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G-TOTAL CURVATURE OF IMMERSED MANIFOLDS

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Given an immersion $x: M \to E^m$ of a bounded manifold M of dimension n in a euclidean space E^m of dimension m, we define what we call the G-total curvature with respect to a given vector-valued function g on the normal boundle B_v as the integral over B_v of g times a power of a general mean curvature, i.e., $\int_{B_v} g(K_i)^m dV \wedge d\sigma$. We also define the G-total absolute curvatures in a similar way. The main purpose of this paper is to give the relations between different G-total curvatures or G-total absolute curvatures depending on g, i and m, first for a fixed immersion and later for different immersions. In particular, our results generalize many well-known results in differential geometry such as Gauss-Bonnet's formula, Chern-Lashof's theorems, Minkowski-Hsiung's formulas, etc.

1. Definitions

Throughout this paper, a bounded manifold means a compact manifold with or without smooth boundary. A closed manifold is a (compact) bounded manifold without boundary. Let M be a bounded manifold of dimension n, and $x: M \to E^m$ an immersion of M into a euclidean space E^m of dimension m. Suppose that E^m is oriented. By a frame $P, e_1, \cdots, e_m$ in the space E^m we mean a point $P \in E^m$ and an ordered set of mutually perpendicular unit vectors $e_1, \cdots, e_m$ with an orientation coherent with that of the space E^m. Let $F(E^m)$ be the set of all frames in the space E^m, and $F(M)$ be the set of all (orthonormal) frames in M with respect to the induced metric on M.

To avoid confusion, we shall use the following ranges of indices throughout this paper unless otherwise stated:

$$1 \leq i, j, k, \cdots \leq n ; \quad n+1 \leq r, s, t, \cdots \leq m ; \quad 1 \leq A, B, C, \cdots \leq m .$$

In $F(E^m)$ we introduce the 1-forms θ_A, θ_{AB} by

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$$dx = \sum \theta_A e_A \,, \quad de_A = \sum \theta_{AB} e_B \,, \quad \theta_{AB} + \theta_{BA} = 0 \,. \tag{1.1}$$

Since

$$d(dx) = 0 \,, \qquad d(de_A) = 0 \,, \tag{1.2}$$

from (1.1) we have that

$$d\theta_A = \sum \theta_B \wedge \theta_{BA} \,, \qquad d\theta_{AB} = \sum \theta_{AC} \wedge \theta_{CB} \,, \tag{1.3}$$

where $\wedge$ denotes the exterior product.

Let B_v denote the bundle of unit normal vectors of $x(M)$ so that a point of B_v is a pair (P, e) where e is a unit normal vector at $x(P)$. Then B_v is a bundle of $(m-n-1)$-dimensional spheres over M and is a (smooth) manifold of dimension $m-1$. Let B be the set of elements $b = (P, e_1, \cdots, e_m)$ such that

$$(P, e_1, \cdots, e_n) \in F(M) \,, \qquad (x(P), e_1, \cdots, e_m) \in F(E^m) \,,$$

where we have identified e_i with $dx(e_i)$. Then $B \to M$ may be regarded as a principal bundle with fibre $0(n) \times SO(m-n)$, and $\tilde{x}\colon B \to F(E^m)$ is naturally defined by $\tilde{x}(b) = (x(P), e_1, \cdots, e_m)$. Let ω_A, ω_{AB} be the induced 1-forms from θ_A, θ_{AB} by the mapping $\tilde{x}$. Then we have $\omega_r = 0$, and $\omega_1, \cdots, \omega_n$ are linearly independent. Hence the first equation of (1.3) gives $\sum \omega_i \wedge \omega_{ir} = 0$. By Cartan's lemma we may write

$$\omega_{ir} = \sum A^r_{ij} \omega_j \,. \qquad A^r_{ij} = A^r_{ji} \,. \tag{1.4}$$

The eigenvalues $k_1(P, e_r), \cdots, k_n(P, e_r)$ of the symmetric matrix (A^r_{ij}) (which is called the second fundamental form at (P, e_r)) are called the principal curvatures of M at (P, e_r). The *i-th mean curvature* $K_i(P, e_r)$ at (P, e_r) are defined by the elementary symmetric functions as follows:

$$\binom{n}{i} K_i(P, e_r) = \sum k_1(P, e_r) \cdots k_i(P, e_r) \,, \qquad i = 1, \cdots, n \,, \tag{1.5}$$

where $\binom{n}{i} = n!/[i!\,(n-i)!]$.

In the following, let $dV = \omega_1 \wedge \cdots \wedge \omega_n$ and $d\sigma = \omega_{m,n+1} \wedge \cdots \wedge \omega_{m,m-1}$. Then dV is the volume element of M, and $d\sigma$ is a differential $(m-n-1)$-form on B_v such that its restriction to a fibre S_P^{m-n-1} of B_v over $P \in M$ is the volume element of S_P^{m-n-1}. Furthermore, $d\sigma \wedge dV$ can be regarded as the volume element of B_v (for the detail, see [10]).

Let V be a finite dimensional vector space over $\mathbf{R}$, and let

$$g\colon B_v \to V \tag{1.6}$$

be a V-valued continuous function on the normal bundle B_v. The integral

$$G_i(\boldsymbol{x}, P, \boldsymbol{g}, m) = \int_{S_P^{m-n-1}} \boldsymbol{g}(P, \boldsymbol{e})(K_i(P, \boldsymbol{e}))^m d\sigma \tag{1.7}$$

is called the *i-th G-total curvature of rank m at P with respect to* $\boldsymbol{g}$, and the integral

$$T_i(\boldsymbol{x}, \boldsymbol{g}, m) = \int_M G_i(\boldsymbol{x}, P, \boldsymbol{g}, m) dV \tag{1.8}$$

is called the *i-th G-total curvature of rank m with respect to* $\boldsymbol{g}$ if the right hand side of (1.8) exists. The integral

$$K_i^*(\boldsymbol{x}, P, \boldsymbol{g}, m) = \int_{S_P^{m-n-1}} \boldsymbol{g}(P, \boldsymbol{e}) |K_i(P, \boldsymbol{e})|^m d\sigma \tag{1.9}$$

is called the *i-th G-total absolute curvature of rank m with respect to* $\boldsymbol{g}$ *at P*, and the integral

$$TA_i(\boldsymbol{x}, \boldsymbol{g}, m) = \int_M K_i^*(\boldsymbol{x}, P, \boldsymbol{g}, m) dV \tag{1.10}$$

is called the *i-th G-total absolute curvature of rank m with respect to* $\boldsymbol{g}$ if the right hand side of (1.10) exists.

In this paper, let $\boldsymbol{X}$ denote the E^m-valued function on B_v which maps $(P, \boldsymbol{e}) \in B_v$ onto $\boldsymbol{x}(P)$, and $\boldsymbol{e}$ the E^m-valued function on B_v which maps $(P, \boldsymbol{e}) \in B_v$ onto $\boldsymbol{e}$. We also denote by $\boldsymbol{X}$ the position vector field on M in E^m.

The G-total absolute curvatures have been studied previously by Chen [4], [6] and Santaló [17] for arbitrary i-th G-total absolute curvatures, and by Chern-Lashof [10], Chen [2] and many others for the last G-total absolute curvature. For the relations between i-th G-total absolute curvatures and integral geometry, see Chern [8], Santaló [17].

2. Elementary formulas

Through a point in E^m, let $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{m-1}, \boldsymbol{v}$ be m vectors in E^m, and let $\boldsymbol{v}_1 \times \cdots \times \boldsymbol{v}_{m-1}$ denote the vector product of the $m - 1$ vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{m-1}$. Then

$$\boldsymbol{v} \cdot (\boldsymbol{v}_1 \times \cdots \times \boldsymbol{v}_{m-1}) = (-1)^{m-1} |\boldsymbol{v}, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_{m-1}| , \tag{2.1}$$

where $|\boldsymbol{v}, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_{m-1}|$ denotes the determinant of $\boldsymbol{v}, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_{m-1}$. From (2.1) we have

$$\boldsymbol{e}_1 \times \cdots \times \hat{\boldsymbol{e}}_A \times \cdots \times \boldsymbol{e}_m = (-1)^{m+A} \boldsymbol{e}_A , \tag{2.2}$$

where the roof means the omitted term. In the following, let $\langle\ ,\ \rangle$ denote the scalar product in E^m, and $\hat{\times}$ the combined operation of the vector product and the exterior product. We list a few formulas for later use:

$$d^2x = d^2e_A = 0\ , \tag{2.3}$$

$$\underbrace{dx\ \hat{\times} \cdots \hat{\times}\ dx}_{n}\ \hat{\times}\ e_{n+1}\ \hat{\times} \cdots \hat{\times}\ \hat{e}_r\ \hat{\times} \cdots \hat{\times}\ e_m = n!\,(-1)^{m+r}e_r dV\ , \tag{2.4}$$

$$\begin{aligned} &|\underbrace{de_r, \cdots, de_r}_{i}, \underbrace{dx, \cdots, dx}_{n-i}, e_{n+1}, \cdots, e_m| \\ &\quad = (-1)^i n!\, K_i(P, e_r)dV \qquad (i \text{ not summed})\ , \end{aligned} \tag{2.5}$$

$$p(P, e) = x(P)\cdot e\ , \qquad K_0(P, e) = 1\ . \tag{2.6}$$

In the following, if there is no danger of confusion, we shall simply denote $K_i(P, e)$ by K_i, e_m by e and A^m_{ij} by A_{ij}.

