J. DIFFERENTIAL GEOMETRY
**78** (2008) 369-428

# THE THEORY OF SUPERSTRING WITH FLUX ON NON-KÄHLER MANIFOLDS AND THE COMPLEX MONGE-AMPÈRE EQUATION

Ji-Xiang Fu & Shing-Tung Yau

## Abstract

The purpose of this paper is to solve a problem posed by Strominger in constructing smooth models of superstring theory with flux. These are given by non-Kähler manifolds with torsion.

## 1. Introduction

The purpose of this paper is twofold. The first purpose is to solve an old problem posed by Strominger in constructing smooth models of superstring theory with flux. These are given by non-Kähler manifolds with torsion. To achieve this, we solve a nonlinear Monge-Ampère equation which is more complicated than the equation in the Calabi conjecture. The estimate of the volume form gives extra complication, for example. The second purpose is to point out the connection of the newly constructed geometry based on Strominger's equations in realizing the proposal of M. Reid [**19**] on connecting one Calabi-Yau manifold to another one with different topology. In Reid's proposal, the construction of Clemens-Friedman (see [**9**]) is needed where a Calabi-Yau manifold is deformed to complex manifolds diffeomorphic to connected sums of $S^3 \times S^3$. These are non-Kähler manifolds.

There is a rich class of non-Kähler complex manifolds for dimensions greater than two. It is therefore important to construct canonical geometry on such manifolds. Since for non-Kähler geometry, the complex structure is not quite compatible with the Riemannian metric, it has been difficult to find a reasonable class of Hermitian metric that exhibits rich geometry. We believe that metrics motivated by theoretic physics should have good properties. This is especially true for those metrics which admit parallel spinors. The work of Strominger provided such a candidate. In this paper, we provide a smooth solution to the Strominger system. This has been an important open problem through the past twenty years. Our method is based on a priori estimates which

Received 05/24/2007.

can be generalized to elliptic fibration over general Calabi-Yau manifolds. However, in this paper, for the sake of importance in string theory, we shall restrict ourselves to complex three-dimensional manifolds. The structure of the equations for higher-dimensional Calabi-Yau manifolds is a little bit different. They are also more relevant to algebraic geometry and hence will be treated in a later occasion.

The physical context of the solutions is discussed in a companion paper [**3**] written jointly with K. Becker, M. Becker, and L.-S. Tseng.

**Acknowledgement.** The authors would like to thank K. Becker, M. Becker, and L.-S. Tseng for useful discussions. J.-X. Fu would also like to thank J. Li and X.-P. Zhu for useful discussions. J.-X. Fu is supported in part by NSFC grant 10471026. S.-T. Yau is supported in part by NSF grants DMS-0244462, DMS-0354737 and DMS-0306600.

## 2. Motivation from string theory

In the original proposal for compactification of superstring [**5**], Candelas, Horowitz, Strominger, and Witten constructed the metric product of a maximal symmetric four-dimensional spacetime $M$ with a six-dimensional Calabi-Yau vacuum $X$ as the ten-dimensional spacetime; they identified the Yang-Mills connection with the $SU(3)$ connection of the Calabi-Yau metric and set the dilaton to be a constant. Adapting the second author's suggestion of using Uhlenbeck-Yau's theorem [**22**] on constructing Hermitian-Yang-Mills connections over stable bundles, Witten [**23**] and later Horava-Witten [**13**] proposed to use higher rank bundles for strong coupled heterotic string theory so that the gauge groups can be $SU(4)$ or $SU(5)$.

At around the same time, Strominger [**20**] analyzed heterotic superstring background with spacetime supersymmetry and non-zero torsion by allowing a scalar "warp factor" for the spacetime metric. He considered a ten-dimensional spacetime that is a warped product of a maximal symmetric four-dimensional spacetime $M$ and an internal space $X$; the metric on $M \times X$ takes the form

$$g^0 = e^{2D(y)} \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{ij}(y) \end{pmatrix}, \qquad x \in M, \quad y \in X;$$

the connection on an auxiliary bundle is Hermitian-Yang-Mills connection over $X$:

$$F \wedge \omega^2 = 0, \quad F^{2,0} = F^{0,2} = 0.$$

Here $\omega$ is the Hermitian form $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ defined on the internal space $X$. In this system, the physical relevant quantities are

$$h = -\sqrt{-1}(\bar{\partial} - \partial)\omega,$$

$$\phi = -\frac{1}{2}\log \|\Omega\| + \phi_0,$$

and

$$g^0_{ij} = e^{2\phi_0}\|\Omega\|^{-1} g_{ij},$$

for a constant $\phi_0$.

In order for the ansatz to provide a supersymmetric configuration, one introduces a Majorana-Weyl spinor $\epsilon$ so that

$$\delta\psi_M = \nabla_M \epsilon - \frac{1}{8} h_{MNP}\gamma^{NP}\epsilon = 0,$$
$$\delta\lambda = \gamma^M \partial_M \phi\epsilon - \frac{1}{12} h_{MNP}\gamma^{MNP}\epsilon = 0,$$
$$\delta\chi = \gamma^{MN} F_{MN}\epsilon = 0,$$

where $\psi_M$ is the gravitino, $\lambda$ is the dilatino, $\chi$ is the gluino, $\phi$ is the dilaton and $h$ is the Kalb-Ramond field strength obeying

$$dh = \frac{\alpha'}{2}(\mathrm{tr} F \wedge F - \mathrm{tr} R \wedge R),$$

where $\alpha'$ is positive. Strominger [**20**] showed that in order to achieve spacetime supersymmetry, the internal six manifold $X$ must be a complex manifold with a non-vanishing holomorphic three-form $\Omega$; and the anomaly cancellation demands that the Hermitian form $\omega$ obey[1]

$$\sqrt{-1}\partial\bar\partial\omega = \frac{\alpha'}{4}(\mathrm{tr} R \wedge R - \mathrm{tr} F \wedge F)$$

and supersymmetry requires[2]

$$d^*\omega = \sqrt{-1}(\bar\partial - \partial)\log\|\Omega\|_\omega.$$

Accordingly, he proposed the system

$$F_H \wedge \omega^2 = 0; \tag{2.1}$$

$$F_H^{2,0} = F_H^{0,2} = 0; \tag{2.2}$$

$$\sqrt{-1}\partial\bar\partial\omega = \frac{\alpha'}{4}(\mathrm{tr} R \wedge R - \mathrm{tr} F_H \wedge F_H); \tag{2.3}$$

$$d^*\omega = \sqrt{-1}(\bar\partial - \partial)\ln\|\Omega\|_\omega. \tag{2.4}$$

This system gives a solution of a superstring theory with flux that allows a non-trivial dilaton field and a Yang-Mills field. (It turns out $D(y) = \phi$ and is the dilaton field.) Here $\omega$ is the Hermitian form and $R$ is the curvature tensor of the Hermitian metric $\omega$; $H$ is the Hermitian metric and $F$ is its curvature of a vector bundle $E$; tr is the trace of the endomorphism bundle of either $E$ or $TX$.

In [**17**], Li and Yau observed the following:

---

[1]The curvature $F$ of the vector bundle $E$ in ref.[**20**] is real, i.e., $c_1(E) = \frac{F}{2\pi}$. But we are used to taking the curvature $F$ such that $c_1(E) = \frac{\sqrt{-1}}{2\pi}F$. So this equation corrects eq. (2.18) of ref. [**20**] by a minus sign.

[2]See eq. (56) of ref.[**21**], which corrects eq. (2.30) of ref.[**20**] by a minus sign.

**Lemma 1.** *Equation* (2.4) *is equivalent to*

$$d(\parallel \Omega \parallel_\omega \omega^2) = 0. \tag{2.5}$$

In fact, Li and Yau gave the first irreducible non-singular solution of the supersymmetric system of Strominger for $U(4)$ and $U(5)$ principle bundle. They obtained their solutions by perturbing around the Calabi-Yau vacuum coupled with the sum of tangent bundle and trivial line bundles. In this paper, we consider the solution on complex manifolds which do *not* admit Kähler structures. Study of non-Kähler manifolds should be useful to understand the speculation of M. Reid that all Calabi-Yau manifolds can be deformed to each other through conifold transition.

An example of non-Kähler manifolds $X$ is given by $T^2$-bundles over Calabi-Yau varieties [**2**, **4**, **10**, **12**, **14**]. Since we demand that the internal six manifold $X$ is a complex manifold with a non-vanishing holomorphic three form $\Omega$, we consider the $T^2-$bundle $(X, \omega, \Omega)$ over a complex surface $(S, \omega_S, \Omega_S)$ with a non-vanishing holomorphic 2-form $\Omega_S$. According to the classification of complex surfaces by Enriques and Kodaira, such complex surfaces must be finite quotients of K3 surface, complex torus (Kähler), and Kodaira surface (non-Kähler). If $(X, \omega, \Omega)$ satisfies Strominger's equation (2.4), Lemma 1 shows that $d(\parallel \Omega \parallel_\omega \omega^2) = 0$. Let $\omega' = \parallel \Omega \parallel_\omega^{\frac{1}{2}} \omega$. Then $d\omega'^2 = 0$, i.e., $\omega'$ is a balanced metric [**18**]. The balanced metric was studied extensively by Michelsohn. She proved that the balanced condition is preserved under proper holomorphic submersions. Note that Alessandrini and Bassanelli [**1**] proved that this condition is also preserved under modifications of complex manifolds. Hence if a holomorphic submersion $\pi$ from a balanced manifold $X$ to a complex surface $S$ is proper, $S$ is also balanced (actually $\pi_*\omega'^2$ is the balanced metric on $S$, see Proposition 1.9 in [**18**]). When the dimension of complex manifold is two, the conditions of being balanced and Kähler coincide. Hence there is no solution to Strominger's equation (1.4) on $T^2$ bundles over Kodaira surface and we consider $T^2$-bundles over $K3$ surface and complex torus only.

On the other hand, duality from $M$-theory suggests that there is no supersymmetric solution when the base manifold is a complex torus (see [**3**]). This class of three manifolds includes the Iwasawa manifold. But the solution to Strominger's system should exist when the base is $K3$ surface. In this paper we prove the existence of solutions to Strominger's system on such torus bundles over $K3$ surfaces.

## 3. Statement of main result

Let $(S, \omega_S, \Omega_S)$ be a K3 surface or a complex torus with a Kähler form $\omega_S$ and a non-vanishing holomorphic (2,0)-form $\Omega_S$. Let $\omega_1$ and $\omega_2$ be anti-self-dual (1,1)-forms such that $\frac{\omega_1}{2\pi}$ and $\frac{\omega_2}{2\pi}$ represent integral cohomology classes. Using these two forms, Goldstein and Prokushkin [**10**] constructed a non-Kähler manifold $X$ such that $\pi : X \to S$ is a holomorphic $T^2$-fibration over $S$ with a Hermitian form $\omega_0 = \pi^*\omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}$ and a holomorphic (3,0)-form $\Omega = \Omega_S \wedge \theta$ (for the definition of $\theta$, see section 3). Note that $(\omega_0, \Omega)$ satisfies equation (2.5). (Sethi pointed out that in papers [**6**] and [**2**] similar ansatz was discussed. However the major problem of solving equations was not addressed in the literature.)

Let $u$ be any smooth function on $S$ and let

$$\omega_u = \pi^*(e^u\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}.$$

Then $(\omega_u, \Omega)$ also satisfies equation (2.5) (see [**10**] or Lemma 12), i.e., $\omega_u$ is conformal balanced. The stability concept can be defined on a vector bundle over a complex manifold using the Gauduchon metric [**16**], and hence for complex manifolds with balanced metrics. Note that the stability concept of the vector bundle depends only on the conformal class of metric. Let $V \to X$ be a stable bundle over $X$ with degree zero with respect to the metric $\omega_u$. (Such bundles can be obtained by pulling back stable bundles over a $K3$ surface or a complex torus, see Lemma 16.) According to Li-Yau's theorem [**16**], there is a Hermitian-Yang-Mills metric $H$ on $V$, which is unique up to positive constants. The curvature $F_H$ of the Hermitian metric $H$ satisfies equation (2.1) and (2.2). So $(V, F_H, X, \omega_u)$ satisfies Strominger's equations (2.1), (2.2) and (2.4). Therefore we only need to consider equation (2.3). As $\omega_1$ and $\omega_2$ are harmonic, $\bar{\partial}\omega_1 = \bar{\partial}\omega_2 = 0$. According to $\bar{\partial}$-Poincaré Lemma, we can write $\omega_1$ and $\omega_2$ locally as

$$\omega_1 = \bar{\partial}\xi = \bar{\partial}(\xi_1 dz_1 + \xi_2 dz_2)$$

and

$$\omega_2 = \bar{\partial}\zeta = \bar{\partial}(\zeta_1 dz_1 + \zeta_2 dz_2),$$

where $(z_1, z_2)$ is a local coordinate on $S$. Let

$$B = \begin{pmatrix} \xi_1 + \sqrt{-1}\zeta_1 \\ \xi_2 + \sqrt{-1}\zeta_2 \end{pmatrix}.$$

We can use $B$ to compute $\mathrm{tr}R_0 \wedge R_0$ of the metric $\omega_0$ (see Proposition 8) and $\mathrm{tr}R_u \wedge R_u$ of the metric $\omega_u$ (see Lemma 14). Then we reduce equation (2.3) to

$$\begin{aligned}(3.1)\quad & \sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - \frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u \\ &= \frac{\alpha'}{4}\mathrm{tr}R_S \wedge R_S - \frac{\alpha'}{4}\mathrm{tr}F_H \wedge F_H - \frac{1}{2}(\| \omega_1 \|^2_{\omega_S} + \| \omega_2 \|^2_{\omega_S})\frac{\omega_S^2}{2!},\end{aligned}$$

where $g = (g_{i\bar{j}})$ is the Ricci-flat metric on $S$ associated to the Kähler form $\omega_S$ and $g^{-1}$ is the inverse matrix of $g$; $R_S$ is the curvature of $g$.

Taking wedge product with $\omega_u$ and integrating both sides of the above equation over $X$, we obtain

(3.2)
$$\alpha'\int_X \{\mathrm{tr}R_S\wedge R_S-\mathrm{tr}F_H\wedge F_H\}\wedge\omega_u-2\int_X(\|\,\omega_1\,\|^2_{\omega_S}+\|\,\omega_2\,\|^2_{\omega_S})\frac{\omega_S^2}{2!}\wedge\omega_u=0.$$

When $S=T^4$, $R_S=0$. We obtain immediately

**Proposition 2.** *There is no solution of Strominger's system on the torus bundle $X$ over $T^4$ if the metric has the form $e^u\omega_S+\frac{\sqrt{-1}}{2}\theta\wedge\bar\theta$.*

This situation is different if the base is a $K3$ surface. If $E$ is a stable bundle over $S$ with degree 0 with respect to the metric $\omega_S$, then $V=\pi^*E$ is also a stable bundle with degree 0 over $X$ with respect to the Hermitian metric $\omega_u$. In this case, equation (3.1) on $X$ can be considered as an equation on $S$. Integrating equation (3.1) over $S$,

$$\alpha'\int_S\{\mathrm{tr}R_S\wedge R_S-\mathrm{tr}F_H\wedge F_H\}=2\int_S(\|\,\omega_1\,\|^2_{\omega_S}+\|\,\omega_2\,\|^2_{\omega_S})\frac{\omega_S^2}{2!}. \tag{3.3}$$

As $\int_S \mathrm{tr}R_S\wedge R_S=8\pi^2c_2(V)=8\pi^2\times 24$, and $\int_S\mathrm{tr}F_H\wedge F_H=8\pi^2\times(c_2(E)-\frac{1}{2}c_1^2(E))\geq 0$, we can rewrite equation (3.3) as

$$\alpha'\left(24-\left(c_2(E)-\frac{1}{2}c_1^2(E)\right)\right)=\int_S\left(\left\|\frac{\omega_1}{2\pi}\right\|^2_{\omega_S}+\left\|\frac{\omega_2}{2\pi}\right\|^2_{\omega_S}\right)\frac{\omega_S^2}{2!}. \tag{3.4}$$

For a compact, oriented, simply connected four-manifold $S$, the Poincaré duality gives rise to a pairing

$$Q: H_2(S;\mathbb{Z})\times H_2(S;\mathbb{Z})\to\mathbb{Z}$$

defined by

$$Q(\beta,\gamma)=\int_S\beta\wedge\gamma.$$

We shall denote $Q(\beta,\beta)$ by $Q(\beta)$. Then for an integral anti-self-dual (1,1)-form $\frac{\omega_1}{2\pi}$, the intersection number $Q(\frac{\omega_1}{2\pi})$ can be expressed as $-\int_S\|\,\frac{\omega_1}{2\pi}\,\|^2\,\frac{\omega_S^2}{2!}$. On the other hand, the intersection form on $K3$ surface is given by [**7**]

$$3\begin{pmatrix}0&1\\1&0\end{pmatrix}\oplus 2(-E_8),$$

where

$$E_8=\begin{pmatrix}2&0&-1&&&&&\\0&2&0&-1&&&&\\-1&0&2&-1&&&&\\&-1&-1&2&-1&&&\\&&&-1&2&-1&&\\&&&&-1&2&-1&\\&&&&&-1&2&-1\\&&&&&&-1&2\end{pmatrix}.$$

Hence $Q(\frac{\omega_1}{2\pi}) \in \{-2, -4, -6, \ldots\}$.

We shall use the following convention for vector bundles over a compact oriented four-manifold:

$$\begin{aligned} \kappa(E) &= c_2(E) && \text{for } SU(r) \text{ bundle } E, \\ &= c_2(E) - \frac{1}{2}c_1^2(E) && \text{for } U(r) \text{ bundle } E, \\ &= -\frac{1}{2}p_1(E) && \text{for } SO(r) \text{ bundle } E. \end{aligned}$$

Then (3.4) implies

$$\alpha'(24 - \kappa(E)) + \left(Q\left(\frac{\omega_1}{2\pi}\right) + Q\left(\frac{\omega_2}{2\pi}\right)\right) = 0, \tag{3.5}$$

which means that there is a smooth function $\mu$ such that

$$\frac{\alpha'}{4}\mathrm{tr}R_S \wedge R_S - \frac{\alpha'}{4}\mathrm{tr}F_H \wedge F_H - \frac{1}{2}(\parallel \omega_1 \parallel^2 + \parallel \omega_2 \parallel^2)\frac{\omega_S^2}{2!} = -\mu\frac{\omega_S^2}{2!} \tag{3.6}$$

and $\int_S \mu\frac{\omega^2}{2!} = 0$. Inserting (3.6) into (3.1), we obtain the following equation:

$$\sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - \frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2!} = 0, \tag{3.7}$$

where $\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})$ is a smooth well-defined $(1,1)$-form on $S$. In particular, when $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$,

$$\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1}) = \sqrt{-1}\frac{1+n^2}{4} \parallel \omega_1 \parallel^2_{\omega_S} \omega_S$$

(see Proposition 11). Hence if we set $f = \frac{1+n^2}{4} \parallel \omega_1 \parallel^2_{\omega_S}$, we can rewrite equation (3.7) as the standard complex Monge-Ampère equation:

$$\Delta(e^u - \frac{\alpha'}{2}fe^{-u}) + 4\alpha'\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0, \tag{3.8}$$

where $u_{i\bar{j}}$ denotes $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$ and $\triangle = 2g^{i\bar{j}}\frac{\partial^2}{\partial z_i \partial \bar{z}_j}$. We shall solve equation (3.7) by the continuity method [**24**]. Our main theorem is

**Theorem 3.** *The equation* (3.7) *has a smooth solution $u$ such that*

$$\omega' = e^u\omega_S - \frac{\alpha'}{2}\sqrt{-1}e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1}) + \alpha'\sqrt{-1}\partial\bar{\partial}u$$

*defines a Hermitian metric on $S$.*

Our solution $u$ satisfies $\left(\int_S e^{-4u}\right)^{\frac{1}{4}} = A \ll 1$. Actually we can prove that $\inf u \geq -\ln(C_1 A)$ (see Proposition 21) where $A$ must be *very small* (see Proposition 22) and our solution $u$ must be *very big*.

**Theorem 4.** *Let $S$ be a $K3$ surface with a Ricci-flat metric $\omega_S$. Let $\omega_1$ and $\omega_2$ be anti-self-dual $(1,1)$-forms on $S$ such that $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^2(S,\mathbb{Z})$. Let $X$ be a $T^2$-bundle over $S$ constructed by $\omega_1$ and $\omega_2$. Let $E$ be a stable bundle over $S$ with degree $0$. Suppose $\omega_1$, $\omega_2$ and $\kappa(E)$ satisfy condition (3.5). Then there exist a smooth function $u$ on $S$ and a Hermitian-Yang-Mills metric $H$ on $E$ such that $(V = \pi^*E, \pi^*F_H, X, \omega_u)$ is a solution of Strominger's system.*

Since it is easy to find $(\omega_1, \omega_2, \kappa(E))$ which satisfies condition (3.5), this theorem provides first examples of solutions to Strominger's system on non-Kähler manifolds.

## 4. Geometric model

In this section, we take the geometric model of Goldstein and Prokushkin for complex non-Kähler manifolds with an $SU(3)$ structure [**10**]. We summarize their results as follows:

**Theorem 5.** [**10**] *Let $(S, \omega_S, \Omega_S)$ be a Calabi-Yau 2-fold with a non-vanishing holomorphic $(2,0)-$form $\Omega_S$. Let $\omega_1$ and $\omega_2$ be anti-self-dual $(1,1)$-forms on $S$ such that $\frac{\omega_1}{2\pi} \in H^2(S,\mathbb{Z})$ and $\frac{\omega_2}{2\pi} \in H^2(S,\mathbb{Z})$. Then there is a Hermitian 3-fold $X$ such that $\pi : X \to S$ is a holomorphic $T^2$-fibration over $S$ and the following holds:*

1. *For any real 1-forms $\alpha_1$ and $\alpha_2$ defined on some open subset of $S$ that satisfy $d\alpha_1 = \omega_1$ and $d\alpha_2 = \omega_2$, there are local coordinates $x$ and $y$ on $X$ such that $dx + idy$ is a holomorphic form on $T^2$-fibers and a metric on $X$ has the following form:*

$$g_0 = \pi^*g + (dx + \pi^*\alpha_1)^2 + (dy + \pi^*\alpha_2)^2, \tag{4.1}$$

   *where $g$ is a Calabi-Yau metric on $S$ corresponding to the Kähler form $\omega_S$.*
2. *$X$ admits a nowhere vanishing holomorphic $(3,0)$-form with unit length:*

$$\Omega = ((dx + \pi^*\alpha_1) + i(dy + \pi^*\alpha_2)) \wedge \pi^*\Omega_S.$$

3. *If either $\omega_1$ or $\omega_2$ represents a non-trivial cohomological class then $X$ admits no Kähler metric.*
4. *$X$ is a balanced manifold. The Hermitian form*

$$\omega_0 = \pi^*\omega_S + (dx + \pi^*\alpha_1) \wedge (dy + \pi^*\alpha_2) \tag{4.2}$$

   *corresponding to the metric (4.1) is balanced, i.e., $d\omega_0^2 = 0$.*
5. *Furthermore, for any smooth function $u$ on $S$, the Hermitian metric*

$$\omega_u = \pi^*(e^u\omega_S) + (dx + \pi^*\alpha_1) \wedge (dy + \pi^*\alpha_2)$$

   *is conformal balanced. Actually $(\omega_u, \Omega)$ satisfies equation (2.5).*

Goldstein and Prokushkin also studied the cohomology of this non-Kähler manifold $X$:

$$h^{1,0}(X) = h^{1,0}(S),$$
$$h^{0,1}(X) = h^{0,1}(S) + 1;$$

In particular,

$$h^{0,1}(X) = h^{1,0}(X) + 1.$$

Moreover,

$$\begin{aligned} b_1(X) &= b_1(S) + 1, \quad \text{when} \quad \omega_2 = n\omega_1, \\ b_1(X) &= b_1(S), \quad \text{when} \quad \omega_2 \neq n\omega_1; \\ b_2(X) &= b_2(S) - 1, \quad \text{when} \quad \omega_2 = n\omega_1, \\ b_2(X) &= b_2(S) - 2, \quad \text{when} \quad \omega_2 \neq n\omega_1 \end{aligned}$$

and

$$\chi(X) = 0.$$

The above topological results can be explained as follows. Let $L_1$ be a holomorphic line bundle over $S$ with the first Chern class $c_1(L_1) = [-\frac{\omega_1}{2\pi}]$. Then we can choose a Hermitian metric $h_1$ on $L_1$ such that its curvature is $\sqrt{-1}\omega_1$. Let $S_1 = \{v \in L_1 \mid h_1(v, v) = 1\}$ which is a circle bundle over $S$. Locally we write $\omega_1 = d\alpha_{1U}$ for some real 1-form $\alpha_{1U}$ on some open subset $U$ on $S$. Such $\alpha_{1U}$ define a connection on $S_1$, i.e., there is a section $\xi_U$ on $S_1$ such that

$$\nabla\xi_U = \sqrt{-1}\alpha_{1U} \otimes \xi_U.$$

The section $\xi_U$ defines a local coordinate $x_U$ on fibers of $S_1 |_U$, i.e., we can describe the circle $S^1$ by $e^{\sqrt{-1}x_U}\xi_U$. If we write $\omega_1 = d\alpha_{1V}$ on another open set $V$ of $S$, then there is another section $\xi_V$ such that

$$\nabla\xi_V = \sqrt{-1}\alpha_{1V} \otimes \xi_V \tag{4.3}$$

and this section $\xi_V$ defines another coordinate $x_V$ on fiber of $S_1 |_V$. On $U \cap V$, $d(\alpha_{1U} - \alpha_{1V}) = 0$ and there is a function $f_{UV}$ such that

$$df_{UV} = \alpha_{1U} - \alpha_{1V}. \tag{4.4}$$

On the other hand, on $U \cap V$, there is also a function $g_{UV}$ on $U \cap V$ such that $\xi_V = e^{\sqrt{-1}g_{UV}}\xi_U$. We compute

$$\begin{aligned} \nabla\xi_V &= \nabla(e^{\sqrt{-1}g_{UV}}\xi_U) \\ &= (\sqrt{-1}dg_{UV} + \sqrt{-1}\alpha_{1U}) \otimes (e^{\sqrt{-1}g_{UV}}\xi_U) \\ &= (\sqrt{-1}dg_{UV} + \sqrt{-1}\alpha_{1U}) \otimes \xi_V. \end{aligned}$$

Comparing the above equality with (4.3), we get

$$-dg_{UV} = \alpha_{1U} - \alpha_{1V}. \tag{4.5}$$

So combining (4.4), we find

$$g_{UV} = f_{UV} + c_{UV}, \tag{4.6}$$

where $c_{UV}$ is some constant on $U \cap V$. On $U \cap V$, from

$$e^{ix_U}\xi_U = e^{ix_V}\xi_V = e^{\sqrt{-1}x_V}e^{\sqrt{-1}g_{UV}}\xi_U,$$

we obtain

$$x_U = x_V + g_{UV} + 2k\pi = x_V + f_{UV} + c_{UV} + 2k\pi. \tag{4.7}$$

(4.4) and (4.7) imply

$$dx_U - dx_V = df_{UV} = -\alpha_{1U} + \alpha_{1V}. \tag{4.8}$$

So $dx_U + \alpha_{1U}$ is a globally defined 1-form on $X$. We denote it by $dx + \alpha_1$.

We construct another line bundle $L_2$ with the first Chern class $[-\frac{\omega_2}{2\pi}]$. Similarly, we write locally $\omega_2 = d\alpha_2$, and define a coordinate $y$ on fibers such that $dy + \alpha_2$ is a well-defined 1-form on the circle bundle $S_1$ of $L_2$. On $X$, $\omega_1 = d(dx + \alpha_1)$ and $\omega_2 = d(dy + \alpha_2)$, and so $[\omega_1] = [\omega_2] = 0 \in H^2(X, \mathbb{R})$. When $\omega_2 = n\omega_1$, $d(n(dx + \alpha_1) - (dy + \alpha_2)) = 0$. So $[n(dx + \alpha_1) - (dy + \alpha_2)] \in H^1(X, \mathbb{R})$. Finally we define

$$\theta = dx + \alpha_1 + \sqrt{-1}(dy + \alpha_2).$$

Then $\theta$ is a $(1,0)$-form on $X$ , see [**10**] or the next section. Because $d\bar{\theta} = \omega_1 - \sqrt{-1}\omega_2$ is a $(1,1)$-form on $X$, its $(0,2)$-component $\bar{\partial}\bar{\theta} = 0$. So $[\bar{\theta}] \in H^{0,1}_{\bar{\partial}}(X) \cong H^1(X, \mathcal{O})$.

