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CLASSIFICATION OF THE SIMPLE SEPARABLE REAL L^*-ALGEBRAS

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Introduction

A *real* (complex) L^*-algebra is a Lie algebra L over the real (complex) numbers such that the underlying vector space is a Hilbert space (throughout this work the Hilbert space is assumed to be separable) and such that, for each $x \in L$, there is an $x^* \in L$ satisfying $([x, y], z) = (y, [x^*, z])$ for all y, z in L. L^*-subalgebras and L^*-ideals are defined in the usual way, with the additional property of being closed subspaces, invariant under the map $x \to x^*$. These algebras were introduced by J. R. Schue [11], [12], who obtained a complete classification of all simple separable complex L^*-algebras. V. K. Balachandran [1], [2], [3], [4], [5] gave more general settings to the techniques used by Schue for not necessarily separable L^*-algebras; he also defined the notions of real form and compact real form.

The main result of this work is the classification[1] of the simple separable real L^*-algebras up to L^*-automorphism.

We show in § 1 that the complexification $\tilde{L}$ of a simple real L^*-algebra is not simple if and only if $L = M^R$ (M^R denotes the real L^*-algebra obtained from M by restriction of scalars). Therefore, the classification reduces essentially, aside from simple real L^*-algebras having a complex structure which are in a one-to-one correspondence with the simple complex L^*-algebras, to the study of the real forms of all simple complex L^*-algebras.

If L is a real form of a semisimple L^*-algebra $\tilde{L}$, the decomposition $L = K + M$ (Hilbert direct sum), where $K = \{a \in L: a^* = -a\}$ and $M = \{a \in L: a^* = a\}$, defines an involutive L^*-automorphism S of L ($S|K = \text{id}$ and $S|M = -\text{id}$.) which can be extended to $\tilde{L}$ by linearity. S is called the involution of L associated to $\tilde{L}$. Conversely, if S is an involutive L^*-automorphism of L, then S leaves the unique compact form U (set of all self-adjoint elements of $\tilde{L}$) invariant and we have $U = K + iM$, the decomposition of U into eigenspaces of S. The real form $L = K + M$ is said to be associated to S.

There is a one-to-one correspondence between isomorphism classes of real

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[1] The classification was also obtained, independently, by Mr. Pierre de la Harpe.

forms of $\tilde{L}$ and conjugacy classes of L^*-automorphisms of $\tilde{L}$ containing an involutive element.

Following an idea of S. Murakami [9], [10] we show that of S is an involution of $\tilde{L}$ we can find a Cartan subalgebra $\tilde{H}$ and a regular self-adjoint element h in it such that $S\tilde{H} = \tilde{H}$, $Sh = h$, and the 1-eigenspace of S in $\tilde{H}$ is a maximal abelian L^*-subalgebra of $\tilde{K}$ (the complexification of K),

Having such a Cartan subalgebra we are able to compute explicitly the structure of $\tilde{K}$ in terms of the roots of $\tilde{L}$ relative to $\tilde{H}$.

Next (§§ 2, 3, 4), we show case by case, that if an involutive rotation leaves a regular self-adjoint element fixed, then it is a "particular" rotation (i.e., it leaves some system of simple roots invariant).

It is known [4] that in the case of simple complex L^*-algebras of types A and C all Cartan subalgebras are conjugate, and in case B, the Cartan subalgebras fall into two conjugacy classes. Thus, if we fix in cases A and C a Cartan subalgebra $\tilde{H}$ and a system Π of simple roots there exists in each conjugacy class of L^*-automorphisms containing an involutive element, an involution leaving $\tilde{H}$ and Π invariant. In case B we have to take two non-conjugate Cartan subalgebras in order to get a similar result.

The classification follows easily by reducing such an involution to a normal form.

At the end of § 5 we discuss natural realizations of all the real forms.

The result we obtain is exactly what we expect as an infinite dimensional analogue of classical real simple Lie algebras.

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1. Reduction of the problem

1.1. Preliminaires. Throughout this work, all L^*-algebras are assumed to be separable. Let L be an L^*-algebra. L is *semisimple* if $[L, L] = L$ (where $[A, B]$ = closed subspace spanned by $\{[a, b], a \in A, b \in B]\}$). This is equivalent to saying that the map $x \to \operatorname{ad}(x)$ is one-to-one. If L is semisimple, x^* is uniquely determined by x and satisfies $x^{**} = x$, $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$, $[x, y]^* = [y^*, x^*]$ and $(x^*, y^*) = (y, x)$. L is *simple* if there are no nontrivial ideals. Let L_1 and L_2 be L^*-algebras. A map $T: L_1 \to L_2$ is a L^*-isomorphism, if T is a Lie algebra isomorphism and is an isometry and $T(x^*) = T(x)^*$.

Let $\tilde{L}$ be a complex L^*-algebra. The real L^*-algebra obtained from $\tilde{L}$ by restriction of scalars and by taking the real part of the inner product of $\tilde{L}$ is denoted by $\tilde{L}^R$ and called the *real L^*-algebra obtained from $\tilde{L}$ by restriction of scalars*. The map J from $\tilde{L}^R$ onto itself, defined by $Jx = ix$, is an orthogonal map satisfying $J(x^*) = -(Jx)^*$, $[Jx, y] = J[x, y] = [x, Jy]$ and $J^2 = \text{id}$. J is called a *complex structure of* $\tilde{L}^R$. Conversely, let L be a real L^*-algebra with a complex structure J. L together with the complex multiplication defined by $(r_1 + ir_2)x = r_1x + r_2Jx$ $(r_1, r_2 \in \boldsymbol{R}, x \in L)$ and the Hermitian inner product $(x, y) + i(x, Jy)$ (where (,) is the inner product in L) becomes a complex L^*-algebra. Let L be a real L^*-algebra. Then the complexification $\tilde{L} = L + iL$ of L together with the Hermitian inner product $(x + iy, u + iv) = (x, u) + (y, v) + i((x, u) - (x, v))$, the conjugation $(x + iy)^* = x^* - iy^*$ and the Lie bracket extended by linearity, becomes a complex L^*-algebra called the *complexification of L*. If L and M are L^*-algebras over the same field, the vector space $L \times M$ together with the inner product $((x, y), (u, v)) = (x, u)_L + (y, v)_M$, the Lie bracket defined by $[(x, y), (u, v)] = ([x, u], [y, v])$ and the conjugation $(x, y)^* = (x^*, Y^*)$ becomes an L^*-algebra called the *product L^*-algebra*. From now on we will denote a real L^*-algebra by L, its complexification by $\tilde{L}$, and the real L^*-algebra obtained from $\tilde{L}$ by $\tilde{L}^R$. It is trivial to see that L, $\tilde{L}$, $\tilde{L}^R$ are all semisimple if and only if one of them is.

