LATTICE COUNTING FOR DEFORMATIONS
OF CONVEX DOMAINS

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Abstract

Let $D_0 = \{x \in \mathbb{R}^n, H_0(x) \leq 1\}$ be a strictly convex domain in $\mathbb{R}^n$ with $n \leq 3$ and $D_u = \{x \in \mathbb{R}^n, H_u(x) \leq 1\}$, $u \in [\eta, \eta]$ be a continuous one-parameter deformation of D_0 with lattice-point counting function $N_u(T) := \{m \in \mathbb{Z}^n : H_u(m) \leq T^2\}$. The main result of this paper is an estimate for large values of T of the variation of the counting function, $N_u(T)$, over generic volume-preserving deformations D_u.

1. Introduction

The problem of constructing manifolds and planar domains which are isospectral, i.e., have the same spectrum for the Laplace operator but are not isometric, has attracted a lot of attention over the past thirty years. Much less is known about isospectral deformations: continuous families of metrics or domains with the same spectrum. Guillemin and Kazhdan [**6**] proved that negatively curved surfaces are rigid. Min-Oo [**10**] extended the result to higher dimensional negatively curved manifolds with restrictions on the curvature and Croke and Sharafutdinov [**1**] proved the result for all negatively curved manifolds. Recently, Zelditch [**15**] has shown that analytic convex surfaces of revolution are spectrally determined (i.e., any two such surfaces with analytic profile curve and with the same Laplace spectrum are isometric).

This is not the case for manifolds with mixed positive and negative curvature or for manifolds with boundary: Gordon and Szabo [**5**] constructed isospectral deformations of negatively curved manifolds with boundary. They are open domains in solvmanifolds. On spheres of dimension $n \geq 8$ Gordon [**3**] constructed isospectral deformations, which can be chosen arbitrary close to the standard metric. She also contructed examples of isospectral deformations on balls for the Laplace operator with Dirichlet or Neumann data. See also [**4**].

On the other hand, flat tori Λ in 2 and 3 dimensions are spectrally determined (the case $n = 3$ being difficult, see [13]). The eigenvalues of flat tori are essentially the squared lengths of the dual lattice vectors and by a linear transformation, the spectral function

$$N(T) = \#\{j, \lambda_j \le T^2\}$$

equals the counting function for lattice points inside elliptic regions in the plane. For this case we have the famous (unproven) Hardy conjecture on $N(T)$:

$$N(T) = \frac{\text{vol}(\Lambda)}{4\pi} T^2 + O_\delta(T^{1/2+\delta})$$

for all $\delta > 0$. The exponent $1/2$ is optimal, see [7]. There has been a great deal of work on this conjecture, most of which deals with various estimates for spectral averages of the error term. In previous work [11] we proved that metric averaging also behaves in a consistent fashion with the Hardy conjecture. We showed that, for $\epsilon > 0$ small,

$$\int_{[1-\epsilon,1+\epsilon]^2} \left| N(T) - \frac{\text{vol}(\Lambda_{u,v})}{4\pi} T^2 \right|^2 \, du \, dv \ll T^{1+\delta},$$

where $\Lambda_{u,v}$ is a family of flat tori close to a given flat torus. For a somewhat simpler proof of this result, see [8].

In this note we are interested in the case of general convex domains, which correspond to more general toric integrable systems. We address the following question related to rigidity:

Question. Given a generic, one-parameter deformation of convex domains in $\mathbb{R}^n$ with the same volume, how much does the lattice-point counting function vary over the deformation for large values of T?

When $n = 2, 3$, we give explicit lower bounds for the variation of the lattice-point counting function in any such deformation. Before we state our results in detail, we discuss the simplest kind of one-parameter deformations.

Example. Consider the special case of deformations of the form $H_u(x) = H_0(x) + u\kappa(x)$ for $|u|$ small. The condition that area is preserved is just

$$\int_{\{H_u=1\}} \kappa(x) d\sigma_u(x) = 0; \quad \forall u \in (-\eta, \eta),$$

where, $d\sigma_u$ is surface measure on ∂D_u. So, in particular, $\{\kappa = 0\}$ must intersect $\{H_0 = 1\}$. The points on $H_u = 1$, where $\partial_u H_u = 0$ are these intersection points. The condition we need for our results (see 1.3 below) says that the intersection of these two sets is transversal.

Notation. Throughout the paper, v denotes a row vector and v^t a column vector.

