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MANIFOLD MAPS COMMUTING WITH THE LAPLACIAN

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The commutative algebra $\mathscr{D}(G/H)$ of isometry-invariant differential operators on a Riemannian symmetric space always contains the Laplace-Beltrami operator Δ. In fact, Δ is the generator of $\mathscr{D}(G/H)$ exactly when G/H is of rank one. Therefore it is natural to ask which manifold maps $\varphi: G_1/H_1 \to G_2/H_2$ commute with the Laplacian on C^∞ functions of G_2/H_2, i.e., $\varphi^*\Delta_2 f = \Delta_1\varphi^* f$ for all $f \in C^\infty(G_2/H_2)$. Helgason [3, p. 387] showed for a general pseudo-Riemannian manifold M that the only diffeomorphisms $\Phi: M \to M$ which commute with Δ are the isometries. Recalling the powerful de Rham-Hodge theorem (classical real pth cohomology group $\cong$ pth de Rham cohomology group $\cong$ space of harmonic p-forms) on compact Riemannian manifolds, the above question should be: which surjective maps $\varphi: M \to N$ commute with Δ on differential p-forms for compact M and N?

Our main results are:

(1) Every such Laplacian-commuting map is a Riemannian submersion, and therefore is a locally trivial differentiable Riemannian fibre space.

(2) If there exists such a map $\varphi: M \to N$ commuting with Δ on p-forms for compact M and fixed p, then $b_p(N) \leq b_p(M)$.

(3) For compact $M, \varphi: M \to N$ commutes with the Laplacian on functions if and only if φ is a harmonic Riemannian submersion.

An analogous question "which compact fibre space mappings $\pi: E \to B$ commute with the codifferential operator δ on forms of all degrees simultaneously" has been answered in certain specific cases [5], but our result is more general.

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1. Definitions

All manifolds will be real, compact, connected, smooth, oriented and Riemannian. Let M (resp. N) have dimension m (resp. n) and Riemannian

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structure g (resp. h). Since we will never consider powers of the Laplacian operator, it should cause no confusion to let Δ_M^p denote the Laplacian operator $-(d\delta + \delta d) = \Delta: \Lambda^p(M) \to \Lambda^p(M)$ on the differential p-forms of M.

We define the set of *pth Laplacian commuters* to be

$$\Omega^p(M, N) = \begin{Bmatrix} \varphi: M \to N \text{ is a } C^3 \text{ surjective manifold map and} \\ \varphi^* \Delta_N^p \alpha = \Delta_M^p \varphi^* \alpha \text{ for all } \alpha \in \Lambda^p(N) \end{Bmatrix} .$$

If $\Omega^p(M,N)$ is empty, we say $\Omega^p(M,N)$ is *trivial*. Similarly, $\Omega^p(M,N)$ is *trivial* if $p > \min\{m, n\}$.

Proposition 1.1. *If $\varphi \in \Omega^p(M,N)$ and $\psi \in \Omega^p(N,N')$, then $\psi \circ \varphi \in \Omega^p(M,N')$.*

Proof. Easy.

Recall from [1] that a mapping $\varphi: M \to N$ is *harmonic* if $\tau(\varphi) = 0$, where the *tension field* $\tau(\varphi)$ is locally given by

$$\tau(\varphi)^r = -\Delta_M^0(\varphi^r) + (\bar{\Gamma}_{\alpha\beta}^r \circ \varphi) g^{ij} \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_j} .$$

(The superior bar indicates parameters of the target manifold N.) In that paper, Eells and Sampson defined certain vector-valued differential forms associated to the vector bundle $\pi: \varphi^{-1}(T(N)) \to M$ and a Laplacian operator $\tilde{\Delta}$ on these forms. For instance, $\varphi_*: T(M) \to T(N)$ is a differential vector-valued 1-form and $\tau(\varphi): M \to T(N)$ is a vector-valued 0-form.

It is known [1, p. 123] that φ is a harmonic mapping if and only if $\tilde{\Delta}\varphi_* = 0$.

