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# TOTAL POSITIVE CURVATURE OF HYPERSURFACES WITH CONVEX BOUNDARY

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## Abstract

We prove that if $\Sigma$ is a compact hypersurface in Euclidean space $\mathbf{R}^n$, its boundary lies on the boundary of a convex body $C$, and meets $C$ orthogonally from the outside, then the total positive curvature of $\Sigma$ is bigger than or equal to half the area of the sphere $\mathbf{S}^{n-1}$. Also, we obtain necessary and sufficient conditions for the equality to hold.

## 1. Introduction

It is well-known that the total positive curvature $\tau^+$ of a smooth closed hypersurfaces in Euclidean space $\mathbf{R}^n$ is bigger than or equal to the area of the sphere $\mathbf{S}^{n-1}$. Further, the case of equality has been extensively studied within the context of the theory of *tight immersions* [**6**, **5**]. Motivated by applications to isoperimetric problems [**10**], we obtain in this paper an analogous sharp inequality for hypersurface whose boundary lies on a convex body, and meets that convex body orthogonally from the outside, as we describe below.

First, we give a general definition for $\tau^+$. Let $\Sigma$ be a compact $\mathcal{C}^0$ hypersurface with boundary $\partial\Sigma$ in $\mathbf{R}^n$, which is $\mathcal{C}^1$-immersed on a neighborhood of $\partial\Sigma$ (it is not required that $\Sigma$ be locally embedded away from $\partial\Sigma$). A hyperplane $\Pi \subset \mathbf{R}^n$ is called a *restricted support hyperplane* of $\Sigma$ at a point $p$, if $p \in \Pi \cap \Sigma$, $\Sigma$ lies on one side of $\Pi$, and $\Pi$ is tangent to $\Sigma$ when $p \in \partial\Sigma$. An *outward normal* of $\Pi$ is a normal vector to $\Pi$ which points towards a side of $\Pi$ not containing $\Sigma$. If $\Pi$ is a restricted support hyperplane for an open neighborhood $U_p$ of $p$ in $\Sigma$, then $\Pi$ is called a *restricted local support hyperplane*; furthermore, $p$ is a *locally strictly convex point* of $\Sigma$, or $p \in \Sigma^+$, provided that $\Pi \cap U_p = \{p\}$. The *total positive curvature* $\tau^+$ of $\Sigma$ is defined as the algebraic area of the unit normals to restricted local support hyperplanes of $\Sigma$ at points of $\Sigma^+$, where by *area*, we mean the $n-1$ dimensional Hausdorff measure.

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Our definition of $\tau^+$ is validated by the fact that when $\Sigma^+$ is $\mathcal{C}^{1,1}$, the outward unit normal vector field $\nu\colon \Sigma^+ \to \mathbf{S}^{n-1}$ is well-defined and Lipschitz continuous; thus, by the *area formula* [**11**, Theorem 3.2.3],

$$\tau^+(\Sigma) = \int_{\Sigma^+} |GK|,$$

where $GK := \det(d\nu)$ is the *Gauss–Kronecker curvature* of $\Sigma$. To state our main result, it only remains to set $\mathbf{c}_n := \mathrm{area}(\mathbf{S}^n)$, and recall that the *inward conormal* at $p \in \partial\Sigma$ is a unit normal vector of $\partial\Sigma$ at $p$ which is tangent to $\Sigma$ and points inside $\Sigma$.

**Theorem 1.1.** *Let $\Sigma$ be a compact $\mathcal{C}^0$ hypersurface in $\mathbf{R}^n$ which is $\mathcal{C}^1$ immersed on a neighborhood of its boundary $\partial\Sigma$. Suppose that $\partial\Sigma$ lies on the boundary of a convex set $C \subset \mathbf{R}^n$, and at each point $p \in \partial\Sigma$, the inward conormal $\sigma(p)$ is an outward unit normal to a support hyperplane of $C$. Then,*

$$\tau^+(\Sigma) \geq \frac{\mathbf{c}_{n-1}}{2}. \tag{1}$$

*Equality holds if and only if*

(i) *$\partial\Sigma$ lies in a hyperplane $\Pi$,*
(ii) *$\sigma(p) \perp \Pi$ for all $p \in \partial\Sigma$,*
(iii) *$\Sigma$ lies strictly on one side of $\Pi$, and*
(iv) *every restricted local support hyperplane of $\Sigma$ at each point of $\Sigma^+$ is a restricted support hyperplane of $\Sigma$.*

Note that when $\partial C$ is $\mathcal{C}^1$, the boundary hypothesis in the above theorem is equivalent to the requirement that $\Sigma$ meet $\partial C$ orthogonally along $\partial\Sigma$, and a neighborhood of $\partial\Sigma$ in $\Sigma$ lie outside of the interior of $C$. Further, when $\Sigma$ is $\mathcal{C}^1$, condition (iv) above may be replaced by the requirement that $\Sigma^+$ lie on the boundary of the convex hull of $\Sigma$.

A pair of surfaces which satisfy conditions (i)–(iv) of Theorem 1.1 are illustrated in Figure 1. The example on the left is a Möbius strip and the other is an annulus with a bridge and a handle attached. Similarly, one may construct surfaces of every topological genus which satisfy conditions (i)–(iv) by adding bridges or handles to an annulus or a Möbius strip. In short, equality in (1) does not restrict the topology of $\Sigma$ or force it to be embedded.

As we mentioned earlier, the above theorem mirrors well-known results for closed hypersurfaces. In particular, recall that when $\Sigma$ is closed, $\tau^+(\Sigma) \geq \mathbf{c}_{n-1}$, because for almost any $u \in \mathbf{S}^{n-1}$, $\Sigma$ has a strict support hyperplane with outward unit normal $u$ [**18**, Theorem 2.2.9]. Further, it is easy to see that $\tau^+(\Sigma) = \mathbf{c}_{n-1}$ if and only if every local support hyperplane of $\Sigma$ at each point of $\Sigma^+$ is a support hyperplane of $\Sigma$. Such surfaces are called *0-tight*; they satisfy Banchoff's two-piece-property (TPP) [**4**], and, when they are smooth, have minimal total absolute curvature $\int_\Sigma |GK|$ as studied by Chern and Lashof [**7**, **8**], Kuiper [**13**, **14**],

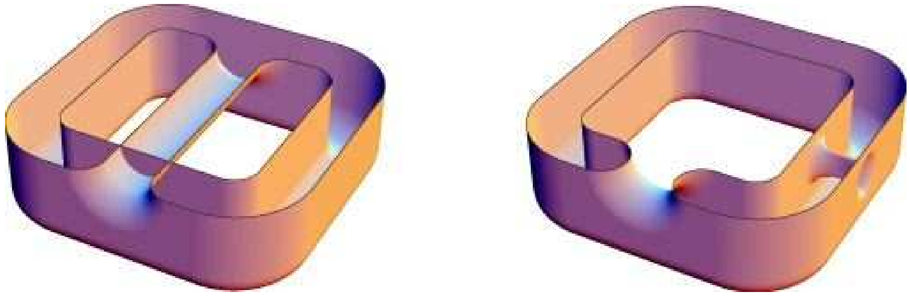

**Figure 1.**

and others [**6**, **5**]. In particular, we should mention papers of Rodriguez [**17**] and Kühnel [**12**] where they prove that (in contrast to the examples illustrated in Figure 1) a surface with boundary and TPP lies embedded on the boundary of its convex hull, and therefore has restricted topology. The earliest study of closed surfaces with $\tau^+ = \mathbf{c}_{n-1}$ is due to Alexandrov [**3**], see Nirenberg [**16**].

The prime motivation for this work, however, stems from applications to isoperimetric problems. In particular, Theorem 1.1 is used in [**10**], to show that the area of a hypersurface $\Sigma$ which traps a given volume outside of a convex body in $\mathbf{R}^n$ must be greater than or equal to the area of a hemisphere trapping the given volume on one side of a hyperplane, and equality holds only when $\Sigma$ is itself a hemisphere. See also [**9**], for a generalization of this result to Cartan–Hadamard 3-manifolds. Other recent results on the structure of hypersurfaces whose boundary lies on a convex body have been obtained in [**1**, **2**]; also see [**15**].