3. Mean curvature form

Let

$$\Theta = \sum (-1)^{i-1}\omega_1 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \omega_n e_i\ . \tag{3.1}$$

Then Θ is a well-defined vector-valued $(n-1)$-form on M, and is called the *mean curvature form* of the immersion x: $M \to E^m$. Since we have, from a direct computation, that

$$\Theta = \frac{(-1)^{m-1}}{(n-1)!}\underbrace{dx\ \hat{\times} \cdots \hat{\times}\ dx}_{n-1}\ \hat{\times}\ e_{n+1}\ \hat{\times} \cdots \hat{\times}\ e_m\ , \tag{3.2}$$

by taking exterior derivative of (3.2) we obtain

$$d\Theta = nHdV\ , \tag{3.3}$$

where $H = (1/n)\sum A^r_{ii}e_r$ is called the mean curvature vector. If the mean curvature vector $H = 0$ identically on M, then M is called a minimal submanifold of E^m. From (3.3) we see that

Observation. *M is a minimal submanifold of E^m when and only when the mean curvature form Θ is closed.*

Moreover, by (3.3) and Stokes' theorem, we have

Proposition 3.1. *Let x: $M \to E^m$ be an immersion of an n-dimensional bounded manifold M in E^m. Then*

$$n\int_M HdV = \int_{\partial M}\Theta\ , \tag{3.4}$$

where ∂M denotes the boundary of M.

Proposition 3.2. *Under the hypothesis of Proposition* 3.1, *we have*

$$nv(M) + n\int_M \langle X, H\rangle dV = \int_{\partial M}\langle X, \Theta\rangle\,, \tag{3.5}$$

where $v(M)$ and X denote the volume and position vector field of M, respectively.

Proof. By taking exterior derivative of $\langle X, \Theta\rangle$ and applying (3.3) we obtain

$$d\langle X, \Theta\rangle = ndV + n\langle X, H\rangle dV\,. \tag{3.6}$$

Integrating both sides of (3.6) over M and applying Stokes' theorem, we obtain (3.5).

Remark 3.1. Proposition 3.2 was obtained by Hsiung [13] for $n = 2$ and by Chern-Hsiung [9] for closed M.

Corollary 3.3 (*Chern-Hsiung* [9]). *There exist no closed minimal submanifolds in a euclidean space.*

Corollary 3.4. *If M is a minimal submanifold of E^m, then*

$$n\,v(M) = \int_{\partial M}\langle X, \Theta\rangle\,. \tag{3.7}$$

These two corollaries follow immediately from (3.5). By Corollary 3.4 we have

Corollary 3.5. *Let M and M' be two bounded minimal submanifolds of E^m such that* (a) *there exist two neighborhoods U and U' of ∂M and $\partial M'$ respectively such that $U = U'$ and* (b) $\dim U = \dim U' = \dim M = \dim M'$. *Then $v(M) = v(M')$.*

4. Differential formulas

Let $\boldsymbol{a} = \sum a_i \boldsymbol{e}_i$ be a smooth vector field on M. Then

$$d\boldsymbol{a} = \sum (da_j + \sum a_j\omega_{ij})\boldsymbol{e}_j + \sum a_i\omega_{ir}\boldsymbol{e}_r\,, \tag{4.1}$$

and therefore

$$d\langle \boldsymbol{a}, \Theta\rangle = \sum (-1)^{j-1}(da_j + \sum a_i\omega_{ij}) \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_n\,.$$

Thus, if we put $da_j = \sum (a_j)_k\omega_k$ and $\omega_{ij} = \sum \Gamma^j_{ik}\omega_k$, then

$$d\langle \boldsymbol{a}, \Theta\rangle = (\operatorname{div} \boldsymbol{a})dV\,, \tag{4.2}$$

where $\operatorname{div} \boldsymbol{a} = \sum (a_j)_j + \sum a_i\Gamma^j_{ij}$. From (4.2) follows immediately

Proposition 4.1. *Under the hypothesis of Proposition* 3.1, *we have*

$$\int_M (\operatorname{div} \boldsymbol{a})dV = \int_{\partial M} \langle \boldsymbol{a}, \boldsymbol{\Theta} \rangle , \tag{4.3}$$

for any tangent vector field $\boldsymbol{a}$ *on* M.

Let f be a smooth function on M. By $\boldsymbol{grad}\, f$, we mean $\boldsymbol{grad}\, f = \sum f_i \boldsymbol{e}_i$, where $df = \sum f_i \omega_i$. Since

$$d(f\boldsymbol{\Theta}) = (\boldsymbol{grad}\, f)dV + nf\boldsymbol{H}dV , \tag{4.4}$$

from Stokes' theorem we obtain

Proposition 4.2. *Under the hypothesis of Proposition* 3.1, *we have*

$$n \int_M f\boldsymbol{H}dV + \int_M (\boldsymbol{grad}\, f)dV = \int_{\partial M} f\boldsymbol{\Theta} . \tag{4.5}$$

Let g be a smooth function on the normal bundle B_v. Put $dg = \sum g_i \omega_i + \sum g_r \omega_{mr}$ and $\boldsymbol{\nabla} g = \sum g_i \boldsymbol{e}_i$. Then we have

Lemma 4.3. *Under the hypothesis of Proposition* 3.1, *we have*

$$d(g\boldsymbol{\Theta} \wedge d\sigma) = (\boldsymbol{\nabla} g)dV \wedge d\sigma + ng\boldsymbol{H}dV \wedge d\sigma . \tag{4.6}$$

Proof. By taking exterior derivative of $g\boldsymbol{\Theta} \wedge d\sigma$ and applying (3.3), we obtain (4.6) immediately.

There exists a self-adjoint linear transformation A of the tangent space $T_P(M)$ of M at P into itself defined by

$$A\boldsymbol{e}_i = -\sum A_{ij}\boldsymbol{e}_j , \tag{4.7}$$

where (A_{ij}) denotes the second fundamental form at (P, e). It follows that

$$A(d\boldsymbol{x}) = A(\sum \omega_i \boldsymbol{e}_i) = \sum \omega_{mj}\boldsymbol{e}_j = (d\boldsymbol{e})^t , \tag{4.8}$$

where $(d\boldsymbol{e})^t$ is the tangential component of $d\boldsymbol{e}$. Let $A^{(j)}(d\boldsymbol{x})$ denote the tangent vector obtained from $d\boldsymbol{x}$ by applying A repeatedly j times, and $*$ the Hodge star operator defined by

$$*(\sum f_i \omega_i \boldsymbol{e}_i) = \sum (-1)^{i-1} f_i \omega_1 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \omega_n \boldsymbol{e}_i . \tag{4.9}$$

For convenience we put $\boldsymbol{U}_0 = d\boldsymbol{x}$ and $\boldsymbol{U}_j = A^{(j)}(d\boldsymbol{x})$, $j = 1, 2, \cdots$.

Lemma 4.4. *Let* $\boldsymbol{e} = \boldsymbol{e}_m$. *Then*

$$\begin{aligned} &\underbrace{d\boldsymbol{x} \hat{\times} \cdots \hat{\times} d\boldsymbol{x}}_{n-i-1} \hat{\times} \underbrace{d\boldsymbol{e} \hat{\times} \cdots \hat{\times} d\boldsymbol{e}}_{i} \hat{\times} \boldsymbol{e}_{n+1} \hat{\times} \cdots \hat{\times} \boldsymbol{e}_m \\ &= (-1)^{m+1+i} i!\,(n-i-1)! \sum_{j=0}^{i} \binom{n}{i-j} K_{i-j} {*}\boldsymbol{U}_j . \end{aligned} \tag{4.10}$$

This lemma can be proved in the same way as Lemma 2.1 was proved in [1], so we omit the proof here.

Lemma 4.5. *Let*

$$\Delta_i = e \mathbin{\hat{\times}} \underbrace{de \mathbin{\hat{\times}} \cdots \mathbin{\hat{\times}} de}_{m-n+i-1} \mathbin{\hat{\times}} \underbrace{dx \mathbin{\hat{\times}} \cdots \mathbin{\hat{\times}} dx}_{n-i-1}, \quad i = 0, 1, \cdots, n-1 . \tag{4.11}$$

Then

$$\begin{aligned} \Delta_i = &-(m-n+i-1)!\,(n-i-1)!\,(-1)^i \sum_{h=0}^{i} \binom{n}{i-h} K_{i-h} {}^*U_h \wedge d\sigma \\ &+ \frac{n!\,(m-n+i-1)!}{(i+1)!} \sum_{s=n+1}^{m-1} (-1)^{i+s+1} K_{i+1} dV \wedge \omega_{m,n+1} \\ &\wedge \cdots \wedge \hat{\omega}_{m,s} \wedge \cdots \wedge \omega_{m,m-1} e_s . \end{aligned} \tag{4.12}$$

This lemma can be proved by a direct computation of the left hand side of (4.12); we omit the proof.

