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SINGULARITIES OF HOLOMORPHIC FOLIATIONS

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To S. S. Chern & D. C. Spencer on their 60th birthdays

0. Introduction

The purpose of this note is twofold. First, we give a simpler and more natural proof of our meromorphic vector-field theorem of [5]; and second, we give a theorem on singularities of holomorphic foliations which includes the meromorphic vector-field theorem as a special case. We have tried to make the exposition as elementary and self-contained as possible.

To recall the result of [5], let M be a complex analytic manifold. Set $n = \dim_C M$. Assume $n \geq 2$. Let T be the holomorphic tangent bundle of M, L be a holomorphic line bundle on M, and $\eta\colon L \to T$ be a holomorphic vector-bundle map. Let $X_1, \cdots, X_n$ be indeterminates, and φ be a polynomial in $X_1, \cdots, X_n$ with complex coefficients:

$$\varphi \in C[X_1, \cdots, X_n] \ . \tag{0.1}$$

Assume that φ is symmetric and homogeneous of degree n. Given an isolated zero p of η, define a number $\varphi(\eta, p)$ as follows. About p choose a nonvanishing holomorphic section s_p of L. Also about p, choose a complex-analytic coordinate system $z_1, \cdots, z_n$ with origin at p. The vector-field $\eta(s_p)$ is then well-defined near p, and there has the expansion

$$\eta(s_p) = \sum_{i=1}^{n} a_i \partial / \partial z_i \ , \tag{0.2}$$

where the a_i are holomorphic functions near p.

Form the matrix A of partial derivatives: $A = \|\partial a_i / \partial z_j\|$. Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be the elementary symmetric functions in $X_1, \cdots, X_n$. Define $\sigma_i(A)$ by

$$\det (I + tA) = 1 + t\sigma_1(A) + \cdots + t^n \sigma_n(A) \ . \tag{0.3}$$

Thus each $\sigma_i(A)$ is a function near p. Since φ is symmetric, there is a unique polynomial $\tilde{\varphi}$ such that

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$$(0.4)\qquad \varphi = \tilde{\varphi}(\sigma_1, \cdots, \sigma_n) .$$

Define $\varphi(A)$ by

$$(0.5)\qquad \varphi(A) = \tilde{\varphi}(\sigma_1(A), \cdots, \sigma_n(A)) .$$

Then $\varphi(\eta, p)$ is defined to be the value at p of the Grothendieck residue symbol.

$$(0.6)\qquad \varphi(\eta, p) = \mathrm{Res}_p \begin{bmatrix} \varphi(A)dz_1 \cdots dz_n \\ a_1, \cdots, a_n \end{bmatrix} .$$

If p is a nondegenerate zero of η, i.e., if $\det \|(\partial a_i/\partial z_j)(p)\| \neq 0$, let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $\|(\partial a_i/\partial z_j)(p)\|$. From the general properties of the Grothendieck residue given in [18] it then follows that in this case,

$$(0.7)\qquad \varphi(\eta, p) = \varphi(\lambda_1, \cdots, \lambda_n)/(\lambda_1 \cdots \lambda_n) .$$

More generally, here is an explicit algorithm for computing the right-hand side of (0.6).

Since the origin is an isolated zero of the a_i, there exist positive integers $\alpha_1, \cdots, \alpha_n$ with $z_i^{\alpha_i}$ in the ideal generated by $a_1, \cdots, a_n$. Hence there exist holomorphic functions b_{ij} near p with

$$(0.8)\qquad z_i^{\alpha_i} = \sum_{j=1}^{n} b_{ij} a_j .$$

One then has

$$(0.9)\qquad \mathrm{Res}_p \begin{bmatrix} \varphi(A)dz_1 \cdots dz_n \\ a_1, \cdots, a_n \end{bmatrix} = \mathrm{Res}_p \begin{bmatrix} \varphi(A) \det \|b_{ij}\| \, dz_1 \cdots dz_n \\ z_1^{\alpha_1}, \cdots, z_n^{\alpha_n} \end{bmatrix} .$$

The right-hand side of (0.9) is now evaluated by expanding $\varphi(A) \det \|b_{ij}\|$ in a power series in the z_i. The coefficient of $dz_1 \cdots dz_n/(z_1 \cdots z_n)$ in the resulting Laurent series for $\varphi(A) \det \|b_{ij}\| \, dz_1 \cdots dz_n/(z_1^{\alpha_1} \cdots z_n^{\alpha_n})$ is the desired answer.

This algorithm was derived for us by R. Hartshorne. It is an immediate consequence of the general properties of the Grothendieck residue given in [18].

It can be easily checked that $\varphi(\eta, p)$ does not depend on the choices made in defining it. Hence $\varphi(\eta, p)$ is a well-defined local number depending only on φ and the local behavior of η near p.

The result of [5] is:

Theorem 1. *Let M be a compact complex-analytic manifold, $\eta\colon L \to T$ be a holomorphic vector-bundle map with isolated zeroes, and φ be symmetric and homogeneous of degree n. Consider the virtual bundle $T - L$. Then*

$$(0.10)\qquad \varphi(T - L)[M] = \sum_{p \in \mathrm{Zero}(\eta)} \varphi(\eta, p) .$$

Remarks. (a) Let $c_1(T - L), \cdots, c_n(T - L)$ be the Chern classes of

$T - L$, taken in $H^*(M; C)$. Then, as is customary, $\varphi(T - L)$ is defined by

$$\varphi(T - L) = \tilde{\varphi}(c_1(T - L), \cdots, c_n(T - L)) , \tag{0.11}$$

where $\tilde{\varphi}$ is as in (0.4). Since φ is homogeneous of degree n, $\varphi(T-L) \in H^{2n}(M; C)$. $\varphi(T - L)[M]$ denotes $\varphi(T - L)$ evaluated on the canonical generator of $H_{2n}(M; C)$.

(b) If M is a submanifold of complex projective space, then by tensoring T with a sufficiently high power of the hyperplane bundle H, $\dim_C \Gamma(T \otimes H^r)$ can be made arbitrarily large. Here $\Gamma(T \otimes H^r)$ denotes the vector-space of all holomorphic sections of $T \otimes H^r$. Furthermore, almost all sections of $T \otimes H^r$ will have only isolated zeroes when r is large enough. A section of $T \otimes H^r$ gives a vector-bundle map $(H^r)^* \to T$. Thus there are many examples to which Theorem 1 applies.

We now take the point of view that Theorem 1 is really a theorem about holomorphic foliations with singularities. To see this, let us use the notation convention that whenever E is a holomorphic vector-bundle, $\underline{E}$ shall denote the sheaf of germs of holomorphic sections of E. Then at the sheaf level η is injective.

$$\underline{\eta}\colon \underline{L} \to \underline{T} \qquad \text{is injective} . \tag{0,12}$$

Set $\xi = \underline{\eta}(\underline{L})$ and $Q = \underline{T}/\xi$. Observe that ξ is an *integrable* subsheaf of $\underline{T}$ in the sense that for each $x \in M$, the stalk ξ_x is closed under the bracket operation for vector-fields. On M-Zero (η) we have a one-dimensional foliation, in the usual sense, of M-Zero (η). On M, however, we have a foliation with singularities. ξ can be thought of as the tangent sheaf of the foliation with singularities. If $\mathcal{O}$ is the structure sheaf of M, then the singularities occur precisely at those points $p \in M$ such that Q_p is not a free $\mathcal{O}_p$-module.

The exactness of

$$0 \to \underline{L} \to \underline{T} \to Q \to 0 \tag{0.13}$$

implies that $c_i(Q) = c_i(T - L)$, $i = 1, \cdots, n$. Hence (0.10) can be rewritten

$$\varphi(Q)[M] = \sum_{p \in \mathrm{Zero}(\eta)} \varphi(\eta, p) . \tag{0.14}$$

So we conclude that Theorem 1 computes the Chern numbers of Q in terms of local information at the singularities of the foliation.

Pass now to higher dimensional foliations. Define a subsheaf $\xi \subset \underline{T}$ to be *integrable* if

(i) ξ is coherent,

(ii) for each $x \in M$, ξ_x is closed under the bracket operation for vector-fields.

Set $Q = \underline{T}/\xi$ and $S = \{x \in M \mid Q_x$ is not a free $\mathcal{O}_x$-module$\}$. S is a closed holomorphic subvariety of M. S will be referred to as the *singular set*. On $M - S$ there is a unique holomorphic sub-vector-bundle F of $T \mid M - S$ such that

$$(0.15) \qquad \underline{F} = \xi \mid M - S \, .$$

We assume that $\dim_C F_x$ is constant throughout $M - S$. This is automatically the case if M is connected. $\dim_C F_x$ will be denoted by k and will be referred to as the leaf dimension of ξ. We shall always assume

$$(0.16) \qquad 1 \leq k < n \, .$$

Given $p \in M - S$, the well-known theorem of Frobenius asserts that there exists a complex-analytic coordinate system $z_1, \cdots, z_n$ defined on an open neighborhood U_p of p such that

$$(0.17) \qquad \partial/\partial z_1, \cdots, \partial/\partial z_k \qquad \text{is a frame of } F \mid U_p \, .$$

A leaf of this foliation of $M - S$ will be called a leaf of ξ.

It is natural to assume that ξ satisfies the following condition:

(0.18) Let U be an open subset of M, and γ a holomorphic section of $T \mid U$. Suppose that $\gamma(x) \in F_x$ for each $x \in U \cap (M - S)$. Then at each $p \in U \cap S$ the germ of the holomorphic vector-field γ is in ξ_p.

A ξ which satisfies this condition will be said to be *full*. In the situation of Theorem 1, $\eta(\underline{L})$ is full.

If ξ is integrable and $\dim_C S \leq n - 2$, then there is a unique sheaf $\hat{\xi}$ such that $\hat{\xi}$ is both full and integrable, and

$$(0.19) \qquad \hat{\xi} \mid M - S = \xi \mid M - S \, .$$

To define $\hat{\xi}$, let F be as in (0.15). Define $\hat{\xi}$ by

$$(0.20) \qquad \Gamma(\hat{\xi}, U) = \{\gamma \in \Gamma(T \mid U) \mid \gamma(x) \in F_x \text{ whenever } x \in U \cap (M - S)\} \, .$$

In (0.20), U is any open set of M, $\Gamma(\hat{\xi}, U)$ denotes the continuous sections of $\hat{\xi} \mid U$, and $\Gamma(T \mid U)$ denotes the holomorphic sections of $T \mid U$. Thus by restricting attention to full integrable sheaves we rule out artificial singularities and deal only with genuine foliation singularities.

Given a full integrable subsheaf ξ of $\underline{T}$ we would like to compute Chern polynomials $\varphi(Q)$ in terms of local information at the singular set S. Let Z be a connected component of S. Recall that if M is compact, there is the homomorphism μ_*:

$$(0.21) \qquad \mu_* \colon H_j(Z; C) \to H^{2n-j}(M; C) \qquad j = 0, 1, \cdots, 2n \, .$$

$\mu_* = \alpha i_*$ where $i_*: H_j(Z; C) \to H_j(M; C)$ is induced by the inclusion of Z in M, and $\alpha: H_j(M; C) \to H^{2n-j}(M; C)$ is the Poincaré duality isomorphism.

We then have:

Theorem 2 (*Residue existence theorem*). *Let M be a complex-analytic manifold, ξ be a full integrable subsheaf of $\underline{T}$, and k be the leaf dimension of ξ. Set $Q = \underline{T}/\xi$, and let $\varphi \in C[X_1, \cdots, X_n]$ be a symmetric polynomial which is homogeneous of degree l, where $n = \dim_C M$ and $n - k < l \leq n$. Let Z be a connected component of the singular set S, and assume that Z is compact. Then there exists a homology class* $\mathrm{Res}_\varphi(\xi, Z) \in H_{2n-2l}(Z; C)$ *such that*

(0.22) $\mathrm{Res}_\varphi(\xi, Z)$ *depends only on φ and on the local behavior of the leaves of ξ near Z* ,

(0.23) *if M is compact, then* $\sum_Z \mu_* \mathrm{Res}_\varphi(\xi, Z) = \varphi(Q)$.

Remarks. (a) If M is compact then clearly every connected component of S must be compact. In (0.23) the sum is taken over all the connected components of S.

(b) Let $\sigma_1, \cdots, \sigma_n$ be the elementary symmetric functions of $X_1, \cdots, X_n$. Since φ is symmetric and homogeneous of degree l, there is a unique polynomial $\tilde{\varphi}$ in $\sigma_1, \cdots, \sigma_l$ with $\tilde{\varphi}(\sigma_1, \cdots, \sigma_l) = \varphi$. Let $c_1(Q), \cdots, c_n(Q)$ be the Chern classes of Q. Then $\varphi(Q)$ is defined by setting $\varphi(Q) = \tilde{\varphi}(c_1(Q), \cdots, c_l(Q))$.

(c) Let U be an open subset of M with $U \supset Z$. $\mathrm{Res}_\varphi(\xi, Z)$ is a local matter so $\mathrm{Res}_\varphi(\xi, Z)$ depends only on φ and on $\xi | U$.

(d) This is just an existence theorem. It asserts that $\mathrm{Res}_\varphi(\xi, Z)$ exists and has the desirable properties (0.22) and (0.23). But it does not give an explicit formula for $\mathrm{Res}_\varphi(\xi, Z)$ in terms of local information near Z.

To think about the problem of explicitly computing $\mathrm{Res}_\varphi(\xi, Z)$, one must confront the question: "What is the 'generic' singularity of a foliation?" Put otherwise: "What sort of a singularity is it reasonable to expect?" This appears to be a delicate question whose complete answer has eluded us. We have therefore only considered the case when the singular set satisfies certain natural dimension conditions. When $k = 1$, these conditions reduce to asserting that the singular set consists of isolated points.

In general, observe that a connected component Z of S comes endowed with a filtration. For given $p \in Z$ choose holomorphic vector-fields $\gamma_1, \cdots, \gamma_r$ defined on an open neighborhood U_p of p in M such that

(0.24) For all $x \in U_p$, the germs at x of the holomorphic vector-fields $\gamma_1, \cdots, \gamma_r$ are in ξ_x and span ξ_x as an $\mathcal{O}_x$-module.

Define a subspace $V_p(\xi) \subset T_p$ by letting $V_p(\xi)$ be the sub-vector-space of T_p spanned by $\gamma_1(p), \cdots, \gamma_r(p)$. $V_p(\xi)$ depends only on p and ξ, and is independent of the choice of $\gamma_1, \cdots, \gamma_r$. Set

(0.25) $$Z^{(i)} = \{p \in Z \,|\, \dim_C V_p(\xi) \le k - i\}\,, \qquad i = 1, \cdots, k\,.$$

Then

(0.26) $$Z = Z^{(1)} \supset \cdots \supset Z^{(k)}$$

is a filtration of Z. Each $Z^{(i)}$ is a closed holomorpic subvariety of M.

Our dimension conditions on Z are:

(0.27) $$\dim_C Z = k - 1\,,$$

(0.28) $$\dim_C Z^{(2)} < k - 1\,.$$

If (0.27) is valid for Z, a point $p \in Z$ will be said to be *regular* if there exist an open neighborhood U_p of p in M and complex-analytic coordinates $z_1 \cdots$, z_n defined on U_p such that

(0.29) $$U_p \cap Z = \{x \in U_p \,|\, z_k(x) = \cdots = z_n(x) = 0\}\,.$$

Let N be the set of all points p in Z, which are not regular. N is a closed holomorphic subvariety of M with $\dim_C N < k - 1$.

Elsewhere [4] a proof will be given of the following theorem which to some extent describes the structure of a singularity for which (0.27) and (0.28) are valid: Given such a Z, let $p \in Z - (Z^{(2)} \cup N)$. The theorem asserts that in the vicinity of p the foliation singularity is the "pull-back" via a submersion of an isolated zero of a holomorphic vector field. The submersion maps a neighborhood of p in M onto a neighborhood of the origin in C^{n-k+1}.

(0.30) Theorem. *Let M be a complex-analytic manifold, ξ be a full integrable subsheaf of $\underline{T}$, and Z be a connected component of the singular set S. Assume that $\dim_C Z = k - 1$ and $\dim_C Z^{(2)} < k - 1$. Let $p \in Z - (Z^{(2)} \cup N)$. Then there exist an open neighborhood U_p of p in M, complex-analytic coordinates $z_1, \cdots, z_n$ defined on U_p, holomorphic functions $a_k, \cdots, a_n$ on U_p, and a positive real uumber ε such that:*

(0.31) $$Z \cap U_p = \{x \in U_p \,|\, z_k(x) = \cdots = z_n(x) = 0\}\,.$$

(0.32) $$x \longrightarrow (z_1(x), \cdots, z_n(x)) \qquad \textit{maps } U_p \textit{ onto}$$
$$\{(\zeta_1, \cdots, \zeta_n) \in C^n \,|\, |\zeta_i| < \varepsilon,\ i = 1, \cdots, n\}\,.$$

(0.33) $$Z \cap U_p = \{x \in U_p \,|\, a_k(x) = \cdots = a_n(x) = 0\}\,.$$

(0.34) *If $1 \le j \le k - 1$ and $k \le i \le n$, then $\partial a_i / \partial z_j$ vanishes throughout U_p,*

(0.35) *At each $x \in U_p$ the germs of the holomorphic vector-fields $\partial/\partial z_1, \cdots$, $\partial/\partial z_{k-1}$, $\sum_{i=k}^{n} a_i \partial/\partial z_i$ are in ξ_x and span ξ_x as on $\mathcal{O}_x$-module.*

Remarks. (a) (0.34) implies that for $x \in U_p$, $a_i(x)$ depends only on $z_k(x)$, $\cdots, z_n(x)$. Thus the submersion referred to above is

$$x \longrightarrow (z_k(x), \cdots, z_n(X)) .$$

(b) Several examples of foliation singularities for which (0.27) and (0.28) are valid will be described in § 11 below.

Let $\deg \varphi = n - k + 1$. Assume that Z is compact and satisfies (0.27) and (0.28). Let $Z_1, \cdots, Z_s$ be the irreducible complex-analytic components of Z of dimension $k - 1$. Denote by $[Z_i]$ the element of $H_{2k-2}(Z; \boldsymbol{C})$ given by the fundamental cycle of Z_i. Then $[Z_1], \cdots, [Z_s]$ is a vector-space basis for $H_{2k-2}(Z; \boldsymbol{C})$. To each Z_i associate a complex number $\sharp(\varphi, \xi, Z_i)$ as follows. Choose $p \in Z_i - (Z^{(2)} \cup N)$. Choose a neighborhood U_p of p and $z_1, \cdots, z_n$, $a_k, \cdots, a_n$, ε as in (0.31)–(0.35). Form the $(n - k + 1) \times (n - k + 1)$ matrix A of partial derivatives:

$$A = \|\partial a_i / \partial z_j\| , \qquad k \leq i, j \leq n . \tag{0.36}$$

If $\det \|(\partial a_i/\partial z_j)(p)\| \neq 0$, then let $\lambda_1, \cdots, \lambda_{n-k+1}$ be the eigenvalues of $\|(\partial a_i/\partial z_j)(p)\|$. In this case,

$$\sharp(\varphi, \xi, Z_i) = \varphi(\lambda_1, \cdots, \lambda_{n-k+1}, 0, \cdots, 0)/(\lambda_1 \cdots \lambda_{n-k+1}) . \tag{0.37}$$

More generally, let $\sigma_1, \cdots, \sigma_n$ be the elementary symmetric functions in the indeterminates $X_1, \cdots, X_n$. For $i = 1, \cdots, n - k + 1$ define $\sigma_i(A)$ by

$$\det (I + tA) = 1 + t\sigma_1(A) + \cdots + t^{n-k+1}\sigma_{n-k+1}(A) . \tag{0.38}$$

Since $\deg \varphi = n - k + 1$, there is a polynomial $\tilde{\varphi}$ in $\sigma_1, \cdots, \sigma_{n-k+1}$ with

$$\varphi = \tilde{\varphi}(\sigma_1, \cdots, \sigma_{n-k+1}) , \qquad \deg \varphi = n - k + 1 . \tag{0.39}$$

Define $\varphi(A)$ by

$$\varphi(A) = \tilde{\varphi}(\sigma_1(A), \cdots, \sigma_{n-k+1}(A)) . \tag{0.40}$$

Thus $\varphi(A)$ is a holomorphic function on U_p.

Let $D_p = \{x \in U_p \mid z_1(x) = z_1(p), \cdots, z_{k-1}(x) = z_{k-1}(p)\}$. D_p is a holomorphic normal disc to Z_i at p. Restrict $\varphi(A)$ to D_p and define $\sharp(\varphi, \xi, Z_i)$ to be the value at p of the Grothendieck residue symbol

$$\sharp(\varphi, \xi, Z_i) = \operatorname{Res}_p \begin{bmatrix} \varphi(A) dz_k \cdots dz_n \\ a_k, \cdots, a_n \end{bmatrix} . \tag{0.41}$$

Then $\sum_{i=1}^s \sharp(\varphi, \xi, Z_i)[Z_i]$ is a well-defined homology class depending only on φ and the local behavior of the leaves of ξ near Z.

Theorem 3. *Let M be a complex manifold, ξ be a full integrable sub sheaf of $\underline{T}$, S be the singular set of ξ, and Z be a connected component of S. Assume that Z is compact,* $\dim_C Z = k - 1$, $\dim_C Z^{(2)} < k - 1$. *Let $Z_1, \cdots, Z_s$ be the* *irreducible complex-analytic components of Z of dimension $k-1$. Let* $\deg \varphi = n-k+1$. *Then*

$$\operatorname{Res}_\varphi(\xi, Z) = \sum_{i=1}^{s} \#(\varphi, \xi, Z_i)[Z_i] \;. \tag{0.42}$$

Remark. Suppose $k = 1$. Then from (0.14) it is clear that Theorem 2 and Theorem 3 combine to imply Theorem 1. Hence Theorem 2 and Theorem 3 together constitute a result on holomorphic foliations, which includes Theorem 1 as a special case.

We turn now to the question of computing $\operatorname{Res}_\varphi(\xi, Z)$ when $n-k+1 < \deg\varphi \leq n$. Here we have been unable to find an explicit formula for $\operatorname{Res}_\varphi(\xi, Z)$. However, we have discovered that $\operatorname{Res}_\varphi(\xi, Z)$ has a rigidity property. This rigidity[1] appears to be the most relevant fact about these $\operatorname{Res}_\varphi(\xi, Z)$.