Let D_u be a family of (strictly) convex domains in $\mathbb{R}^n$ with the same volume (i.e., $\mathrm{vol}(D_u)) = A$). We assume that the domains are defined through a one-parameter family $H_u(x)$, $u \in (-\eta, \eta)$, of smooth functions on $\mathbb{R}^n$, such that

$$D_u = \{x \in \mathbb{R}^n;\ H_u(x) \leq 1\}.$$

In particular, the origin belongs to all the domains. We assume that $H_u(x)$ is homogeneous of degree 2. Let

$$N_u(T) := \#\{k \in \mathbb{Z}^n;\ H_u(k) \leq T^2\}$$

be the lattice-point counting function associated with the T-dilate, TD_u. We will make a non-degeneracy assumption on the one-parameter family of defining functions which can be described as follows: Consider the unfolded open surface in $\mathbb{R}^{n+1}$ defined by:

$$(1.1) \qquad\qquad \mathcal{D} = \cup_{u \in (-\eta, \eta)} D_u \times \{u\}.$$

We denote by $H(x, u) = H_u(x)$, the boundary defining function of $\mathcal{D}$ and let $\psi_u^+ : \mathbb{R}^n \setminus \{0\} \longrightarrow \partial D_u$ (resp. $\psi_u^- : \mathbb{R}^n \setminus \{0\} \longrightarrow \partial D_u$) be the inverse exterior (resp. interior) Gauss map of the boundary ∂D_u.

Definition 1.1. We say that the deformation D_u, $u \in (-\eta, \eta)$ is *non-degenerate* if for any $(p, u) \in \partial \mathcal{D}$ with $\psi_u^\pm(\xi) = p, \partial_u \psi_u^\pm(\xi) = v$ satisfying

$$(1.2) \qquad\qquad \frac{\partial H}{\partial u}(p, u) = 0,$$

we have that

$$(1.3) \qquad\qquad (v, 1) \cdot \nabla_{x,u}^2 H(p, u) \cdot (v, 1)^t \gg 1.$$

Example. Consider the one-parameter family of ellipses

$$H_u(x_1, x_2) = (1 + u)x_1^2 + (1 + u)^{-1}x_2^2, \quad u \in (-\eta, \eta).$$

A straightforward computation shows that this deformation is non-degenerate in the sense of Definition 1.1 for η sufficiently small. Indeed, the critical point equation reduces to:

$$x_1^2 - x_2^2 + 2x_2^2 u = O(u^2),$$

where, $x_1^2 + x_2^2 = 1 + O(u)$. The equation $x_1^2 + x_2^2 + u(x_1^2 - x_2^2) + O(u^2) = 1$ gives for the Gauss map $G_u(p) = \nabla_x H(p, u)/|\nabla_x H(p, u)|$

$$\frac{\partial G_u}{\partial u}(x_1, x_2) = (x_1, -x_2) + O(u).$$

This implies for the inverse Gauss map

$$\frac{\partial \psi_u^+}{\partial u}(x_1, x_2) = (-x_1, x_2) + O(u).$$

Since

$$\nabla^2_{x,u}H(p,u) = \begin{pmatrix} 2(1+u) & 0 & 2x_1 \\ 0 & 2(1+u)^{-1} & -2x_2 \\ 2x_1 & -2x_2 & 2x_2^2 \end{pmatrix}$$

modulo higher order terms in u, the Hessian condition (1.3) is computed to be

$$(v^+,1)(\nabla^2_{x,u}H)(p,u)(v^+,1)^t = -2x_1^2 + O(u) = -1 + O(u).$$

A similar calculation shows that the non-degeneracy condition is satisfied for ψ_u^-.

As usual, $f(T) = \Omega(g(T))$ means that there exists a sequence $T_k; k = 1, 2, \ldots$ with $T_k \to \infty$ as $k \to \infty$ such that $f(T_k) \gg g(T_k)$ when $k \to \infty$. Our main results are the following:

Theorem 1.2. *Let H_u, $u \in [-\eta, \eta]$ be a non-degenerate family of domains D_u, $u \in [-\eta, \eta]$, as in Def. 1.1. Then, for $n = 2$, we have*

$$\int_{-\eta}^{\eta} |N_u(T) - N_0(T)|^2 \, du = \Omega(T)$$

as $T \to \infty$. For $n = 3$ and any $\alpha, \beta \in \mathbb{R}$ with $0 \le \alpha < \beta$

$$\int_{\alpha}^{\beta} \int_{-\eta}^{\eta} |N_u(aT) - N_0(aT)|^2 \, du \, da = \Omega(T^2).$$

Theorem 1.2 implies that the domains D_u are rigid in the sense of the lattice-counting problem.

