpendicular, $\langle \xi, V(v)\rangle = 0$, and $\xi \in X_s(v) \cap X_u(v)$ by Proposition 2.12. For any $t \in R$, $dT_t\xi \in X_s(T_tv) \cap X_u(T_tv)$ and $\|K \circ dT_t\xi\| \leq k\,\|d\pi \circ dT_t\xi\| \leq kc$ by Proposition 2.11. Hence $\|dT_t\xi\| \leq c(1 + k^2)^{1/2}$ for all t. By the definition of an Anosov flow, we may write $\xi = \xi_1 + \xi_2 + \xi_3$, where $\xi_1 \in X_s^*(v)$, $\xi_2 \in X_u^*(v)$, and $\xi_3 = eV(v)$ for some number $e \in R$. It is easy to show that $\|dT_t\xi_3\| = \|\xi_3\|$ for all $t \in R$, since $dT_tV(w) = V(T_tw)$ and $\|V(w)\| = 1$ for all $w \in SM$. By the properties of the spaces $X_s^*(v)$ and $X_u^*(v)$, it follows that if $\xi_1 \neq 0$, then $\|dT_t\xi\| \to +\infty$ as $t \to -\infty$, while if $\xi_2 \neq 0$, then $\|dT_t\xi\| \to +\infty$ as $t \to +\infty$. Therefore $\xi_1 = \xi_2 = 0$, but this contradicts the assumption that ξ is nonzero and orthogonal to $V(v)$.

Corollary 3.14. *Let the geodesic flow in SM be of Anosov type. Then $X_u^*(v) = X_u(v)$, and $X_s^*(v) = X_s(v)$ for every $v \in SM$. In particular, the spaces $X_s^*(v)$, $X_u^*(v)$ are always of dimension $n - 1$.*

Proof. Let $\xi \in X_s^*(v)$ be given. By Propositions 3.8 and 3.13, we can write $\xi = \xi_1 + \xi_2 + \xi_3$ where $\xi_1 \in X_s(v)$, $\xi_2 \in X_u(v)$ and $\xi_3 = eV(v)$ for some $e \in R$. By Proposition 1.7, 5), $\langle dT_t\xi_1, V(T_tv)\rangle = \langle dT_t\xi_2, V(T_tv)\rangle = 0$ for any $t \in R$, and hence $\|\xi_3\|^2 = \|dT_t\xi_3\|^2 \leq \|dT_t\xi_3\|^2 + \|dT_t(\xi_1 + \xi_2)\|^2 = \|dT_t\xi\|^2$. $\xi_3 = 0$ since $\xi \in X_s^*(v)$. If $\xi_2 \neq 0$, then by Proposition 3.13 and the proof of Proposition 3.10, $\|dT_t\xi_2\| \to \infty$ and $\|dT_t\xi_1\| \to 0$ as $t \to +\infty$. It would follow that $\|dT_t\xi\| \to \infty$ as $t \to +\infty$, contradicting the hypothesis that $\xi \in X_s^*(v)$. Therefore $\xi = \xi_1 \in X_s(v)$ and $X_s^*(v) \subseteq X_s(v)$. Similarly one shows that $X_u^*(v) \subseteq X_u(v)$. Since $2n - 2 = \dim X_u^*(v) + \dim X_s^*(v) \leq \dim X_u(v) + \dim X_s(v) = 2n - 2$, we see that $X_s(v) = X_s^*(v)$ and $X_u(v) = X_u^*(v)$.

Proposition 3.15. *If the geodesic flow in SM is of Anosov type, then property* 5) *of Theorem* 3.2 *holds.*

Proof. Part 1) of property 5) is a consequence of Propositions 3.7 and 3.13. To establish part 2) of property 5), it suffices to show that $g(t) \geq \alpha e^{ct}$

for $t \geq t_0 > 0$, where α, c, and t_0 are suitably chosen positive constants. This result has also been proved by W. Klingenberg. By the remarks in the proof of Proposition 2.15, it suffices to prove this property for the function $g(t)$ defined relative to the Jacobi fields of the universal Riemannian covering H of M.