Theorem 4 (*Rigidity thorem*). *Let M be a complex manifold. Assume that $n-k+1 < \deg\varphi \leq n$. Let U be an open subset of M, and $[a, b]$ be a closed interval of real numbers. For $t \in [a, b]$ let $\{\xi_t\}$ be a C^∞ 1-parameter family of full integrable subsheaves of $\underline{T}\,|\,U$. Let $Z_t = \{x \in U \,|\, (\underline{T}/\xi_t)_x$ is not a free $\mathcal{O}_x$-module$\}$. Assume that each Z_t is compact and connected, and also that there is a fixed compact subset B of U with*

$$Z_t \subset B \qquad \textit{for all } t \in [a, b] \;. \tag{0.43}$$

Let $i_*\colon H_*(Z_t; C) \to H_*(U; C)$ be induced by the inclusion of Z_t in U. Then

$$i_* \operatorname{Res}_\varphi(\xi_a, Z_a) = i_* \operatorname{Res}_\varphi(\xi_b, Z_b) \;. \tag{0.44}$$

An immediate corollary of Theorem 4 is

(0.45) Corollary. *Let $M, U, [a, b], \{\xi_t\}, \varphi$ be as above. Assume, in addition, that there is a fixed compact connected subvariety Z of U with*

$$Z_t = Z \qquad \textit{for all } t \in [a, b] \;. \tag{0.46}$$

Then

$$\operatorname{Res}_\varphi(\xi_a, Z) = \operatorname{Res}_\varphi(\xi_b, Z) \;. \tag{0.47}$$

Remarks. (a) In Theorem 4 and Corollary (0.45) no special assumption is made on Z_t other than that Z_t be compact and connected. In particular, it is *not* required that (0.27) and (0.28) be valid for Z_t.

(b) Theorem 4 and Corollary (0.45) show that the two cases $\deg\varphi = n-k+1$ and $\deg\varphi > n-k+1$ are quite different. If $\deg\varphi = n-k+1$, then there are many examples where $\operatorname{Res}_\varphi(\xi_t, Z)$ is not constant in t.

[1] Independently and in a slightly different context, a similar rigidity theorem has been recently noted by James L. Heitsch, *Deformations of secondary characteristic classes,* to appear in Topology.

Thorem 4 suggests a conjecture. Let Q denote the rational numbers. The inclusion $Q \subset C$ gives inclusions

$$Q[X_1, \cdots, X_n] \subset C[X_1, \cdots, X_n] , \tag{0.48}$$

$$H_*(Z; Q) \subset H_*(Z; C) . \tag{0.49}$$

Rationality conjecture. *Let M be a complex manifold, ξ be a full integrable subsheaf of $\underline{T}$, and Z be a compact connected component of the singular set S. If $n - k + 1 < \deg \varphi \leq n$ and $\varphi \in Q[X_1, \cdots, X_n]$, then*

$$\operatorname{Res}_\varphi (\xi, Z) \in H_*(Z; Q) . \tag{0.50}$$

Remark. This conjecture, if true, would again point up a very sharp difference between the two cases where $\deg \varphi = n - k + 1$ and $\deg \varphi > n - k + 1$.

Two special situations deserve special comment. If the singular set S is empty, then Theorem 2 becomes the vanishing theorem of [5] and [9].

Vanishing theorem. *Let M be a complex manifold. On M, let F be an integrable holomorphic sub-vector-bundle of T. Then*

$$\varphi(T/F) = 0 \tag{0.51}$$

for all φ with $n - k < \deg \varphi \leq n$.

Remarks. (a) In this vanishing theorem, M is not required to be compact.

(b) If the foliation of M is a fibration, then (0.51) is obvious. For in this case let X be the base of the fibration and let $\pi: M \to X$ be the projection of M onto X. Then

$$T/F = \pi^!(TX) , \tag{0.52}$$

where TX is the holomorphic tangent bundle of X and $\pi^!(TX)$ is the pull-back by π of TX. Let $\pi^*: H^*(X; C) \to H^*(M; C)$ be the cohomology map induced by π. Then (0.52) implies

$$\varphi(T/F) = \pi^*\varphi(TX) . \tag{0.53}$$

Since $\dim_C X = n - k$, $\varphi(TX)$ vanishes whenever $\deg \varphi > n - k$. Hence (0.51) is evident in this case.

(c) Compact complex manifolds very rarely foliate without singularities. For example, (0.51) can be used to prove that there is no holomorphic foliation (without singularities) of CP^n. Foliations with singularities, however, exist in great abundance.

A second special case of interest is the case when (0.27) and (0.28) are valid for Z and in addition to this $Z^{(2)}$ and N are empty. Here we can give an explicit formula for $\operatorname{Res}_\varphi (\xi, Z)$ for all φ with $n - k < \deg \varphi \leq n$. See § 11 below.

Finally, let us remark that the local classes $\operatorname{Res}_\varphi (\xi, Z)$ are functorial in an appropriate sense. Once this is made precise, it becomes clear that the work of this note is very closely related to the problem of computing the homotopy and homology of the foliation classifying spaces $B\Gamma_q^C$ introduced by A. Haefliger [15]. This will be commented on in § 12 below.

The paper is divided into 12 sections with the following titles:

We thank P. Griffiths, R. Hartshorne, and R. MacPherson for many helpful comments and suggestions. L. Illusie [19] has, independently, done some work quite analogous to ours in the algebraic category.

1. Connections and curvature

Some standard facts on connections and curvature are very briefly reviewed here. For a careful detailed treatment see [10]. The matters considered here are purely C^∞, so in this section let M be a C^∞ manifold. Set $m = \dim_R M$, and let n be the largest integer with $n \leq m/2$. Let $T_R M$ be the usual C^∞ tangent bundle of M, which is a real vector bundle. We wish to consider only complex vector-bundles, so let τ be the complexification of $T_R M$, i.e.,

$$\tau = C \bigotimes_R T_R M \, . \tag{1.1}$$

If E is a C^∞ complex vector-bundle on M, then $C^\infty(E)$ denotes the space of all C^∞ sections of E. E^* denotes the bundle dual to E. $\Lambda^i E$ denotes the i-th exterior power of E.

On M we have the de Rham complex of all C^∞ complex-valued differential forms on M:

$$0 \longrightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \xrightarrow{d} A^m \longrightarrow 0 \, . \tag{1.2}$$

A^0 is the set of all smooth functions from M to C. For $i \geq 1$, $A^i = C^\infty(\Lambda^i \tau^*)$. d is the usual de Rham differentiation operator.

(1.3) $$H^i(M; C) = \text{Kernel}\,\{d\colon A^i \to A^{i+1}\}/\text{Image}\,\{d\colon A^{i-1} \to A^i\}\ .$$

If $\omega \in A^i$ has $d\omega = 0$, then we denote by $[\omega]$ the element of $H^i(M; C)$ determined by ω:

(1.4) $$[\omega] \in H^i(M; C)\ .$$

(1.5) Definition. Let E be a C^∞ complex vector-bundle on M. *A connection* for E is a C-linear map D from $C^\infty(E)$ to $C^\infty(\tau^* \otimes E)$ such that

(1.6) $$D(fs) = df \otimes s + fDs\ ,$$

whenever $f \in A^0$ and $s \in C^\infty(E)$.

Remark. E always has many connections.

If D is a connection for E, then for each $i > 0$, D induces a unique C-linear map, also denoted by D:

(1.7) $$D\colon C^\infty(\Lambda^i\tau^* \otimes E) \to C^\infty(\Lambda^{i+1}\tau^* \otimes E)$$

such that

(1.8) $$D(\omega \otimes s) = d\omega \otimes s + (-1)^i\omega Ds\ ,$$

whenever $\omega \in A^i$ and $s \in C^\infty(E)$.

There is a unique C^∞ vector-bundle map $K(D)$:

(1.9) $$K(D)\colon E \to \Lambda^2\tau^* \otimes E$$

such that for all $s \in C^\infty(E)$,

(1.10) $$DDs = K(D)s\ .$$

$K(D)$ is the curvature of D.

Let U be an open subset of M. If $s \in C^\infty(E)$ vanishes on U, then Ds also vanishes on U. From this remark it follows immediately that D restricts to give a connection for $E \mid U$:

(1.11) $$D\colon C^\infty(E \mid U) \to C^\infty(\tau^* \otimes E \mid U)\ .$$

On U, let $e_1, \cdots, e_r$ be a C^∞ frame of E. A matrix $\theta = \|\theta_{ij}\|$ of 1-forms is determined by

(1.12) $$De_i = \sum_{j=1}^{r} \theta_{ij} \otimes e_j\ .$$

θ is the connection matrix of D with respect to the frame $e_1, \cdots, e_r$. Set $\kappa = d\theta - \theta \wedge \theta$. Then

(1.13) $$\kappa_{ij} = d\theta_{ij} - \sum_{\alpha=1}^{r} \theta_{i\alpha} \wedge \theta_{\alpha j} .$$

$\kappa = \|\kappa_{ij}\|$ is the curvature matrix of D with rsepect to $e_1, \cdots, e_r$, so that

(1.14) $$K(D)e_i = DDe_i = \sum_{j=1}^{r} \kappa_{ij} \otimes e_j .$$

If $e'_1, \cdots, e'_r$ is another C^∞ frame of E on U, let $A = \|a_{ij}\|$ be determined by

(1.15) $$e'_i = \sum_{j=1}^{r} a_{ij} e_j .$$

Let κ' be the curvature matrix of D with respect to $e'_1, \cdots, e'_r$. Then

(1.16) $$\kappa' = A\kappa A^{-1} .$$

Let $\sigma_1, \cdots, \sigma_n$ be the elementary symmetric functions of $X_1, \cdots, X_n$. Define $\sigma_1(\kappa), \cdots, \sigma_n(\kappa)$ by

(1.17) $$\det (I + t\kappa) = 1 + t\sigma_1(\kappa) + \cdots + t^n \sigma_n(\kappa) .$$

$\sigma_j(\kappa)$ is then a $2j$-form on U. Note that if $r < n$, then $\sigma_j(\kappa) = 0$ whenever $r < j \leq n$. (1.16) implies

(1.18) $$\sigma_j(\kappa) = \sigma_j(\kappa') , \qquad j = 1, \cdots, n .$$

Hence by choosing local frames for E throughout M a well-defined differential form $\sigma_j(K(D))$ is obtained on M. $\sigma_j(K(D))$ is closed, i.e.,

(1.19) $$d\sigma_j(K(D)) = 0 .$$

Let $c_1(E), \cdots, c_n(E)$ be the Chern classes of E taken in $H^*(M; C)$. Note that if $r < n$, then $c_j(E) = 0$ whenever $r < j \leq n$. The Chern-Weil theory of characteristic classes [10] asserts that the element of $H^{2j}(M; C)$ determined by $\sigma_j(K(D))$ is $(2\pi/\sqrt{-1})^j c_j(E)$, i.e.,

(1.20) $$[\sigma_j(K(D))] = (2\pi/\sqrt{-1})^j c_j(E) , \qquad j = 1, \cdots, n .$$

In particular, if $\tilde{D}$ is another connection for E, then $[\sigma_j(K(D))] = [\sigma_j(K(\tilde{D}))]$.

Assume $l \leq n$. If $\varphi \in C[X_1, \cdots, X_n]$ is symmetric and homogeneous of degree l, set $\varphi = \tilde{\varphi}(\sigma_1, \cdots, \sigma_l)$. Define $\varphi(E) \in H^{2l}(M; C)$ by

(1.21) $$\varphi(E) = \tilde{\varphi}(c_1(E), \cdots, c_l(E)) .$$

Let D be a connection for E, and set $K = K(D)$. On M define a $2l$-form $\varphi(K)$ by

$$(1.22)\qquad \varphi(K) = \tilde{\varphi}(\sigma_1(K), \cdots, \sigma_l(K)) .$$

(1.9) implies

$$(1.23)\qquad d\varphi(K) = 0 .$$

(1.20) implies

$$(1.24)\qquad [\varphi(K)] = (2\pi/\sqrt{-1})^l \varphi(E) .$$

2. Partial connections

As in § 1, let M be a C^∞ manifold, and E a C^∞ complex vector-bundle on M. If H is a C^∞ sub-vector-bundle of τ, then H^* is a quotient bundle of τ^*. Denote by $\rho\colon \tau^* \to H^*$ the projection of τ^* onto H^*.

(2.1) Definition. A *partial connection* for E is a pair (H, δ) where H is a C^∞ sub-vector-bundle of τ and δ is a C-linear map from $C^\infty(E)$ to $C^\infty(H^* \otimes E)$ such that

$$(2.2)\qquad \delta(fs) = \rho(df) \otimes s + f\delta s ,$$

whenever $f \in A^0$ and $s \in C^\infty(E)$.

Remark. Let (H, δ) be a partial connection for E, and U an open subset of M. If $s \in C^\infty(E)$ vanishes on U, then δs also vanishes on U. From this it follows that (H, δ) restricts to give a partial connection for $E \mid U$:

$$(2.3)\qquad \delta\colon C^\infty(E \mid U) \to C^\infty(\tau^* \otimes E \mid U) .$$

(2.4) Definition. Let (H, δ) be a partial connection for E, and D a connection for E. D *extends* δ if the diagram

$$\begin{tikzcd} C^\infty(E) \arrow[r, "D"] \arrow[dr, "\delta"'] & C^\infty(\tau^* \otimes E) \arrow[d, "\rho \otimes 1"] \\ & C^\infty(H^* \otimes E) \end{tikzcd}$$

is commutative.

(2.5) Lemma. *Let (H, δ) be a partial connection for E. Then there exists a connection D for E such that D extends δ.*

Proof. Cover M by open sets $\{U_\alpha\}$ such that on each U_α there is a C^∞ frame $e_1^\alpha, \cdots, e_r^\alpha$ of E. Define $\gamma_{ij}^\alpha \in C^\infty(H^* \mid U_\alpha)$ by

$$(2.6)\qquad \delta e_i^\alpha = \sum_{j=1}^{r} \gamma_{ij}^\alpha \otimes e_j^\alpha .$$

Choose $\theta_{ij}^\alpha \in C^\infty(\tau^* \mid U_\alpha)$ such that

$$\rho(\theta^\alpha_{ij}) = \gamma^\alpha_{ij} . \tag{2.7}$$

Define a connection D_α for $E \mid U_\alpha$ by

$$D_\alpha e^\alpha_i = \sum_{j=1}^{r} \theta^\alpha_{ij} \otimes e^\alpha_j . \tag{2.8}$$

Then on U_α the diagram

$$\begin{tikzcd} C^\infty(E \mid U_\alpha) \arrow[r, "D_\alpha"] \arrow[dr, "\delta"'] & C^\infty(\tau^* \otimes E \mid U_\alpha) \arrow[d, "\rho \otimes 1"] \\ & C^\infty(H^* \otimes E \mid U_\alpha) \end{tikzcd} \tag{2.9}$$

is commutative.

Let $\{\psi_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$. Define a connection D for E by

$$D = \sum_\alpha \psi_\alpha D_\alpha . \tag{2.10}$$

D extends δ.

(2.11) Lemma. *Let (H, δ) be a partial connection for E, and $s \in C^\infty(E)$ be such that*:

$$s(x) \neq 0 \qquad \textit{for all } x \in M , \tag{2.12}$$

$$\delta s = 0 . \tag{2.13}$$

Then there exists a connection D for E with

$$D \textit{ extends } \delta , \tag{2.14}$$

$$Ds = 0 . \tag{2.15}$$

Proof. Proceed as in the proof of Lemma (2.5) except that e^α_1 is required to be $s \mid U_\alpha$, and θ^α_{ij} is required to be zero.

Remarks. We have the evident pairing $C^\infty(H) \times C^\infty(H^*) \to A^0$. Hence $u \in C^\infty(H)$ determines a map $i(u)$ from $C^\infty(H^*)$ to A^0:

$$i(u): C^\infty(H^*) \to A^0 . \tag{2.16}$$

Similarly, u determines a map, also denoted by $i(u)$, from $C^\infty(H^* \otimes E)$ to $C^\infty(E)$:

$$i(u): C^\infty(H^* \otimes E) \to C^\infty(E) . \tag{2.17}$$

Note also that if $f \in A^0$, then by applying u to f we obtain $u[f] \in A^0$:

$$u[f] = i(u)\rho(df) \ . \tag{2.18}$$

Let (H, δ) be a partial connection for E. Then

$$i(u_1 + u_2)\delta s = i(u_1)\delta s + i(u_2)\delta s \ , \tag{2.19}$$

$$i(fu)\delta s = fi(u)\delta s \ , \tag{2.20}$$

$$i(u)\delta(s_1 + s_2) = i(u)\delta s_1 + i(u)\delta s_2 \ , \tag{2.21}$$

$$i(u)\delta(fs) = u[f]s + fi(u)\delta s \ , \tag{2.22}$$

whenever u, u_1, $u_2 \in C^\infty(H)$, s, s_1, $s_2 \in C^\infty(E)$, and $f \in A^0$.

3. Proof of the vanishing theorem

Let M be a complex-analytic manifold. As in (1.1) set $\tau = \boldsymbol{C} \underset{\boldsymbol{R}}{\otimes} T_{\boldsymbol{R}}M$. Then there are the standard splittings:

$$\tau = T \oplus \bar{T} \ , \tag{3.1}$$

$$\tau^* = T^* \oplus \bar{T}^* \ . \tag{3.2}$$

T is the holomorphic tangent bundle of M. $\bar{T}$ is the anti-holomorphic tangent bundle of M. A C^∞ section of T^* is a 1-form of type (1,0). A C^∞ section of $\bar{T}^*$ is a 1-form of type (0,1).

Let U be an open subset of M. On U let $z_1, \cdots, z_n$ be a complex-analytic coordinate system. Then on U:

(3.3) $\partial/\partial z_1, \cdots, \partial/\partial z_n$ is a holomorphic frame of T,

(3.4) $dz_1, \cdots, dz_n$ is a holomorphic frame of T^*.

Let E be a holomorphic vector-bundle on M. If U is an open subset of M, then $\Gamma(E \mid U)$ will denote the space of all holomorphic sections of $E \mid U$. Since E is holomorphic there is the $\bar{\partial}$ operator:

$$\bar{\partial}\colon C^\infty(E) \to C^\infty(\bar{T}^* \otimes E) \ . \tag{3.5}$$

Setting $H = \bar{T}$ and $\delta = \bar{\partial}$, we then have a partial connection $(\bar{T}, \bar{\partial})$ for E. Note that:

$$\Gamma(E \mid U) = \text{Kernel}\,\{\bar{\partial}\colon C^\infty(E \mid U) \to C^\infty(\bar{T}^* \otimes E \mid U)\} \ . \tag{3.6}$$

A connection for E which extends $(\bar{T}, \bar{\partial})$ is said to be a connection of type (1,0). A straightforward argument shows that a connection D for E is of type (1,0) if and only if D has the following property:

(3.7) Whenever $e_1, \cdots, e_r$ is a holomorphic frame for E, the connection matrix $\|\theta_{ij}\|$ of D with respect to this frame has each θ_{ij} of type (1,0).

The bracket operation for C^∞ sections of T satisfies:

$$[u_1 + u_2, u_3] = [u_1, u_3] + [u_2, u_3] \ , \tag{3.8}$$

$$[fu_1, u_2] = -u_2[f]u_1 + f[u_1, u_2] \ , \tag{3.9}$$

$$[u_1, u_2 + u_3] = [u_1, u_2] + [u_1, u_3] \ , \tag{3.10}$$

$$[u_1, fu_2] = u_1[f]u_2 + f[u_1, u_2] \ , \tag{3.11}$$

whenever $u_1, u_2, u_3 \in C^\infty(T)$ and $f \in A^0$.

Recall also that if U is an open subset of M, and $z_1, \cdots, z_n$ is a complex-analytic coordinate system on U, then

$$[\partial/\partial z_i, \partial/\partial z_j] = 0 \qquad 1 \leq i, j \leq n \ . \tag{3.12}$$

(3.13) Definition. A holomorphic sub-vector-bundle F of T is *integrable* if $C^\infty(F)$ is closed under the bracket operation.

Remark. A holomorphic sub-vector-bundle F of T is integrable if and only if:

(3.14) whenever U is an open subset of M, and $\gamma_1, \gamma_2 \in \Gamma(F \mid U)$, then $[\gamma_1, \gamma_2] \in \Gamma(F \mid U)$.

Assume now that F is an integrable holomorphic sub-vector-bundle of T. Form the quotient bundle T/F and denote by $\eta\colon T \to T/F$ the projection of T onto T/F. Let $u \in C^\infty(F)$ and $s \in C^\infty(T/F)$. Choose $\tilde{s} \in C^\infty(T)$ such that

$$\eta(\tilde{s}) = s \ . \tag{3.15}$$

Then, since $C^\infty(F)$ is closed under bracket,

$$\eta[u, \tilde{s}] \qquad \text{depends only on } u \text{ and } s. \tag{3.16}$$

Denote $\eta[u, \tilde{s}]$ by $\langle u, s\rangle$. Then from (3.8)–(3.11) it is clear that:

$$\langle u_1 + u_2, s\rangle = \langle u_1, s\rangle + \langle u_2, s\rangle \ , \tag{3.17}$$

$$\langle fu, s\rangle = f\langle u, s\rangle \ , \tag{3.18}$$

$$\langle u, s_1 + s_2\rangle = \langle u, s_1\rangle + \langle u, s_2\rangle \ , \tag{3.19}$$

$$\langle u, fs\rangle = u[f]s + f\langle u, s\rangle \ , \tag{3.20}$$

whenever $u, u_1, u_2 \in C^\infty(F)$, $s, s_1, s_2 \in C^\infty(T/F)$, and $f \in A^0$.

Comparing (3.17)–(3.20) to (2.19)–(2.22) and noting that T/F is a holomorphic vector-bundle on M, we then have

(3.21) Proposition. *Let F be an integrable holomorphic sub-vector-bundle of T. Then there exists a unique partial connection $(F \oplus \overline{T}, \delta)$ for T/F such that*

$$(3.22) \qquad i(u)\delta s = \langle u, s \rangle ,$$

$$(3.23) \qquad i(v)\delta s = i(v)\bar{\partial} s ,$$

whenever $u \in C^\infty(F)$, $v \in C^\infty(\bar{T})$, *and* $s \in C^\infty(T/F)$.

(3.24) Definition. A *basic connection* for T/F is a connection for T/F which extends $(F \oplus \bar{T}, \delta)$.

Remarks. (a) A connection D for T/F is basic if and only if

$$(3.25) \qquad i(u)D(\eta\gamma) = \eta[u, \gamma] \qquad \text{whenever } u \in C^\infty(F) \text{ and } \gamma \in C^\infty(T),$$

$$(3.26) \qquad D \text{ is of type } (1,0) .$$

(b) By Lemma (2.5) a basic connection D exists for T/F.

(3.27) Proposition. *Let M be a complex manifold, and F an integrable holomorphic sub-vector-bundle of T. Set $n = \dim_C M$, $k = \dim_C F_x$. Let $\varphi \in C[X_1, \cdots, X_n]$ be symmetric and homogeneous of degree l, where $n - k < l \leq n$. Let D be a basic connection for T/F, and set $K = K(D)$. Then*

$$(3.28) \qquad \varphi(K) = 0 .$$

Proof. Given $p \in M$, let U be an open neighborhood of p in M such that on U there is a complex-analytic coordinate system $z_1, \cdots, z_n$ with

$$(3.29) \qquad \partial/\partial z_1, \partial/\partial z_2, \cdots, \partial/\partial z_k \in \Gamma(F \mid U) .$$

Let $A(U)$ be the set of all C^∞ complex-valued differential forms on U. $A(U)$ is a ring under the usual addition and wedge product of differential forms. In $A(U)$, let $I(F, U)$ be the ideal generated by $dz_{k+1}, \cdots, dz_n$. This ideal has the properties:

(3.30) If $\omega \in I(F, U)$, then $d\omega \in I(F, U)$.