Remark 1.3. Our method does not work for domains in $\mathbb{R}^n$ with $n \ge 4$. Although it is desirable to remove the averaging in a in dimension 3, we are currently unable to do so.

The first five sections are devoted to the proof of Theorem 1.2. In section 5 we prove Theorem 1.2 for $n = 3$.

2. The case $n = 2$

Since $N_u(T) = AT^2 + R_u(T)$ for all $u \in I = (-\eta, \eta)$, it suffices to prove Theorem 1.2 for $R_u(T)$. As is standard, one has to mollify R_u appropriately to be able to compute effectively and, in particular, to apply Poisson summation formula. This is done as follows.

Let ρ be a nonnegative C^∞ function compactly supported in, say, $[1/2, 2]^2$, with $\int_{\mathbb{R}^2} \rho = 1$. As usual we define the family of mollifiers constructed out of ρ by $\rho_\delta(x) = \delta^{-2}\rho(x/\delta)$. We define the mollified counting function for TD_u by

$$N_u^\delta(T) = \sum_{k \in \mathbb{Z}^2} \chi_{TD_u} \star \rho_\delta(k),$$

which, using the Poisson summation formula, equals

$$(2.1) \qquad N_u^\delta(T) = AT^2 + T^2 \sum_{k \neq 0} \hat{\chi}_{D_u}(Tk)\,\hat{\rho}(\delta k) = AT^2 + R_u^\delta(T).$$

Since ρ is compactly-supported, there exists a constant C (independent of D_u) such that

$$(2.2) \qquad N_u^\delta(T - C\delta) \leq N_u(T) \leq N_u^\delta(T + C\delta).$$

We henceforth fix $\delta = T^{-1}$ once and for all. It is possible to choose δ even smaller in our argument, but one gains nothing by doing this. With this choice of T we get from (2.2) and (2.1) that

$$(2.3) \qquad R_u^\delta(T - C\delta) + O(1) \leq R_u(T) \leq R_u^\delta(T + C\delta) + O(1).$$

By using Green's formula and stationary phase [9, Corollary 7.7.15, p. 229], one can show that the Fourier transform of the characteristic function of a convex domain D_u in $\mathbb{R}^2$ has the following asymptotic expansion as $\xi \to \infty$:

$$
\begin{aligned}
\hat{\chi}_{D_u}(\xi) \;=\; & |\xi|^{-3/2}(2\pi)^{1/2}(K(\psi_u^+(\xi))^{-1/2}e^{-i\langle\psi_u^+(\xi),\xi\rangle + \pi i 3/4} \\
& + K(\psi_u^-(\xi))^{-1/2}e^{-i\langle\psi_u^-(\xi),\xi\rangle + \pi i 3/4}) + O(|\xi|^{-5/2}),
\end{aligned}
$$

where $\psi_u^+(\xi)$ and $\psi_u^-(\xi)$ are the points of ∂D_u where the exterior normal is ξ and $-\xi$ respectively, and $K(x)$ is the Gauss curvature at x. As a result

$$R_u^\delta(T) = T^{1/2} \sum_{k \neq 0} \frac{c_\pm e^{iT\langle\psi_u^\pm(k),k\rangle}}{|k|^{3/2}K(\psi_u^\pm(k))^{1/2}}\hat{\rho}(k/T) + O(1).$$

Since all the domains D_u are strictly convex, we have $1 \ll K(x_u^\pm) \ll 1$. Introduce a cut-off function in the u variable, $\phi(u) \in C_0^\infty([-\eta,\eta])$ and put $d\mu(u) = \phi(u)du$. Then

$$
\begin{aligned}
(2.4) \qquad & \int_{-\eta}^{\eta} R_u^\delta(T)\,d\mu(u) \\
& = \sqrt{T} \sum_{k \neq 0} \int_{-\eta}^{\eta} \frac{c_\pm e^{iT\langle\psi_u^\pm(k),k\rangle}}{|k|^{3/2}K(\psi_u^\pm(k))^{1/2}}\hat{\rho}(k/T)\,d\mu(u) + O(1).
\end{aligned}
$$