## 5. The calculation of tr$R \wedge R$

In order to calculate the curvature $R$ and tr$R \wedge R$, we express the Hermitian metric (4.1) in terms of a basis of holomorphic (1,0) vector fields. Hence we need to write down the complex structure of $X$. Let $\{U, z_j = x_j + \sqrt{-1}y_j, j = 1,2\}$ be a local coordinate in $S$. The horizontal lifts of vector fields $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$, which are in the kernel of $dx + \pi^*\alpha_1$ and $dy + \pi^*\alpha_2$, are

$$X_j = \frac{\partial}{\partial x_j} - \alpha_1\left(\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x} - \alpha_2\left(\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial y} \quad \text{for} \quad j = 1,2,$$

$$Y_j = \frac{\partial}{\partial y_j} - \alpha_1\left(\frac{\partial}{\partial y_j}\right)\frac{\partial}{\partial x} - \alpha_2\left(\frac{\partial}{\partial y_j}\right)\frac{\partial}{\partial y} \quad \text{for} \quad j = 1,2.$$

The complex structure $\tilde{I}$ on $X$ is defined as

$$\begin{aligned} \tilde{I}X_j &= Y_j, \quad \tilde{I}Y_j = -X_j, \quad \text{for} \quad j = 1,2, \\ \tilde{I}\frac{\partial}{\partial x} &= \frac{\partial}{\partial y}, \quad \tilde{I}\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}. \end{aligned}$$

Let

$$\begin{aligned} U_j &= X_j - \sqrt{-1}\tilde{I}X_j = X_j - \sqrt{-1}Y_j, \\ U_0 &= \frac{\partial}{\partial x} - \sqrt{-1}\tilde{I}\frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}. \end{aligned}$$

Then $\{U_j, U_0\}$ is the basis of the $(1,0)$ vector fields on $X$. The metric (4.1) takes the following Hermitian form:

$$\begin{pmatrix} (g_{i\bar{j}}) & 0 \\ 0 & 1 \end{pmatrix} \tag{5.1}$$

as $U_1$ and $U_2$ are in the kernel of $dx + \pi^*\alpha_1$ and $dy + \pi^*\alpha_2$. Let

$$\theta = dx + \sqrt{-1}dy + \pi^*(\alpha_1 + \sqrt{-1}\alpha_2). \tag{5.2}$$

It's easy to check that $\{\pi^* d\bar{z}_j, \overline{\theta}\}$ annihilates the $\{U_j, U_0\}$ and is the basis of $(0,1)$-forms on $X$. So $\{\pi^* dz_j, \theta\}$ are $(1,0)$-forms on $X$. Certainly $\pi^* dz_j$ are holomorphic $(1,0)$-forms and $\theta$ is not. We need to construct another holomorphic $(1,0)$-form on $X$. Because $\omega_1$ and $\omega_2$ are harmonic forms on $S$, $\overline{\partial}\omega_1 = \overline{\partial}\omega_2 = 0$. By $\bar{\partial}$-Poincaré Lemma, locally we can find $(1,0)$-forms $\xi = \xi_1 dz_1 + \xi_2 dz_2$ and $\zeta = \zeta_1 dz_1 + \zeta_2 dz_2$ on $S$ , where $\xi_i$ and $\zeta_j$ are smooth complex functions on some open set of $S$, such that $\omega_1 = \overline{\partial}\xi$ and $\omega_2 = \overline{\partial}\zeta$. Let

$$\begin{aligned} \theta_0 &= \theta - \pi^*(\xi + \sqrt{-1}\zeta) \\ &= (dx + \sqrt{-1}dy) + \pi^*(\alpha_1 + \sqrt{-1}\alpha_2) - \pi^*(\xi + \sqrt{-1}\zeta). \end{aligned}$$

We claim that $\theta_0$ is a holomorphic $(1,0)$-form. By our construction, $\theta_0$ is the $(1,0)$-form. But $d\theta = d(dx + \sqrt{-1}dy + \pi^*(\alpha_1 + \sqrt{-1}\alpha_2)) = \pi^*(\omega_1 + \sqrt{-1}\omega_2)$ is a $(1,1)$-form on $X$. So

$$\partial\theta = 0 \quad\text{and}\quad \overline{\partial}\theta = d\theta = \pi^*(\omega_1 + i\omega_2). \tag{5.3}$$

Thus,

$$\begin{aligned} \overline{\partial}\theta_0 &= \overline{\partial}\theta - \overline{\partial}\pi^*(\xi + \sqrt{-1}\zeta) \\ &= \pi^*(\omega_1 + \sqrt{-1}\omega_2) - \pi^*(\omega_1 + \sqrt{-1}\omega_2) = 0. \end{aligned}$$

So $\theta_0$ is a holomorphic $(1,0)$-form and $\{\pi^* dz_j, \theta_0\}$ forms a basis of holomorphic $(1,0)$-forms on $X$. Let

$$\varphi_j = \xi_j + \sqrt{-1}\zeta_j \quad\text{for}\quad j = 1, 2$$

and

$$\tilde{U}_j = U_j + \varphi_j U_0 \quad\text{for}\quad j = 1, 2.$$

Then $\{\tilde{U}_j, U_0\}$ is dual to $\{\pi^* dz_j, \theta_0\}$ because $U_j$ is in the kernel of $\theta$. It's the basis of holomorphic $(1,0)$-vector fields. The metric $g_0$ then becomes the following Hermitian matrix:

(5.4)
$$H_X = \begin{pmatrix} g_{1\bar{1}} + \mid \varphi_1 \mid^2 & g_{1\bar{2}} + \varphi_1\overline{\varphi}_2 & \varphi_1 \\ g_{2\bar{1}} + \varphi_2\overline{\varphi}_1 & g_{2\bar{2}} + \mid \varphi_2 \mid^2 & \varphi_2 \\ \overline{\varphi}_1 & \overline{\varphi}_2 & 1 \end{pmatrix} = \begin{pmatrix} g + B \cdot B^* & B \\ B^* & 1 \end{pmatrix},$$

where $g$ is the Calabi-Yau metric on $S$ and $B = (\varphi_1, \varphi_2)^t$.

According to Strominger's explanation in [**20**], when the manifold is not Kähler, we should take the curvature of Hermitian connection on the holomorphic tangent bundle $T'X$. Using the metric (5.4), we compute the curvature to be

$$R = \overline{\partial}(\partial H_X \cdot H_X^{-1}) = \begin{pmatrix} R_{1\bar{1}} & R_{1\bar{2}} \\ R_{2\bar{1}} & R_{2\bar{2}} \end{pmatrix},$$

where

$$\begin{aligned} R_{1\bar{1}} &= R_S + \overline{\partial} B \wedge (\partial B^* \cdot g^{-1}) + B \cdot \overline{\partial}(\partial B^* \cdot g^{-1}), \\ R_{1\bar{2}} &= -R_S B + (\partial g \cdot g^{-1}) \wedge \overline{\partial} B - \overline{\partial} B \wedge (\partial B^* \cdot g^{-1}) B \\ &\quad - B\overline{\partial}(\partial B^* \cdot g^{-1}) B + B(\partial B^* \cdot g^{-1}) \wedge \overline{\partial} B + \overline{\partial}\partial B, \\ R_{2\bar{1}} &= \overline{\partial}(\partial B^* \cdot g^{-1}), \\ R_{2\bar{2}} &= -\overline{\partial}(\partial B^* \cdot g^{-1}) B + (\partial B^* \cdot g^{-1}) \wedge \overline{\partial} B, \end{aligned}$$

and $R_S$ is the curvature of Calabi-Yau metric $g$ on $S$. It is easy to check that $\mathrm{tr}(\overline{\partial} B \wedge (\partial B^* \cdot g^{-1}) + B \cdot \overline{\partial}(\partial B^* \cdot g^{-1})) - \overline{\partial}(\partial B^* \cdot g^{-1}) B + (\partial B^* \cdot g^{-1}) \wedge \overline{\partial} B = 0$. So $\mathrm{tr}R = \pi^* \mathrm{tr} R_S$.

**Proposition 6** ([**11**]). *The Ricci forms of the Hermitian connections on $X$ and $S$ have the relation* $\mathrm{tr}R = \pi^* \mathrm{tr} R_S$.

**Remark 7.** In the above calculation, we don't use the condition that the metric $g$ on $S$ is Calabi-Yau.

**Proposition 8.**

$$\mathrm{tr} R \wedge R = \pi^*(\mathrm{tr} R_S \wedge R_S + 2\mathrm{tr}\partial\overline{\partial}(\overline{\partial} B \wedge \partial B^* \cdot g^{-1})). \tag{5.5}$$

*Proof.* Fix any point $p \in S$, we pick $B$ such that $B(p) = 0$. Otherwise, $B(p) \neq 0$ and we simply replace $B$ by $B - B(p)$. Hence in the calculation of $\mathrm{tr} R \wedge R$ at $p$, all terms containing the factor $B$ will vanish. Thus

$$\begin{aligned} &\mathrm{tr} R \wedge R \\ &= \mathrm{tr} R_S \wedge R_S + 2\mathrm{tr} R_S \wedge \overline{\partial} B \wedge (\partial B^* \cdot g^{-1}) \\ &\quad + 2\mathrm{tr}\partial g \cdot g^{-1} \wedge \overline{\partial} B \wedge \bar{\partial}(\partial B^* \cdot g^{-1}) + 2\mathrm{tr}\overline{\partial}\partial B \wedge \overline{\partial}(\partial B^* \cdot g^{-1}). \end{aligned}$$

Proposition 8 follows from the next two lemmas. q.e.d.

**Lemma 9.**

$$\begin{aligned}\operatorname{tr}\partial\overline{\partial}(\overline{\partial}B\wedge\partial B^*\cdot g^{-1}) &= \operatorname{tr}R_S\wedge\overline{\partial}B\wedge(\partial B^*\cdot g^{-1})\\ &\quad+\operatorname{tr}\partial g\cdot g^{-1}\wedge\overline{\partial}B\wedge\bar{\partial}(\partial B^*\cdot g^{-1})\\ &\quad+\operatorname{tr}\overline{\partial}\partial B\wedge\overline{\partial}(\partial B^*\cdot g^{-1}).\end{aligned}$$

*Proof.*

$$\begin{aligned}&\operatorname{tr}\partial\overline{\partial}(\overline{\partial}B\wedge\partial B^*\cdot g^{-1})\\ &=-\operatorname{tr}\partial(\overline{\partial}B\wedge\overline{\partial}(\partial B^*\cdot g^{-1}))\\ &=\operatorname{tr}\overline{\partial}\partial B\wedge\overline{\partial}(\partial B^*\cdot g^{-1})+\operatorname{tr}\overline{\partial}B\wedge\overline{\partial}(\partial B^*\wedge\partial g^{-1})\\ &=\operatorname{tr}\overline{\partial}\partial B\wedge\overline{\partial}(\partial B^*\cdot g^{-1})-\operatorname{tr}\overline{\partial}B\wedge\overline{\partial}(\partial B^*\cdot g^{-1}\wedge\partial g\cdot g^{-1})\\ &=\operatorname{tr}\overline{\partial}\partial B\wedge\overline{\partial}(\partial B^*\cdot g^{-1})-\operatorname{tr}\overline{\partial}B\wedge\overline{\partial}(\partial B^*\cdot g^{-1})\wedge\partial g\cdot g^{-1}\\ &\quad+\operatorname{tr}\overline{\partial}B\wedge(\partial B^*\cdot g^{-1})\wedge\bar{\partial}(\partial g\cdot g^{-1})\\ &=\operatorname{tr}\overline{\partial}\partial B\wedge\overline{\partial}(\partial B^*\cdot g^{-1})-\operatorname{tr}\overline{\partial}B\wedge\overline{\partial}(\partial B^*\cdot g^{-1})\wedge\partial g\cdot g^{-1}\\ &\quad+\operatorname{tr}\overline{\partial}B\wedge(\partial B^*\cdot g^{-1})\wedge R_S\\ &=\operatorname{tr}(\overline{\partial}\partial B\wedge\overline{\partial}(\partial B^*\cdot g^{-1}))+\operatorname{tr}(R_S\wedge\overline{\partial}B\wedge\partial B^*\cdot g^{-1})\\ &\quad+\operatorname{tr}(\partial g\cdot g^{-1}\wedge\overline{\partial}B\wedge\overline{\partial}(\partial B^*\cdot g^{-1}).\end{aligned}$$

q.e.d.

**Lemma 10.** $\operatorname{tr}(\bar{\partial}B\wedge\partial B^*\cdot g^{-1})$ *is a well-defined* $(1,1)$*-form on* $S$.

*Proof.* We take local coordinates $(U,z_i)$ and $(W,w_j)$ on $S$ such that $U\cap W\neq\emptyset$. Let $J=(\frac{\partial w_i}{\partial z_j})$ and

$$(\omega_1+\sqrt{-1}\omega_2)\mid_U=\overline{\partial}(\varphi_1dz_1+\varphi_2dz_2)=\overline{\partial}\varphi_1\wedge dz_1+\bar{\partial}\varphi_2\wedge dz_2,$$

$$(\omega_1+\sqrt{-1}\omega_2)\mid_W=\overline{\partial}(\gamma_1dw_1+\gamma_2dw_2)=\overline{\partial}\gamma_1\wedge dw_1+\bar{\partial}\gamma_2\wedge dw_2.$$

Then on $U\cap W$,

$$\begin{pmatrix}\bar{\partial}\gamma_1 & \bar{\partial}\gamma_2\end{pmatrix}\wedge\begin{pmatrix}dw_1\\ dw_2\end{pmatrix}=\begin{pmatrix}\bar{\partial}\varphi_1 & \bar{\partial}\varphi_2\end{pmatrix}\wedge\begin{pmatrix}dz_1\\ dz_2\end{pmatrix}.$$

So

$$\begin{pmatrix}\bar{\partial}\varphi_1 & \bar{\partial}\varphi_2\end{pmatrix}=\begin{pmatrix}\bar{\partial}\gamma_1 & \bar{\partial}\gamma_2\end{pmatrix}J. \tag{5.6}$$

On the other hand, we have

$$g(z)=J^tg(w)\overline{J}, \tag{5.7}$$

where $g(z) = (g_{i\bar{j}}(z))$ and $g(w) = (g_{i\bar{j}}(w))$. Then on $U \cap W$, using (5.6), (5.7), we have

$$\begin{aligned}
&\operatorname{tr}\begin{pmatrix} \bar\partial\gamma_1 \\ \bar\partial\gamma_2 \end{pmatrix} \wedge \begin{pmatrix} \partial\bar\gamma_1 & \partial\bar\gamma_2 \end{pmatrix} \cdot g^{-1}(w) \\
&= \operatorname{tr}(J^t)^{-1} \begin{pmatrix} \bar\partial\varphi_1 \\ \bar\partial\varphi_2 \end{pmatrix} \wedge \begin{pmatrix} \overline{\bar\partial\varphi_1} & \overline{\bar\partial\varphi_2} \end{pmatrix} \bar{J}^{-1} \cdot \bar{J} \cdot g^{-1}(z) \cdot J^t \\
&= \operatorname{tr}\begin{pmatrix} \bar\partial\varphi_1 \\ \bar\partial\varphi_2 \end{pmatrix} \wedge \begin{pmatrix} \partial\bar\varphi_1 & \partial\bar\varphi_2 \end{pmatrix} \cdot g^{-1}(z),
\end{aligned}$$

which proves that $\operatorname{tr}(\bar\partial B \wedge \partial B^* \cdot g^{-1})$ is a well-defined $(1,1)$-form on $S$. q.e.d.

Although $\operatorname{tr}(\bar\partial B \wedge \partial B^* \cdot g^{-1})$ is a well-defined (1,1)-form on $S$, we can not express it by $\omega_1$ and $\omega_2$. But in some particular cases, we can.

**Proposition 11.** *When $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$,*

$$\operatorname{tr}(\bar\partial B \wedge \partial B^* \cdot g^{-1}) = \frac{\sqrt{-1}}{4}(1+n^2) \parallel \omega_1 \parallel^2_{\omega_S} \omega_S, \tag{5.8}$$

*where $g$ is the given Calabi-Yau metric on $S$ and $\omega_S$ is the corresponding Kähler form.*

*Proof.* We recall that locally,

$$\begin{aligned}
\omega_1 &= \bar\partial\xi, \quad \xi = \xi_1 dz_1 + \xi_2 dz_2, \\
\omega_2 &= \bar\partial\zeta, \quad \zeta = \zeta_1 dz_1 + \zeta_2 dz_2, \\
\varphi_j &= \xi_j + \sqrt{-1}\zeta_j, \quad \text{for } \ j = 1, 2, \\
B &= \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \ B^* = \begin{pmatrix} \bar\varphi_1 & \bar\varphi_2 \end{pmatrix}.
\end{aligned}$$

When $\omega_2 = n\omega_1$,we take $\zeta = n\xi$. Then $\bar\partial\zeta_j = n\bar\partial\xi_j$,

$$\bar\partial B = \begin{pmatrix} \bar\partial\varphi_1 \\ \bar\partial\varphi_2 \end{pmatrix} = (1 + n\sqrt{-1}) \begin{pmatrix} \bar\partial\xi_1 \\ \bar\partial\xi_2 \end{pmatrix}$$

and

$$\partial B^* = \begin{pmatrix} \partial\bar\varphi_1 & \partial\bar\varphi_2 \end{pmatrix} = (1 - n\sqrt{-1}) \begin{pmatrix} \partial\bar\xi_1 & \partial\bar\xi_2 \end{pmatrix}.$$

Using above equalities, we find

(5.9)

$$\begin{aligned}
&\operatorname{tr}(\bar\partial B\wedge\partial B^*\cdot g^{-1})\\
&=(1+n^2)\operatorname{tr}\begin{pmatrix}\bar\partial\xi_1\\ \bar\partial\xi_2\end{pmatrix}\wedge\begin{pmatrix}\partial\bar\xi_1 & \partial\bar\xi_2\end{pmatrix}\cdot g^{-1}\\
&=\frac{1+n^2}{\det g}\operatorname{tr}\begin{pmatrix}\frac{\partial\xi_1}{\partial\bar z_i}d\bar z_i\\ \frac{\partial\xi_2}{\partial\bar z_i}d\bar z_i\end{pmatrix}\wedge\begin{pmatrix}\overline{\frac{\partial\xi_1}{\partial\bar z_j}}dz_j & \overline{\frac{\partial\xi_2}{\partial\bar z_j}}dz_j\end{pmatrix}\cdot\begin{pmatrix}g_{2\bar 2} & -g_{1\bar 2}\\ -g_{2\bar 1} & g_{1\bar 1}\end{pmatrix}\\
&=\frac{1+n^2}{\det g}\operatorname{tr}\begin{pmatrix}\frac{\partial\xi_1}{\partial\bar z_i}\\ \frac{\partial\xi_2}{\partial\bar z_i}\end{pmatrix}\wedge\begin{pmatrix}\overline{\frac{\partial\xi_1}{\partial\bar z_j}} & \overline{\frac{\partial\xi_2}{\partial\bar z_j}}\end{pmatrix}\cdot\begin{pmatrix}g_{2\bar 2} & -g_{1\bar 2}\\ -g_{2\bar 1} & g_{1\bar 1}\end{pmatrix}d\bar z_i\wedge dz_j.
\end{aligned}$$

In order to get the global formula, we need to calculate $\omega_1$. As $\omega_1$ is real,

$$\overline{\frac{\partial\xi_i}{\partial\bar z_j}}=-\frac{\partial\xi_j}{\partial\bar z_i}\quad\text{for}\quad i,j=1,2.\tag{5.10}$$

Since $\omega_1$ is anti-self-dual, i.e., $\omega_1\wedge\omega_S=0$, we have

$$g_{1\bar 1}\frac{\partial\xi_2}{\partial\bar z_2}+g_{2\bar 2}\frac{\partial\xi_1}{\partial\bar z_1}-g_{1\bar 2}\frac{\partial\xi_2}{\partial\bar z_1}-g_{2\bar 1}\frac{\partial\xi_1}{\partial\bar z_2}=0.\tag{5.11}$$

Because

$$\omega_1\wedge\omega_1=-\omega_1\wedge*\omega_1=-\omega_1*\bar\omega_1=-\parallel\omega_1\parallel^2_{\omega_S}\frac{\omega_S^2}{2!},\tag{5.12}$$

locally we also have

$$\frac{1}{\det(g)}\left(\frac{\partial\xi_1}{\partial\bar z_1}\frac{\partial\xi_2}{\partial\bar z_2}-\frac{\partial\xi_1}{\partial\bar z_2}\frac{\partial\xi_2}{\partial\bar z_1}\right)=\frac{1}{8}\parallel\omega_1\parallel^2_{\omega_S}.\tag{5.13}$$

Now using above (5.10), (5.11) and (5.13), we calculate the component of $d\bar z_1\wedge dz_1$ in (5.9) to be

(5.14)

$$\begin{aligned}
&\frac{1+n^2}{\det(g)}\left(g_{2\bar 2}\frac{\partial\xi_1}{\partial\bar z_1}\overline{\frac{\partial\xi_1}{\partial\bar z_1}}-g_{2\bar 1}\frac{\partial\xi_1}{\partial\bar z_1}\overline{\frac{\partial\xi_2}{\partial\bar z_1}}-g_{1\bar 2}\frac{\partial\xi_2}{\partial\bar z_1}\overline{\frac{\partial\xi_1}{\partial\bar z_1}}-g_{1\bar 1}\frac{\partial\xi_2}{\partial\bar z_1}\overline{\frac{\partial\xi_2}{\partial\bar z_1}}\right)\\
&=\frac{1+n^2}{\det(g)}\left(\frac{\partial\xi_1}{\partial\bar z_1}\left(g_{1\bar 1}\frac{\partial\xi_2}{\partial\bar z_2}+g_{2\bar 2}\frac{\partial\xi_1}{\partial\bar z_1}\right)-g_{2\bar 2}\left(\frac{\partial\xi_1}{\partial\bar z_1}\right)^2-g_{1\bar 1}\frac{\partial\xi_2}{\partial\bar z_1}\frac{\partial\xi_1}{\partial\bar z_2}\right)\\
&=\frac{1+n^2}{\det(g)}g_{1\bar 1}\left(\frac{\partial\xi_1}{\partial\bar z_1}\frac{\partial\xi_2}{\partial\bar z_2}-\frac{\partial\xi_2}{\partial\bar z_1}\frac{\partial\xi_1}{\partial\bar z_2}\right)\\
&=\frac{1+n^2}{8}\parallel\omega_1\parallel^2_{\omega_S}g_{1\bar 1}.
\end{aligned}$$

Similarly, the components of $d\bar z_2\wedge dz_1$, $d\bar z_1\wedge dz_2$ and $d\bar z_2\wedge dz_2$ in (5.9) are $\frac{1+n^2}{8}\parallel\omega_1\parallel^2_{\omega_S}g_{1\bar 2}$, $\frac{1+n^2}{8}\parallel\omega_1\parallel^2_{\omega_S}g_{2\bar 1}$ and $\frac{1+n^2}{8}\parallel\omega_1\parallel^2_{\omega_S}g_{2\bar 2}$ respectively.

So we obtain

$$\operatorname{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1}) = \sqrt{-1}\frac{1+n^2}{4} \parallel \omega_1 \parallel^2_{\omega_S} \omega_S.$$

q.e.d.

## 6. Reduction of Strominger's system

Consider a 3-dimensional Hermitian manifold $(X, \omega_0, \Omega)$ as described in the section 2. Let $\omega_S$ be the Calabi-Yau metric on $S$. Let

$$\theta = dx + \alpha_1 + \sqrt{-1}(dy + \alpha_2),$$

then the Hermitian form $\omega_0$ in (4.2) is

$$\omega_0 = \pi^*\omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}.$$

Because $\parallel \Omega \parallel_\omega = 1$, and $\omega_1$ and $\omega_2$ are anti-self-dual,we use (5.3) to compute

$$\begin{aligned}(6.1)\qquad & d(\parallel \Omega \parallel_{\omega_0} \omega_0^2) \\ &= d(\pi^*\omega_S^2 + \sqrt{-1}\pi^*\omega_S \wedge \theta \wedge \bar{\theta}) \\ &= \sqrt{-1}\pi^*\omega_S \wedge d\theta \wedge \bar{\theta} - \sqrt{-1}\pi^*\omega_S \wedge \theta \wedge d\bar{\theta} \\ &= \sqrt{-1}\pi^*\omega_S \wedge ((\omega_1 + \sqrt{-1}\omega_2) \wedge \bar{\theta} - (\omega_1 - \sqrt{-1}\omega_2) \wedge \theta) \\ &= 0.\end{aligned}$$

According to Lemma 1, $(\omega_0, \Omega)$ is the solution of equation (2.4). Let $u$ be any smooth function on $S$ and let

$$\omega_u = \pi^*(e^u\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}. \tag{6.2}$$

Then

$$\parallel \Omega \parallel^2_{\omega_u} = \frac{\omega_0^3}{\omega_u^3} = \frac{1}{e^{2u}}$$

and

$$\parallel \Omega \parallel_{\omega_u} \omega_u^2 \quad = \quad \omega_0^2 + (e^u - 1)\omega_S^2.$$

Using (6.1), we obtain

$$d(\parallel \Omega \parallel_{\omega_u} \omega_u^2) = d\omega_0^2 + d(e^u - 1) \wedge \omega_S^2 = 0$$

because $e^u$ is a function on $S$. Hence we have proven the following

**Lemma 12** ([**10**])**.** *The metric* (6.2) *defined on $X$ satisfies equation* (2.5) *and so satisfies equation* (2.4)*.*

Let $V$ be a stable vector bundle over $X$ with degree 0 with respective to the metric $\omega_u$. According to Li-Yau's theorem [**16**], there is a Hermitian-Yang-Mills metric $H$ on $V$, which is unique up to constant.

Then $(V, H, X, \omega_u)$ satisfies equation (2.1), (2.2) and (2.4) of the Strominge's system. Hence to look for a solution to Strominger's system, we need only to consider equation (2.3):

$$\sqrt{-1}\partial\bar{\partial}\omega_u = \frac{\alpha'}{4}(\mathrm{tr}R_u \wedge R_u - \mathrm{tr}F_H \wedge F_H), \tag{6.3}$$

where $R_u$ is the curvature of Hermitian connection of metric $\omega_u$ on the holomorphic tangent bundle $T'X$. Define the Laplacian operator $\triangle$ with respective to the metric $\omega_S$ as

$$\triangle\psi\frac{\omega_S^2}{2!} = \sqrt{-1}\partial\bar{\partial}\psi \wedge \omega_S.$$

**Lemma 13.** $\sqrt{-1}\partial\bar{\partial}\omega_u = \triangle e^u \cdot \frac{\omega_S^2}{2!} + \frac{1}{2}(\| \omega_1 \|^2_{\omega_S} + \| \omega_2 \|^2_{\omega_S})\frac{\omega_S^2}{2!}$.

*Proof.* Using (5.3) and (5.12), we compute

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}\omega_u &= \sqrt{-1}\partial\bar{\partial}(e^u\omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}) \\ &= \sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - \frac{1}{2}\bar{\partial}\theta \wedge \partial\bar{\theta} \\ &= \triangle e^u \cdot \frac{\omega_S^2}{2!} + \frac{1}{2}(\| \omega_1 \|^2_{\omega_S} + \| \omega_2 \|^2_{\omega_S})\frac{\omega_S^2}{2!}.\end{aligned}$$

q.e.d.

**Lemma 14.**
$\mathrm{tr}R_u \wedge R_u = \pi^*\mathrm{tr}R_S \wedge R_S + 2\sqrt{-1}\pi^*(\partial\bar{\partial}u \wedge \partial\bar{\partial}u) + 2\pi^*(\partial\bar{\partial}(e^{-u}\rho))$, *where* $\rho = -\sqrt{-1}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})$.