(3.31) If $\omega_1, \cdots, \omega_{n-k+1}$ are any $n - k + 1$ elements of $I(F, U)$, then $\omega_1 \wedge \cdots \wedge \omega_{n-k+1} = 0$.

Let $\eta\colon T \to T/F$ be the projection, and $\theta = \|\theta_{ij}\|$ the connection matrix of D with respect to the frame $\eta\partial/\partial z_{k+1}, \cdots, \eta\partial/\partial z_n$. D is basic, so (3.26) implies that each θ_{ij} is of type (1,0). (3.25) and (3.12) imply that for each θ_{ij}

$$(3.32) \qquad 0 = i(\partial/\partial z_1)\theta_{ij} = \cdots = i(\partial/\partial z_k)\theta_{ij} .$$

Hence each θ_{ij} is in $I(F, U)$. Let $\kappa = \|\kappa_{ij}\|$ be the curvature matrix of D with respect to $\eta\partial/\partial z_{k+1}, \cdots, \eta\partial/\partial z_n$. From (3.30) and (1.13) it is clear that each κ_{ij} is in $I(F, U)$:

$$(3.33) \qquad \kappa_{ij} \in I(F, U) .$$

As in (1.17) define $\sigma_1(\kappa), \cdots, \sigma_n(\kappa)$ by

$$\det(I + t\kappa) = 1 + t\sigma_1(\kappa) + \cdots + t^n\sigma_n(\kappa) . \tag{3.34}$$

Set $\varphi = \tilde{\varphi}(\sigma_1, \cdots, \sigma_l)$, and $l = \deg \varphi$. Then on U,

$$\varphi(K) \,|\, U = \tilde{\varphi}(\sigma_1(\kappa), \cdots, \sigma_l(\kappa)) . \tag{3.35}$$

Since $l \geq n - k + 1$, (3.31) and (3.33) now imply that $\varphi(K)$ vanishes on U. This proves (3.28).

Due to (1.24), (0.51) is now evident.

4. Exact sequences

Some well-known facts about connections and exact sequences of vector bundles are collected here. As in § 1, the matters considered here are purely C^∞. So in this section let M be a C^∞ manifold. Let $m = \dim_{\boldsymbol{R}} M$, and let n be the largest integer with $n \leq m/2$.

If E is a C^∞ complex vector-bundle on M, let $c(E)$ denote the total Chern class of E in $H^*(M; \boldsymbol{C})$, so that

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) . \tag{4.1}$$

Note that in the ring $H^*(M; \boldsymbol{C}) = H^0(M; \boldsymbol{C}) \oplus H^1(M; \boldsymbol{C}) \oplus \cdots \oplus H^m(M; \boldsymbol{C})$, $c(E)$ is invertible.

If E_1, E_0 are two C^∞ vector-bundles on M, then the total Chern class of the virtual bundle $E_0 - E_1$ is defined by

$$c(E_0 - E_1) = c(E_0)/c(E_1) . \tag{4.2}$$

Thus the Chern classes $c_1(E_0 - E_1), \cdots, c_n(E_0 - E_1)$ are determined by

$$c_j(E_0 - E_1) \in H^{2j}(M; \boldsymbol{C}) , \tag{4.3}$$

$$c(E_0)/c(E_1) = 1 + c_1(E_0 - E_1) + \cdots + c_n(E_0 - E_1) . \tag{4.4}$$

More generally, let $E_q, E_{q-1}, \cdots, E_0$ be C^∞ complex vector-bundles on M. Set $\varepsilon(i) = (-1)^i$. Then the total Chern class of the virtual bundle $\sum_{i=0}^q (-1)^i E_i$ is defined by

$$c\left(\sum_{i=0}^q (-1)^i E_i\right) = \prod_{i=0}^q (c(E_i))^{\varepsilon(i)} , \qquad \varepsilon(i) = (-1)^i . \tag{4.5}$$

Set $\zeta = \sum_{i=0}^q (-1)^i E_i$. Thus the Chern classes $c_1(\zeta), \cdots, c_n(\zeta)$ are determined by

$$c_j(\zeta) \in H^{2j}(M; \boldsymbol{C}) , \tag{4.6}$$

$$\prod_{i=0}^q (c(E_i))^{\varepsilon(i)} = 1 + c_1(\zeta) + \cdots + c_n(\zeta) . \tag{4.7}$$

Let $\varphi \in C[X_1, \cdots, X_n]$ be symmetric and homogeneous of degree l. Assume $l \leq n$. Set $\varphi = \tilde{\varphi}(\sigma_1, \cdots, \sigma_l)$. Then $\varphi(\zeta)$ is defined by

$$\varphi(\zeta) = \tilde{\varphi}(c_1(\zeta), \cdots, c_l(\zeta)) . \tag{4.8}$$

Hence $\varphi(\zeta) \in H^{2l}(M; C)$.

Suppose now that $D_q, D_{q-1}, \cdots, D_0$ are connections for $E_q, E_{q-1}, \cdots, E_0$ respectively. Set $K_i = K(D_i)$, and define differential forms $\sigma_j(K_q | K_{q-1} | \cdots | K_0)$ by

$$\sigma_j(K_q | K_{q-1} | \cdots | K_0) \in A^{2j} , \qquad j = 1, \cdots, n , \tag{4.9}$$

$$\begin{aligned} &\prod_{i=0}^{q} (\det (I + K_i))^{\epsilon(i)} \\ &\quad = 1 + \sigma_1(K_q | K_{q-1} | \cdots | K_0) + \cdots + \sigma_n(K_q | K_{q-1} | \cdots | K_0) . \end{aligned} \tag{4.10}$$

(1.19) implies

$$d\sigma_j(K_q | K_{q-1} | \cdots | K_0)) = 0 . \tag{4.11}$$

(1.20) and (4.7) imply

$$[\sigma_j(K_q | K_{q-1} | \cdots | K_0)] = (2\pi/\sqrt{-1})^j c_j(\zeta) . \tag{4.12}$$

As above let φ be symmetric and homogeneous of degree $l \leq n$. Set $\varphi = \tilde{\varphi}(\sigma_1, \cdots, \sigma_l)$ and $\omega_j = \sigma_j(K_q | K_{q-1} | \cdots | K_0)$. Define a $2l$-form $\varphi(K_q | K_{q-1} | \cdots | K_0)$ on M by

$$\varphi(K_q | K_{q-1} | \cdots | K_0) = \tilde{\varphi}(\omega_1, \cdots, \omega_l) . \tag{4.13}$$

Then

$$d\varphi(K_q | K_{q-1} | \cdots | K_0) = 0 , \tag{4.14}$$

$$[\varphi(K_q | K_{q-1} | \cdots | K_0)] = (2\pi/\sqrt{-1})^l \varphi(\zeta) , \tag{4.15}$$

where as above $\zeta = \sum_{i=0}^{q} (-1)^i E_i$.

(4.16) Definition. Let $0 \to E_q \to E_{q-1} \to \cdots \to E_0 \to E_{-1} \to 0$ be an exact sequence of C^∞ vector-bundles on M. Denote by η_i the map from E_i to E_{i-1}. Let $D_q, D_{q-1}, \cdots, D_0, D_{-1}$ be connections for $E_q, E_{q-1}, \cdots, E_0, E_{-1}$ respectively. Then $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ is *compatible* with the exact sequence if for each $i = q, q-1, \cdots, 0$ the diagram

$$\begin{array}{ccc} C^\infty(E_i) & \longrightarrow & C^\infty(\tau^* \otimes E^i) \\ {\scriptstyle \eta_i}\big\downarrow & & \big\downarrow{\scriptstyle 1 \otimes \eta_i} \\ C^\infty(E_{i-1}) & \longrightarrow & C^\infty(\tau^* \otimes E_{i-1}) \end{array}$$

is commutative.

(4.17) Lemma. *Let $0 \to E_q \to E_{q-1} \to \cdots \to E_0 \to E_{-1} \to 0$ be an exact sequence of C^∞ vector bundles on M, and D_{-1} be a connection for E_{-1}. Then there exist connections $D_q, D_{q-1}, \cdots, D_0$ for $E_q, E_{q-1}, \cdots, E_0$ such that $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ is compatible with the exact sequence.*

Proof. Proceed by induction on q. If $q = 0$, the exact sequence is $0 \to E_0 \to E_{-1} \to 0$. D_{-1} then determines a unique connection D_0 for E_0 such that (D_0, D_{-1}) is compatible with the exact sequence.

Assume now that the lemma is valid for $q - 1$. Consider an exact sequence $0 \to E_q \to E_{q-1} \to \cdots \to E_0 \to E_{-1} \to 0$. Let $\eta_q(E_q)$ be the image of $\eta_q \colon E_q \to E_{q-1}$. Choose a C^∞ sub-vector-bundle J of E_{q-1} such that

$$E_{q-1} = J \oplus \eta_q(E_q) \ . \tag{4.18}$$

By the induction hypotheses there exist connections $D_{q-1}, D_{q-2}, \cdots, D_0$ for $J, E_{q-2}, \cdots, E_0$ such that

(4.19) $(D_{q-1}, D_{q-2}, \cdots, D_0, D_{-1})$ is compatible with the exact sequence $0 \to J \to E_{q-2} \to \cdots \to E_0 \to E_{-1} \to 0$.

Choose a connection D for $\eta_q(E_q)$. Let D_q be the unique connection for E_q such that

(4.20) (D_q, D) is compatible with the exact sequence $0 \to E_q \to \eta_q(E_q) \to 0$.

On $E_{q-1} = J \oplus \eta_q(E_q)$ let $D_{q-1} \oplus D$ be the direct sum connection. Thus

$$(D_{q-1} \oplus D)(s_1 + s_2) = D_{q-1}s_1 + Ds_2 \ , \tag{4.21}$$

whenever $s_1 \in C^\infty(J)$ and $s_2 \in C^\infty(\eta_q(E_q))$. Then $(D_q, D_{q-1} \oplus D, D_{q-2}, \cdots, D_0, D_{-1})$ is compatible with the exact sequence $0 \to E_q \to E_{q-1} \to \cdots \to E_0 \to E_1 \to 0$. This proves the lemma.

(4.22) Lemma. *Let $0 \to E_q \to E_{q-1} \to \cdots \to E_0 \to E_{-1} \to 0$ be an exact sequence of C^∞ vector bundles on M, and $D_q, D_{q-1}, \cdots, D_0, D_{-1}$ be connections for $E_q, E_{q-1}, \cdots, E_0, E_{-1}$. Assume that $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ is compatible with the exact sequence. Let φ be symmetric and homogeneous of degree $l \leq n$. Set $K_i = K(D_i)$. Then*

$$\varphi(K_{-1}) = \varphi(K_q \,|\, K_{q-1} \,|\, \cdots \,|\, K_0) \ . \tag{4.23}$$

Proof. Set $\varepsilon(i) = (-1)^i$. To prove (4.23) it suffices to show

$$\det(I + K_{-1}) = \prod_{i=0}^{q} (\det(I + K_i))^{\varepsilon(i)} \ . \tag{4.24}$$

To prove (4.24) proceed by induction on q. If $q = 0$, the exact sequence is $0 \to E_0 \to E_{-1} \to 0$ and (4.24) is obvious in this case.

Assume now that (4.24) is valid for $q-1$, and consider an exact sequence $0 \to E_q \to E_{q-1} \to \cdots \to E_0 \to E_{-1} \to 0$. Let $\eta_q(E_q)$ be the image of $\eta_q\colon E_q \to E_{q-1}$. Choose a C^∞ sub-vector-bundle J of E_{q-1} such that

(4.25) $$E_{q-1} = J \oplus \eta_q(E_q)\ .$$

Let $\rho\colon E_{q-1} \to J$ be the projection of E_{q-1} onto J resulting from this direct sum decomposition. So we have

(4.26) $$1 \otimes \rho\colon \tau^* \otimes E_{q-1} \to \tau^* \otimes J\ .$$

Define a connection ∇ for J by

(4.27) $$\nabla = (1 \otimes \rho)D_{q-1}\ .$$

Then

(4.28) $$\det(I + K_{q-1}) = \det(I + K_q)\det(I + K(\nabla))\ ,$$

(4.29) $(\nabla, D_{q-2}, \cdots, D_0, D_{-1})$ is compatible with the exact sequence $0 \to J \to E_{q-2} \to \cdots \to E_0 \to E_{-1} \to 0$.

The induction hypotheses and (4.29) imply

(4.30) $$\det(I + K_{-1}) = \det(I + K(\nabla))^{\varepsilon(q-1)} \prod_{i=0}^{q-2} (\det(I + K_i))^{\varepsilon(i)}\ .$$

(4.30) and (4.28) combine to give

(4.31) $$\det(I + K_{-1}) = \prod_{i=0}^{q} (\det(I + K_i))^{\varepsilon(i)}\ .$$

This completes the inductive step and the proof.

(4.32) Lemma. *On M let*

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & E'_q & \to & E'_{q-1} & \to & \cdots & \to & E'_0 & \to & E'_{-1} & \to & 0\\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & E_q & \to & E_{q-1} & \to & \cdots & \to & E_0 & \to & E_{-1} & \to & 0\\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & E''_q & \to & E''_{q-1} & \to & \cdots & \to & E''_0 & \to & E''_{-1} & \to & 0\\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

be a commutative diagram of C^∞ vector bundles in which each row and each *column are exact. Let $\iota_j(E'_j)$ be the image of the map $\iota_j\colon E'_j \to E_j$. Then there exist C^∞ sub-vector-bundles $F_q, F_{q-1}, \cdots, F_0, F_{-1}$ of $E_q, E_{q-1}, \cdots, E_0, E_{-1}$ such that*

$$E_j = F_j \oplus \iota_j(E'_j) , \qquad j = q, q-1, \cdots, 0, -1 , \tag{4.33}$$

(4.34) *the map $\eta_j\colon E_j \to E_{j-1}$ maps F_j into F_{j-1}, $j = q, q-1, \cdots, 0$.*

Proof. Construct $F_q, F_{q-1}, \cdots, F_0, F_{-1}$ by a downward induction. First, let F_q be any C^∞ sub-vector-bundle of E_q such that

$$E_q = F_q \oplus \iota_q(E'_q) . \tag{4.35}$$

Next, suppose that $F_q, F_{q-1}, \cdots, F_r$ have been constructed so that

$$E_j = F_j \oplus \iota_j(E'_j) , \qquad j = q, q-1, \cdots, r , \tag{4.36}$$

$$\eta_j(F_j) \subset F_{j-1} , \qquad j = q, q-1, \cdots, r+1 . \tag{4.37}$$

$\eta_r(F_r)$ is then a C^∞ sub-vector-bundle of E_{r-1}. A diagram chase shows that

$$\eta_r(F_r) \cap \iota_{r-1}(E'_{r-1}) = \{0\} . \tag{4.38}$$

Hence there exists a C^∞ sub-vector-bundle F_{r-1} of E_{r-1} such that

$$E_{r-1} = F_{r-1} \oplus \iota_{r-1}(E'_{r-1}) , \tag{4.39}$$

$$\eta_r(F_r) \subset F_{r-1} . \tag{4.40}$$

This completes the inductive step and the proof.

(4.41) Lemma. *Let E be a C^∞ vector-bundle on M, and B a closed subset of M. On $M - B$ let D be a connection for $E|M - B$. Let Σ be a closed subset of M such that B is contained in the interior of Σ. Then on M there exists a connection $\tilde{D}$ for E such that*

$$D \text{ and } \tilde{D} \text{ agree on } E|M - \Sigma . \tag{4.42}$$

Proof. On M let ∇ be a connection for E. Let $\psi\colon M \to \boldsymbol{R}$ be a C^∞ function such that

(4.43) ψ vanishes on a neighborhood of B,

$$\psi = 1 \quad \text{on} \quad M - \Sigma . \tag{4.44}$$

Set $\tilde{D} = \psi D + (1 - \psi)\nabla$. $\tilde{D}$ satisfies (4.42).

(4.45) Lemma. *On M let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of C^∞ vector-bundles. Denote by $\iota(E')$ the image of $\iota\colon E' \to E$. Let B be a closed subset of M. On $M - B$ let F be a C^∞ sub-vector-bundle of $E|M - B$ such that*

(4.46) $$E\,|\,M - B = F \oplus \iota(E'\,|\,M - B)\ .$$

Let Σ be a closed subset of M such that B is contained in the interior of Σ. Then on M there exists a C^∞ sub-vector-bundle $\tilde{F}$ of E such that

(4.47) $$E = \tilde{F} \oplus \iota(E')\ ,$$

(4.48) $$\tilde{F}\,|\,M - \Sigma = F\,|\,M - \Sigma\ .$$

Proof. Denote by $\mu\colon E \to E''$ the map from E to E''. On $M - B$ there is a unique map $\alpha\colon E''\,|\,M - B \to E\,|\,M - B$ such that

(4.49) $$\alpha(E''\,|\,M - B) = F\ ,$$

(4.50) $$\mu\alpha = 1\ .$$

On M let $\beta\colon E'' \to E$ be a map of C^∞ vector-bundles such that

(4.51) $$\mu\beta = 1\ .$$

Let $\psi\colon M \to \boldsymbol{R}$ be a C^∞ function with (4.43) and (4.44) valid for ψ. On M define $\tilde{\alpha}\colon E'' \to E$ by

(4.52) $$\tilde{\alpha} = \psi\alpha + (1 - \psi)\beta\ .$$

Set $\tilde{F} = \tilde{\alpha}(E'')$. $\tilde{F}$ satisfies (4.47) and (4.48).

(4.53) Remark. Let M, X be C^∞ manifolds, E be a C^∞ vector-bundle on X, and $g\colon M \to X$ be a C^∞ map. Then on M there is the pull-back bundle $g^!(E)$:

(4.54) $$g^!(E)_p = E_{gp}\ , \qquad p \in M\ .$$

Let U be an open subset of X, and let $s \in C^\infty(E\,|\,U)$. Then on $g^{-1}(U)$ there is $g^!(s) \in C^\infty(g^!(E)\,|\,g^{-1}(U))$:

(4.55) $$g^!(s)p = s(gp)\ , \qquad p \in M\ .$$

If D is a connection for E, then there is the pull-back connection $g^!(D)$ for $g^!(E)$.

Let $e_1, \cdots, e_r$ be a C^∞ frame of $g^!(E)$, $\theta = \|\theta_{ij}\|$ be the connection matrix of D with respect to $e_1, \cdots, e_r$, and $\omega = \|\omega_{ij}\|$ be the connection matrix of $g^!(D)$ with respect to $g^!(e_1), \cdots, g^!(e_r)$. Then for each θ_{ij},

(4.56) $$\omega_{ij} = g^*\theta_{ij}\ .$$

(4.56) characterizes $g^!(D)$. Here g^* is the usual map

(4.57) $$g^*\colon C^\infty(\tau^*X\,|\,U) \to C^\infty(\tau^*M\,|\,g^{-1}(U))\ .$$

If E' is another C^∞ vector-bundle on X, and $\eta\colon E' \to E$ is a map of C^∞ vector-bundles, then there is

$$g^!(\eta)\colon g^!(E') \to g^!(E) \ . \tag{4.58}$$

For $p \in M$, $g^!(\eta)\colon g^!(E')_p \to g^!(E)_p$ is $\eta\colon E'_{gp} \to E_{gp}$.

Example. Suppose that M is complex-analytic, and that F is a holomorphic integrable sub-vector-bundle of T. Assume that the foliation determined by F is a fibration. Let X be the base of this fibration, and $\pi\colon M \to X$ the projection of M onto X. Then

$$T/F = \pi^!(TX) \ . \tag{4.59}$$

On X let D be a connection of type (1,0) for TX. $\pi^!(D)$ is then a basic connection for T/F.

5. Z-sequences

As in the introduction let M be complex-analytic, and ξ an integrable subsheaf of $\underline{T}$. Let k be the leaf dimension of ξ, and S the singular set. On $M - S$, let F be the unique holomorphic sub-vector-bundle of T such that

$$\underline{F} = \xi | M - S \ . \tag{5.1}$$

Here $\underline{F}$ denotes the sheaf of germs of holomorphic sections of F. On $M - S$ set

$$\nu = T/F \ . \tag{5.2}$$

(5.3) Lemma. *Let W be an open subset of $M - S$. On W, let D and D' be two basic connections for $\nu | W$. Set $\tilde W = W \times [0, 1]$. Let $\rho\colon \tilde W \to W$ and $t\colon \tilde W \to [0, 1]$ be the projections. On $\tilde W$ define a connection ∇ for $\rho^!(\nu | W)$ by*

$$\nabla = t\rho^!(D') + (1 - t)\rho^!(D) \ . \tag{5.4}$$

Set $K = K(\nabla)$. Let $\varphi \in C[X_1, \cdots, X_n]$ be symmetric and homogeneous of degree l. Assume $n - k < l \leq n$. Then

$$\varphi(K) = 0 \ . \tag{5.5}$$

Proof. The proof is very much like the proof of (3.28). Given $p \in W$, let W_p be an open neighborhood of p in W such that W_p is the domain of a complex-analytic coordinate system $z_1, \cdots, z_n$ with

$$\partial/\partial z_1, \cdots, \partial/\partial z_k \in \Gamma(F | W_p) \tag{5.6}$$

Set $\tilde W_p = W_p \times [0, 1]$. Let $A(\tilde W_p)$ be the ring of all C^∞ complex-valued differential forms on $\tilde W_p$. In $A(\tilde W_p)$ let $I(F, \tilde W_p)$ be the ideal generated by

$\rho^*(dz_{k+1}), \cdots, \rho^*(dz_n)$. This ideal has the properties:

(5.7) $\quad$ If $\omega \in I(F, \tilde{W}_p)$, then $d\omega \in I(F, \tilde{W}_p)$.

(5.8) If $\omega_1, \cdots, \omega_{n-k+1}$ are any $n-k+1$ elements of $I(F, \tilde{W}_p)$, then $\omega_1 \wedge \cdots \wedge \omega_{n-k+1} = 0$.

On W_p let $\eta \colon T \to \nu$ be the projection of T onto ν. Let $\theta = \|\theta_{ij}\|$ and $\theta' = \|\theta'_{ij}\|$ be the connection matrices of D and D' with respect to the frame $\eta\partial/\partial z_{k+1}, \cdots, \eta\partial/\partial z_n$. Let $\omega = \|\omega_{ij}\|$ be the connection matrix of ∇ with respect to the frame $\rho^!\eta\partial/\partial z_{k+1}, \cdots, \rho^!\eta\partial/\partial z_n$. Then according to (5.4), for each ω_{ij}:

(5.9)
$$\omega_{ij} = t\rho^*(\theta'_{ij}) + (1-t)\rho^*(\theta_{ij}) .$$

Since D and D' are basic, (3.25) and (3.26) now imply that each ω_{ij} is in $I(F, \tilde{W}_p)$:

(5.10)
$$\omega_{ij} \in I(F, \tilde{W}_p) .$$

Let $\kappa = \|\kappa_{ij}\|$ be the curvature matrix of ∇ with respect to the frame $\rho^!\eta\partial/\partial z_{k+1}, \cdots, \rho^!\eta\partial/\partial z_n$. Then (5.7) and (1.13) imply that each κ_{ij} is in $I(F, \tilde{W}_p)$:

(5.11)
$$\kappa_{ij} \in I(F, \tilde{W}_p) .$$

Let $\sigma_1, \cdots, \sigma_n$ be the elementary symmetric functions of $X_1, \cdots, X_n$. Set $\varphi = \tilde{\varphi}(\sigma_1, \cdots, \sigma_l)$. On $\tilde{W}_p$ define differential forms $\sigma_1(\kappa), \cdots, \sigma_n(\kappa)$ by requiring:

(5.12) $\quad \sigma_j(\kappa)$ in a $2j$-form on $\tilde{W}_p$, $\quad j = 1, \cdots, n$,

(5.13)
$$\det(I + \kappa) = 1 + \sigma_1(\kappa) + \cdots + \sigma_n(\kappa) .$$

Then

(5.14)
$$\varphi(K) \,|\, \tilde{W}_p = \tilde{\varphi}(\sigma_1(\kappa), \cdots, \sigma_l(\kappa)) .$$

Since $\deg \varphi > n-k$, (5.14), (5.11) and (5.8) imply that $\varphi(K)$ vanishes on $\tilde{W}_p$. This proves the lemma.