We would like to apply a stationary phase (with parameters) argument this time in the u-variable to estimate the integral sum in (2.4). To do this, fix $\omega_k = k/|k|$ and note that, since $\psi_u^\pm(k)$ is the inverse Gauss map, it is positive homogeneous of degree zero in the k-variable, i.e., $\psi_u^\pm(k)$ depends only on the direction of k. Thus, we can define $f(u, \omega_k)$ as

$$(2.5) \qquad f(u, \omega_k) = \langle\psi_u^\pm(k), k\rangle/|k|.$$

To decompose the sum in (2.4) we need to consider the solutions of the critical point equation

$$(2.6) \qquad \frac{\partial}{\partial u} f(u, \omega_k) = 0.$$

We claim that near each point $(u^{(0)}, \omega_k^{(0)})$ satisfying (2.6), there exists a locally unique C^∞ solution $u = u(\omega_k)$ to (2.6). To see this, we differentiate $H(u, \psi_u^\pm(\omega)) = 1$ with respect to the u-variable to get

$$(2.7) \qquad \nabla_x H(u, \psi_u^\pm(\omega)) \cdot \frac{\partial \psi_u^\pm(\omega)}{\partial u} + \frac{\partial H}{\partial u}(u, \psi_u^\pm(\omega)) = 0.$$

Since $\nabla_x H(u, \psi_u^\pm(\omega_k))$ is parallel to ω_k, it follows from (2.7) that $\frac{\partial}{\partial u}\langle \psi_u^\pm(\omega_k), \omega_k \rangle = 0$ is equivalent to $\frac{\partial H}{\partial u}(u, \psi_u^\pm(\omega_k)) = 0$. We differentiate (2.7) again in u to get

$$(2.8) \qquad \left\langle \nabla_x^2 H(u, \psi_u^\pm(\omega_k)) \cdot \frac{\partial \psi_u^\pm}{\partial u}(\omega_k), \frac{\partial \psi_u^\pm}{\partial u}(\omega_k) \right\rangle$$
$$+ 2\nabla_x \frac{\partial H}{\partial u}(u, \psi_u^\pm(\omega_k)) \cdot \frac{\partial \psi_u^\pm(\omega_k)}{\partial u}$$
$$+ \frac{\partial^2 \psi_u^\pm}{\partial u^2}(\omega_k) \cdot \nabla_x H(u, \psi_u^\pm(\omega_k)) + \frac{\partial^2 H}{\partial u^2}(u, \psi_u^\pm(\omega_k)) = 0.$$

We rewrite (2.8) as

$$(2.9) \qquad \frac{\partial^2}{\partial u^2} \psi_u^\pm(\omega_k) \cdot \nabla_x H_u(\psi_u^\pm(\omega_k))$$
$$= -(\partial_u \psi_u(\omega_k), 1) \cdot \nabla_{x,u}^2 H(u, \psi_u(\omega_k)) \cdot (\partial_u \psi_u(\omega_k), 1)^t.$$

At points where $\partial_u H(u, \psi_u^\pm(\omega_k) = 0$ the non-degeneracy assumption on the deformation implies that the expression in (2.9) is bounded away from zero. But, by the definition of the map ψ_u,

$$(2.10) \qquad \frac{\partial^2}{\partial u^2} \psi_u^\pm(\omega_k) \cdot \nabla_x H_u(\psi_u^\pm(\omega_k))$$
$$= \pm |\nabla_x H(u, \psi_u^\pm(\omega_k))| \cdot \left\langle \frac{\partial^2}{\partial u^2} \psi_u^\pm(\omega_k), \omega_k \right\rangle.$$

Thus, since by convexity $1 \ll |\nabla_x H| \ll 1$ near $H = 1$, we get from (2.10) and (2.9) that

$$(2.11) \qquad \frac{\partial^2}{\partial u^2} \langle \psi_u^\pm(\omega_k), \omega_k \rangle \gg 1.$$

The nondegeneracy condition (2.11) on the phase function in (2.4) implies that by the implicit function theorem the solution $u = u(\omega_k)$ to (2.6) is locally C^∞. By possibly decomposing the parameter space $u \in [0, 1]$ via a partition of unity, we get C^∞ solutions $u = u_j(\omega_k); j = 1, ..., M$ for some $M \geq 1$. Without loss of generality, we assume that

$M = 1$. The nondegeneracy condition (2.11) allows us to apply stationary phase in (2.4). We actually only need the following simple consequence of the stationary phase lemma (see [**9**, Th. 7.7.5, p. 220]):