Let $\beta(\varphi)$ be the fundamental form of the mapping φ. If φ is a Riemannian submersion, and T and A denote its structure tensors [6], then essentially $\beta(\varphi)$ has the matrix form:

$$\begin{pmatrix} 0 & -A \\ -A & -T \end{pmatrix}$$

corresponding to the orthogonal decomposition $T(M) = H \oplus V$ induced by φ_* on the tangent bundle of M. Since $\mathrm{Tr}\,(\beta(\varphi)) = \tau(\varphi)$, we infer

Proposition 1.2. *A Riemannian submersion $\varphi: M \to N$ is harmonic if and only if* $\mathrm{Tr}\,(T) = 0$.

We also note that T is actually the second fundamental form of the fibres $\varphi^{-1}(y)$ of the Riemannian submersion φ. Recall, too, that a submanifold F of M is *minimal* (resp. *totally geodesic*) if the trace of the second fundamental form (resp. the entire second fundamental form) of the immersion of F into M is identically zero.

2. 0-forms

Theorem 2.1. *A c^3 mapping $\varphi \in \Omega^0(M,N)$ if and only if* (a) *φ is a Riemannian submersion and* (b) *one of the following equivalent conditions holds*:

(i) *φ is a harmonic mapping,*

(ii) $\tilde{\Delta}\varphi_* = 0$,

(iii) *each fibre $\varphi^{-1}(y)$ of φ is a (compact) minimal submanifold of M,*

(iv) $\mathrm{Tr}\,(\beta(\varphi)) = \mathrm{Tr}\,(T) = 0$.

Proof. Let $\varphi \in C^3(M,N)$. We take local coordinates about $x \in M$ and $\varphi(x) \in N$. Choose $f \in \Lambda^0(N) = C^\infty(N)$ arbitrarily. Then $\varphi \in \Omega^0(M,N)$ if and only if $\varphi^*\Delta_N^0 f = \Delta_M^0 \varphi^* f$. In local coordinates $\Delta_M^0 = -g^{ij}\nabla_i\nabla_j$, and similarly for Δ_N^0. Expressing the commutation condition in local coordinates and comparing the corresponding terms which contain $\partial f/\partial x_i$ and $\partial^2 f/\partial x_i \partial x_j$, we obtain

$$(1) \qquad h^{\alpha\beta} \circ \varphi = g^{ij} \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_j} \qquad \text{for all } \alpha, \beta \,,$$

$$(2) \qquad \Delta_M^0(\varphi^\gamma) = (h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}) \circ \varphi \qquad \text{for all } \alpha, \beta, \gamma \,.$$

(1) is equivalent to φ being a Riemannian submersion. Substitution of (1) into (2) and a glance at the definition of the tension field $\tau(\varphi)$ yield that φ must be a harmonic mapping. Then the statement of the theorem follows from Proposition 1.2 and its accompanying remarks.

Corollary 2.1.1. *If $\dim M < \dim N$, then $\Omega^0(M,N)$ is trivial.*

Proof. A Riemannian submersion is an open map.

Corollary 2.1.2. *A C^3 mapping $\varphi\colon M \to N$ which commutes with Δ on 0-forms is actually C^∞.*

Proof. C^3 harmonic mappings are smooth [1, p. 117].

In the cases

(a) $\dim M = \dim N$,

(b) $\dim M = \dim N + 1$,

we are able to completely classify the manifold pairs (M, N) which admit nontrivial sets $\Omega^0(M,N)$. First, we have

Theorem 2.2. *Suppose M and N have the same dimension. Then $\Omega^0(M,N)$ is nontrivial if and only if M is a Riemannian covering manifold of N.*

Proof. Let $\varphi \in \Omega^0(M,N)$. Then the Riemannian submersion criterion is equivalent to

$$J(\varphi) h J(\varphi)^t = g$$

as $(m \times m)$-matrices. Hence

$$\det\,(J(\varphi))^2 = \frac{\det g}{\det h} > 0 \,.$$

So φ has a nonvanishing Jacobian determinant and is a local isometry. For complete manifolds, it is well-known [7, p. 254] that this implies φ is a Riemannian covering map. Conversely, Riemannian coverings are clearly harmonic Riemannian submersions.

Corollary 2.2.1. $\Omega^0(M, M) = I(M)$.