(5.15) Definition. Let Z be a connected component of the singular set S. A Z-sequence β is a triple $\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$ such that the following five conditions are satisfied:

(5.16) U is an open subset of M such that $U \cap S = Z$ and Z is a deformation retract of U.

(5.17) $E_q, E_{q-1}, \cdots, E_0$ are C^∞ complex vector-bundles on U.

(5.18) For $i = q, q-1, \cdots, 1$, η_i is a C^∞ vector-bundle map from $E_i \,|\, U - Z$ to $E_{i-1} \,|\, U - Z$.

(5.19) η_0 is a C^∞ vector-bundle map from $E_0 \,|\, U - Z$ to $\nu \,|\, U - Z$.

(5.20) On $U - Z$ the sequence

$$0 \to E_q | U - Z \to E_{q-1} | U - Z \to \cdots \to E_0 | U - Z \to \nu | U - Z \to 0$$

is exact.

Remark. Note that although each E_i is a vector-bundle on all of U, η_i exists only on $U - Z$:

(5.21) $\eta_i \colon E_i | U - Z \to E_{i-1} | U - Z , \qquad i = q, q-1, \cdots, 1 ,$

(5.22) $\eta_0 \colon E | U - Z \to \nu | U - Z .$

(5.23) Definition. Let Z be a connected component of S. Assume that Z is compact. Let $\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$ be a Z-sequence. On U let $D_q, D_{q-1}, \cdots, D_0$ be connections for $E_q, E_{q-1}, \cdots, E_0$. On $U - Z$, let D_{-1} be a connection for $\nu | U - Z$. Then $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ is *fitted to* β *if*

(5.24) D_{-1} is a basic connection for $\nu | U - Z$,

(5.25) there exists a compact subset Σ of U with Z contained in the interior of Σ such that on $U - \Sigma$, $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ is compatible with the exact sequence

$$0 \to E_q | U - \Sigma \to E_{q-1} | U - \Sigma \to \cdots \to E_0 | U - \Sigma \to \nu | U - \Sigma \to 0 .$$

(5.26) Lemma. *Let Z be a connected component of S. Assume that Z is compact. Let $\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$ be a Z-sequence, and D_{-1} a basic connection for $\nu | U - Z$. Then on U there exist connections $D_q, D_{q-1}, \cdots, D_0$ for $E_q, E_{q-1}, \cdots, E_0$ such that*

(5.27) $\qquad (D_q, D_{q-1}, \cdots, D_0, D_{-1}) \qquad$ *is fitted to* β.

Proof. According to Lemma (4.17) on $U - Z$ there exist connections ∇_q, $\nabla_{q-1}, \cdots, \nabla_0$ for $E_q | U - Z, E_{q-1} | U - Z, \cdots, E_0 | U - Z$ such that

(5.28) $(\nabla_q, \nabla_{q-1}, \cdots, \nabla_0, D_{-1})$ is compatible with the exact sequence

$$0 \to E_q | U - Z \to E_{q-1} | U - Z \to \cdots \to E_0 | U - Z \to \nu | U - Z \to 0 .$$

Let Σ be a compact subset of U with Z contained in the interior of Σ. According to Lemma (4.41) on U there exist connections D_q, $D_{q-1}, \cdots, D_0$ for E_q, $E_{q-1}, \cdots, E_0$ such that

(5.29) D_i and ∇_i agree on $E_i | U - Z$, $i = q, q-1, \cdots, 0$.

$(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ is then fitted to β.

(5.30) Remark. Note that given $\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$ and given any compact subset Σ of U with Z contained in the interior of

Σ, one can then construct $D_q, D_{q-1}, \cdots, D_0, D_{-1}$ such that (5.25) is valid for these D_i and the given Σ.

(5.31) Proposition. *Let Z be a connected component of S. Assume that Z is compact. Let $\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$ be a Z-sequence. Assume that $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ is fitted to β. Let $\varphi \in C[X_1, X_2, \cdots, X_n]$ be symmetric and homogeneous of degree l, where $n - k < l \le n$. Set $K_i = K(D_i)$. On U consider the $2l$-form $\varphi(K_q | K_{q-1} | \cdots | K_0)$. Then*

$$(5.32) \qquad \varphi(K_q | K_{q-1} | \cdots | K_0) \qquad \textit{has compact support.}$$

Moreover, suppose $(D'_q, D'_{q-1}, \cdots, D'_0, D'_{-1})$ is also fitted to β. Set $K'_i = K(D'_i)$. Then there exists a $2l - 1$ form ω on U such that

(5.33) *ω has compact support,*

$$(5.34) \qquad d\omega = \varphi(K'_q | K'_{q-1} | \cdots | K'_0) - \varphi(K_q | K_{q-1} | \cdots | K_0) \ .$$

Proof. Given $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ let Σ be as in (5.25). Then according to (4.23) on $U - \Sigma$,

$$(5.35) \qquad \varphi(K_q | K_{q-1} | \cdots | K_0) | U - \Sigma = \varphi(K_{-1}) | U - \Sigma \ .$$

Hence by (3.28), $\varphi(K_q | K_{q-1} | \cdots | K_0)$ vanishes on $U - \Sigma$. This proves (5.32).

To prove (5.33) and (5.34) we may assume that on $U - \Sigma$, $(D'_q, D'_{q-1}, \cdots, D'_0, D'_{-1})$ is also compatible with the exact sequence

$$(5.36) \quad 0 \to E_q | U - \Sigma \to E_{q-1} | U - \Sigma \to \cdots \to E_0 | U - \Sigma \to \nu | U - \Sigma \to 0 \ .$$

Define $\tilde{U}$, $\tilde{\Sigma}$, $\tilde{Z}$ by

$$(5.37) \qquad \tilde{U} = U \times [0, 1] \ ,$$

$$(5.38) \qquad \tilde{\Sigma} = \Sigma \times [0, 1] \ ,$$

$$(5.39) \qquad \tilde{Z} = Z \times [0, 1] \ .$$

Let $\rho : \tilde{U} \to U$ and $t : \tilde{U} \to [0, 1]$ be the projections. On $\tilde{U}$ there is the pull-back bundle $\rho^!(E_i)$, and there are the pull-back connections $\rho^!(D'_i)$ and $\rho^!(D_i)$ for $\rho^!(E_i)$. On $\tilde{U}$ define a connection ∇_i for $\rho^!(E_i)$ by

$$(5.40) \qquad \nabla_i = t\rho^!(D'_i) + (1 - t)\rho^!(D_i) \ , \qquad i = q, q-1, \cdots, 0 \ .$$

Set $\tilde{K}_i = K(\nabla_i)$. On $\tilde{U} - \tilde{Z}$ set

$$(5.41) \qquad \tilde{\nu} = \rho^!(\nu) \ ,$$

$$(5.42) \qquad \nabla_{-1} = t\rho^!(D'_{-1}) + (1 - t)\rho^!(D_{-1}) \ ,$$

$$(5.43) \qquad \tilde{K}_{-1} = K(\nabla_{-1}) \ .$$

Define $i_0 : U \to \tilde{U}$ and $i_1 : U \to \tilde{U}$ by

(5.44) $$i_0(x) = (x, 0) , \qquad x \in U ,$$

(5.45) $$i_1(x) = (x, 1) , \qquad x \in U ,$$

Then (5.40) and (4.56) imply

(5.46) $$i_0^* \varphi(\tilde{K}_q | \tilde{K}_{q-1} | \cdots | \tilde{K}_0) = \varphi(K_q | K_{q-1} | \cdots | K_0) ,$$

(5.47) $$i_1^* \varphi(\tilde{K}_q | \tilde{K}_{q-1} | \cdots | \tilde{K}_0) = \varphi(K'_q | K'_{q-1} | \cdots | K'_0) .$$

Hence in order to prove (5.33) and (5.34) it suffices to prove

(5.48) $$\varphi(\tilde{K}_q | \tilde{K}_{q-1} | \cdots | \tilde{K}_0) \quad \text{vanishes on} \quad \tilde{U} - \tilde{\Sigma} .$$

If the exact sequence (5.36) is pulled back by $\rho^!$ to $\tilde{U} - \tilde{\Sigma}$, then on $\tilde{U} - \tilde{\Sigma}$ $(\tilde{V}_q, \tilde{V}_{q-1}, \cdots, \tilde{V}_0, \tilde{V}_{-1})$ is compatible with the pulled-back exact sequence. So according to (4.23), on $\tilde{U} - \tilde{\Sigma}$

(5.49) $$\varphi(\tilde{K}_q | \tilde{K}_{q-1} | \cdots | \tilde{K}_0) | \tilde{U} - \tilde{\Sigma} = \varphi(\tilde{K}_{-1}) | \tilde{U} - \tilde{\Sigma} .$$

By (5.5), $\varphi(\tilde{K}_{-1}) = 0$. This completes the proof.

Remark. Let Z be compact, and $\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$ be a Z-sequence. Let $H_c^*(U; \boldsymbol{C})$ denote the cohomology of U with compact supports and coefficients $\boldsymbol{C}$. Then there are isomorphisms:

(5.50) $$H_c^j(U; \boldsymbol{C}) \to H_{2n-j}(U; \boldsymbol{C}) \leftarrow H_{2n-j}(Z; \boldsymbol{C}) .$$

The isomorphism $H_c^j(U; \boldsymbol{C}) \to H_{2n-j}(U; \boldsymbol{C})$ is the usual Poincaré duality isomorphism. The isomorphism $H_{2n-j}(U; \boldsymbol{C}) \leftarrow H_{2n-j}(Z; \boldsymbol{C})$ is given by the inclusion of Z in U. Recall that by (5.16), Z is a deformation retract of U. Thus a closed j-form ω on U with compact support determines an element of $H_{2n-j}(Z; \boldsymbol{C})$.

We come now to the main definition of this section.

(5.51) Definition. Let Z be a connected component of S. Assume that Z is compact. Let $\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$ be a Z-sequnce. Choose connections $D_q, D_{q-1}, \cdots, D_0, D_{-1}$ such that $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ is fitted to β. Set $K_i = K(D_i)$. Let φ be symmetric and homogeneous of degree l, where $n - k < l \leq n$. Define $\operatorname{Res}_\varphi(\xi, Z, \beta) \in H_{2n-2l}(Z; \boldsymbol{C})$ to be the element of $H_{2n-2l}(Z; \boldsymbol{C})$ determined by $(\sqrt{-1}/(2\pi))^l \varphi(K_q | K_{q-1} | \cdots | K_0)$.

Remarks. (a) (5.33) and (5.34) imply that $\operatorname{Res}_\varphi(\xi, Z, \beta)$ depends only on φ, ξ, Z, and β. $\operatorname{Res}_\varphi(\xi, Z, \beta)$ does not depend on the choice of $D_q, D_{q-1}, \cdots, D_0, D_{-1}$.

(b) Since Z is a compact holomorphic subvariety of M, Z has the property:

(5.52) Let V be any open subset of M with $V \supset Z$. Then there exists an open subset V_1 of M such that $V \supset V_1 \supset Z$ and Z is a deformation retract of V_1.

(c) For $\mathrm{Res}_\varphi(\xi, Z, \beta)$ only the local structure of ξ and β near Z is relevant. Let $\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$. Let W be an open subset of M with $U \supset W \supset Z$, and Z a deformation retract of W. Set $\beta | W = (W, (E_q | W, E_{q-1} | W, \cdots, E_0 | W), (\eta_q | W, \eta_{q-1} | W, \cdots \eta_0 | W))$. Then

$$\mathrm{Res}_\varphi(\xi, Z, \beta) = \mathrm{Res}_\varphi(\xi, Z, \beta | W) . \tag{5.53}$$

(5.53) can be proved by applying Remark (5.30). Choose $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ fitted to β so that the Σ of (5.25) is contained in W. $(D_q | W, D_{q-1} | W, \cdots, D_0 | W, D_{-1} | W)$ is then fitted to $\beta | W$. Hence the element of $H_{2n-l}(Z; C)$ determined by $(\sqrt{-1}/(2\pi))^l \varphi(K_q | K_{q-1} | \cdots | K_0)$ is $\mathrm{Res}_\varphi(Z, \xi, \beta)$ and is also $\mathrm{Res}_\varphi(Z, \xi, \beta | W)$.

The next proposition will be used in § 7.

(5.54) Definition. Let β and γ be two Z-sequences. Set

$$\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$$

and

$$\gamma = (V, (I_s, I_{s-1}, \cdots, I_0), (\mu_s, \mu_{s-1}, \cdots, \mu_0)) .$$

Assume $s = q$. An *admissible epimorphism* or γ onto β is a pair of consisting of an open subset W of M and a diagram

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & L_q & \to & L_{q-1} & \to & \cdots & \to & L_0 & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & I_q & \to & I_{q-1} & \to & \cdots & \to & I_0 & \to & \nu & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & E_q & \to & E_{q-1} & \to & \cdots & \to & E_0 & \to & \nu & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array} \tag{5.55}$$

of C^∞ vector-bundles such that the following six conditions are satisfied:

$$U \cap V \supset W \supset Z . \tag{5.56}$$

(5.57) Each column $0 \to L_j \to I_j \to E_j \to 0$ is defined and exact on all of W. $j = q, q-1, \cdots, 0$.

(5.58) The map $\nu \to \nu$ is the identity of $\nu | W - Z$.

(5.59) The top row $0 \to L_q \to L_{q-1} \to \cdots \to L_0 \to 0 \to 0$ is defined and exact on all of W.

(5.60) The middle row is $\gamma \mid W$. The bottom row is $\beta \mid W$.

(5.61) The diagram commutes on $W - Z$.

(5.62) Proposition. *If there is an admissible eipimorphism of γ onto β, then for all φ with $n - k < \deg \varphi \leq n$*

$$\mathrm{Res}_\varphi (Z, \xi, \beta) = \mathrm{Res}_\varphi (Z, \xi, \gamma) \ . \tag{5.63}$$

Proof. Set $\beta = (U, (E_q, E_{q-1}, \cdots, E_0), (\eta_q, \eta_{q-1}, \cdots, \eta_0))$ and $\gamma = (V, (I_q, I_{q-1}, \cdots, I_0), (\mu_q, \mu_{q-1}, \cdots, \mu_0))$. Let W be as in Definition (5.54), and consider the diagram (5.55). In view (5.52) and of (5.53) we may assume

$$U = V = W \ . \tag{5.64}$$

On U choose connections $\nabla_q, \nabla_{q-1}, \cdots, \nabla_0$ for $L_q, L_{q-1}, \cdots, L_0$ such that on U

(5.65) $(\nabla_q, \nabla_{q-1}, \cdots, \nabla_0)$ is compatible with the exact sequence $0 \to L_q \to L_{q-1} \to \cdots \to L_0 \to 0$.

Let D_{-1} be a basic connection for $\nu \mid U - Z$. On U choose connections $D_q, D_{q-1}, \cdots, D_0$ for $E_q, E_{q-1}, \cdots, E_0$ such that

$$(D_q, D_{q-1}, \cdots, D_0) \qquad \text{is fitted to } \beta. \tag{5.66}$$

Hence there exists a compact subset Σ of U with (5.25) valid for Σ.

Denote by $\iota_j(L_j)$ the image of $\iota_j \colon L_j \to I_j$. According to Lemma (4.32) on $U - Z$ there exist C^∞ sub-vector-bundles $F_q, F_{q-1}, \cdots, F_0$ of $I_q \mid U - Z, I_{q-1} \mid U - Z, \cdots, I_0 \mid U - Z$ such that

$$I_j \mid U - Z = F_j \oplus \iota_j(L_j \mid U - Z) \ , \qquad j = q, q-1, \cdots, 0 \ , \tag{5.67}$$

$$\mu_j(F_j) \subset F_{j-1} \ , \qquad j = q, q-1, \cdots, 1 \ . \tag{5.68}$$

According to Lemma (4.45) on U there exist C^∞ sub-vector-bundles $\tilde{F}_q, \tilde{F}_{q-1}, \cdots, \tilde{F}_0$ of $I_q, I_{q-1}, \cdots, I_0$ such that

$$I_j = \tilde{F}_j \oplus \iota_j(L_j) \ , \qquad j = q, q-1, \cdots, 0 \ , \tag{5.69}$$

$$\tilde{F}_j \mid U - \Sigma = F_j \mid U - \Sigma \ , \qquad j = q, q-1, \cdots, 0 \ . \tag{5.70}$$

On U let $\hat{D}_j$ be the unique connection for $\tilde{F}_j$ such that

(5.71) $(\hat{D}_j, D_j)$ is compatible with the exact sequence

$$0 \to \tilde{F}_j \to E_j \to 0 \ , \qquad j = q, q-1, \cdots, 0 \ .$$

On U let $\hat{\nabla}_j$ be the unique connection for $\iota_j(L_j)$ such that

(5.72) $(\nabla_j, \hat{\nabla}_j)$ is compatible with the exact sequence

$$0 \to L_j \to \iota_j(L_j) \to 0\ , \qquad j = q, q-1, \cdots, 0\ .$$

Let $\tilde{D}_j$ be the direct sum connection for I_j:

$$\tilde{D}_j = \hat{D}_j \oplus \hat{\nabla}_j\ . \tag{5.73}$$

Thus

$$\tilde{D}_j(s_1 + s_2) = \hat{D}_j s_1 + \hat{\nabla}_j s_2\ , \qquad s_1 \in C^\infty(\tilde{F}_j),\ s_2 \in C^\infty(\iota_j(L_j))\ . \tag{5.74}$$

(5.65) and (5.66) imply

$$(\tilde{D}_q, \tilde{D}_{q-1}, \cdots, \tilde{D}_0, D_{-1}) \qquad \text{is fitted to } \gamma\ . \tag{5.75}$$

Set $\tilde{K}_j = K(\tilde{D}_j)$, $K_j = K(D_j)$. (5.73) implies

$$\det(I + \tilde{K}_j) = \det(I + K_j)\det(I + K(\nabla_j))\ . \tag{5.76}$$

Set $\varepsilon(i) = (-1)^i$, (5.65) and (4.24) imply

$$\prod_{i=0}^{q} \det(I + K(\nabla_i))^{\varepsilon(i)} = 1\ . \tag{5.77}$$

Therefore

$$\prod_{i=0}^{q} (\det(I + \tilde{K}_i))^{\varepsilon(i)} = \prod_{i=0}^{q} (\det(I + K_i))^{\varepsilon(i)}\ . \tag{5.78}$$

(5.78) and (4.13) imply

$$\varphi(\tilde{K}_q | \tilde{K}_{q-1} | \cdots | \tilde{K}_0) = \varphi(K_q | K_{q-1} | \cdots | K_0)\ . \tag{5.79}$$

Due to (5.66) and (5.75), (5.63) has been proved.

6. Coherent real-analytic sheaves

In order to prove the residue existence theorem stated in the introduction, we shall have to use real-analytic sheaves. Following [1] let us state the basic facts which we need.

Let M be a complex-analytic manifold, and $n = \dim_C M$. Denote by $\mathcal{O}$ the sheaf of germs of holomorphic functions on M, and by $\mathcal{A}$ the sheaf of germs of real-analytic functions on M. Given $x \in M$, let $z_1, \cdots, z_n$ be complex-analytic coordinates defined about x with $z_i(x) = 0$. Then $\mathcal{A}_x$ is isomorphic to the ring $C\{z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n\}$ of convergent power series in z_i, $\bar{z}_i$. Any module over $\mathscr{A}_x$ has a projective resolution of length $\leq 2n$. $\mathcal{O}$ and $\mathscr{A}$ are both sheaves of rings, and there is the natural injection $\mathcal{O} \to \mathscr{A}$. If $\mathscr{F}$ is a sheaf of $\mathcal{O}$-modules, then $\mathscr{A} \otimes_{\mathcal{O}} \mathscr{F}$ is a sheaf of $\mathscr{A}$-modules.

(6.1) Proposition. *Let $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}$-modules. Then $\mathscr{A} \otimes_{\mathcal{O}} \mathscr{F}$ is a coherent sheaf of $\mathscr{A}$-modules. Moreover, if $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3$ is an exact sequence of coherent sheaves of $\mathcal{O}$-modules, then $\mathscr{A} \otimes_{\mathcal{O}} \mathscr{F}_1 \to \mathscr{A} \otimes_{\mathcal{O}} \mathscr{F}_2 \to \mathscr{A} \otimes_{\mathcal{O}} \mathscr{F}_3$ is an exact sequence of coherent sheaves of $\mathscr{A}$-modules.*

Proof. See [1, Proposition 2.9, p. 30] and also [2, Proposition (1.5), p. 153].

(6.2) Definition. Let U be open in M. On U, let $\mathscr{F}$ be a coherent sheaf of $\mathscr{A}$-modules. A *resolution* of $\mathscr{F}$ is an exact sequence

$$0 \to H_{2n} \to H_{2n-1} \to \cdots \to H_0 \to \mathscr{F} \to 0$$

of coherent sheaves of $\mathscr{A}$-modules on U such that each H_i is locally free.

(6.3) Proposition (*Existence of resolutions*). *Let U be an open subset of M, and $\mathscr{F}$ a coherent sheaf of $\mathscr{A}$-modules on U. Let W be an open subset of U such that there is a compact B with $U \supset B \supset W$. Then on W, $\mathscr{F} \mid W$ has a resolution.*

Proof. See [1, Proposition 2.6, p. 29].

(6.4) Definition. On U, let R_1, R_2 be two resolutions of $\mathscr{F}$. A *morphism* of R_1 to R_2 is a commutative diagram

$$\begin{array}{ccccccccccc} 0 \to & H_{2n} & \to & H_{2n-1} & \to \cdots \to & H_0 & \to & \mathscr{F} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & J_{2n} & \to & J_{2n-1} & \to \cdots \to & J_0 & \to & \mathscr{F} & \to 0 \end{array}$$

of sheaves of $\mathscr{A}$-modules on U such that the upper row is R_1, the lower row is R_2, and the vertical arrow farthest to the right is the identity map. The morphism is said to be a morphism of R_1 *onto* R_2 if the vertical arrows are all surjections.