The proof of Theorem 1.1 presented here is based on successive generalizations of the simple observation that if $X \subset \mathbf{S}^{n-1}$ is any *convex spherical set*, then the intersection of $X$ with any hemisphere centered at a point of $X$ contains at least half of $X$. This fact is proved in Section 3, and is then extended to a result for *normal cones* of finite sets in Section 4. The latter result is used in turn to prove a still more general proposition for support cones of general sets in Section 5. Applying the last result to $\partial\Sigma$ and its conormal vector field leads to the proof of Theorem 1.1 in Section 6.

In the appendix, we discuss a relatively short analytic proof of inequality (1) when $\partial\Sigma$ is $\mathcal{C}^2$.

**Note 1.2.** Inequality (1) has an easy proof when $C$ is a sphere. Indeed, in this case, it can be shown that for every $u \in \mathbf{S}^{n-1}$, $\Sigma$ has a restricted support hyperplane which is orthogonal to $u$. To see this, let $\widetilde{\Sigma}$ be the surface obtained from $\Sigma$ by connecting all points of $\partial\Sigma$ to the center $o$ of the sphere. Then, $\widetilde{\Sigma}$ is $\mathcal{C}^1$ immersed near $\partial\Sigma$. For $u \in \mathbf{S}^{n-1}$, let $h_u \colon \widetilde{\Sigma} \to \mathbf{R}$ be the height function $h_u(\cdot) := \langle \cdot, u \rangle$. Note that since $\widetilde{\Sigma}$ is a closed $\mathcal{C}^0$-immersed hypersurface, it does not lie entirely in a hyperplane, by the theorem on invariance of domain. Thus, for every $u \in \mathbf{S}^{n-1}$, $h_u$ has a maximum point and a minimum point on $\widetilde{\Sigma}$ which are distinct. In particular, at least one of these extremum points, which we denote by $v_u$, must be different from $o$. So, either $v_u \in \Sigma$, or $v_u \in \widetilde{\Sigma} - \{o\} - \Sigma$. In the former case, $T_{v_u}\Sigma$ is orthogonal to $u$. In the latter case, $v_u$ lies in the interior of a line segment $oq$ for some $q \in \Sigma$. Thus, since $T_{v_u}\widetilde{\Sigma}$ is a support hyperplane of $\widetilde{\Sigma}$, it follows that $T_{v_u}\widetilde{\Sigma}$ is tangent to $\Sigma$ at $q$. So $T_q\Sigma = T_{v_u}\widetilde{\Sigma}$ is the desired hyperplane.

**Note 1.3.** Unlike the case where $C$ is a sphere, which was addressed in Note 1.2, there are surfaces which satisfy the hypothesis of Theorem 1.1, but do not have restricted support hyperplanes orthogonal to every direction. See Figure 2 for one such surface whose boundary lies on a cylinder.

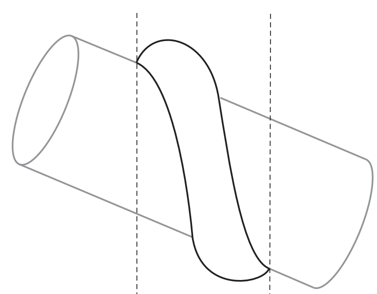

**Figure 2.**

**Note 1.4.** Inequality (1) is an easy consequence of the Gauss–Bonnet theorem when $n = 3$, $\Sigma$ is homeomorphic to a disk, and $C$ has $\mathcal{C}^2$ positively curved boundary. To see this, let $\gamma\colon (-\epsilon, \epsilon) \to \partial\Sigma$ be a local parametrization of $\partial\Sigma$ with $\gamma(0) = p$ and $\|\gamma'\| = 1$. Let $\nu(p)$ be the unit normal to $\partial C$, which is parallel to the mean curvature vector of $\partial C$ at $p$, and $\sigma(p)$ be the inward conormal of $\partial\Sigma$ at $p$. Then, the geodesic curvature of $\partial\Sigma$ at $p$ is given by

$$\kappa_g(p) = \big\langle \gamma''(0), \sigma(p) \big\rangle = \big\langle \gamma''(0), -\nu(p) \big\rangle = -\,\mathrm{II}_p\left(\gamma'(0), \gamma'(0)\right),$$

where $\mathrm{II}_p$ is the second fundamental form of $\partial C$ with respect to $\nu(p)$. Since $\partial C$ has positive curvature at $p$, and $\nu(p)$ is parallel to the mean curvature vector, $\mathrm{II}_p$ is positive definite. So $\kappa_g < 0$, and consequently

$$\int_{\Sigma^+} GK \geq \int_{\Sigma} GK = 2\pi\chi(\Sigma) - \int_{\partial\Sigma} \kappa_g \geq 2\pi\chi(\Sigma) = 2\pi.$$

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## 3. Convex Spherical Sets

We say that a subset $X \subset \mathbf{S}^{n-1}$ is *convex* if every pair of points of $X$ may be joined by a distance minimizing geodesic which lies in $X$. For every $u \in \mathbf{S}^{n-1}$, we define the (closed) *hemisphere centered at* $u$ as

$$\mathbf{H}_u := \left\{ p \in \mathbf{S}^{n-1} \mid \langle p, u \rangle \geq 0 \right\}.$$

The *distance* between any pairs of sets $X$, $Y \subset \mathbf{R}^n$ is given by

$$\operatorname{dist}(X, Y) := \inf \left\{ \|x - y\| \mid x \in X, y \in Y \right\}.$$

If $p \in \mathbf{R}^n$, we adopt the common convention $\operatorname{dist}(X, p) := \operatorname{dist}(X, \{p\})$.

**Proposition 3.1.** *Let $X \subset \mathbf{S}^{n-1}$ be a closed convex set with interior points and $u \in X$. Then,*

$$\operatorname{area}(X \cap \mathbf{H}_u) \geq \frac{1}{2} \operatorname{area}(X). \tag{2}$$

*Equality holds if and only if $-u \in X$. Further, for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\text{if} \quad \operatorname{area}(X \cap \mathbf{H}_u) \leq \left(\frac{1}{2} + \delta\right) \operatorname{area}(X), \quad \text{then} \quad \operatorname{dist}(X, -u) \leq \epsilon. \tag{3}$$

*Proof.* Let $A$ consist of all geodesic segments connecting $u$ to the points of $X \cap \partial\mathbf{H}_u$, where $\partial\mathbf{H}_u$ is the set of points of $\mathbf{S}^{n-1}$ which are orthogonal to $u$, and let $B$ be the complement of $A$ in $\mathbf{H}_u$. Further, let $A'$ and $B'$ be the reflections of $A$ and $B$ with respect to the hyperplane which is orthogonal to $u$ and passes through the origin. Since $A \subset X$,

$$\operatorname{area}(X \cap A) = \operatorname{area}(A) = \operatorname{area}(A') \geq \operatorname{area}(X \cap A').$$

Further, note that if $B'$ contains any point $p$ of $X$, then the geodesic connecting $p$ to $u$ belongs to $X$ and crosses $\partial\mathbf{H}_u$ at a point $x$. But this would imply that the geodesic $ux$ belongs to $A$, which can happen only if $p \in A'$. Thus,

$$\operatorname{area}(X \cap B') = 0.$$

So, it follows that

$$\begin{aligned} \operatorname{area}(X \cap \mathbf{H}_u) &= \operatorname{area}(X \cap A) + \operatorname{area}(X \cap B) \\ &\geq \operatorname{area}(X \cap A') + \operatorname{area}(X \cap B') \\ &= \operatorname{area}(X \cap \mathbf{H}_{-u}), \end{aligned}$$

which establishes the desired inequality (2).