Lemma 4.6. *Let*

$$\pi_i = dp \wedge \langle X, \Delta_i \rangle , \quad x_A = \langle X, e_A \rangle , \quad i = 0, 1, \cdots, n-1 . \tag{4.13}$$

Then

$$\begin{aligned} \pi_i = &(m-n+i-1)!\,(n-i-1)! \sum_{h=0}^{i} \sum_{j_0,\cdots,j_h=1}^{n} (-1)^{i+h} \\ &\cdot \binom{n}{i-h} K_{i-h} \sum_{j=1}^{n} x_j x_{j_h} A_{jj_0} \left(\prod_{k=1}^{h} A_{j_{k-1}j_k} \right) dV \wedge d\sigma \\ &+ (-1)^i \frac{n!\,(m-n+i-1)!}{(i+1)!} K_{i+1} \left(\sum_{s=n+1}^{m-1} (x_s)^2 \right) dV \wedge d\sigma . \end{aligned} \tag{4.14}$$

Proof. By (4.8) and (4.9), we have

$$U_i = \sum_{j_0,\cdots,j_i=1}^{n} (-1)^i \left(\prod_{k=1}^{i} A_{j_{k-1}j_k} \right) \omega_{j_0} e_{j_i} , \tag{4.15}$$

and therefore

$${}^*U_i = \sum_{j_0,\cdots,j_i=1}^{n} (-1)^{j_0+i+1} \left(\prod_{k=1}^{i} A_{j_{k-1}j_k} \right) \omega_1 \wedge \cdots \wedge \hat{\omega}_{j_0} \wedge \cdots \wedge \omega_n e_{j_i} . \tag{4.16}$$

By Lemma 4.5, we obtain

$$\pi_i = (m-n+i-1)!\,(n-i-1)! \sum_{h=0}^{i} \sum_{j_0,\cdots,j_h=1}^{n} (-1)^{h+j_0+i} \binom{n}{i-h} K_{i-h} \cdot \sum_{A=1}^{n} x_A x_{j_h} \left(\prod_{k=1}^{h} A_{j_{k-1}j_k}\right) \omega_{m,A} \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_{j_0} \wedge \cdots \wedge \omega_n \wedge d\sigma + (-1)^i \frac{n!\,(m-n+i-1)!}{(i+1)!} K_{i+1} \left(\sum_{s=n+1}^{m-1} (x_s)^2\right) dV \wedge d\sigma .$$

From this we can easily derive (4.14).

Lemma 4.7.

$$dp \wedge \Delta_i = (m-n+i-1)!\,(n-i-1)! \sum_{h=0}^{i} \sum_{j_0,\cdots,j_h=1}^{n} (-1)^{i+h} \cdot \binom{n}{i-h} K_{i-h} \sum_{j=1}^{n} x_j e_{j_h} A_{jj_0} \left(\prod_{k=1}^{h} A_{j_{k-1}j_k}\right) dV \wedge d\sigma + (-1)^i \frac{n!\,(m-n+i-1)!}{(i+1)!} K_{i+1} \left(\sum_{s=n+1}^{m-1} x_s e_s\right) dV \wedge d\sigma . \tag{4.17}$$

This lemma can be proved in the same way as we prove Lemma 4.6.

Lemma 4.8.

$$d(p^{l-1}\langle X, \Delta_i\rangle) = (l-1)p^{l-2}\pi_i - (-1)^i \frac{n!\,(m-n+i-1)!}{i!} p^{l-1} K_i dV \wedge d\sigma - (-1)^i \frac{n!\,(m-n+i)!}{(i+1)!} p^l K_{i+1} dV \wedge d\sigma . \tag{4.18}$$

Proof. Since

$$\underbrace{de \mathbin{\hat{\times}} \cdots \mathbin{\hat{\times}} de}_{m-n+i-1} \mathbin{\hat{\times}} \underbrace{dx \mathbin{\hat{\times}} \cdots \mathbin{\hat{\times}} dx}_{n-i} = (-1)^i \frac{n!\,(m-n+i-1)!}{i!} K_i e dV \wedge d\sigma , \tag{4.19}$$

$$\langle dX, \Delta_i\rangle = (-1)^{i-1} \frac{n!\,(m-n+i-1)!}{i!} K_i dV \wedge d\sigma , \tag{4.20}$$

by using (4.11), (4.13), (4.19) and (4.20) we can prove (4.18) without difficulty.

Lemma 4.9.

$$\binom{n}{i} \left\{ \frac{(n-i)(m-n+i)}{i+1} p^l K_{i+1} + i p^{l-1} K_i - (l-1) p^{l-2} K_i \sum x_j x_k A_{jk} \right\} dV \wedge d\sigma$$

$$
\begin{aligned}
(4.21) \quad & + \binom{n}{i+1}(l-1)p^{l-2}\sum_{s=n+1}^{m-1}(x_s)^2 K_{i+1}dV \wedge d\sigma \\
& = (-1)^{i+1} d\left\{\frac{p^{l-1}\langle \boldsymbol{X}, \boldsymbol{\Delta}_i\rangle}{(m-n+i-1)!\,(n-i-1)!}\right\} \\
& \quad - \sum_{h=1}^{i}\binom{n}{i-h}\{d(p^{l-1}K_{i-h}\langle \boldsymbol{X}, {}^*\boldsymbol{U}_h\rangle) - p^{l-1}\langle \boldsymbol{X}, d(K_{i-h}{}^*\boldsymbol{U}_h)\rangle\} \wedge d\sigma .
\end{aligned}
$$

Proof. By Lemma 4.4, (2.5) and (3.1), we have

$$
\begin{aligned}
\sum_{h=1}^{i}\binom{n}{i-h}K_{i-h}{}^*\boldsymbol{U}_h &= -\binom{n}{i}K_i\boldsymbol{\Theta} + \sum_{h=0}^{i}\binom{n}{i-h}K_{i-h}{}^*\boldsymbol{U}_h \\
&= -\binom{n}{i}K_i\boldsymbol{\Theta} + \frac{(-1)^{m+i-1}}{i!\,(n-i-1)!} \\
&\quad \cdot \underbrace{d\boldsymbol{x} \,\hat{\times}\, \cdots \,\hat{\times}\, d\boldsymbol{x}}_{n-i-1} \,\hat{\times}\, \underbrace{d\boldsymbol{e} \,\hat{\times}\, \cdots \,\hat{\times}\, d\boldsymbol{e}}_{i} \,\hat{\times}\, \boldsymbol{e}_{n+1} \,\hat{\times}\, \cdots \,\hat{\times}\, \boldsymbol{e}_m ,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\sum_{h=1}^{i}\binom{n}{i-h}K_{i-h}\langle d\boldsymbol{x}, {}^*\boldsymbol{U}_h\rangle &= -\binom{n}{i}K_i\langle d\boldsymbol{x}, \boldsymbol{\Theta}\rangle \\
& + \frac{(-1)^i}{i!\,(n-i-1)!}\underbrace{d\boldsymbol{x} \,\hat{\times}\, \cdots \,\hat{\times}\, d\boldsymbol{x}}_{n-i} \,\hat{\times}\, \underbrace{d\boldsymbol{e} \,\hat{\times}\, \cdots \,\hat{\times}\, d\boldsymbol{e}}_{i} \,\hat{\times}\, \boldsymbol{e}_{n+1} \,\hat{\times}\, \cdots \,\hat{\times}\, \boldsymbol{e}_m \\
&= -n\binom{n}{i}K_i dV + \binom{n}{i}(n-i)K_i dV = -i\binom{n}{i}K_i dV .
\end{aligned}
$$

On the other hand, from (4.12), (4.13) it follows immediately

$$
\begin{aligned}
\sum_{h=1}^{i}\binom{n}{i-h}K_{i-h}dp \wedge \langle \boldsymbol{X}, {}^*\boldsymbol{U}_h\rangle \wedge d\sigma &= -\binom{n}{i}K_i \sum x_j x_k A_{jk} dV \wedge d\sigma \\
& + \frac{(-1)^{i+1}\pi_i}{(m-n+i-1)!\,(n-i-1)!} - \binom{n}{i+1}\sum_{s=n+1}^{m-1}(x_s)^2 K_{i+1}dV \wedge d\sigma .
\end{aligned}
$$