*Proof.* In the proof of Proposition 8 we don't use the condition that $\omega_S$ is Kähler. So if we replace metric $g$ by $e^u g$, we can still obtain:

$$\mathrm{tr}R_u \wedge R_u = \pi^*(\mathrm{tr}R_S^u \wedge R_S^u + 2\sqrt{-1}\partial\bar{\partial}(e^{-u}\rho)), \tag{6.4}$$

here $R_S^u$ denotes the curvature of Hermitian connection of the metric $e^u g$ on holomorphic tangent bundle $T'S$. So

$$R_S^u = \bar{\partial}\partial u \cdot I + R_S$$

and

$$\mathrm{tr}R_S^u \wedge R_S^u = \mathrm{tr}R_S \wedge R_S + 2\partial\bar{\partial}u \wedge \partial\bar{\partial}u, \tag{6.5}$$

here we use the fact that $\mathrm{tr}R_S = 0$ because the Hermitian metric $g$ is the Calabi-Yau metric on $S$. Inserting (6.5) into (6.4), we have proven the lemma. q.e.d.

From Lemma 13 and 14, we can rewrite equation (6.3) as

$$\begin{aligned}&\sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - \sqrt{-1}\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u \\ &= \frac{\alpha'}{4}\mathrm{tr}R_S \wedge R_S - \frac{\alpha'}{4}\mathrm{tr}F_H \wedge F_H - \frac{1}{2}(\| \omega_1 \|^2 + \| \omega_2 \|^2_{\omega_S})\frac{\omega_S^2}{2!}.\end{aligned} \tag{6.6}$$

**Proposition 15.** *There is no solution of Strominger's system on the torus bundle $X$ over $T^4$ if the metric is $e^u\omega_S + \frac{\sqrt{-1}}{2}\theta\wedge\bar\theta$.*

*Proof.* Wedging the left-hand side of equation (6.6) by $\omega_u$ and integrating over $X$, we get

$$\int_X \left\{\sqrt{-1}\partial\bar\partial e^u\wedge\omega_S - \frac{\alpha'}{2}\partial\bar\partial(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar\partial u\wedge\partial\bar\partial u\right\}\wedge\omega_u = 0 \tag{6.7}$$

because $\partial\omega_u = \partial(e^u)\wedge\omega_S + 2\theta\wedge(\omega_1 - \sqrt{-1}\omega_2)$. When $S = T^4$, $R_{T^4} = 0$. Integrating both sides of (6.6) and applying (6.7), we get

$$\alpha'\int_X \mathrm{tr} F_H\wedge F_H\wedge\omega_u + \frac{1}{2}\int_X(\|\,\omega_1\,\|^2_{\omega_S} + \|\,\omega_2\,\|^2_{\omega_S})\frac{\omega^2}{2!}\wedge\omega_u = 0. \tag{6.8}$$

Certainly

$$2\int_X(\|\,\omega_1\,\|^2_{\omega_S} + \|\,\omega_2\,\|^2_{\omega_S})\frac{\omega_S^2}{2!}\wedge\omega_u > 0. \tag{6.9}$$

On the other hand, it is well-known that

$$\frac{\mathrm{tr}F_H^2}{8\pi^2} = \frac{1}{2}c_1^2(V) - c_2(V) = \frac{1}{2r}c_1^2(V) - \frac{1}{2r}(2rc_2(V) - (r-1)c_1^2(V)),$$

where $r$ is a rank of the bundle $V$ and that

$$(2r(c_2(V) - (r-1)c_1^2(V))\wedge\omega_u = \frac{r}{4\pi^2}\mid F_0\mid^2\frac{\omega_u^3}{3!},$$

where $F_0 = F_H - \frac{1}{r}\mathrm{tr}F_H\cdot \mathrm{id}_V$. So

$$\mathrm{tr}F_H^2\wedge\omega_u = \frac{8\pi^2}{2r}c_1^2(V) - \mid F_0\mid^2\frac{\omega_u^3}{3!}.$$

Now according to equation (2.2), $F_H\wedge\omega_u^2 = 0$ and $c_1(V)\wedge\omega_u^2 = 0$. Thus

$$c_1^2(V)\wedge\omega_u = -\mid c_1(V)\mid^2\frac{\omega_u^3}{3!}$$

and

$$\int_X \mathrm{tr}F_H^2\wedge\omega_u = -\frac{4\pi^2}{r}\int_X\mid c_1(V)\mid^2\frac{\omega_u^3}{3!} - \int_X\mid F_0\mid^2\frac{\omega_u^3}{3!}\le 0. \tag{6.10}$$

Inserting (6.9) and (6.10) into (6.8), we get a contradiction. q.e.d.

This situation is different if the base is a $K3$ surface. At first we observe

**Lemma 16.** *Let $E$ be a stable vector bundle over $S$ with degree 0 with respective to the Calabi-Yau metric $\omega_S$. Then $V = \pi^*E$ is also a stable vector bundle over $X$ with degree 0 with respective to Hermitian metric $\omega_u$ for any smooth function $u$ on $S$.*

*Proof.* According to the Donaldson-Uhlenbeck-Yau theorem, there is a unique Hermitian-Yang-Mills metric $H$ on $E$ up to constant. Since we assume that the degree of $E$ is zero, the curvature $F_H$ of $H$ satisfies the equation

$$F_H \wedge \omega_S = 0.$$

For the metric $\pi^* H$ on $V = \pi^* E$, the curvature $\pi^*(F_H)$ satisfies

$$\pi^* F_H \wedge \omega_u^2 = \pi^*(F_H \wedge \omega_S) \wedge (\pi^*(e^{2u}\omega_S) + \pi^*(e^u)\theta \wedge \bar{\theta}) = 0.$$

So $\pi^* H$ is also the Hermitian-Yang-Mills metric on $V = \pi^* E$ with degree 0. Thus $V$ is a stable vector bundle over $X$ with respective to the Hermitian metric $\omega_u$ for any smooth function $u$. q.e.d.

We also have the following observation:

**Proposition 17** ([**3**]). *Let $(V, H)$ be a Hermitain-Yang-Mills vector bundle over $(X, \omega_u)$ with gauge group $SU(r)$. If $(X, \omega_u, V, H)$ is the solution to Strominger's system, then there is a Hermitian-Yang-Mills vector bundle $(E, H')$ over $S$ and a flat line bundle $L$ over $X$ such that*

$$V = \pi^* E \otimes L.$$

When we restrict ourselves to consider such a vector bundle ($V = \pi^* E, \pi^* F_H$) over $X$, we see that equation (6.6) on $X$ can be considered as an equation on $S$. Integrating equation (6.6) over $S$, we get

$$\text{(6.11)}\quad \alpha' \int_S \{\mathrm{tr} R_S \wedge R_S - \mathrm{tr} F_H \wedge F_H\} = 2 \int_S (\| \omega_1 \|^2_{\omega_S} + \| \omega_2 \|^2_{\omega_S}) \frac{\omega_S^2}{2!}.$$

As $\int_S \mathrm{tr} R_S \wedge R_S = 8\pi^2 c_2(V) = 8\pi^2 \times 24$, and $\int_S \mathrm{tr} F_H \wedge F_H = 8\pi^2 \times (c_2(E) - \frac{1}{2} c_1^2(E)) \geq 0$, we can rewrite equation (6.11) as

$$\text{(6.12)}\quad \alpha'(24 - (c_2(E) - \frac{1}{2} c_1^2(E))) = \int_S (\| \frac{\omega_1}{2\pi} \|^2_{\omega_S} + \| \frac{\omega_2}{2\pi} \|^2_{\omega_S}) \frac{\omega_S^2}{2!}.$$

Using notations of section 1, above equation implies:

$$\text{(6.13)}\quad \alpha'(24 - \kappa(E)) + \left( Q\left(\frac{\omega_1}{2\pi}\right) + Q\left(\frac{\omega_2}{2\pi}\right) \right) = 0.$$

This equation implies that there is a smooth function $\mu$ such that

$$\text{(6.14)}\quad \frac{\alpha'}{4} \mathrm{tr} R_S \wedge R_S - \alpha' \mathrm{tr} F_H \wedge F_H - \frac{1}{2} (\| \omega_1 \|^2 + \| \omega_2 \|^2_{\omega_S}) \frac{\omega_S^2}{2!} = -\mu \frac{\omega_S^2}{2!}$$

and $\int_S \mu \frac{\omega_S^2}{2!} = 0$. Inserting (6.14) into (6.6), we obtain the following equation:

$$\text{(6.15)}\quad \sqrt{-1}\partial\bar{\partial} e^u \wedge \omega_S - \sqrt{-1}\frac{\alpha'}{2}\partial\bar{\partial}(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu \frac{\omega_S^2}{2!} = 0$$

where $\mu$ is a smooth function satisfying the integrable condition $\int_S \mu = 0$ and $\rho = \sqrt{-1}\mathrm{tr}(\bar{\partial} B \wedge \partial B^* \cdot g^{-1})$ is a smooth well-defined real $(1,1)$-form on $S$. In the next section we will use the continuity method to solve equation (6.15). We will prove that equation (6.15) has a smooth solution $u$.

**Theorem 18.** *Let $S$ be a $K3$ surface with a Calabi-Yau metric $\omega_S$. Let $\omega_1$ and $\omega_2$ be anti-self-dual $(1,1)$-forms on $S$ such that $\frac{\omega_1}{2\pi} \in H^2(S,\mathbb{Z})$ and $\frac{\omega_2}{2\pi} \in H^2(S,\mathbb{Z})$. Let $X$ be a $T^2$-bundle over $S$ constructed by $\omega_1$ and $\omega_2$. Let $E$ be a stable bundle over $S$ with degree 0. Suppose that $\omega_1$, $\omega_2$ and $\kappa(E)$ satisfy the condition (6.13). Then there exists a smooth function $u$ on $S$ and a Hermitian-Yang-Mills metric $H$ on $E$ such that $(V = \pi^*E, \pi^*F_H, X, \omega_u)$ is a solution of Strominger's system.*

*Proof.* Because we assume that $E$ is a stable bundle over $S$ with degree 0 with respect to the Calabi-Yau metric $\omega_S$, according to the Donaldson-Uhlenbeck-Yau theorem, there is a unique Hermitian-Yang-Mills metric $H$ on $E$ up to constant such that the curvature $F_H$ of metric $H$ satisfies

$$F_H^{2,0} = F_H^{0,2} = 0, \qquad F_H \wedge \omega_S = 0.$$

So we have $\pi^*F_H^{2,0} = \pi^*F_H^{0,2} = 0$ and according to Lemma 16, we also have $\pi^*F_H \wedge \omega_u^2 = 0$. Now according to our assumption, $(\omega_1, \omega_2, E)$ satisfies the condition (6.13), and hence there is a function $\mu$ satisfying equation (6.14). Then we solve equation (6.15). According to Theorem 19 in the next section, there exists a smooth solution $u$ of equation (6.15). Combining equation (6.15) with (6.14), we know that $u$ is the solution of equation (6.6). So $(\pi^*F_H, \omega_u)$ satisfies equation (2.3). On the other hand, according to Lemma 12, the metric $\omega_u = e^u\omega_S + \frac{\sqrt{-1}}{2}\theta\wedge\bar\theta$ on $X$ satisfies equation (2.4). Thus we have proven that $(V = \pi^*E, \pi^*F_H, X, \omega_u)$ satisfy all equations of Strominger's system. q.e.d.

## 7. Solving the equation

As above section, we let

$$\rho = -\sqrt{-1}\mathrm{tr}(\bar\partial B \wedge \partial B^* \cdot g^{-1}).$$

In this section, we want to prove

**Theorem 19.** *The equation*

$$\sqrt{-1}\partial\bar\partial e^u \wedge \omega_S - \sqrt{-1}\frac{\alpha'}{2}\partial\bar\partial(e^{-u}\rho) - \frac{\alpha'}{2}\partial\bar\partial u \wedge \partial\bar\partial u + \mu\frac{\omega_S^2}{2!} = 0 \tag{7.1}$$

*has a smooth solution $u$ such that*

$$\omega' = e^u\omega_S + \frac{\alpha'}{2}e^{-u}\rho + \alpha'\sqrt{-1}\partial\bar\partial u$$

*defines a Hermitian metric on $S$.*

*Proof.* We solve equation (7.1) by the continuity method. More precisely, we introduce a parameter $t \in [0,1]$ and consider the following equation

$$\sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - t\alpha\sqrt{-1}\partial\bar{\partial}(e^{-u}\rho) - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + t\mu\frac{\omega_S^2}{2!} = 0. \tag{7.2}$$

We shall impose the following:

$$\text{Elliptic condition}: \quad \omega' = e^u\omega_S + t\alpha e^{-u}\rho + 2\alpha\sqrt{-1}\partial\bar{\partial}u > 0 \tag{7.3}$$

and

$$\text{Normalization}: \quad \Big(\int_S e^{-4u}\frac{\omega_S^2}{2!}\Big)^{\frac{1}{4}} = A, \qquad \int_S 1\frac{\omega_S^2}{2!} = 1. \tag{7.4}$$

Let $C^{k,\alpha_0}(S)$ be the space of functions whose $k$-derivatives are Hölder continuous with exponent $0 < \alpha_0 < 1$. We consider the solution in the following space

$$B_A = \{u \in C^{2,\alpha_0}(S) \mid u \ \text{ satisfies the normalization (7.4)}\} \tag{7.5}$$

and

$$B_{A,t} = \{u \in B_A \mid u \ \text{ also satisfies the elliptic condition (7.3)}\}. \tag{7.6}$$

Let

$$\mathbf{T} = \{s \in [0,1] \mid \text{for } \ t \in [0,s] \ \text{ equation (7.2) admits a solution in } B_{A,t}\}. \tag{7.7}$$

Obviously $0 \in \mathbf{T}$ with a solution $u = -\ln A$. Hence we need only to show that $\mathbf{T}$ is both closed and open in $[0,1]$. This will imply that $1 \in \mathbf{T}$ and that our original equation has a solution in $C^{2,\alpha_0}$. To see that the set $\mathbf{T}$ is open, we use the standard implicity function theorem.

Let $t_0 \in \mathbf{T}$ and $u_{t_0}$ be a solution of equation (7.2). Let $B_{[0,1]} = \{(t,u) \in [0,1] \times B_A \mid u \in B_{A,t}\}$. Then $B_{[0,1]}$ is an open set of $[0,1] \times B_A$. Let $C_0^{0,\alpha_0}(S) = \{\psi \in C^{0,\alpha_0} \mid \int_S \psi\frac{\omega_S^2}{2!} = 0\}$. We have a map: $\tilde{L} : B_{[0,1]} \to C_0^{0,\alpha_0}(S)$,

$$\tilde{L}(t,u) = *_{\omega_S}(\sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - \sqrt{-1}t\alpha\partial\bar{\partial}(e^{-u}\rho) - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + t\mu\omega_S^2/2!). \tag{7.8}$$

According to the definition of $t_0$, $\tilde{L}(t_0, u_{t_0}) = 0$. The differential $d\tilde{L}$ of $\tilde{L}$ at $u_{t_0}$ evaluated at $\varphi$ is $L(\varphi)$, where the linear operator $L$ from $C^{2,\alpha_0}(S)$ to $C^{0,\alpha_0}(S)$ is defined as:

$$L(\varphi) = *_{\omega_S}(\sqrt{-1}\partial\bar{\partial}(e^{u_{t_0}}\varphi) \wedge \omega_S + \sqrt{-1}t_0\alpha\partial\bar{\partial}(e^{-u_{t_0}}\varphi\rho) - 2\alpha\partial\bar{\partial}u_{t_0} \wedge \partial\bar{\partial}\varphi). \tag{7.9}$$

So $d\tilde{L} = L \mid_{T_{u_{t_0}}B_A}$, where $T_{u_{t_0}}B_A = \{\varphi \in C^{2,\alpha_0}(S) \mid \int e^{-4u_{t_0}}\varphi = 0\}$ is the tangent space of $B_A$ at $u_{t_0}$. The principle part of the operator $*_{\omega_S}L$ is

$$\sqrt{-1}\partial\bar{\partial}\varphi \wedge (e^{u_{t_0}}\omega_S + t_0\alpha e^{-u_{t_0}}\rho + 2\alpha\sqrt{-1}\partial\bar{\partial}u_{t_0}). \tag{7.10}$$

From the elliptic condition (7.3), we get:

$$\omega'_{t_0} = e^{u_{t_0}}\omega + t_0\alpha e^{-u_{t_0}}\rho + 2\alpha\sqrt{-1}\partial\bar{\partial}u_{t_0} > 0. \tag{7.11}$$

$\omega'_{t_0}$ can be taken as a Hermitian (not Kähler !) metric on $S$. Let

$$P = \sqrt{-1}\Lambda_{\omega'_{t_0}}\partial\bar{\partial}. \tag{7.12}$$

Then $P$ is an elliptic operator on $S$. Because $u_{t_0}$ is a solution in $C^{2,\alpha_0}$ and our $\mu$ and $\rho$ are smooth, according to Schauder theory, $u_{t_0}$ is smooth. So the operator $P$ is smooth and can be defined by

$$\sqrt{-1}\partial\bar{\partial}\psi \wedge \omega'_{t_0} = P(\psi)\omega'^2_{t_0}/2! \tag{7.13}$$

for any $C^2(S)$ function $\psi$ on $S$. For any $\phi, \psi \in C^{2,\alpha_0}(S,\mathbb{R})$, we compute

$$\begin{aligned}
&\int L^*(\psi)\varphi\frac{\omega_S^2}{2!}\\
&= \int \psi\cdot\{\sqrt{-1}\partial\bar{\partial}(e^{u_{t_0}}\varphi)\wedge\omega_S + \sqrt{-1}t_0\alpha\partial\bar{\partial}(e^{-u_{t_0}}\varphi\rho) - 2\alpha\partial\bar{\partial}u_{t_0}\wedge\partial\bar{\partial}\varphi\}\\
&= \int \varphi\sqrt{-1}\bar{\partial}\partial\psi\wedge(e^{u_{t_0}}\omega_S + t_0\alpha e^{-u_{t_0}}\rho + 2\alpha\sqrt{-1}\partial\bar{\partial}u_{t_0})\\
&= \sqrt{-1}\int \varphi\partial\bar{\partial}\psi\wedge\omega'_{t_0}\\
&= \int P^*(\varphi)\psi\frac{\omega'^2_{t_0}}{2!}.
\end{aligned}$$

Thus using the Corollary in page 227 of [**15**], we obtain

$$\ker L^* = \ker P = \mathbb{R}$$

and

$$\begin{aligned}
\ker L &= \ker P^*\\
&= \{\mathbb{R}\varphi_0 \mid \varphi_0 \text{ is a nonzero function that has constant sign}\}.
\end{aligned}$$

Now we are ready to prove $d\tilde{L}$ is invertible. Because $d\tilde{L} = L\,|_{T_{u_{t_0}}B_A}$, we only need to prove $L\,|_{T_{u_{t_0}}B_A}\colon T_{u_{t_0}}B_A \to C_0^{0,\alpha_0}(S)$ is invertible. It is clear that $\ker L \cap T_{u_{t_0}}B_A = 0$. So $d\tilde{L} = L\,|_{T_{u_{t_0}}B_A}$ is injective. Next we prove that $d\tilde{L} = L\,|_{T_{u_{t_0}}B_A}$ is surjective. For any $\psi \in C_0^{0,\alpha_0}(S)$, we have $\psi \perp \ker L^*$. It is well known that there is a weak solution $\varphi_1$ of linear elliptic equation $L(\varphi_1) = \psi$. The Schauder theory shows that $\varphi \in C^{2,\alpha_0}(S)$ when $\psi \in C^{0,\alpha_0}(S)$. Take $c_0 = -\frac{\int e^{-4u_{t_0}}\varphi_1}{\int e^{-4u_{t_0}}\varphi_0}$, then $\varphi_1 + c_0\varphi_0 \in T_{u_{t_0}}B_A$ and $L(\varphi_1 + c_0\varphi_0) = \psi$. So $d\tilde{L} = L\,|_{T_{u_{t_0}}B_A}$ is surjective. Hence $d\tilde{L}$ of $\tilde{L}$ at $u_{t_0}$ is invertible and $\tilde{L}$ maps an open neighborhood of $(t_0, u_{t_0})$ in $B_{[0,1]}$ to an open neighborhood of $\tilde{L}(t_0, u_{t_0})$ in $C_0^{0,\alpha_0}(S)$. This proves the set $\mathbf{T}$ is open.

It remains to prove that $\mathbf{T}$ is closed. Let $\rho = \frac{\sqrt{-1}}{2}\rho_{i\bar{j}}dz_i \wedge d\bar{z}_j$, then we can write $g'_{i\bar{j}}$ as

$$g'_{i\bar{j}} = e^u g_{i\bar{j}} + t\alpha e^{-u}\rho_{i\bar{j}} + 4\alpha u_{i\bar{j}}.$$

By directly computation, we get

(7.14)

$$\begin{aligned}\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} =& e^{2u} + 2\alpha e^u \triangle u + t\alpha g^{i\bar{j}}\rho_{i\bar{j}} + 2t\alpha^2 e^{-u}(\sqrt{-1}\partial\bar{\partial}u \wedge \rho, \frac{\omega_S^2}{2!}) \\ &+ t^2\alpha^2 e^{-2u}\frac{\det \rho_{i\bar{j}}}{\det g_{i\bar{j}}} + 16\alpha^2 \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}.\end{aligned}$$

We can rewrite equation (7.2) as

(7.15)

$$\begin{aligned}8\alpha\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} =& - e^u \triangle u - 2e^u \mid \nabla u \mid^2 - t\mu - t\alpha e^{-u}(\sqrt{-1}\partial\bar{\partial}u \wedge \rho, \frac{\omega_S^2}{2!}) \\ &+ t\alpha e^{-u}(\sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \rho, \frac{\omega_S^2}{2!}) - t\alpha e^{-u}(\sqrt{-1}\partial u \wedge \bar{\partial}\rho, \frac{\omega_S^2}{2!}) \\ &+ t\alpha e^{-u}(\sqrt{-1}\bar{\partial}u \wedge \partial\rho, \frac{\omega_S^2}{2!}) + t\alpha e^{-u}(\sqrt{-1}\partial\bar{\partial}\rho, \frac{\omega_2^2}{2!}).\end{aligned}$$

Then inserting (7.15) into (7.14), we find the Monge-Ampère-type equation:

$$\frac{\det(e^u g_{i\bar{j}} + t\alpha e^{-u}\rho_{i\bar{j}} + 4\alpha u_{i\bar{j}})}{\det g_{i\bar{j}}} = F_{t,u_t} \tag{7.16}$$

where

$$\begin{aligned}F_{t,u_t} =& e^{2u} + t\alpha g^{i\bar{j}}\rho_{i\bar{j}} + t^2\alpha^2 e^{-2u}\frac{\det \rho_{i\bar{j}}}{\det g_{i\bar{j}}} - 2e^u \mid \nabla u \mid^2 \\ &+ 2t\alpha^2 e^{-u}(\sqrt{-1}\partial u \wedge \bar{\partial}u \wedge \rho, \frac{\omega_S^2}{2!}) - 2t\alpha^2 e^{-u}(\sqrt{-1}\partial u \wedge \bar{\partial}\rho, \frac{\omega_S^2}{2!}) \\ &+ 2t\alpha^2 e^{-u}(\sqrt{-1}\bar{\partial}u \wedge \partial\rho, \frac{\omega_S^2}{2!}) + 2t\alpha^2 e^{-u}(\sqrt{-1}\partial\bar{\partial}\rho, \frac{\omega_2^2}{2!}) - 2t\alpha\mu.\end{aligned}$$

In particular, when $\omega_2 = n\omega_1$,

$$\begin{aligned}F_{t,u_t} =& (e^u + t\alpha f e^{-u})^2 - 2\alpha(e^u - t\alpha f e^{-u}) \mid \nabla u \mid^2 \\ &- 4t\alpha^2 e^{-u} \nabla u \cdot \nabla f + 2t\alpha^2 e^{-u} \triangle f - 2t\alpha\mu.\end{aligned}$$

If $t_q$ is a sequence in $\mathbf{T}$, then we have a sequence $u_q \in C^{2,\alpha_0}(S)$ such that

$$\frac{\det(e^{u_q} g_{i\bar{j}} + t_q\alpha e^{-u_q}\rho_{i\bar{j}} + 4\alpha\frac{\partial^2 u_q}{\partial z_i \partial \bar{z}_j})}{\det g_{i\bar{j}}} = F_{t_q,u_{t_q}}. \tag{7.17}$$

Differentiating equation (7.17), we have

(7.18)

$$4\alpha \det\Big(e^{u_q} g_{i\bar{j}} + t_q \alpha e^{-u_q}\rho_{i\bar{j}} + 4\alpha \frac{\partial^2 u_q}{\partial z_i \partial \bar{z}_j}\Big) \cdot \sum g_q'^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}\Big(\frac{\partial u_q}{\partial z_k}\Big)$$
$$= \frac{\partial}{\partial z_k}\{\det g_{i\bar{j}} \cdot F_{t_q, u_{t_q}}\}$$
$$-\det\Big(e^{u_q} g_{i\bar{j}} + t_q \alpha e^{-u_q}\rho_{i\bar{j}} + 4\alpha \frac{\partial^2 u_q}{\partial z_i \partial \bar{z}_j}\Big) \cdot \sum g_q'^{i\bar{j}} \frac{\partial}{\partial z_k}(e^{u_q} g_{i\bar{j}} + t_q \alpha e^{-u_q}\rho_{i\bar{j}})$$

Proposition 25 (and Proposition 21–23 for a special case $\omega_2 = n\omega_1$) shows that the operator on the left-hand side of (7.18) is uniformly elliptic. Proposition 26 (and Proposition 24 for the special case) shows that the coefficients are Hölder continuous with exponent $\alpha$ for any $0 \le \alpha_0 \le 1$. The Schauder estimate then gives an estimate for the $C^{2,\alpha_0}$-estimates of $\partial u_q/\partial z_k$. Similarly we can find the $C^{2,\alpha_0}$-norm of $\partial u_q/\partial \bar{z}_k$. Therefore the sequence $\{u_q\}$ converges in the $C^{2,\alpha_0}$-norm to a solution of the equation

$$\frac{\det(e^u g_{i\bar{j}} + t_0 \alpha e^{-u}\rho_{i\bar{j}} + 4\alpha \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j})}{\det g_{i\bar{j}}} = F_{t_0},$$

where $t_0 = \lim_{q\to\infty} t_q$. Thus we find a $C^{2,\alpha_0}(S)$ solution $u_{t_0}$ of equation (7.16). But equation (7.16) is equivalent to equation (5.3). Certainly, we also have $\int_S e^{-4u_{t_0}} \frac{\omega^2}{2!} = A$. Hence $\mathbf{T}$ is closed. So there is a solution $u$ of equation (7.1) in $C^{2,\alpha_0}(S)$. Since $\mu$ and $(1,1)$-form $-\sqrt{-1}\mathrm{tr}(\bar{\partial} B \wedge \partial B^* \cdot g^{-1})$ are smooth, the Schauder theory provides the smooth solution of equation (7.1). q.e.d.

## 8. Zeroth order estimate

From this section to the section 11, we give a priori estimates of $u$ up to the third order. In these sections, we deal with the simpler case $\omega_2 = n\omega_1$, where $\omega_1$ is an anti-self-dual $(1,1)$-form on $S$. Let $f = \frac{1+n^2}{4} \parallel \omega_1 \parallel^2_{\omega_S}$. Then the equation is

$$\triangle(e^u - t\alpha f e^{-u}) + 8\alpha \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + t\mu = 0,$$

where $f$ and $\mu$ are smooth functions on $S$ such that $f \ge 0$ and $\int_S \mu \frac{\omega_S^2}{2!} = 0$. According to our assumption, $u \in C^{2,\alpha_0}(S)$. So by the Schauder theory, the solution $u$ is smooth. We denote partial derivatives by $u_{i\bar{j}} = \partial_{i\bar{j}} u = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$. If we replace $t\alpha f$ by $f$ and $t\mu$ by $\mu$, then the equation can be written as

$$\triangle(e^u - fe^{-u}) + 8\alpha\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0. \tag{8.1}$$

We impose the elliptic condition

$$\omega' = (e^u + fe^{-u})\omega_S + 2\alpha\sqrt{-1}\partial\bar{\partial}u > 0$$

and the normalization condition

$$\Big(\int_S e^{-4u}\frac{\omega_S^2}{2!}\Big)^{\frac{1}{4}} = A, \qquad \int_S 1\frac{\omega_S^2}{2!} = 1. \tag{8.2}$$

In this section we prove that if $A$ is small enough, then the solution $u$ has an upper bound and a lower bound depending only on $\alpha$, $f$, $\mu$, Sobolev constant of metric $\omega_S$, and $A$. In the next section, we shall prove that if $A$ is small enough, then the determinant of $\omega'$ has a lower bound greater than 0 and the metric $\omega'$ is uniformly positive.