Now, suppose that equality holds in (2). Then, the first and the last quantities in the above expression are equal. So, the intermediate quantities must be equal as well. Thus, we have

$$\begin{aligned} \operatorname{area}(X \cap A) &\leq \operatorname{area}(X \cap A) + \operatorname{area}(X \cap B) \\ &= \operatorname{area}(X \cap A') + \operatorname{area}(X \cap B') \\ &= \operatorname{area}(X \cap A'). \end{aligned}$$

So, it follows that

$$\operatorname{area}(X \cap A') = \operatorname{area}(X \cap A) = \operatorname{area}(A) = \operatorname{area}(A').$$

Since $A'$ and $X$ are both closed convex sets, the last equality above yields that $X \cap A' = A'$. In particular, $-u \in X$.

Conversely, if $-u \in X$, then the convexity of $X$ implies that $X \cap A' = A'$. Furthermore, if $p \in X \cap B$, then by convexity of $X$, the geodesic $p(-u)$ must also be contained in $X$. But, since $p(-u)$ lies partly in $B'$, that would imply that $X \cap B' \neq \emptyset$, which is a contradiction. So, $X \cap B = \emptyset$. Since $X \cap A = A$, we conclude then that

$$\operatorname{area}(X \cap \mathbf{H}_u) = \operatorname{area}(A) = \operatorname{area}(A') = \operatorname{area}(X \cap \mathbf{H}_{-u}).$$

So, equality holds in (2).

Finally, note that if left-hand side of (3) holds, then

$$\operatorname{area}(X \cap \mathbf{H}_{-u}) \geq \left(\frac{1}{2} - \delta\right) \operatorname{area}(X),$$

which yields

$$\begin{aligned} 2\delta \operatorname{area}(X) &\geq \operatorname{area}(X \cap \mathbf{H}_u) - \operatorname{area}(X \cap \mathbf{H}_{-u}) \\ &= \operatorname{area}(X \cap A) + \operatorname{area}(X \cap B) - \operatorname{area}(X \cap A') \\ &\geq \operatorname{area}(X \cap A) - \operatorname{area}(X \cap A') \\ &= \operatorname{area}(A') - \operatorname{area}(X \cap A'). \end{aligned}$$

In particular, if $B^n_\epsilon(-u)$ denotes the $n$-dimensional closed ball of radius $\epsilon$ centered at $-u$, and we set

$$\delta \leq \frac{\operatorname{area}\left(B^n_\epsilon(-u) \cap A'\right)}{2 \operatorname{area}(X)},$$

it follows that

$$\operatorname{area}(X \cap A') \geq \operatorname{area}(A') - \operatorname{area}\left(B^n_\epsilon(-u) \cap A'\right).$$

So $X \cap B^n_\epsilon(-u) \neq \emptyset$, which yields $\operatorname{dist}(X, -u) \leq \epsilon$, as desired. q.e.d.

**Note 3.2.** The proof of Proposition 3.1 shows that if $X \subset \mathbf{S}^{n-1}$ is *any* convex spherical set of Hausdorff dimension $d$, then

$$\mathcal{H}^d(X \cap \mathbf{H}_u) \geq \frac{1}{2}\mathcal{H}^d(X),$$

where $\mathcal{H}^d$ is the $d$-dimensional Hausdorff measure, and again equality holds if and only if $-u \in X$.

## 4. Restricted Normal Cones of Finite Sets

For any subset $X \subset \mathbf{R}^n$ and point $p \in \mathbf{R}^n$, the (unit) *normal cone* of $X$ at $p$ is defined as

$$N_pX := \left\{\, u \in \mathbf{S}^{n-1} \mid \langle x - p, u\rangle \leq 0,\ \forall x \in X \,\right\},$$

i.e., the set of outward unit normals to support hyperplanes of $X \cup \{p\}$ at $p$. We also set

$$NX := \underset{p\in X}{\cup} N_pX.$$

**Lemma 4.1.** *For any set $X \subset \mathbf{R}^n$, and point $p \in \mathbf{R}^n$, $N_pX$ is either a convex spherical set or consists exactly of a pair of antipodal points.*

*Proof.* Let $u_0$, $u_1 \in N_pX$. If $u_0 \neq -u_1$, then the geodesic segment between $u_0$ and $u_1$ may be parametrized by

$$u(\lambda) := \frac{(1-\lambda)u_0 + \lambda u_1}{\|(1-\lambda)u_0 + \lambda u_1\|},$$

where $\lambda \in [0, 1]$. Since $\langle x - p, u_0\rangle \leq 0$ and $\langle x - p, u_1\rangle \leq 0$ for all $x \in X$, it follows that $\langle x - p, u(\lambda)\rangle \leq 0$ as well, which yields that $u(\lambda) \in N_pX$.

If $u_0 = -u_1$, and $N_pX$ contains no other points, then we are done. Otherwise, let $x$ be a point of $N_pX$ distinct from $u_0$ and $u_1$. Then, $N_pX$ contains the geodesic segments $u_0x$ and $xu_1$. Let $\Pi$ be the two dimensional plane spanned by $u_0$ and $x$. Then, $u_0x$ and $xu_1$ both lie on $\Pi$, since $\Pi$ is a plane of symmetry of $\mathbf{S}^{n-1}$ and geodesics of length less than $\pi$ are unique in $\mathbf{S}^{n-1}$. Thus, $u_0x \cup xu_1$ is a geodesic, and so we conclude that $N_pX$ contains a geodesic connecting $u_0$ and $u_1$. q.e.d.

For any subset $X \subset \mathbf{R}^n$ and mapping $\sigma\colon X \to \mathbf{S}^{n-1}$, we define the *restricted normal cone* of $X$ at $p$ with respect to $\sigma$ as

$$N_pX/\sigma := N_pX \cap \mathbf{H}_{\sigma(p)},$$

and set

$$NX/\sigma := \underset{p\in X}{\cup} N_pX/\sigma.$$

We say that a point $p \in X$ is *exposed* provided that there passes through $p$ a support hyperplane $\Pi$ of $X$ such that $\Pi \cap X = \{p\}$. The set of exposed points of $X$ is denoted by $X^E$. The *width* of a subset $X \subset \mathbf{R}^n$ is the distance between the closest pairs of parallel hyperplanes which contain $X$ in between them.

**Proposition 4.2.** *Let $X := \{x_1, \dots, x_k\} \subset \mathbf{R}^n$ be a finite set which lies on the boundary of a convex body. Choose $\sigma(x_i) \in N_{x_i}X$. Then,*

$$\operatorname{area}\big(NX/\sigma\big) \geq \frac{\mathbf{c}_{n-1}}{2}. \tag{4}$$

*Equality holds if and only if $X$ lies in a hyperplane $\Pi$, and $\sigma(x_i)$ is orthogonal to $\Pi$ for all $x_i \in X^E$. Further, for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\text{if} \quad \operatorname{area}(NX/\sigma) \leq \left(\frac{1}{2} + \delta\right)\mathbf{c}_{n-1}, \quad \text{then} \quad \operatorname{width}(X) \leq \epsilon. \tag{5}$$

*Proof.* First note that, since $X$ is compact, for every $u \in \mathbf{S}^{n-1}$, the height function $\langle \cdot, u\rangle$ has a maximum point in $X$, which means that $X$ has a support hyperplane with outward normal $u$. Thus,

$$NX = \mathbf{S}^{n-1}.$$

Further, a point $u \in \mathbf{S}^{n-1}$ belongs to the interior of some $N_{x_i}X$, as a subset of $\mathbf{S}^{n-1}$, if and only if there exists a support hyperplane of $X$ at $x_i$ with outward normal $u$ which intersects $X$ only at $x_i$. Thus,

$$\operatorname{int}_{\mathbf{S}^{n-1}}\big(N_{x_i}X\big) \cap \operatorname{int}_{\mathbf{S}^{n-1}}\big(N_{x_j}X\big) = \emptyset,$$

for all $i \neq j$. Since, by Lemma 4.1, each $N_{x_i}X$ with non-vanishing area is a convex spherical set, and X is a finite set, Proposition 3.1 together with the two equalities displayed above yields that

$$\operatorname{area}\big(NX/\sigma\big) = \sum_i \operatorname{area}\big(N_{x_i}X/\sigma\big) \geq \sum_i \frac{1}{2}\operatorname{area}\big(N_{x_i}X\big) = \frac{\mathbf{c}_{n-1}}{2}.$$

Now, suppose that equality holds in (4). Then, the middle two quantities in the above expression are equal. This together with Proposition 3.1 yields that

$$\operatorname{area}\big(N_{x_i}X/\sigma\big) = \frac{1}{2}\operatorname{area}\big(N_{x_i}X\big),$$

whenever $N_{x_i}X$ has interior points. Since $X$ is finite, this can happen if and only if $x_i \in X^E$. So, again by Proposition 3.1, $N_{x_i}X$ contains a pair of antipodal points $\pm\sigma(x_i)$ for all $x_i \in X^E$. This yields that $X$ lies in a hyperplane orthogonal to $\pm\sigma(x_i)$.