Secondly, we have

Theorem 2.3. *Suppose* $\dim M = \dim N + 1$. *Then* $\Omega^0(M, N)$ *is nontrivial if and only if M is a fibre bundle over N, which has totally geodesic fibres and has the Lie group of isometries of a fibre as structural group.*

Proof. Necessity is clear because fibre bundle mappings are Riemannian submersions, and, for fibre dimension 1, the notions of minimal and totally geodesic submanifolds coincide. Sufficiency follows from Hermann's theorem [4, Theorem 1].

For $\dim M \geq \dim N + 2$, the problem of a complete classification reduces to classifying those locally trivial compact fibre spaces whose fibre map has fundamental form

$$\beta(\pi) = \begin{pmatrix} 0 & * \\ * & \mathrm{Tr} = 0 \end{pmatrix} .$$

This is a decidedly nontrivial question. We may, however, produce several nonexistence theorems.

Theorem 2.4. *Suppose that the Ricci tensor* $\{R_{ij}(x)\}$ *is positive semidefinite everywhere on the compact manifold M and is positive definite at least at one point of M. If the Riemannian curvature of N is nonpositive, then* $\Omega^0(M, N)$ *is trivial.*

Proof. A result of Eells and Sampson [1, p. 124] implies that under these curvature conditions, every harmonic mapping $\varphi\colon M \to N$ is a constant, and the constants are not in $\Omega^0(M, N)$.

Theorem 2.5. *Suppose* $\dim N > 1$ *and N has everywhere negative Riemannian curvature. If M has a positive semidefinite Ricci tensor, then* $\Omega^0(M, N)$ *is trivial.*

Proof. The result of Eells and Sampson just mentioned above implies here that each harmonic mapping $\varphi\colon M \to N$ is either a constant or $\varphi(M)$ is a geodesic of N. Constants are excluded as we have remarked previously, and a Laplacian commuter on 0-forms is a surjective mapping by Theorem 2.1. Hence $\varphi(M) = N$ whose dimension is greater than 1, and $\varphi(M)$ cannot then be a geodesic of N.

Let H^n denote any n-dimensional compact manifold of negative Riemannian curvature. Then we have

Corollary 2.5.1. $\Omega^0(T^m, H^n)$ *and* $\Omega^0(S^m, H^n)$ *are trivial.*

3. Examples

There are many more nontrivial examples of sets $\Omega^0(M, N)$ besides those which have already been exhibited in Theorem 2.2. From Theorem 2.3 we see that $\Omega^0(S^{2n+1}, P_n(C))$ is nontrivial. Clearly, $\Omega^0(M_1 \times \cdots \times M_r, M_i)$ is nontrivial because product projection maps are totally geodesic Riemannian submersions, i.e., $\beta(\pi_i) = 0$. The Hopf map $\lambda: S^7 \to S^4$ is a fibre bundle map with totally geodesic fibres. Hence $\Omega^0(S^7, S^4)$ is nontrivial. Let G/H be a compact oriented Riemannian homogeneous coset space with compact Lie group G, and give G the usual bi-invariant metric. Then $\pi: G \to G/H$ is a principal fibre bundle with $T = 0$. So $\Omega^0(G, G/H)$ is nontrivial. In fact, $T = 0$ for any principal fibre bundle $\pi: P \to M$. Thus $\Omega^0(P, M)$ is nontrivial in general. Let $\pi: \mathscr{F}(M) \to M$ be the orthogonal frame bundle of M. Then $\Omega^0(\mathscr{F}(M), M)$ is nontrivial.

4. 1-forms

Assume a differential 1-form α on N is expressed locally as $\alpha = a_i dy^i$. Then $\Delta_N^1 \alpha = (\Delta_N^0 a_i + a_j \bar{R}_i^j) dy^i$. The proof of the following theorem is straightforwardly modeled upon that of Theorem 2.1, incorporating the above formula for Δ^1.