Let $g' = \frac{\sqrt{-1}}{2}g'_{i\bar{j}}dz_i \wedge dz_{\bar{j}}$, where

$$g'_{i\bar{j}} = (e^u + fe^{-u})g_{i\bar{j}} + 4\alpha u_{i\bar{j}}.$$

We note that

$$\frac{\omega'^2}{2!} = \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}\frac{\omega_S^2}{2!}.$$

We shall denote the inverse matrix of $(g'_{i\bar{j}})$ by $(g'^{i\bar{j}})$. Then from the definition (7.13) of the operator $P$, we have

$$P(\varphi) = 2g'^{i\bar{j}}\varphi_{i\bar{j}}.$$

From (8.1), we have

$$\begin{aligned} P(u)\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} &= (e^u + fe^{-u}) \triangle u + 16\alpha\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} \\ &= (e^u + fe^{-u}) \triangle u - 2 \triangle (e^u - fe^{-u}) - 2\mu. \end{aligned} \tag{8.3}$$

We shall denote the volume form to be $\frac{\omega_S^2}{2!}$ unless it is clear from the context. Then form (8.3),

$$\begin{aligned} &\int P(e^{-ku})\frac{\omega'^2}{2!} \\ &= k^2\int e^{-ku}(2g'^{i\bar{j}}\partial_i u\partial_{\bar{j}}u)\frac{\omega'^2}{2!} - k\int e^{-ku}(2g'^{i\bar{j}}\partial_{i\bar{j}}u)\frac{\omega'^2}{2!} \\ &\geq -k\int e^{-ku}P(u)\frac{\omega'^2}{2!} \\ &= -k\int e^{-ku}(e^u + fe^{-u}) \triangle u \\ &\quad + 2k\int e^{-ku} \triangle (e^u - fe^{-u}) + 2k\int e^{-ku}\mu. \end{aligned} \tag{8.4}$$

On the other hand, from (7.13),

$$
\begin{aligned}
(8.5)\quad \int P(e^{-ku})\frac{\omega'^2}{2!} &= \sqrt{-1}\int \partial\bar\partial(e^{-ku})\wedge\omega' \\
&= \int (e^u + fe^{-u}) \triangle (e^{-ku}) \\
&= -k\int e^{-ku}(e^u + fe^{-u}) \triangle u + k^2\int e^{-ku}(e^u + fe^{-u}) \mid \bigtriangledown u \mid^2,
\end{aligned}
$$

where $\mid \bigtriangledown u \mid^2 = 2g^{i\bar j}u_i u_{\bar j}$. Combing (8.4) and (8.5),

$$
\begin{aligned}
(8.6)\quad & k\int (e^u + fe^{-u})e^{-ku} \mid \bigtriangledown u \mid^2 \\
&\geq 2\int e^{-ku}(e^u + fe^{-u}) \triangle u + 2\int e^{-ku}(e^u - fe^{-u}) \mid \bigtriangledown u \mid^2 \\
&\quad - 2\int e^{-(k+1)u} \triangle f + 4\int e^{-(k+1)u} \bigtriangledown u \cdot \bigtriangledown f + 2\int e^{-ku}\mu,
\end{aligned}
$$

where $\bigtriangledown u \cdot \bigtriangledown f = g^{i\bar j}(u_i f_{\bar j} + u_{\bar j} f_i)$.

When $k \geq 2$, we integrate by part and obtain

$$
\begin{aligned}
(8.7)\quad & 2\int e^{-ku}(e^u + fe^{-u}) \triangle u \\
&= 2(k-1)\int e^{-(k-1)u} \mid \bigtriangledown u \mid^2 + 2(k+1)\int fe^{-(k+1)u} \mid \bigtriangledown u \mid^2 \\
&\quad + \frac{2}{k+1}\int e^{-(k+1)u} \triangle f \frac{\omega_S^2}{2!} - 4\int e^{-(k+1)u} \bigtriangledown u \cdot \bigtriangledown f.
\end{aligned}
$$

Inserting (8.7) into (8.6),

$$
\begin{aligned}
& k\int e^{-(k-1)u} \mid \bigtriangledown u \mid^2 + k\int fe^{-(k+1)u} \mid \bigtriangledown u \mid^2 \\
&\leq 2(1 - \frac{1}{k+1})\int e^{-(k+1)u} \triangle f - 2\int e^{-ku}\mu.
\end{aligned}
$$

Because $f \geq 0$, the above inequality implies

$$
(8.8)\quad k\int e^{-(k-1)u} \mid \bigtriangledown u \mid^2 \leq C_0\int e^{-(k+1)u} + C_0\int e^{-ku},
$$

where $C_0$ depends only on $f$ (so also depends on $\alpha$) and $\mu$. In the following, $C_0$ may depend on $\alpha$, $f$, $\mu$ and the Sobolev constant of $S$ about the metric $\omega_S$. We use the constant $C_0$ in the generic sense. So $C_0$ may mean different constants in different equations.

From the above inequality, if we replace $k-1$ by $k$, then when $k \geq 1$,

$$
(8.9)\quad \int \mid \bigtriangledown (e^{-u})^{\frac{k}{2}} \mid^2 \leq C_0 k\int e^{-(k+2)u} + C_0 k\int e^{-(k+1)u}.
$$

The Sobolev inequality shows

$$\| e^{-\frac{k}{2}u} \|_{L^r} \leq C_0(\| e^{-\frac{k}{2}u} \|_{L^p} + \| \bigtriangledown e^{-\frac{k}{2}u} \|_{L^p}$$

where $r = \frac{4p}{4-p} = 4$.

In the case $p = 2$, we have

$$\Big(\int (e^{-u})^{2k}\Big)^{\frac{1}{2}} \leq C_0 \int (e^{-u})^k + C_0 \int | \bigtriangledown (e^{-u})^{\frac{k}{2}} |^2 .$$

Inserting (8.9) into above inequality, we get

$$\Big(\int (e^{-u})^{2k}\Big)^{\frac{1}{2}} \leq C_0 \int (e^{-u})^k + C_0 k \int (e^{-u})^{k+2} + C_0 k \int (e^{-u})^{k+1}.$$

Since we normalize $\omega_S$ by $\int_S 1 \frac{\omega_S^2}{2!} = 1$, we can apply Hölder inequality to the above inequality to obtain

$$(8.10) \quad \Big(\int (e^{-u})^{2k}\Big)^{\frac{1}{2}} \leq C_0 \Big(\int (e^{-u})^{k+2}\Big)^{\frac{k}{k+2}} + C_0 k \Big(\int (e^{-u})^{k+2}\Big)^{\frac{k+1}{k+2}} + C_0 k \int (e^{-u})^{k+2}.$$

Note that when $k = 2$, above inequality has no use. This explains why we need the normalization (8.2).

In the following we assume that

$$(8.11) \qquad A < 1.$$

There are two cases:

*Case* (1). For any $k \geq 4$, $\int (e^{-u})^k \leq 1$. Then (8.10) implies

$$(8.12) \qquad \Big(\int (e^{-u})^{2k}\Big)^{\frac{1}{2}} \leq C_0 k \Big(\int (e^{-u})^{k+2}\Big)^{\frac{k}{k+2}}.$$

Applying the Hölder inequality,

$$(8.13) \qquad \int (e^{-u})^{k+2} = \int (e^{-u})^{k-2} (e^{-u})^4 \leq \Big(\int (e^{-u})^k\Big)^{\frac{k-2}{k}} \Big(\int (e^{-u})^{2k}\Big)^{\frac{2}{k}}.$$

Inserting above inequality into (8.12), we see

$$(8.14) \qquad \int (e^{-u})^{2k} \leq C_0 k^{2\frac{k+2}{k-2}} \Big(\int (e^{-u})^k\Big)^2 \leq C_0 k^2 \Big(\int (e^{-u})^k\Big)^2.$$

Take $k = 2^\beta$ for $\beta \geq 2$. Then $\beta \geq 2$ and rewrite (8.14) as

$$\int (e^{-u})^{2^{\beta+1}} \leq C_0 2^{2\beta} \Big(\int (e^{-u})^{2^\beta}\Big)^2.$$

Iterating above inequality, we get

$$(8.15) \qquad \Big(\int (e^{-u})^{2^{\beta+1}}\Big)^{\frac{1}{2^{\beta+1}}} \leq C_0 \Big(\int (e^{-u})^4\Big)^{\frac{1}{4}}.$$

We fix the constant $C_0$ and denote it by $C_1$, which depends only on $f$, $\mu$, $\alpha$, and the Sobolev constant of $S$ with respect to the metric $\omega_S$. Letting $\beta \to \infty$, we find

$$\exp(-\inf u) = \| e^{-u} \|_\infty \leq C_1 A. \tag{8.16}$$

*Case* (2). There is an integer $k$ such that $\int (e^{-u})^k > 1$. Let $k_0$ be the first such an integer. According to the assumption (8.11), $k_0 > 4$. Then for any $k \geq k_0$, Hölder inequality shows $\int (e^{-u})^k > 1$. For any $k \geq k_0 > 4$, inequality (8.10) and (8.13) imply

$$\begin{aligned}\Big(\int (e^{-u})^{2k}\Big)^{\frac{1}{2}} &\leq C_0 k \int (e^{-u})^{k+2} \\ &\leq C_0 k \Big(\int (e^{-u})^k\Big)^{\frac{k-2}{k}} \Big(\int (e^{-u})^{2k}\Big)^{\frac{2}{k}}.\end{aligned}$$

We see from the above inequality:

$$\int (e^{-u})^{2k} \leq C_0 k^2 \Big(\int (e^{-u})^k\Big)^{2\frac{k-2}{k-4}} \quad \text{for} \quad k \geq k_0 > 4.$$

Using the above inequality for $k \geq k_0$ and the inequality (8.14) for $k < k_0$, we get the estimate (8.16) of $\inf u$, because $A^a < A$ when $A < 1$ and $a > 1$.

Next we estimate $\sup_S u$. We also compute $\int_S P(e^{pu}) \frac{\omega'^2}{2!}$ by two methods and get

$$p \int (e^u + f e^{-u}) e^{pu} \mid \triangledown u \mid^2 \geq -2 \int e^{pu} \triangle (e^u - f e^{-u}) - 2 \int e^{pu} \mu. \tag{8.17}$$

Integrating by part, when $p \geq 2$,

$$\begin{aligned}&\int e^{pu} \triangle (e^u - f e^{-u}) \\ &= -p \int e^{(p+1)u} \mid \triangledown u \mid^2 - p \int e^{(p-1)u} f \mid \triangledown u \mid^2 \\ &\quad - \Big(1 + \frac{1}{p-1}\Big) \int e^{(p-1)u} \triangle f\end{aligned} \tag{8.18}$$

and when $p = 1$,

$$\int e^u \triangle (e^u - f e^{-u}) = -\int e^{2u} \mid \triangledown u \mid^2 - \int f \mid \triangledown u \mid^2 - \int u \triangle f. \tag{8.19}$$

Inserting (8.18) or (8.19) into (8.17), because $f > 0$, we get

$$p \int e^{(p+1)u} \mid \triangledown u \mid^2 \leq C_0 \int e^{pu} + C_0 \int e^{(p-1)u} \quad \text{for} \quad p \geq 2, \tag{8.20}$$

and when $p = 1$,

$$\int e^{2u} \mid \triangledown u \mid^2 \leq 2 \int e^u \mu - 2 \int u \triangle f \leq C_0 \int e^u + C_0 \int \mid u \mid. \tag{8.21}$$

**Remark 20.** When $t = 0$, $f$ and $\mu$ (actually $t\alpha f$ and $t\mu$) are equal to zero. From above inequality we have

$$\int e^{2u} \mid \triangledown u \mid^2 \leq 2\int e^u \mu - 2\int u \triangle f = 0,$$

which implies $\mid \triangledown u \mid^2 \equiv 0$. So when $t = 0$, there is an unique constant solution under the normalization and the elliptic condition.

We choose $A$ small enough such that

$$A < C_1^{-1}. \tag{8.22}$$

Then from $e^{-\inf u} \leq C_1 A < 1$, $u > 0$. (8.21) implies

$$\int \mid \triangledown e^u \mid^2 \leq C_0 \int e^u \tag{8.23}$$

and (8.20) implies

$$\int \mid \triangledown e^{\frac{p}{2}u} \mid^2 \leq C_0 p \int e^{pu} \quad \text{when} \quad p \geq 3. \tag{8.24}$$

Applying the Sobolov inequality and using (8.23), (8.24), we obtain

$$\Big(\int (e^u)^{2p}\Big)^{\frac{1}{2}} \leq C_0 p \int e^{pu}, \quad \text{for} \quad p \geq 2.$$

Take $p = 2^\beta$ for $\beta \geq 1$. Then

$$\int (e^u)^{2^{\beta+1}} \leq C_0 2^{2\beta} \Big(\int (e^u)^{2^\beta}\Big)^2.$$

Iterating the above inequality and taking the limit $\beta \to \infty$, we get

$$\exp(\sup u) \leq C_0 \Big(\int e^{2u}\Big)^{\frac{1}{2}}. \tag{8.25}$$

Let $\int e^u = M_u$, then $\int (e^u - M_u) = 0$. The Poincaré inequality and (8.23) imply

$$\int (e^u)^2 - \Big(\int e^u\Big)^2 \leq C_0 \int \mid \triangledown (e^u - M_u) \mid^2 \leq C_0 \int e^u. \tag{8.26}$$

Let $U_1 = \{x \in S \mid e^{-u(x)} \geq \frac{A}{2}\}$ and $U_2 = \{x \in S \mid e^{-u(x)} < \frac{A}{2}\}$. Then

$$\begin{aligned} A^4 = \int_S e^{-4u} &= \int_{U_1} e^{-4u} + \int_{U_2} e^{-4u} \\ &\leq \int_{U_1} e^{-4\inf u} + \int_{U_2} (A/2)^4 \\ &= e^{-4\inf u}\mathrm{Vol}(U_1) + \Big(A/2\Big)^4 \mathrm{Vol}(U_2) \\ &= \Big[(e^{-\inf u})^4 - (A/2)^4\Big]\,\mathrm{Vol}(U_1) + \Big(A/2\Big)^4. \end{aligned}$$

So

$$\begin{aligned}\mathrm{Vol}(U_1) \geq & \frac{A^4 - (A/2)^4}{(e^{-\inf u})^4 - (A/2)^4} \geq \frac{A^4 - (A/2)^4}{(C_1 A)^4 - (A/2)^4} \\ = & \frac{2^4 - 1}{(2C_1)^4 - 1} = m_0 > 0.\end{aligned}$$

Thus

$$\mathrm{Vol}(U_2) = 1 - \mathrm{Vol}(U_1) \leq 1 - m_0 < 1.$$

Applying the Young inequality, the Hölder inequality, and then using (8.26), we find

(8.27)

$$\begin{aligned}\Big(\int e^u\Big)^2 = & \Big(\int_{U_1} e^u + \int_{U_2} e^u\Big)^2 \\ \leq & \Big(1 + \frac{1}{\epsilon_0}\Big)\Big(\int_{U_1} e^u\Big)^2 + (1+\epsilon_0)\Big(\int_{U_2} e^u\Big)^2 \\ \leq & \Big(1 + \frac{1}{\epsilon_0}\Big)\Big(\int_{U_1} e^{2u}\Big)\mathrm{Vol}(U_1) + (1+\epsilon_0)\mathrm{Vol}(U_2)\int_{U_2} e^{2u} \\ \leq & \Big(1 + \frac{1}{\epsilon_0}\Big)\Big(\frac{2}{A}\Big)^2 + (1+\epsilon_0)\mathrm{Vol}(U_2)\int_S e^{2u} \\ \leq & \Big(1 + \frac{1}{\epsilon_0}\Big)\Big(\frac{2}{A}\Big)^2 + (1+\epsilon_0)(1-m_0)\Big(\Big(\int e^u\Big)^2 + C_0\int e^u\Big).\end{aligned}$$

Take $\epsilon_0$ small enough such that

$$(1+\epsilon_0)(1-m_0) < 1.$$

Then from (8.27),

(8.28)

$$\Big(\int_S e^u\Big)^2 - \frac{(1+\epsilon_0)(1-m_0)C_0}{1-(1+\epsilon_0)(1-m_0)}\int e^u + \frac{\Big(1+\frac{1}{\epsilon_0}\Big)\Big(\frac{2}{A}\Big)^2}{1-(1+\epsilon_0)(1-m_0)} \leq 0,$$

which implies an upper bound of $\int e^u$. Now the estimate of $\int e^{2u}$ follows from (8.26) and the estimate of $\sup u$ then follows from (8.25). We summarize above discussion in the following:

**Proposition 21.** *Let $t \in \mathbf{T}$ and $u$ is a solution of equation* (8.1) *under the elliptic condition $\omega' = (e^u + t\alpha f e^{-u})\omega_S + 2\alpha\sqrt{-1}\partial\bar{\partial}u > 0$ and normalization $\left(\int e^{-4u}\right)^{\frac{1}{4}} = A$ and $\int 1\frac{\omega_S^2}{2!} = 1$. If $A < 1$, then there is a constant $C_1$ which depends on $\alpha$, $f$, $\mu$, and the Sobolev constant of $\omega_S$ such that*

$$\inf_S u \geq -\ln(C_1 A).$$

*Moreover, if $A$ is small enough such that $A < (C_1)^{-1}$, then there is an upper bound of $\sup_S u$ which depends on $\alpha$, $f$, $\mu$, the Sobolev constant of $\omega_S$, and $A$.*

## 9. An estimate of the determinant

In this section, we want to obtain a lower bound of the determinant, which is equal to

(9.1)
$$F = \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = (e^u + t\alpha f e^{-u})^2 + 2\alpha(e^u + t\alpha f e^{-u}) \triangle u + 16\alpha^2 \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}$$
$$= (e^u + t\alpha f e^{-u})^2 - 2\alpha(e^u - t\alpha f e^{-u}) \mid \triangledown u \mid^2$$
$$- 4t\alpha^2 e^{-u} \triangledown u \cdot \triangledown f + 2t\alpha^2 e^{-u} \triangle f - 2t\alpha\mu.$$

From (9.1), we see

$$e^{-2u} F = 1 - 2\alpha e^{-u} \mid \triangledown u \mid^2 + e^{-2u} O(1), \tag{9.2}$$

where

$$O(1) = 2t\alpha f + t^2\alpha^2 f^2 e^{-2u} + 2t\alpha^2 f e^{-u} \mid \triangledown u \mid^2$$
$$- 4t\alpha^2 e^{-u} \triangledown u \cdot \triangledown f + 2t\alpha^2 e^{-u} \triangle f - 2t\alpha\mu. \tag{9.3}$$

The elliptic condition $\omega' > 0$ implies that $F > 0$.

The first step is to derive an upper bound of $\mid \triangledown u \mid^2$.

In the above section we have proven that $e^{-\inf u} \leq C_1 A$ and have assumed that $C_1 A < 1$. Using this assumption,

(9.4)
$$e^{-2u} F$$
$$= 1 - 2\alpha e^{-u} \mid \triangledown u \mid^2 + e^{-2u} O(1)$$
$$\leq 1 - 2\alpha e^{-u} \mid \triangledown u \mid^2 + (2t\alpha^2 f e^{-3u} + 2t\alpha^2 e^{-3u}) \mid \triangledown u \mid^2$$
$$+ e^{-2u}\{2t\alpha f + t^2\alpha^2 f^2 e^{-2u} + 2t\alpha^2 e^{-u} \mid \triangledown f \mid^2 + 2t\alpha^2 e^{-u} \triangle f - 2t\alpha\mu\}$$
$$\leq 1 - 2\alpha\{1 - \alpha(1 + \sup f)(C_1 A)^2\} e^{-u} \mid \triangledown u \mid^2 + C_2 (C_1 A)^2,$$

where

$$C_2 = 2\alpha \sup f + \alpha^2 (\sup f)^2 + 2\alpha^2 \sup \mid \triangledown f \mid^2$$
$$+ 2\alpha^2 \sup \mid \triangle f \mid + 2\alpha \sup \mid \mu \mid . \tag{9.5}$$

Applying $F > 0$ to (9.4), we get

$$1 - 2\alpha\{1 - \alpha(1 + \sup f)(C_1 A)^2\} e^{-u} \mid \triangledown u \mid^2 + C_2 (C_1 A)^2 > 0. \tag{9.6}$$

If we take

$$A \leq \{2\alpha(1 + \sup f)\}^{-\frac{1}{2}} C_1^{-1},$$

then we obtain

$$1 - \alpha(1 + \sup f)(C_1 A)^2 \geq \frac{1}{2} > 0. \tag{9.7}$$

From (9.6) and (9.7), we can get

$$\mid \triangledown u \mid^2 \leq \frac{1+C_2(C_1A)^2}{2\alpha\cdot\frac{1}{2}} e^u \leq \frac{1+C_2}{\alpha} e^u. \tag{9.8}$$

Hence we have an upper estimate of $\mid \triangledown u \mid^2$.

In the following we prove that for any given constant $\kappa$ satisfying $0 < \kappa < 1$, there is a choice of small constant $A$ (depending on $\kappa$) so that $e^{-2u}F(t,\cdot) > \kappa$. In the above section, we have seen that when $t = 0$, the equation has an unique solution $u = -\ln A$. So $e^{-2u}F(0,\cdot) \equiv 1$.

By the continuity assumption (7.7), we only need to prove that there is no $t = t_0 \in \mathbf{T}$ such that $\inf(e^{-2u}F(t_0,\cdot)) = \kappa$. If not, there is a $t_0 \in \mathbf{T}$ and $q_1$ such that $F(t_0, q_1) = \inf(e^{-2u}F(t_0,\cdot)) = \kappa$. We fix this $t_0$ and will get the contradiction if we choose small enough value for $A$.

When $t = t_0$, we assume

$$\inf(e^{-2u}F) = \kappa. \tag{9.9}$$

Applying (9.9) to (9.4), we get

$$1 - 2\alpha\{1-\alpha(1+\sup f)(C_1A)^2\}e^{-u} \mid \triangledown u \mid^2 + C_2(C_1A)^2 \geq \kappa. \tag{9.10}$$

Then (9.7) and (9.10) imply

$$\begin{aligned} e^{-u} \mid \triangledown u \mid^2 &\leq \frac{1-\kappa+C_2(C_1A)^2}{2\alpha\{1-\alpha(1+\sup f)(C_1A)^2\}} \\ &\leq \frac{1-\kappa}{2\alpha} + \Big(\frac{C_2}{\alpha} + 1 + \sup f\Big)(C_1A)^2. \end{aligned} \tag{9.11}$$

We shall now apply the maximum principle to the function

$$G_1 = 1 - 2\alpha e^{-u} \mid \triangledown u \mid^2 + 2\alpha e^{-\epsilon u} - 2\alpha e^{-\varepsilon \inf u}, \tag{9.12}$$

where $\varepsilon$ is some constant satisfying $0 < \varepsilon < 1$ which will be determined later. Comparing (9.12) and (9.2), we get

$$e^{-2u}F - G_1 = e^{-2u}O(1) - 2\alpha e^{-\varepsilon u} + 2\alpha e^{-\varepsilon \inf u}. \tag{9.13}$$

From $\inf(e^{-2u}F) = \kappa$, we see

$$\begin{aligned} &\kappa - \sup(e^{-2u} \mid O(1) \mid) - 2\alpha e^{-\varepsilon \inf u} \\ &\leq \inf G_1 \leq \kappa + \sup(e^{-2u} \mid O(1) \mid) + 2\alpha e^{-\varepsilon \inf u}. \end{aligned} \tag{9.14}$$

We can use (9.5) and (9.8) to estimate

$$\begin{aligned} \sup \mid O(1) \mid &\leq 2\alpha \sup f + \alpha^2(\sup f)^2(C_1A)^2 + 2\alpha \sup f\,\{(1+C_2)\} \\ &\quad + 2\alpha\,\{(1+C_2)\} + 2\alpha^2(C_1A)\sup \mid \triangledown f \mid^2 \\ &\quad + 2\alpha^2(C_1A)\sup \mid \triangle f \mid + 2\alpha \sup \mid \mu \mid \\ &\leq C_2 + 2\alpha(1+\sup f)(1+C_2) \end{aligned}$$

and

$$\sup(e^{-2u} \mid O(1) \mid) + 2\alpha e^{-\varepsilon \inf u} \leq C_2'(C_1A)^{\varepsilon}, \tag{9.15}$$

where

$$C_2' = 2\{C_2 + 2\alpha(1 + \sup f)\}$$

depends only on $\alpha$, $f$ and $\mu$.