Let $v \in SH$ and $\xi \in (SH)_v$ be given such that $\langle \xi, V(v) \rangle = 0$, $d\pi\xi = 0$, and $\|\xi\| = 1$. By the remark preceding Proposition 3.7, the geodesic flow in SH is of Anosov type, and one can show, either directly or by means of Remark 2.3, 2) and Corollary 3.14, that the subspaces $X_s^*(v)$ and $X_u^*(v)$ are orthogonal to $V(v)$. We may therefore write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in X_s^*(v)$, and $\xi_2 \in X_u^*(v)$. $\xi_1 \neq 0$ and $\xi_2 \neq 0$, by either Proposition 2.9 and the Anosov conditions, or Propositions 2.6 and 3.14. We show that there exist constants $\varepsilon > 0$ and $\delta > 0$, independent of ξ, ξ_1 and ξ_2, such that $\|\xi_2\| \geq \varepsilon$ and $\|\xi_1\| \leq \delta$.

Suppose that no such constant $\varepsilon > 0$ exists. Then we can find sequences $v_n \subseteq SH$ and $\xi_n, \xi_{1,n}, \xi_{2,n} \subseteq (SH)_{v_n}$, such that for every n, $\xi_n = \xi_{1,n} + \xi_{2,n}$, $\xi_{1,n} \in X_s^*(v_n)$, $\xi_{2,n} \in X_u^*(v_n)$, $\langle \xi_n, V(v_n) \rangle = 0$, $d\pi\xi_n = 0$, $\|\xi_n\| = 1$, and $\|\xi_{2,n}\| \to 0$ as $n \to \infty$. Since $\|\xi_{1,n}\| \to 1$, it follows from Proposition 1.12 that there exist isometries ϕ_n of H and vectors $\xi^*, \xi_1^* \in T(SH)$ such that by passing to a subsequence, $\xi_n^* = dT_{\phi_n}\xi_n \to \xi^*$, $\xi_{1,n}^* = dT_{\phi_n}\xi_{1,n} \to \xi_1^*$, and $\xi_{2,n}^* = dT_{\phi_n}\xi_{2,n} \to 0$, where $T_{\phi_n} = d\phi_n$. Since $T_{\phi_n}: SH \to SH$ is an isometry, $\xi_{1,n}^* \in X_s^*(T_{\phi_n}v_n)$ for every integer n by Proposition 1.5, 2). $\xi_1^* \in A_s^* = \bigcup_{v \in SH} X_s^*(v)$, since A_s^* is closed in $T(SH)$. If ξ^* is tangent to SH at v, then by continuity $\xi^* = \xi_1^* \in X_s^*(v)$, $\langle \xi^*, V(v) \rangle = 0$, $d\pi\xi^* = 0$, and $\|\xi^*\| = 1$. This contradicts the remarks of the preceding paragraph.

We next show that the angle between the subspaces $X_s^*(v)$ and $X_u^*(v)$ is uniformly bounded away from zero, that is, there exists a number σ $(0 \leq \sigma < 1)$ such that for any $v \in SH$ and any unit vectors $\xi \in X_s^*(v)$ and $\eta \in X_u^*(v)$, we have $|\langle \xi, \eta \rangle| \leq \sigma$. If this were false, then by arguing as in the previous paragraph we would be able to find a vector $v \in SH$ and unit vectors $\xi \in X_s^*(v)$, $\eta \in X_u^*(v)$ such that $|\langle \xi, \eta \rangle| = 1$. This would imply that $\xi = \pm\eta$, which contradicts the fact that $X_s^*(v) \cap X_u^*(v) = 0$.

Let $\xi = \xi_1 + \xi_2$ be defined as in the second paragraph of the proof. Let $\varepsilon > 0$ and $\sigma > 0$ be the constants chosen above and let $\delta = 1/(2\varepsilon(1 - \sigma))$. Then $1 = \|\xi_1 + \xi_2\|^2 = (\|\xi_1\| - \|\xi_2\|)^2 + 2\|\xi_1\|\,\|\xi_2\|\,\{1 + [\langle \xi_1, \xi_2 \rangle / (\|\xi_1\|\,\|\xi_2\|)]\} \geq 2\varepsilon\,\|\xi_1\|\,(1 - \sigma)$, or $\|\xi_1\| \leq \delta$.