Let $\tilde{L}$ be a complex L^*-algebra. A real form L [3] is an L^*-subalgebra of $\tilde{L}^R$ such that, the inner product of $\tilde{L}$ restricted to $L \times L$ is real-valued and $\tilde{L}$ is the complexification of L, i.e., $\tilde{L} = L + iL$. The map $\sigma\colon \tilde{L} \to \tilde{L}$ defined by $\sigma(x + iy) = x - iy$ is an involutive L^*-automorphism of $\tilde{L}^R$ such that $(\sigma x, \sigma y) = (y, x)$ and $\sigma(\alpha x) = \bar{\alpha}\sigma(x)$ $(\alpha \in \boldsymbol{C})$. σ is called the *conjugation* of $\tilde{L}$ with respect to L. Conversely, if σ is a map of $\tilde{L}$ onto itself with the above properties, the set L of fixed points of σ is a real form of $\tilde{L}$ having σ as the associated conjugation. A real form U of $\tilde{L}$ is a *compact* real form of $\tilde{L}$ if $(x, x^*) < 0$ for all $x \in U$. Every complex L^*-algebra has a unique compact real form [3]; indeed, $U = \{x \in \tilde{L}\colon x^* = -x\}$. We always denote the unique compact real form by U and the conjugation of $\tilde{L}$ with respect to U by τ.

1.2. Real forms and involutive L^*-automorphisms. Let $\tilde{L}$ be a semisimple complex L^*-algebra. If L is a real form of $\tilde{L}$, then $L = K + M$ (Hilbert direct sum) where K and M are the skew-adjoint and self-adjoint parts of L, i.e., $K = \{x \in L\colon x^* = -x\}$ and $M = M\{x \in L\colon x^* = x\}$. They are orthogonal closed subspaces, and K is an L^*-subalgebra of L called the *characteristic subalgebra* of L, also $[K, M] \subset M$ and $[M, M] \subset K$. The map S of L onto itself defined by $S(x + y) = x - y$ $(x \in K, y \in M)$ is an involutive L^*-automorphism of $L(S = \tau|L$, τ as above). The extension of S by linearity to an L^*-automorphism of $\tilde{L}$, which we also denote by S, is involutive, and we say that S is the *involution of L associated with real with real form L*. Conversely, let S be an involution (an involutive L^*-automorphism) of $\tilde{L}$. Since $(Sx)^* = Sx^*$, S leaves U (the unique compact form of $\tilde{L}$) invariant. Let $K + iM$ be the decomposition of U into eigenspaces of S corresponding to the eigenvalues $+1$ and -1. Then $L = K + M$ is a real form of $\tilde{L}$ having S as its associated involution. L is said to be the *real form of $\tilde{L}$ associated to the invoultion S*. So the real forms of L are in a one-to-one correspondence with the involutive L^*-automorphisms of $\tilde{L}$.

Denote the graup of all L^*-automorphisms of $\tilde{L}$ by Aut$(\tilde{L})$. The next theorem shows that there is a one-to-one correspondence between isomorphism classes of real forms of a semisimple complex L^*-algebra $\tilde{L}$ and all conjugacy classes in Aut $(\tilde{L})$ containing an involutive element.

Theorem 1.2.1. *Let L_1, L_2 be real forms of a semisimple L^*-algebra $\tilde{L}$, and S_1, S_2 be the associated involutions of $\tilde{L}$. Then L_1 and L_2 are L^*-isomorphic if and only if S_1 and S_2 are conjugate in* Aut $(\tilde{L})$.

Proof. Suppose that L_1 and L_2 are L^*-isomorphic, and that T is an L^*-isomorphism between them. In the decompositions $L_1 = K_1 + M_1$ and $L_2 = K_2 + M_2$ into skew-adjoint and self-adjoint elements, we have $TK_1 = K_2$ and $TM_1 = M_2$. Since $S_j \mid (K_j + iK_j) = \text{id}$ and $S_j \mid (M_j + iM_j) = -\text{id}$ $(j = 1, 2)$, the extension of T to $\tilde{L}$ by linearity satisfies $S_2 = TS_1T^{-1}$. T gives the desired conjugation.

Suppose that S_1 and S_2 are conjugate, i.e., there exists $T \in \text{Aut}(\tilde{L})$ such that $S_2 = TS_1T^{-1}$. Since T, S_1 and S_2 leave U invariant, $T(K_1) = K_2$ and $T(iM_1) = iM_2$ ($U = K_i + iM_j$ is the eigenspace decomposition of U with respect to S_j). Then $T \mid L_1$ is an L^*-isomorphism between L_1 and L_2.

1.3. Reduction of the problem. In this section, we show that the simple real L^*-algebras fall into two classes, one class containing all the simple real L^*-algebras with a complex structure and the other containing the real forms of all simple complex L^*-algebras.