Conversely, suppose that $X$ lies in a hyperplane and $\sigma(x_i)$ are orthogonal to that hyperplane for all $x_i \in X^E$. Then $-\sigma(x_i) \in N_{x_i}X$, for all $x_i \in X^E$. Thus, by Proposition 3.1, the above equality holds for all $x_i \in X^E$, which yields that equality holds in (4).

Finally, suppose that the left-hand side of (5) holds. Then, since $\operatorname{area}\big(NX/\sigma\big)$ is the sum of $\operatorname{area}\big(N_{x_i}X/\sigma\big)$, which have disjoint interiors, there must exist an $i$ such that

$$\operatorname{area}\big(N_{x_i}X/\sigma\big) \leq \left(\frac{1}{2} + \delta\right)\operatorname{area}\big(N_{x_i}X\big).$$

In particular, by Proposition 3.1, we may choose $\delta$ so small that, for any $\epsilon > 0$,

$$\operatorname{dist}\big(N_{x_i}X, -\sigma(x_i)\big) \leq 2\sin\left(\frac{\epsilon}{2\operatorname{diam}(X)}\right),$$

for some $i$, where $\operatorname{diam}(X)$ denotes the distance between the farthest points of $X$. This implies that there exists an element $\widetilde{\sigma}_i \in N_{x_i}X$ such that the angle between $-\sigma(x_i)$ and $\widetilde{\sigma}_i$ is less than or equal to $\epsilon/\operatorname{diam}(X)$. Consequently, the angle of the 'wedge' containing $X$ generated by the support hyperplanes of $X$ at $x_i$, with outward unit normals $\sigma(x_i)$ and $\widetilde{\sigma}_i$, is less than or equal to $\epsilon/\operatorname{diam}(X)$. So,

$$\operatorname{width}(X) \leq \frac{\epsilon}{\operatorname{diam}(X)} \cdot \operatorname{diam}(X) = \epsilon,$$

as desired. q.e.d.

## 5. Restricted Normal Cones of General Sets

For any subset $X \subset \mathbf{R}^n$, let $B_r(X)$ denote the union of all closed balls of radius $r$ centered at points of $X$. The *Hausdorff distance* between any pairs of subsets $X$, $Y$ of $\mathbf{R}^n$ is defined as

$$\operatorname{dist}_H(X, Y) := \inf\big\{\, r \geq 0 \mid X \subset B_r(Y) \text{ and } Y \subset B_r(X) \,\big\}.$$

We say that a sequence of sets $X_i \subset \mathbf{R}^n$ converges to $X \subset \mathbf{R}^n$, and write $\lim_{i\to\infty} X_i = X$, provided that for every $\epsilon > 0$, there exists an integer $k$ such that $\operatorname{dist}_H(X_i, X) \leq \epsilon$ whenever $i \geq k$.

**Lemma 5.1.** *Let $X \subset \mathbf{R}^n$ be compact, and $p_i \in \mathbf{R}^n$ be a sequence of points which converges to a point $p \in \mathbf{R}^n$. Then,* $\lim_{i\to\infty} N_{p_i}(X) \subset N_p(X)$*. Further, if $p \in \mathbf{R}^n - X$, then* $\lim_{i\to\infty} N_{p_i}(X) = N_p(X)$*.*

*Proof.* Since $X$ is compact, the set of hyperplanes in $\mathbf{R}^n$ with respect to which $X$ lies on both sides or are disjoint from $X$ is open. This implies that the set of support hyperplanes of $X$ are closed. Thus, $\lim_{i\to\infty} N_{p_i}(X) \subset N_p(X)$.

Now, suppose that $p \in \mathbf{R}^n - X$. Then, $N_p(X)$ has non-empty interior (as a subset of $\mathbf{S}^{n-1}$). Let $u \in \operatorname{int}_{\mathbf{S}^{n-1}}(N_p(X))$, and $\Pi$ be the hyperplane through $p$ and orthogonal to $u$. Then, $\Pi \cap X = \emptyset$, and consequently, since $X$ is compact, $\operatorname{dist}(X, \Pi) > 0$. In particular, we may choose $i$ so large that the $\operatorname{dist}(p_i, \Pi) < \operatorname{dist}(p_i, X)$. Then, the hyperplane $\Pi_i$ which passes through $p_i$ and is orthogonal to $u$ has $X$ entirely on one side. Thus, $u \in N_{p_i}X$ for $i$ sufficiently large, and so we conclude that

$$\operatorname{int}_{\mathbf{S}^{n-1}}\big(N_p(X)\big) \subset \lim_{i\to\infty} N_{p_i}(X).$$

But $\lim_{i\to\infty} N_{p_i}(X)$ is closed, because the space of compact subsets of $\mathbf{R}^n$ is locally compact with respect to the Hausdorff metric [**18**, Theorem 1.8.4]. So, $N_p(X) \subset \lim_{i\to\infty} N_{p_i}(X)$. q.e.d.

**Lemma 5.2.** *Let $X \subset \mathbf{R}^n$ be compact. Then, except for a set of zero area, every $u \in \mathbf{S}^{n-1}$ is the outward normal to a support hyperplane of $X$ which intersects $X$ only at a single point.*

*Proof.* See [**18**, Theorem 2.2.9]. q.e.d.

**Proposition 5.3.** *Let $X \subset \mathbf{R}^n$ be a compact set which is disjoint from the relative interior of its convex hull. Suppose there exists a continuous mapping $\sigma \colon X \to \mathbf{S}^{n-1}$ such that $\sigma(p) \in N_pX$ for all $p \in X$. Then,*

$$\text{area}\left(NX/\sigma\right) \geq \frac{\mathbf{c}_{n-1}}{2}. \tag{6}$$

*Equality holds if and only if $X$ lies in a hyperplane $\Pi$, and $\sigma(p)$ is orthogonal to $\Pi$ for all $p \in X^E$.*

*Proof.* First, we show that $NX/\sigma$ is closed. To see this, note that, by Lemma 5.1, if $x_i$ is a sequence of points of $X$ which converges to $x$, then the limit of $N_{x_i}X$ is a subset of $N_xX$. Further, since $\sigma$ is continuous, the hemispheres $\mathbf{H}_{\sigma(x_i)}$ converge to $\mathbf{H}_{\sigma(x)}$. So, the limit of $N_{x_i}X/\sigma$ is a subset of $N_xX/\sigma$. Now, suppose that we have a sequence of elements $u_i \in NX/\sigma$ which converges to a point $u \in \mathbf{S}^{n-1}$. Then $u_i \in N_{x_i}X/\sigma$, for some $x_i \in X$. Since $X$ is compact, $x_i$ has an accumulation point $x \in X$. Consequently, as we just argued, the limit of $N_{x_i}X/\sigma$ lies in $N_xX/\sigma$. So $u \in N_xX/\sigma$, and we conclude that $NX/\sigma$ is closed.

Next note that, since $X$ is bounded, for any $i = 1, 2, \ldots$, we may cover it by finitely many balls in $\mathbf{R}^n$ of radius $1/i$ centered at points of $X$. Let $X_i$ be the set of the centers of these balls. As $i \to \infty$, $X_i$ converges to $X$ with respect to the Hausdorff metric, consequently, for any $p \in X$, $N_pX_i$ converges to $N_pX$. We claim that, since $X$ is compact, for every $\delta > 0$, there exists $k > 0$ such that for all $i \geq k$, $N_pX_i$ is within a Hausdorff distance $\delta$ of $N_pX$ for all $p \in X$.