Let $\xi = \xi_1 + \xi_2$ be as above, and let a, b, c be the positive constants appearing in the definition of Anosov flow. For $t \geq 0$, $\|dT_t\xi\| \geq \|dT_t\xi_2\| - \|dT_t\xi_1\| \geq b\,\|\xi_2\|\,e^{ct} - a\,\|\xi_1\|\,e^{-ct} \geq b\varepsilon e^{ct} - a\delta e^{-ct} \geq (b\varepsilon/2)e^{ct}$ for sufficiently large $t > 0$. If $k > 0$ is chosen so that the curvature satisfies the condition $K > -k^2$, then $\|dT_t\xi\| \leq (1 + k^2 \coth^2(kt))^{1/2}\,\|d\pi \circ dT_t\xi\| \leq (1 + 4k^2)^{1/2}\,\|d\pi \circ dT_t\xi\|$ for sufficiently large $t > 0$ by Proposition 2.7. Therefore there exists a number $t_0 > 0$, depending only on a, b, c, δ, ε and k, such that $\|d\pi \circ dT_t\xi\| \geq \alpha e^{ct}$ for $t \geq t_0$, where $\alpha = b\varepsilon/[2(1 + 4k^2)^{1/2}]$. Since ξ was an arbitrary unit vector in

$T(SH)$ satisfying $\langle \xi, V(v)\rangle = 0$, $\|\xi\| = 1$, and $d\pi\xi = 0$, it follows that $g(t) \geq \alpha e^{ct}$ for $t \geq t_0$.

Proposition 3.16. *If condition* 5) *of Theorem* 3.2 *holds, then the geodesic flow in SM is of Anosov type.*

Proof. By Proposition 2.15, $0 < 1/g(t) < \infty$ for any $t > 0$, and for any number $\alpha > 0$, $\{s > 0\colon 1/g(s) > \alpha\} = \{s > 0\colon g(s) < 1/\alpha\}$ is an open set in $(0, \infty)$. Since $1/g(t)$ is semi-continuous and finite, the Lebesgue integral $\int_1^\infty [1/g(t)]dt$ makes sense. For $s > 0$ let $\psi(s) = \int_s^\infty [1/g(t)]dt$. Then it follows that $\psi(s) \to 0$ as $s \to +\infty$. Clearly $\psi(s)$ is monotone decreasing in s, and the Lebesgue convergence theorem shows that for any integer $n > 0$, $\psi(n) = \int_1^\infty [1/g(t)]\chi_{(n,\infty)}(t)dt \to 0$ as $n \to \infty$.

If $s_0 \geq 0$ is the constant of property 5), we show that there exists a constant $B > 0$ such that if $\xi \in A_s = \bigcup_{v \in SM} X_s(v)$, then $\|dT_s\xi\| \leq B\psi(s)\,\|\xi\|$ for $s \geq s_0$.

Given $v \in SM$, construct an adapted frame field along γ_v, and consider the matrix equation (J). We show that for some number $B' > 0$, independent of v and the frame field, $\|D(s)\|_\infty \leq B'\psi(s)$ for $s \geq s_0$. For $s > 0$, $D(s) = A(s)\int_s^\infty A^{-1}(u)A^{-1}(u)^*du = \int_s^\infty A(s)A^{-1}(u)A^{-1}(u)^*du$, since $A(s)$ is a constant matrix. The following relations hold: a) For $s_0 \leq s \leq u$, $\|A(s)A^{-1}(u)\|_\infty \leq 1/A$, where $A > 0$ is the other constant of property 5); b) $\|A^{-1}(u)\|_\infty \leq 1/g(u)$ for any number $u > 0$. To see this, let $x \in R^{n-1}$ be an arbitrary unit vector, and let $y = A^{-1}(u)x$. If $Y(t) = A(t)y$, relative to the adapted frame field, then a) $\|A(s)A^{-1}(u)x\| = \|Y(s)\| \leq (1/A)\,\|Y(u)\| = (1/A)\,\|x\| = 1/A$ for $s_0 \leq s \leq u$; and b) $1 = \|x\| = \|A(u)y\| = \|y\|\,\|A(u)(y/\|y\|)\| \geq \|y\|\,g(u) = \|A^{-1}(u)x\|\,g(u)$ for $u > 0$. Since x is arbitrary, the results a) and b) follow. Recall that for any positive integer k we may define a norm $\|\ \ \|_1$ on the set of $k \times k$ matrices by setting $\|A\|_1 = \max\{|a_{ij}|\colon 1 \leq i, j \leq k\}$, for any matrix $A = (a_{ij})$. It is easy to show that $(1/k)\,\|A\|_\infty \leq \|A\|_1 \leq k^{1/2}\,\|A\|_\infty$. If $*$ denotes the transpose operation, then $\|A^*\|_\infty \leq k\|A^*\|_1 = k\,\|A\|_1 \leq k^{3/2}\|A\|_\infty$. Hence for $s \geq s_0$, $\|D(s)\|_\infty$

$$= \left\| \int_s^\infty A(s)A^{-1}(u)A^{-1}(u)^*\,du \right\|_\infty \leq (n-1)\left\| \int_s^\infty A(s)A^{-1}(u)A^{-1}(u)^*\,du \right\|_1 \leq$$