Theorem 1.3.1. *Let L be a simple real L^*-algebra. Then the complexification $\tilde{L}$ of L is not simple if and only if $L = M^R$, where M is a simple complex L^*-algebra, i.e., L has a complex structure.*

We break the proof in several lemmas.

Lemma 1.3.2. *Let L be a simple L^*-algebra, S be the involution of $\tilde{L}$ with respect to L, and σ be the conjugation of $\tilde{L}$ with respect to L. Then $\tilde{L}$ is either simple or the sum of two nonzero L^*-ideals interchanged by σ, and U is the sum of two nonzero L^*-ideals interchanged by S.*

Proof. Since $\tilde{L}$ is semisimple, let $\tilde{L} = \sum_j \tilde{L}_j$ be the decomposition of $\tilde{L}$ into simple L^*-ideals [11]. σ interchanges the L_j's. If for some index 1 one has $\sigma\tilde{L}_1 = \tilde{L}_1$, then $\tilde{L}_1 = (L \cap \tilde{L}_1) + (iL \cap \tilde{L}_1)$. Since L_1^R is an L^*-ideal in $\tilde{L}^R$ and L is an L^*-subalgebra, $\tilde{L}_1 \cap L$ is an L^*-ideal in L. By assumption it must either be $\{0\}$ or L; if $\tilde{L}_1 \cap L = \{0\}$, then $i(L \cap \tilde{L}_1) = (iL \cap \tilde{L}_1) = \{0\}$ because $\tilde{L}_1$ is a complex vector space, and $\tilde{L}_1$ reduces to $\{0\}$, which is impossible. So $L \subset \tilde{L}_1$ and $iL \subset \tilde{L}_1$, and $\tilde{L} = \tilde{L}_1$ is a simple L^*-algebra. On the other hand, if σ interchanges two of them, say $\sigma\tilde{L}_1 = \tilde{L}_2$, then $\sigma(\tilde{L}_1 + \tilde{L}_2) = \tilde{L}_1 + \tilde{L}_2$ and, along the same lines as in the above argument, $\tilde{L} = \tilde{L}_1 + \tilde{L}_2$, i.e., $\tilde{L}$ is the sum of two nonzero L^*-ideals interchanged by σ, and $U = U_1 + U_2$ (U_i: unique compact real form of $\tilde{L}_i$) is the sum of two nonzero L^*-ideals interchanged by $S(S \mid U = \tau \mid U)$.

Lemma 1.3.3. *Let $\tilde{L}$ be a semisimple L^*-algebra, L be a noncompact real form of $\tilde{L}$, and S be the involution of $\tilde{L}$ associated with $L(S \neq \text{id})$. If $U =$*

$U_1 + U_2$, two nonzero L^*-ideals interchanged by S, then L has a complex structure.

Proof. The map $T: U_1^R \to L$ defined by $T(u + iv) = ((u + Sv) + i(v - Sv))$ satisfies all the conditions, but is not an isometry, i.e., $(Tx, Ty)\tilde{U}_1^R = 2(x, y)_L$, as is trivial to see. Anyway, the pull-back of the complex structure on $\tilde{U}_1^R$ gives the desire complex structure on L.

Lemma 1.3.4. *Let $\tilde{L}$ be a semisimple L^*-algebra, L be a real form of $\tilde{L}$, and S be the involution of $\tilde{L}$ associated to L. If L is noncompact and carries a complex structure J, then U is the sum of two nonzero L^*-ideals interchanged by S.*

This lemma is the converse of Lemma 1.3.3.

Proof. Let $\tilde{L}_1$ and $\tilde{L}_2$ be the eigenspaces of the extension of J to $\tilde{L}$ by linearity corresponding to the eigenvalues i and $-i$. Then $\tilde{L}_1$ and $\tilde{L}_2$ are L^*-ideals of $\tilde{L}$ interchanged by σ (conjugation of $\tilde{L}$ with respect to L) because if $x + iy \in \tilde{L}_1$ $(x, y \in L)$ we have $(x - iy) \in \tilde{L}_2$. So $U = U_1 + U_2$ (the compact real forms of $\tilde{L}_1$ and $\tilde{L}_2$ respectively) and $\sigma = S\,|\,U$ interchanges them.

Proof of Theorem 1.3.1. Trivial.

1.4. Simple real L^*-algebras having a complex structure. In this section we classify all simple real L^*-algebras having a complex structure.

Proposition 1.4.1. *If $\tilde{L}$ is a simple L^*-algebra, then $\tilde{L}^R$ is a simple L^*-algebra.*

Proof. Suppose $\tilde{L}^R$ is not simple (in any case is semisimple). We can find a nontrivial L^*-ideal A properly contained in $\tilde{L}^R$; its orthogonal complement $A^\perp$ is also an L^*-ideal, and $\tilde{L}^R = A + A^\perp$. Since $\tilde{L}^R$ is semisimple, both A and $A^\perp$ are semisimple. A is invariant under complex multiplication; if $x \in A$, then $ix = a + b$ $(a \in A,\, b \in A^\perp)$, and for every $y \in A^\perp$ we have $[b, y] = [ix, y] - [a, y] = i[x, y] - [a, y] = 0$. By the semisimplicity of $A^\perp$ the component b of ix must be zero and $ix \in A$. So A is a nontrivial simple complex L^*-subalgebra properly contained in $\tilde{L}$, which is a contradiction. q.e.d.

In the next proposition we prove that if two simple complex L^*-algebras induce L^*-isomorphic real L^*-algebras by restriction of scalars they are also L^*-isomorphic.

Proposition 1.4.2. *Let L be a simple L^*-algebra having two complex structures J and I. Then the two complex simple L^*-algebras obtained from L through these complex structures are L^*-isomorphic.*

Proof. We indicate the complex L^*-algebras obtained from L through J and I by (L, J) and (L, I); the corresponding inner products by $(\,,\,)_J$ and $(\;)_I$.