Substituting the right side of the above two equations in the following equation and simplifying the resulting equation by using (4.18) we can easily reach (4.21):

$$
\begin{aligned}
&\sum_{h=1}^{i}\binom{n}{i-h}\{d(p^{l-1}K_{i-h}\langle \boldsymbol{X}, {}^*\boldsymbol{U}_n\rangle) - p^{l-1}\langle \boldsymbol{X}, d(K_{i-h}{}^*\boldsymbol{U}_h)\rangle\} \wedge d\sigma \\
&\quad = \sum_{h=1}^{i}\binom{n}{i-h}\{(l-1)p^{l-2}K_{i-h}dp \wedge \langle \boldsymbol{X}, {}^*\boldsymbol{U}_h\rangle \\
&\qquad + p^{l-1}K_{i-h}\langle d\boldsymbol{X}, {}^*\boldsymbol{U}_h\rangle\} \wedge d\sigma .
\end{aligned}
$$

Lemma 4.10. *Let*

$$(4.22)\quad \boldsymbol{\Psi} = \sum_{s=n+1}^{m-1} (-1)^s dV \wedge \omega_{m,n+1} \wedge \cdots \wedge \hat{\omega}_{m,s} \wedge \cdots \wedge \omega_{m,m-1}\boldsymbol{e}_s \,.$$

Then

$$(4.23)\quad \begin{aligned} p^{l-1}\langle \boldsymbol{X}, d(K_{i+1}\boldsymbol{\Psi})\rangle &= d\langle p^{l-1}K_{i+1}\boldsymbol{X}, \boldsymbol{\Psi}\rangle \\ &\quad - (l-1)p^{l-2} \sum_{s=n+1}^{m-1} (x_s)^2 K_{i+1} dV \wedge d\sigma \,, \\ &\qquad i = 0, 1, \cdots, n-1 \,. \end{aligned}$$

Proof. By using $\langle d\boldsymbol{x}, K_{i+1}\boldsymbol{\Psi}\rangle = 0$, we have

$$\begin{aligned} &d\langle p^{l-1}K_{i+1}\boldsymbol{X}, \boldsymbol{\Psi}\rangle - p^{l-1}\langle \boldsymbol{X}, d(K_{i+1}\boldsymbol{\Psi})\rangle \\ &\quad = (l-1)p^{l-2}K_{i+1}dp \wedge \langle \boldsymbol{X}, \boldsymbol{\Psi}\rangle + p^{l-1}K_{i+1}\langle d\boldsymbol{x}, \boldsymbol{\Psi}\rangle \\ &\quad = (l-1)p^{l-2}\left(\sum_{s=n+1}^{m-1} (x_s)^2\right)K_{i+1}dV \wedge d\sigma \,. \end{aligned}$$

Lemma 4.11.

$$(4.24)\quad \begin{aligned} &\sum_{i=0}^{n-1} \sum_{j_0,\cdots,j_i=1}^{n} (-1)^{i-1}\binom{n}{n-i-1} K_{n-i-1} \sum_{j=1}^{n} x_j x_{j_i} A_{jj_0}\left(\prod_{k=1}^{i} A_{j_{k-1}j_k}\right) \\ &\quad = -K_n\left(\sum_{i=1}^{n} (x_i)^2\right) \,. \end{aligned}$$

Proof. For simplicity, we choose the principal frame with respect to $\boldsymbol{e} = \boldsymbol{e}_m$, so that

$$\omega_{mi} = -k_i\omega_i \qquad (i \text{ not summed}) \,.$$

By a direct calculation we can easily obtain (4.24).

Similarly we can prove

Lemma 4.12.

$$(4.25)\quad \begin{aligned} &\sum_{i=0}^{n-1} (-1)^{i-1}\binom{n}{n-i-1} K_{n-i-1} \sum x_j \boldsymbol{e}_{j_i} A_{jj_0}\left(\prod_{k=1}^{i} A_{j_{k-1}j_k}\right) \\ &\quad = -K_n\left(\sum_{i=1}^{n} x_i \boldsymbol{e}_i\right) \,. \end{aligned}$$

5. Integral formulas and their applications

Theorem 5.1. *Let $\boldsymbol{x}\colon M \to E^m$ be an immersion of an n-dimensional bounded manifold M in E^m. Then we have*

$$(5.1)\quad \begin{aligned} &T_0(\boldsymbol{x}, p^{l-1}\langle \boldsymbol{X}, \nabla K_i\rangle, 1) + nT_i(\boldsymbol{x}, p^{l-1}\langle \boldsymbol{X}, \boldsymbol{H}\rangle, 1) + iT_i(\boldsymbol{x}, p^{l-1}, 1) \\ &\quad - (l-1)T_i(\boldsymbol{x}, \textstyle\sum x_j x_k A_{jk} p^{l-2}, 1) = \displaystyle\int_{\partial B_v} p^{l-1}K_i\langle \boldsymbol{X}, \boldsymbol{\Theta}\rangle \wedge d\sigma \,, \end{aligned}$$

for all $i = 0, 1, \cdots, n-1$ *and an integer* l.

Proof. By Lemma 4.5 and (4.22), we have

$$\begin{aligned}\frac{(-1)^{i+1}\boldsymbol{\Delta}_i}{(m-n+i-1)!\,(n-i-1)!} &= \binom{n}{i}K_i\boldsymbol{\Theta}\wedge d\sigma + \binom{n}{i+1}K_{i+1}\boldsymbol{\Psi} \\ &\quad + \sum_{h=1}^{i}\binom{n}{i-h}K_{i-h}{*\boldsymbol{U}_h}\wedge d\sigma\,. \end{aligned} \tag{5.2}$$

By first taking exterior derivative of (5.2), using (4.19) and applying Lemma 4.3, and then taking scalar product of X with both sides of the resulting equation and multiplying by p^{l-1}, we obtain

$$\begin{aligned}(m-n+i)\binom{n}{i+1}p^lK_{i+1}dV\wedge d\sigma &= \binom{n}{i}p^{l-1}\langle \boldsymbol{X}, \boldsymbol{\nabla} K_i\rangle dV\wedge d\sigma \\ &\quad + n\binom{n}{i}p^{l-1}K_i\langle \boldsymbol{X}, \boldsymbol{H}\rangle dV\wedge d\sigma \\ + \sum_{h=1}^{i}\binom{n}{i-h}p^{l-1}\langle \boldsymbol{X}, d(K_{i-h}{*\boldsymbol{U}_h}\wedge d\sigma)\rangle &+ \binom{n}{i+1}p^{l-1}\langle \boldsymbol{X}, d(K_{i+1}\boldsymbol{\Psi})\rangle\,.\end{aligned}$$

Substituting (4.21), (4.23) in the above equation for the last two terms and simplifying the resulting equation by using (4.12), (4.13), (4.22) we can easily obtain

$$\begin{aligned}&\binom{n}{i}\{p^{l-1}\langle \boldsymbol{X}, \boldsymbol{\nabla} K_i\rangle + np^{l-1}K_i\langle \boldsymbol{X}, \boldsymbol{H}\rangle \\ &\qquad + ip^{l-1}K_i - (l-1)p^{l-2}K_i\sum x_jx_kA_{jk}\}dV\wedge d\sigma \\ &= -\binom{n}{i+1}d(p^{l-1}K_{i+1}\langle \boldsymbol{X}, \boldsymbol{\Psi}\rangle) \\ &\qquad + (-1)^{i+1}\frac{d(p^{l-1}\langle \boldsymbol{X}, \boldsymbol{\Delta}_i\rangle)}{(m-n+i-1)!\,(n-i-1)!} \\ &\qquad - \sum_{h=1}^{i}\binom{n}{i-h}d(p^{l-1}K_{i-h}\langle \boldsymbol{X}, {*\boldsymbol{U}_h}\wedge d\sigma\rangle) \\ &= \binom{n}{i}d\langle p^{l-1}K_i\boldsymbol{X}, \boldsymbol{\Theta}\wedge d\sigma\rangle\,.\end{aligned}$$

Integration of both sides of the above equation and application of Stokes' theorem give immediately (5.1).

Theorem 5.2. *Under the hypothesis of Theorem* 5.1, *we have*

$$\begin{aligned}&(l-1)\sum_{h=0}^{i}\sum_{j_0,\cdots,j_h=1}^{n}(-1)^{h-1}\binom{n}{i-h}T_{i-h}\left(\boldsymbol{x}, p^{l-2}\sum_{j=1}^{n}x_jx_{j_h}A_{jj_0}\left(\prod_{k=1}^{h}A_{j_{k-1}j_k}\right), 1\right) \\ &\qquad + (m-n+i)\binom{n}{i+1}T_{i+1}(\boldsymbol{x}, p^l, 1) + (n-i)\binom{n}{i}T_i(\boldsymbol{x}, p^{l-1}, 1)\end{aligned}$$

$$
\begin{aligned}
&= (l-1)\binom{n}{i+1}T_{i+1}\Big(\boldsymbol{x}, p^{l-2}\sum_{s=n+1}^{m-1}(x_s)^2, 1\Big) \\
(5.3)\qquad &+ \frac{(-1)^{i+1}}{(m-n+i-1)!\,(n-i-1)!}\int_{\partial B_v} p^{l-1}\langle \boldsymbol{X}, \boldsymbol{\Delta}_i\rangle\,, \\
&\qquad\qquad i = 0, 1, \cdots, n-1.
\end{aligned}
$$

This theorem follows from Lemmas 4.6 and 4.8.