$$(n-1)\int_s^\infty \|A(s)A^{-1}(u)A^{-1}(u)^*\|_1 du \leq (n-1)^{3/2}\int_s^\infty \|A(s)A^{-1}(u)\|_\infty \|A^{-1}(u)^*\|_\infty du$$

$$\leq [(n-1)^3/A]\int_s^\infty \|A^{-1}(u)\|_\infty du \leq [(n-1)^3/A]\int_s^\infty [1/g(u)]du = B'\psi(s),$$

where $B' = (n-1)^3/A$. In the inequalities above, the real valued integrals may be viewed as either improper Riemann integrals or Lebesgue integrals. If $\xi \in X_s(v)$, then $Y_\xi(s) = D(s)x$ for some vector $x \in R^{n-1}$. Therefore, if $s \geq s_0$, then $\|d\pi \circ dT_s\xi\| = \|D(s)x\| \leq \|D(s)\|_\infty\|x\| \leq B'\psi(s)\,\|d\pi\xi\| \leq B'\psi(s)\,\|\xi\|$. Since

$\|K \circ dT_s\xi\| \leq k\|d\pi \circ dT_s\xi\|$ by Proposition 2.11 (the curvature satisfies the condition $K > -k^2$), $\|dT_s\xi\| \leq B\psi(s)\|\xi\|$ for $s \geq s_0$, where $B = B'(1 + k^2)^{1/2}$. For any $s \geq 0$, let $\phi(s) = \sup\{\|dT_s\xi\| : \xi \in A_s, \|\xi\| = 1\}$. The previous paragraphs show that $\phi(s) \leq B\psi(s)$ for $s \geq s_0$, and thus $\phi(s) \to 0$ as $s \to \infty$. By Lemma 3.11, $\phi(s)$ is bounded above for $s \geq 0$, and $\phi(s + t) \leq \phi(s) \cdot \phi(t)$ for any numbers $s \geq 0$, $t \geq 0$. By Lemma 3.12, $\phi(s) \leq ae^{-cs}$ for all $s \geq 0$ and suitable numbers $a > 0$ and $c > 0$. Using the same argument found in the last paragraph of the proof of Proposition 3.10, we see that for every vector $v \in SM$, $(SM)_v = X_s(v) \oplus X_u(v) \oplus Z(v)$, where $X_s(v)$ and $X_u(v)$ satisfy the Anosov conditions 2) i) and ii) of Definition 3.1.

This completes the proof of Theorem 3.2. Propositions 3.15 and 3.16 together show that if condition 5) holds, then $g(t) \geq \alpha e^{ct}$ for $t \geq t_0 > 0$, where $\alpha > 0$, $c > 0$ and $t_0 > 0$ are suitably chosen.

Proof of Corollary 3.3. By the discussion in § 1, condition 5), i) of Theorem 3.2 holds when $A = 1$ and $s_0 = 0$. Hence condition 3) of Corollary 3.3 and condition 5) of Theorem 3.2 are equivalent for manifolds without focal points. We now prove that condition 2) of Corollary 3.3 and condition 4) of Theorem 3.2 are also equivalent for manifolds without focal points. If Y is a nonzero perpendicular Jacobi vector field on a unit speed geodesic γ such that $Y'(t) = 0$ for every real number t, then $\|Y(t)\|$ is a constant function. Thus condition 4) of Theorem 3.2 implies condition 2) of Corollary 3.3. Before proving the converse we need some preliminary observations.

Let γ be any unit speed geodesic in M. If a Jacobi vector field Z is in $J_s(\gamma)(J_u(\gamma))$, then the function $\|Z(t)\|$ is nonincreasing (nondecreasing) in t. Let a vector field $Z \in J_s(\gamma)$ and arbitrary real numbers $a < b$ be given. Relative to the constants $A = 1$ and $s_0 = 0$, the condition of Proposition 2.13 is satisfied. Therefore $\|Z(t)\| \leq \|Z(0)\|$ for all real numbers $t \geq 0$ by 1) of Proposition 2.13. For each real number u, let $\gamma^*(u) = \gamma(a + u)$ and $Z^*(u) = Z(a + u)$. Since $\|Z^*(u)\| \leq \|Z(0)\|$ for $u \geq -a$, it follows from Proposition 2.12 that $Z^* \in J_s(\gamma^*)$. The reasoning above then implies that $\|Z(b)\| = \|Z^*(b - a)\| \leq \|Z^*(0)\| = \|Z(a)\|$. Therefore $\|Z(t)\|$ is nonincreasing in t. If $Z \in J_u(\gamma)$, then a similar argument shows that $\|Z(t)\|$ is nondecreasing in t.