To establish this claim note that, since by assumption $X$ is disjoint from the relative interior of its convex hull, there exists for every $i$, a convex set $\overline{X}_i$ such that $\overline{X}_i \subset \text{conv}(X_i)$, $\overline{X}_i \cap X = \emptyset$, and $\text{dist}(\overline{X}_i, X) \leq 2/i$. Further, after passing to a subsequence, we may assume that $\overline{X}_i \subset \overline{X}_{i+1}$. For every $i$, define $f_i \colon X \to \mathbf{R}$ by

$$f_i(p) := \text{dist}_H\left(N_pX, N_p\overline{X}_i\right).$$

Since $p \in X$ and $\overline{X}_i \cap X = \emptyset$, the mapping $p \mapsto N_p\overline{X}_i$ is continuous, by Lemma 5.1, with respect to the Hausdorff metric. Further, recall that, again by Lemma 5.1, that if $p_k$ converges to $p$, then the limit of $N_{p_k}X$ is a subset of $N_pX$. Thus, $f_i$ is lower semicontinuous. Consequently, since $X$ is compact, $f_i$ achieves its supremum on $X$, i.e., there exists $p_i \in X$ such that $\sup(f_i) = f_i(p_i)$. But $f_{i+1}(p_{i+1}) \leq f_i(p_{i+1})$, because $N_p\overline{X}_i \subset N_p\overline{X}_{i+1} \subset N_pX$, since $\overline{X}_i \subset \overline{X}_{i+1} \subset \text{conv}(X)$. Thus, $\sup(f_i)$

is a decreasing sequence:

$$\sup(f_{i+1}) = f_{i+1}(p_{i+1}) \leq f_i(p_{i+1}) \leq \sup(f_i).$$

So, since $\sup(f_i) > 0$, $\lim_{i\to\infty} \sup(f_i)$ exists. Since $\overline{X}_i \to X$, this limit must be zero. This proves the claim, because, since $\overline{X}_i \subset \mathrm{conv}(X_i)$, we have $f_i(p) \geq \mathrm{dist}_H\left(N_pX, N_pX_i\right)$.

Since $\sigma$ is continuous, it follows that, for any $\delta > 0$, $N_pX/\sigma$ is within a (Hausdorff) distance $\delta$ of $N_pX_i/\sigma$ for all $p \in X$, provided that $i$ is sufficiently large. This yields that $NX/\sigma$ is within an arbitrarily small $\delta$ distance of $NX_i/\sigma$, once $i$ is large.

Now, suppose towards a contradiction that the area of $NX/\sigma$ is less than $\mathbf{c}_{n-1}/2$. Then, the area of the complement of $NX/\sigma$ is bigger than $\mathbf{c}_{n-1}/2$. Since $NX/\sigma$ is closed, its complement is open, and therefore, the complement contains a compact subset, say $A$, whose area is also bigger than $\mathbf{c}_{n-1}/2$. Since $A$ is at a finite distance away from $NX/\sigma$, by the above discussion, it is disjoint from $NX_i/\sigma$ as well once $i$ is sufficiently large; therefore, $NX_i/\sigma$ has area less than $\mathbf{c}_{n-1}/2$. But since $X_i$ is a finite set, by Proposition 4.2, the area of $NX_i/\sigma$ is at least $\mathbf{c}_{n-1}/2$, and we have our contradiction.

Next, suppose that equality holds in (6). Then, choosing $i$ large enough, we can make sure that the area of $NX_i/\sigma$ is as close to $\mathbf{c}_{n-1}/2$ as desired. So, by Proposition 4.2, the upper bound for the width of $X_i$ becomes arbitrarily small as $i$ grows large. But

$$\mathrm{width}(X) \leq \mathrm{width}(X_i) + \frac{2}{i}.$$

So, we conclude that $X$ lies in a hyperplane.

Finally, we show that for all $p \in X^E$, $\sigma(p)$ is orthogonal to the hyperplane, say $\Pi$, which contains $X$. To see this, suppose that $\Pi$ is the set of points in $\mathbf{R}^n$ whose $n^{th}$ coordinate is zero. Let $e_n := (0, 0, \ldots, 1)$ denote the 'north pole' of $\mathbf{S}^{n-1}$, and $A \subset X$ be the set of points $p$ where $\langle \sigma(p), e_n\rangle < 0$. Define $\overline{\sigma}\colon X \to \mathbf{S}^{n-1}$, by $\overline{\sigma}(p) = \sigma(p)$ if $p \in X - A$, and let $\overline{\sigma}(p)$ be the reflection of $\sigma(p)$ with respect to $\Pi$ otherwise. Note that $NA/\overline{\sigma}$ is the reflection of $NA/\sigma$ with respect to $\Pi$. Thus,

$$\mathrm{area}(NX/\overline{\sigma}) = \mathrm{area}(NX/\sigma) = \frac{\mathbf{c}_{n-1}}{2}.$$

Let $u \in \mathbf{H}_{e_n}$ and $p$ be a maximum point of the height function $\langle \cdot, u\rangle$. Then, $u \in N_pX$. So $u \in N_pX \cap \mathbf{H}_{e_n}$. But $\overline{\sigma}(p) \in N_pX \cap \mathbf{H}_{e_n}$ as well. This yields that $\langle u, \overline{\sigma}(p)\rangle \geq 0$, because, since $X \subset \Pi$, $\{e_n, -e_n\} \subset N_p(X)$; consequently, either $N_pX = \{e_n, -e_n\}$ or $N_p(X)$ is a 'lune' with vertices at $e_n$ and $e_{-n}$, i.e., $N_pX$ is the intersection of two (closed) hemispheres the boundaries of which pass through $e_n$ and $-e_n$. So, we conclude that $u \in N_pX/\overline{\sigma}$, which yields

$$\mathbf{H}_{e_n} \subset NX/\overline{\sigma}.$$

But area $\big(NX/\overline{\sigma}\big) = \mathbf{c}_{n-1}/2$. So $NX/\overline{\sigma} \subset \mathbf{H}_{e_n}$ except for a subset of area 0. In particular, $NX^E/\overline{\sigma} \subset \mathbf{H}_{e_n}$ except for a subset of area 0. But if there exists a point $u \in NX^E/\overline{\sigma}$ such that $u \notin \mathbf{H}_{e_n}$, then since $\mathbf{H}_{e_n}$ is closed, $\overline{\sigma}$ is continuous, and $NX^E$ is dense in $\mathbf{S}^{n-1}$, it follows that there exists an open neighborhood $U$ of $u$ in $NX^E$ which is disjoint from $\mathbf{H}_{e_n}$. But almost every point of $\mathbf{S}^{n-1}$ belongs to $NX^E$, thus $U$ has non-zero area, which is a contradiction. So, it follows that

$$NX^E/\overline{\sigma} \subset \mathbf{H}_{e_n}.$$

In particular, for all $p \in X^E$, $N_pX/\overline{\sigma} \subset \mathbf{H}_{e_n}$, which can happen only if $\overline{\sigma}(p) = e_n$. So, we conclude that, when equality holds in (6), $\sigma(p) = \pm e_n$, i.e., $\sigma(p)$ is orthogonal to $\Pi$ for all $p \in X^E$.

Conversely, suppose that $X$ lies in a hyperplane $\Pi$ and $\sigma(p)$ is orthogonal to $\Pi$ for all $p \in X^E$. Then, we claim the equality holds in (6). To see this, first note that, by Lemma 5.2, area$(NX^E/\sigma) =$ area$(NX/\sigma)$. Let $\overline{\sigma}(p) := \sigma(p)$ if $\sigma(p) = e_n$, and $\overline{\sigma}(p) := -\sigma(p)$ otherwise. Then, area$(NX^E/\sigma) =$ area$(NX^E/\overline{\sigma})$. Next, recall that, as we argued above, $N_pX^E$ is a lune with vertices at $\pm e_n$. Thus, $N_pX^E/\overline{\sigma}(p) = N_pX^E \cap \mathbf{H}_{e_n}$. So, $NX^E/\overline{\sigma} = NX^E \cap \mathbf{H}_{e_n} = \mathbf{H}_{e_n}$, which yields that area$(NX^E/\overline{\sigma}) = \mathbf{c}_{n-1}/2$. q.e.d.