Theorem 4.1. *A c^3 mapping $\varphi \in \Omega^1(M, N)$ if and only if* (a) *φ is a Riemannian submersion and*

$$\text{(b)}\quad \frac{\partial \varphi^i}{\partial x_k} \tau(\varphi)^\gamma = -2 g^{\alpha\beta} \frac{\partial \varphi^\gamma}{\partial x_\alpha} \frac{\partial^2 \varphi^i}{\partial x_\beta \partial x_k} \qquad \forall i, k, \gamma ,$$

$$\text{(c)}\quad \Delta_M^0 \left(\frac{\partial \varphi^i}{\partial x_k} \right) = \frac{\partial \varphi^j}{\partial x_k} (\bar{R}_j^i \circ \varphi) - \frac{\partial \varphi^i}{\partial x_j} R_k^j \qquad \forall i, k .$$

The Riemannian submersion criterion (a) produces the obvious analogues of Corollaries 2.1.1 and 2.1.2. Condition (c) of Theorem 4.1 is not unnatural, although it looks so at the first glance. In their paper [1], Eells and Sampson found it unnecessary to explicitly compute $\tilde{\Delta}\varphi_*$ for $\varphi: M \to N$ because they were studying harmonic mappings ($\tilde{\Delta}\varphi_* \equiv 0$). However an elementary calculation shows

Lemma 4.2. *The term corresponding to dx^k in the γ-th component of the vector-valued 1-form $\tilde{\Delta}\varphi_*$ is*

$$(\tilde{\Delta}\varphi_*)_k^\gamma = -\Delta_M^0 \left(\frac{\partial \varphi^\gamma}{\partial x_k} \right) + \frac{\partial}{\partial x_k} \{ (h^{\alpha\beta} \bar{\Gamma}_{\alpha\beta}^\gamma) \circ \varphi \} + \tau(\varphi)^\alpha (\bar{\Gamma}_{\alpha\beta}^\gamma \circ \varphi) \frac{\partial \varphi^\beta}{\partial x_k} .$$

Theorem 4.3. *Suppose M is Einsteinian and N is flat. If $\varphi \in \Omega^1(M, N)$, then*

$$\tilde{\Delta}\varphi_* = \lambda \varphi_* ,$$

where

$$\lambda = R/m .$$

Proof. If M is Einsteinian, $R_j^i = \delta_j^i R/m$. N flat implies $\bar{R} = 0 = \bar{\Gamma}_{\alpha\beta}^\gamma$. Assume $\varphi \in \Omega^1(M, N)$. Then Theorem 4.1 (c) gives

$$\Delta_M^0\left(\frac{\partial\varphi^\gamma}{\partial x_k}\right) = -\frac{R}{m}\frac{\partial\varphi^\gamma}{\partial x_k} .$$

So

$$(\tilde{\Delta}\varphi_*)_k^\gamma = -\Delta_M^0\left(\frac{\partial\varphi^\gamma}{\partial x_k}\right) = \frac{R}{m}\frac{\partial\varphi^\gamma}{\partial x_k} = \frac{R}{m}(\varphi_*)_k^\gamma .$$

Corollary 4.3.1. *If M is Einsteinian, N is flat, and $\Omega^1(M,N)$ is nontrivial, then the constant Ricci scalar curvature of M is nonpositive.*

Proof. The eigenvalues of the elliptic self-adjoint operator $\tilde{\Delta}$ are discrete and nonpositive.

Notice that if both M and N are arbitrary Einstein spaces and $\varphi \in \Omega_0^1(M,N)$, then

$$(\tilde{\Delta}\varphi_*)_k^\gamma = \left(\frac{R}{m} - \frac{\bar{R}}{m}\right)(\varphi_*)_k^\gamma + (h^{\alpha\beta}\circ\varphi)\frac{\partial}{\partial x_k}(\bar{\Gamma}^\gamma_{\alpha\beta}\circ\varphi)$$
$$+ (\Gamma^\gamma_{\alpha\beta}\circ\varphi)\frac{\partial g^{ij}}{\partial x_k}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j} .$$

The possibility that the last two terms on the right side might collapse and so produce a global invariant condition of an eigen-1-form type: $\tilde{\Delta}\varphi_* = \lambda\varphi_*$ has not escaped the author. Theorem 4.3 is promising in this regard.