Let γ be any unit speed geodesic in M, and consider the Jacobi equation (J) relative to some adapted frame field along γ. We assert that the symmetric matrix $U(s) = D'(s)D^{-1}(s)$ is negative semidefinite for every real number s. Let a real number s and a nonzero vector $y \in R^{n-1}$ be given. Since $D(s)$ is nonsingular by Lemma 1 of [6], there exists a vector $x \in R^{n-1}$ such that $D(s)x = y$. If Z is the vector field in $J_s(\gamma)$ represented by the curve $s \to D(s)x$, then the previous paragraph implies that $\langle U(s)y, y\rangle = \langle U(s)D(s)x, D(s)x\rangle = \langle D'(s)x, D(s)x\rangle = \frac{1}{2}\|Dx\|^{2\prime}(s) \leq 0$. (In this case, $\langle\ ,\ \rangle$ denotes the usual inner product in R^{n-1}). Therefore $U(s)$ is negative semidefinite.

Suppose now that Y is a nonzero perpendicular Jacobi vector field on a unit speed geodesic γ such that $\|Y(t)\|$ is uniformly bounded above for all $t \in R$.

When we show that $Y'(t) = 0$ for all $t \in R$, it will follow that condition 2) of Corollary 3.3 implies condition 4) of Theorem 3.2. It follows from Proposition 2.12 that $Y \in J_s(\gamma) \cap J_u(\gamma)$. Remarks above imply that the function $\| Y(t)\|$ is both nonincreasing and nondecreasing in t, and hence is constant. Relative to an adapted frame field along γ, we may choose a nonzero vector $x \in R^{n-1}$ such that Y corresponds to the curve $s \to D(s)x$. We find that $\langle U(s)D(s)x, D(s)x\rangle = \langle D'(s)x, D(s)x\rangle = \frac{1}{2}\| Y\|^{2\prime}(s) = 0$. Since $U(s)$ is negative semidefinite, it follows that $D'(s)x = U(s)D(s)x = 0$ for every number s. Since $D(0)$ is the identity matrix, $D(s)x = x$ for every s. Therefore Y is a parallel vector field.

Proof of Corollary 3.4. If Y is a nonzero perpendicular parallel Jacobi vector field along a unit speed geodesic γ, then $\| Y(t)\|$ is a constant function and $Y \in J_s(\gamma) \cap J_u(\gamma)$ by Proposition 2.12. Therefore the geodesic flow in SM is not of Anosov type by Theorem 3.2. Corollary 3.4 is now an immediate consequence of the following:

Proposition 3.17. *Let M be a complete manifold without conjugate points. Let E be a nonzero perpendicular parallel vector field along a unit speed geodesic γ such that $K(E, \gamma')(t) \geq 0$ for every real number t. Then $K(E, \gamma')(t) = 0$ for every real number t, and E is a Jacobi vector field along γ.*

Proof. Relative to an adapted frame field along γ, let the constant curve $x \in R^{n-1}$ represent the parallel vector field E. Without loss of generality, we may assume that $\|E(s)\| \equiv 1$ so that x is a unit vector. The curvature hypothesis and the definition of the matrix $R(s)$ together imply that $\langle R(s)x, x\rangle = K(E, \gamma')(s) \geq 0$ for every real number s. Let $y(s) = \langle U(s)x, x\rangle$, where $U(s) = D'(s)D^{-1}(s)$. The function $g(s) = \{\| U(s)x\|^2 - \langle U(s)x, x\rangle^2 + \langle R(s)x, x\rangle\}$ is ≥ 0 for every number s by the Schwarz inequality and the curvature hypothesis. Since $U(s)$ is a symmetric solution of the Riccati equation