## 6. Proof of Theorem 1.1

**6.1. The inequality.** Let $RN\Sigma$ denote the set of outward unit normals to restricted support hyperplanes of $\Sigma$. By Lemma 5.2, almost every element of $RN\Sigma$ is an outward normal to a support hyperplane of $\Sigma$ which intersects $\Sigma$ at a point of $\Sigma^+$. Thus,

$$\tau^+(\Sigma) \geq \text{area}\,\big(RN\Sigma\big). \tag{7}$$

By assumption, $\sigma(p) \in N_p\partial\Sigma$ for all $p \in \partial\Sigma$. Thus, if $u \in N_p\partial\Sigma/\sigma$, the height function $\langle \cdot, u\rangle$ either has a maximum point in the interior of $\Sigma$, or $u \perp \sigma(p)$. In either case, $u \in RN\Sigma$, which yields that $N\partial\Sigma/\sigma \subset RN\Sigma$.

Thus, by Proposition 5.3,

$$\text{area}\,\big(RN\Sigma\big) \geq \text{area}\,\big(N\partial\Sigma/\sigma\big) \geq \frac{\mathbf{c}_{n-1}}{2}, \tag{8}$$

which establishes inequality (1).

**6.2. Necessary conditions for equality.** Suppose that equality holds in (1). We then show that the following conditions hold.

**6.2.1. $\partial\Sigma$ lies in a hyperplane.** If equality holds in (1), then the last two quantities in (8) are equal. So, by Proposition 5.3, $\partial\Sigma$ lies in a hyperplane $\Pi$.

For convenience, we assume from now on that $\Pi$ is the hyperplane of the first $n-1$ coordinates in $\mathbf{R}^n$. In particular, $\Pi$ is orthogonal to $e_n :=$

$(0, 0, \ldots, 1)$, the 'north pole' of $\mathbf{S}^{n-1}$. Further, we may assume that $\Sigma \cap \Pi^+ \neq \emptyset$, where $\Pi^+$ denotes the half-space where the $n^{th}$ coordinate of points of $\mathbf{R}^n$ is non-negative.

**6.2.2. Every restricted local support hyperplane of $\Sigma$ at a point of $\Sigma^+$ is a restricted support hyperplane of $\Sigma$.** Let $A \subset \mathbf{S}^{n-1}$ be the set of unit normals to restricted local support hyperplanes of $\Sigma$, and $A^+ \subset A$ be the set of unit normals to restricted local support hyperplanes of $\Sigma$ at points of $\Sigma^+$. It follows from Lemma 5.2 that $\operatorname{area}(A - A^+) = 0$. In particular, every non-empty open subset of $A^+$ has positive area. Now, let $\widetilde{A^+}$ be those elements of $A^+$ which are not unit normals to restricted support hyperplanes of $\Sigma$. Then, $\widetilde{A^+}$ is open in $A^+$. So, if $\widetilde{A^+} \neq \emptyset$, then $\operatorname{area}(\widetilde{A^+}) > 0$. On the other hand, if equality holds in (1), then it follows from (7) and (8) that $\tau^+(\Sigma) = \operatorname{area}(RN\Sigma)$, which means that $\operatorname{area}(\widetilde{A^+}) = 0$. So, we conclude that $\widetilde{A^+} = \emptyset$.

**6.2.3. $\Sigma \subset \Pi^+$.** Let $\Sigma'$ be the reflection of $\Sigma$ with respect to $\Pi$, and $\sigma'$ be the inward conormal of $\partial\Sigma'$. Then if, for some $p \in \partial\Sigma$, $\sigma(p)$ lies in the 'northern hemisphere' $\mathbf{H}_{e_n}$, $\sigma'(p)$ must lie in the 'southern hemisphere' $\mathbf{H}_{-e_n}$ and vice versa. Suppose that there exists a support hyperplane $\Pi'$ of $\Sigma \cup \Sigma'$ at a point $p \in \partial\Sigma = \partial\Sigma'$. Let $u$ be the outward normal of $\Pi'$. Then, $\langle \sigma(p), u \rangle \leq 0$, and $\langle \sigma'(p), u \rangle \leq 0$. Now, recall that $\sigma(p)$, $\sigma'(p)$, and $u$ are all outward unit normals to support hyperplanes of $\partial\Sigma$, i.e., they are elements of $N_p\partial\Sigma$. Further, since $\partial\Sigma$ is $\mathcal{C}^1$, has codimension 2, and lies in a hyperplane, $N_p\partial\Sigma$ is half of a great circle connecting the north and south poles of $\mathbf{S}^{n-1}$. So, it follows that $\langle \sigma(p), u \rangle = 0 = \langle \sigma'(p), u \rangle$. Thus if a support hyperplane of $\Sigma \cup \Sigma'$ intersects a point of $\partial\Sigma$, then it is tangent to $\Sigma$. In other words, every support hyperplane of $\Sigma \cup \Sigma'$ is a restricted support hyperplane of $\Sigma$ or $\Sigma'$. So, $\tau^+(\operatorname{bd}\operatorname{conv}(\Sigma \cup \Sigma')) = \tau^+(\Sigma \cup \Sigma')$. Consequently, $\Sigma^+ \subset \operatorname{bd}\operatorname{conv}(\Sigma \cup \Sigma')$. So, $\Sigma^+ \cap \operatorname{int}\operatorname{conv}\Sigma' = \emptyset$, because $\operatorname{int}\operatorname{conv}\Sigma' \subset \operatorname{int}\operatorname{conv}(\Sigma \cup \Sigma')$. This yields that $S := \operatorname{bd}\operatorname{conv}\Sigma \cap \operatorname{int}\operatorname{conv}\Sigma' = \emptyset$, because otherwise $S$ is a non-flat convex cap, and so it must have strictly convex points. Thus, we conclude that $\operatorname{conv}\Sigma = \operatorname{conv}\Sigma'$, or $\operatorname{int}\operatorname{conv}\Sigma \cap \operatorname{int}\operatorname{conv}\Sigma' = \emptyset$.

Suppose that $\operatorname{conv}\Sigma = \operatorname{conv}\Sigma'$. Then $\operatorname{conv}\Sigma$ is symmetric with respect to $\Pi$. If there exists a point $p \in \partial\Sigma \cap \operatorname{bd}\operatorname{conv}\Sigma$, let $u$ be the outward normal of $\partial\Sigma$ at $p$ in $\Pi$, and note that, since $\sigma(p) \in N_p\partial\Sigma$, $\langle \sigma(p), u \rangle \geq 0$. On the other hand, since $\operatorname{conv}\Sigma$ is symmetric with respect to $\Pi$, $u$ is an outward unit normal to a support hyperplane of $\operatorname{conv}\Sigma$ at $p$, which yields $\langle \sigma(p), u \rangle \leq 0$. So, $\langle \sigma(p), u \rangle = 0$. Thus, we conclude that if $p \in \partial\Sigma \cap \operatorname{bd}\operatorname{conv}\Sigma$, then any support hyperplane of $\Sigma$ at $p$ is orthogonal to $\Pi$, and is therefore tangent to $\Sigma$ at $p$. So, any support hyperplane of $\operatorname{bd}\operatorname{conv}\Sigma$ is a restricted support hyperplane of $\Sigma$. But since $\operatorname{bd}\operatorname{conv}\Sigma$ is a closed surface, $\tau^+(\operatorname{bd}\operatorname{conv}\Sigma) \geq \mathbf{c}_{n-1}$, whereas, by assumption, $\tau^+(\Sigma) = \mathbf{c}_{n-1}/2$. Hence, we have a contradiction.