Theorem 5.3. *Under the hypothesis of Theorem* 5.1, *we have*

$$
\begin{aligned}
&(m-n+i)\binom{n}{i+1}T_{i+1}(\boldsymbol{x}, p^l\boldsymbol{e}, 1) - l\binom{n}{i+1}T_{i+1}\Big(\boldsymbol{x}, p^{l-1}\Big(\sum_{s=n+1}^{m-1}x_s\boldsymbol{e}_s\Big), 1\Big) \\
(5.4)\qquad &+ l\sum_{h=0}^{i}(-1)^{h-1}\binom{n}{i-h}T_{i-h}\Big(\boldsymbol{x}, p^{l-1}\sum x_j\boldsymbol{e}_{j_h}A_{jj_0}\Big(\prod_{k=1}^{h}A_{j_{k-1}j_k}\Big), 1\Big) \\
&= \frac{(-1)^{i+1}}{(m-n+i-1)!\,(n-i-1)!}\int_{\partial B_v} p^l\boldsymbol{\Delta}_i\,, \qquad i = 0, 1, \cdots, n-1\,.
\end{aligned}
$$

Proof. Substituting (4.17), (4.19) in

$$
d(p^l\boldsymbol{\Delta}_i) = p^l d\boldsymbol{\Delta}_i + lp^{l-1}dp \wedge \boldsymbol{\Delta}_i\,,
$$

we can easily obtain

$$
\begin{aligned}
&(m-n+i)\binom{n}{i+1}p^lK_{i+1}\boldsymbol{e}dV \wedge d\sigma - lp^{l-1}\binom{n}{i+1}K_{i+1}\Big(\sum_{s=n+1}^{m-1}x_s\boldsymbol{e}_s\Big)dV \wedge d\sigma \\
&\quad + lp^{l-1}\sum_{h=0}^{i}\sum_{j_0,\cdots,j_h=1}^{n}(-1)^{h-1}K_{i-h}\binom{n}{i-h}\sum_{j=1}^{n}x_j\boldsymbol{e}_{j_h}A_{jj_0} \\
&\qquad\qquad \cdot\Big(\prod_{k=1}^{h}A_{j_{k-1}j_k}\Big)dV \wedge d\sigma \\
&= (-1)^{i+1}d(p^l\boldsymbol{\Delta}_i)/[(m-n+i-1)!\,(n-i-1)!]\,.
\end{aligned}
$$

Integrating both sides of the above equation and applying Stokes' theorem, we hence have (5.4).

Theorem 5.4. *Under the hypothesis of Theorem* 5.1, *we have*

$$
\begin{aligned}
&(l-1)T_n(\boldsymbol{x}, p^{l-2}\langle \boldsymbol{X}, \boldsymbol{X}\rangle, 1) + nT_{n-1}(\boldsymbol{x}, p^{l-1}, 1) \\
(5.5)\qquad &= (m+l-2)T_n(\boldsymbol{x}, p^l, 1) + \frac{(-1)^{n+1}}{(n-2)!}\int_{\partial B_v} p^{l-1}\langle X, \Delta_{n-1}\rangle\,.
\end{aligned}
$$

Proof. This theorem follows from Lemma 4.11, Theorem 5.2 for $i = n-1$ and the following identity $\sum_{A=1}^{m-1} x_Ax_A = \boldsymbol{X}\cdot\boldsymbol{X} - p^2$.

For $l = 1$, Theorem 5.2 reduces to

Corollary 5.5. *If M is closed, then we have*

$$(i+1)T_i(x,1,1)+(m-n+i)T_{i+1}(x,p,1)=0\,,\qquad i=0,1,\cdots,n-1\,. \tag{5.6}$$

Remark 5.1. If the codimension $m-n=1$, then (5.6) are Minkowski-Hsiung's formulas [12].

If i is odd, then $G_i(x,P,1,1)=0$; if i is even, then $G_i(x,P,1,1)$ depends only on the Riemannian structure of M with the induced metric (see Remark 8.2). Hence from Corollary 5.5 we obtain

Corollary 5.6. *If M is closed, then the i-th G-total curvatures $T_i(x,p,1)$ for all $i=1,\cdots,n$ depend only on the Riemannian structure of M with respect to the induced metric. In other word, $T_i(x,p,1)$ is an isometric invariant for all $i=1,\cdots,n$.*

From Corollary 5.5 follows

Corollary 5.7. *If M is a complete submanifold of E^m with $G_1(x,P,p,1)=0$ everywhere on M, then M is not compact.*

Putting $l=0$ in Theorem 5.3 we obtain

Corollary 5.8. *If M is closed, then $T_i(x,e,1)=0$ for all $i=1,\cdots,n$.*

Corollary 5.9. *If M is closed, then*

$$(m+l-1)T_n(x,p^le,1)=T_n(x,p^{l-1}X,1)\,. \tag{5.7}$$

Corollary 5.9 follows from Lemma 4.12 and the identity $X-pe=\sum_{A=1}^{m-1}x_Ae_A$.

Corollary 5.10. *If M is closed, then*

$$nT_i(x,H,1)+T_0(x,\nabla K_i,1)=0\,,\qquad i=0,1,\cdots,n-1\,. \tag{5.8}$$

In particular, if $G_i(x,P,1,1)$ is a constant, then

$$T_0(x,\nabla K_i,1)=0\,,\qquad i=0,1,\cdots,n-1\,. \tag{5.9}$$

The first part of this corollary can be obtained by applying to (5.1) for $l=1$ a translation $x\to x+c$ where c is any constant vector in E^m, and the second part follows from Proposition 3.1 and (5.8).

Corollary 5.11. *Let M be an n-dimensional oriented closed submanifold in E^m such that M does not contain the origin and the n-th mean curvature $K_n(P,e)$ is nonnegative everywhere on B_v. Then*

$$nT_{n-1}(x,p^{-1},1)\geq(m-2)T_n(x,1,1)\,. \tag{5.10}$$

Proof. This corollary follows from Theorem 5.4 for $l=0$ and the assumption $K_n(P,e)\geq 0$.

6. Gauss-Bonnet's formula

In this section, we shall assume that M is an n-dimensional oriented closed manifold imbedded in E^m.

Proposition 6.1. *Let $\alpha_1, \cdots, \alpha_h$ be h nonnegative integers, and $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_h$ be h fixed vectors in E^m. Then*

$$\begin{aligned}
&\sum_{i=1}^{h-1} \alpha_i T_n\left(\boldsymbol{x}, \prod_{j=1}^{h-1} \langle \boldsymbol{a}^j, \boldsymbol{e}\rangle^{\alpha_j} \langle \boldsymbol{X}, \boldsymbol{e}\rangle^{\alpha_h} \boldsymbol{a}_i / \langle \boldsymbol{a}_i, \boldsymbol{e}\rangle, 1\right) \\
&\qquad + \alpha_h T_n\left(\boldsymbol{x}, \prod_{j=1}^{h-1} \langle \boldsymbol{a}_j, \boldsymbol{e}\rangle^{\alpha_j} \langle \boldsymbol{X}, \boldsymbol{e}\rangle^{\alpha_h - 1} \boldsymbol{X}, 1\right) \\
&\qquad = (m + \alpha_1 + \cdots + \alpha_h - 1) T_n\left(\boldsymbol{x}, \prod_{j=1}^{h-1} \langle \boldsymbol{a}_j, \boldsymbol{e}\rangle^{\alpha_j} \langle \boldsymbol{X}, \boldsymbol{e}\rangle^{\alpha_h} \boldsymbol{e}, 1\right) .
\end{aligned} \tag{6.1}$$

Proof. Put

$$\boldsymbol{Q} = \sum_{A=1}^{m-1} (-1)^{A-1} \omega_{m,1} \wedge \cdots \wedge \hat{\omega}_{m,A} \wedge \cdots \wedge \omega_{m,m-1} \boldsymbol{e}_A . \tag{6.2}$$

Then

$$\underbrace{d\boldsymbol{e} \hat{\times} \cdots \hat{\times} d\boldsymbol{e}}_{m-2} \hat{\times} \boldsymbol{e} = (m-2)!\,(-1)^{m-1} \boldsymbol{Q} .$$

On the other hand, from (4.19) we have

$$\underbrace{d\boldsymbol{e} \hat{\times} \cdots \hat{\times} d\boldsymbol{e}}_{m-1} = (-1)^n (m-1)!\, K_n \boldsymbol{e} dV \wedge d\sigma . \tag{6.3}$$