$$\text{(R)} \qquad U'(s) + U(s)^2 + R(s) = 0 ,$$

it follows that $y(s)$ satisfies the equation

$$y'(s) + y(s)^2 + g(s) = 0 .$$

For a given positive number ε, $g(s) > -\varepsilon^2$ for every real number s. Since $y(s)$ is defined for every real number s, it follows that $|y(s)| \leq \varepsilon$ by either Lemma 2.1 of [5] or the essentially identical argument in [9]. Since ε is arbitrary, $y(s) \equiv 0$ and therefore $g(s) \equiv 0$. It follows that $K(E, \gamma')(s) = \langle R(s)x, x\rangle = 0$ and $\| U(s)x\|^2 - \langle U(s)x, x\rangle^2 = 0$ for every number s, since both terms are nonnegative and their sum is $g(s)$. Since $0 = y'(s) = \langle U'(s)x, x\rangle$, it follows from equation (R) that $\| U(s)x\|^2 = \langle U(s)^2x, x\rangle \equiv 0$. Therefore $U'(s)x \equiv 0$, and it follows again from equation (R) that $R(s)x \equiv 0$. By the definition of $R(s)$, $R_{\gamma'(s)}E(s)\gamma'(s) \equiv 0$. Finally, $E''(s) + R_{\gamma'(s)}E(s)\gamma'(s) \equiv 0$, and E is a Jacobi vector field.

Proof of Corollary 3.5. Suppose that M has no focal points and satisfies the condition of Corollary 3.4. If the geodesic flow in SM is not of Anosov type, then Corollary 3.3 implies that there exists a unit speed geodesic γ in M which admits a nonzero perpendicular parallel Jacobi vector field Y along γ. Choose an adapted frame field along γ. Since $Y \in J_s(\gamma)$, we may choose a vector $x \in R^{n-1}$ such that Y corresponds to the curve $s \to D(s)x$ relative to the adapted frame field. Since Y is parallel, $D(s)x = x$ for every number s, and therefore $D'(s)x \equiv 0$. We see that $U(s)x = D'(s)D^{-1}(s)x = D'(s)x \equiv 0$, and it follows from equation (R) that $R(s)x \equiv 0$. Therefore $K(E, \gamma')(s) = \langle R(s)x, x\rangle \equiv 0$, which contradicts the hypothesis that M satisfies the condition of Corollary 3.4. This contradiction completes the proof of Corollary 3.5.

We conclude by constructing examples of compact Riemannian manifolds with curvature $K \leq 0$, Anosov geodesic flow, and open subsets where the sectional curvature is zero on all tangent planes.

We first construct a special convex function. The construction is due to H. Karcher.

Lemma 3.18. *Let numbers $b > 0$ and $\varepsilon > 0$ be given. Then there exist a number $0 < a < b$ and a C^∞ function $f: R \to R$ such that $f''(x) \geq 0$ for all x, $f(x) = x$ for $x \leq a$, and $f(x) = \sinh x$ for $x \geq b + \varepsilon$.*

Proof. Because of the convexity condition, the function f must always lie on one side of its tangent line, so the number a cannot be chosen arbitrarily close to b. The tangent line at b of the curve $y(x) = \sinh x$ is given by the equation $t(s) = \sinh b + (\cosh b)(s - b)$. Let $0 < a^* < b$ be chosen so that $t(a^*) = a^*$. If we consider the broken line segment defined by the condition that $y(x) = x$ for $x \leq a^*$, and $y(x) = t(x)$ for $x \geq a^*$, then we shall obtain the desired function f by smoothing the first derivative of $y(x)$ in the neighborhood of a^* and b. More exactly, we shall find a C^∞ function $K(x)$ such that $f(x) = \int_0^x K(x)dx$. We need a function $K(x)$ such that $K(x) = 1$ for $0 \leq x \leq \alpha$, where α is some number near a^*; $K(x) = \cosh b$ in some interval between a^* and b, and $K(x) = \cosh x$ for $x \geq \beta$, where β is some number near b.

Given $\varepsilon > 0$, we may assume that ε is so small that $a^* + \varepsilon < b - 2\varepsilon$. We first construct a nonnegative C^∞ function $\phi_\varepsilon(x)$ such that $\phi_\varepsilon(x) = 1$ for $x \geq \varepsilon$, and $\phi_\varepsilon(x) = 0$ for $x \leq -\varepsilon$. If $h(x) = 0$ for $x \leq 0$, and $h(x) = e^{-1/x^2}$ for $x \geq 0$, then $h(x)$ is C^∞. We define a C^∞ function $g_\varepsilon(x)$ by requiring that $g_\varepsilon(x) = e^{-\varepsilon^2/(\varepsilon^2 - x^2)}$ for $-\varepsilon < x < \varepsilon$, and $g_\varepsilon(x) = 0$ for $|x| \geq \varepsilon$. Let $\phi_\varepsilon(x) = \int_{-\varepsilon}^x g_\varepsilon(y)dy \Big/ \int_{-\varepsilon}^\varepsilon g_\varepsilon(y)dy$.