So, we conclude that $\operatorname{int}\operatorname{conv}\Sigma \cap \operatorname{int}\operatorname{conv}\Sigma' = \emptyset$, which yields that $\Sigma$ lies on one side of $\Pi$. In particular, since by assumption a point of $\Sigma$ lies in $\Pi^+$, we have $\Sigma \subset \Pi^+$.

**6.2.4.** $\Sigma \cap \Pi = \partial\Sigma$ **and** $\sigma(p) \perp \Pi$**.** Let $\overline{\Sigma}$ be the closure of $\operatorname{bd}\operatorname{conv}\Sigma \cap \operatorname{int}\Pi^+$. Then, since $\Sigma$ lies on one side of $\Pi$, $\partial\overline{\Sigma} = \operatorname{bd}\operatorname{conv}\partial\Sigma$. Let $\overline{\sigma}$ be the inward unit normal of $\partial\overline{\Sigma}$, and $\Pi'$ be the support hyperplane of $\overline{\Sigma}$ which passes through a point $p \in \partial\overline{\Sigma}$ and contains $\overline{\sigma}(p)$. Then, since $\Pi \cap \Pi'$ is a support hyperplane of $\partial\overline{\Sigma}$ as a subset of $\Pi$, and $\partial\overline{\Sigma} = \operatorname{bd}\operatorname{conv}\partial\Sigma$, $\Pi'$ must contain an *extreme* point $q$ of $\operatorname{conv}\partial\Sigma$, i.e., a point which does not lie in the relative interior of any line segment of $\operatorname{conv}\partial\Sigma$. This is due to the general fact that any support hyperplane of a convex body contains an extreme point of that body (which is proved easily by induction on the dimension of $C$). By Carathéodory's theorem [**18**, Theorem 1.1.4], every point of $\operatorname{conv}\partial\Sigma$ lies in a simplex with vertices on $\Sigma$. Thus, any extreme point of $\operatorname{conv}\partial\Sigma$ must belong to $\partial\Sigma$. In particular, $q \in \partial\Sigma$. But by Straszewicz's theorem [**18**, Theorem 1.4.7], each extreme point of $\operatorname{conv}\partial\Sigma$ is a limit of its exposed points (which again must be elements of $\partial\Sigma$, since each exposed point is extreme). Further, by Proposition 5.3 and since $\Sigma \subset \Pi^+$, $\sigma = e_n$ at exposed points of $\partial\Sigma$. So, since $\sigma$ is continuous, it follows that $\sigma(q) = e_n$. Since $\Pi'$ supports $\Sigma$ at $q$, it follows then that $\langle u, e_n\rangle \leq 0$, where $u$ is the outward unit normal to $\Pi'$. This yields that $\mathbf{H}_{e_n} \subset RN\overline{\Sigma}$. But $RN\overline{\Sigma} \subset RN\Sigma$, and recall that $\operatorname{area}(RN\Sigma) = \tau^+(\Sigma) = \mathbf{c}_{n-1}/2$. Thus $RN\overline{\Sigma} = \mathbf{H}_{e_n}$, which yields that $\overline{\sigma}$ is orthogonal to $\Pi$. Further, $RN\Sigma = \mathbf{H}_{e_n}$, which yields that $\Sigma \cap \Pi = \partial\Sigma$. Thus, every point of $\partial\Sigma$ which lies on $\operatorname{bd}\operatorname{conv}\partial\Sigma$ is a point of $\partial\overline{\Sigma}$. So $\sigma(p) = e_n$ at all such points. This completes the proof because if a point of $\partial\Sigma$ lies in $\operatorname{int}\operatorname{conv}\partial\Sigma$, then, since $\sigma$ is by assumption an outward normal of $\partial\Sigma$, it follows that $\sigma(p) = e_n$.

**6.3. Sufficient conditions for equality.** Suppose that the conditions, we established above hold. Let $\Sigma'$ be the reflection of $\Sigma$ with respect to $\Pi$. Then at each locally strictly convex point of $\Sigma \cup \Sigma'$, every local support hyperplane of $\Sigma \cup \Sigma'$ is a support hyperplane of $\Sigma \cup \Sigma'$. Thus

$$2\,\tau^+(\Sigma) = \tau^+(\Sigma \cup \Sigma') = \operatorname{area}\big(N(\Sigma \cup \Sigma')\big) = \mathbf{c}_{n-1},$$

which completes the proof. q.e.d.

## Appendix: Analytic Proof of Inequality (1) When $\partial\Sigma$ is $\mathcal{C}^2$

Let

$$U\partial\Sigma := \big\{\, (p, u) \mid p \in \partial\Sigma, u \in \mathbf{S}^{n-1}, u \perp T_p\partial\Sigma \,\big\}$$

denote the *unit normal bundle* of $\partial\Sigma$, and $\nu\colon U\partial\Sigma \to \mathbf{S}^{n-1}$, given by

$$\nu(p,u) := u,$$

be its *Gauss map*. Define $I \subset J \subset U\partial\Sigma$ by

$$I := \big\{\, (p,u) \in U\partial\Sigma \mid \langle x-p, u\rangle \le 0, \quad \forall x \in \Sigma \,\big\},$$
$$J := \big\{\, (p,u) \in U\partial\Sigma \mid \langle x-p, u\rangle \le 0, \quad \forall x \in \partial\Sigma \,\big\}.$$

Note that if $(p,u) \in J - I$, then the height function $x \mapsto \langle x-p, u\rangle$ achieves its maximum in the interior of $\Sigma$, and thus, $\Sigma$ has a restricted support hyperplane with outward normal $u$. Hence,

$$\tau^+(\Sigma) \ge \operatorname{area}\nu(J-I),$$

since almost every support hyperplane of $\Sigma$ intersects $\Sigma$ at a single point [**18**, Theorem 2.2.9]. So to prove (1) it suffices to show that

$$\operatorname{area}\nu(J-I) \ge \frac{\mathbf{c}_{n-1}}{2}. \tag{9}$$

To this end, note that, since, again by [**18**, Theorem 2.2.9], almost every element of $\nu(I-J)$ has multiplicity one,

$$\operatorname{area}\nu(J-I) = \int_{J-I} \operatorname{Jac}\nu = \int_J \operatorname{Jac}\nu - \int_I \operatorname{Jac}\nu,$$

where $\operatorname{Jac}\nu$ denotes the Jacobian of $\nu$, which may be defined as the pull back via $\nu$ of the volume element of $\mathbf{S}^{n-1}$. Further, note that, since every unit vector $u \in \mathbf{S}^{n-1}$ is the outward normal to some support hyperplane of $\partial\Sigma$,

$$\int_J \operatorname{Jac}\nu = \operatorname{area}\nu(J) = \mathbf{c}_{n-1}.$$

Thus, to establish (9), it suffices to show that

$$\int_I \operatorname{Jac}\nu \le \frac{1}{2}\int_J \operatorname{Jac}\nu.$$

In particular, if $I_p$ and $J_p$ denote the fibers of $I$ and $J$ respectively, then, by Fubini's theorem, it suffices to show that

$$\int_{I_p} \operatorname{Jac}\nu \le \frac{1}{2}\int_{J_p} \operatorname{Jac}\nu, \tag{10}$$

for all $p \in \partial\Sigma$.

The above inequality is trivially satisfied whenever $I_p = \emptyset$ or $\nu(I_p)$ consists only of a pair of antipodal points of $\mathbf{S}^{n-1}$. Thus, by Lemma 4.1, we may assume that $I_p$ is non-empty and connected, which in turn yields that $J_p$ is non-empty and connected as well.