Let $\tilde{L}$ be the complexification of L. $\tilde{L} = \tilde{L}_1 + \tilde{L}_2$, where $\tilde{L}_1$ are $\tilde{L}_2$ are the eigenspaces of the extension of J to $\tilde{L}$ by linearity corresponding to the eigenvalues i and $-i$. $\tilde{L}_1$ and $\tilde{L}_2$ are L^*-ideals of $\tilde{L}$. If $x \in \tilde{L}$, then $x = \frac{1}{2}(x - Jix) + \frac{1}{2}(x + iJx)$ with respect to the decomposition $\tilde{L} = \tilde{L}_1 + \tilde{L}_2$. The map T_1: $(L, J) \to L_1$ defined by $T(x) = \frac{1}{2}(x - iJx)$ satisfies all the conditions for an L^*-isomorphism except that it is not an isometry, i.e., $(x, y)_J = \frac{1}{2}(Tx, Ty)$. The map Tz: $(L, J) \to \tilde{L}_2$ defined by $T(x) = \frac{1}{2}(x + iJx)$ satisfies all the conditions for an anti-L^*-isomorphisms except again that it is not an isometry i.e., $(x, y)_I = \frac{1}{2}(Tx, Ty)$. In any case, we see that $\tilde{L}_1$ and $\tilde{L}_2$ are simple, and the decomposition $\tilde{L} = \tilde{L}_1 + \tilde{L}_2$ is that of $\tilde{L}$ into simple L^*-ideals. Doing exactly the same with I, we get that the decomposition of $\tilde{L}$ is also $\tilde{L}_1 + \tilde{L}_2$ by the uniqueness of the decomposition into simple L^*-ideals. Thus (L, J) and (L, I) are either L^*-isomorphic or anti-L^*-isomorphic; in the second case in order to get the desired L^*-isomorphism we need only to compose the given anti-L^*-isomorphism with, for instance, the map $x \to x^*$. q.e.d.

Thus the isomorphism classes of simple real L^*- algebras having a complex structure are in a one-to-one correspondence with the isomorphism classes of simple L^*-algebras.

1.5. Cartan subalgebras and involutive L^*-automorphisms. It remains to classify, up to L^*-isomorphisms, the real forms of all simple complex L^*-algebras. According to Theorem 1.2.1, it is enough to classify the conjugacy classes in Aut $(\tilde{L})$ containing an involutive element, for all simple complex L^*-algebras. We already know [11] that there are essentially three different kinds of simple separable complex L^*-algebras, and we called them of types A, B, and C. In this section, we show that in the case of L^*-algebras of types A and C, if we fix a Cartan subalgebra, we can choose in each conjugacy class containing an involution, an element leaving invariant the Cartan subalgebra and a regular self-adjoint element in it. To get a similar result in case B we need two Cartan subalgebras.

Theorem 1.5.1. *Let $\tilde{L}$ be a semisimple L^*-algebra, and S an involution of $\tilde{L}$. Then we can find a Cartan subalgebra (a maximal abelian L^*-subalgebra) $\tilde{H}$ and a regular self-adjoint element h in $\tilde{H}$ such that $S\tilde{H} = \tilde{H}$ and $Sh = h$.*

Proof. Since S leaves U invariant, let $U = K + M$ be the decomposition of U into eigenspaces of S. Then the complexifications of K and M provide the decomposition of $\tilde{L}$ with respect to S, i.e., $\tilde{L} = \tilde{K} + \tilde{M}$. K is an L^*-subalgebra of U which may not be semisimple, but it can be written as the sum of two L^*-ideals, its center Z and its semisimple derived L^*-subalgebra $K_1 = [K, K]$ (L^*-algebras are reductive [11]). The corresponding decomposition of $\tilde{K}$ is $\tilde{K} = \tilde{Z} + \tilde{K}_1$. It is easy to see that an abelian L^*-subalgebra H_1 of K is maximal if and only if $H_1 = Z + H_1^1$, where H_1^1 is a maximal abelian L^*-subalgebra of K_1. Let H_1 be a maximal abelian L^*-subalgebra of K, and H a maximal abelian L^*-subalgebra of U containing H_1. H is invariant under S: if $x \in H$ and $h \in H_1$, then

$$[x + Sx, h] = [x, h] + S[x, Sh] = S[x, h] = 0 .$$

Since $x + Sx \in K$ and H_1 is maximal abelian in K, we have that $x + Sx \in H_1$ and $x \in H$. In other words, we can write $H = H_1 + H_{-1}$ where $H_1 = K \cap H$ $H_{-1} = M \cap H$.

H_{-1} is completely determined by H_1, i.e., $H_{-1} = \{x \in M\colon [x, h] = 0$ for all $h \in H_1\}$. Suppose that $x \in M$ and $[x, h] = 0$ for all $h \in H_1$. Let a be any element in H. Then

$$a = \tfrac{1}{2}(a + Sa) + \tfrac{1}{2}(a - Sa) = a' + a'' ,$$

in the decomposition of H mentioned above.

If $a' \in H_1$, then $[a', x] = 0$. If $a'' \in H_{-1}$, then $[a'', x] \in K$. Actually, $[a'', x] \in H_1$ because it commutes with all elements in H_1, i.e., if $y \in H_1$, then we have

$$[a'', y] = 0 \qquad \text{because } a'' \text{ and } y \text{ are in } H ,$$
$$[y, x] = 0 \qquad \text{because } y \in H_1 .$$

Thus $[y, [a'', x]] = -[a'', [x, y]] - [x, [y, a'']] = 0$. Since H_1 is maximal abelian in K, we conclude that $[a'', x] \in H_1$. Hence $[x, a] = [x, a'] \in H$ for all $a \in H$, and x is in the normalizer of H, which is exactly H because $\tilde{H} = H + iH$ is a Cartan subalgebra of $\tilde{L}$.