$\phi_\varepsilon(x)$ has the properties desired, and moreover $\int_{-\varepsilon}^\varepsilon \phi_\varepsilon(x)dx = \varepsilon$, since for any $x \geq 0$, $1 - \phi_\varepsilon(x) = \phi_\varepsilon(-x)$.

Let $m = \cosh b$, and let $F(x) = 1 + (m - 1)\phi_\varepsilon(x - a^*)$. Then $F(x)$ is a

C^∞ function such that $F(x) = 1$ for $x \leq a^* - \varepsilon$, and $F(x) = m$ for $x \geq a^* + \varepsilon$. If $G(x) = \int_0^x F(y)dy$, then $G(x)$ is C^∞, $G(x) = x$ for $x \leq a^* - \varepsilon$, and $G(x) = t(x)$ for $x \geq a^* + \varepsilon$. This completes the first step in the construction of the function $K(x)$.

To define $K(x)$ near b, we need a C^∞ function $H(x) \geq \max\{m, \cosh x\}$, which agrees with $\max\{m, \cosh x\}$ outside a small neighborhood of b. For any number r, let $\psi_r(x) = (\sinh x)\phi_\varepsilon(x - r)$. Then $0 \leq \psi_r(x) \leq \sinh x$ for $x \geq 0$. If $r = b - \varepsilon$, then $\psi_r(x) = 0$ for $x \leq b - 2\varepsilon$, and $\psi_r(x) = \sinh x$ for $x \geq b$. If $r = b + \varepsilon$, then $\psi_r(x) = 0$ for $x \leq b$, and $\psi_r(x) = \sinh x$ for $x \geq b + 2\varepsilon$. If we consider the continuous function $r \to \int_{r-\varepsilon}^{b} \psi_r(x)dx - \int_b^{r+\varepsilon} [\sinh x - \psi_r(x)]dx$, then we see that for some $r_0 \in (b - \varepsilon, b + \varepsilon)$, $\int_{r_0-\varepsilon}^{b} \psi_{r_0}(x)dx = \int_b^{r_0+\varepsilon} [\sinh x - \psi_{r_0}(x)]dx$. If $H(x) = m + \int_0^x \psi_{r_0}(y)dy$, then $H(x) \geq \max\{m, \cosh x\}$ for $x \geq 0$, and $H(x) = \max\{m, \cosh x\}$ for $|x - b| \geq 2\varepsilon$.

Finally, for any number r let $K_r(x) = F(x) + (H(x) - F(x))\phi_\varepsilon(x - r)$. Then $F(x) \leq K_r(x) \leq H(x)$ for $x \geq 0$, $K_r(x) = F(x)$ for $x \leq r - \varepsilon$, and $K_r(x) = H(x)$ for $x \geq r + \varepsilon$. If $r = b - \varepsilon$, then $K_r(x) \geq \max\{m, \cosh x\}$ for $x \geq a^* + \varepsilon$; if $r = b + 3\varepsilon$, then $K_r(x) \leq \max\{m, \cosh x\}$ for $x \geq a^* + \varepsilon$. By continuity we can find a number $r_1 \in [b - \varepsilon, b + 3\varepsilon]$ such that $\int_{b-2\varepsilon}^{b+4\varepsilon} K_{r_1}(x)dx = \int_{b-2\varepsilon}^{b+4\varepsilon} \max\{m, \cosh x\}dx$. If $K(x) = K_{r_1}(x)$, then $f(x) = \int_0^x K(y)dy$ is a C^∞ function such that $f(x) = x$ for $x \leq a^* - \varepsilon$ and $f(x) = \sinh x$ for $x \geq b + 4\varepsilon$. Since $f'(x) = K_{r_1}(x)$ is monotone nondecreasing, $f''(x) \geq 0$ for all x. Replacing ε by $\varepsilon/4$ and setting $a = a^* - \varepsilon$, we obtain a function with the properties described in the statement of the Lemma.

Let M be a compact C^∞ Riemannian manifold of dimension $n \geq 2$, metric g, and sectional curvature $K \equiv -1$. Fix a point $p \in M$. It is known that the exponential map $k\colon M_p \to M$ is a covering map. Let N be a covering neighborhood of p, and d be a positive number such that k maps the closed ball of radius d and center $\{0\}$ diffeomorphically onto its image $V \subseteq N$. By modifying the metric g inside V_0, the interior of V, we obtain a new Riemannian manifold M^* with Anosov geodesic flow and curvature $K^* \leq 0$ such that $K^*(\pi) = 0$ for each 2-plane π tangent to an arbitrary point of a small p-neighborhood $U_0 \subseteq V_0$.