Therefore

$$(-1)^n d\boldsymbol{Q} = -(m-1) K_n \boldsymbol{e} dV \wedge d\sigma . \tag{6.4}$$

Moreover, we can prove that

$$(-1)^n \langle \boldsymbol{X}, d\boldsymbol{e}\rangle \wedge \boldsymbol{Q} = (\boldsymbol{X} - \langle \boldsymbol{X}, \boldsymbol{e}\rangle \boldsymbol{e}) K_n dV \wedge d\sigma , \tag{6.5}$$

$$(-1)^n \langle \boldsymbol{a}, d\boldsymbol{e}\rangle \wedge \boldsymbol{Q} = (\boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{e}\rangle, \boldsymbol{e}) K_n dV \wedge d\sigma , \tag{6.6}$$

where $\boldsymbol{a}$ is a fixed vector in E^m. Hence by taking exterior derivative of $\langle \boldsymbol{a}_1, \boldsymbol{e}\rangle^{\alpha_1} \cdots \langle \boldsymbol{a}_{h-1}, \boldsymbol{e}\rangle^{\alpha_{h-1}} \langle \boldsymbol{X}, \boldsymbol{e}\rangle^{\alpha_h} \boldsymbol{Q}$ and applying (6.3), $\cdots$, (6.6) and Stokes' theorem, we can obtain (6.1).

Proposition 6.2. *Let $\boldsymbol{a}$ be a fixed vector in E^m perpendicular to $\boldsymbol{a}_2, \cdots, \boldsymbol{a}_{h-1}$. Then*

$$\begin{aligned}
&\alpha_1 T_n(\boldsymbol{x}, \langle \boldsymbol{a}, \boldsymbol{a}_1\rangle \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle^{\alpha_1 - 1} \langle \boldsymbol{a}_2, \boldsymbol{e}\rangle^{\alpha_2} \cdots \langle \boldsymbol{a}_{h-1}, \boldsymbol{e}\rangle^{\alpha_{h-1}} \langle \boldsymbol{X}, \boldsymbol{e}\rangle^{\alpha_h}, 1) \\
&\qquad + \alpha_h T_n(\boldsymbol{x}, \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle^{\alpha_1} \cdots \langle \boldsymbol{a}_{h-1}, \boldsymbol{e}\rangle^{\alpha_{h-1}} \langle \boldsymbol{X}, \boldsymbol{e}\rangle^{\alpha_h - 1} \langle \boldsymbol{X}, \boldsymbol{a}\rangle, 1) \\
&\quad = (m + \alpha_1 + \cdots + \alpha_h - 1) T_n(\boldsymbol{x}, \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle^{\alpha_1} \\
&\qquad\qquad \cdots \langle \boldsymbol{a}_{h-1}, \boldsymbol{e}\rangle^{\alpha_{h-1}} \langle \boldsymbol{X}, \boldsymbol{e}\rangle^{\alpha_h} \langle \boldsymbol{a}, \boldsymbol{e}\rangle, 1) .
\end{aligned} \tag{6.7}$$

By taking scalar product of (6.1) with $\boldsymbol{a}$, we obtain (6.7).

Proposition 6.3. *If $\boldsymbol{a}_1$ is a fixed unit vector in E^m perpendicular to $\boldsymbol{a}_2, \cdots, \boldsymbol{a}_h$, and α_1 is a positive even integer, then*

$$(6.8) \quad T_n(\boldsymbol{x}, \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle^{\alpha_1} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h}, 1) = \gamma T_n(\boldsymbol{x}, \langle \boldsymbol{a}_2, \boldsymbol{e}\rangle^{\alpha_2} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h}, 1) ,$$

where

$$(6.9) \qquad \gamma = 2c_{m+\alpha_1+\cdots+\alpha_h-1}/(c_{\alpha_1}c_{m+\alpha_2+\cdots+\alpha_h-1}) ,$$

and $c_k = 2\pi^{\frac{1}{2}(k+1)}/\Gamma(\frac{1}{2}(k+1))$ is the area of the unit k-sphere.

Proof. Setting $\alpha_h = 0$ and $\boldsymbol{a} = \boldsymbol{a}_1$ in (6.7) we readily obtain

$$(6.10) \quad \begin{aligned} &(\alpha_1 - 1)T_n(\boldsymbol{x}, \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle^{\alpha_1-2}\langle \boldsymbol{a}_2, \boldsymbol{e}\rangle^{\alpha_2} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h}, 1) \\ &\quad = (m + \alpha_1 + \cdots + \alpha_h - 2)T_n(\boldsymbol{x}, \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle^{\alpha_1} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h}, 1) . \end{aligned}$$

Repeating (6.10) for $\frac{1}{2}\alpha_1 - 1$ times thus gives (6.8).

Proposition 6.4. *Let $\chi(M)$ denote the Euler characteristic of M. Then*

$$(6.11) \qquad T_n(\boldsymbol{x}, 1, 1) = c_{m-1}\chi(M) .$$

Proof. If $\dim M = n$ is odd, then we have $G_n(\boldsymbol{x}, P, 1, 1) = 0$, so that $T_n(\boldsymbol{x}, 1, 1) = 0$. On the other hand, by the Poincaré duality, we have $\chi(M) = 0$. Thus we obtain (6.11). Now assume n to be even. If the codimension $m - n$ is odd, then the normal bundle B_v has dimension $m - 1$. Since B_v is closed and oriented, from Gauss-Bonnet's formula we have

$$(6.12) \qquad T_n(\boldsymbol{x}, 1, 1) = \tfrac{1}{2}c_{m-1}\chi(B_v) .$$

Since B_v is a bundle space of $(m - n - 1)$-dimensional sphere over M, we have $\chi(M) = \chi(S^{m-n-1})\chi(M) = 2\chi(M)$. Hence (6.12) reduces to (6.11). If the codimension $m - n$ is even, then we define an immersion $\bar{\boldsymbol{x}}\colon M \to E^{m+1}$ by $\bar{\boldsymbol{x}}(P) = \boldsymbol{x}(P)$ for all P in M. By a direct computation, we obtain $c_m T_n(\boldsymbol{x}, 1, 1) = c_{m-1}T_n(\bar{\boldsymbol{x}}, 1, 1) = c_{m-1}c_m\chi(M)$, which implies (6.11).

The main purpose of this section is to prove the following generalization of Gauss-Bonnet's formula.

Theorem 6.5. *Let $\alpha_1, \cdots, \alpha_h$ be h nonnegative integers, and $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_h$ be h orthonormal vectors in E^m. Then*

$$(6.13) \quad T_n(\boldsymbol{x}, \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle^{\alpha_1} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h}, 1) = \begin{cases} t\chi(M), & \textit{if } \alpha_1, \cdots, \alpha_h \textit{ are even,} \\ 0, & \textit{otherwise,} \end{cases}$$

where

$$(6.14) \qquad t = 2^h c_{m+\alpha_1+\cdots+\alpha_h-1}/(c_{\alpha_1} \cdots c_{\alpha_h}) .$$

Proof. If $\alpha_1, \cdots, \alpha_h$ are all even, then by applying Proposition 6.3 for h times and using (6.11) we obtain (6.13).

If at least one of $\alpha_1, \cdots, \alpha_h$ is odd, then without loss of generality we can assume α_1 to be odd. Application of (6.10) for $\frac{1}{2}(\alpha_1 - 1)$ times thus gives

$$(6.15)\qquad \begin{aligned} &T_n(\boldsymbol{x}, \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle^{\alpha_1} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h}, 1) \\ &\quad = cT_n(\boldsymbol{x}, \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle\langle \boldsymbol{a}_2, \boldsymbol{e}\rangle^{\alpha_2} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h}, 1) , \end{aligned}$$

where c is a constant. On the other hand, by Proposition 6.1 we have

$$(6.16)\qquad \begin{aligned} &\sum_{i=0}^{h} \alpha_i T_n(\boldsymbol{x}, \langle \boldsymbol{a}_2, \boldsymbol{e}\rangle^{\alpha_2} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h} \boldsymbol{a}_i, 1) \\ &\quad = (m + \alpha_2 + \cdots + \alpha_h - 1) T_n(\boldsymbol{x}, \langle \boldsymbol{a}_2, \boldsymbol{e}\rangle^{\alpha_2} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h} \boldsymbol{e}, 1) . \end{aligned}$$

Thus by taking scalar product of (6.16) with $\boldsymbol{a}_1$ we obtain

$$(6.17)\qquad T_n(\boldsymbol{x}, \langle \boldsymbol{a}_1, \boldsymbol{e}\rangle\langle \boldsymbol{a}_2, \boldsymbol{e}\rangle^{\alpha_2} \cdots \langle \boldsymbol{a}_h, \boldsymbol{e}\rangle^{\alpha_h}, 1) = 0 .$$

Combination of (6.15) and (6.17) hence gives (6.13).