Theorem 4.4. *Assume* $\dim M = \dim N$. *Then $\Omega^1(M,N)$ is nontrivial if and only if N is a Riemannian covering manifold of N.*

Proof. Since $\dim M = \dim N$, by Theorem 2.2 a Riemannian submersion $\varphi\colon M \to N$ is a Riemannian covering map. Lichnerowicz [4] showed that mappings of totally geodesic locally trivial fibre spaces commute with δ and so with Δ on forms of all degrees. A Riemannian covering map is clearly a map of totally geodesic locally trivial fibre spaces.

Corollary 4.4.1. $\Omega^1(M,M) = I(M)$.

5. Forms of degree ≥ 2

Due to the complexity of the expression in local coordinates for the Laplacian on forms of degree ≥ 2, (a complete either local or global) classification is lacking. This complexity is attributable to the introduction of $R^{ij}{}_{kl}$ terms into the formula for Δ. Recalling that since for spaces of constant sectional curvature, $R^i{}_j$ and $R^{ij}{}_{kl}$ are usually zero and at least constants, there appears hope that a local theorem may be obtained for Laplacian commuters between such spaces. The tantalizing connection between $\Omega^p(M,N)$ and $\tilde{\Delta}$ for $p = 0, 1$ indicates that a necessary and sufficient invariant condition for commuting with the Laplacian Δ lies in the operator $\tilde{\Delta}$.

However, a necessary condition for a map to commute with Δ on arbitrary manifolds can be found, and from that condition (Theorem (5.1)) an equal-dimension classification theorem and several nonexistence results will follow.

Theorem 5.1. *For all p the nontriviality of $\Omega^p(M, N)$ implies that each map $\varphi \in \Omega^p(M, N)$ is a Riemannian submersion.*

Proof. Since we already have the theorem for $p = 0, 1$, we assume $p \geq 2$. If $\alpha \in \Lambda^p(N)$ is expressed locally as

$$\alpha = \frac{1}{p!} a_{i_1 \cdots i_p} dy^{i_1} \wedge \cdots \wedge dy^{i_p} ,$$

then

$$\Delta_N^p \alpha = \frac{1}{p!} b_{i_1 \cdots i_p} dy^{i_1} \wedge \cdots \wedge dy^{i_p} ,$$

where

$$b_{i_1 \cdots i_p} = \Delta_N^0 (a_{i_1 \cdots i_p}) + \sum_{r=1}^{p} a_{i_1 \cdots i_{r-1} j i_{r+1} \cdots i_p} \bar{R}^j{}_{i_r} + \tfrac{1}{2} \sum_{r=1}^{p} \sum_{s=1}^{p} a_{i_1 \cdots i_{r-1} j i_{r+1} \cdots i_{s-1} k i_{s+1} \cdots i_p} \bar{R}^{kj}{}_{i_r i_s} .$$

Let $\varphi \in \Omega_0^p(M, N)$. Upon calculating the equation

$$\varphi^* \Delta_N^p \alpha = \Delta_M^p \varphi^* \alpha$$

in local coordinates, we find

$(\varphi^*(\Delta_N^0(a_{i_1 \cdots i_p}))) \dfrac{\partial \varphi^{i_1}}{\partial x_{j_1}} \cdots \dfrac{\partial \varphi^{i_p}}{\partial x_{j_p}}$ + (other terms in $\varphi^* a_{i_1 \cdots i_p}$ and φ^*-pullbacks of the curvature tensors of N, but *not* containing terms which involve the second partial derivatives of $\varphi^* a_{i_1 \cdots i_p}$)

$= (\Delta_M^0(\varphi^* a_{i_1 \cdots i_p})) \dfrac{\partial \varphi^{i_1}}{\partial x_{j_1}} \cdots \dfrac{\partial \varphi^{i_p}}{\partial x_{j_p}} + (\varphi^* a_{i_1 \cdots i_p}) \Delta_M^0 \left(\dfrac{\partial \varphi^{i_1}}{\partial x_{j_1}} \cdots \dfrac{\partial \varphi^{i_p}}{\partial x_{j_p}} \right)$ + (other terms *not* containing second partial derivatives of the $a_{i_1 \cdots i_p}$).