It should be remarked that $\tilde{H} = \{h \in \tilde{L}\colon [h, x] = 0$ for all $x \in H_1\}$.

Let us see now that iH_1 contains a regular element [2]. Let Δ be the root system of $\tilde{L}$ with respect to $\tilde{H}$. The elements of Δ are real-valued linear functionals on iH. For any $\gamma \in \Delta$ set $M_\gamma = \{h \in iH\colon \gamma(h) = 0\}$, and assume that iH_1 contains no regular elements, i.e., $iH_1 \subset \bigcup_{\gamma \in \Delta} M_\gamma$. Since a separable metric space is not the union of a countable number of nowhere dense subsets, we conclude that $iH_1 \subset M_\gamma$ for some γ. In other words $\gamma \mid iH_i \equiv 0$. Since γ is $\boldsymbol{C}$-linear, $\gamma \mid H_1 \equiv 0$. If e_γ is a root vector of γ, then $[h, e_\gamma] = \gamma(h)e_\gamma = 0$ for all $h \in H_1$. By the above remark, $e_\gamma \in \tilde{H}$ which is a contradiction, so iH_1 contains a regular element, say h. Since $H_1 \subset K$, $S \mid H_1 \equiv \mathrm{id}$, and therefore $Sh = h$. Hence our proof is complete.

In the case of simple complex L^*-algebras of type A and C, all Cartan subalgebras being conjugate, we can restate the theorem as follows:

Corollary 1.5.2. *Let $\tilde{L}$ be a simple L^*-algebra of type A or C, and $\tilde{H}$ a Cartan subalgebra. Then every conjugacy class of L^*-automorphisms containing an involutive L^*-automorphism has an element leaving $\tilde{H}$ and a regular self-adjoint element in it invariant.*

In the case of simple complex L^*-algebras of type B, the Cartan subalgebras fall into two classes such that any two in the same class are conjugate, while no two from different classes are [4]. We call those in one class Cartan subalgebras of type I and those in the other class Cartan subalgebras of type II.

Corollary 1.5.3. *Let $\tilde{L}$ be a simple L^*-algebra of type B, and $\tilde{H}_{\mathrm{I}}$, $\tilde{H}_{\mathrm{II}}$ be Cartan subalgebras of type* I *and* II *respectively. Then every conjugacy class of L^*-automorphisms containing an involutive L^*-automorphism has an element leaving one of the Cartan subalgebras $\tilde{H}_{\mathrm{I}}$, $\tilde{H}_{\mathrm{II}}$ and a regular self-adjoint element in it invariant.*

1.6. Characteristic subalgebras. Let $\tilde{L}$ be a semisimple L^*-algebra, and $L = K + iM$ (skew-adjoint part and self-adjoint part of L respectively) be a real form of $\tilde{L}$. K is an L^*-subalgebra of L called the characteristic subalgebra of L. If L and L_1 are L^*-isomorphic real forms of $\tilde{L}$, their characteristic subalgebras are also L^*-isomorphic. The classification will show that the converse is also true, i.e., a simple real L^*-algebra is determined by its complexification and the structure of the characteristic subalgebra. In this section, we develop some techniques which will allow us to compute the structure of the complexification of the characteristic subalgebra associated to an involution of $\tilde{L}$.

Theorem 1.6.1. *Let $\tilde{L}$ be a simple L^*-algebra, and S an involution of $\tilde{L}$. Then $\tilde{K}$ (1-eigenspace of S in $\tilde{L}$) is a maximal L^*-subalgebra of $\tilde{L}$.*

Proof. It is enough to show that K is a maximal proper L^*-subalgebra in U (simple real L^*-algebra). Suppose that K is contained properly in some L^*-subalgebra of $U = K + M$, i.e., there exists a nontrivial closed subspace M_1 in M such that $[K, M_1] \subset M_1$. If $M_2 = M_1^{\perp}$ (in M), then $[K, M_2] \subset M_2$.

$[M_1, M_2] = \{0\}$: If $a_1 \in M_1$, $a_2 \in M_2$, and $x \in K$, then we have

$$(x, [a_1, a_2]) = ([a_1^*, x], a_2) = ([x, a_1], a_2) = 0 .$$

Since $[a_1, a_2] \in K$ and x is arbitrary in K, we have $[a_1, a_2] = 0$ and they generate $[M_1, M_2] = \{0\}$.

Denote $K_i = [M_i, M_i]$ $(i = 1, 2)$, and $K_0 = (K_1 + K_2)^{\perp}$.

K_0, K_1, and K_2 are L^*-ideals in K: Since $[K, M_i] \subset M_i$ and $[M_i, M_i] \subset K_i$, we have $[K, K_i] \subset K_i$ $(i = 1, 2)$. Thus K_1 and K_2 are L^*-ideals in K together with $K_1 + K_2$. K_0 is an L^*-ideal because it is the orthogonal complement of an L^*-ideal.

$[K_0, M_1] = [K_0, M_2] = \{0\}$: If $x \in K_0$ and a, b, $\in M_1$, we have $([x, a], b) = (x, [a, b]) = 0$ $(a^* = -a)$ because $x \in K_0$ and $[a, b] \in K_1$. So $[x, a]$ is orthogonal to M_1 and belongs to M_1, it must be zero. The same for $[K_0, M_2]$.

From this we see that K_0 is an ideal in U, a simple L^*-algebra, and K_0 reduces to 0.

$(K_1, K_2) = \{0\}$: If $a_i b_i \in M_i$, then $[a_i, b_i] \in K_i$ $(i = 1, 2)$ and $([a_1, b_1], [a_2, b_2]]) = (a_1 - [a_2, [b_2, b_1]] - [b_2[b_2, a_2]]) = 0$ because $[M_1, M_2] = 0$. Thus $K = K_1 + M_2$ is a Hilbert direct sum.