Lemma 2.1. *Let $f \in C^\infty(\mathbb{R}; \mathbb{R})$ and $\chi \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(\chi) \subset [0,1]$. Given*

$$I(f) = \int_{\mathbb{R}} e^{iTf(x)} \chi(x)\, dx,$$

we have for $T > 0$ large and some constant $C > 0$,

$$|I(f)| \leq C \frac{(||\chi||_{L^\infty} + ||\nabla\chi||_{L^1}) T^{-1/2}}{\inf\{|\nabla^2 f(x)|^{1/2}, x \in \mathrm{supp}(\chi)\}}.$$

Let $\chi(x) \in C_0^\infty(\mathbb{R})$ be a cut-off function equal to 1 for $|x| \leq \frac{1}{C_0}$ and zero when $|x| \geq \frac{2}{C_0}$ where $C_0 > 0$ is sufficiently large. Then, choosing $N > 0$ large we define the arcs

$$(2.12) \qquad \Omega_j = \left\{ \omega \in S^1, \frac{j-1}{N} \leq \omega \leq \frac{j}{N} \right\}, \quad j = 1, \ldots, N.$$

We write the sum in (2.4) by grouping the $k \in \mathbb{Z}^2$ with the same length and summing first over the lengths of k. We denote this first summation by $\sum_{r=|k|\neq 0}$. First, we decompose the sum:

$$(2.13) \quad T^{1/2} \sum_{r=|k|\neq 0} \sum_{j=1}^{N} \sum_{\omega_k \in \Omega_j} \int_{-\eta}^{\eta} \frac{c_\pm e^{iT|k|\langle \psi_u^\pm(\omega_k), \omega_k \rangle}}{|k|^{3/2} K(\psi_u^\pm(\omega_k))^{1/2}} \hat{\rho}(k/T)\, d\mu(u)$$

$$= T^{1/2} \sum_{r=|k|\neq 0} \sum_{j=1}^{N} \sum_{\omega_k \in \Omega_j} \int_{-\eta}^{\eta} \frac{c_\pm e^{iT|k|\langle \psi_u^\pm(\omega_k), \omega_k \rangle}}{|k|^{3/2} K(\psi_u^\pm(\omega_k))^{1/2}}$$
$$\cdot \chi(u - u(\omega_k))\, \hat{\rho}(k/T)\, d\mu(u)$$

$$+ T^{1/2} \sum_{r=|k|\neq 0} \sum_{j=1}^{N} \sum_{\omega_k \in \Omega_j} \int_{-\eta}^{\eta} \frac{c_\pm e^{iT|k|\langle \psi_u^\pm(\omega_k), \omega_k \rangle}}{|k|^{3/2} K(\psi_u^\pm(\omega_k))^{1/2}}$$
$$\cdot [1 - \chi(u - u(\omega_k))]\, \hat{\rho}(k/T)\, d\mu(u).$$

Then, an application of Lemma 2.1 together with convexity implies that the first integral sum on the right-hand side of (2.13) is bounded by

$$\sum_{0 < r=|k|\leq T} |k|^{-2} \ll \int_1^T \frac{dr}{r} = O(\log T).$$

To estimate the second integral sum on the right-hand side of (2.13), we repeatedly integrate by parts in the u-variable to get

$$\int_{-\eta}^{\eta} \frac{c_\pm e^{iT\langle \psi_u^\pm(k), k \rangle} \phi(u)}{|k|^{3/2} K(\psi_u^\pm(k))^{1/2}} [1 - \chi(u - u(k))]\, \hat{\rho}(k/T)\, du = O(T^{-\infty}).$$

Thus, (2.13) gives

$$\int_{-\eta}^{\eta} R_u^{\delta}(T \pm C\delta) \, du = O(\log T),$$

and since from (2.3) we have that $R_u^{\delta}(T - C\delta) + O(1) \leq R_u(T) \leq R_u^{\delta}(T + C\delta) + O(1)$ as $T \to \infty$, we have proved:

Proposition 2.2. *Under the non-degeneracy assumption* (1.3) *on the deformation,*

$$\int_{-\eta}^{\eta} R_u(T)\phi(u)du = O(\log T) + O(1) = O(\log T).$$

3. Lower spectral bounds for $R_u(t)$

We only need such bounds for a fixed u, say $u = 0$. We follow here a simple argument due to Sarnak [**12**, p. 226]: Consider the quantum Hamiltonian $P = H(\partial_{\theta_1}, \partial_{\theta_2})$ on the 2-torus. Let T_0 be a nonzero period for the Hamilton flow of $p(x, \xi) = H(\xi_1, \xi_2)$ and let $\phi \in S(\mathbb{R})$ satisfy $\phi > 0$ and so that its Fourier transform $\hat{\phi}(T) \in C_0^{\infty}(\mathbb{R})$ contains only the period T_0 in its support. Then, the Duistermaat-Guillemin wave-trace formula [**2**] gives