Remark 6.1. Theorem 6.5 is the well-known Gauss-Bonnet formula when $\alpha_1 = \cdots \alpha_h = 0$, and was proved in [3] when $h = 1$ and $m - n = 1$.

7. Immersions with Lipschitz-Killing curvature ≥ 0

For an immersion of an n-dimensional manifold M in E^m, the n-th mean curvature $K_n(P, \boldsymbol{e})$ is also called the *Lipschitz-Killing curvature*. In [10], S. S. Chern and R. K. Lashof studied the n-th total absolute curvature of rank 1 with respect to 1, i.e., $TA_n(\boldsymbol{x}, 1, 1)$, and proved the following interesting inequality for closed M:

$$(7.1)\qquad TA_n(\boldsymbol{x}, 1, 1) \geq \beta(M) c_{m-1} ,$$

where $\beta(M) = \max\{\sum_{i=0}^{n} \dim H_i(M; F) \colon F \text{ fields}\}$, and $H_i(M, F)$ denotes the i-th homology group of M over F. If we denote the i-th betti number of M by $b_i(M)$, then it is obvious that $\beta(M) \geq \sum_{i=0}^{n} b_i(M)$. In this paper, an immersion of an n-dimensional closed manifold M in E^m is called a *minimal imbedding* if $TA_n(\boldsymbol{x}, 1, 1) = \beta(M) c_{m-1}$. In the following, let

$$(7.2)\qquad \lambda(P) = \max\{K_n(P, \boldsymbol{e}) ; \boldsymbol{e} \in S_P^{m-n-1}\} ,$$

$$(7.3)\qquad \mu(P) = \min\{K_n(P, \boldsymbol{e}) \colon \boldsymbol{e} \in S_P^{m-n-1}\} ,$$

$$(7.4)\qquad \begin{aligned} A_+ &= \{(P, \boldsymbol{e}) \in B_v \colon K_n(P, \boldsymbol{e}) > 0\} , \\ A_- &= \{(P, \boldsymbol{e}) \in B_v \colon K_n(P, \boldsymbol{e}) < 0\} , \end{aligned}$$

$$(7.5)\qquad \lambda^+(P) = \max\{\lambda(P), 0\} , \qquad \mu^-(P) = \min\{\mu(P), 0\} ,$$

$$t(M) = \tfrac{1}{2}\left(\beta(M) - \sum_{i=0}^{n} b_i(M)\right), \tag{7.6}$$

where S_P^{m-n-1} denotes the unit $(m - n - 1)$-sphere of unit normal vectors to $\boldsymbol{x}(M)$ at $\boldsymbol{x}(P)$ in E^m. We call λ and μ the *principal curvature* and *secondary curvature* of M in E^m. If is clear that M has no torsion when and only when $t(M) = 0$.

Proposition 7.1. *Let M be an n-dimensional oriented closed manifold imbedded in E^m. Then*

$$\int_M \lambda^+ dV \geq \left(t(M) + \sum_{i=0}^{\frac{1}{2}n} b_{2i}(M)\right)\frac{c_{m-1}}{c_{m-n-1}}, \tag{7.7}$$

$$\int_M \mu^- dV \leq -\left(t(M) + \sum_{i=1}^{\frac{1}{2}n} b_{2i-1}(M)\right)\frac{c_{m-1}}{c_{m-n-1}}. \tag{7.8}$$

Equality sign of (7.7) *holds when and only when the codimension $m - n = 1$ and $\boldsymbol{x}\colon M \to E^m$ is a minimal imbedding. Moreover, equality sign of* (7.8) *holds when and only when either* (a) $\dim M = n$ *is even and the Lipschitz-Killing curvature $K_n(P, \boldsymbol{e}) \geq 0$ everywhere, or* (b) *the codimension $m - n = 1$ and $\boldsymbol{x}\colon M \to E^m$ is a minimal imbedding.*

Proof. From Theorem 6.5 it follows that

$$\int_{A_+} K_n(P, \boldsymbol{e})dV \wedge d\sigma + \int_{A_-} K_n(P, \boldsymbol{e})dV \wedge d\sigma = \sum_{i=0}^{n} (-1)^i b_i(M) c_{m-1}. \tag{7.9}$$

On the other hand, by (7.1) and (7.6) we have

$$\begin{aligned} &\int_{A_+} K_n(P, \boldsymbol{e})dV \wedge d\sigma - \int_{A_-} K_n(P, \boldsymbol{e})dV \wedge d\sigma \\ &\qquad \geq \left(\sum_{i=0}^{n} b_i(M) + 2t(M)\right)c_{m-1}. \end{aligned} \tag{7.10}$$

Combination of (7.9) and (7.10) yields

$$\int_{A_+} K_n(P, \boldsymbol{e})dV \wedge d\sigma \geq \left(t(M) + \sum_{i=0}^{\frac{1}{2}n} b_{2i}(M)\right)c_{m-1}, \tag{7.11}$$

$$\int_{A_-} K_n(P, \boldsymbol{e})dV \wedge d\sigma \leq -\left(t(M) + \sum_{i=1}^{\frac{1}{2}n} b_{2i-1}(M)\right)c_{m-1}. \tag{7.12}$$

Therefore by (7.2), (7.3), (7.4) and (7.5) we obtain (7.7) and (7.8). Now suppose that equality sign of (7.7) holds. Then the inequalities (7.10), (7.11) and (7.12) are actually equalities, so that $\boldsymbol{x}\colon M \to E^m$ is a minimal imbedding.

Next suppose that the codimension $m-n>1$. It is easy to see that if $\lambda(P)>0$ at $P \in M$, then $K_n(P, \boldsymbol{e}) = \lambda(P)$ for all $(P, \boldsymbol{e}) \in S_P^{m-n-1}$. In particular, this implies that $\dim M = n$ is even. Since the set $\{(P, \boldsymbol{e}) \in B_v$: the second fundamental form at $(P, \boldsymbol{e})$ is positive definite$\}$ is of positive measure, by choosing a point $(\bar{P}, \bar{\boldsymbol{e}})$ in this set we have $\lambda(\bar{P}) > 0$. Thus we obtain $K_n(\bar{P}, \boldsymbol{e}) = \lambda(\bar{P})$ for all $\boldsymbol{e} \in S_{\bar{P}}^{m-n-1}$. On the other hand, by definition we see that the second fundamental form at $(\bar{P}, -\bar{\boldsymbol{e}})$ is negative definite, and the continuity of the second fundamental form implies that the Lipschitz-Killing curvature $K_n(\bar{P}, \boldsymbol{e}) = 0$ for some points in $S_{\bar{P}}^{m-n-1}$. Since this is a contradiction, we get $m-n=1$. Conversely, if $m-n=1$ and $\boldsymbol{x}\colon M \to E^m$ is a minimal imbedding, then the equality sign holds in (7.11) and (7.12). On the other hand, $K_n(P, \boldsymbol{e}) = \lambda^+(P)$ on A_+ and $K_n(P, \boldsymbol{e}) = \mu^-(P)$ on A_-. Moreover, $A_+ = \{P \in M\colon \lambda^+(P) \neq 0\}$ and $A_- = \{P \in M\colon \mu^-(P) \neq 0\}$. Consequently, the equality sign holds in (7.7) and (7.8).

Now suppose that the equality sign of (7.8) holds, and the Lipschitz-Killing curvature $K_n(P, \boldsymbol{e}) < 0$ for some points $(P, \boldsymbol{e})$ in B_v. Then $\mu^-(P) < 0$ for some P in M, and $K_n(P, \boldsymbol{e}) = \mu^-(P)$ for all $(P, \boldsymbol{e}) \in S_P^{m-n-1}$ whenever $\mu^-(P) < 0$. This is impossible by the continuity of the second fundamental form on the fibre S_P^{m-n-1} if the codimension $m-n>1$. Thus we get $m = n+1$. On the other hand, from the equality of (7.8) and the inequality of (7.10) it follows that the equality sign holds in (7.11) and (7.12). This implies that the immersion of M in E^m is a minimal imbedding. Consequently, either the Lipschitz-Killing curvature is nowhere negative, or $m = n+1$ and $\boldsymbol{x}\colon M \to E^m$ is a minimal imbedding. In the first case, we have $t(M) = 0$ and $b_i(M) = 0$ for all odd i. Thus (a) if $K_n(P, \boldsymbol{e})$ is nowhere negative, then by the inequality (7.8) we have $t(M) = 0$ and $b_i(M) = 0$ for all odd i, and therefore by (7.3) and (7.5) we get the equality sign of (7.8); and (b) if $m = n+1$ and $\boldsymbol{x}\colon M \to E^m$ is a minimal imbedding, then the equality sign of (7.8) follows immediately from the equality sign of (7.10) and the definition of μ. This completes the proof of the proposition.