(6.5) Proposition (*Comparison of resolutions*). *Let U be open in M. On U, let $\mathscr{F}$ be a coherent sheaf of $\mathscr{A}$-modules, and R_1, R_2 be two resolutions of $\mathscr{F}$. Let W be an open subset of U such that there is a compact set B with $U \supset B \supset W$. Let $R_1 \mid W, R_2 \mid W$ be the restrictions of R_1, R_2 to W. Then on W there is a resolution R_3 of $\mathscr{F} \mid W$ such that*

(6.6) *there exists a morphism of R_3 onto $R_1 \mid W$,*

(6.7) *there exists a morphism of R_3 onto $R_2 \mid W$.*

Proof. See [7, Lemmas 13 and 14, p. 107]. In [7] these are proved on an algebraic variety using coherent algebraic sheaves. But due to [1, Corollary 2.5, p. 29] the same reasoning is valid in the real-analytic framework.

(6.8) Remarks. (a) If H is a coherent sheaf of $\mathscr{A}$-modules, then H is locally free if and only if each stalk H_x is a free $\mathscr{A}_x$-module.

(b) If E is a real-analytic vector-bundle, let $\underline{\underline{E}}$ denote the sheaf of germs of real-analytic sections of E. Then $E \rightsquigarrow \underline{\underline{E}}$ is a functor which gives an equivalence between the category of real-analytic vector-bundles on M and the category of locally free coherent sheaves of $\mathscr{A}$-modules on M.

(c) If M is compact and $\mathscr{F}$ is a coherent sheaf of $\mathscr{O}$-modules, then Propositions (6.1),(6.3), and (6.5) can be used to define the Chern classes of $\mathscr{F}$. To do this, on M let

$$0 \to H_{2n} \to H_{2n-1} \to \cdots \to H_0 \to \mathscr{A} \underset{\mathscr{O}}{\otimes} \mathscr{F} \to 0$$

be a resolution of $\mathscr{A} \otimes_{\mathscr{O}} \mathscr{F}$. Let E_i be the real-analytic vector-bundle with $\underline{\underline{E}}_i = H_i$. In $K(M)$ let ζ be the virtual bundle:

$$\zeta = \sum_{i=0}^{2n} (-1)^i E_i \, . \tag{6.9}$$

$c_i(\mathscr{F})$ is defined by

$$c_i(\mathscr{F}) = c_i(\zeta) \, , \qquad i = 1, \cdots, n \, . \tag{6.10}$$

It follows easily from Proposition (6.5) that ζ depends only on $\mathscr{F}$. Hence $c_i(\mathscr{F})$ is well-defined. For a detailed proof of this see [7, Lemma 11, p. 106].

More generally, let φ be symmetric and homogeneous with $\deg \varphi \leq n$. $\varphi(\mathscr{F})$ is defined by

$$\varphi(\mathscr{F}) = \tilde{\varphi}(c_1(\mathscr{F}), \cdots, c_l(\mathscr{F})) \, , \tag{6.11}$$

where $\varphi = \tilde{\varphi}(\sigma_1, \cdots, \sigma_l)$, and $l = \deg \varphi$.

(d) We are forced to use real-analytic sheaves because it is not known whether the propositions on existence and comparison of resolutions are true in the holomorphic category. By Proposition (6.1) a resolution in the holomorphic category, when tensored with $\mathscr{A}$, gives a resolution in the real-analytic category.

7. Proof of the residue existence theorem

As in the statement of Theorem 2 let M be complex-analytic, and ξ a full integrable sub-sheaf of $\underline{T}$. Set $Q = \underline{T}/\xi$. Let φ be symmetric and homogeneous of degree l where $n - k < l \leq n$.

(7.1) Definition. Let Z be a connected component of the singular set S. Assume that Z is compact. Choose an open subset U of M with $U \cap S = Z$ and Z a deformation retract of U such that on U there exists a resolution R:

$$0 \to \underline{\underline{E}}_{2n} \to \underline{\underline{E}}_{2n-1} \to \cdots \to \underline{\underline{E}}_0 \to \mathscr{A} \bigotimes_{\mathcal{O}} Q \to 0$$

of $\mathscr{A} \otimes_{\mathcal{O}} Q \mid U$.

For $i = 2n, 2n-1, \cdots, 1$ let $\eta_i \colon E_i \mid U - Z \to E_{i-1} \mid U - Z$ be the vector-bundle map which gives $\underline{\underline{E}}_i \mid U - Z \to \underline{\underline{E}}_{i-1} \mid U - Z$. Let $\eta_0 \colon E_0 \mid U - Z \to \nu \mid U - Z$ be the vector-bundle map which gives $\underline{\underline{E}}_0 \mid U - Z \to \mathscr{A} \otimes_{\mathcal{O}} Q \mid U - Z$. Then $(U, (E_{2n}, E_{2n-1}, \cdots, E_0), (\eta_{2n}, \eta_{2n-1}, \cdots, \eta_0))$ is a Z-sequence. Call this Z-sequence $\beta(R)$, and define $\mathrm{Res}_\varphi(\xi, Z)$ by

$$\mathrm{Res}_\varphi(\xi, Z) = \mathrm{Res}_\varphi(\xi, Z, \beta(R)) \ . \tag{7.2}$$

(7.3) Remarks. (a) The existence of U, R as in Definition (7.1) is implied by (5.52) and Proposition (6.3).

(b) In order for (7.2) to be legitimate it must be shown that $\mathrm{Res}_\varphi(\xi, Z, \beta(R))$ does not depend on the choice of U and R. This is implied by (5.62) and Proposition (6.5). A morphism of a resolution R_1 onto a resolution R_2 gives an admissible epimorphism of $\beta(R_1)$ onto $\beta(R_2)$.

Proof of (0.22). From (7.3)b and (5.53) it is clear that $\mathrm{Res}_\varphi(\xi, Z)$ depends only on φ and on the local structure of Q near Z. Since ξ is full, (0.18) implies that the local structure of Q near Z is determined by the local behavior of the leaves of ξ near Z. (0.22) is now evident.

Proof of (0.23). If M is compact, then on M let

$$0 \to \underline{\underline{E}}_{2n} \to \underline{\underline{E}}_{2n-1} \to \cdots \to \underline{\underline{E}}_0 \to \mathscr{A} \bigotimes_{\mathcal{O}} Q \to 0$$

be a resolution of $\mathscr{A} \otimes_{\mathcal{O}} Q$. Let $Z_1, \cdots, Z_r$ be the connected components of the singular set S. Choose open subsets $U_1, \cdots, U_r$ of M such that

$$U_i \cap S = Z_i \ , \qquad i = 1, \cdots, r \ , \tag{7.4}$$

$$U_i \cap U_j = \phi \qquad \text{if } i \neq j \ , \tag{7.5}$$

$$Z_i \text{ is a deformation retract of } U_i. \tag{7.6}$$

Choose compact subsets $\Sigma_1, \cdots, \Sigma_r$ of M such that

$$U_i \supset \Sigma_i \ , \qquad i = 1, \cdots, r \ , \tag{7.7}$$

$$Z_i \text{ is contained in the interior of } \Sigma_i,\ i = 1, \cdots, r. \tag{7.8}$$

Set $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$. On $M - S$ let D_{-1} be a basic connection for ν. On M let $D_{2n}, D_{2n-1}, \cdots, D_0$ be connections for $E_{2n}, E_{2n-1}, \cdots, E_0$ such that

(7.9) On $M - \Sigma$, $(D_{2n}, D_{2n-1}, \cdots, D_0, D_{-1})$ is compatible with the exact sequence

$$0 \to E_{2n} \mid M - \Sigma \to E_{2n-1} \mid M - \Sigma \to \cdots \to E_0 \mid M - \Sigma \to \nu \mid M - \Sigma \to 0 \ .$$

Set $K_i = K(D_i)$. According to (4.15) and (6.11),

$$(7.10) \qquad [\varphi(K_{2n} \mid K_{2n-1} \mid \cdots \mid K_0)] = (2\pi/\sqrt{-1})^l \varphi(Q) .$$

(7.9), (4.23), and (3.28) imply

$$(7.11) \qquad \varphi(K_{2n} \mid K_{2n-1} \mid \cdots \mid K_0) \quad \text{vanishes on } M - \Sigma .$$

Therefore $\varphi(K_{2n} \mid K_{2n-1} \mid \cdots \mid K_0) \mid U_i$ is a $2l$-form on U_i with compact support. By (7.2) the element of $H_{2n-2l}(Z; \boldsymbol{C})$ determined by $\varphi(K_{2n} \mid K_{2n-1} \mid \cdots \mid K_0) \mid U_i$ is $(2\pi/\sqrt{-1})^l \operatorname{Res}_\varphi(\xi, Z_i)$. Let ω_i be the $2l$-form on M defined by

$$(7.12) \qquad \omega_i \mid U_i = \varphi(K_{2n} \mid K_{2n-1} \mid \cdots \mid K_0) \mid U_i ,$$

$$(7.13) \qquad \omega_i \text{ vanishes on } M - U_i .$$

From definition (0.21) of μ_* we then have

$$(7.14) \qquad [\omega_i] = (2\pi/\sqrt{-1})^l \mu_* \operatorname{Res}_\varphi(\xi, Z_i) , \qquad i = 1, \cdots, r .$$

But

$$(7.15) \qquad \varphi(K_{2n} \mid K_{2n-1} \mid \cdots \mid K_0) = \omega_1 + \cdots + \omega_r .$$

(7.15), (7.14), and (7.10) imply (0.23). This completes the proof of Theorem 2.

Remark. In Definition (7.1) the exact sequence of sheaves has $2n$ locally free sheaves. The next lemma asserts that $\operatorname{Res}_\varphi(\xi, Z)$ can be obtained from a suitable exact sheaf sequence of any length.

(7.16) Lemma. *Let U be an open set containing Z with $U \cap S = Z$ and Z a deformation retract of U. On U let*

$$(7.17) \qquad 0 \to \underline{\underline{E}}_q \to \underline{\underline{E}}_{q-1} \to \cdots \to \underline{\underline{E}}_0 \to \mathscr{A} \otimes_{\mathcal{O}} Q \to 0$$

be an exact sequence of sheaves of $\mathscr{A}$-modules. Let β be the resulting Z-sequence. Then

$$(7.18) \qquad \operatorname{Res}_\varphi(\xi, Z, \beta) = \operatorname{Res}_\varphi(\xi, Z) .$$

Proof. If $q < 2n$, add on $2n - q$ zeroes to the left of (7.17) to obtain

$$(7.19) \quad 0 \to 0 \to \cdots \to 0 \to \underline{\underline{E}}_q \to \underline{\underline{E}}_{q-1} \to \cdots \to \underline{\underline{E}}_0 \to \mathscr{A} \otimes_{\mathcal{O}} Q \to 0 .$$

Let β' be the Z-sequence resulting from (7.19). β' has length $2n$ so by (7.1)

$$(7.20) \qquad \operatorname{Res}_\varphi(\xi, Z) = \operatorname{Res}_\varphi(\xi, Z, \beta') .$$

But it is obvious that

$$\text{(7.21)} \qquad \operatorname{Res}_\varphi(\xi, Z, \beta) = \operatorname{Res}_\varphi(\xi, Z, \beta') .$$

This proves (7.18) when $q < 2n$.

If $q > 2n$, then by the syzygy theorem [13, Chapter VIII, Theorem 6.5′, p. 158], the kernel of $\underline{\underline{E}}_{2n-1} \to \underline{\underline{E}}_{2n-2}$ is locally free. Denote this kernel by $\underline{\underline{E}}$. Thus

$$\text{(7.22)} \qquad 0 \to \underline{\underline{E}} \to \underline{\underline{E}}_{2n-1} \to \underline{\underline{E}}_{2n-2} \to \cdots \to \underline{\underline{E}}_0 \to \mathscr{A} \bigotimes_{\mathcal{O}} Q \to 0$$

is a resolution of $\mathscr{A} \otimes_{\mathcal{O}} Q$. Denote this resolution by R. The map $\underline{\underline{E}}_{2n} \to \underline{\underline{E}}_{2n-1}$ gives a surjection $\underline{\underline{E}}_{2n} \to \underline{\underline{E}} \to 0$. Consider the commutative diagram:

$$\begin{array}{ccccccccccccccccccc}
0 & \to & \underline{\underline{E}}_q & \to & \underline{\underline{E}}_{q-1} & \to & \cdots & \to & \underline{\underline{E}}_{2n+1} & \to & \underline{\underline{E}}_{2n} & \to & \underline{\underline{E}}_{2n-1} & \to & \cdots & \to & \underline{\underline{E}}_0 & \to & \mathscr{A} \otimes_{\mathcal{O}} Q \to 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
0 & \to & 0 & \to & 0 & \to & \cdots & \to & 0 & \to & \underline{\underline{E}} & \to & \underline{\underline{E}}_{2n-1} & \to & \cdots & \to & \underline{\underline{E}}_0 & \to & \mathscr{A} \otimes_{\mathcal{O}} Q \to 0
\end{array}$$

This diagram gives an admissible epimorphism of Z-sequences. Therefore by (5.63)

$$\text{(7.23)} \qquad \operatorname{Res}_\varphi(\xi, Z, \beta) = \operatorname{Res}_\varphi(\xi, Z, \beta(R)) .$$

This proves (7.18) when $q > 2n$. If $q = 2n$, then (7.18) is immediate from (7.1).

(7.24) Corollary. *Suppose that $E_q, E_{q-1}, \cdots, E_0$ are holomorphic vector-bundles on U. Let*

$$\text{(7.25)} \qquad 0 \to \underline{E}_q \to \underline{E}_{q-1} \to \cdots \to \underline{E}_0 \to Q \to 0$$

be an exact sequence of sheaves of $\mathcal{O}$-modules on U. Denote the resulting Z-sequence by β. Then

$$\text{(7.26)} \qquad \operatorname{Res}_\varphi(\xi, Z) = \operatorname{Res}_\varphi(\xi, Z, \beta) .$$

Proof. View each E_i as a real-analytic vector-bundle. According to Proposition (6.1), (7.25) gives an exact sequence

$$\text{(7.27)} \qquad 0 \to \underline{\underline{E}}_q \to \underline{\underline{E}}_{q-1} \to \cdots \to \underline{\underline{E}}_0 \to \mathscr{A} \bigotimes_{\mathcal{O}} Q \to 0$$

of sheaves of $\mathscr{A}$-modules on U. Hence Lemma (7.16) applies, and the corollary is proved.

8. Proof of Theorem 1

Recall the data of the theorem. M is compact and complex-analytic. L is a holomorphic line bundle on M. $\eta\colon L \to T$ is a holomorphic vector-bundle map.

Each zero of η is isolated. φ is symmetric and homogeneous of degree $n = \dim_C M$. Set $\xi = \eta(\underline{L})$. For $p \in \text{Zero}(\eta)$ there is the usual identification $H_0(p; C) = C$. Let $\varphi(\eta, p)$ be as in the introduction. Then due to (0.23) and (0,14), (0.10) will be implied by

$$(8.1) \qquad \varphi(\eta, p) = \text{Res}_\varphi(\xi, p) .$$

The remainder of this section will be devoted to proving (8.1). The proof will have two steps:

Step 1. Replace a situation involving several vector-bundles and several connections by a much simpler situation involving only one vector-bundle and one connection.

Step 2. An explicit computation using one vector-bundle and one connection.

To begin Step 1, let W be an open subset of M. On W let X be a holomorphic section of $T \mid W$ such that X has no zeroes. According to (3.10) and (3.11),

$$(8.2) \qquad [X, s_1 + s_2] = [X, s_1] + [X, s_2] ,$$

$$(8.3) \qquad [X, fs] = (Xf)s + f[X, s] ,$$

whenever s, s_1, $s_2 \in C^\infty(T \mid W)$, and $f\colon W \to C$ is a C^∞ function.

If (X) denotes the sub-line-bundle of $T \mid W$ spanned by X, then (8.2) and (8.3) imply that there is a unique partial connection $((X) \oplus \bar{T} \mid W, \delta)$ for $T \mid W$ such that

$$(8.4) \qquad i(X)\delta s = [X, s] , \qquad s \in C^\infty(T \mid W) ,$$

$$(8.5) \qquad i(\gamma)\delta s = i(\gamma)\bar{\partial} s , \qquad s \in C^\infty(T \mid W) , \quad \gamma \in C^\infty(\bar{T} \mid W) .$$

This partial connection for $T \mid W$ will be referred to as the partial connection for $T \mid W$ determined by X. Note that (8.4) implies

$$(8.6) \qquad i(fX)\delta s = f[X, s] , \qquad f\colon W \to C .$$

(8.7) Definition. Let X be a holomorphic section of $T \mid W$ such that X has no zeroes. An *X-connection* for $T \mid W$ is a connection for $T \mid W$, which extends the partial connection for $T \mid W$ determined by X.

(8.8) Remarks. (a) A connection D for $T \mid W$ is an X-connection if and only if

$$(8.9) \qquad i(X)Ds = [X, s] , \qquad s \in C^\infty(T \mid W) ,$$

$$(8.10) \qquad D \text{ is of type } (1,0) .$$

(b) Lemma (2.5) guarantees the existence of X-connections. Since $[X, X] = 0$ and $\bar{\partial} X = 0$, Lemma (2.11) implies that there exist X-connections D with $DX = 0$.

(8.11) Lemma. *Let X be a holomorphic section of $T\,|\,W$ such that X has no zeroes. Let D be an X-connection for $T\,|\,W$. Set $K = K(D)$. Assume* $\deg \varphi = n$. *Then*

$$\varphi(K) = 0 \; . \tag{8.12}$$

Proof. Given $p \in W$, let W_p be an open neighborhood of p in W such that W_p is the domain of a complex-analytic coordinate system $z_1, \cdots, z_n$ with

$$\partial/\partial z_1 = X\,|\,W_p \; . \tag{8.13}$$

Let $A(W_p)$ be the ring of all C^∞ complex-valued differential forms on W_p. In $A(W_p)$ let $I(X, W_p)$ be the ideal generated by $dz_2, \cdots, dz_n$. This ideal has the properties:

(8.14) If $\omega \in I(X, W_p)$, then $d\omega \in I(X, W_p)$.

(8.15) If $\omega_1, \cdots, \omega_n$ are any n elements of $I(X, W_p)$, then $\omega_1 \wedge \cdots \wedge \omega_n = 0$.

Let $\theta = \|\theta_{ij}\|$ be the connection matrix of D with respect to the frame $\partial/\partial z_1, \cdots, \partial/\partial z_n$. Then (8.9), (8.10), and (3.12) imply that each θ_{ij} is in $I(X, W_p)$:

$$\theta_{ij} \in I(X, W_p) \; . \tag{8.16}$$

Let $\kappa = \|\kappa_{ij}\|$ be the curvature matrix of D with respect to the frame $\partial/\partial z_1, \cdots, \partial/\partial z_n$. (8.16), (8.14), and (1.13) imply that each κ_{ij} is in $I(X, W_p)$:

$$\kappa_{ij} \in I(X, W_p) \; . \tag{8.17}$$

Since $\deg \varphi = n$, (8.17) and (8.15) imply that $\varphi(K)$ vanishes on W_p. This proves the lemma.

(8.18) Lemma. *Let X be a holomorphic section of $T\,|\,W$ such that X has no zeroes. Let D and D' be two X-connections for $T\,|\,W$. Set $\tilde{W} = W \times [0,1]$. Let $\rho\colon \tilde{W} \to W$ and $t\colon \tilde{W} \to [0,1]$ be the projections. On $\tilde{W}$ define a connection ∇ for $\rho^!(T\,|\,W)$ by*

$$\nabla = t\rho^!(D') + (1 - t)\rho^!(D) \; . \tag{8.19}$$

Set $K = K(\nabla)$. Assume $\deg \varphi = n$. *Then*

$$\varphi(K) = 0 \; . \tag{8.20}$$

Proof. Given $p \in W$, let W_p be an open neighborhood of p in W such that W_p is the domain of a complex-analytic coordinate system $z_1, \cdots, z_n$ with

$$\partial/\partial z_1 = X\,|\,W_p \; . \tag{8.21}$$

Set $\tilde{W}_p = W_p \times [0,1]$. Let $A(\tilde{W}_p)$ be the ring of all C^∞ complex-valued differential forms on $\tilde{W}_p$. In $A(\tilde{W}_p)$ let $I(X, \tilde{W}_p)$ be the ideal generated by $\rho^*(dz_2), \cdots, \rho^*(dz_n)$.

Let $\theta = \|\theta_{ij}\|$ and $\theta' = \|\theta'_{ij}\|$ be the connection matrices of D and D' with respect to the frame $\partial/\partial z_1, \cdots, \partial/\partial z_n$. Let $\omega = \|\omega_{ij}\|$ be the connection matrix of ∇ with respect to the frame $\rho^!(\partial/\partial z_1), \cdots, \rho^!(\partial/\partial z_n)$. Then (8.19) and (4.56) imply for each ω_{ij},

$$\omega_{ij} = t\rho^*(\theta'_{ij}) + (1 - t)\rho^*(\theta_{ij}) \ . \tag{8.22}$$

Since D and D' are both X-connections, it now follows that each ω_{ij} is in $I(X, \tilde{W}_p)$:

$$\omega_{ij} \in I(X, \tilde{W}_p) \ . \tag{8.23}$$

Let $\kappa = \|\kappa_{ij}\|$ be the curvature matrix of ∇ with respect to the frame $\rho^!(\partial/\partial z_1), \cdots, \rho^!(\partial/\partial z_n)$. $I(X, \tilde{W}_p)$ is closed under d, so each κ_{ij} is in $I(X, \tilde{W}_p)$:

$$\kappa_{ij} \in I(X, \tilde{W}_p) \ . \tag{8.24}$$

The wedge product of any n elements of $I(X, \tilde{W}_p)$ is zero. Since $\deg \varphi = n$, (8.24) now implies that $\varphi(K)$ vanishes on $\tilde{W}_p$. This proves the lemma.

(8.25) Definition. Let U be an open subset of M. On U let X be a holomorphic section of $T \mid U$. Set $Z = \{p \in U \mid X(p) = 0\}$. Assume that Z is compact and connected. On U let D be a connection for $T \mid U$. Then D is *fitted to* X if

(8.26) there exists a compact subset Σ of U with Z contained in the interior of Σ such that on $U - \Sigma$, D is an X-connection for $T \mid U - \Sigma$.

Remark. Given U, X as in Definition (8.25), choose any compact subset Σ of U with Z contained in the interior of Σ. Then according to Lemma (4.41) there exists a connection D for $T \mid U$ such that (8.26) is valid for D and the chosen Σ.

(8.27) Lemma. *Let U be an open subset of M. On U let X be a holomorphic section of $T \mid U$. Assume that the zero set Z of X is compact and connected. Let D be a connection for $T \mid U$ such that D is fitted to X. Set $K = K(D)$. Assume* $\deg \varphi = n$. *Then*

(8.28) $\varphi(K)$ *has compact support.*

Moreover, suppose that D' is another connection for $T \mid U$ such that D' is also fitted to X. Set $K' = K(D')$. Then

$$\int_U \varphi(K) = \int_U \varphi(K') \ . \tag{8.29}$$

Proof. To prove (8.28), let Σ be as in (8.26). Then on $U - \Sigma$, D is an X-connection for $T \mid U - \Sigma$. Hence by (8.12), $\varphi(K)$ vanishes on $U - \Sigma$. This proves (8.28).