$$(3.1) \qquad \int_{-\infty}^{\infty} \phi(T - x) \, dN(x) \sim \sum_{j=0}^{\infty} c_j T^{1/2-j}, \quad T \to \infty,$$

with $c_0 \gg 1$. Recall that A is the common area of all D_u. From the Weyl formula we have

$$(3.2) \qquad \int_{-\infty}^{\infty} \phi(T - x) \, dN(x)$$
$$= 2A \int_{-\infty}^{\infty} \phi(T - x) \, x \, dx + \int_{-\infty}^{\infty} \phi(T - x) \, dR(x)$$
$$= \int_{-\infty}^{\infty} \phi'(T - x) \, R(x) \, dx + O(1),$$

since $\hat{\phi}(\xi) = 0$ near $\xi = 0$. Then, by integrating by parts in (3.2) and using that ϕ' is Schwartz and $R(T) = O(T)$ by the Hörmander bound, it follows that

Proposition 3.1.

$$\frac{1}{T} \int_T^{2T} |R(x)| dx \gg T^{1/2}.$$

4. Variance estimates over the deformation

We set $V(T) = \int_{-\eta}^{\eta} |N_u(T) - N_0(T)|^2 du$. Since $A = \text{area}(D_u)$ is constant,

$$(4.1) \quad V(T) = \int_{-\eta}^{\eta} |R_u(T) - R_0(T)|^2 \, du$$

$$= \int_{-\eta}^{\eta} |R_u(T)|^2 \, du + 2\eta |R_0(T)|^2 - 2R_0(T) \cdot \int_{-\eta}^{\eta} R_u(T) du$$

$$\geq 2\eta |R_0(T)| \cdot \left(|R_0(T)| + O\left(\int_{-\eta}^{\eta} R_u(T) du \right) \right).$$

From Proposition 2.2,

$$\int_{-\eta}^{\eta} R_u(T) du = O(\log T).$$

On the other hand, by Proposition 3.1, $|R_0(T)|^2 = \Omega(T)$, and so, from (4.1), we get that

$$(4.2) \qquad V(T) \geq 2\eta |R_0(T)| \cdot [|R_0(T)| + O(\log T)] = \Omega(T).$$

This completes the proof of Theorem 1.2.   q.e.d.

5. The case $n = 3$

We assume now that $D_u, u \in (-\eta, \eta)$ is a non-degenerate deformation of convex domains. We indicate only the differences between the case of $n = 3$ and $n = 2$. Let the mollifier scale be $\delta = \delta(T)$ be chosen later. We make a polar variable decomposition in the summation indices $k \in \mathbb{R}^3$: we write $r = |k|$ and $\omega_k = k/|k| \in S^2$. Then, by exactly the same Poisson summation formula argument as in section 2, Eq. (2.13) we get that for any $a \in \mathbb{R}$,

$$(5.1) \qquad \int_{-\eta}^{\eta} R_u^\delta(aT) \, d\mu(u)$$

$$= T \sum_{r=1}^{\infty} \sum_{j=1}^{N} \sum_{\omega_k \in \Omega_j} \int_{-\eta}^{\eta} \frac{c_\pm e^{iaT|k|\langle \psi_u^\pm(\omega_k), \omega_k \rangle}}{|k|^2 K(\psi_u^\pm(\omega_k))^{1/2}} \hat{\rho}(k\delta) \, d\mu(u)$$

$$= T \sum_{r=1}^{\infty} \sum_{j=1}^{N} \sum_{\omega_k \in \Omega_j} \int_{-\eta}^{\eta} \frac{c_\pm e^{iaT|k|\langle \psi_u^\pm(\omega_k), \omega_k \rangle}}{|k|^2 K(\psi_u^\pm(\omega_k))^{1/2}}$$

$$\cdot \chi(u - u(\omega_k)) \, \hat{\rho}(k\delta) \, d\mu(u)$$

$$+ T \sum_{r=1}^{\infty} \sum_{j=1}^{N} \sum_{\omega_k \in \Omega_j} \int_{-\eta}^{\eta} \frac{c_\pm e^{iaT|k|\langle \psi_u^\pm(\omega_k), \omega_k \rangle}}{|k|^2 K(\psi_u^\pm(\omega_k))^{1/2}}$$