From the corresponding terms containing $\partial^2 a_{i_1 \cdots i_p} / \partial x_\delta \partial x_\tau$ we obtain

$$\left(\frac{\partial \varphi^{i_1}}{\partial x_{j_1}} \cdots \frac{\partial \varphi^{i_p}}{\partial x_{j_p}} \right) h^{\delta \tau} = \left(\frac{\partial \varphi^{i_1}}{\partial x_{j_1}} \cdots \frac{\partial \varphi^{i_p}}{\partial x_{j_p}} \right) g^{kl} \frac{\partial \varphi^\delta}{\partial x_k} \frac{\partial \varphi^\tau}{\partial x_l} .$$

Since $\varphi \in \Omega^p(M, N)$, it is surjective and not all of its first partial derivatives are locally zero. Thus

$$h^{\delta \tau} = g^{kl} \frac{\partial \varphi^\delta}{\partial x_k} \frac{\partial \varphi^\tau}{\partial x_l} ,$$

and therefore φ is a Riemannian submersion.

Corollary 5.1.1. *If $\Omega^p(M, N)$ is nontrivial and $\varphi \in \Omega^p(M, N)$, then $\varphi: M \to N$ is a compact locally trivial fibre space in the sense of Ehresmann.*

Corollary 5.1.2. *If* $\dim M < \dim N$, *then $\Omega^p(M, N)$ is trivial for all p.*

As before, we have

Theorem 5.2. *Suppose* $\dim M = \dim N$. *Then $\Omega^p(M, N)$ is nontrivial if and only if M is a Riemannian covering manifold of N.*

Corollary 5.2.1. *$\Omega^p(M, M) = I(M)$ for all p.*

Theorem 5.3. *Let M and N be spaces of constant sectional curvature K and $\bar{K}$, respectively. In order that $\Omega^p(M, N)$ be nontrivial for any $p \geq 0$ it is necessary that $\bar{K} \geq K$.*

Proof. It is well-known [2, p. 724] that for any Riemannian submersion $\varphi: M \to N$ between arbitrary complete Riemannian manifolds, $(\bar{K}_{\tilde{X}\tilde{Y}} \circ \varphi) \geq K_{XY}$ where $\tilde{X}$ is the φ-related vector field on N associated to the vector field X on M.

Let $S^m(r)$ denote the m-sphere of radius r. For $S^m(r), K = 1/r^2$.

Corollary 5.3.1. *If $r < r'$, then $\Omega^p(S^m(r), S^n(r'))$ is trivial for all p.*

Corollary 5.3.2. *$\Omega^p(S^m, T^n)$ is trivial for all p.*

As we have seen (Corollary 5.1.1), all Laplacian commuters are locally trivial fibre spaces. Lichnerowicz [5] has shown that any mapping of locally trivial compact fibre spaces with minimal fibres (Tr $(T) = 0$) commutes with the codifferential operator δ and so with $\Delta = -(d + \delta)^2$ on forms of all degrees if and only if the horizontal distribution is completely integrable ($A = 0$). Since from the proof of Theorem 4 any totally geodesic Riemannian submersion commutes with Δ^p for all p, we have

(a) $\Omega^p(M_1 \times \cdots \times M_r, M_i)$ is nontrivial for all $p \leq \dim M_i$,

(b) $\Omega^p(T^m, T^n)$ is nontrivial for $m \geq n$ and all $p \leq n$.

6. Cohomology

The following result is obvious.

Theorem 6.1. *For a fixed p suppose $H^p(N, R)$ is nontrivial, while $H^p(M, R) = \{0\}$. Then $\Omega^p(M, N)$ is trivial.*

Proof. Let $\mathscr{H}^p(M)$ denote the space of harmonic p-forms of M. If $\varphi \in \Omega^p(M, N)$, then $\varphi^*\{\mathscr{H}^p(N)\} \subseteq \mathscr{H}^p(M)$ and $\varphi^*\{\mathscr{H}^p(N)\}$ is nontrivial. Hence the result follows immediately from Hodge's theorem.

A more powerful cohomology result is possible.