To prove (8.29) we may assume that on $U - \Sigma$, D' is also an X-connection for $T \mid U - \Sigma$. Let $\tilde{U} = U \times [0,1]$. Let $\rho: \tilde{U} \to U$ and $t: \tilde{U} \to [0,1]$ be the projections. On $\tilde{U}$ define a connection ∇ for $\rho^!(T \mid U)$ by

$$(8.30) \qquad \nabla = t\rho^!(D') + (1 - t)\rho^!(D) .$$

Set $\tilde{K} = K(\nabla)$, and define $i_0, i_1: U \to \tilde{U}$ by

$$(8.31) \qquad i_0(x) = (x, 0) , \qquad x \in U ,$$

$$(8.32) \qquad i_1(x) = (x, 1) , \qquad x \in U .$$

Then

$$(8.33) \qquad \varphi(K) = i_0^* \varphi(\tilde{K}) ,$$

$$(8.34) \qquad \varphi(K') = i_1^* \varphi(\tilde{K}) .$$

According to (8.20), $\varphi(\tilde{K})$ vanishes on $\tilde{U} - \Sigma \times [0,1]$. Therefore

$$(8.35) \qquad \varphi(\tilde{K}) \text{ has compact support.}$$

(8.33), (8.34), and (8.35) imply that there exists a $(2n - 1)$-form ω on U with

$$(8.36) \qquad \omega \text{ has compact support,}$$

$$(8.37) \qquad d\omega = \varphi(K') - \varphi(K) .$$

(8.29) is now evident.

(8.38) Proposition. *Let U be an open subset of M. On U let X be a holomorphic section of $T \mid U$. Assume that the zero set Z of X is compact and connected. Let D be a connection for $T \mid U$ such that D is fitted to X. Set $K = K(D)$. Let ξ be the subsheaf of $T \mid U$ spanned by X. Assume* $\deg \varphi = n$. *Then*

$$(8.39) \qquad \operatorname{Res}_\varphi (\xi, Z) = (\sqrt{-1}/(2\pi))^n \int_U \varphi(K) .$$

Proof. Due to (8.29) it will suffice to exhibit a connection D' for $T \mid U$ such that D' is fitted to X and such that for $K' = K(D')$ it is immediate and obvious that

$$(8.40) \qquad \operatorname{Res}_\varphi (\xi, Z) = (\sqrt{-1}/(2\pi))^n \int_U \varphi(K') .$$

To construct such a D', let V be an open subset of U, and Δ be a compact subset of U with

(8.41) $$U \supset V \supset \Delta \supset Z\,,$$

(8.42) Z is contained in the interior of Δ,

(8.43) Z is a deformation retract of V.

Let D' be a connection for $T\,|\,U$ such that

(8.44) on $U - \Delta$, D' is an X-connection for $T\,|\,U - \Delta$,

(8.45) $$\text{on} \quad U - \Delta\,, \quad D'X = 0\,.$$

The existence of such a D' is implied by Lemma (2.11) and (4.41). Set $K' = K(D')$.

To verify (8.40), let (1) denote the trivial line bundle $U \times \boldsymbol{C}$. Define $\eta\colon (1) \to T\,|\,U$ by

(8.46) $$\eta(p, z) = zX(p)\,, \qquad p \in U,\ z \in \boldsymbol{C}\,.$$

Define a section s of (1) by

(8.47) $$s(p) = (p, 1)\,, \qquad p \in U\,.$$

Thus

(8.48) $$\eta s = X\,.$$

Let D_1 be the unique connection for (1) with

(8.49) $$D_1 s = 0\,.$$

On $U - Z$ set $\nu = T/\eta(1)$. Let $\mu\colon T\,|\,U - Z \to \nu$ be the projection. On $V - Z$ consider

(8.50) $$0 \to (1) \to T \to \nu \to 0\,.$$

Let β denote the Z-sequence obtained from (8.50).

For $\nu\,|\,U - \Delta$ there is a unique connection D_{-1} such that

(8.51) $$D_{-1}(\mu\gamma) = (1 \otimes \mu)D'\gamma\,,$$

whenever $\gamma \in C^\infty(T\,|\,U - \Delta)$. D_{-1} is a basic connection for $\nu\,|\,U - \Delta$. By enlarging Δ slightly and applying Lemma (4.41) we may assume that on $U - Z$ there is a basic connection $\tilde{D}_{-1}$ for ν such that

(8.52) D_{-1} and $\tilde{D}_{-1}$ agree on $\nu\,|\,U - \Delta$.

Thus $(D_1, D', \tilde{D}_{-1})$ is fitted to the Z-sequence β. Set $Q = \underline{T}/\xi$. At the sheaf level (8.50) gives on V an exact sequence of sheaves of $\mathcal{O}$-modules:

$$0 \to (\underline{1}) \to \underline{T} \,|\, V \to Q \,|\, V \to 0 \,. \tag{8.53}$$

Set $K_1 = K(D_1)$. Then from (7.26) it follows directly and immediately that

$$\operatorname{Res}_\varphi (\xi, Z) = (\sqrt{-1}/(2\pi))^n \int_U \varphi(K_1 \,|\, K') \,. \tag{8.54}$$

But (8.49) implies

$$K_1 = 0 \,. \tag{8.55}$$

Therefore

$$\varphi(K_1 \,|\, K') = \varphi(K') \,. \tag{8.56}$$

(8.56) and (8.54) imply (8.40), so the proof is complete.

Remark. Definition (7.1) of $\operatorname{Res}_\varphi (\xi, Z)$ requires choosing a resolution and then choosing connections for the vector-bundles in the resolution. Thus several vector-bundles and several connections are involved. The point of (8.39) is that it replaces this complicated situation involving several vector-bundles and several connections by a much simpler situation involving only one vector-bundle and one connection.

The next proposition will use (8.39) to explicitly compute $\operatorname{Res}_\varphi (\xi, p)$ when p is an isolated zero of X. But first, a lemma which will permit an application of the Lebesgue bounded convergence theorem.

(8.57) Lemma. *Let $g\colon [0,1) \to [0,1)$ be a nondecreasing C^∞ function with*

$$g(r) = r \,, \qquad \text{for } 0 \le r \le 1/3 \,, \tag{8.58}$$

$$g(r) = 1 \,, \qquad \text{for } 2/3 \le r < 1 \,. \tag{8.59}$$

For $m = 1, 2, \cdots$ define $g_m\colon [0,1) \to [0,1)$ by

$$g_m(r) = \sqrt[m]{g(r^m)} \,, \qquad r \in [0,1) \,. \tag{8.60}$$

Then there exists a positive real number b such that for all $r \in [0,1)$ and all $m = 1, 2, \cdots$

$$|(dg_m/dr)(r)| < b \,. \tag{8.61}$$

Proof. Choose a real number b such that

$$3\,|(dg/dr)(r)| < b \,, \qquad \text{for all } r \in [0,1) \,. \tag{8.62}$$

Then (8.61) will be implied by

$$|(dg_m/dr)(r)| \le 3\,|(dg/dr)(r^m)| \,, \qquad \text{for all } r \in [0,1) \,. \tag{8.63}$$

On $[0, \sqrt[m]{1/3}]$ $g_m(r) = r$. So on $[0, \sqrt[m]{1/3}]$,

$$(8.64) \quad 1 = |(dg_m/dr)(r)| < 3\,|(dg/dr)(r^m)| = 3\ , \qquad r \in [0, \sqrt[m]{1/3}]\ .$$

On $[\sqrt[m]{1/3}, 1)$ differentiation of g_m gives

$$(8.65) \qquad (dg_m/dr)(r) = (dg/dr)(r^m)g_m(r)r^{m-1}/g(r^m)\ .$$

g is nondecreasing so $g(r^m) \geq 1/3$ for all $r \in [\sqrt[m]{1/3}, 1)$. Hence (8.65) implies

$$(8.66) \qquad |(dg_m/dr)(r)| \leq 3\,|(dg/dr)(r^m)|\ , \qquad r \in [\sqrt[m]{1/3}, 1)\ .$$

(8.66) and (8.64) combine to give (8.63). The lemma is proved.

(8.67) Proposition. *Let U be an open subset of M. On U let X be a holomorphic section of $T\,|\,U$. Assume that the zero set of X consists of one point p. Let $z_1, \cdots, z_n$ be a complex-analytic coordinate system with domain U and origin p. Denote by ξ the subsheaf of $\underline{T}\,|\,U$ spanned by X. Assume* $\deg \varphi = n$. *Then*

$$(8.68) \qquad \operatorname{Res}_\varphi(\xi, p) = \operatorname{Res}_p \begin{bmatrix} \varphi(A)dz_1 \cdots dz_n \\ a_1, \cdots, a_n \end{bmatrix}\ ,$$

where $X = \sum_{i=1}^n a_i \partial/\partial z_i$ and $A = \|\partial a_i/\partial z_j\|$.

Proof. Since p is an isolated zero of the a_i, there exist positive integers $\alpha_1, \cdots, \alpha_n$ such that $z_i^{\alpha_i}$ is in the ideal generated by $a_1, \cdots, a_n$. So there exist holomorphic functions b_{ij} defined about p with

$$(8.69) \qquad z_i^{\alpha_i} = \sum_{j=1}^n b_{ij}a_j\ .$$

Passing to a smaller U, if necessary, it may be assumed that each b_{ij} is defined on all of U. Hence (8.69) holds throughout U.

Let $z\colon U \to C^n$ be

$$(8.70) \qquad z(x) = (z_1(x), \cdots, z_n(x))\ , \qquad x \in U\ .$$

In C^n denote by B_α the set

$$(8.71) \qquad B_\alpha = \left\{(\zeta_1, \cdots, \zeta_n) \in C^n \,\middle|\, \sum_{i=1}^n (\zeta_i\bar{\zeta}_i)^{\alpha_i} < 1\right\}\ .$$

We may assume

$$(8.72) \qquad B_\alpha \subset z(U)\ .$$

For if $B_\alpha \not\subset z(U)$, then replace $z_1, \cdots, z_n$ by $bz_1, \cdots, bz_n$ where b is a large positive real number. Set $w_i = bz_i$. Then

$$(8.73) \qquad X = \sum_{i=1}^n (ba_i)\partial/\partial w_i\ ,$$

$$\|\partial(ba_i)/\partial w_j\| = \|\partial a_i/\partial z_j\| , \tag{8.74}$$

$$\mathrm{Res}_p \begin{bmatrix} \varphi(A)dw_1 \cdots dw_n \\ ba_1, \cdots, ba_n \end{bmatrix} = \mathrm{Res}_p \begin{bmatrix} \varphi(A)dz_1 \cdots dz_n \\ a_1, \cdots, a_n \end{bmatrix} . \tag{8.75}$$

So it is legitimate to assume that (8.72) is valid.

In U let B denote the subset

$$B = \{x \in U \mid z(x) \in B_\alpha\} . \tag{8.76}$$

On B define a 1-form ω by

$$\omega = \sum_{1 \le i,j \le n} (\bar{z}_i)^{\alpha_i} b_{ij} dz_j . \tag{8.77}$$

On B let D be the connection for $T \mid B$ given by

$$D(\partial/\partial z_i) = \omega \otimes [X, \partial/\partial z_i] , \qquad i = 1, \cdots, n . \tag{8.78}$$

Set $K = K(D)$. Then (8.68) will be proved if it can be shown that

$$\mathrm{Res}_\varphi (\xi, p) = (\sqrt{-1}/(2\pi))^n \int_B \varphi(K) , \tag{8.79}$$

$$(\sqrt{-1}/(2\pi))^n \int_B \varphi(K) = \mathrm{Res}_p \begin{bmatrix} \varphi(A)dz_1 \cdots dz_n \\ a_1, \cdots, a_n \end{bmatrix} . \tag{8.80}$$

To prove (8.79) construct a sequence of connections $D_1, D_2, \cdots$ for $T \mid B$ as follows. On B denote by $(z, z)^\alpha$ the function

$$(z, z)^\alpha = \sum_{i=1}^n (z_i \bar{z}_i)^{\alpha_i} . \tag{8.81}$$

Let π be the 1-form on $B - \{p\}$ defined by

$$\pi = \omega/(z, z)^\alpha . \tag{8.82}$$

Then on $B - \{p\}$

$$\pi \text{ is of type } (1, 0) , \tag{8.83}$$

$$i(X)\pi = 1 , \tag{8.84}$$

Note that (8.84) is implied by (8.69). For $m = 1, 2, \cdots$, let $g_m : [0, 1) \to [0, 1)$ be as in (8.60). Define $\psi_m : B \to [0, 1)$ by

$$\psi_m(x) = g_m((z, z)^\alpha x) , \qquad x \in B . \tag{8.85}$$

Then on B take D_m to be the connection for $T \mid B$ such that

$$(8.86)\qquad D_m(\partial/\partial z_i) = \psi_m \pi \otimes [X, \partial/\partial z_i]\ , \qquad i = 1, \cdots, n\ .$$

Note that near p $\psi_m \pi$ agrees with ω, so D_m is well-defined on all of B. Set $K_m = K(D_m)$. (8.83), (8.84), and (8.59) imply

$$(8.87)\qquad D_m \text{ is fitted to } X.$$

So by (8.39),

$$(8.88)\qquad \operatorname{Res}_\varphi(\xi, p) = (\sqrt{-1}/(2\pi))^n \int_B \varphi(K_m)\ .$$

Now for all $r \in [0, 1)$,

$$(8.89)\qquad \lim_{m\to\infty} g_m(r) = r\ .$$

Hence for all $x \in B$,

$$(8.90)\qquad \lim_{m\to\infty} \psi_m(x) = (z, z)^\alpha x\ .$$

So if $x \in B$ and $v \in T_x \oplus \overline{T}_x$, then

$$(8.91)\qquad \lim_{m\to\infty} i(v)\psi_m \pi = i(v)\omega\ .$$

Moreover, if $x \in B$ and $v_1, \cdots, v_{2n} \in T_x \oplus \overline{T}_x$, then

$$(8.92)\qquad \lim_{m\to\infty} i(v_1, \cdots, v_{2n})\varphi(K_m) = i(v_1, \cdots, v_{2n})\varphi(K)\ .$$

Due to (8.61) the Lebesgue bounded convergence theorem [17, Chapter V, Theorem D, p. 110] applies to give

$$(8.93)\qquad \lim_{m\to\infty} \int_B \varphi(K_m) = \int_B \varphi(K)\ .$$

(8.93) and (8.88) imply (8.79), so (8.79) has been proved.

To prove (8.80), let θ, κ denote respectively the connection and curvature matrices of D with respect to the frame $\partial/\partial z_1, \cdots, \partial/\partial z_n$. Then

$$(8.94)\qquad \theta = -\omega A\ .$$

From (8.94) it is clear that $\theta \wedge \theta = 0$, so $\kappa = d\theta$,

$$(8.95)\qquad \kappa = -(d\omega)A + (\omega)dA\ .$$

ω is of type (1,0), so by (8.95) each entry of κ is a sum of 2-forms of type (1,1) and type (2,0). In $\varphi(\kappa)$, which is of type (n, n), the terms of type (2,0) will play no role. Set $d\omega = d'\omega + d''\omega$ where

(8.96) $$d'\omega \text{ is of type } (2,0)\ ,$$

(8.97) $$d''\omega \text{ is of type } (1,1)\ .$$

Then

(8.98) $$\varphi(\kappa) = \varphi(-(d''\omega)A)\ ,$$

(8.99) $$d''\omega = \sum_{i,j} \alpha_i(\bar{z}_i)^{\alpha_i-1}d\bar{z}_i b_{ij} dz_j\ .$$

Since φ is homogeneous of degree n, (8.98) implies

(8.100) $$\varphi(\kappa) = (-d''\omega)^n \varphi(A)\ .$$

Set $\Omega = dz_1 d\bar{z}_1 \cdots dz_n d\bar{z}_n$. A straightforward calculation from (8.99) shows

(8.101) $$(-d''\omega)^n = n!\alpha_1\cdots\alpha_n(\bar{z}_1)^{\alpha_1-1}\cdots(\bar{z}_n)^{\alpha_n-1} \det \|b_{ij}\|\, \Omega\ .$$

Therefore

(8.102) $$\varphi(\kappa) = n!\alpha_1\cdots\alpha_n(\bar{z}_1)^{\alpha_1-1}\cdots(\bar{z}_n)^{\alpha_n-1}\varphi(A) \det \|b_{ij}\|\, \Omega\ .$$

Now (8.72) implies

(8.103) $$\int_B (z_1\bar{z}_1)^{\alpha_1-1}\cdots(z_n\bar{z}_n)^{\alpha_n-1}\Omega = (n!\alpha_1\cdots\alpha_n)^{-1}(2\pi/\sqrt{-1})^n\ .$$

Also, if $\beta = (\beta_1, \cdots, \beta_n)$ is an n-tuple of nonnegative integers with $(\beta_1, \cdots, \beta_n) \neq (\alpha_1 - 1, \cdots, \alpha_n - 1)$, then (8.72) implies

(8.104) $$\int_B \bar{z}_1^{\alpha_1-1} z_1^{\beta_1}\cdots \bar{z}_n^{\alpha_n-1} z_n^{\beta_n}\Omega = 0\ .$$

Expand $\varphi(A) \det \|b_{ij}\|$ in a power series in $z_1, \cdots, z_n$. Denote by λ the coefficient of $z_1^{\alpha_1-1}\cdots z_n^{\alpha_n-1}$. Then (8.102)–(8.104) imply

(8.105) $$\lambda = (\sqrt{-1}/(2\pi))^n \int_B \varphi(K)\ .$$

If (8.105) is compared to the algorithm for computing $\operatorname{Res}_p \begin{bmatrix} \varphi(A)dz_1\cdots dz_n \\ a_1, \cdots, a_n \end{bmatrix}$ given by (0.9), then it is evident that (8.80) has been proved.

The proof of the proposition is complete.

From (0.2) and (0.6) it is clear that (8.68) implies (8.1). Theorem 1 is proved.

9. Proof of Theorem 3

Let U be an open subset of M. On U, let ξ be a full integrable subsheaf of $\underline{T}\,|\,U$. Set $Z = \{x \in U \,|\, T_x/\xi_x$ is not a free $\mathcal{O}_x$-module$\}$. Assume that Z is compact and connected. Assume also that (0.27) and (0.28) are valid for Z. Throughout this section $\deg \varphi = n - k + 1$.

Let $Z_1, \cdots, Z_s$ be the irreducible complex-analytic components of Z of dimension $k - 1$. If $[Z_i]$ denotes the element of $H_{2k-2}(Z; \boldsymbol{C})$ given by the fundamental cycle of Z_i, then $[Z_1], \cdots, [Z_s]$ is a vector-space basis for $H_{2k-2}(Z; \boldsymbol{C})$. Hence there exist complex numbers $\lambda_1, \cdots, \lambda_s$ with

$$\operatorname{Res}_\varphi(\xi, Z) = \sum_{i=1}^{s} \lambda_i [Z_i] \,. \tag{9.1}$$

In order to prove (0.42) we must compute $\lambda_1, \cdots, \lambda_s$.

Let V be an open subset of U such that

(9.2) V contains Z, and Z is a deformation retract of V,

(9.3) on V there is an exact sequence

$$0 \to \underline{\underline{E}}_q \to \underline{\underline{E}}_{q-1} \to \cdots \to \underline{\underline{E}}_1 \to \mathscr{A} \underset{\mathcal{O}}{\otimes} \xi \to 0$$

of sheaves of $\mathscr{A}$-modules.

On V there is the short exact sequence

$$0 \to \mathscr{A} \underset{\mathcal{O}}{\otimes} \xi \to \underline{\underline{T}}\,|\,V \to \mathscr{A} \underset{\mathcal{O}}{\otimes} Q \to 0 \,. \tag{9.4}$$

Combining (9.3) and (9.4) gives

$$0 \to \underline{\underline{E}}_q \to \underline{\underline{E}}_{q-1} \to \cdots \to \underline{\underline{E}}_1 \to \underline{\underline{T}}\,|\,V \to \mathscr{A} \underset{\mathcal{O}}{\otimes} Q \to 0 \,. \tag{9.5}$$

Denote by β the Z-sequence resulting from (9.5). On V, let $D_q, D_{q-1}, \cdots, D_0, D_{-1}$ be connections for $E_q, E_{q-1}, \cdots, E_1, T\,|\,V, \nu$ such that

$$(D_q, D_{q-1}, \cdots, D_0, D_{-1}) \qquad \text{is fitted to } \beta. \tag{9.6}$$

Set $K_i = K(D_i)$. As in (5.50) a closed j-form ω on Z with compact support determines an element of $H_{2n-j}(Z; \boldsymbol{C})$. (9.5) is exact, so by (7.18)

$$(\sqrt{-1}/(2\pi))^{n-k+1}\varphi(K_q \,|\, K_{q-1} \,|\, \cdots \,|\, K_0) \qquad \text{determines } \operatorname{Res}_\varphi(\xi, Z) \,. \tag{9.7}$$

Since (0.27) and (0.28) are valid for Z, Theorem (0.30) applies. Let $p \in Z_i - (Z_i \cap (Z^{(2)} \cup N))$. Let U_p be an open neighborhood of p in V such that on U_p there are defined a complex-analytic coordinate system $z_1, \cdots, z_n$ and holomorphic functions $a_k, \cdots, a_n$ as in (0.31)–(0.35). Define a holomorphic normal disc D_p by

(9.8) $$\{D_p = x \in U_p \,|\, z_1(x) = z_1(p), \cdots, z_{k-1}(x) = z_{k-1}(p)\} .$$

Let $i\colon D_p \to V$ be the inclusion. It may be assumed that V and U_p have been chosen so that i is proper. Hence there is the induced map of cohomology with compact supports

(9.9) $$i^*\colon H_c^*(V; C) \to H_c^*(D_p; C) .$$

Consider the homomorphism $I_p\colon H_{2k-2}(Z; C) \to C$ given by

(9.10) $$H_{2k-2}(Z; C) \cong H_c^{2n-2k+2}(V, C) \to H_c^{2n-2k+2}(D_p; C) \cong C .$$

(9.1) implies

(9.11) $$I_p(\mathrm{Res}_\varphi\,(\xi, Z)) = \lambda_i .$$

But then (9.7) implies

(9.12) $$\lambda_i = (\sqrt{-1}/(2\pi))^{n-k+1} \int_{D_p} i^*\varphi(K_q | K_{q-1} | \cdots | K_0) .$$

On D_p let A be the $(n-k+1) \times (n-k+1)$ matrix

(9.13) $$A = \|\partial a_i/\partial z_j\| , \qquad k \leq i, j \leq n .$$

Due to (9.12), (0.42) will be implied by

(9.14) $$(\sqrt{-1}/(2\pi))^{n-k+1} \int_{D_p} i^*\varphi(K_q | K_{q-1} | \cdots | K_0) = \mathrm{Res}_p \begin{bmatrix} \varphi(A) dz_k \cdots dz_n \\ a_k, \cdots, a_n \end{bmatrix} .$$

To prove (9.14), observe that D_p is itself a complex manifold with

(9.15) $$\dim_C D_p = n - k + 1 .$$

Let $T(D_p)$ denote the holomorphic tangent bundle of D_p. On D_p set

(9.16) $\underline{T}(D_p) =$ sheaf of germs of holomorphic sections of $T(D_p)$,

(9.17) $$\dot{X} = \sum_{i=k}^{n} a_i \partial/\partial z_i ,$$

(9.18) $$\dot{\xi} = \text{subsheaf of } \underline{T}(D_p) \text{ spanned by } \dot{X}.$$

Then according to (8.68),

(9.19) $$\mathrm{Res}_\varphi\,(\dot{\xi}, p) = \mathrm{Res}_p \begin{bmatrix} \varphi(A) dz_k \cdots dz_n \\ a_k, \cdots, a_n \end{bmatrix} .$$

So (9.14) will follow from

$$\text{(9.20)} \qquad \operatorname{Res}_\varphi(\dot{\xi}, p) = (\sqrt{-1}/(2\pi))^{n-k+1} \int_{D_p} i^*\varphi(K_q | K_{q-1} | \cdots | K_0) \, .$$

To prove (9.20), on D_p set

(9.21) $\dot{\mathcal{O}}$ = sheaf of germs of holomorphic functions,

(9.22) $\dot{\mathscr{A}}$ = sheaf of germs of real-analytic functions,

$$\text{(9.23)} \qquad \dot{Q} = \underline{T}(D_p)/\dot{\xi} \, ,$$

$$\text{(9.24)} \qquad \dot{E}_j = i^!(E_j) \, , \qquad j = q, q-1, \cdots, 1 \, ,$$

$$\text{(9.25)} \qquad \dot{T} = i^!(T) \, .$$

Thus $\underline{\underline{\dot{E}}}_j$ is a sheaf of $\dot{\mathscr{A}}$ modules on D_p. Now use (9.5) to construct on D_p an exact sequence

$$\text{(9.26)} \qquad 0 \to \underline{\underline{\dot{E}}}_q \to \underline{\underline{\dot{E}}}_{q-1} \to \cdots \to \underline{\underline{\dot{E}}}_1 \to \underline{\underline{\dot{T}}} \to \dot{\mathscr{A}} \otimes_{\dot{\mathcal{O}}} \dot{Q} \to 0$$

of sheaves of $\dot{\mathscr{A}}$-modules.