Now $K_1 + M_1$ and $K_2 + M_2$ are L^*-ideals in U; hence one of them must be zero, i.e., $M_2 = \{0\}$, $K_1 = K$ and $M_1 = M$. q.e.d.

Let $\tilde{L}$ be a semisimple L^*-algebra, $\tilde{H}$ be a Cartan subalgebra, $\Delta = \{\gamma\}$ be the root system of $\tilde{L}$ with respect to $\tilde{H}$, and $\Pi = \{P_i\}$ be a system of simple roots [3]. Suppose that S is an involution of $\tilde{L}$ leaving $\tilde{H}$ invariant and inducing a particular rotation $\sigma(\sigma = S | iH$, $\sigma\Pi = \Pi$, $\tilde{H} = H + iH)$. Let $\{e_\gamma : \gamma \in \Delta\}$ be a Weyl basis [3], i.e., $\|e_\gamma\| = 1$, $e_\gamma^* = e_\gamma$, $[e_\gamma, e_{-\gamma}] = \gamma$ (we assume $\Delta \subset iH$ through the inner product), $[e_\gamma, e_\delta] = 0$ if $\gamma + \delta$ is not a root, and $[e_\gamma, e_\delta] = N_{\gamma,\delta} e_{\gamma+\delta}$ if $\gamma + \delta \in \Delta$, where the $N_{\gamma,\delta}$ are real numbers satisfying $N_{\gamma,\delta} = -N_{-\gamma\ -\delta}$

and $N_{\gamma,\gamma}^2 = \frac{1}{2}(1-p)q(\gamma,\gamma)$. Set $Se_\gamma = \nu_\gamma e_{\sigma\gamma}$, $(\gamma \in \Delta)$. It is easy to see that $|\nu_\gamma| = 1$, $\nu_\gamma \nu_{\sigma\gamma} = 1$, $\nu_{-\gamma} = \bar{\nu}_\gamma$. Setting

$$\Delta_1 = \{\alpha \in \Delta : \sigma\alpha = \alpha,\ \nu_\alpha = 1\}\ ,$$
$$\Delta_2 = \{\beta \in \Delta : \sigma\beta = \beta,\ \nu_\beta = -1\}\ ,$$
$$\Delta_3 = \{\xi \in \Delta : \sigma\xi \neq \xi\}\ ,$$

we have $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$. We denote, from now on, any root in Δ by γ, δ, μ, π, any root in Δ_1 by α, any root in Δ_2 by β, and any root in Δ_3 by ξ.

If $H = H_1 + H_{-1}$ is the decomposition of H into the (± 1)-eigenspaces of S, then $\tilde{H} = \tilde{H}_1 + \tilde{H}_{-1}$ is the corresponding decomposition of $\tilde{H}$.

Lemma 1.6.2. $iH_1 = \{h \in iH : (\xi - \sigma\xi)h = 0$ *for all* $\xi \in \Delta_3\}$.

Proof. Suppose that $(\xi, h) = (\sigma\xi, h)$ for every $\xi \in \Delta_3$, and consider the element $h - Sh$.

$$(h - Sh, \alpha) = (h, \alpha) - (Sh, \alpha) = (h, \alpha) - (h, \alpha) = 0\ , \qquad \alpha \in \Delta_1\ ,$$
$$(h - Sh, \beta) = (h, \beta) - (Sh, \beta) = (h, \beta) - (h, \beta) = 0\ , \qquad \beta \in \Delta_2\ ,$$
$$(h - Sh, \xi) = (h, \xi) - (h, S\xi) = 0\ , \qquad \xi \in \Delta_3\ .$$

Since Δ is total in iH_1, $Sh = h$ and $h \in iH_1$. The converse is clear.

Lemma 1.6.3. *For any root* $\xi \in \Delta_3$, $\xi - \sigma\xi$ *is not a root.*

Proof. Since σ leaves Π invariant, we write

$$\Pi = \{\alpha_1, \alpha_2, \cdots, \beta_1, \beta_2, \cdots, \xi_1, \sigma\xi_1, \xi_2, \sigma\xi_2, \cdots\}\ .$$

Then

$$\xi = a_1\alpha_1 + a_2\alpha_2 + \cdots + b_1\beta_1 + b_2\beta_2 + \cdots + c_1\xi_1 + c_2\xi_2 + c_3\xi_3 + c_4\sigma\xi_2 + \cdots\ ,$$

where the coefficients are integers, all nonpositive or all nonnegative.

$$\sigma\xi = a_1\alpha_1 + a_2\alpha_2 + \cdots + b_1\beta_1 + b_2\beta_2 + \cdots + c_1\sigma\xi_1 + c_2\xi_1 + c_3\sigma\xi_2 + c_4\xi_2 + \cdots\ .$$

If $\xi - \sigma\xi$ were a root, its expression in terms of the elements of Π would be

$$\xi - \sigma\xi = (c_1 - c_2)\xi_1 + (c_2 - c_1)\sigma\xi_1 + (c_3 - c_4)\xi_2 + (c_4 - c_3)\sigma\xi_2 + \cdots\ ,$$

and all the coefficients should be either nonnegative or nonpositive, i.e., $0 = c_1 = c_2 = c_3 = c_4 = \cdots$ to get $\sigma\xi = \xi$ which is a contradiction. q.e.d.