$$\cdot [1 - \chi(u - u(\omega_k))] \, \hat{\rho}(k\delta) \, d\mu(u).$$

Just as in section 2, the last integral in (5.1) is $O(T^{-\infty})$ by repeated integration by parts in the u-variable. We apply stationary phase with parameters, see [**9**, Th. 7.7.6, p 222] to the first integral on the right-hand side of (5.1) and, up to a bounded error, $O(1)$, we get

$$(5.2) \qquad \int_{-\eta}^{\eta} R_u^\delta(aT)\, d\mu(u)$$

$$= \sqrt{T} \sum_{r=1}^{\infty} \sum_{\substack{\omega_k \in \Omega_j \\ j \le N}} c_{\pm} e^{iaTr\langle \psi^{\pm}(u(\omega_k),\omega_k),\omega_k\rangle} \frac{g(r,\omega_k)}{r^{5/2}}$$

where $g(r,\omega) = (K(\psi(u(\omega),\omega) \det(\langle \nabla_u^2 \psi(u(\omega),\omega),\omega\rangle)))^{-1/2} \hat{\rho}(r\delta\omega)$. Let $a \in [\alpha,\beta]$ and form the double variation,

$$(5.3) \qquad V(T) = \int_{\alpha}^{\beta} \int_{-\eta}^{\eta} |N_u(aT) - N_0(aT)|^2\, d\mu(u)\, da.$$

Since by assumption $\mathrm{vol}(D_u) = A$ for all $u \in (-\eta,\eta)$, it follows from the Cauchy-Schwarz inequality that

$$V(T) = \int_{\alpha}^{\beta} \int_{-\eta}^{\eta} |R_u(aT) - R_0(aT)|^2\, d\mu(u)\, da$$

$$\gg \int_{\alpha}^{\beta} |R_0(aT)|^2 da$$

$$+ O\left(\left| \int_{\alpha}^{\beta} |R_0(aT)|^2 da \right|^{1/2} \cdot |M(T)|^{1/2} \right),$$

where

$$(5.4) \qquad M(T) = \int_{\alpha}^{\beta} \left| \int_{-\eta}^{\eta} R_u(aT)\, d\mu(u) \right|^2\, da.$$

Again, just as in the $n = 2$ case, we work with the mollified mean square,

$$(5.5) \qquad M^\delta(T) = \int_{\alpha}^{\beta} \left| \int_{-\eta}^{\eta} R_u^\delta(aT)\, d\mu(u) \right|^2\, da.$$

It follows from (2.3) that

$$(5.6) \qquad M(T) - M^\delta(T) = O(\delta^2 T^4),$$

(recall $R_u(T) \ll T^2$ in $n = 3$). Put

$$(5.7) \qquad \Phi(\omega,\omega',r,r') = r\langle \psi^{\pm}(u(\omega),\omega),\omega\rangle - r'\langle \psi^{\pm}(u(\omega'),\omega'),\omega'\rangle.$$

Then, inserting the estimate (5.2) into (5.5) implies that

(5.8)
$$M^\delta(T) = T \sum_{\substack{r,j,\omega_k \in \Omega_j \\ r',j',\omega_k' \in \Omega_j}} \int_\alpha^\beta c_\pm e^{iaT\,\Phi(\omega,\omega',r,r')} r^{-5/2}(r')^{-5/2} f(r,r',\omega_k,\omega_k') da$$
$$+ O(T^{1/2}\delta^{-1/2}),$$

where

(5.9)
$$f(r,r',\omega,\omega')$$
$$= [K(\psi(u(\omega)),\omega) \det(\langle \nabla_u^2 \psi(u(\omega),\omega),\omega \rangle$$
$$\cdot K(\psi(u(\omega')),\omega') \det(\langle \nabla_u^2 \psi(u(\omega'),\omega'),\omega' \rangle]^{-1/2} \hat{\rho}(r\delta\omega)\hat{\rho}(r'\delta\omega').$$

We split this multiple integral sum up into three pieces. For simplicity we first study the diagonal, although it is actually included in the third piece.