For any positive number α, let $B_\alpha(0)$ denote the vectors in M_p whose g-norm is less than α. $B_d(0) - \{0\}$ can be represented diffeomorphically as a product $(0, d) \times S^{n-1}$ by means of the projections $\pi\colon M_p - \{0\} \to (0, \infty)$, $(\pi(x) = \|x\|)$, and $\eta: M_p - \{0\} \to S^{n-1}$, $(\eta(x) = x/\|x\|)$. Let g also denote the unique metric in $B_d(0)$ such that $k: B_d(0) \to V_0$ is an isometry. Let $x \in B_d(0)$ be given, and let v and w be vectors in $(M_p)_x$. If $x = 0$, then $\langle v, w \rangle_g$ is the usual Euclidean inner product. If $x \neq 0$, then

$$(*) \qquad \langle v, w \rangle_g = \langle \pi_* v, \pi_* w \rangle + (\sinh^2 \|x\|) \langle \eta_* v, \eta^* w \rangle ,$$

where the inner products on $(0, \infty)$ and S^{n-1} are the usual ones. By Lemma 3.18 we may choose positive numbers a and b such that $a < b < d$, and a C^∞ function $f: R \to R$ such that $f''(t) \geq 0$ for $t \geq 0$, $f(t) = t$ for $0 \leq t \leq a$, and $f(t) = \sinh t$ for $t \geq b$. By substituting the quantity $f^2(\|x\|)$ for the quantity $\sinh^2 \|x\|$ in the expression $(*)$, we obtain a new C^∞ metric $\bar{g}$ on $B_d(0)$ (see [3, pp. 33–34]). Define a metric g^* in M as follows: Let $g^* = g$ in $M - V_0$; in V_0 let g^* be the unique metric such that $k: B_d(0) \to V_0$ is an isometry with respect to the metrics $\bar{g}$ and g^*. The metric g^* is C^∞ on M, since it is C^∞ inside $V_0 = k(B_d(0))$, and it is equal to g in $V_0 - k(\overline{B_b(0)})$. Let q be an arbitrary point of M, and π be an arbitrary 2-plane tangent to M at q. Using the curvature formula of [3, p. 27] we see that 1) $K^*(\pi) \leq 0$, 2) $K^*(\pi) = 0$ if $q \in U_0 = k(B_a(0)) \subseteq V_0$, and 3) $K^*(\pi) = -1$ if $q \in M - k(\overline{B_b(0)})$.

If M^* denotes the manifold M furnished with the metric g^*, then we assert that the geodesic flow in SM^* is of Anosov type. Since $K^* \leq 0$, by Corollary 3.5 it suffices to show that every geodesic of M^* meets $M^* - V_0$. Suppose that some geodesic γ^* of M^* is contained in V_0. Let g^* also denote the unique metric in M_p such that $k: M_p \to M^*$ is a local isometry. Since V_0 is contained in a covering neighborhood of the map k, there exists a g^*-geodesic γ in M_p such that $k \circ \gamma = \gamma^*$ and γ is contained in $B_d(0)$. Since M_p with the metric g^* is complete and has curvature $K^* \leq 0$, the geodesic γ realizes the distance between any two of its points and cannot be contained in the compact set $\overline{B_d(0)}$. This contradiction shows that γ^* meets $M^* - V_0$.

We may further modify the metric g^* on M so that the g^*-geodesic flow is of Anosov type and all g^*-sectional curvatures at points of a larger open subset of M are identically zero. Let $\{U_\alpha\}$, $\alpha \in S$, be a collection of pairwise disjoint open subsets of M, and for each α let p_α be a point of U_α. Let $k_\alpha: M_{p_\alpha} \to M$ denote the g-exponential map at p_α, and choose a number $d_\alpha > 0$ so that $k_\alpha: B_{d_\alpha}(0) \to V_\alpha \subseteq U_\alpha$ is a diffeomorphism. Obtain a new metric g^* on M by modifying the original metric g inside each set V_α in the manner described above. The manifold M^* with the metric g^* has sectional curvature $K^* \leq 0$, and the sectional curvature vanishes in a neighborhood of the set $\{p_\alpha\}$. The geodesic flow in SM^* is of Anosov type; a connectedness argument shows that no maximal g^*-geodesic is contained in the union of the sets V_α since these sets are disjoint covering neighborhoods and $\overline{B_{d_\alpha}(0)} \subseteq M_{p_\alpha}$ is compact.

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