

COMPACT CONFORMALLY FLAT RIEMANNIAN MANIFOLDS

SHŪKICHI TANNO

1. Introduction

Sufficient conditions for a conformally flat Riemannian manifold to be a space of constant curvature were given by M. Tani [15] and S. I. Goldberg [5], etc. In this paper we study compact conformally flat Riemannian manifolds with finite fundamental group, and obtain the following theorems. Throughout this paper manifolds are assumed to be connected, of class C^∞ , and of dimension $m \geq 3$.

Theorem A. *Let (M^m, g) be a compact conformally flat Riemannian manifold with finite fundamental group. If the scalar curvature S of (M^m, g) is constant, then S is positive and (M^m, g) is of constant curvature.*

If a complete Riemannian manifold (M^m, g) is of positive Ricci curvature ($\geq \epsilon > 0$), M^m is compact and has finite fundamental group (S. B. Myers [10]). Hence, as a natural consequence of Theorem A, we have M. Tani's theorem:

Corollary [15]. *Let (M^m, g) be a compact orientable conformally flat Riemannian manifold. Then (M^m, g) is of constant curvature if its Ricci curvature is positive and scalar curvature is constant.*

By $S^m(K)$ we denote a Euclidean m -sphere of constant curvature K .

Theorem B. *Let (M^m, g) be a compact conformally flat Riemannian manifold with finite fundamental group and constant scalar curvature S . If (M^m, g) admits a nonisometric conformal transformation, then (M^m, g) is isometric to $S^m(K)$ where $K = S/[m(m-1)]$.*

It is well known that there are no harmonic p -forms for $0 < p < m$ on a compact orientable conformally flat Riemannian manifold (M^m, g) of positive Ricci curvature; thus (M^m, g) is a rational homology sphere ([3], [9], see also [18, Theorem 4.1]). A more precise statement is as follows.

*If a complete conformally flat Riemannian manifold (M^m, g) is of positive Ricci curvature ($\geq \epsilon > 0$), then the universal covering Riemannian manifold $(^*M^m, ^*g)$ of (M^m, g) is conformorphic to $S^m(1)$.*

With respect to this we have

Theorem C. *The following three are equivalent:*

(i) *On a compact conformally flat Riemannian manifold (M^m, g) with finite*

fundamental group, we have a conformally related Riemannian metric g^* of constant curvature.

(ii) On a compact conformally flat Riemannian manifold (M^m, g) with finite fundamental group, we have a conformally related Riemannian metric g^* whose scalar curvature is constant.

(iii) For a finite group $\{\varphi_0 = \text{identity}, \varphi_1, \dots, \varphi_{k-1}\}$ of conformal transformations acting on $S^m(1)$ without fixed point, there are a conformal transformation θ and isometries ϕ_i on $S^m(1)$ such that $\varphi_i = \theta^{-1} \cdot \phi_i \cdot \theta$, $i = 1, \dots, k-1$.

With respect to the integral of the scalar curvature, we have

Theorem D. Let (M^m, g) be a compact conformally flat Riemannian manifold with finite fundamental group. Then

$$\int_M S dM > 0,$$

where dM denotes the volume element of $(M, {}^m g)$.

2. Proof of theorems

To prove our theorems we need to apply the following lemmas.

Lemma 1 (Kuiper [7]). A conformally flat simply connected Riemannian manifold (M^m, g) is conformomorphic to an open submanifold of $S^m(1)$. In particular, if M^m is compact, then (M^m, g) is conformomorphic to $S^m(1)$.

Lemma 2 (Obata [12, Theorem 1]). There is no conformal transformation between compact (M^m, g) of constant scalar curvature $S \geq 0$ and (M^m, g') of constant scalar curvature $S' \leq 0$ except for $S = S' = 0$.

Lemma 3 (Obata [14, Proposition 6.1]). Let g_0 be the Riemannian metric of $S^m(1)$, and g^* another Riemannian metric on S^m conformal to g_0 . Then g^* has constant scalar curvature $S^* = m(m-1)$ if and only if g^* has constant curvature 1.

Lemma 4 (Nagano [11]). Let (M^m, g) , $m \geq 3$, be a complete Riemannian manifold with parallel Ricci tensor. If (M^m, g) admits a conformal transformation f , then one of the three cases occurs: (1) f is an isometry, (2) f is homothetic, and (M^m, g) is isometric to the Euclidean space, (3) (M^m, g) is isometric to $S^m(K)$.

Lemma 5 (Trudinger [14, Theorem 2, Corollary 1]). Let (M^m, g) be a compact Riemannian manifold with nonpositive total scalar curvature, i.e.,

$$\int_M S dM \leq 0,$$

and $m \geq 3$. Then there is a conformally related Riemannian metric g^* whose scalar curvature is nonpositive and constant.