We can write for $\tilde{K}$ and $\tilde{M}$:

$$\tilde{K} = \tilde{H}_1 + \sum_{\alpha \in \Delta_1} \{e_\alpha\}_C + \sum_{\xi \in \Delta_3} \{e_\xi + \nu_\xi e_{\sigma\xi}\}_C\ ,$$

$$\tilde{M} = \tilde{H}_{-1} + \sum_{\beta \in \Delta_2} \{e_\beta\}_C + \sum_{\xi \in \Delta_3} \{e_\xi - \nu_\xi e_{\sigma\xi}\}_C \ .$$

The sum $iH = iH_1 + iH_{-1}$ is an orthogonal direct sum. Denote by $h \to h'$ the orthogonal projection onto iH_1. In other words $h' = \frac{1}{2}(h + \sigma h)$. For the roots

$$\alpha' = \alpha \ , \quad \beta' = \beta \ , \quad \xi' = \tfrac{1}{2}(\xi + \sigma\xi) \ .$$

Since iH_1 contains a regular element, $\gamma' \neq 0$ for all $\gamma \in \Delta$. In $\tilde{K}$ we have the following relationships:

$$\begin{aligned} &[h, e_\alpha] = (h, \alpha)e_\alpha \ , \qquad [e_\alpha, e_{-\alpha}] = \alpha \ , \\ &[h, e_\xi + \nu_\xi e_{\sigma\xi}] = (h, \xi)(e_\xi + \nu_\xi e_{\sigma\xi}) = (h, \xi')(e_\xi + \nu_\xi e_{\sigma\xi}) \ , \\ &[e_\xi + \nu_\xi e_{\sigma\xi}, e_{-\xi} + \nu_{-\xi} e_{-\sigma\xi}] = [e_\xi, e_{-\xi}] + \nu_\xi \nu_{-\xi}[e_{\sigma\xi}, e_{\sigma\xi}] \\ &\qquad\qquad = \xi + \sigma\xi = 2\xi' \ . \end{aligned}$$

So we can see that $\tilde{K}$ is generated by the elements

$$\begin{array}{lll} \alpha, & e_\alpha, & \alpha \in \Delta_1 ; \\ \beta, & & \beta \in \Delta_2 ; \\ \xi', & e_\gamma + \nu_\xi e_{\sigma\xi}, & \xi \in \Delta_3 ; \end{array}$$

and the derived algebra $\tilde{K}_1$ is generated by

$$\begin{array}{lll} \alpha, & e_\alpha, & \alpha \in \Delta_1 ; \\ \xi', & e_\xi + \nu_\xi e_{\sigma\xi}, & \xi' \in \Delta_3'. \end{array}$$

As we mentioned before $\tilde{H}_1 = \tilde{Z} + \tilde{H}_1'$, where $\tilde{H}_1'$ is a Cartan subalgebra of $\tilde{K}_1'$. Then the corresponding root system of $\tilde{K}_1'$ relative to $\tilde{H}_1'$ is $\Delta_1' \cup \Delta_3'$.

So in the next sections, in order to compute the structure of $\tilde{K}$ we are going to compute in each case the center $\tilde{Z}$ and a system of simple roots in $\Delta_1' \cup \Delta_3'$.

Theorem 1.6.4. *Let $\tilde{L}$ be a simple L^*-algebra, $\tilde{H}$ be a Cartan subalgebra, and S be an involution of $\tilde{L}$ leaving $\tilde{H}$ invariant. If $\sigma \neq \mathrm{id}$ (σ being the rotation induced by S), then $\tilde{K}$ is semisimple.*

Proof. The center $\tilde{Z}$ of $\tilde{K}$ is contained in $\tilde{H}_1$. The centralizer of $\tilde{Z}$ in $\tilde{L}$ contains $\tilde{K}$ and $\tilde{H}_{-1}$ which is not zero by assumption. Since $\tilde{K}$ is a maximal proper subalgebra of $\tilde{L}$ (Theorem 1.6.1), the centralizer of $\tilde{Z}$ must be $\tilde{L}$. Thus $\tilde{Z}$ is contained in the center of $\tilde{L}$ which reduces to zero, and $\tilde{K}$ is semisimple.

2. Real forms in simple complex L^*-algebras of type A

2.1. Description of L_A. Let E be a separable Hilbert space over the complex numbers, and $\{e_i \colon i \in Z\}$ be an o.n.b. (orthonormal basis) which we keep fixed throughout this section. We consider every bounded linear transformation of E into itself as a matrix, i.e., $a = (a_{ij})$ where $a_{ij} = (ae_j, e_i)$. The set L_2 of all Hilbert-Schmidt operators (bounded linear transformations $a = (a_{ij})$ such that $\sum_{i,j} |a_{ij}|^2 < \infty$) with the positive definite hermitian form $(a, b) = \sum_{i,j} a_{ij}\bar{b}_{ij}$ becomes a Hilbert space over the complex numbers (the inner product just defined is independent of the particular o.n.b. in E, [6]). Let L_A be the complex L^*-algebra arising out of L_2 by introducing $a^* = {}^t\bar{a}$, and $[a, b] = ab - ba$. L_A is a simple complex L^*-algebra of type A.

Let e_{ij} denote the element of L_A having 1 in the (i, j) entry and 0 elsewhere. The set $\{e_{ij} : i,j \in Z\}$ is an o.n.b. of L_A, and if $a = (a_{ij})$, then $a = \sum_{ij} a_{ij}e_{ij}$.

Given a Cartan subalgebra $\tilde{H}$ in L_A we can find an o.n.b. in E such that $\tilde{H}$ consists of all diagonal elements in L_A relative to the o.n.b. Conversely, given an o.n.b. in E, all diagonal elements in L_A form a Cartan subalgebra.

Let $\tilde{H}$ be a Cartan subalgebra of all diagonal elements in L_A, i.e., $\tilde{H} = \{h \in L_A : h = \sum_i h_i e_{ii}\}$. The linear functional $\lambda_i : \tilde{H} \to C$ defined by $\lambda_i(h) = h_i$ for all $h \in \tilde{H}$ is bounded, and the system of nonzero roots Δ of L_A relative to $\tilde{H}$ is:

root	*root vector*
$\lambda_i - \lambda_j = e_{ii} - e_{jj}$ $(i \neq j)$	e_{ij}

(We identify the linear functional λ_i with the element e_{ii} through the inner product.) We denote $\lambda_i - \lambda_j$ by γ_{ij} for brevity. A system of simple roots in Δ is

$$\Pi = \{\cdots, \gamma_{-n,-n+1}, \cdots, \gamma_{-1,0}, \gamma_{0,1}, \gamma_{1,2}, \cdots, \gamma_{n,n+1}, \cdots\} .$$

The following family of L^*-automorphisms will be used frequently in this section as well as the next two. If μ is a unitary operator of E, then the map $T : L_A \to L_A$ defined by $Ta = a^{-1}$ is an L^*-automorphism. We say that T is the L^*-automorphism of L_A inplemented by the unitary operator μ or simply that T is implemented by u.