The sequence (9.26) is obtained by first noting that (0.35) implies

(9.27) ξ_x is a free $\mathcal{O}_x$-module for all $x \in U_p$.

Let E be the unique holomorphic vector bundle on U_p with

$$\text{(9.28)} \qquad \underline{E} = \xi \,|\, U_p \, .$$

Then (9.3) gives on U_p an exact sequence of vector-bundles

$$\text{(9.29)} \qquad 0 \to E_q \to E_{q-1} \to \cdots \to E_1 \to E \to 0 \, .$$

Set $\dot{E} = i^!(E)$. Applying $i^!$ to (9.29) gives on D_p an exact sequence of vector-bundles

$$\text{(9.30)} \qquad 0 \to \dot{E}_q \to \dot{E}_{q-1} \to \cdots \to \dot{E}_1 \to \dot{E} \to 0 \, .$$

So on D_p the sequence

$$\text{(9.31)} \qquad 0 \to \underline{\underline{\dot{E}}}_q \to \underline{\underline{\dot{E}}}_{q-1} \to \cdots \to \underline{\underline{\dot{E}}}_1 \to \underline{\underline{\dot{E}}} \to 0$$

is exact.

On U_p the inclusion $\xi \,|\, U_p \subset \underline{T} \,|\, U_p$ gives a vector-bundle map $\eta: E \to T \,|\, U_p$ such that

(9.32) on U_p there is a holomorphic frame $e_1, \cdots, e_k$ of E with

$$\eta e_i = \partial/\partial z_i \text{ for } i = 1, \cdots, k-1 \text{ and } \eta e_k = \sum_{i=k}^{n} a_i \partial/\partial z_i \, ,$$

$$(9.33) \qquad 0 \to \underline{E} \to \underline{T} \,|\, U_p \to \underline{Q} \,|\, U_p \to 0 \qquad \text{is exact .}$$

Restricting η to D_p gives $\dot{E} \to \dot{T}$ and thus gives a map of $\dot{\mathcal{O}}$-modules $\underline{\dot{E}} \to \underline{\dot{T}}$. On D_p let J be the holomorphic sub-vector-bundle of $\dot{T}$ spanned by $\partial/\partial z_1, \cdots, \partial/\partial z_{k-1}$. Then

$$(9.34) \qquad \dot{T} = J \oplus T(D_p) \; .$$

This direct sum decomposition gives a projection $\dot{T} \to T(D_p)$. Map $\underline{\dot{T}}$ to $\dot{Q}$ by

$$(9.35) \qquad \underline{\dot{T}} \to \underline{T}(D_p) \to \dot{Q} \; .$$

Then (9.32) and (9.23) imply

$$(9.36) \qquad 0 \to \underline{\dot{E}} \to \underline{\dot{T}} \to \dot{Q} \to 0 \qquad \text{is exact.}$$

Hence by (6.1)

$$(9.37) \qquad 0 \to \underline{\underline{\dot{E}}} \to \underline{\underline{\dot{T}}} \to \dot{\mathscr{A}} \otimes_{\dot{\mathcal{O}}} \dot{Q} \to 0 \qquad \text{is exact.}$$

Now (9.26) is the exact sequence obtained by combining (9.37) and (9.31).

On D_p let $\dot{\beta}$ be the p-sequence resulting from (9.26). Set $\dot{D}_j = i^!(D_j)$. (9.26) implies

$$(9.38) \qquad (\dot{D}_q, \dot{D}_{q-1}, \cdots, \dot{D}_0, \dot{D}_{-1}) \qquad \text{is fitted to } \dot{\beta}.$$

Set $K_j = K(D_j)$. Since (9.26) is exact, (7.18) implies

$$(9.39) \qquad \operatorname{Res}_\varphi(\dot{\xi}, p) = (\sqrt{-1}/(2\pi))^{n-k+1} \int_{D_p} \varphi(\dot{K}_q \,|\, \dot{K}_{q-1} \,|\, \cdots \,|\, \dot{K}_0) \; .$$

But

$$(9.40) \qquad \varphi(\dot{K}_q \,|\, \dot{K}_{q-1} \,|\, \cdots \,|\, \dot{K}_0) = i^*\varphi(K_q \,|\, K_{q-1} \,|\, \cdots \,|\, K_0) \; .$$

So (9.20) has been proved. This concludes the proof of Theorem 3.

Remark. The argument of this section really verifies a very special case of the functoriality of $\operatorname{Res}_\varphi(\xi, Z)$.

10. Proof of the rigidity theorem

Let F be a holomorphic integrable sub-vector-bundle of T, $k = \dim_C F_x$, and $\nu = T/F$. If U is an open subset of M, let $A(U)$ denote the ring of all C^∞ complex-valued differential forms on U. In $A(U)$ let $I(F, U)$ be the ideal generated by all C^∞ 1-forms ω on U such that

$$\omega \text{ is of type } (1,0)\ , \tag{10.1}$$

$$i(\gamma)\omega = 0 \qquad \text{for every } \gamma \in C^\infty(F \mid U)\ . \tag{10.2}$$

(10.3) Lemma. *Let D be a basic connection for ν. On U let $e_1, \cdots, e_{n-k}$ be a C^∞ frame of ν. Let $\kappa = \|\kappa_{ij}\|$ be the curvature matrix of D with respect to the frame $e_1, \cdots, e_{n-k}$. Then for each κ_{ij},*

$$\kappa_{ij} \in I(F, U)\ . \tag{10.4}$$

Proof. Let $\eta\colon T \to T/F$ be the projection. Given $p \in U$, let U_p be an open neighborhood of p in U such that on U_p there is a complex-analytic coordinate system $z_1, \cdots, z_n$ with

$$\partial/\partial z_1, \cdots, \partial/\partial z_k \in \Gamma(F \mid U_p)\ . \tag{10.5}$$

Let $\kappa' = \|\kappa'_{ij}\|$ be the curvature matrix of D with respect to the frame $\eta\partial/\partial z_{k+1}, \cdots, \eta\partial/\partial z_n$. Then according to (3.33) each κ'_{ij} is in $I(F, U)$:

$$\kappa'_{ij} \in I(F, U)\ . \tag{10.6}$$

(10.4) is now implied by (1.16), and the lemma is proved.

Next, let U be open in M, and $[a, b]$ a closed interval of real numbers. Set $\tilde U = U \times [a, b]$, and let $\rho\colon \tilde U \to U$, $t\colon \tilde U \to [a, b]$ be the projections. For each $r \in [a, b]$ define $i_r\colon U \to \tilde U$ by

$$i_r(x) = (x, r)\ , \qquad x \in U,\ r \in [a, b]\ . \tag{10.7}$$

(10.8) Definition. A C^∞ 1-parameter family of holomorphic foliations of U is a 1-parameter family $\{F_r\}$, $a \le r \le b$, such that

(10.9) for each $r \in [a, b]$, F_r is a holomorphic integrable sub-vector-bundle of $T \mid U$,

(10.10) on $\tilde U$ there exists a C^∞ sub-vector-bundle $\tilde F$ of $\rho^!(T \mid U)$ with $i_r^!(\tilde F) = F_r$ for each $r \in [a, b]$.

(10.11) Lemma. *Let $\{F_r\}$, $a \le r \le b$, be a C^∞ 1-parameter family of holomorphic foliations of U. On $\tilde U$ let D be a connection for $\rho^!(T)/\tilde F$ such that*

(10.12) *for each $r \in [a, b]$, $i_r^!(D)$ is a basic connection for T/F_r.*

Set $K = K(D)$, and assume $n - k + 1 < \deg\varphi \le n$. Then

$$\varphi(K) = 0\ . \tag{10.13}$$

Proof. Given $p \in U$, let U_p be an open neighborhood of p in U such that

p is a deformation retract of U_p. Set $\tilde{U}_p = U_p \times [a, b]$. In $A(\tilde{U}_p)$ let I be the ideal

(10.14) $$I = \{\omega \in A(\tilde{U}_p) \mid \text{For each } r \in [a, b],\ i_r^*(\omega) \in I(F_r, U)\} .$$

On $\tilde{U}_p$ let $u_1, \cdots, u_{n-k}$ be a C^∞ frame of $\rho^!(T)/\tilde{F}$. Let $\kappa = \|\kappa_{ij}\|$ be the curvature matrix of D with respect to $u_1, \cdots, u_{n-k}$. Then (10.12), (10.4), and (4.56) imply that each κ_{ij} is in I:

(10.15) $$\kappa_{ij} \in I .$$

Hence (10.13) will follow from

(10.16) if $\omega_1, \cdots, \omega_{n-k+2}$ are any $n - k + 2$ elements of I, then $\omega_1 \wedge \cdots \wedge \omega_{n-k+2} = 0$.

To prove (10.16) let $T_R\tilde{U}$ and $T_R[a, b]$ be the C^∞ tangent bundles of $\tilde{U}$ and $[a, b]$. Define $T_C\tilde{U}$, $T_C[a, b]$ by

(10.17) $$T_C\tilde{U} = C \otimes_R T_R\tilde{U} ,$$

(10.18) $$T_C[a, b] = C \otimes_R T_R[a, b] .$$

Then

(10.19) $$T_C\tilde{U} = \rho^!(T) \oplus \rho^!(\overline{T}) \oplus t^!T_C[a, b] .$$

On $\tilde{U}_p$ let $v_1, \cdots, v_{2n+1}$ be a C^∞ frame of $T_C\tilde{U}$ such that

(10.20) $$v_1, \cdots, v_k \in C^\infty(\tilde{F} \mid \tilde{U}_p) ,$$

(10.21) $$v_{k+1}, \cdots, v_n \in C^\infty(\rho^!(T) \mid \tilde{U}_p) ,$$

(10.22) $$v_{n+1}, \cdots, v_{2n} \in C^\infty(\rho^!(\overline{T}) \mid \tilde{U}_p) ,$$

(10.23) $$v_{2n+1} \in C^\infty(t^!T_C[a, b] \mid \tilde{U}_p) .$$

Let $v_1^*, \cdots, v_{2n+1}^*$ denote the dual frame for the dual bundle $(T_C\tilde{U})^* \mid \tilde{U}_p$. Then (10.14) implies that I is the ideal in $A(\tilde{U}_p)$ generated by $v_{k+1}^*, \cdots, v_n^*$ and v_{2n+1}^*. Since there are $n - k + 1$ of these, (10.16) is clear. This completes the proof.

Remark. If $\{F_r\}$ is as in Lemma (10.11), then there always exist connections D for $\rho^!(T)/\tilde{F}$ such that (10.12) is valid for D. To see this, set $\tilde{\nu} = \rho^!(T)/\tilde{F}$ and $\nu_r = T/F_r$. As in Proposition (3.21) for each ν_r there is a partial connection

(10.24) $$\delta_r \colon C^\infty(\nu_r) \to C^\infty((F_r \oplus \overline{T})^* \otimes \nu_r) .$$

These δ_r fit together to give a partial connection δ for $\tilde{\nu}$:

(10.25) $$\delta\colon C^\infty(\tilde{\nu}) \to C^\infty((\tilde{F} \oplus \rho^! T)^* \otimes \tilde{\nu}) .$$

A connection D for $\tilde{\nu}$ which extends this δ will satisfy (10.12).

(10.26) Definition. On U let ξ be a full integrable subsheaf of $\underline{T} \mid U$. Let $E_q, E_{q-1}, \cdots, E_1$ be real-analytic vector-bundles on U such that there is an exact sequence of sheaves of $\mathscr{A}$-modules

(10.27) $$0 \to \underline{\underline{E}}_q \to \underline{\underline{E}}_{q-1} \to \cdots \to \underline{\underline{E}}_1 \to \mathscr{A} \underset{\mathcal{O}}{\otimes} \xi \to 0$$

on U. From (10.27) a complex

(10.28) $$0 \to E_q \to E_{q-1} \to \cdots \to E_1 \to T \mid U$$

of real-analytic vector-bundles on U is obtained. By viewing each E_i and T as C^∞ vector-bundles, (10.28) may then be taken to be a complex of C^∞ vector-bundles on U. Any complex of C^∞ vector-bundles on U which arises in this way will be referred to as *a complex for* ξ.

Remark. Up to this point we have not precisely defined a C^∞ 1-parameter family of sheaves. This is made precise by

(10.29) Definition. A C^∞ *1-parameter family* $\{\xi_r\}$, $a \le r \le b$, of full integrable subsheaves of $\underline{T} \mid U$ is a 1-parameter family such that

(10.30) for each $r \in [a, b]$, ξ_r is a full integrable subsheaf of $\underline{T} \mid U$,

(10.31) on $\tilde{U} = U \times [a, b]$ there exists a complex

$$0 \to E_q \to E_{q-1} \to \cdots \to E_0 \to \rho^!(T) \to 0$$

of C^∞ vector-bundles such that for each $r \in [a, b]$,

$$0 \to i_r^!(E_q) \to i_r^!(E_{q-1}) \to \cdots \to i_r^!(E_1) \to T \mid U$$

is a complex for ξ_r.

Proof of Theorem 4. Let $\{\xi_r\}$, $a \le r \le b$, be a C^∞ 1-parameter family of full integrable subsheaves of $\underline{T} \mid U$. For $r \in [a, b]$, let $Z_r = \{x \in U \mid (\underline{T}/\xi_r)_x$ is not a free $\mathcal{O}_x$-module$\}$. Assume that each Z_r is compact and connected. As in (0.43) assume that there is a compact subset B of U with

(10.32) $$Z_r \subset B \qquad \text{for all } r \in [a, b] .$$

Let $i_*\colon H_*(Z_r; C) \to H_*(U; C)$ be the homology map induced by the inclusion of Z_r in U. If $n - k + 1 < \deg \varphi \le n$, we then wish to prove

(10.33) $$i_* \operatorname{Res}_\varphi (\xi_a, Z_a) = i_* \operatorname{Res}_\varphi (\xi_b, Z_b) .$$

To prove (10.33) set $\tilde{U} = U \times [a, b]$, and on $\tilde{U}$ let

(10.34) $$0 \to E_q \to E_{q-1} \to \cdots \to E_1 \to \rho^!(T)$$

be as in (10.31). On $U - Z_r$ let F_r be the unique holomorphic sub-vector-bundle of $T \mid U - Z_r$ such that

(10.35) $$\underline{F}_r = \xi_r \mid U - Z_r \ .$$

On $U - Z_r$, set $\nu_r = T/F_r$. Let V_r be an open subset of U with Z_r contained in V_r and Z_r a deformation retract of V_r. Then

(10.36) when restricted to V_r,
$$0 \to i_r^!(E_q) \to i_r^!(E_{q-1}) \to \cdots \to i_r^!(E_1) \to T \to \nu_r \to 0$$
is a Z_r-sequence.

Set $\tilde{Z} = \{(x, r) \in \tilde{U} \mid x \in Z_r\}$. On $\tilde{U} - \tilde{Z}$, let $\tilde{F}$ be the unique C^∞ sub-vector-bundle of $\rho^!(T)$ such that

(10.37) $$i_r^!(\tilde{F}) = F_r \qquad \text{for each } r \in [a, b] \ .$$

On $\tilde{U} - \tilde{Z}$, set $\tilde{\nu} = \rho^!(T)/\tilde{F}$. Let D_{-1} be a connection for $\tilde{\nu}$ with

(10.38) $i_r^!(D_{-1})$ is a basic connection for ν_r for each $r \in [a, b]$.

With B as in (10.32) choose a compact subset Σ of U with B contained in the interior of Σ. Set $\tilde{\Sigma} = \Sigma \times [a, b]$. On $\tilde{U}$ let $D_q, D_{q-1}, \cdots, D_1, D_0$ be connections for $E_q, E_{q-1}, \cdots, E_1, \rho^!(T)$ such that

(10.39) on $\tilde{U} - \tilde{\Sigma}$, $(D_q, D_{q-1}, \cdots, D_0, D_{-1})$ is compatible with the exact sequence
$$0 \to E_q \to E_{q-1} \to \cdots \to E_1 \to \rho^!(T) \to \tilde{\nu} \to 0 \ .$$

Set $K_i = K(D_i)$. Then according to (4.23),

(10.40) $$\text{On} \quad \tilde{U} - \tilde{\Sigma}, \varphi(K_q \mid K_{q-1} \mid \cdots \mid K_0) = \varphi(K_{-1}) \ .$$

Since $n - k + 1 < \deg \varphi \leq n$, (10.38) and (10.13) imply

(10.41) $$\varphi(K_{-1}) \quad \text{vanishes on} \quad \tilde{U} - \tilde{\Sigma} \ .$$

Hence

(10.42) $\varphi(K_q \mid K_{q-1} \mid \cdots \mid K_0)$ is a closed form on $\tilde{U}$ with compact support.

Set $D_i^r = i_r^!(D_i)$, $K_i^r = K(D_i^r)$. Let $l = \deg \varphi$. On U there is the Poincaré duality isomorphism:

(10.43) $$\alpha\colon H_{2n-2l}(U; C) \to H_c^{2l}(U; C) \ .$$

(10.36)–(10.39) imply

(10.44) for each $r \in [a, b]$, $\varphi(K_q^r | K_{q-1}^r | \cdots | K_0^r)$ is a $2l$-form on U with compact support,

(10.45) the element of $H_c^{2l}(U; \boldsymbol{C})$ given by $(\sqrt{-1}/(2\pi))^l \varphi(K_q^r | K_{q-1}^r | \cdots | K_0^r)$ is $\alpha i_* \operatorname{Res}_\varphi(\xi_r, Z_r)$.

Since α is an isomorphism, (10.33) will be proved if it can be shown that $\varphi(K_q^a | K_{q-1}^a | \cdots | K_0^a)$ and $\varphi(K_q^b | K_{q-1}^b | \cdots | K_0^b)$ give the same element of $H_c^{2l}(U)$. But (4.56) implies

$$(10.46) \qquad i_r^* \varphi(K_q | K_{q-1} | \cdots | K_0) = \varphi(K_q^r | K_{q-1}^r | \cdots | K_0^r) \qquad \text{for each } r \in [a, b] \ .$$

From (10.46) and (10.42) it is clear that the proof is complete.

Proof of Corollary 0.44. Let Z be as in Corollary (0.45). Choose an open subset V of U with Z contained in V and Z a deformation retract of V. Let $i_*: H_*(Z; \boldsymbol{C}) \to H_*(V; \boldsymbol{C})$ be the homology map induced by the inclusion of Z in V. Then according to (0.43),

$$(10.47) \qquad i_* \operatorname{Res}_\varphi(\xi_a, Z) = i_* \operatorname{Res}_\varphi(\xi_b, Z) \ .$$

Since $i_*: H_*(Z; \boldsymbol{C}) \to H_*(V; \boldsymbol{C})$ is an isomorphism, (10.47) implies

$$(10.48) \qquad \operatorname{Res}_\varphi(\xi_a, Z) = \operatorname{Res}_\varphi(\xi_b, Z) \ .$$

This proves the corollary.

11. Examples

Example 1. Let $\lambda_1, \cdots, \lambda_n$ be nonzero complex numbers. On $\boldsymbol{C}^n$, with its usual coordinate system, let X be the homomorphic vector-field:

$$(11.1) \qquad X = \sum_{i=1}^{n} \lambda_i z_i \partial / \partial z_i \ .$$

The origin is the only zero of X. Let ξ be the subsheaf of $\underline{T}$ spanned by X. Assume $\deg \varphi = n$. Identify, as usual, $H_0(0, \boldsymbol{C}) = \boldsymbol{C}$. Then

$$(11.2) \qquad \operatorname{Res}_\varphi(\xi, 0) = \varphi(\lambda_1, \lambda_2, \cdots, \lambda_n)/(\lambda_1 \lambda_2 \cdots \lambda_n) \ .$$

Example 2. Let $a_0, \cdots, a_n$ be $n + 1$ distinct complex numbers. Define a holomorphic flow

$$(11.3) \qquad \boldsymbol{C} \times \boldsymbol{CP}^n \to \boldsymbol{CP}^n$$

by

$$(11.4) \qquad (z, [z_0 : z_1 : \cdots : z_n]) \to [e^{a_0 z} z_0 : e^{a_1 z} z_1 : \cdots : e^{a_n z} z_n] \ .$$

Let X be the holomorphic vector field on CP^n, which generates this flow. The zeroes of X are the $n+1$ points $p_0, p_1, \cdots, p_n$ where

$$\begin{aligned} p_0 &= [1:0:0:\cdots:0] , \\ p_1 &= [0:1:0:\cdots:0] , \\ \vdots & \qquad\qquad \vdots \\ p_n &= [0:0:0:\cdots:1] . \end{aligned}$$

Each p_i is a non-degenerate zero of X. Let ξ be the subsheaf of $\underline{T}$ spanned by X. Identify $H_0(p_i, C) = C$, and assume $\deg \varphi = n$. Then

$$\begin{aligned} &\operatorname{Res}_\varphi(\xi, p_i) \\ (11.5) \qquad &= \frac{\varphi(a_0 - a_i, a_1 - a_i, \cdots, a_{i-1} - a_i, a_{i+1} - a_i, \cdots, a_n - a_i)}{(a_0 - a_i)(a_1 - a_i)\cdots(a_{i-1} - a_i)(a_{i+1} - a_i)\cdots(a_n - a_i)} . \end{aligned}$$

Example 3. Fix integers k and n with $1 < k < n$. Let A be a $k \times (n+1)$ matrix of complex numbers

$$(11.6) \qquad A = \begin{bmatrix} a_{10} & a_{11} & \cdots & a_{1n} \\ a_{20} & a_{21} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{k0} & a_{k1} & \cdots & a_{kn} \end{bmatrix} .$$

For each $i = 0, 1, \cdots, n$ denote by A_i the $k \times n$ matrix obtained by subtracting the i-th column of A from all the other columns of A

$$(11.7) \qquad A_i = \begin{bmatrix} a_{10} - a_{1i} & a_{11} - a_{1i} & \cdots & a_{1n} - a_{1i} \\ a_{20} - a_{2i} & a_{21} - a_{2i} & \cdots & a_{2n} - a_{2i} \\ \vdots & \vdots & & \vdots \\ a_{k0} - a_{ki} & a_{k1} - a_{ki} & \cdots & a_{kn} - a_{ki} \end{bmatrix} .$$

Assume

(11.8) for each $i = 0, 1, \cdots, n$ all the $k \times k$ sub-matrices of A_i are non-singular.