5.0.1. The diagonal. Consider the set

$$\Delta = \{(r,\omega,r',\omega'), r = r' \text{ and } \omega = \omega'\}.$$

Then, since $f(r,r',\omega,\omega') = O(1)$, the sum of the terms in (5.8) with $(r,\omega,r',\omega') \in \Delta$ is bounded by

(5.10)
$$T \sum_{|k|=1}^{1/\delta} |k|^{-5} \ll T \int_1^{1/\delta} r^{-3} dr \ll T.$$

5.0.2. The off-diagonal. Part I. Fix $M > 0$ and consider the set of subindices

$$G = \{(r,\omega,r',\omega'); \Phi(\omega,\omega',r,r') \gg r^M; r,r' \leq 1/\delta\}.$$

Then, by carrying out the integration in the a-variable, this part of the sum in (5.8) is

(5.11)
$$\ll \frac{T}{T} \sum_{(r,\omega,r',\omega') \in G} \frac{1}{r^{5/2+M} r'^{5/2}}$$
$$\ll \int_1^{1/\delta} \int_1^{1/\delta} \frac{1}{r^{1/2+M} r'^{1/2}} \, dr dr' \ll \delta^{M-1}.$$

5.0.3. The off-diagonal. Part II. Here we estimate the piece of the sum coming from the complement of G. This includes the diagonal. This is just the set

$$B = \{(r, \omega, r', \omega'); \; \Phi(\omega, \omega', r, r') \ll r^M; r, r' \leq 1/\delta\}.$$

The main tool in estimating this multiple sum is the implicit function theorem, which allows one to reduce effectively the number of indices in the summation. First we note that $\langle \nabla_u \psi(u(\omega), \omega), \omega \rangle = 0$, and that $\langle \psi(u, \cdot), \cdot \rangle$ is positive, homogeneous of degree one. Thus, the chain rule gives

$$(5.12) \qquad \nabla_\omega \Phi(\omega, \omega', r, r') = r \, \nabla_\omega \langle \psi^\pm(u(\omega), \omega), \omega \rangle \gg r,$$

where, the last estimate is a consequence of Euler homogeneity, since $\omega \cdot \nabla_\omega \langle \psi^\pm(u, \omega), \omega \rangle = \psi^\pm(u, \omega)$ and $\psi^\pm$ takes its values on the boundary of the convex domain, $D_{u(\omega)}$, containing $0 \in \mathbb{R}^3$.

Let r, r' be parameters and consider solutions $(\omega^{(0)}, (\omega')^{(0)})$ of the equation

$$(5.13) \qquad\qquad\qquad \Phi(\omega, \omega'; r, r') = 0.$$

Then, given the lower bound in (5.12) for $\nabla_\omega \Phi$ it suffices to assume that

$$\frac{\partial}{\partial \omega_1} \Phi(\omega, \omega', r, r') \gg r.$$

By the implicit function theorem, locally near the point $(\omega^{(0)}, (\omega')^{(0)})$ there exists a function $F \in C^\infty(\mathbb{R} \times S^2 \times \mathbb{R}^+ \times \mathbb{R}^+)$ such that for all (ω, ω', r, r') satisfying (5.13), we have

$$(5.14) \qquad\qquad\qquad \omega_1 = F(\omega_2, \omega', r, r').$$

The Taylor expansion of F implies that for $(r, \omega, r', \omega') \in B$,

$$(5.15) \qquad\qquad\qquad \omega_1 = F(\omega_2, \omega', r, r') + O(r^{M-1}).$$

Thus, the part of the sum in (5.8) coming from the set B is
(5.16)

$$\ll T \left(\sum_{k \in \mathbb{Z}^3; |k|=1}^{1/\delta} |k|^{-5/2} \right) \left(\sum_{l \in \mathbb{Z}^2; |l|=1}^{1/\delta} |l|^{-5/2+M-1} \right) \ll T \left(\frac{1}{\delta} \right)^{M-1}.$$

Consequently, from (5.16), (5.6), (5.10), and (5.11) it follows that

$$(5.17) \qquad |M(T)| \ll \max\{T, T^{1/2}\delta^{-1/2}, \delta^{M-1}, T\delta^{1-M}, \delta^2 T^4\}.$$

Choosing $M = 3/4$ and $\delta = T^{-2}$ we get

$$(5.18) \qquad\qquad\qquad |M(T)| \ll T^{3/2}.$$

On the other hand, by the same trace formula argument as in section 3 with $n = 3$, one gets

$$\int_\alpha^\beta |R_0(aT)|^2 da = \Omega(T^2).$$

This completes the proof of Theorem 1.2 for $n = 3$. q.e.d.

Remark 5.1. It is likely that Theorem 1.2 yields generic quantitative rigidity results for volume-preserving deformations, $-\Delta_u; u \in I$, of quantum completely integrable Laplacians. These would include surfaces and tori of revolution. We hope to discuss this point in more detail elsewhere.

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