Theorem 7.2. *Let $\boldsymbol{x}\colon M \to E^m$ be an imbedding of an n-dimensional oriented closed manifold M in E^m.* (a) *The Lipschitz-Killing curvature $K_n(P, \boldsymbol{e}) \geq 0$ everywhere if and only if* (i) *M has no torsion,* (ii) *all odd-dimensional betti numbers of M vanish, and* (iii) *the imbedding $\boldsymbol{x}\colon M \to E^m$ is minimal.* (b) *If the Lipschitz-Killing curvature $K_n(P, \boldsymbol{e}) > 0$ everywhere, then* $\dim M$ *is even, and either* $\dim M = 0$ *or the codimension $m-n=1$, M has no torsion, and $\boldsymbol{x}(M)$ is a convex hypersurface in E^{n+1}.*

Proof. (a) If the Lipschitz-Killing curvature $K_n(P, \boldsymbol{e}) \geq 0$ everywhere, then $\mu^-(P) = 0$. Thus by Proposition 7.1, we obtain $t(M) = 0$ and $b_i(M) = 0$ for all odd i. Moreover, $A_- = \emptyset$. These imply that

$$TA_n(\boldsymbol{x}, 1, 1) = T_n(\boldsymbol{x}, 1, 1) = \chi(M)c_{m-1} = \beta(M)c_{m-1}\,,$$

i.e., the imbedding $\boldsymbol{x}$ is minimal. Conversely, if $\boldsymbol{x}$ is a minimal imbedding, M has no torsion, and odd-dimensional betti numbers of M vanish, then

$$TA_n(\boldsymbol{x}, 1, 1) = \chi(M)c_{m-1} = T_n(\boldsymbol{x}, 1, 1) .$$

By the continuity of $K_n(P, \boldsymbol{e})$ on the normal bundle B_v and the definitions of $TA_n(\boldsymbol{x}, 1, 1)$ and $T_n(\boldsymbol{x}, 1, 1)$, we thus obtain $K_n(P, \boldsymbol{e}) \geq 0$ everywhere.

(b) Suppose that $K_n(P, \boldsymbol{e}) > 0$ everywhere, and $n > 0$. Then from $K_n(P, -\boldsymbol{e}) = (-1)^n K_n(P, \boldsymbol{e})$ it follows that $\dim M = n$ is even. Let $(\bar{P}, \bar{\boldsymbol{e}})$ be a point in B_v such that the second fundamental form at $(\bar{P}, \bar{\boldsymbol{e}})$ is positive definite. Then the second fundamental form at $(\bar{P}, -\bar{\boldsymbol{e}})$ is negative definite. By the continuity of the second fundamental form on the fibre $S_{\bar{P}}^{m-n-1}$ we see that if the codimension $m - n > 1$, then the Lipschitz-Killing curvature $K_n(\bar{P}, \boldsymbol{e}) = 0$ at some points in $S_{\bar{P}}^{m-n-1}$. This is impossible by the assumption. Thus we have $m - n = 1$. In this case, the condition that $K_n(P, \boldsymbol{e}) > 0$ everywhere implies that Gauss-Kronecker curvature of M in E^{n+1} is positive everywhere. Hence $\boldsymbol{x}(M)$ is a convex hypersurface in E^{n+1}.

Remark 7.1. If the codimension $m - n = 1$, then the sufficiency of Theorem 7.2, Part (a) was proved by Chern-Lashof [10, II], and Theorem 7.2, Part (b) was the well-known Hadamard theorem. In [10, II], Chern and Lashof gave an example of nonconvex hypersurface in E^{n+1} with $K_n(P, \boldsymbol{e}) \geq 0$ everywhere. In [15], Kobayashi gave an example of a minimal imbedding of complex projective spaces in higher dimensional euclidean space; in his example, the Lipschitz-Killing curvature $K_n(P, \boldsymbol{e}) \geq 0$ everywhere.

If C is a closed curve in E^3, then we have the so-called curvature k and torsion τ. If the torsion $\tau = 0$ identically on C, then C is a plane curve in E^3. Moreover, if the curvature k is constant and the torsion $\tau = 0$ identically, then C is a circle in a plane of E^3. By using Theorem 7.2 and a result of Chern-Lashof [10, I], we have

Corollary 7.3. *Let $\boldsymbol{x}\colon M \to E^m$ be an imbedding of an even-dimensional topological sphere in E^m with $m - n > 1$. Then the secondary curvature $\mu = 0$ when and only when M is imbedded as a convex hypersurface in an $(n + 1)$-dimensional linear subspace of E^m. Moreover, the secondary curvature $\mu = 0$, and the principal curvature λ is constant when and only when M is imbedded as a hypersphere in an $(n + 1)$-dimensional linear subspace of E^m.*

8. Product immersion and immersion with constant G-total curvature

Proposition 8.1. *Let $\boldsymbol{x}_1\colon M_1 \to E^{m_1}$ and $\boldsymbol{x}_2\colon M_2 \to E^{m_2}$ be immersions of M_1 and M_2 in E^{m_1} and E^{m_2} respectively, and $\boldsymbol{x}_1 \times \boldsymbol{x}_2$ be the product immersion of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Then*

$$\begin{aligned} G_{n_1+n_2}(\boldsymbol{x}_1 \times \boldsymbol{x}_2, (P_1, P_2), 1, 1)c_{m_1-1}c_{m_2-1} \\ = G_{n_1}(\boldsymbol{x}_1, P_1, 1, 1)G_{n_2}(\boldsymbol{x}_2, P_2, 1, 1)c_{m_1+m_2-1} , \end{aligned} \tag{8.1}$$

for all $(P_1, P_2) \in M_1 \times M_2$, where $\dim M_1 = n_1$ *and* $\dim M_2 = n_2$.

This proposition can be proved in the same way as Theorem 10 was proved in [2, I], so we omit the proof.

Corollary 8.2. *Let M_1 and M_2 be two oriented closed manifolds. Then the Euler characteristics of M_1 and M_2 satisfy*

$$\chi(M_1 \times M_2) = \chi(M_1) \times \chi(M_2) . \tag{8.2}$$

This corollary follows immediately from Theorem 6.5 and Proposition 8.1.

Proposition 8.3. *Let $\boldsymbol{x}\colon M \to E^m$ be an immersion of an oriented closed even-dimensional manifold M in E^m such that $\boldsymbol{x}(M)$ is contained in an $(n+1)$-dimensional linear subspace E^{n+1} and the n-th G-total curvature $G_n(\boldsymbol{x}, P, 1, 1) > 0$ everywhere on M. Then $\boldsymbol{x}(M)$ is a convex hypersphere in E^{n+1}, and there exists an oriented closed even-dimensional nonconvex submanifold in E^{n+2} with positive constant n-th G-total curvature $G_n(\boldsymbol{x}, P, 1, 1)$.*

Proof. The first part follows from Proposition 8.1 and Theorem 7.2. Let $S^{\frac{1}{2}n} \times S^{\frac{1}{2}n} \subset E^{n+2}$ be the natural product manifold of two unit $\frac{1}{2}n$-spheres in E^{n+2}. Then this product manifold in E^{n+2} has constant n-th G-total curvature $G_n(\boldsymbol{x}, P, 1, 1)$ everywhere.

By Proposition 8.3 we have

Corollary 8.4. *If M is an exotic n-sphere, then M cannot be immersed in E^{n+1} as a hypersphere with $G_n(\boldsymbol{x}, P, 1, 1) > 0$.*

Remark 8.1. Every compact homogeneous space M can be immersed in a euclidean space with constant i-th G-total curvature $G_i(\boldsymbol{x}, P, 1, 1)$. This immersion can be done by using equivariant immersion of M in the euclidean space.

Remark 8.2. Let M be an n-dimensional manifold immersed in E^m. If i is an even positive integer, $2 \leq i \leq n$, then we have

$$G_i(\boldsymbol{x}, P, 1, 1) = \text{const.} \sum \delta\begin{pmatrix} j_1, \cdots, j_i \\ k_1, \cdots, k_i \end{pmatrix} R_{j_1 j_2 k_1 k_2} \cdots R_{j_{i-1} j_i k_{i-1} k_i} ,$$

in which R_{jklh} are the components of the Riemannian-Christoffel tensor (relative to orthonormal frames) of the induced Riemannian metric on M, and $\delta\begin{pmatrix} j_1, \cdots, j_i \\ k_1, \cdots, k_i \end{pmatrix}$ does not vanish if and only if $j_1, \cdots, j_i$ are rearrangement of $k_1, \cdots, k_i$; its value is 1 if the permutation is even and -1 if odd. Hence we see that $G_i(\boldsymbol{x}, P, 1, 1)$ are isometric scalar invariants. In fact, $G_i(\boldsymbol{x}, P, 1, 1)$ are among the most important scalar invariants of the Riemannian metric. For example, $G_2(\boldsymbol{x}, P, 1, 1) = \text{const.} \sum R_{j_1 j_2 k_1 k_2}$ is called the scalar curvature of the Riemannian metric (see, for instance, Chern [7], Nagano [16]).

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