Combining (9.14) and (9.15), we get

$$\kappa - C_2'(C_1 A)^\varepsilon \le \inf G_1 \le \kappa + C_2'(C_1 A)^\varepsilon. \tag{9.16}$$

Let $G_1$ achieve the minimum at the point $q_2 \in S$ where we can apply (9.15) and (9.16) to (9.13) to obtain

(9.17)

$$\begin{aligned} e^{-2u(q_2)}F(q_2) &= G_1(q_2) + e^{-2u(q_2)}O(1)(q_2) - 2\alpha e^{-\varepsilon u(q_2)} + 2\alpha^{-\varepsilon \inf u} \\ &\le \inf G_1 + \sup(e^{-2u} \mid O(1) \mid) + 2\alpha e^{-\varepsilon \inf u} \\ &\le \kappa + 2C_2'(C_1 A)^\varepsilon. \end{aligned}$$

We can also apply (9.16) to (9.12) to obtain

$$\begin{aligned} & e^{-u(q_2)} \mid \bigtriangledown u \mid^2 (q_2) \\ &= (2\alpha)^{-1}\{1 - G_1(q_2) + 2\alpha e^{-\varepsilon u(q_2)} - 2\alpha e^{-\varepsilon \inf u}\} \\ &\ge (2\alpha)^{-1}\{1 - \inf G_1 - 2\alpha e^{-\varepsilon \inf u}\} \\ &\ge (1-\kappa)/(2\alpha) - (1 + (2\alpha)^{-1}C_2')(C_1 A)^\varepsilon. \end{aligned} \tag{9.18}$$

Let

$$C_3 = \max\{\alpha^{-1}C_2 + 1 + \sup f, 2C_2', 1 + (2\alpha)^{-1}C_2'\}.$$

Then (9.9) and (9.17) imply

$$\kappa \le e^{-2u(q_2)}F(q_2) \le \kappa + C_3(C_1 A)^\varepsilon; \tag{9.19}$$

(9.11) and (9.18) imply

$$\begin{aligned} (1-\kappa)/(2\alpha) - C_3(C_1 A)^\varepsilon &\le e^{-u(q_2)} \mid \bigtriangledown u \mid^2 (q_2) \\ &\le (1-\kappa)/(2\alpha) + C_3(C_1 A)^\varepsilon. \end{aligned} \tag{9.20}$$

We now compute $F \cdot P(G_1)$ at the point $q_2$. In the following we replace $t\alpha f$ by $f$ and $t\mu$ by $\mu$. At the point $q_2$, from $\bigtriangledown G_1(q_2) = 0$, we have

$$\bigtriangledown(\mid \bigtriangledown u \mid^2) = (\mid \bigtriangledown u \mid^2 - \varepsilon e^{(1-\varepsilon)u}) \bigtriangledown u. \tag{9.21}$$

Because $\omega_S$ is Kähler, we can choose normal coordinate $(z_1, z_2)$ at the point $q_2$, i.e., $g_{i\bar{j}} = \delta_{ij}$ and $dg_{i\bar{j}} = 0$. At the same time , we can assume $\frac{\partial u}{\partial z_1} \neq 0$ and $\frac{\partial u}{\partial z_2} = 0$. As $u$ is real, we can assume that $\frac{\partial u}{\partial x_1} \ge 0$ and $\frac{\partial u}{\partial y_1} = 0$. So at the point $q_2$, $u_1 = u_{\bar{1}}$ and

$$2u_1 u_1 = 2u_1 u_{\bar{1}} = 2u_{\bar{1}} u_{\bar{1}} = \mid \bigtriangledown u \mid^2 . \tag{9.22}$$

If we assume

$$A < \Big(\frac{1-\kappa}{2\alpha C_3}\Big)^{\frac{1}{\varepsilon}} C_1^{-1},$$

then

$$\frac{1-\kappa}{2\alpha} - C_3(C_1A)^{\varepsilon} > 0.$$

Hence (9.20) implies $|\bigtriangledown u|^2 (q_2) > 0$ and (9.21) implies

$$u_{11} + u_{1\bar{1}} = u_{1\bar{1}} + u_{\bar{1}\bar{1}} = (|\bigtriangledown u|^2 - \varepsilon^{(1-\varepsilon)u})/2, \quad u_{12} + u_{\bar{1}2} = u_{1\bar{2}} + u_{\bar{1}\bar{2}} = 0. \tag{9.23}$$

In the following we denote

$$L(u) = e^u + fe^{-u} + \alpha \bigtriangleup u,$$

$$L_1(u) = e^u + fe^{-u} + 4\alpha u_{1\bar{1}}.$$

and

$$L_2(u) = e^u + fe^{-u} + 4\alpha u_{2\bar{2}}.$$

From (8.3) and (9.1), we can see

$$FP(u) = \alpha^{-1}F - \alpha^{-1}(e^u + fe^{-u})L(u). \tag{9.24}$$

We should compute $FP(G_1)$ at the point $q_2$. At first we deal with

(9.25)

$$\begin{aligned} &P(2\alpha e^{-\varepsilon u})F \\ &= -2\alpha\varepsilon e^{-\varepsilon u}P(u)F + 2\alpha\varepsilon^2 e^{-\varepsilon u} \cdot g'^{1\bar{1}} |\bigtriangledown u|^2 \cdot F \\ &= -2\varepsilon e^{-\varepsilon u}F + 2\varepsilon e^{-\varepsilon u}(e^u + fe^{-u})L(u) + 2\alpha\varepsilon^2 e^{-\varepsilon u} |\bigtriangledown u|^2 L_2(u). \end{aligned}$$

Using (9.21), we derive

$$\begin{aligned} &P(-2\alpha e^{-u} |\bigtriangledown u|^2)F \\ &= -2e^{-u}(e^u + fe^{-u}) |\bigtriangledown u|^2 L(u) \\ &\quad + \{2\alpha e^{-u} |\bigtriangledown u|^4 - 4\alpha\varepsilon e^{-\varepsilon u} |\bigtriangledown u|^2\}L_2(u) \\ &\quad + 2e^{-u} |\bigtriangledown u|^2 F - 2\alpha e^{-u}P(|\bigtriangledown u|^2)F. \end{aligned} \tag{9.26}$$

Combining (9.25) and (9.26), we get

(9.27)

$$\begin{aligned} FP(G_1) = &\{2e^{-u} |\bigtriangledown u|^2 - 2\varepsilon e^{-\varepsilon u}\}F \\ &- \{2e^{-u}(e^u + fe^{-u}) |\bigtriangledown u|^2 - 2\varepsilon e^{-\varepsilon u}(e^u + fe^{-u})\}L(u) \\ &+ \{2\alpha e^{-u} |\bigtriangledown u|^4 + (2\alpha\varepsilon^2 - 4\alpha\varepsilon)e^{-\varepsilon u} |\bigtriangledown u|^2\}L_2(u) \\ &- 2\alpha e^{-u}P(|\bigtriangledown u|^2)F. \end{aligned}$$

We now estimate the term

$$\begin{aligned} \alpha P(|\bigtriangledown u|^2)F \geq\ &4\alpha g'^{i\bar{j}}\{u_{i\bar{j}k}u_{\bar{k}} + u_{i\bar{j}\bar{k}}u_k\}F + 4\alpha g'^{i\bar{j}}\{u_{i\bar{k}}u_{k\bar{j}}\}F \\ &+ 4\alpha g^{i\bar{j}}\{u_{i1}u_{\bar{1}\bar{j}}\}F + 4\alpha g'^{i\bar{j}}\{\partial_i\partial_{\bar{j}}(g^{1\bar{1}})u_1u_{\bar{1}}\}F. \end{aligned} \tag{9.28}$$

But the first term of (9.28) is equal to

$$4\alpha g'^{i\bar{j}}\{u_{i\bar{j}k}u_{\bar{k}} + u_{i\bar{j}\bar{k}}u_k\}F$$
$$= 2\alpha(e^u + fe^{-u}) \bigtriangledown \triangle u \cdot \bigtriangledown u + 16\alpha^2 \bigtriangledown \left(\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}\right) \cdot \bigtriangledown u.$$

Applying the equation to the last term, we find

$$\begin{aligned}
&4\alpha g'^{i\bar{j}}\{u_{i\bar{j}k}u_{\bar{k}} + u_{i\bar{j}\bar{k}}u_k\}F \\
&= -2\alpha(e^u - fe^{-u}) \mid \bigtriangledown u \mid^2 \triangle u - 2\alpha(e^u + fe^{-u}) \mid \bigtriangledown u \mid^4 \\
&\quad - 2\alpha(e^u - fe^{-u})\bigtriangledown \mid \bigtriangledown u \mid^2 \cdot \bigtriangledown u - 4\alpha e^{-u} \bigtriangledown (\bigtriangledown u \cdot \bigtriangledown f) \cdot \bigtriangledown u \\
&\quad + 6\alpha e^{-u} \mid \bigtriangledown u \mid^2 \bigtriangledown u \cdot \bigtriangledown f - 2\alpha e^{-u} \mid \bigtriangledown u \mid^2 \triangle f \\
&\quad + 2\alpha e^{-u} \bigtriangledown \triangle f \cdot \bigtriangledown u - 2\alpha \bigtriangledown \mu \cdot \bigtriangledown u - 2\alpha e^{-u}(\bigtriangledown u \cdot \bigtriangledown f) \triangle u.
\end{aligned}$$

From (9.8), we see $e^{-u} \mid \bigtriangledown u \mid^2 < C_4$, where $C_4$ only depends on $\alpha$, $f$ and $\mu$ and does not depend on $A$. In the following we use $C_4$ in the generic sense. We have gotten $\mid \bigtriangledown u \mid^2 \leq C_4 e^u$. Our assumptions of $A$ imply $e^u > 1$, $\mid \bigtriangledown u \mid \leq C_4 e^u$. In the following we will deal with such small terms and obtain

(9.29)
$$\begin{aligned}
&4\alpha g'^{i\bar{j}}\{u_{i\bar{j}k}u_{\bar{k}} + u_{i\bar{j}\bar{k}}u_k\}F \\
&\geq -2\alpha(e^u - fe^{-u}) \mid \bigtriangledown u \mid^2 \triangle u - 2\alpha(e^u + fe^{-u}) \mid \bigtriangledown u \mid^4 \\
&\quad - 2\alpha(e^u - fe^{-u})\bigtriangledown \mid \bigtriangledown u \mid^2 \cdot \bigtriangledown u - 4\alpha e^{-u} \bigtriangledown (\bigtriangledown u \cdot \bigtriangledown f) \cdot \bigtriangledown u \\
&\quad - C_4 e^u - C_4 L(u).
\end{aligned}$$

From (9.23),
(9.30)
$$\begin{aligned}
&- 4\alpha e^{-u} \bigtriangledown (\bigtriangledown u \cdot \bigtriangledown f) \cdot \bigtriangledown u \\
&\geq -4\alpha e^{-u}\{(u_{i1} + u_{i\bar{1}})f_{\bar{i}} + (u_{\bar{i}1} + u_{\bar{i}\bar{1}})f_i\}u_1 - C_4 \\
&= -2\alpha e^{-u}\{(\mid \bigtriangledown u \mid^2 - e^{(1-\varepsilon)u})f_{\bar{1}} + (\mid \bigtriangledown u \mid^2 - e^{(1-\varepsilon)u})f_1\}u_1 - C_4 \\
&\geq -C_4 e^u.
\end{aligned}$$

Inserting (9.30) into (9.29) and applying (9.21), we find

(9.31)
$$\begin{aligned}
&4\alpha g'^{i\bar{j}}\{u_{i\bar{j}k}u_{\bar{k}} + u_{i\bar{j}\bar{k}}u_k\}F \\
&\geq 2 \mid \bigtriangledown u \mid^2 F - 2(e^u - fe^{-u}) \mid \bigtriangledown u \mid^2 L(u) \\
&\quad + 2\alpha\varepsilon e^{(2-\varepsilon)u} \mid \bigtriangledown u \mid^2 - C_4 e^u - C_4 L(u).
\end{aligned}$$

Next we deal with the second term in (9.28):

(9.32)
$$\begin{aligned}
&4\alpha g'^{i\bar{j}}(u_{i\bar{k}}u_{k\bar{j}})F \\
&= \alpha(e^u + fe^{-u})(\triangle u)^2 \\
&\quad - 8\alpha(e^u + fe^{-u})\Big(\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}\Big) + 8\alpha^2 \triangle u\Big(\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}\Big).
\end{aligned}$$

Using the equation, we have the following estimates

$$
\begin{aligned}
(9.33)\qquad & 8\alpha^2 \triangle u\Big(\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}\Big) \\
& \geq -\alpha(e^u + fe^{-u})(\triangle u)^2 - \alpha(e^u - fe^{-u}) \mid \nabla u \mid^2 \triangle u \\
& \quad - C_4 e^u - C_4 L(u),
\end{aligned}
$$

and

$$
\begin{aligned}
(9.34)\qquad & -8\alpha(e^u + fe^{-u})\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} \\
& \geq (e^u + fe^{-u})^2 \triangle u + (e^u + fe^{-u})(e^u - fe^{-u}) \mid \nabla u \mid^2 - C_4 e^u.
\end{aligned}
$$

Inserting (9.33) and (9.34) into (9.32), we find the following estimate for the second term:

$$
\begin{aligned}
(9.35)\qquad & 4\alpha g'^{i\bar{j}}(u_{i\bar{k}}u_{k\bar{j}})F \\
& \geq \{\alpha^{-1}(e^u + fe^{-u})^2 - (e^u - fe^{-u}) \mid \nabla u \mid^2\}L(u) \\
& \quad - \alpha^{-1}(e^u + fe^{-u})F - C_4 e^u - C_4 L(u).
\end{aligned}
$$

Then we compute the third term in (9.28). Let

$$
a = \frac{1}{2}(\mid \nabla u \mid^2 - \varepsilon e^{(1-\varepsilon)u}).
$$

We can use (9.23) to prove

$$
\begin{aligned}
(9.36)\qquad & 4\alpha g'^{i\bar{j}}(u_{i1}u_{\bar{1}\bar{j}})F \\
& = 4\alpha(e^u + fe^{-u})a^2 - 8\alpha a(e^u + fe^u)u_{1\bar{1}} \\
& \quad + 4\alpha(e^u + fe^{-u})u_{1\bar{1}}^2 + 16\alpha^2 a^2 u_{2\bar{2}} \\
& \quad + 4\alpha(e^u + fe^{-u})u_{1\bar{2}}u_{2\bar{1}} + 16\alpha^2 u_{1\bar{1}}\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} - 32\alpha^2 a\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}.
\end{aligned}
$$

Using the equation again, we have the following estimates

$$
\begin{aligned}
(9.37)\qquad & 16\alpha^2 u_{1\bar{1}}\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} \\
& \geq -4\alpha(e^u + fe^{-u})u_{1\bar{1}}^2 - 4\alpha(e^u + fe^{-u})u_{1\bar{1}}u_{2\bar{2}} \\
& \quad - \alpha(e^u - fe^{-u}) \mid \nabla u \mid^2 \triangle u + 2\alpha(e^u - fe^{-u}) \mid \nabla u \mid^2 u_{2\bar{2}} \\
& \quad - C_4 e^u - C_4 L_1(u)
\end{aligned}
$$

and

$$
\begin{aligned}
(9.38)\qquad -32\alpha^2 a\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} & \geq 8\alpha a(e^u + fe^{-u})u_{1\bar{1}} + 8\alpha a(e^u + fe^{-u})u_{2\bar{2}} \\
& \quad + 4\alpha a(e^u - fe^{-u}) \mid \nabla u \mid^2 - C_4 e^u.
\end{aligned}
$$

Inserting (9.37) and (9.38) into (9.36) and simplifying, we get

$$\begin{aligned}
&4\alpha g'^{i\bar{j}}u_{i1}u_{\bar{1}\bar{j}}F\\
&\geq \frac{1}{2\alpha}FL(u)-\frac{1}{2\alpha}(e^u+fe^{-u})F-2aF-C_4e^u-C_4L(u)\\
&\quad+\{4\alpha a^2+2a(e^u+fe^{-u})+\frac{1}{2}(e^u-fe^{-u})\mid\triangledown u\mid^2\}L_2(u)
\end{aligned}$$

Recalling that $a=\frac{1}{2}(\mid\triangledown u\mid^2-\varepsilon e^{(1-\varepsilon)u})$, we can simplify above inequality and find

$$\begin{aligned}
(9.39)\quad &4\alpha g'^{i\bar{j}}u_{i1}u_{\bar{1}\bar{j}}F\\
&\geq \frac{1}{2\alpha}FL(u)+\Big\{\varepsilon e^{(1-\varepsilon)u}-\frac{1}{2\alpha}(e^u+fe^{-u})-\mid\triangledown u\mid^2\Big\}F\\
&\quad+\Big\{\Big(\frac{3}{2}e^u+\alpha\mid\triangledown u\mid^2-2\alpha\varepsilon e^{(1-\varepsilon)u}\Big)\mid\triangledown u\mid^2-\varepsilon e^{(2-\varepsilon)u}\Big\}L_2(u)\\
&\quad-C_4e^u-C_4L(u).
\end{aligned}$$

The last term of (9.28) is

$$(9.40)\qquad 4\alpha g'^{i\bar{j}}\partial_{i\bar{j}}(g^{1\bar{1}})u_1u_{\bar{1}}F\geq -C_4\mid\triangledown u\mid^2 L(u)$$

where $C_4$ also depends on the curvature bound of the given metric $\omega_S$. Inserting (9.31, 9.35, 9.39, 9.40) into (9.28) and simplifying, we get

$$\begin{aligned}
(9.41)\quad &\alpha P(\mid\triangledown u\mid^2)F\\
&\geq\Big\{\mid\triangledown u\mid^2-\frac{3}{2\alpha}e^u+\varepsilon e^{(1-\varepsilon)u}\Big\}F+2\alpha\varepsilon e^{(2-\varepsilon)u}\mid\triangledown u\mid^2\\
&\quad+\Big\{\frac{1}{2\alpha}F+\frac{1}{\alpha}(e^u+fe^{-u})^2-3(e^u-fe^{-u})\mid\triangledown u\mid^2\Big\}L(u)\\
&\quad+\Big\{\Big(\frac{3}{2}e^u+\alpha\mid\triangledown u\mid^2-2\alpha\varepsilon e^{(1-\varepsilon)u}\Big)\mid\triangledown u\mid^2-\varepsilon e^{(2-\varepsilon)u}\Big\}L_2(u)\\
&\quad-C_4e^u-C_4e^uL(u).
\end{aligned}$$

where we used $e^{-u}F\leq C_4e^u$.

Now inserting (9.41) into (9.27), we finally obtain

(9.42)

$$\begin{aligned}
&FP(G_1)\\
&\leq\frac{3}{\alpha}F-2\varepsilon e^{-\varepsilon u}F-2\varepsilon e^{(2-\varepsilon)u}+C_4\\
&\quad-\Big\{\frac{3}{2\alpha}e^{-u}F-\varepsilon e^{(1-\varepsilon)u}-C_4\Big\}L_1(u)\\
&\quad-\Big\{\frac{3}{2\alpha}e^{-u}F-3\varepsilon e^{(1-\varepsilon)u}+(3-2\alpha\varepsilon^2e^{-\varepsilon u})\mid\triangledown u\mid^2-C_4\Big\}L_2(u).
\end{aligned}$$

Let

$$(9.43)\quad \begin{aligned} a_1 &= \frac{3}{\alpha}F - 2\varepsilon e^{-\varepsilon u}F - 2\varepsilon e^{(2-\varepsilon)u} + C_4 \\ a_2 &= \frac{3}{\alpha}e^{-u}F - 2\varepsilon e^{(1-\varepsilon)u} - C_4 \\ a_3 &= \frac{3}{\alpha}e^{-u}F - 6\varepsilon e^{(1-\varepsilon)u} + 6 \mid \bigtriangledown u \mid^2 - 4\alpha\varepsilon^2 e^{-\varepsilon u} \mid \bigtriangledown u \mid^2 - C_4. \end{aligned}$$

Then (9.42) implies

$$(9.44)\qquad a_1 \geq a_2 \frac{e^u + fe^{-u} + 4\alpha u_{1\bar{1}}}{2} + a_3 \frac{e^u + fe^{-u} + 4\alpha u_{2\bar{2}}}{2}$$

at the point $q_2$ where $FP(G_1) \geq 0$.

Let $0 < \kappa < 1$ and $\varepsilon > 0$ be chosen to satisfy

$$(9.45)\qquad \varepsilon < \min\left\{1, \alpha^{-1/2}, (2\alpha)^{-1}\kappa\right\}$$

Then

$$3 - 2\alpha\varepsilon^2 > 0$$

and

$$\frac{3}{\alpha}\kappa - 6\varepsilon > 0.$$

Choose $A$ so that

$$(9.46)\qquad A < \frac{\frac{3}{\alpha}\kappa - 6\varepsilon}{C_4} C_1^{-1}.$$

Then $\kappa, \varepsilon$ and $A$ satisfy

$$\frac{3}{\alpha}\kappa - 6\varepsilon - C_4 C_1 A > 0.$$

We find

$$\begin{aligned} a_2 &\geq e^u\left\{\frac{3}{\alpha}e^{-2u}F - 2\varepsilon e^{-\varepsilon u} - C_4 e^{-u}\right\} \\ &\geq e^u\left\{\frac{3}{\alpha}\kappa - 2\varepsilon(C_1 A)^{\varepsilon} - C_4 C_1 A\right\} \\ &\geq e^u\{\frac{3}{\alpha}\kappa - 6\varepsilon - C_4 C_1 A\} > 0 \end{aligned}$$

and

$$\begin{aligned} a_3 &\geq e^u\left\{\frac{3}{\alpha}\kappa - 6\varepsilon(C_1 A)^{\varepsilon} - C_4 C_1 A\right\} + 2 \mid \bigtriangledown u \mid^2 (3 - 2\alpha\varepsilon^2 (C_1 A)^{\varepsilon}) \\ &\geq e^u\left\{\frac{3}{\alpha}\kappa - 6\varepsilon - C_4 C_1 A\right\} + 2 \mid \bigtriangledown u \mid^2 (3 - 2\alpha\varepsilon^2) > 0. \end{aligned}$$

Applying arithmetic-geometric inequality to (9.44), we find

$$(9.47)\qquad \begin{aligned} a_1^2 &\geq \left(a_2 \frac{e^u + fe^{-u} + 4u_{1\bar{1}}}{2} + a_3 \frac{e^u + fe^{-u} + 4\alpha u_{2\bar{2}}}{2}\right)^2 \\ &\geq a_2 a_3 (e^u + fe^{-u} + 4u_{1\bar{1}})(e^u + fe^{-u} + 4\alpha u_{2\bar{2}}) \\ &\geq a_2 a_3 F. \end{aligned}$$

Using (9.2), we can write $a_3$ as

$$(9.48)\quad a_3 = \frac{3}{\alpha}e^{-u}F + 6 \mid \bigtriangledown u \mid^2 - 6\varepsilon e^{(1-\varepsilon)u} - 4\alpha\varepsilon^2 e^{-\varepsilon u} \mid \bigtriangledown u \mid^2 - C_4$$
$$= \frac{3}{\alpha}e^{u} - 6\varepsilon e^{(1-\varepsilon)u} - 4\alpha\varepsilon^2 e^{-\varepsilon u} \mid \bigtriangledown u \mid^2 - e^{-u}O(1) - C_4.$$

Inserting (9.43) and (9.48) into (9.47) and simplifying, we get

$$(9.49)\quad 4\varepsilon^2 e^{-2\varepsilon u}F^2 + 4\varepsilon^2 e^{2(2-\varepsilon)u} + 12\varepsilon^2 e^{-(1+\varepsilon)u} \mid \bigtriangledown u \mid^2 F^2 + C_4' e^{2u}$$
$$\geq \frac{6}{\alpha}\varepsilon e^{(2-\varepsilon)u}F - \frac{6}{\alpha}\varepsilon e^{-\varepsilon u}F^2 + 4\varepsilon^2 e^{2(1-\varepsilon)u}F$$
$$\geq 12\varepsilon e^{(1-\varepsilon)u} \mid \bigtriangledown u \mid^2 F - C_4' e^{2u},$$

where $C_4'$ may be bigger than $C_4$ which we shall still denote by $C_4$. Dividing (9.49) by $4\varepsilon e^{-\varepsilon u}e^{2u}F$, we get

(9.50)

$$\varepsilon e^{-\varepsilon u}(e^{-2u}F) + \varepsilon\frac{e^{-\varepsilon u}}{e^{-2u}F} + 3\varepsilon(e^{-u} \mid \bigtriangledown u \mid^2)(e^{-2u}F) + C_4\frac{e^{-(2-\varepsilon)u}}{\varepsilon e^{-2u}F}$$
$$\geq 3(e^{-u} \mid \bigtriangledown u \mid^2).$$

Using the inequalities (9.19) and (9.20) to two sides of above inequality, we obtain

(9.51)

$$\varepsilon e^{-\varepsilon u}(e^{-2u}F) + \varepsilon\frac{e^{-\varepsilon u}}{e^{-2u}F} + 3\varepsilon(e^{-u} \mid \bigtriangledown u \mid^2)(e^{-2u}F) + C_4\frac{e^{-(2-\varepsilon)u}}{\varepsilon e^{-2u}F}$$
$$\leq \varepsilon(C_1A)^{\varepsilon}(\kappa + C_3(C_1A)^{\varepsilon}) + \varepsilon\frac{(C_1A)^{\varepsilon}}{\kappa}$$
$$+ 3\varepsilon(\kappa + C_3(C_1A)^{\varepsilon})\Big(\frac{1-\kappa}{2\alpha} + C_3(C_1A)^{\varepsilon}\Big) + C_4\frac{(C_1A)^{2-\varepsilon}}{\varepsilon\kappa}$$
$$\leq \left\{1 + \frac{\varepsilon}{\kappa} + \varepsilon\Big(1 + 3\kappa + \frac{3}{2\alpha} + 3C_3\Big)C_3 + \frac{C_4}{\varepsilon\kappa}\right\}(C_1A)^{\varepsilon} + \frac{3\varepsilon\kappa}{2\alpha}(1-\kappa)$$

and

$$(9.52)\quad 3(e^{-u} \mid \bigtriangledown u \mid^2) \geq \frac{3}{2\alpha}(1-\kappa) - 3C_3(C_1A)^{\varepsilon}.$$

Applying (9.51) and (9.52) to (9.50), we see

$$\left\{1 + \frac{\varepsilon}{\kappa} + 3C_3 + \varepsilon\Big(1 + 3\kappa + \frac{3}{2\alpha} + 3C_3\Big)C_3 + \frac{C_4}{\varepsilon\kappa}\right\}(C_1A)^{\varepsilon}$$
$$\geq \frac{3}{2\alpha}(1-\kappa)(1-\varepsilon\kappa).$$

So finally we obtain that at $(t_0, q_2)$,

$$(9.53)\quad A \geq \left(\frac{\frac{3}{2\alpha}(1-\kappa)(1-\varepsilon\kappa)}{1 + \frac{\varepsilon}{\kappa} + 3C_3 + \varepsilon\Big(1 + 3\kappa + \frac{3}{2\alpha} + 3C_3\Big)C_3 + \frac{C_4}{\varepsilon\kappa}}\right)^{\frac{1}{\varepsilon}} C_1^{-1}.$$

Now it is easy to prove the following

**Proposition 22.** *Let $t \in \mathbf{T}$ and $u$ is a solution of equation* (8.1) *under the elliptic condition* $(e^u + t\alpha f e^{-u})\omega_S + 2\alpha\sqrt{-1}\partial\bar{\partial}u > 0$ *and the normalization* $\left(\int e^{-4u}\right)^{\frac{1}{4}} = A$ *and* $\int 1 \frac{\omega_S^2}{2!} = 1$. *Given any constant* $\kappa \in (0,1)$, *we fix some positive constant* $\varepsilon$ *satisfying*

$$\varepsilon < \min\{1, \alpha^{-\frac{1}{2}}, (2\alpha)^{-1}\kappa\}. \tag{9.54}$$

*Suppose that $A$ satisfies*

$$A < \min\left\{1, C_1^{-1}, \{2\alpha(1+\sup f)\}^{-\frac{1}{2}}C_1^{-1}, \Big(\frac{1-\kappa}{2\alpha C_3}\Big)^{\frac{1}{\varepsilon}}C_1^{-1}, \frac{\frac{3}{\alpha}-6\varepsilon}{C_4}C_1^{-1}\right\} \tag{9.55}$$

*and*

$$A < \left(\frac{\frac{3}{2\alpha}(1-\kappa)(1-\varepsilon\kappa)}{1+\frac{\varepsilon}{\kappa}+3C_3+\varepsilon\Big(1+3\kappa+\frac{3}{2\alpha}+3C_3\Big)C_3+\frac{C_4}{\varepsilon\kappa}}\right)^{\frac{1}{\varepsilon}}C_1^{-1}, \tag{9.56}$$

*where $C_1$ is determined in above section and depends on $\alpha$, $f$ and $\mu$, and also depends on the Sobolev constant; $C_3$ and $C_4$ are determined in above discussion and depend on $\alpha$, $f$, $\mu$, and $C_4$ also depends the curvature bound of $\omega_S$. Then $F > \kappa e^{2u} \geq \kappa(C_1A)^{-2}$.*

*Proof.* When $t=0$, the equation has an unique solution $u = -\ln A$ and so $e^{-2u}F(0,\cdot) \equiv 1$. According to our continuity assumption, we claim that for any $t \in \mathbf{T}$, $e^{-2u}F(t,\cdot) > \kappa$. Otherwise if there is a $t_0 \in \mathbf{T}$ such that the equation has a solution $u$ and $\inf(e^{-2u}F) = \kappa$. Fix this $t_0$ and apply the maximum principle to the function $G_1 = 1 - 2\alpha e^{-u} \mid \bigtriangledown u \mid^2 + 2\alpha e^{-\varepsilon u} - 2\alpha e^{-\varepsilon \inf u}$. Let $G_1$ achieve the minimum at the point $q_2$. Then at point $q_2$, $P(G_1)F > 0$. From above discussion, we have gotten the inequality (9.53) at point $q_2$ under assumptions (9.54) and (9.55), which contradict the assumption (9.56). So $e^{-2u}F > \kappa$ and $F > \kappa e^{2u} > \kappa(C_1A)^{-2}$. q.e.d.