Proof of Theorem A. Let $(^*M^m, ^*g)$ be the universal covering Riemannian manifold of (M^m, g) . Since the fundamental group of M^m is finite, M^m is compact. Since by the hypothesis $(^*M^m, ^*g)$ has constant scalar curvature $^*S = S$, by Lemma 1 we have a conformal diffeomorphism f of $(^*M^m, ^*g)$ to $S^m(1)$, and therefore by Lemma 2 the scalar curvature S cannot be nonpositive. If we consider a homothetic deformation $g \rightarrow [S/(m(m-1))]g$, it follows that $[S/(m(m-1))]g$ has constant scalar curvature $m(m-1)$. Now define a Riemannian metric g^* on S^m by $g^* = [S/(m(m-1))]f^{-1*}g$. Then g^* is conformal to g_0 and $S^* = m(m-1)$. By Lemma 3, g^* is of constant curvature 1, so that *g and hence g are of constant curvature $S/[m(m-1)]$.

Proof of Theorem B. By S we denote the scalar curvature of (M^m, g) . Then (M^m, g) is of positive constant curvature $K = S/[m(m-1)]$ by Theorem A, and is isometric to $S^m(K)$ by Lemma 4.

Proof of Theorem C. (i) \Rightarrow (ii) is obvious.

We prove (ii) \Rightarrow (iii). Let $G = \{\varphi_0, \varphi_1, \dots, \varphi_{k-1}\}$ be a finite group of conformal transformations on $S^m(1)$ acting without fixed point. Since G is a compact group, we have a conformally related Riemannian metric g' such that G is isometric with respect to g' (cf. Ishihara [6], more precisely, $(g')_x = [g_x + (\varphi_1^*g)_x + \dots + (\varphi_{k-1}^*g)_x]/k, x \in S^m$). (S^m, g') is factorized by G and we get a Riemannian manifold (M^m, g') . Denote the projection by π . Because (M^m, g') is compact conformally flat with finite fundamental group, by (ii) we have a conformally related Riemannian metric g^* whose scalar curvature S^* is constant, so that we can assume that $S^* = m(m-1)$. Denoting π^*g^* by g^* again, (S^m, g^*) is the universal covering Riemannian manifold of (M^m, g^*) . Since φ_i induces an isometry φ_i^* of (S^m, g^*) , $\theta': x \in (S^m, g') \rightarrow x \in (S^m, g^*)$ is a conformal transformation, and therefore $\theta'': (S^m, g_0) \rightarrow (S^m, g') \rightarrow (S^m, g^*)$ is also so. Thus we have $\varphi_i = \theta''^{-1} \cdot \varphi_i^* \cdot \theta''$, $i = 1, \dots, k-1$. By Lemma 3, (S^m, g^*) is of constant curvature 1. Hence we obtain an isometry $\gamma: (S^m, g^*) \rightarrow (S^m, g_0)$. Putting $\theta = \gamma \cdot \theta''$ and $\phi_i = \gamma \cdot \varphi_i^* \cdot \gamma^{-1}$, we have $\varphi_i = \theta^{-1} \cdot \phi_i \cdot \theta$.

Finally we prove (iii) \Rightarrow (i). Let (M^m, g) be a compact conformally flat Riemannian manifold with finite fundamental group G , and $(^*M^m, ^*g)$ its universal covering Riemannian manifold. Then we have a conformal transformation f' of $(^*M^m, ^*g)$ to $S^m(1)$. Let $\xi_i \in G$ be a covering transformation, and put $\varphi'_i = f' \cdot \xi_i \cdot f'^{-1}$. Then φ'_i is conformal with respect to g_0 on $S^m(1)$. By (iii) we have a conformal transformation θ and isometries ϕ_i on $S^m(1)$ such that $\varphi'_i = \theta^{-1} \cdot \phi_i \cdot \theta$ for $i = 1, \dots, k-1$. Thus by defining φ_i by $f \cdot \xi_i \cdot f^{-1}$ for $f = \theta \cdot f'$, we obtain $\varphi_i = f \cdot \xi_i \cdot f^{-1} = \theta \cdot \varphi'_i \cdot \theta^{-1} = \phi_i$ isometry. Now we show that we can define a conformally related Riemannian metric on M^m from a conformally related metric f^*g_0 on $^*M^m$. Clearly, f^*g_0 is of constant curvature 1. Since $\xi_i = f^{-1} \cdot \phi_i \cdot f$, we have $\xi_i^*(f^*g_0) = f^* \cdot \phi_i^* \cdot f^{-1*} \cdot f^*g_0 = f^* \cdot \phi_i^* g_0 = f^*g_0$. Therefore G is isometric with respect to f^*g_0 , and hence f^*g_0 is projectable on M^m .

Remark. So-called Yamabe's theorem [17] is in doubt (cf. Aubin [2],

Eliasson [4], Trudinger [16]). If it is true, it gives some important property (iii) by Theorem C.

Proof of Theorem D. If the theorem is false, then the total scalar curvature is nonpositive, and therefore by Lemma 5 we have a conformally related Riemannian metric g^* on (M^m, g) such that the scalar curvature S^* is nonpositive and constant. However, this is impossible by Theorem A. Hence the theorem.

Remark. In Theorem D, “with finite fundamental group” is essential. In fact, if N^2 is a compact 2-dimensional Riemannian manifold (with genus ≥ 2) of negative constant curvature, and M^3 is a Riemannian product manifold $N^2 \times S^1$, where S^1 is a circle, then M^3 is compact conformally flat and has negative total scalar curvature.

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