Theorem 6.2. *For a fixed p suppose $\Omega^p(M, N)$ is nontrivial. Then $b_p(N) \leq b_p(M)$.*

Proof. Since both M and N are connected, $H^0(M, R) = H^0(N, R) = R$. Thus $b_0(N) = 1 = b_0(M)$. So we can assume that we have fixed $p \geq 1$. As we have seen in the proof of Theorem 6.1, $\varphi \in \Omega^p(M, N)$ implies

$$\{0\} \neq \varphi^*\{\mathscr{H}^p(N)\} \subseteq \mathscr{H}^p(M) .$$

Thus

$$\dim \varphi^*\{\mathscr{H}^p(N)\} \leq \dim \mathscr{H}^p(M) .$$

Since $\varphi^* : \mathscr{H}^p(N) \to \mathscr{H}^p(M)$ is a distance-preserving linear mapping, Ker $(\varphi^*) = \{0\}$ and therefore

$$\dim \varphi^*\{\mathscr{H}^p(N)\} = \dim \mathscr{H}^p(N) .$$

Hence, from Hodge's theorem,

$$b_p(N) = \dim \mathscr{H}^p(N) \leq \dim \mathscr{H}^p(M) = b_p(M) .$$

We begin our applications of Theorem 6.2 by showing that the nontriviality of Ω^0 does not, in general, force the nontriviality of $\Omega^p, p \geq 1$. For instance,

Theorem 6.3. *$\Omega^{2k}(S^{2n+1}, P_n(C))$ is trivial for $k \geq 1$.*

Proof. $b_{2k}(P_n(C)) = 1$ and $b_{2k}(S^{2n+1}) = 0$.

Theorem 6.4.

(i) *$\Omega^0(U(5), G_{5,3}(C))$ is nontrivial,*

(ii) *$\Omega^4(U(5), G_{5,3}(C))$ is trivial.*

Proof. (i) follows from our remarks in § 3 regarding homogeneous coset spaces. To see (ii) we need only to notice that $b_4(U(5)) = 1$, while $b_4(G_{5,3}(C)) = 2$.

The examples of Ω^p-triviality produced so far involve even-degree forms. However, this is not necessary, as we see in

Theorem 6.5.

(i) *$\Omega^0(SO(6), \tilde{G}_{6,3}(R))$ is nontrivial,*

(ii) *$\Omega^9(SO(6), \tilde{G}_{6,3}(R))$ is trivial.*

Proof. $\pi: SO(6) \to SO(6)/(SO(3) \times SO(3)) \cong \tilde{G}_{6,3}(R)$ is a compact oriented reductive homogeneous coset space. Hence (i). The 9-th Betti number of the compact Lie group $SO(6)$ is 0. To calculate $b_9(\tilde{G}_{6,3}(R))$ we note that the dimension of the simply connected Grassmann manifold in 9. Poincaré duality then implies $b_9 = 1$, and hence (ii) follows from Theorem 6.2.

Use of Poincaré duality also gives

Theorem 6.6. *Assume N is a compact manifold of dimension $n < \dim M$. Then $\Omega^n(S^m, N)$ is trivial.*

Corollary 6.6.1. *$\Omega^n(S^m, S^n)$ is trivial for $m > n$.*

Thus the Hopf maps $\lambda: S^7 \to S^4$ and $\mu: S^3 \to S^2$ do not commute with Δ on 4-forms and 2-forms, respectively, although they both commute with the Laplacian on functions.

7. Appendix: The noncompact case

If M is not compact but only complete, Theorem 5.1 is still true with only basic assumptions (e.g., connectedness and completeness) on N. Thus Laplacian-commuting maps are locally trivial Riemannian fibre spaces in the noncompact case also. In this way, the case where $\dim M < \dim N$ is excluded, and we again obtain the theorem for the case of equal dimensions. However, the results on betti numbers are not to be expected because of the lack of a Hodge theorem relating the topological betti numbers with the de Rham cohomology theory through the harmonic forms.

It is straightforward to prove, using the maximum modulus principle on harmonic functions, that:

(a) $\Omega^p(R^m, R^n)$ is trivial if $m < n$,

(b) $\Omega^p(R^m, R^m) =$ the Euclidean group $E(m)$ on R^m,

(c) $\Omega^p(R^m, R^n) = \pi \circ E(m)$ if $m > n$, where $\pi: R^m \to R^n$ is the canonical projection mapping.

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