## 10. Second order estimate

We now consider the second order a priori estimate of $u$. Since we have proved $F > \kappa(C_1A)^{-2} > 0$, $e^u + fe^{-u} + \alpha\triangle u \geq F^{\frac{1}{2}} > \kappa^{\frac{1}{2}}(C_1A)^{-1} > 0$. It is sufficient to find an upper estimate of $e^u + fe^{-u} + \alpha \triangle u$. We fix some point and choose the normal coordinate $(z_1, z_2)$ at this point for the given metric $g_{i\bar{j}}$, i.e., at this point, $g_{i\bar{j}} = \delta_{ij}$ and $dg_{i\bar{j}} = 0$. We replace $t\alpha f$ by $f$ and $t\mu$ by $\mu$ and rewrite the equation as

$$\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = F, \tag{10.1}$$

where

$$F = (e^u + fe^{-u})^2 - 2\alpha(e^u - fe^{-u}) \mid \triangledown u \mid^2 \\ - 4\alpha e^{-u} \triangledown u \cdot \triangledown f + 2\alpha e^{-u} \triangle f - 2\alpha\mu.$$

Following the method of [**24**], we compute

(10.2)

$$4\alpha g'^{i\bar{j}} \frac{\partial^4 u}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l} = g'^{i\bar{q}} g'^{p\bar{j}} \frac{\partial g'_{p\bar{q}}}{\partial \bar{z}_l} \frac{\partial g'_{i\bar{j}}}{\partial z_k} - g'^{i\bar{j}} \frac{\partial^2 [(e^u + fe^{-u}) g_{i\bar{j}}]}{\partial z_k \partial \bar{z}_l} \\ - g^{i\bar{q}} g^{p\bar{j}} \frac{\partial g_{p\bar{q}}}{\partial \bar{z}_l} \frac{\partial g_{i\bar{j}}}{\partial z_k} + g^{i\bar{j}} \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} \\ + \frac{1}{F} \frac{\partial^2 F}{\partial z_k \partial \bar{z}_l} - \frac{1}{F^2} \frac{\partial F}{\partial z_k} \frac{\partial F}{\partial \bar{z}_l}.$$

As $\omega_S$ is Ricci-flat, we see

(10.3)

$$\alpha P(\triangle u) F = - 2^{-1} \triangle (e^u + fe^{-u}) \sum g'^{i\bar{i}} \cdot F + 4\alpha g'^{i\bar{j}} (g^{k\bar{l}})_{i\bar{j}} \cdot u_{k\bar{l}} \cdot F \\ + 2^{-1} \triangle F - (2F)^{-1} \mid \triangledown F \mid^2 + g^{k\bar{l}} g'^{i\bar{q}} g'^{p\bar{j}} g'_{i\bar{j}k} g'_{p\bar{q}\bar{l}} \cdot F \\ = - L(u) \triangle (e^u + fe^{-u}) + 4\alpha g'^{i\bar{j}} (g^{k\bar{l}})_{i\bar{j}} \cdot u_{k\bar{l}} \cdot F \\ + 2^{-1} \triangle F - (2F)^{-1} \mid \triangledown F \mid^2 + g^{k\bar{l}} g'^{i\bar{q}} g'^{p\bar{j}} g'_{i\bar{j}k} g'_{p\bar{q}\bar{l}} \cdot F.$$

As above section, we denote

$$L(u) = e^u + fe^{-u} + \alpha \triangle u.$$

We shall apply the maximum principle to the function

$$G_2 = e^{-\lambda_1 u + \lambda_2 |\triangledown u|^2} \cdot L(u),$$

where $\lambda_1$ and $\lambda_2$ are some positive constants which will be determined later.

The Schwarz' inequality implies

$$\text{(10.4)} \qquad P(G_2) \cdot e^{-(-\lambda_1 u + \lambda_2 |\triangledown u|^2)} \\ \geq L(u) \cdot (-\lambda_1 P(u) + \lambda_2 P(\mid \triangledown u \mid^2)) + P(L(u)) \\ - L(u)^{-1} \mid \triangledown' L(u) \mid^2_{g'},$$

where we denote $2g'^{i\bar{j}} \psi_i \psi_{\bar{j}}$ by $\mid \triangledown' \psi \mid^2_{g'}$.

In computing the last term of (10.4), we assume that $g_{i\bar{j}} = \delta_{ij}$ and $u_{i\bar{j}} = u_{i\bar{i}} \delta_{ij}$ at a point. Using the method of [**24**], we get

$$\text{(10.5)} \qquad L(u)^{-1} \mid \triangledown' L(u) \mid^2_{g'} \leq \sum_{ik} g'^{i\bar{i}} g'^{k\bar{k}} g'_{k\bar{k}i} g'_{k\bar{k}\bar{i}}.$$

Note that when $i \neq k$,

$$\text{(10.6)} \quad g'_{k\bar{k}i} = g'_{i\bar{k}k} + (e^u + fe^{-u})_i - [(e^u + fe^{-u}) g_{i\bar{k}}]_k = g'_{i\bar{k}k} + (e^u + fe^{-u})_i$$

and

(10.7) $g'_{k\bar{k}\bar{i}} = g'_{k\bar{i}\bar{k}} + (e^u + fe^{-u})_{\bar{i}} - [(e^u + fe^{-u})g_{k\bar{i}}]_{\bar{k}} = g'_{k\bar{i}\bar{k}} + (e^u + fe^{-u})_{\bar{i}}.$

Inserting (10.6) and (10.7) into (10.5), and then applying Schwarz inequality, we estimate

$$\begin{aligned}(10.8)\qquad & L(u)^{-1} \mid \nabla' L(u) \mid^2_{g'} \\ & \leq g'^{i\bar{i}} g'^{k\bar{k}} g'_{i\bar{k}k} g'_{k\bar{i}\bar{k}} + g'^{2\bar{2}} g'^{2\bar{2}} g'_{2\bar{2}1} g'_{2\bar{2}\bar{1}} + g'^{1\bar{1}} g'^{1\bar{1}} g'_{1\bar{1}2} g'_{1\bar{1}\bar{2}} \\ & \quad + g'^{1\bar{1}} g'^{1\bar{1}} (e^u + fe^{-u})_1 (e^u + fe^{-u})_{\bar{1}} \\ & \quad + g'^{2\bar{2}} g'^{2\bar{2}} (e^u + fe^{-u})_2 (e^u + fe^{-u})_{\bar{2}} \\ & \leq g'^{i\bar{i}} g'^{k\bar{k}} g'_{i\bar{k}j} g'_{k\bar{i}\bar{j}} + C_5 (g'^{1\bar{1}} g'^{1\bar{1}} + g'^{2\bar{2}} g'^{2\bar{2}}) \\ & \leq g'^{i\bar{i}} g'^{k\bar{k}} g'_{i\bar{k}j} g'_{k\bar{i}\bar{j}} + C_5 F^{-2} (g'_{1\bar{1}} + g'_{2\bar{2}})^2 \\ & \leq g'^{i\bar{i}} g'^{k\bar{k}} g'_{i\bar{k}j} g'_{k\bar{i}\bar{j}} + C_5 L(u)^2,\end{aligned}$$

where $C_5$ is some constant. In this section we will use the constant $C_5$ in the generic sense which depends on $f$, $\alpha$, $\mu$, the curvature bound of the metric $\omega_S$, and $u$ up to first order derivation. It can also depend on the lower bound of $F$ as we have proven that $F \geq \kappa e^{2u} \geq \kappa (C_1 A)^{-2}$. Note when we assume that $g_{i\bar{j}} = \delta_{ij}$ and $u_{i\bar{j}} = u_{i\bar{i}} \delta_{ij}$, the last term of (10.3) is $g'^{i\bar{i}} g'^{k\bar{k}} g'_{i\bar{k}j} g'_{k\bar{i}\bar{j}} F$. Multiplying (10.4) by $F$ and then inserting (10.3) and (10.8) into it, we obtain

$$\begin{aligned}(10.9)\quad & P(G_2) \cdot e^{-(-\lambda_1 u + \lambda_2 |\nabla u|^2)} \cdot F \\ & \geq -\lambda_1 L(u) P(u) \cdot F + \lambda_2 L(u) P(\mid \nabla u \mid^2) \cdot F \\ & \quad - L(u) \triangle (e^u + fe^{-u}) + 4\alpha g'^{i\bar{j}} (g^{k\bar{l}})_{i\bar{j}} u_{k\bar{l}} \cdot F \\ & \quad + 2^{-1} \triangle F - (2F)^{-1} \mid \nabla F \mid^2 + P(e^u + fe^{-u}) \cdot F - C_5 L(u)^2.\end{aligned}$$

We assume that the function $G_2$ achieve its maximum at the point $q_3$. Taking the normal coordinate $(z_1, z_2)$ at the point $q_3$ with respect to the given metric $\omega_S$, we shall estimate every term in (10.9).

At the point $q_3$, $\nabla G_2 = 0$ implies

$$(10.10)\quad \nabla \triangle u = \alpha^{-1} L(u)(\lambda_1 \nabla u - \lambda_2 \nabla \mid \nabla u \mid^2) - \alpha^{-1} \nabla (e^u + fe^{-u}).$$

At first we derive some inequalities which will be used to estimate terms in (10.9).

Using the equation we compute

$$\begin{aligned}(10.11)\qquad 4\alpha^2 g^{i\bar{j}} g^{k\bar{l}} u_{i\bar{l}} u_{k\bar{j}} & = 4\alpha^2 (u_{1\bar{1}} + u_{2\bar{2}})^2 - 8\alpha^2 \det u_{i\bar{j}} \\ & = \alpha^2 (\triangle u)^2 + \alpha \triangle (e^u - fe^{-u}) + \alpha\mu \\ & \leq L(u)^2 + C_5.\end{aligned}$$

Let

$$\Gamma = 4g^{i\bar{j}}g^{k\bar{l}}u_{,ik}u_{,\bar{j}\bar{l}},$$

where indices preceded by a comma, e.g., $u_{,ik}$ indicate covariant differentiation with respect to the given metric $\omega_S$. At the point $q_3$, we use the normal coordinate. Therefore at $q_3$, $u_{,ik} = u_{ik}$ and $u_{,\bar{j}\bar{l}} = u_{\bar{j}\bar{l}}$ (see p. 345 of [**24**] paper or the next section). Hence

$$\Gamma = 4u_{ik}u_{\bar{i}\bar{k}} = 4\sum_{ik} \mid u_{ik} \mid^2 .$$

As was done in above section, we take the normal coordinate at the point $q_3$ such that $u_1 = u_{\bar{1}}$ and $u_2 = u_{\bar{2}} = 0$. Applying the Schwarz inequality and (10.11),

$$\begin{aligned} \mid \bigtriangledown \mid \bigtriangledown u \mid^2\mid^2 &= 4(u_{1p}u_{1\bar{p}} + u_{\bar{1}\bar{p}}u_{\bar{1}p} + u_{1p}u_{\bar{1}\bar{p}} + u_{\bar{1}p}u_{1\bar{p}}) \mid \bigtriangledown u \mid^2 \\ &\leq 2 \mid \bigtriangledown u \mid^2 \left\{\Gamma + \alpha^{-2}L(u)^2\right\} + C_5. \end{aligned} \tag{10.12}$$

So,

$$\mid \bigtriangledown \mid \bigtriangledown u \mid^2 \mid \leq \sqrt{2} \mid \bigtriangledown u \mid \{\Gamma^{\frac{1}{2}} + \alpha^{-1}L(u)\} + C_5. \tag{10.13}$$

We also need to estimate

$$\begin{aligned} &\mid \bigtriangledown(\bigtriangledown u \cdot \bigtriangledown f) \mid^2 \\ &= 2(u_{ip}f_{\bar{i}} + u_i f_{\bar{i}p} + u_{\bar{i}p}f_i + u_{\bar{i}}f_{ip})(u_{k\bar{p}}f_{\bar{k}} + u_k f_{\bar{k}\bar{p}} + u_{\bar{k}\bar{p}}f_k + u_{\bar{k}}f_{k\bar{p}}). \end{aligned}$$

Changing the indices $i$ and $k$ in some terms and then applying the Schwarz inequality, we get

(10.14)
$$\begin{aligned} \mid \bigtriangledown(\bigtriangledown u \cdot \bigtriangledown f) \mid^2 &\leq C_5(2 \mid u_{ip} \mid\mid u_{k\bar{p}} \mid + \mid u_{ip} \mid\mid u_{\bar{k}\bar{p}} \mid + \mid u_{\bar{i}p} \mid\mid u_{k\bar{p}} \mid) \\ &\quad + C_5(\mid u_{ip} \mid + \mid u_{i\bar{p}} \mid) + C_5 \\ &\leq C_5\Gamma + C_5L(u)^2 + C_5. \end{aligned}$$

Hence

$$\mid \bigtriangledown(\mid \bigtriangledown u \cdot \bigtriangledown f) \mid \leq C_5\Gamma^{\frac{1}{2}} + C_5L(u) + C_5. \tag{10.15}$$

Applying (10.11), (10.10), and (10.13), we estimate

(10.16)
$$\begin{aligned}\triangle \mid \triangledown u \mid^2 &= 2g^{k\bar{l}}\{(2g^{i\bar{j}}u_{i\bar{j}})_k u_{\bar{l}} + (2g^{i\bar{j}}u_{i\bar{j}})_{\bar{l}} u_k\} \\ &\quad + 4g^{i\bar{j}}g^{k\bar{l}}u_{i\bar{l}}u_{k\bar{j}} + 4g^{i\bar{j}}g^{k\bar{l}}u_{ik}u_{\bar{j}\bar{l}} \\ &\leq 2 \triangledown \triangle u \cdot \triangledown u + \Gamma + \alpha^{-2}L(u)^2 + C_5 \\ &= 2\alpha^{-1}L(u)(\lambda_1 \mid \triangledown u \mid^2 - \lambda_2 \triangledown \mid \triangledown u \mid^2 \cdot \triangledown u) \\ &\quad - 2\alpha^{-1} \triangledown (e^u + fe^{-u}) \cdot \triangledown u + \Gamma + \alpha^{-2}L(u)^2 + C_5 \\ &\leq \{\alpha^{-2} + 2\sqrt{2}\alpha^{-2} \mid \triangledown u \mid^2 \lambda_2\}L(u)^2 \\ &\quad + 2\sqrt{2}\alpha^{-1} \mid \triangledown u \mid^2 \lambda_2 L(u)\Gamma^{\frac{1}{2}} + \Gamma \\ &\quad + C_5\lambda_1 L(u) + C_5 \\ &\leq (C_5\lambda_2^2 + C_5\lambda_2 + C_5)L(u)^2 + 2\Gamma + C_5\lambda_1 L(u) + C_5.\end{aligned}$$

For the same reason we can also get the following estimate

$$\begin{aligned}\triangle(\triangledown u \cdot \triangledown f) \leq \Gamma &+ (C_5\lambda_2^2 + C_5\lambda_2 + C_5)L(u)^2 \\ &+ C_5\lambda_1 L(u) + C_5.\end{aligned} \tag{10.17}$$

We now deal with every term in (10.9). For the first term, we use (9.24) to obtain

$$\begin{aligned}&- \lambda_1 L(u)P(u)F \\ &= -\lambda_1 L(u)(\alpha^{-1}F - \alpha^{-1}(e^u + fe^{-u})L(u)) \\ &\geq (\alpha C_1 A)^{-1}\lambda_1 L(u)^2 - C_5\lambda_1 L(u).\end{aligned} \tag{10.18}$$

Next we deal with the second term $\lambda_2 L(u)P(\mid \triangledown u \mid^2)F$:

$$\begin{aligned}P(\mid \triangledown u \mid^2)F \geq 4g'^{i\bar{j}}&(u_{i\bar{j}k}u_{\bar{k}} + u_{i\bar{j}\bar{k}}u_k)F \\ &+ 4g'^{i\bar{j}}u_{ik}u_{\bar{k}\bar{j}}F - C_5 L(u).\end{aligned} \tag{10.19}$$

Applying (9.29), (10.13) and (10.15), we estimate

$$\begin{aligned}&4g'^{i\bar{j}}(u_{i\bar{j}k}u_{\bar{k}} + u_{i\bar{j}\bar{k}}u_k)F \\ &\geq -2(e^u - fe^{-u})\triangledown \mid \triangledown u \mid^2 \cdot \triangledown u - 4e^{-u} \triangledown (\triangledown u \cdot \triangledown f) \cdot \triangledown u \\ &\geq -C_5\Gamma^{\frac{1}{2}} - C_5 L(u) - C_5.\end{aligned} \tag{10.20}$$

Inserting (10.20) into (10.19), we obtain

$$\begin{aligned}\lambda_2 L(u)P(\mid \triangledown u \mid^2)F \geq \lambda_2 L(u)&(4g'^{i\bar{j}}u_{ik}u_{\bar{k}\bar{j}})F - \Gamma \\ &- C_5(\lambda_2^2 + \lambda_2)L(u)^2 - C_5\lambda_2 L(u).\end{aligned} \tag{10.21}$$

We assume that $g_{i\bar{j}} = \delta_{ij}$ and $u_{i\bar{j}} = u_{i\bar{i}}\delta_{ij}$ at the point $q_3$. Then

$$
\begin{aligned}
(10.22)\qquad & \lambda_2 L(u)(4g'^{i\bar{j}}u_{ik}u_{\bar{k}\bar{j}})F \\
& = 4\lambda_2 F\frac{1}{g'_{i\bar{i}}}\Big(\frac{g'_{1\bar{1}}+g'_{2\bar{2}}}{2}\Big)u_{ik}u_{\bar{k}\bar{i}} \\
& \geq 2\lambda_2 F u_{ik}u_{\bar{k}\bar{i}} \geq \frac{1}{2}(C_1A)^{-2}\kappa\lambda_2\Gamma.
\end{aligned}
$$

Inserting (10.22) into (10.21), we find an estimate of the second term in (10.9)

$$
\begin{aligned}
(10.23)\quad & \lambda_2 L(u)P(\mid \triangledown u \mid^2)F \\
& \geq (2^{-1}(C_1A)^{-2}\kappa\lambda_2 - 1)\Gamma - C_5(\lambda_2^2+\lambda_2)L(u)^2 - C_5\lambda_2 L(u).
\end{aligned}
$$

The third term is

$$
\begin{aligned}
(10.24)\qquad & -L(u)\triangle(e^u + fe^{-u}) \\
& \geq -L(u)\{(e^u - fe^{-u})\triangle u + C_5\} \\
& \geq -C_5L(u)^2 - C_5L(u)
\end{aligned}
$$

and the fourth term is

$$
\begin{aligned}
(10.25)\quad & 4\alpha g'^{i\bar{j}}(g^{k\bar{l}})_{i\bar{j}}u_{k\bar{l}}F \\
& = 4\alpha(g'_{1\bar{1}}(g^{k\bar{l}})_{2\bar{2}} + g'_{2\bar{2}}(g^{k\bar{l}})_{1\bar{1}} - g'_{1\bar{2}}(g^{k\bar{l}})_{2\bar{1}} - g'_{2\bar{1}}(g^{k\bar{l}})_{1\bar{2}})u_{k\bar{l}} \\
& = 4\alpha(e^u + fe^{-u})g^{i\bar{j}}(g^{k\bar{l}})_{i\bar{j}}u_{k\bar{l}} \\
& \quad + 16\alpha^2(u_{1\bar{1}}(g^{k\bar{l}})_{2\bar{2}} + u_{2\bar{2}}(g^{k\bar{l}})_{1\bar{1}} - u_{1\bar{2}}(g^{k\bar{l}})_{2\bar{1}} - u_{2\bar{1}}(g^{k\bar{l}})_{1\bar{2}})u_{k\bar{l}} \\
& \geq -64\alpha^2 \max \mid R_{i\bar{j}k\bar{l}} \mid \sum \mid u_{i\bar{j}} \mid^2 \\
& \geq -C_5L(u)^2 - C_5,
\end{aligned}
$$

where $C_5$ depends the curvature of $\omega_S$. Next we deal with the fifth term. From the definition of $F$, we can easily get
(10.26)

$$
\begin{aligned}
2^{-1}\triangle F \geq & - C_5 \mid \triangle \mid \triangledown u \mid^2 \mid - C_5 \mid \triangle(\triangledown u \cdot \triangledown f) \mid - C_5 \mid \triangledown \mid \triangledown u \mid^2 \mid \\
& - C_5 \mid \triangledown(\triangledown u \cdot \triangledown f) \mid - C_5L(u) - C_5.
\end{aligned}
$$

We note that the inequalities (10.16) and (10.17) are also true for $|\triangle|\triangledown u|^2|$ and $|\triangle(\triangledown u \cdot \triangledown f)|$. Applying (10.16), (10.17), (10.13) and (10.15), we get

$$
\begin{aligned}
(10.27)\qquad 2^{-1}\triangle F \geq & -C_5\Gamma - (C_5\lambda_2^2 + C_5\lambda_2 + C_5)L(u)^2 \\
& - (C_5\lambda_1 + C_5)L(u) - C_5.
\end{aligned}
$$

We also observe that

$$
\triangledown F = -C_5\triangledown \mid \triangledown u \mid^2 - C_5 \triangledown (\triangledown u \cdot \triangledown f) - C_5 \triangledown u - C_5.
$$

Then applying the Schwarz inequality, and applying (10.12)-(10.15), we get

$$
\begin{aligned}
(10.28)\qquad & -(2F)^{-1} \mid \triangledown F \mid^2 \\
& \geq -C_5 \mid \triangledown \mid \triangledown u \mid^2 \mid^2 - C_5 \mid \triangledown \mid \triangledown u \mid^2 \mid \cdot \mid \triangledown(\triangledown u \cdot \triangledown f) \mid \\
& \quad - C_5 \mid \triangledown(\triangledown u \cdot \triangledown f) \mid^2 - C_5 \mid \triangledown \mid \triangledown u \mid^2 \mid \\
& \quad - C_5 \mid \triangledown(\triangledown u \cdot \triangledown f) \mid - C_5 \\
& \geq -C_5\Gamma - C_5 L(u)^2 - C_5 L(u) - C_5.
\end{aligned}
$$

The last term is

$$
\begin{aligned}
(10.29)\qquad & P(e^u + fe^{-u})F \\
& = (e^u - fe^{-u})P(u)F + (e^u + fe^{-u}) \cdot 2g'^{i\bar{j}} u_i u_{\bar{j}} F \\
& \quad - e^{-u} \cdot 2g'^{i\bar{j}}(u_i f_{\bar{j}} + u_{\bar{j}} f_i)F + e^{-u}P(f)F \\
& \geq -C_5 L(u) - C_5.
\end{aligned}
$$

Inserting (10.18), (10.23), (10.24), (10.25),(10.27), (10.28), and (10.29) into (10.9), at last we get

$$
\begin{aligned}
(10.30)\qquad & FP(G_2) \cdot e^{-(-\lambda_1 u + \lambda_2 |\triangledown u|^2)} \\
& \geq \left\{(\alpha C_1 A)^{-1}\lambda_1 - C_5(\lambda_2^2 + \lambda_2 + 1)\right\} L(u)^2 \\
& \quad - \{C_5\lambda_1 + C_5\lambda_2 + C_5\} L(u) \\
& \quad + (2^{-1}(C_1A)^{-2}\kappa\lambda_2 - C_5)\Gamma - C_5\Gamma^{\frac{1}{2}} - C_5.
\end{aligned}
$$

Fix the constant $C_5$. Take $\lambda_2$ big enough such that

$$2^{-1}(C_1A)^{-2}\kappa\lambda_2 - C_5 > 0$$

and then take $\lambda_1$ big enough such that

$$(\alpha C_1 A)^{-1}\lambda_1 - C_5(\lambda_2^2 + \lambda_2 + 1) > 0.$$

Fix $\lambda_1$ and $\lambda_2$. Then we can now estimate $G_2 = e^{-\lambda_1 u + \lambda_2 |\triangledown u|^2} L(u)$. In fact, it must achieve its maximum at some point $q_3$ so the right-hand side of (10.30) is non-positive. At this point,

$$
\begin{aligned}
0 & \geq \left\{(\alpha C_1 A^{-1}\lambda_1 - C_5(\lambda_2^2 + \lambda_2 + 1)\right\} L(u)^2 \\
& \quad - \{C_5\lambda_1 + C_5\lambda_2 + C_5\} L(u) \\
& \quad + (2^{-1}(C_1A)^{-2}\kappa\lambda_2 - C_5)\Gamma - C_5\Gamma^{\frac{1}{2}} - C_5 \\
& \geq \left\{(\alpha C_1 A)^{-1}\lambda_1 - C_5(\lambda_2^2 + \lambda_2 + 1)\right\} L(u)^2 \\
& \quad - C_5(\lambda_1 + \lambda_2 + 1) L(u) \\
& \quad - \frac{C_5}{4(2^{-1}(C_1A)^{-2}\kappa\lambda_2 - C_5)} - C_5.
\end{aligned}
$$

Hence $L(u)(q_3)$ has an upper bound $C_5'$ depending on $\alpha$, $f$, $\mu$, the curvature bound of metric $\omega_S$, $A$. Since $G_2$ achieves its maximum at the point $q_3$, we get the estimate

$$L(u) \le C_5' \frac{\sup(e^{-\lambda_1 u+\lambda_2|\bigtriangledown u|^2})}{\inf(e^{-\lambda_1 u+\lambda_2|\bigtriangledown u|^2})} \le C_5' \frac{e^{-\lambda_1 \inf u+\lambda_2 \sup|\bigtriangledown u|^2}}{e^{-\lambda_1 \sup u}}.$$

As $|\bigtriangledown u|^2$ has the upper bound (7.8), we get an upper bound of $e^u + fe^{-u} + \alpha \triangle u$. In conclusion, we have proved the following:

**Proposition 23.** *Let $S$ be a $K3$ surface with Calabi-Yau metric $\omega_S$ such that $\int_S 1\frac{\omega_S^2}{2!} = 1$. Let $u \in C^4(S)$ be the solution of the equation $\triangle(e^u - t\alpha fe^{-u}) + 8\alpha\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} + t\mu = 0$ which satisfies the condition $(e^u + t\alpha fe^{-u})\omega_S + 2\alpha\sqrt{-1}\partial\bar{\partial}u > 0$ and $\left(\int_S e^{-4u}\right)^{\frac{1}{4}} = A \ll 1$ (see (9.55) and (9.56)). Then $e^u + t\alpha fe^{-u} + \alpha \triangle u$ has an upper bound depending only on $\alpha$, $f$, $\mu$, $\omega_S$ and $A$. Moreover, combing with the Proposition 22, $e^u + t\alpha fe^{-u} + 4\alpha u_{i\bar{i}}$, for $i = 1, 2$, have the positive lower and upper bounds depending only on $\alpha$, $f$, $\mu$, $\omega_S$ (both Sobolev constant and curvature bound) and $A$.*

## 11. Third order estimate

In this section we use indices to denote partial derivatives, e.g., $u_i = \partial_i u = \frac{\partial u}{\partial z_i}$, $u_{i\bar{j}} = \partial_{i\bar{j}} u = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$. Indices preceded by a comma, e.g., $u_{,ik}$ indicate covariant differentiation with respect to the given metric $\omega_S$. Let

$$\begin{aligned}
\Gamma &= g^{i\bar{j}} g^{k\bar{l}} u_{,ik} u_{,\bar{j}\bar{l}} \\
\Theta &= g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} u_{,i\bar{j}k} u_{,\bar{r}s\bar{t}} \\
\Xi &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} u_{,ikp} u_{,\bar{j}\bar{l}\bar{q}} \\
\Phi &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} g'^{r\bar{s}} u_{,i\bar{l}pr} u_{,\bar{j}k\bar{q}\bar{s}} \\
\Psi &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} g'^{r\bar{s}} u_{,i\bar{l}p\bar{s}} u_{,\bar{j}k\bar{q}r}.
\end{aligned}$$

We shall apply the maximum principle to the function

$$G_3 = (\lambda_3 + \alpha \triangle u)\Theta + \lambda_4(m + \alpha \triangle u)\Gamma + \lambda_5 |\bigtriangledown u|^2 \Gamma + \lambda_6 \Gamma, \tag{11.1}$$

where all $\lambda_i$ for $i = 3, 4, 5, 6$ are positive constants and will be determined later; $m$ is a fixed constant such that $m + \alpha \triangle u > 0$. At first we assume that $\lambda_3 + \alpha \triangle u > 1$. We shall use $C_6$ as a constant in the generic sense which depends only on $\alpha$, $f$, $\mu$, $\omega_S$, and $u$ up to the second order derivations.

Let the function $G_3$ achieve the maximum at a point $q_4 \in S$. Before computing $P(G_3)$ at $q_4$, we need to derive some relations between partial derivatives and covariant differentiations.