For every $u \in \mathbf{S}^{n-1}$, let $h_u \colon \partial\Sigma \to \mathbf{S}^{n-1}$ be the height function given by

$$h_u(p) := \langle p, u\rangle.$$

Then, we have the following well-known identity

$$\operatorname{Jac}\nu_{(p,u)} = \big|\det(\operatorname{Hess} h_u)_p\big|,$$

where $(\operatorname{Hess} h_u)_p \colon T_p\partial\Sigma \times T_p\partial\Sigma \to \mathbf{R}$ denotes the Hessian of $h_u$ at $p$. (To see this one may note that $U\partial\Sigma$ can be identified with a hypersurface $\overline{U\partial\Sigma}$ of $\mathbf{R}^n$ via the endpoint map $(p,u) \mapsto p+u$. Then, the Gauss map $\overline{\nu}$ of $\overline{U\partial\Sigma}$, is given by $\overline{\nu}(p+u) := u = \nu(p,u)$. Consequently, $\operatorname{Jac}\nu_{(p,u)} = \operatorname{Jac}\overline{\nu}_{p+u} = |\det(\mathrm{II}_{p+u})|$, where $\mathrm{II}_{p+u}$ is the second fundamental form of $\overline{U\partial\Sigma}$ at $p+u$. But $\mathrm{II}_{p+u} = (\operatorname{Hess}\overline{h}_u)_{p+u}$, where $\overline{h}_u \colon \overline{U\partial\Sigma} \to \mathbf{R}$ is the height function $\overline{h}_u(p+u) := \langle p+u, u\rangle$. In particular, $\overline{h}_u(p+u) = h_u(p)+1$, which yields that $\det(\operatorname{Hess}\overline{h}_u)_{p+u} = \det(\operatorname{Hess} h_u)_p$.)

Next, let $\sigma^{\perp}(p)$ be a unit normal vector of $\partial\Sigma$ at $p$ which is orthogonal to $\sigma(p)$, and is chosen so that the function $\langle x-p, \sigma^{\perp}(p)\rangle$ is positive for some $x \in \Sigma$ or vanishes for all $x \in \Sigma$. For $\theta \in [-\pi, \pi]$, define

$$u(\theta) := \cos\theta\,\sigma(p) + \sin\theta\,\sigma^{\perp}(p), \quad \text{and} \quad H_\theta := (\operatorname{Hess} h_{u(\theta)})_p.$$

Then, since $\|\partial u/\partial\theta\| = 1$, the change of variables formula allows us to rewrite (10) as

$$\int_{\theta_0}^{\theta_1} |\det(H_\theta)|\,d\theta \le \frac{1}{2}\int_{\phi_0}^{\phi_1} |\det(H_\theta)|\,d\theta,$$

where $[\theta_0, \theta_1] \subset [\phi_0, \phi_1] \subset [-\pi, \pi]$, and $u([\theta_0, \theta_1]) = \nu(I_p)$, $u([\phi_0, \phi_1]) = \nu(J_p)$.

Note that if $u \in \nu(I_p)$, then $\langle u, \sigma(p)\rangle$ and $\langle u, \sigma^{\perp}(p)\rangle$ must both be non-positive. Thus, $[\theta_0, \theta_1] \subset [-\pi, -\frac{\pi}{2}]$. Further, since $0 \in [\phi_0, \phi_1]$, it follows that $[\theta_0, 0] \subset [\phi_0, \phi_1]$. Hence, to prove the above inequality, it is enough to show that

$$\int_{\theta_0}^{-\frac{\pi}{2}} |\det(H_\theta)|\,d\theta \le \int_{-\frac{\pi}{2}}^{0} |\det(H_\theta)|\,d\theta. \tag{11}$$

To this end, note that for any tangent vectors $X_p, Y_p \in T_p\partial\Sigma$, with local extensions $X$, $Y$,

$$\begin{aligned} H_\theta(X_p, Y_p) &= X_p\big(Y h_{u(\theta)}\big) = \big\langle D_{X_p}Y, u(\theta)\big\rangle \\ &= \cos\theta\big\langle D_{X_p}Y, u(0)\big\rangle + \sin\theta\left\langle D_{X_p}Y, u\left(\frac{\pi}{2}\right)\right\rangle \\ &= \cos\theta\, H_0(X_p, Y_p) + \sin\theta\, H_{\frac{\pi}{2}}(X_p, Y_p), \end{aligned}$$

where $D$ denotes the standard covariant derivative, or Levi–Civita connection on $\mathbf{R}^n$.

Also note that $H_0$ is negative semidefinite because by assumption $u(0) = \sigma(p) \in \nu(J_p)$. Further, since $\theta_0 \in [-\pi, -\frac{\pi}{2}] \cap [\phi_0, \phi_1]$, and $0 \in [\phi_0, \phi_1]$, it follows that $-\frac{\pi}{2} \in [\phi_0, \phi_1]$. So $u(-\frac{\pi}{2}) \in \nu(J_p)$, which yields that $H_{\frac{\pi}{2}}$ is positive semidefinite. For any $\theta \in [-\pi, -\frac{\pi}{2}]$, let $\theta' := -\pi - \theta \in [-\frac{\pi}{2}, 0]$. Then, $\cos\theta' = -\cos\theta < 0$, and $\sin\theta' = \sin\theta < 0$. Thus,

$$-H_{\theta'}(X_p, X_p) \geq -H_\theta(X_p, X_p).$$

Hence, the eigenvalues of $-H_{\theta'}$ are bigger than or equal to those of $-H_\theta$. But for all $\theta \in [\theta_0, -\frac{\pi}{2}]$, $H_\theta$ and $H_{\theta'}$ are both negative semidefinite, because $u(\theta)$, $u(\theta') \in \nu(I_p)$. So $-H_\theta$ and $-H_{\theta'}$ are positive semidefinite. Consequently,

$$|\det(H_{\theta'})| = \det(-H_{\theta'}) \geq \det(-H_\theta) = |\det(H_\theta)|,$$

which yields that

$$\int_{\theta_0}^{-\frac{\pi}{2}} |\det(H_\theta)|\, d\theta \leq \int_{-\frac{\pi}{2}}^{\theta_0'} |\det(H_\theta)|\, d\theta. \tag{12}$$

Since $\theta_0' \leq 0$, this yields (11), which in turn completes the proof of (1).

Now, suppose that equality holds in (1), then equality holds in the above inequalities. In particular, equalities hold in (11) and (12), which yields

$$\int_{-\frac{\pi}{2}}^{\theta_0'} |\det(H_\theta)|\, d\theta = \int_{-\frac{\pi}{2}}^{0} |\det(H_\theta)|\, d\theta.$$

So, we conclude

$$\int_{\theta_0'}^{0} |\det(H_\theta)|\, d\theta = 0.$$

This implies that $(\operatorname{Hess} h_{u(\theta)})_p \equiv 0$ for all $\theta_0'(p) \leq \theta \leq 0$, as $p$ ranges over $\partial\Sigma$. But it is a well-known consequence of Sard's theorem that $h_u$ is a Morse function [**6**], i.e., it has non-degenerate Hessian, for almost all $u \in \mathbf{S}^{n-1}$. So, we must have $\theta_0' = 0$, which yields that $\theta_0 = -\pi$, for some $p$. So $u(-\pi) \in \nu(J_p)$. But $-u(-\pi) = u(0) = \sigma(p) \in \nu(J_p)$ as well. Hence, $\partial\Sigma$ lies in a hyperplane. q.e.d.

**Note 6.1.** If $\partial\Sigma$ is a $\mathcal{C}^3$ closed curve with non-vanishing curvature, and $\gamma\colon \mathbf{R} \to \partial\Sigma$ is a unit parametrization of $\partial\Sigma$, then its unit normals may be parametrized by

$$\nu(t, \theta) := -\cos\theta\, N(t) + \sin\theta\, B(t),$$

where $N(t)$ and $B(t)$ are, respectively, the principal normal and binormal vectors of $\partial\Sigma$ at $\gamma(t)$. A computation, using Frenet–Serret formulas, shows that

$$\operatorname{Jac} \nu_{(t,\theta)} = \left| \frac{\partial n}{\partial t} \times \frac{\partial n}{\partial \theta} \right| = \kappa(t)\, |\cos\theta|,$$

where $\kappa$ is the curvature of $\partial\Sigma$. Then, the observation that $-N(t)$ lies in $\nu(J_{\gamma(t)}) - \nu(I_{\gamma(t)})$ yields a quicker proof of (10).

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