The set of all matrices A for which (11.8) is valid is open and dense in the vector-space of all $k \times (n+1)$ matrices of complex numbers.

So given A satisfying (11.8) let V_i be the holomorphic vector field on CP^n which generates the flow

$$(11.9) \qquad (z, [z_0 : z_1 : \cdots : z_n]) \to (e^{a_{i0}z}z_0 : e^{a_{i1}z}z_1 : \cdots : e^{a_{in}z}z_n) .$$

Let ξ be the subsheaf of $\underline{T}$ spanned by $V_1, \cdots, V_k$. ξ is integrable and full.

To describe the singular set of ξ, let $\alpha = (\alpha_1, \cdots, \alpha_{n-k+1})$ be an $(n-k+1)$-tuple of integers with

$$0 \le \alpha_1 < \cdots < \alpha_{n-k+1} \le n \,. \tag{11.10}$$

Define CP_α^{k-1} by

$$CP_\alpha^{k-1} = \{[z_0 : z_1 : \cdots : z_n] \in CP^n \,|\, 0 = z_{\alpha_1} = \cdots = z_{\alpha_{n-k+1}}\} \,. \tag{11.11}$$

The singular set Z of ξ is

$$Z = \bigcup CP_\alpha^{k-1} \,, \tag{11.12}$$

where the union is taken over all α satisfying (11.10). The CP_α^{k-1} are the irreducible complex-analytic components of Z. (0.27) and (0.28) are valid for Z.

If $\deg \varphi = n - k + 1$, then Theorem 3 applies. Hence according to (0.42)

$$\operatorname{Res}_\varphi(\xi, Z) = \sum_\alpha \#(\varphi, \xi, CP_\alpha^{k-1})[CP_\alpha^{k-1}] \,. \tag{11.13}$$

$\#(\varphi, \xi, CP_\alpha^{k-1})$ can be explicitly computed as follows. First, shuffle the columns of A to obtain a new matrix A_α whose first $n-k+1$ columns are the α_1-th, $\cdots, \alpha_{n-k+1}$-th columns of A. Next, form a $k \times n$ matrix B_α by subtracting the last column of A_α from all the other columns of A_α. Define complex numbers $\lambda_1^\alpha, \cdots, \lambda_{n-k+1}^\alpha$ by letting λ_i^α be the determinant of the $k \times k$ sub-matrix of B_α consisting of the i-th column of B_α and the last $k-1$ columns of B_α. Then by (0.37):

$$\#(\varphi, \xi, CP_\alpha^{k-1}) = \varphi(\lambda_1^\alpha, \cdots, \lambda_{n-k+1}^\alpha, 0, \cdots, 0)/(\lambda_1^\alpha \cdots \lambda_{n-k+1}^\alpha) \,. \tag{11.14}$$

Combining (11.14) and (11.13) gives

$$\operatorname{Res}_\varphi(\xi, Z) = \sum_\alpha \varphi(\lambda_1^\alpha, \cdots, \lambda_{n-k+1}^\alpha, 0, \cdots, 0)/(\lambda_1^\alpha \cdots \lambda_{n-k+1}^\alpha)[CP_\alpha^{k-1}] \,. \tag{11.15}$$

If $n - k + 1 < \deg \varphi \le n$, then the situation is quite different. In this case, set $l = \deg \varphi$ and let $x \in H^2(CP^n; C)$ be the element of $H^2(CP^n; C)$ dual to a hyperplane. Define a complex number $w(\varphi)$ by

$$\varphi(T) = w(\varphi)x^l \,. \tag{11.16}$$

Choose an l-tuple $\beta = (\beta_1, \cdots, \beta_l)$ of integers with

$$0 \le \beta_1 < \cdots < \beta_l \le n \,. \tag{11.17}$$

Define CP_β^{n-l} by

(11.18) $\quad CP_\beta^{n-l} = \{[z_0 : z_1 : \cdots : z_n] \in CP^n \mid 0 = z_{\beta_1} = \cdots = z_{\beta_l}\}$.

Denote by $[CP^{n-l}]$ the element of $H_{2n-2l}(Z, C)$ given by the fundamental cycle of CP_β^{n-l}. $[CP^{n-l}]$ does not depend on the choice of β. Then

(11.19) $\quad \operatorname{Res}_\varphi(\xi, Z) = w(\varphi)[CP^{n-l}]\ , \qquad n - k + 1 < \deg\varphi \leq n$.

Note how (11.15) and (11.19) illustrate the rigidity theorem. If A is varied, then the right-hand side of (11.15) varies, but that of (11.19) remains constant.

Example 4. Fix integers k and n with $1 < k < n$. Let Z be a compact connected complex-analytic manifold with

(11.20) $$\dim_C Z = k - 1\ .$$

Set $r = n - k + 1$. Let L be a holomorphic line bundle on Z. Choose nonzero integers $n_1, \cdots, n_r$. Let M be the total space of the vector-bundle $L^{n_1} \oplus \cdots \oplus L^{n_r}$. Here L^{n_i} denotes the tensor product of L with itself n_i times. Then M is a complex manifold with

(11.21) $$\dim_C M = n\ .$$

Let $\pi\colon L^{n_1} \oplus \cdots \oplus L^{n_r} \to Z$ be the projection. The zero section of the vector-bundle gives an inclusion $Z \subset M$.

Choose a cover $\{U_\alpha\}_{\alpha \in I}$ of Z by open sets with:

(11.22) U_α is the domain of a complex-analytic coordinate system $w_1^\alpha, \cdots, w_{k-1}^\alpha$,

(11.23) on U_α there is a holomorphic section s_α of $L \mid U_\alpha$ such that s_α has no zeroes.

Let $s_\alpha^{n_i}$ denote the tensor product of s_α with itself n_i times. Then $s_\alpha^{n_1}, \cdots, s_\alpha^{n_r}$ is a holomorphic frame of $L^{n_1} \oplus \cdots \oplus L^{n_r}$ on U_α.

Set $\tilde U_\alpha = \pi^{-1}(U_\alpha)$. On $\tilde U_\alpha$ let $z_1^\alpha, \cdots, z_n^\alpha$ be the coordinate system resulting from $w_1^\alpha, \cdots, w_{k-1}^\alpha$ and $s_\alpha^{n_1}, \cdots, s_\alpha^{n_r}$. Thus if $\tilde v \in U_\alpha$, then

(11.24) $$z_i^\alpha(v) = w_i^\alpha(\pi v)\ , \qquad i = 1, \cdots, k - 1\ ,$$

(11.25) $$v = \sum_{i=1}^r z_{i+k-1}^\alpha(v) s_\alpha^{n_i}(\pi v)\ .$$

On $\tilde U_\alpha$ let ξ_α be the subsheaf of $\underline{T} \mid \tilde U_\alpha$ spanned by $\partial/\partial z_1^\alpha, \cdots, \partial/\partial z_{k-1}^\alpha$, $\sum_{i=1}^r n_i z_{i+k-1}^\alpha \partial/\partial z_{i+k-1}^\alpha$. Then

(11.26) $$\xi_\alpha \mid U_\alpha \cap U_\beta = \xi_\beta \mid U_\alpha \cap U_\beta\ .$$

So the $\{\xi_\alpha\}_{\alpha \in I}$ fit together to form a subsheaf ξ of $\underline{T}$, which is integrable and full. Also, (11.26) implies that ξ does not depend on the choice of the cover $\{U_\alpha\}_{\alpha \in I}$. The singular set of ξ is Z.

Set $x = c_1(L)$, so that $x \in H^2(Z; \boldsymbol{C})$. Then for any φ with $n - k + 1 \leq \deg \varphi \leq n$,

(11.27) $\operatorname{Res}_\varphi(\xi, Z)$ is the element of $H_{2n-2l}(Z, \boldsymbol{C})$ which by Poincaré duality in Z is dual to $\dfrac{\varphi(n_1, \cdots, n_r, 0, \cdots, 0)}{n_1 \cdots n_r} x^{l-r}$, $\quad r = n - k + 1$.

Example 5. Let k, n, Z be as in Example 4. Set $r = n - k + 1$. Let M be $Z \times \boldsymbol{C}^r$, and $\pi_1 \colon Z \times \boldsymbol{C}^r \to Z$, $\pi_2 \colon Z \times \boldsymbol{C}^r \to \boldsymbol{C}^r$ be the projections. Then there is the splitting

$$T(Z \times \boldsymbol{C}^r) = \pi_1^!(TZ) \oplus \pi_2^!(T\boldsymbol{C}^r) . \tag{11.28}$$

Let $\lambda_1, \cdots, \lambda_r$ be nonzero complex numbers. On $\boldsymbol{C}^r$ with its usual coordinate system set

$$X = \sum_{i=1}^{r} \lambda_i z_i \partial / \partial z_i . \tag{11.29}$$

On $Z \times \boldsymbol{C}^r$ set

$$\tilde{X} = \pi_2^!(X) . \tag{11.30}$$

Let ξ be the subsheaf of $\underline{T}$ spanned by $\tilde{X}$ and all local holomorphic sections of $\pi_1^!(TZ)$. ξ is integrable and full, and its singular set is $Z \times \{0\}$. Identify $Z \times \{0\} = Z$.

(11.31) If $\deg \varphi = r$, then $\quad \operatorname{Res}_\varphi(\xi, Z) = \dfrac{\varphi(\lambda_1, \cdots, \lambda_r, 0, \cdots, 0)}{\lambda_1 \cdots \lambda_r}[Z]$.

(11.32) If $r < \deg \varphi \leq n$, then $\quad \operatorname{Res}_\varphi(\xi, Z) = 0$.

Example 6. Fix integers d and n with $1 \leq d < n$. Let Z be a compact connected complex manifold with

$$\dim_{\boldsymbol{C}} Z = d . \tag{11.33}$$

Set $s = n - d$. Let $L_1, \cdots, L_s$ be holomorphic line bundles on Z, and M the total space of the vector-bundle $L_1 \oplus \cdots \oplus L_s$. Then M is a complex manifold with

$$\dim_{\boldsymbol{C}} M = n . \tag{11.34}$$

The zero section of the vector bundle gives an inclusion $Z \subset M$. Denote a point of M by $(u_1, \cdots, u_s)$, so that $u_i \in L_i$. Let $\lambda_1, \cdots, \lambda_s$ be nonzero complex numbers. Construct a holomorphic flow

$$C \times M \to M \tag{11.35}$$

by

$$(z, (u_1, \cdots, u_s)) \to (e^{\lambda_1 z} u_1, \cdots, e^{\lambda_s z} u_s) . \tag{11.36}$$

Let X be the vector-field on M, which generates this flow. Let X be the subsheaf of $\underline{T}$ spanned by X. The singular set of ξ is Z. Identify $H_0(Z, C) = C$. Assume $n = \deg \varphi$. Let $x_1, \cdots, x_d$ be the formal Chern roots of TZ. Set $y_i = c_1(L_i)$. Take the $2d$-dimensional component of $\varphi(x_1, \cdots, x_d, \lambda_1 + y_1, \cdots, \lambda_s + y_s)/[(\lambda_1 + y_1) \cdots (\lambda_s + y_s)]$. Evaluate this element of $H^{2d}(Z, C)$ on the fundamental cycle of Z. This gives

$$\operatorname{Res}_\varphi(\xi, Z) = \frac{\varphi(x_1, \cdots, x_d, \lambda_1 + y_1, \cdots, \lambda_s + y_s)}{(\lambda_1 + y_1) \cdots (\lambda_s + y_s)}[Z] . \tag{11.37}$$

For a proof of (11.37) see [8] or [14], and also [6], [3, Theorem (8.11) and Proposition (8.13), pp. 597–599], [20], [21].

Remark. In the general problem of computing $\operatorname{Res}_\varphi(\xi, Z)$ let $Z = Z^{(1)} \supset \cdots \supset Z^{(k)}$ be as (0.26). Consider the special case when $\dim_C Z = k - 1$ and $Z^{(2)} = \phi$. It can be shown that for this special case Examples 4 and 5 above essentially solve the problem of computing $\operatorname{Res}_\varphi(\xi, Z)$.

12. On the space $B\Gamma_q^C$

In the homotopy theory of complex foliations as developed by Haefliger-Phillips-Gromov [15], [16] a complex foliation F on a manifold M determines a classifying map

$$f_F: M \to B\Gamma_q^C . \tag{12.1}$$

Here $q = n - k$ is the codimension of F. In this section we would like to explain the relation of our residue classes $\operatorname{Res}_\varphi(\xi, Z)$ to this homotopy theory.

First recall that there is a natural map

$$\nu: B\Gamma_q^C \to BGL(q) , \tag{12.2}$$

which corresponds to assigning the normal bundle of a foliation. As usual $BGL(q)$ denotes the classifying space of the general linear group $GL(q, C)$. In terms of these concepts the vanishing theorem simply asserts

$$\nu^*: H^{2j}(BGL(q); C) \to H^{2j}(B\Gamma_q^C; C) \qquad \text{vanishes whenever } j > q . \tag{12.3}$$

In contrast to this, it is not difficult to show that

(12.4)
$$\begin{aligned}&\nu^*: H^j(BGL(q); Z) \to H^j(B\Gamma_q^C; C)\\&\text{is injective for all } j = 0, 1, 2, \cdots ,\end{aligned}$$

and (12.3) and (12.4) together imply

(12.5) For each $j > q$, $H_{2j-1}(B\Gamma_q^C; Z)$ is an abelian group which is not finitely generated.

See [12] for details of these first consequences of (12.3). More delicate results arise in the following manner:

Let BGL be the classifying space of the infinite general linear group. If c_1, $c_2, \cdots$ are the universal Chern classes, then

(12.6)
$$H^*(BGL; C) = C[c_1, c_2, \cdots] .$$

Let $\nu_s: B\Gamma_q \to BGL$ be the composition of ν with the inclusion $BGL(q) \subset BGL$. Then (12.3) is equivalent to

(12.7) $\nu_s^*: H^{2j}(BGL; C) \to H^{2j}(B\Gamma_q^C; C)$ vanishes whenever $j > q$.

We are interested in ν_s only up to homotopy type, so ν_s can be taken as an inclusion $B\Gamma_q \subset BGL$. Thus there is the pair of spaces $(BGL, B\Gamma_q)$. In this context the constructions of § 5 (e.g., Definition (5.51)) can be interpreted as lifting each $\varphi \in H^{2j}(BGL; C)$, $j > q$, to a definite and well-defined class $\hat{\varphi} \in H^{2j}(BGL, B\Gamma_q; C)$.

More precisely, let ξ be a full integrable sheaf on a complex manifold M. Set $Q = \underline{T}/\xi$. Let S be the singular set of ξ. The procedure given in § 5 and § 7 (e.g., Definition (7.1)) lifts each $\varphi(Q) \in H^{2j}(M; C)$, $j > n - k$, in a canonical fashion to a class $\hat{\varphi}(Q) \in H^{2j}(M, M - S; C)$. Now ξ determines a homotopy commutative diagram

(12.8)
$$\begin{tikzcd} M - S \arrow[r, "f_F"] \arrow[d] & B\Gamma_q \arrow[d] \\ M \arrow[r, "f_Q"'] & BGL \end{tikzcd}$$

where f_F classifies the foliation of $M - S$, and f_Q classifies the element of $K(M)$ given by Q.

The exactness of a resolution of Q gives on $M - S$ an exact sequence of vector-bundles

(12.9)
$$0 \to E_r \to E_{r-1} \to \cdots \to E_0 \to \nu \to 0 ,$$

which can be thought of as an explicit homotopy between the two maps of $M - S$ into BGL of (12.8). Therefore ξ defines a map of pairs:

$$f_\xi\colon (M, M - U) \to (BGL, B\Gamma_q)\ , \tag{12.10}$$

where U is a small neighborhood of S in M.

The universal liftings $\hat{\varphi} \in H^{2j}(BGL, B\Gamma_q; C)$ are now uniquely characterized by

$$f_\xi^*(\hat{\varphi}) = \hat{\varphi}(Q)\ . \tag{12.11}$$

Quite equivalently this is expressed by the formula

$$\mathrm{Res}_\varphi\,(\xi, Z) = \pi_Z f_\xi^*(\hat{\varphi})\ , \tag{12.12}$$

where

$$\pi_Z\colon H^*(M, M - S; C) \to H_*(Z, C)$$

is induced by excision followed by Poincaré duality.

Granting (12.12) one may use the examples of § 11 to prove the following:

Proposition. *Let $d(n)$ be the dimension of $H^{2n}(BGL(n-1); C)$ over C. Then there exists a surjection of abelian groups*

$$h\colon \pi_{2n-1}(B\Gamma^C_{n-1}) \to C^{d(n)}\ . \tag{12.13}$$

This result was already announced in [9] and is an easy analogue in the complex case of the recent results of Thurston [22], concerning real foliations with varying Godbillion-Vey invariants.

To prove (12.13) we first construct a homomorphism

$$\tilde{h}\colon \pi_{2n-1}(BGL, B\Gamma_{n-1}) \to C^{d(n)} \tag{12.14}$$

in the following manner. Let $\varphi_1, \cdots, \varphi_{d(n)}$ be a basis for the symmetric polynomials in n-variables $X_1, \cdots, X_n$, which are of degree n, and lie in the ideal generated by the first $n-1$ elementary symmetric functions $\sigma_1, \cdots, \sigma_{n-1}$ of the X's. We identify the φ's with classes in $H^*(BGL)$ by interpreting the σ_i as the i-th Chern classes, and denote by $\hat{\varphi}_1, \cdots, \hat{\varphi}_{d(n)}$ the liftings of these classes to $H^*(BGL, B\Gamma^C_{n-1})$. Now then $\tilde{h}$ is defined to be the evaluation of this basis on a relative class:

$$\tilde{h}(\alpha) = \{\hat{\varphi}_1(\alpha), \cdots, \hat{\varphi}_{d(n)}(\alpha)\}\ , \tag{12.15}$$

where α denotes both the element in π_{2n-1} and its image in H_{2n-1} under the Hurewicz map.

We next evaluate $\tilde{h}$ on the relative elements f_λ determined by the foliation F_λ of § 11. Recall that here $\lambda = (\lambda_1, \cdots, \lambda_n)$ is an n-tuple of nonzero complex numbers, and F_λ the foliation of $C^n - \{0\}$ given by the vector-field $\Sigma \lambda_i z_i \partial/\partial z_i$.

According to (11.2) and (12.12) we obtain

$$\tilde{h}(f_\lambda) = \{\varphi_1(\lambda)/\sigma_n(\lambda), \cdots, \hat{\varphi}_{d(n)}(\lambda)/\sigma_n(\lambda)\} .$$

The surjectivity now follows from the folloing proposition whose simple proof is given in an Appendix:

Lemma. *The set* $A \in \boldsymbol{C}^{d(n)}$ *consisting of the values* $\tilde{h}(f_\lambda)$, $\lambda \in (\boldsymbol{C} - \{0\})^n$ *additively generates all of* $\boldsymbol{C}^{d(n)}$.

To proceed to (12.14) consider the diagram:

$$\begin{array}{ccccccc} \pi_{2n}(BGL) & \to & \pi_{2n}(BGL, B\Gamma^{\boldsymbol{C}}_{n-1}) & \to & \pi_{2n-1}(B\Gamma^{\boldsymbol{C}}_{n-1}) & \to & \pi_{2n-1}(BGL) \\ & & \downarrow{\scriptstyle \tilde{h}} & & & & \\ & & \boldsymbol{C}^{d(n)} & & & & \end{array}$$

It is well known that $\pi_{2n-1}(BGL) = 0$, and that $\pi_{2n}(BGL) = \boldsymbol{Z}$. Furthermore any decomposable element vanishes on a spherical class. Hence $\tilde{h}$ is zero on the image of π_{2n} and induces the desired surjection $h: \pi_{2n-1}(B\Gamma^{\boldsymbol{C}}_{n-1}) \to \boldsymbol{C}^{d(n)}$.

Appendix

In § 12 we needed to show that a certain subset of affine space additively generated the whole space. The general principle behind this fact is expressed in the following

Proposition. *Suppose that* $A \subset \boldsymbol{C}^n$ *is a connected complex analytic subset of* $\boldsymbol{C}^n$ *of* dim ≥ 1, *which is not contained in any affine hyperplane of* $\boldsymbol{C}^n$. *Then* A *generates* $\boldsymbol{C}^n$ *additively.*

Proof. Let $M \subset A$ denote the submanifold of nonsingular points in A. M will still satisfy our conditions by well known arguments. Hence it is sufficient to show that M generates $\boldsymbol{C}^n$.

Now let span (M) denote the vector space spanned by the translates to 0 of all the tangent spaces to M. If span (M) does not equal $\boldsymbol{C}^n$, then there is a linear form z on $\boldsymbol{C}^n$ which vanishes identically on span (M). Hence the restriction of the one form dz to M vanishes identically whence—as M is connected, M lies in a hyperplane $z =$ const. contradicting our hypothesis.

Therefore span $(M) = \boldsymbol{C}^n$, and we can find a finite number of points $m_1, \cdots, m_k \in M$ whose tangent spaces already generate $\boldsymbol{C}^n$. Now consider the map

$$M \times \cdots \times M \xrightarrow{F} \boldsymbol{C}^n$$

obtained by sending a k-tuple in M to its sum. Clearly the differential of this map is onto at the point $(m_1, \cdots, m_k)$. Hence the image of F contains an open ball about $m_1 + \cdots + m_k$. But such a ball clearly generates all of $\boldsymbol{C}^n$. q.e.d.

To apply this principle in our case, observe that the image of the map $\lambda \to \tilde{h}(f_\lambda)$ is equal to the image of a map $H\colon C^{n-1} \to C^{d(n)}$, which sends the $(n-1)$-tuple $\{x_i\}$ to the $d(n)$-tuple $\{m_\alpha(x)\}$, where α ranges over the multi-indexes of weight n and $m_\alpha(x)$ denotes the monomial $x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}$. The linear independence of these monomials now clearly implies that the image of M does not lie in a hyperplane.

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