Pick a normal coordinate at $q_4$ such that $g_{i\bar{j}} = \delta_{ij}$, $\partial g_{i\bar{j}}/\partial z_k = \partial g_{i\bar{j}}/\partial \bar{z}_k = 0$. Then at $q_4$, we have

$$\begin{aligned} u_{,i\bar{j}} &= u_{i\bar{j}}, \qquad u_{,ij} = u_{ij}, \qquad u_{,\bar{i}\bar{j}} = u_{\bar{i}\bar{j}}, \\ u_{,i\bar{j}k} &= u_{i\bar{j}k}, \qquad u_{,\bar{i}j\bar{k}} = u_{\bar{i}j\bar{k}}, \quad u_{,i\bar{j}\bar{k}} = u_{i\bar{j}\bar{k}}, \quad u_{,\bar{i}\bar{j}k} = u_{\bar{i}\bar{j}k} \\ \partial_{\bar{k}l}(u_{,ij}) &= u_{,ij\bar{k}l}, \qquad \ldots\ldots \end{aligned}$$

We also have

$$u_{,ik\bar{\gamma}} = u_{,i\bar{\gamma}k} + u_s R^s_{ik\bar{\gamma}}, \quad u_{,\bar{j}\bar{l}\delta} = u_{,\bar{j}\delta\bar{l}} + u_{\bar{s}} R^{\bar{s}}_{\bar{j}\bar{l}\delta}.$$

We shall compute every term in $P(G_3)$:

$$\begin{aligned} (11.2)\quad P(|\bigtriangledown u|^2) &\geq 4g'^{\delta\bar{\gamma}} g^{i\bar{j}} (u_{i\bar{\gamma}\delta} u_{\bar{j}} + u_i u_{\bar{j}\delta\bar{\gamma}} + u_{i\bar{\gamma}} u_{\bar{j}\delta} + u_{i\delta} u_{\bar{j}\bar{\gamma}}) - C_6 \\ &\geq 4g'^{\delta\bar{\gamma}} g^{i\bar{j}} \{u_{,i\delta} u_{,\bar{j}\bar{\gamma}} + u_{,i\bar{\gamma}\delta} u_{\bar{j}} + u_i u_{,\bar{j}\delta\bar{\gamma}}\} - C_6 \\ &\geq m_1\Gamma - C_6 \sum | u_{,i\bar{\gamma}\delta} || u_{\bar{j}} | - C_6 \\ &\geq m_1\Gamma - C_6\Theta^{\frac{1}{2}} - C_6. \end{aligned}$$

Since Proposition 23 shows that the metric $\omega'$ is uniformly equivalent to $\omega_S$, we see that such an $m_1 > 0$ exists. In the following we use $m_2$ and $m_3$ as $m_1$. Next we estimate $\alpha P(\triangle u)$. From (10.3) we know

(11.3)
$$\alpha P(\triangle u) \geq g^{i\bar{j}} g'^{\delta\bar{p}} g'^{q\bar{\gamma}} g'_{\delta\bar{\gamma}i} g'_{\bar{p}q\bar{j}} + (2F)^{-1} \triangle F - (2F^2)^{-1} | \bigtriangledown F |^2 - C_6.$$

We estimate

$$\begin{aligned} (11.4)\qquad & g^{i\bar{j}} g'^{\delta\bar{p}} g'^{q\bar{\gamma}} g'_{\delta\bar{\gamma}i} g'_{\bar{p}q\bar{j}} \\ &\geq 16\alpha^2 g^{i\bar{j}} g'^{\delta\bar{p}} g'^{q\bar{\gamma}} u_{,\delta\bar{\gamma}i} u_{,\bar{p}q\bar{j}} - C_6 \sum | (e^u + fe^{-u})_{\bar{j}} || u_{\delta\bar{\gamma}i} | \\ &\geq m_2\Theta - C_6\Theta^{\frac{1}{2}}. \end{aligned}$$

From (10.26), we also have

$$(11.5)\quad \triangle F \geq -C_6 \sum | u_{,k\bar{l}i} || u_{\bar{j}} | - C_6\Gamma - C_6 \geq -C_6\Theta^{\frac{1}{2}} - C_6\Gamma - C_6.$$

Inserting (11.4), (11.5) and (10.28) into (11.3), we get

$$(11.6)\qquad P(\alpha \triangle u) \geq m_2\Theta - C_6\Theta^{\frac{1}{2}} - C_6\Gamma - C_6 \geq m_2\Theta - C_6\Gamma - C_6,$$

where we have used $m_2$ in the generic sense. We also calculate:

$$\begin{aligned} (11.7)\qquad P(\Gamma) &\geq 2g'^{\delta\bar{\gamma}} g^{i\bar{j}} g^{k\bar{l}} \{(u_{,ik})_{\bar{\gamma}\delta} u_{,\bar{j}\bar{l}} + u_{,ik} (u_{,\bar{j}\bar{l}})_{\delta\bar{\gamma}}\} \\ &\quad + 2g'^{\delta\bar{\gamma}} g^{i\bar{j}} g^{k\bar{l}} \{(u_{,ik})_{\delta} (u_{,\bar{j}\bar{l}})_{\bar{\gamma}} + (u_{,ik})_{\bar{\gamma}} (u_{,\bar{j}\bar{l}})_{\delta}\} - C_6\Gamma \\ &= 2g'^{\delta\bar{\gamma}} g^{i\bar{j}} g^{k\bar{l}} \{u_{,ik\delta} u_{,\bar{j}\bar{l}\bar{\gamma}} + u_{,ik\bar{\gamma}\delta} u_{,\bar{j}\bar{l}} + u_{,ik} u_{,\bar{j}\bar{l}\delta\bar{\gamma}}\} \\ &\quad + 2g'^{\delta\bar{\gamma}} g^{i\bar{j}} g^{k\bar{l}} (u_{,i\bar{\gamma}k} + u_s R^s_{ik\bar{\gamma}})(u_{,\bar{j}\delta\bar{l}} + u_{\bar{s}} R^{\bar{s}}_{\bar{j}\bar{l}\delta}) - C_6\Gamma \\ &\geq m_3\Xi + m_3\Theta - C_6\Phi^{\frac{1}{2}}\Gamma^{\frac{1}{2}} - C_6\Gamma \\ &\geq m_3\Xi + m_3\Theta - \epsilon_1\lambda_6^{-1}\Phi - C_6\lambda_6\epsilon_1^{-1}\Gamma. \end{aligned}$$

Combining (11.2) and (11.7), we find

$$(11.8)\qquad \begin{aligned} &P(|\bigtriangledown u|^2\,\Gamma)\\ &\geq m_1\Gamma^2 - C_6\Theta^{\frac{1}{2}}\Gamma - C_6\Gamma\\ &\quad + |\bigtriangledown u|^2\,(m_3\Xi + m_3\Theta - C_6\Phi^{\frac{1}{2}}\Gamma^{\frac{1}{2}} - C_6\Gamma)\\ &\quad - C_6(\Gamma^{\frac{1}{2}}+1)(\Theta^{\frac{1}{2}}\Gamma^{\frac{1}{2}} + \Xi^{\frac{1}{2}}\Gamma^{\frac{1}{2}} + \Gamma^{\frac{1}{2}})\\ &\geq m_1\Gamma^2 - \epsilon_1\lambda_5^{-1}\Phi - C_6\lambda_5\epsilon_1^{-1}\Gamma - C_6\Xi - C_6\Theta - C_6. \end{aligned}$$

Combining (11.6) and (11.7), we get

$$(11.9)\qquad \begin{aligned} &P((m+\alpha\bigtriangleup u)\Gamma)\\ &\geq (m_2\Theta - C_6\Gamma - C_6)\Gamma - C_6\Theta^{\frac{1}{2}}(\Theta^{\frac{1}{2}} + \Xi^{\frac{1}{2}} + 1)\Gamma^{\frac{1}{2}}\\ &\quad + (m+\alpha\bigtriangleup u)(m_3\Xi + m_3\Theta - C_6\Phi^{\frac{1}{2}}\Gamma^{\frac{1}{2}} - C_6\Gamma)\\ &\geq m_2\Theta\Gamma - C_6\Gamma^2 - \epsilon_1\lambda_4^{-1}\Phi - C_6\lambda_4\epsilon_1^{-1}\Gamma - C_6\Xi - C_6\Theta. \end{aligned}$$

Applying (11.6), we get

$$(11.10)\qquad \begin{aligned} &P((\lambda_3+\alpha\bigtriangleup u)\Theta)\\ &\geq m_2\Theta^2 - C_6\Gamma\Theta - C_6\Theta + (\lambda_3+\alpha\bigtriangleup u)P(\Theta)\\ &\quad + 2\alpha g'^{\delta\bar{\gamma}}\{\partial_\delta(\bigtriangleup u)\partial_{\bar{\gamma}}\Theta + \partial_{\bar{\gamma}}(\bigtriangleup u)\partial_\delta\Theta\}. \end{aligned}$$

So we should deal with the term $2\alpha g'^{\delta\bar{\gamma}}\{\partial_\delta(\bigtriangleup u)\partial_{\bar{\gamma}}\Theta+\partial_{\bar{\gamma}}(\bigtriangleup u)\partial_\delta\Theta\}$. Let $G_3$ achieve the maximum at the point $q_4$. Then at the point $q_4$, we have,

$$\begin{aligned} \partial_{\bar{\gamma}}\Theta = -\frac{1}{\lambda_3+\alpha\bigtriangleup u}\big\{&\Theta\partial_{\bar{\gamma}}(\alpha\bigtriangleup u) + \lambda_4\partial_{\bar{\gamma}}((m+\alpha\bigtriangleup u)\Gamma)\\ &+ \lambda_5\partial_{\bar{\gamma}}(|\bigtriangledown u|^2\,\Gamma) + \lambda_6\partial_{\bar{\gamma}}\Gamma\big\} \end{aligned}$$

and

$$(11.11)\qquad \begin{aligned} &2\alpha g'^{\delta\bar{\gamma}}\big\{\partial_\delta(\bigtriangleup u)\partial_{\bar{\gamma}}\Theta + \partial_{\bar{\gamma}}(\bigtriangleup u)\partial_\delta\Theta\big\}\\ &\geq \frac{-C_6}{\lambda_3+\alpha\bigtriangleup u}\Theta^{\frac{1}{2}}\times\big\{\Theta^{\frac{3}{2}} + \lambda_4\Theta^{\frac{1}{2}}\Gamma + \lambda_5\Gamma^{\frac{3}{2}}\\ &\qquad + (\lambda_4+\lambda_5+\lambda_6)(\Theta^{\frac{1}{2}} + \Xi^{\frac{1}{2}} + \Gamma^{\frac{1}{2}})\Gamma^{\frac{1}{2}}\big\}\\ &\geq \frac{-C_6\{\Theta^2 + (\lambda_4+\lambda_5+\lambda_6)(\Theta\Gamma + \Theta + \Gamma + \Xi) + \lambda_5\Gamma^2\}}{\lambda_3+\alpha\bigtriangleup u} - C_6. \end{aligned}$$

Inserting (11.11) into (11.10), and then combing (11.7)–(11.10), we obtain

(11.12)
$$
\begin{aligned}
P(G_3) \geq{}& (\lambda_3 + \alpha \triangle u) P(\Theta) + \left\{ m_2 - C_6(\lambda_3 + \alpha \triangle u)^{-1} \right\} \Theta^2 \\
&+ \left\{ \lambda_4 m_2 - C_6 - C_6 (\lambda_3 + \alpha \triangle u)^{-1} (\lambda_4 + \lambda_5 + \lambda_6) \right\} \Theta\Gamma \\
&+ \left\{ \lambda_5 m_1 - C_6 \lambda_5 (\lambda_3 + \alpha \triangle u)^{-1} - C_6 \lambda_4 \right\} \Gamma^2 \\
&+ \left\{ \lambda_6 m_3 - C_6(\lambda_4 + \lambda_5) - C_6(\lambda_4 + \lambda_5 + \lambda_6)(\lambda_3 + \alpha \triangle u)^{-1} \right\} \Xi \\
&- 3\epsilon_1 \Phi - C_7 \Theta - C_7 \Gamma - C_7,
\end{aligned}
$$

where $C_7$ also depends on $\lambda_i$ and $\epsilon_1$ at point $q_4$.

At last we should estimate $P(\Theta)$. We follow paper [**24**] to obtain:

(11.13)
$$
\begin{aligned}
P(\Theta) ={}& 2g'^{\delta\bar{\gamma}} \partial_\delta \partial_{\bar{\gamma}} (g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} u_{,i\bar{j}k} u_{,\bar{r}s\bar{t}}) \\
={}& 2g'^{\delta\bar{\gamma}} [\quad 2g'^{i\bar{a}} g'^{b\bar{p}} g'^{q\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} + 2g'^{i\bar{p}} g'^{q\bar{a}} g'^{b\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} \\
& \qquad + 2g'^{i\bar{p}} g'^{q\bar{r}} g'^{s\bar{a}} g'^{b\bar{j}} g'^{k\bar{t}} + 2g'^{i\bar{p}} g'^{q\bar{r}} g'^{s\bar{j}} g'^{k\bar{a}} g'^{b\bar{t}} \\
& \qquad + g'^{i\bar{a}} g'^{b\bar{r}} g'^{s\bar{p}} g'^{q\bar{j}} g'^{k\bar{t}} + g'^{i\bar{r}} g'^{s\bar{a}} g'^{b\bar{p}} g'^{q\bar{j}} g'^{k\bar{t}} \\
& \qquad + g'^{i\bar{r}} g'^{s\bar{p}} g'^{q\bar{a}} g'^{b\bar{j}} g'^{k\bar{t}} + g'^{i\bar{r}} g'^{s\bar{p}} g'^{q\bar{j}} g'^{k\bar{a}} g'^{b\bar{t}}] \\
& \qquad\qquad \times \partial_\delta g'_{b\bar{a}} \partial_{\bar{\gamma}} g'_{\bar{p}q} u_{,i\bar{j}k} u_{,\bar{r}s\bar{t}} \quad \text{(first class)} \\
& - 2g'^{\delta\bar{\gamma}} [2g'^{i\bar{p}} g'^{q\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} + g'^{i\bar{r}} g'^{s\bar{p}} g'^{q\bar{j}} g'^{k\bar{t}}] \\
& \quad \times [\partial_{\bar{\gamma}} g'_{\bar{p}q} u_{,i\bar{j}k\delta} u_{,\bar{r}s\bar{t}} + \partial_\delta g'_{q\bar{p}} u_{,\bar{r}s\bar{t}\bar{\gamma}} u_{,i\bar{j}k}] \quad \text{(second class)} \\
& - 2g'^{\delta\bar{\gamma}} [2g'^{i\bar{p}} g'^{q\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} + g'^{i\bar{r}} g'^{s\bar{p}} g'^{q\bar{j}} g'^{k\bar{t}}] \\
& \quad \times [\partial_{\bar{\gamma}} g'_{\bar{p}q} u_{,i\bar{j}k} u_{,\bar{r}s\bar{t}\delta} + \partial_\delta g'_{q\bar{p}} u_{,i\bar{j}k\bar{\gamma}} u_{,\bar{r}s\bar{t}}] \quad \text{(third class)} \\
& - 2g'^{\delta\bar{\gamma}} [2g'^{i\bar{p}} g'^{q\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} + g'^{i\bar{r}} g'^{s\bar{p}} g'^{q\bar{j}} g'^{k\bar{t}}] \\
& \qquad \times \partial_\delta \partial_{\bar{\gamma}} g'_{\bar{p}q} u_{,i\bar{j}k} u_{,\bar{r}s\bar{t}} \quad \text{(forth class)} \\
& + 2g'^{\delta\bar{\gamma}} g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} \times [u_{,i\bar{j}k\bar{\gamma}\delta} u_{,\bar{r}s\bar{t}} + u_{,i\bar{j}k} u_{,\bar{r}s\bar{t}\delta\bar{\gamma}}] \quad \text{(fifth class)} \\
& + 2g'^{\delta\bar{\gamma}} g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} \times [u_{,i\bar{j}k\bar{\gamma}} u_{,\bar{r}s\bar{t}\delta} + u_{,i\bar{j}k\delta} u_{,\bar{r}s\bar{t}\bar{\gamma}}] \quad \text{(sixth class)} \\
& - C_6 \Theta,
\end{aligned}
$$

when we use normal coordinates, so that at this point we have $\partial_{\bar{\beta}} u_{,i\bar{j}k} = u_{,i\bar{j}k\bar{\beta}}$ and $\partial_\alpha \partial_{\bar{\beta}} u_{,i\bar{j}k} = u_{,i\bar{j}k\bar{\beta}\alpha} + u_{,i\bar{s}k} R^{\bar{s}}_{\bar{j}\bar{\beta}\alpha}$. Comparing with (A.8) in [**24**],

we should deal with the first five classes in (11.13). The first class is:
(11.14)

$$\begin{aligned}
&2g'^{\delta\bar{\gamma}}g'^{i\bar{a}}g'^{b\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\partial_\delta g'_{b\bar{a}}\partial_{\bar{\gamma}}g'_{\bar{p}q}u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}}\\
&= 2g'^{\delta\bar{\gamma}}g'^{i\bar{a}}g'^{b\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}(4\alpha u_{b\bar{a}\delta})(4\alpha u_{\bar{p}q\bar{\gamma}})u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}}\\
&\quad + 4\mathrm{Re}\{g'^{\delta\bar{\gamma}}g'^{i\bar{a}}g'^{a\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}((e^u+fe^{-u})_\delta\cdot(4\alpha u_{\bar{p}q\bar{\gamma}})u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}})\}\\
&\quad + 2g'^{\delta\bar{\gamma}}g'^{i\bar{a}}g'^{a\bar{p}}g'^{p\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}((e^u+fe^{-u})_\delta(e^u+fe^{-u})_{\bar{\gamma}}u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}})\\
&\geq 2g'^{\delta\bar{\gamma}}g'^{i\bar{a}}g'^{b\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}(4\alpha u_{,b\bar{a}\delta})(4\alpha u_{,\bar{p}q\bar{\gamma}})u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}}\\
&\quad - \epsilon_2/12(\lambda_3+\alpha\triangle u)^{-1}\Theta^2 - C_6\epsilon_2^{-1}(\lambda_3+\alpha\triangle u)\Theta.
\end{aligned}$$

The second class is:
(11.15)

$$\begin{aligned}
&-2g'^{\delta\bar{\gamma}}g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\{\partial_{\bar{\gamma}}g'_{\bar{p}q}u_{,i\bar{j}k\delta}u_{,\bar{r}s\bar{t}}+\partial_\delta g'_{q\bar{p}}u_{,\bar{r}s\bar{t}\bar{\gamma}}u_{,i\bar{j}k}\}\\
&= -4\mathrm{Re}\{g'^{\delta\bar{\gamma}}g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\partial_{\bar{\gamma}}((e^u+fe^{-u})g_{\bar{p}q}+4\alpha u_{\bar{p}q})u_{,i\bar{j}k\delta}u_{\bar{r}s\bar{t}}\}\\
&\geq -2g'^{\delta\bar{\gamma}}g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\{(4\alpha u_{,\bar{p}q\bar{\gamma}})u_{,i\bar{j}k\delta}u_{,\bar{r}s\bar{t}}+(4\alpha u_{,q\bar{p}\delta})u_{\bar{r}s\bar{t}\bar{\gamma}}u_{,i\bar{j}k}\}\\
&\quad - \epsilon_1/3(\lambda_3+\alpha\triangle u)^{-1}\Phi - C_6(\lambda_3+\alpha\triangle u)\epsilon_1^{-1}\Theta.
\end{aligned}$$

The third class is:
(11.16)

$$\begin{aligned}
&-2g'^{\delta\bar{\gamma}}g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\{\partial_\delta g'_{q\bar{p}}u_{,i\bar{j}k\bar{\gamma}}u_{,\bar{r}s\bar{t}}+\partial_{\bar{\gamma}}g'_{q\bar{p}}u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}\delta}\}\\
&\geq -2g'^{\delta\bar{\gamma}}g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\{(4\alpha u_{,q\bar{p}\delta})u_{,i\bar{j}k\bar{\gamma}}u_{,\bar{r}s\bar{t}}+(4\alpha u_{,\bar{p}q\bar{\gamma}})u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}\delta}\}\\
&\quad - \epsilon_1/3(\lambda_3+\alpha\triangle u)^{-1}\Psi - C_6(\lambda_3+\alpha\triangle u)\epsilon_1^{-1}\Theta.
\end{aligned}$$

Next we deal with the fourth class. By (10.2),

$$\begin{aligned}
(11.17)\qquad &-2g'^{\delta\bar{\gamma}}g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\partial_\delta\partial_{\bar{\gamma}}g'_{\bar{p}q}u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}}\\
&\geq -2g'^{\delta\bar{a}}g'^{b\bar{\gamma}}g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\partial_{\bar{p}}g'_{\bar{a}b}\partial_q g'_{\delta\bar{\gamma}}u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}} - C_6\Theta\\
&\quad - 2g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\left\{F^{-1}F_{q\bar{p}} - F^{-2}F_qF_{\bar{p}}\right\}u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}}.
\end{aligned}$$

Then from (11.4), (11.5) and (10.30), we can see
(11.18)

$$\begin{aligned}
&-2g'^{\delta\bar{\gamma}}g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}\partial_\delta\partial_{\bar{\gamma}}g'_{\bar{p}q}u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}}\\
&\geq -2g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}g'^{\delta\bar{a}}g'^{b\bar{\gamma}}(4\alpha u_{\bar{a}b\bar{p}})(4\alpha u_{,\delta\bar{\gamma}q})u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}}\\
&\quad - C_6\Theta^{\frac{3}{2}} - C_6\Theta\Gamma^{\frac{1}{2}} - C_6\Gamma\Theta - C_6\Theta\\
&\geq -2g'^{i\bar{p}}g'^{q\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}g'^{\delta\bar{a}}g'^{b\bar{\gamma}}(4\alpha u_{,\bar{a}b\bar{p}})(4\alpha u_{,\delta\bar{\gamma}q})u_{,i\bar{j}k}u_{,\bar{r}s\bar{t}}\\
&\quad - C_6\Theta\Gamma - m_2/24(\lambda_3+\alpha\triangle u)^{-1}\Theta^2 - C_6(\lambda_3+\alpha\triangle u)\Theta - C_6\Gamma.
\end{aligned}$$

Now we deal with the fifth term. By direct calculation, we have

$$\begin{aligned}
u_{,i\bar{j}k\bar{\gamma}\delta} &= u_{i\bar{j}k\bar{\gamma}\delta} + u_{,p\bar{j}\delta}R^p_{ik\bar{\gamma}} + u_{,i\bar{p}k}R^{\bar{p}}_{\bar{j}\bar{\gamma}\delta}\\
&\quad - u_{p\bar{j}}\partial_\delta\partial_{\bar{\gamma}}(g^{p\bar{s}}\partial_k g_{i\bar{s}}) - u_{p\bar{j}\bar{\gamma}}\partial_\delta(g^{p\bar{s}}\partial_k g_{i\bar{s}}).
\end{aligned}$$

So the fifth class can be expressed

$$
(11.19)\qquad \begin{aligned}
&g'^{\delta\bar\gamma}g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\{u_{,i\bar j k\bar\gamma\delta}u_{,\bar r s\bar t}+u_{,i\bar j k}u_{,\bar r s\bar t\delta\bar\gamma}\}\\
&\geq g'^{\gamma\bar\delta}g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\{u_{i\bar j k\bar\gamma\delta}u_{,\bar r s\bar t}+u_{,i\bar j k}u_{\bar r s\bar t\delta\bar\gamma}\}-C_6\Theta.
\end{aligned}
$$

Differentiating (10.2), we can get

$$
(11.20)\quad \begin{aligned}
&4\alpha g'^{\delta\bar\gamma}u_{\delta\bar\gamma i\bar j k}\\
&=4\alpha g'^{\delta\bar p}g'^{q\bar\gamma}g'_{q\bar p k}u_{\delta\bar\gamma i\bar j}+(g'^{\delta\bar p}g'^{q\bar\gamma}g'_{p\bar q\bar j}g'_{\delta\bar\gamma i})_k\\
&\quad+g'^{\delta\bar p}g^{q\bar\gamma}g'_{q\bar p k}((e^u+fe^{-u})g_{\delta\bar\gamma})_{i\bar j}-g'^{\delta\bar\gamma}((e^u+fe^{-u})g_{\delta\bar\gamma})_{i\bar j k}\\
&\quad+F^{-1}F_{i\bar j k}-F^{-2}(F_kF_{i\bar j}+F_iF_{\bar j k}+F_{\bar j}F_{ik})+2F^{-3}F_iF_{\bar j}F_k.
\end{aligned}
$$

Inserting (11.20) into (11.19), we get

$$
(11.21)\quad \begin{aligned}
&g'^{\delta\bar\gamma}g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\{u_{,i\bar j k\bar\gamma\delta}u_{,\bar r s\bar t}+u_{,i\bar j k}u_{,\bar r s\bar t\delta\bar\gamma}\}\\
&\geq g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\delta\bar p}g'^{q\bar\gamma}\{g'_{q\bar p k}u_{\delta\bar\gamma i\bar j}u_{,\bar r s\bar t}+g'_{\bar q p\bar t}u_{\bar\gamma\delta\bar r s}u_{,i\bar j k}\}-C_6\Theta\\
&\quad+(4\alpha)^{-1}g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\{(g'^{\delta\bar p}g'^{q\bar\gamma}g'_{\bar p q\bar j}g'_{\delta\bar\gamma i})_k u_{\bar r s\bar t}\\
&\qquad\qquad+(g'^{\delta\bar p}g'^{q\bar\gamma}g'_{\bar p q\bar s}g'_{\delta\bar\gamma r})_{\bar t}u_{i\bar j k}\}\\
&\quad+(2\alpha)^{-1}\mathrm{Re}\{g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}(F^{-1}F_{i\bar j k}+2F^{-3}F_iF_{\bar j}F_k)u_{,\bar r s\bar t}\}\\
&\quad-(2\alpha)^{-1}\mathrm{Re}\{g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}F^{-2}(F_{i\bar j}F_k+F_iF_{\bar j k}+F_{\bar j}F_{ik})u_{,\bar r s\bar t}\}.
\end{aligned}
$$

We observe

$$
(11.22)\quad \begin{aligned}
&g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\delta\bar p}g'^{q\bar\gamma}\{g'_{q\bar p k}u_{\delta\bar\gamma i\bar j}u_{,\bar r s\bar t}+g'_{\bar q p\bar t}u_{\bar\gamma\delta\bar r s}u_{,i\bar j k}\}\\
&\geq g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\delta\bar p}g'^{q\bar\gamma}\{(4\delta u_{,q\bar p k})u_{,\delta\bar\gamma i\bar j}u_{,\bar r s\bar t}+(4\alpha u_{,\bar p q\bar t})u_{,\bar\gamma\delta\bar r s}u_{,i\bar j k}\}\\
&\quad-C_6\Psi^{\frac12}\Theta^{\frac12}-C_6\Theta,
\end{aligned}
$$

and

$$
(11.23)\quad \begin{aligned}
&(4\alpha)^{-1}g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\{(g'^{\delta\bar p}g'^{q\bar\gamma}g'_{\bar p q\bar j}g'_{\delta\bar\gamma i})_k u_{\bar r s\bar t}+(g'^{\delta\bar p}g'^{q\bar\gamma}g'_{\bar p q\bar s}g'_{\delta\bar\gamma r})_{\bar t}u_{i\bar j k}\}\\
&\geq g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\delta\bar p}g'^{q\bar\gamma}\{[u_{,\bar p q\bar j k}(4\alpha u_{,\delta\bar\gamma i})+(4\alpha u_{,\bar p q\bar j})u_{,\delta\bar\gamma i k}]u_{,\bar r s\bar t}\\
&\qquad\qquad+[(u_{,\bar p q\bar s\bar t}(4\alpha u_{,\delta\bar\gamma r})+(4\alpha u_{,\bar p q\bar s})u_{,\delta\bar\gamma r\bar t}]u_{,i\bar j k}\}\\
&\quad-g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\{g'^{\delta\bar a}g'^{b\bar p}g'^{q\bar\gamma}+g'^{\delta\bar p}g'^{q\bar a}g'^{b\bar\gamma}\}\\
&\qquad\cdot\{(4\alpha u_{,b\bar a k})(4\alpha u_{,\bar p q\bar j})u_{,\delta\bar\gamma i}u_{,\bar r s\bar t}+(4\alpha u_{,\bar a b\bar t})(4\alpha u_{,\delta\bar\gamma r})u_{,\bar p q\bar s}u_{,i\bar j k}\}\\
&\quad-C_6\Theta^{\frac32}-C_6\Psi^{\frac12}\Theta^{\frac12}-C_6\Phi^{\frac12}\Theta^{\frac12}-C_6\Theta-C_6.
\end{aligned}
$$

We also have estimate

$$
\begin{aligned}
(11.24)\quad &+(2\alpha)^{-1}\mathrm{Re}\{g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}(F^{-1}F_{i\bar j k}+2F^{-3}F_iF_{\bar j}F_k)u_{,\bar r s\bar t}\}\\
&-(2\alpha)^{-1}\mathrm{Re}\{g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}F^{-2}(F_{i\bar j}F_k+F_iF_{\bar j k}+F_{\bar j}F_{ik})u_{,\bar r s\bar t}\}\\
&\geq -C_6\Phi^{\frac12}\Theta^{\frac12}-C_6\Psi^{\frac12}\Theta^{\frac12}-C_6\Gamma^{\frac12}\Theta-C_6\Gamma^{\frac32}\Theta^{\frac12}-C_6\Theta-C_6.
\end{aligned}
$$

Inserting (11.22)-(11.24) into (11.21), we get

(11.25)

$$
\begin{aligned}
&g'^{\delta\bar\gamma}g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\{u_{,i\bar j k\bar\gamma\delta}u_{,\bar r s\bar t}+u_{,i\bar j k}u_{,\bar r s\bar t\delta\bar\gamma}\}\\
&\geq g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\delta\bar p}g'^{q\bar\gamma}\{(4\alpha u_{,q\bar p k})u_{,\delta\bar\gamma i\bar j}u_{,\bar r s\bar t}+(4\alpha u_{,\bar p q\bar t})u_{,\bar\gamma\delta\bar r s}u_{,i\bar j k}\}\\
&\quad+g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\delta\bar p}g'^{q\bar\gamma}\{[u_{,\bar p q\bar j k}(4\alpha u_{,\delta\bar\gamma i})+(4\alpha u_{,\bar p q\bar j})u_{,\delta\bar\gamma i k}]u_{,\bar r s\bar t}\\
&\qquad\qquad+[u_{,\bar p q\bar s\bar t}(4\alpha u_{,\delta\bar\gamma r})+(4\alpha u_{,\bar p q\bar s})u_{,\delta\bar\gamma r\bar t}]u_{,i\bar j k}\}\\
&\quad-g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\{g'^{\delta\bar a}g'^{b\bar p}g'^{q\bar\gamma}+g'^{\delta\bar p}g'^{q\bar a}g'^{b\bar\gamma}\}\\
&\qquad\cdot\{(4\alpha u_{,b\bar a k})(4\alpha u_{,\bar p q\bar j})u_{,\delta\bar\gamma i}u_{,\bar r s\bar t}+(4\alpha u_{,\bar a b\bar t})(4\alpha u_{,\delta\bar\gamma r})u_{,\bar p q\bar s}u_{,i\bar j k}\}\\
&\quad-\epsilon_1/2(\lambda_3+\alpha\triangle u)^{-1}(\Phi+\Psi)-C_6\epsilon_1^{-1}(\lambda_3+\alpha\triangle u)\Theta\\
&\quad-m_2/16(\lambda_3+\alpha\triangle u)^{-1}\Theta^2-C_6(\lambda_3+\alpha\triangle u)\Theta-C_6\Theta\Gamma-C_6\Gamma^2.
\end{aligned}
$$

Inserting (11.14)-(11.16), (11.18), and (11.25) into (11.13), diagonalizing, and simplifying, then comparing to (A.8) and (A.9) in [**24**], we obtain

$$
\begin{aligned}
(11.26)\quad P(\Theta)\geq{}&\frac{3\epsilon_1}{\lambda_3+\alpha\triangle u}\Phi-C_6\Theta\Gamma-C_6\Gamma^2\\
&-\Big(\frac{m_2/4+\epsilon_2}{\lambda_3+\alpha\triangle u}+\frac{C_6\epsilon_1}{\lambda_3+\alpha\triangle u-5\epsilon_1}\Big)\Theta^2-C_7(\Theta+\Gamma).
\end{aligned}
$$

Inserting (11.26) into (11.12), at last we obtain

(11.27)

$$
\begin{aligned}
P(G_3)\geq{}&\Big\{m_2-\frac{m_2}{4}-\epsilon_2-\frac{C_6}{\lambda_3+\alpha\triangle u}-C_6\epsilon_1\frac{\lambda_3+\alpha\triangle u}{\lambda_3+\alpha\triangle u-5\epsilon_1}\Big\}\Theta^2\\
&+\Big\{\lambda_4m_2-C_6-\frac{C_6(\lambda_4+\lambda_5+\lambda_6)}{\lambda_3+\alpha\triangle u}-C_6(\lambda_3+\alpha\triangle u)\Big\}\Theta\Gamma\\
&+\Big\{\lambda_5m_1-\frac{C_6\lambda_5}{\lambda_3+\alpha\triangle u}-C_6\lambda_4-C_6(\lambda_3+\alpha\triangle u)\Big\}\Gamma^2\\
&+\Big\{\lambda_6m_3-C_6(\lambda_4+\lambda_5)-\frac{C_6(\lambda_4+\lambda_5+\lambda_6)}{\lambda_3+\alpha\triangle u}\Big\}\Xi\\
&-C_7\Theta-C_7\Gamma-C_7.
\end{aligned}
$$

Note the generic constant $C_6$ does not depend on $\epsilon_i$ and $\lambda_i$, so we can fix it, because we can take the biggest one. Fix $\epsilon_1$ and $\epsilon_2$ such that $\epsilon_2+2C_6\epsilon_1<\frac{m_2}{4}$. Take $\lambda_3$ big enough such that $\frac{C_6}{\lambda_3+\alpha\triangle u}<\frac{m_2}{4}$ and

$\frac{\lambda_3+\alpha\triangle u}{\lambda_3+\alpha\triangle u-5\epsilon_1}<2$, then

(11.28)
$$\left\{m_2-\frac{m_2}{4}-\epsilon_2-\frac{C_6}{\lambda_3+\alpha\triangle u}-C_6\epsilon_1\frac{\lambda_3+\alpha\triangle u}{\lambda_3+\alpha\triangle u-5\epsilon_1}\right\}\Theta^2>\frac{m_2}{4}\Theta^2.$$

Let
$$\tilde{\lambda}_i=\frac{\lambda_i}{\lambda_3+\alpha\triangle u}\quad \text{for}\quad i=4,5,6.$$

We choose $\tilde{\lambda}_4$, $\tilde{\lambda}_5$ and $\tilde{\lambda}_6$ such that
$$\tilde{\lambda}_4>\frac{C_6}{m_2}+1$$
$$\tilde{\lambda}_5>\frac{C_6}{m_1}\tilde{\lambda}_4+\frac{C_6}{m_1}+1$$

and
$$\tilde{\lambda}_6>C_6\frac{\tilde{\lambda}_4+\tilde{\lambda}_5}{m_3}+1.$$

Then if we take $\lambda_3$ big enough such that
$$m_i(\lambda_3+\alpha\triangle u)-C_6(\tilde{\lambda}_4+\tilde{\lambda}_5+\tilde{\lambda}_6)-C_6>m_i,\quad \text{for}\ \ i=1,2,3,$$

we can estimate

(11.29)
$$\begin{aligned}&\left\{\lambda_4 m_2-C_6-\frac{C_6}{\lambda_3+\alpha\triangle u}(\lambda_4+\lambda_5+\lambda_6)-C_6(\lambda_3+\alpha\triangle u)\right\}\Theta\Gamma\\&\geq\{m_2(\lambda_3+\alpha\triangle u)-C_6(\tilde{\lambda}_4+\tilde{\lambda}_5+\tilde{\lambda}_6)-C_6\}\Theta\Gamma>m_2\Theta\Gamma;\end{aligned}$$

$$\begin{aligned}&\left\{\lambda_5 m_1-\frac{C_6\lambda_5}{\lambda_3+\alpha\triangle u}-C_6\lambda_4-C_6(\lambda_3+\alpha\triangle u)\right\}\Gamma^2\\&>\{m_1(\lambda_3+\alpha\triangle u)-C_6\tilde{\lambda}_5\}\Gamma^2>m_1\Gamma^2\end{aligned}\tag{11.30}$$

and

$$\begin{aligned}&\left\{m_3\lambda_6-\frac{C_6}{\lambda_3+\alpha\triangle u}(\lambda_4+\lambda_5+\lambda_6)-C_6(\lambda_4+\lambda_5)\right\}\Xi\\&>\{m_3(\lambda_3+\alpha\triangle u)-C_6(\tilde{\lambda}_4+\tilde{\lambda}_5+\tilde{\lambda}_6)\}\Xi>m_3\Xi.\end{aligned}\tag{11.31}$$

Inserting (11.28), (11.29)-(11.31) into (9.28), we see that
$$0\geq P(G_3)\geq\frac{m_2}{4}\Theta^2+m_2\Theta\Gamma+m_1\Gamma^2+m_3\Xi-C_7\Theta-C_7\Gamma-C_7.$$

The above inequality gives an estimate of the the quantity $\sup_S\Theta$ and $\sup_S\Gamma$. This in turn gives the estimates of $u_{i\bar{j}k}$ and $u_{ij}$ for all $i,j,k$.

**Proposition 24.** *Let $\omega_S$ be a given Calabi-Yau metric on a $K3$ surface with $\int_S 1\frac{\omega_S^2}{2!} = 1$. Let $t \in \mathbf{T}$ and $u \in C^5(S)$ is a solution of the equation $\triangle(e^u - t\alpha f e^{-u}) + 8\alpha\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + t\mu = 0$ under the elliptic condition $\omega' = (e^u + t\alpha f e^{-u})\omega_S + 2\alpha\sqrt{-1}\partial\bar{\partial}u > 0$ and the normalization $\left(\int_S e^{-4u}\right)^{\frac{1}{4}} = A \ll 1$ (see (9.55) and (9.56)). Then there is an estimate of the derivatives $u_{i\bar{j}k}$ in terms of $\alpha$, $f$, $\mu$, $\omega_S$ and $A$.*

## 12. Estimates for the general case

In the general case, the equation is

$$\sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - t\alpha\partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + t\mu\frac{\omega_S^2}{2!} = 0.$$

Let

$$\rho = -\sqrt{-1}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1}),$$

then $\rho$ is a well-defined real $(1,1)$-form on $S$. We replace $t\alpha\rho$ by $\rho$ and $t\mu$ by $\mu$. Then we can rewrite the equation as

$$\sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - \sqrt{-1}\partial\bar{\partial}(e^{-u}\rho) - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu\frac{\omega_S^2}{2!} = 0.$$

The elliptic condition is

$$\omega' = e^u\omega_S + e^{-u}\rho + 2\alpha\sqrt{-1}\partial\bar{\partial}u > 0.$$

If we let $\rho = \frac{\sqrt{-1}}{2}\rho_{i\bar{j}}dz_i \wedge d\bar{z}_j$, then $g'_{i\bar{j}} = e^u g_{i\bar{j}} + e^{-u}\rho_{i\bar{j}} + 4\alpha u_{i\bar{j}}$. Using the definition of $P$ and the equation, we compute

$$\begin{aligned}
&\int_S P(e^{-ku})\frac{\omega'^2}{2!}\\
&\geq -\sqrt{-1}k\int_S e^{-ku}\partial\bar{\partial}u \wedge (e^u\omega_S + e^{-u}\rho + 2\alpha\sqrt{-1}\partial\bar{\partial}u)\\
&= k\int_S e^{-(k-1)u} \triangle u + 2k\int_S e^{-(k-1)u} \mid \bigtriangledown u \mid^2\\
&\quad + \sqrt{-1}k\int_S e^{-(k+1)u}\partial\bar{\partial}u \wedge \rho - 2\sqrt{-1}k\int_S e^{-(k+1)u}\partial u \wedge \bar{\partial}u \wedge \rho\\
&\quad + 2\sqrt{-1}k\int_S e^{-(k+1)u}\partial u \wedge \bar{\partial}\rho - 2\sqrt{-1}k\int_S e^{-(k+1)u}\bar{\partial}u \wedge \partial\rho\\
&\quad - 2\sqrt{-1}k\int_S e^{-(k+1)u}\partial\bar{\partial}\rho + 2k\int_S e^{-ku}\mu.
\end{aligned}$$

On the other hand, we can also compute

$$\int_S P(e^{-ku})\frac{\omega'^2}{2!} = \sqrt{-1}\int_S \partial\bar\partial e^{-ku}\wedge\omega'$$
$$= -k\int_S e^{-(k-1)u}\triangle u + k^2\int_S e^{-(k-1)u}\mid\nabla u\mid^2$$
$$-\sqrt{-1}k\int_S e^{-(k+1)u}\partial\bar\partial u\wedge\rho + \sqrt{-1}k^2\int_S e^{-(k+1)u}\partial u\wedge\bar\partial u\wedge\rho.$$

Combing above two inequalities, we get

$$k\int_S e^{-(k-1)u}\mid\nabla u\mid^2 + \sqrt{-1}k\int_S e^{-(k+1)u}\partial u\wedge\bar\partial u\wedge\rho$$
$$\geq 2\int_S e^{-(k-1)u}\triangle u + 2\int_S e^{-(k-1)u}\mid\nabla u\mid^2 + 2\sqrt{-1}\int_S e^{-(k+1)u}\partial\bar\partial u\wedge\rho$$
$$-2\sqrt{-1}\int_S e^{-(k+1)u}\partial u\wedge\bar\partial u\wedge\rho + 2\sqrt{-1}\int_S e^{-(k+1)u}\partial u\wedge\bar\partial\rho$$
$$-2\sqrt{-1}\int_S e^{-(k+1)u}\bar\partial u\wedge\partial\rho - 2\sqrt{-1}\int_S e^{-(k+1)u}\partial\bar\partial\rho + 2\int_S e^{-ku}\mu.$$

Integrating by part and then simplifying it, when $k \geq 2$, we get

$$k\int_S e^{-(k-1)u}\mid\nabla u\mid^2 + \sqrt{-1}k\int_S e^{-(k+1)u}\partial u\wedge\bar\partial u\wedge\rho \tag{12.1}$$
$$\leq 2\sqrt{-1}(1-\frac{1}{1+k})\int_S e^{-(k+1)u}\partial\bar\partial\rho + 2k\int_S e^{-ku}\mu.$$

Using the notation in section 3, we have

$$\rho = \sqrt{-1}g^{i\bar j}\frac{\partial f_i}{\partial\bar z_l}\cdot\overline{\frac{\partial f_j}{\partial\bar z_k}}dz_k\wedge d\bar z_l$$

and

$$\sqrt{-1}\partial u\wedge\bar\partial u\wedge\rho$$
$$= 4\begin{pmatrix} u_1 & u_2 \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial f_1}{\partial\bar z_1} & \frac{\partial f_2}{\partial\bar z_1} \\ \frac{\partial f_1}{\partial\bar z_2} & \frac{\partial f_2}{\partial\bar z_2} \end{pmatrix}\cdot g\cdot\begin{pmatrix} \frac{\partial f_1}{\partial\bar z_1} & \frac{\partial f_2}{\partial\bar z_1} \\ \frac{\partial f_1}{\partial\bar z_2} & \frac{\partial f_2}{\partial\bar z_2} \end{pmatrix}^*\cdot\begin{pmatrix} u_{\bar 1} \\ u_{\bar 2} \end{pmatrix}\frac{\omega_S^2}{2!}.$$

So

$$\sqrt{-1}k\int_S e^{-(k+1)u}\partial u\wedge\bar\partial u\wedge\rho \geq 0.$$

Then (12.1) implies the inequality (8.8) in Section 6:

$$k\int_S e^{-(k-1)u}\mid\nabla u\mid^2 \leq 2\sqrt{-1}(1-\frac{1}{1+k})\int_S e^{-(k+1)u}\partial\bar\partial\rho + 2\int_S e^{-ku}\mu$$
$$\leq C_0\int_S e^{-(k+1)u} + C_0\int_S e^{-ku}.$$

We follow the discussion in Section 6 to get the estimate $\inf u \geq -\ln(C_1A)$. If $A$ is small enough, we can get $\inf u$ big enough. Then we can check all other estimates that can be derived using the same method because the term $e^u$ can always control terms such as $e^{-u} \mid \mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g) \mid$. Thus we get:

**Proposition 25.** *Propositions* 21, 22, *and* 23 *are also true for the equation of general case:*
(12.2)

$$\sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - t\alpha\partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})) - \alpha\partial\bar{\partial}u \wedge \partial\bar{\partial}u + t\mu\frac{\omega_S^2}{2!} = 0$$

*if we replace* $f$ *by* $-\sqrt{-1}\mathrm{tr}(\bar{\partial}B \wedge \partial B^* \cdot g^{-1})$.

**Proposition 26.** *Proposition* 24 *is also true for the equation* (12.2).

## 13. Further remark–generalization

Let $X$ be an (n+1)-dimensional complex manifold with Hermitian metric $\omega$ and a nowhere vanishing holomorphic $(n+1, 0)$-form $\Omega$. As we stated in the introduction, the string theorists consider the following Strominger's system:

$$F_H \wedge \omega^n = 0; \quad F_H^{2,0} = F_H^{0,2} = 0; \tag{13.1}$$

$$\sqrt{-1}\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\mathrm{tr}R \wedge R - \mathrm{tr}F_H \wedge F_H); \tag{13.2}$$

$$d^*\omega = \sqrt{-1}(\bar{\partial} - \partial)\ln\|\Omega\|_\omega. \tag{13.3}$$

The third equation is equivalent to

$$d(\| \Omega \|_\omega \omega^n) = 0. \tag{13.4}$$

Let $n \geq 2$. Motivated by the constructions in section 2 and 4, we propose to study the following system

$$F_H \wedge \omega^n = 0; \quad F_H^{2,0} = F_H^{0,2} = 0; \tag{13.5}$$

$$\Big\{\sqrt{-1}\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{tr}R \wedge R - \mathrm{tr}F_H \wedge F_H)\Big\} \wedge \omega^{n-2} = 0; \tag{13.6}$$

$$d\Big( \| \Omega \|_\omega^{2\frac{n-1}{n}} \omega^n\Big) = 0. \tag{13.7}$$

Then we can generalize our construction to complex manifolds with dim $\geq 3$. Let $K$ be a Calabi-Yau $n$-fold with a Ricci-flat metric $\omega_K$ and a nowhere vanishing holomorphic $(n, 0)$-form $\Omega_K$. Let $\omega_1, \omega_2$ be a primitive harmonic $(1, 1)$-forms such that $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^{1,1}(K, \mathbb{Z})$. Using these two forms, we can construct an $(n + 1)-$dimensional complex manifold $X$:

1. $\pi : X \to K$ is a $T^2$-fibration over $K$. If we write locally $\omega_1 = d\alpha_1$ and $\omega_2 = d\alpha_2$ for real 1-forms $\alpha_1$ and $\alpha_2$, then there is a coordinate that $x$ and $y$ of fiber $T^2$ such that $dx + \sqrt{-1}dy$ is a holomorphic 1-form on $T^2$-fibers and $dx + \alpha_1$ and $dy + \alpha_2$ are globally defined 1-forms on $X$.
2. Let
$$\theta = (dx + \alpha_1) + \sqrt{-1}(dy + \alpha_2)$$
and let
$$\Omega = \Omega_K \wedge \theta.$$
Then $\Omega$ defines a nowhere vanishing holomorphic $(n+1, 0)$-form on $X$.
3. Let $u \in C^2(K)$ function on $K$ and
$$\omega_u = e^u \omega_K + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}. \tag{13.8}$$
Then $(\Omega, \omega_u)$ satisfies equation (13.7).

*Proof.* As in Section 4, we have
$$\| \Omega \|^2_{\omega_u} = \frac{\| \Omega \|^2_{\omega_u}}{\| \Omega \|^2_{\omega_0}} = \frac{\omega_0^n}{\omega_u^n} = e^{-nu},$$
and
$$\omega_u^n = e^{nu}\omega_K^n + \sqrt{-1}ne^{(n-1)u}\omega_K^{n-1} \wedge \theta \wedge \bar{\theta}.$$
Then
$$\begin{aligned} &d(\| \Omega \|^{2\frac{n-1}{n}} \omega^n) \\ &= \sqrt{-1}n\omega_K^{n-1} \wedge ((\omega_1 + \sqrt{-1}\omega_2) \wedge \bar{\theta} + \theta \wedge (\omega_1 - \sqrt{-1}\omega_2)) = 0, \end{aligned}$$
as $\omega_1, \omega_2$ are primitive $(1,1)$-forms on $K$. So $(\Omega, \omega_u)$ satisfies equation (13.7). q.e.d.

As $\omega_1, \omega_2$ are harmonic, we can find $(1,0)$-forms $\xi_1 = \sum_{i=1}^n \xi_{1i} dz_i$ and $\xi_2 = \sum_{i=1}^n \xi_{2i} dz_i$, locally where $\xi_{1i}$ and $\xi_{2i}$ are smooth complex functions on some open set of $K$, such that $\omega_1 = \bar{\partial}\xi_1$ and $\omega_2 = \bar{\partial}\xi_2$. Let
$$\phi_i = \xi_{1i} + \xi_{2i}, \qquad \text{for} \quad j = 1, 2, \dots, n,$$
and let
$$B = (\phi_1, \phi_2, \dots, \phi_n).$$
Let $R_u$ be the curvature of the Hermitian connection of metric $\omega_u$ of the holomorphic tangent bundle $T'X$ and $R_K$ be the curvature of metric $\omega_K$. Then in section 3, we have
$$\mathrm{tr} R_u \wedge R_u = \mathrm{tr} R_K \wedge R_K + 2\partial\bar{\partial}(e^{-u}\bar{\partial}B \wedge \partial B^* \cdot g^{-1}) + n\partial\bar{\partial}u \wedge \partial\bar{\partial}u,$$
where $g$ is the Calabi-Yau metric associated to Kähler form $\omega_K$. Let $E$ be the stable vector bundle over $(K, \omega_K)$ with degree zero. According

to the Uhlenbeck-Yau theorem, there is a unique Hermitian-Yang-Mills metric $H$ up to constants. Hence

$$(\pi^* E, \pi^* H, X, \omega_u)$$

satisfies equations (13.5) and (13.7). So we only need to consider equation (13.6), which can be decomposed to the following two equations

$$\begin{aligned}0 =&\frac{(n-2)!}{2}\int_K (\| \omega_1 \|^2_{\omega_K} + \| \omega_2 \|^2_{\omega_K})\frac{\omega_K^n}{n!}\\&+\frac{\alpha'}{4}\int_K \mathrm{tr}(F_H \wedge F_H - R_K \wedge R_K) \wedge \omega_K^{n-2}\end{aligned}$$

and

$$\begin{aligned}0 =&\sqrt{-1}\partial\bar{\partial}u \wedge \omega_K^{n-1} - 2\partial\bar{\partial}(e^{-u}\mathrm{tr}\bar{\partial}B \wedge \partial B^* \cdot g^{-1}) \wedge K^{n-2}\\&- n\partial\bar{\partial}u \wedge \partial\bar{\partial}u \wedge K^{n-2} + \mu\frac{\omega_K^n}{n!},\end{aligned}$$

where $\mu$ is a smooth function on $K$ and $\int_K \mu \frac{\omega_K^n}{n!} = 0$. In the next paper, we will continue to consider this problem.

## References


[1] L. Alessandrini & G. Bassanelli, *Modifications of compact balanced manifolds*, R. Acad. Sci. Paris Súr. I Math. **320** (1995) 1517–1522, MR 1340064, Zbl 0842.32024.

[2] K. Becker, M. Becker, K. Dasgupta, & P.S. Green, *Compactifications of heterotic strings on non-Kähler complex manifolds*, I, JHEP **0304** (2003) 007, MR 1989858.

[3] K. Becker, M. Becker, J.-X. Fu, L.-S. Tseng, & S.-T. Yau, *Anomaly cancellation and smooth non-Kähler solutions in heterotic string theory*, Nuclear Physics B **751** (2006) 108–128, MR 2245852.

[4] K. Becker, M. Becker, P.S. Green, K. Dasgupta, & E. Sharpe, *Compactifications of heterotic strings on non-Kähler complex manifolds*, II, Nuclear Physics B **678** (2004) 19–100, MR 2022991.

[5] P. Candelas, G. Horowitz, A. Strominger, & E. Witten, *Vacuum Configurations for Superstrings*, Nuclear Physics B **258** (1985) 46–74, MR 0800347.

[6] K. Dasgupta, G. Rajesh, & S.Sethi, *M theory, orientifolds and G-flux*, JHEP **9908** (1999) 023, MR 1720178, Zbl 1060.81575.

[7] S.K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **50** (1985) 1–26, MR 0765366, Zbl 0529.53018.

[8] S.K. Donaldson & P.B. Kronheimer, *The geometry of Four-Manifolds*, Oxford Science Publication, MR 1079726.

[9] R. Friedman, *On threefolds with trivial canonical bundle*, in 'Complex geometry and Lie theory' (Sundance, UT, 1989), 103–134, Proc. Sympos. Pure Math., **53**, Amer. Math. Soc., Providence, RI, 1991, MR 1141199.

[10] E. Goldstein & S. Prokushkin, *Geometric model for complex non-Kähler manifolds with $SU(3)$ structure*, Comm. Math. Phys. **251** (2004) 65–78, MR 2096734.

[11] D. Grantcharov, G. Grantcharov, & Y. Poon, *Calabi-Yau Connections with Torsion on Toric Bundles*, math.DG/0306207.

[12] S. Gurrieri, J. Louis, A. Micu, & D. Waldram, *Mirror Symmetry in Generalized Calabi-Yau Compactifications*, Nuclear Physics B **654** (2003) 61–113, MR 1965609.

[13] P. Hořava & E. Witten, *Eleven-Dimensional Supergravity on a Manifold with Boundary*, Nuclear Physics B **475** (1996) 94–114, MR 1412097, Zbl 0925.81180.

[14] S. Kachru, M.B. Schulz, P.K. Tripathy, & S.P. Trivedi, *New supersymmetric String Compactfications*, JHEP **0303** (2003) 061, MR 1975671.

[15] M. Lubke & A. Teleman, *The Kobayashi-Hitchin Correspondence, World Scientific*, Singapore, New Jersey, London, Hong Kong, MR 1370660.

[16] J. Li & S.-T. Yau, *Hermitian-Yang-Mills connections on non-Kähler manifolds*, Mathematical aspects of string theory, World Scientific, 1987, 560–573, MR 0915839, Zbl 0664.53011.

[17] ———, *The existence of supersymmetric string theorey with torsion*, J. Differential Geom. **70** (2005) 143–181, MR 2192064, Zbl 1102.53052.

[18] M.L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **149** (1982) 261–295, MR 0688351, Zbl 0531.53053.

[19] M. Reid, *The moduli space of* 3*-folds with* $K = 0$ *may nevertheless be irreducible*, Math. Ann. **278** (1987) 329–334, MR 0909231, Zbl 0649.14021.

[20] A. Strominger, *Superstings with Torsion*, Nuclear Physics B **274** (1986) 253–284, MR 0851702.

[21] ———, *Heterotic solution*, Nuclear Physics B **343** (1990) 167–184, MR 1070937.

[22] K. Uhlenbeck & S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **39**(**S**) (1986) S257–S293, MR 0861491, Zbl 0615.58045.

[23] E. Witten, *New Issuss in Manifolds of $SU(3)$ holonomy*, Nuclear Physics B **268** (1986) 79–112, MR 0832602.

[24] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, I, Comm. Pure Appl. Math. **31** (1978) 339–411, MR 0480350, Zbl 0362.53049.

School of Mathematical Sciences
and Key Laboratory of Mathematics
for Nonlinear Sciences
Fudan University
Shanghai 200433 China

*E-mail address*: majxfu@fudan.edu.cn

Department of Mathematics
Harvard University
Cambridge, MA 02138

*E-mail address*: yau@math.harvard.edu