Let $i \to m_i$ be a permutation of the integers, i.e., an injection of Z onto itself. The map $ue_i = e_{m_i}$ can be extended to an unitary map of E onto itself, which we denote with the same letter. The L^*-automorphism T of L_A implemented by u satisfies $T(\sum_{i,j} a_{ij}e_{ij}) = \sum_{i,j} a_{ij}e_{m_i m_j}$. T leaves $\tilde{H}$ invariant, and the induced rotation in iH (elements in $\tilde{H}$ having real entries) will be said to be "implemented by u".

2.2. Rotations. In this section we characterize the rotations in L_A.

Theorem 2.2.1. *Let σ be a rotation in L_A. Then σ or $-\sigma$ is implemented by a unitary operator of E.*

Proof. Let us study the action of σ on the system Π of simple roots mentioned in § 2.1. Suppose that $\sigma(\gamma_{01}) = \gamma_{mn}$ and $\sigma(\gamma_{12}) = \gamma_{pq}$. Since σ is a one-to-one orthogonal map, it must be either (i) $m \neq q$ and $n = p$ or (ii) $n \neq p$ or

$m = q$. In the first case we keep σ and in the second we consider $-\sigma$ to get $-\sigma(\gamma_{01}) = \gamma_{nm}$ and $-\sigma(\gamma_{12}) = \gamma_{qp}$, i.e., the second subindex of $-\sigma(\gamma_{01})$ is equal to the first subindex of $-\sigma(\gamma_{12})$. So assume

$$\sigma(\gamma_{01}) = \gamma_{mn} , \qquad \sigma(\gamma_{12}) = \gamma_{np} ,$$

and set $m_0 = m$, $m_1 = n$, $m_2 = q$. Suppose now that $\sigma(\gamma_{23}) = \gamma_{rs}$. Again, there are two possibilities, either (i) $m_1 = s$ and $m_2 \neq r$ or (ii) $m_1 \neq s$ and $m_2 = r$. In the first case we have $\sigma(\gamma_{01}) = \gamma_{m_0 m_1}$ and $\sigma(\gamma_{23}) = \gamma_{r m_1}$, which is a contradiction to the fact that σ is an orthogonal map, i.e., $(\sigma(\gamma_{01}), \sigma(\gamma_{23})) = 1$ or 2 and $(\gamma_{01}, \gamma_{23}) = 0$. So it must be the second case, and setting $s \neq m_3$ we have $\sigma(\gamma_{12}) = \gamma_{m_1 m_2}$ and $\sigma(\gamma_{23}) = \gamma_{m_2 m_3}$. Proceeding in the same way to the right of γ_{23} and to the left of γ_{01} we get a map from Z into itself $i \to m_i$ which is one-to-one and onto, Π being a system of simple roots and σ sending Δ onto Δ. Let T be the L^*-automorphism implemented by the extension of the map $u: e_i \to e_{m_i}$ to a unitary operator of E. Then $T \mid iH \equiv \sigma$. q.e.d.

Denote by τ the multiplication by -1 in iH (a rotation), by F the group of all rotations, and by G the supgroup of all rotations implemented by an unitary operator of E. Then $F = G \cup \tau G$, and G is a normal subgroup.

Suppose now that σ is an involutive rotation leaving a regular self-adjoint element $h = \sum h_i e_{ii}$ ($h_i \in R$) fixed. Since h is regular, $\gamma_{ij}(h) = h_i - h_j \neq 0$ ($i \neq j$), i.e., all the components of h are different. According to Theorem 2.2.1, either σ or $-\sigma$ is implemented by an unitary operator of E. In other words, we can find a permutation $i \to m_i$ of Z such that $\sigma e_{ii} = \pm e_{m_i m_i}$ for all $i \in Z$.

(a) σ is implement by an unitary operator of E. Then the equation $\sigma h = h$ is equivalent to $\sum_i h_i e_{ii} = \sum_i h_i e_{m_i m_i}$. Since all the components of h are different, from $h_i = h_{m_i}$ we conclude that $m_i = i$ and σ is the identity.

(b) $-\sigma$ is implemented by an unitary operator of E. Then $\sigma e_{ii} = -e_{m_i m_i}$ for all i, and $\sigma h = h$ implies that $h_i = -h_{m_i}$ for all $i \in Z$. Since all the components of h are different, at most one of them is zero and we have an infinite number of positive components as well as negative components. We distinguish two cases:

(i) One of the components of h is zero. Since $\sum_i |h_i|^2 < 0$ and all the components are different, we can assume, changing σ to be another rotation if necessary, that the components of h satisfy:

$$h_0 = 0 , \quad h_1 > h_2 > h_3 > \cdots > 0 , \quad h_{-1} < h_{-2} < h_{-3} < \cdots < 0 .$$

Then $-h_{m_i} = h_i$ ($i > 0$) implies $m_i = -i$ and $m_0 = 0$. In other words, $\sigma e_{00} = e_{00}$ and $\sigma e_{ii} = e_{-i-i}$. Thus σ (or a conjugate of σ) sends $\gamma_{i,i+1}$ into $\gamma_{-i-1,-i}$ and leaves Π invariant.

(ii) No component of h is equal to zero. As before, we can assume (changing to another rotation conjugate to σ if necessary) that the components of h satisfy:

$$h_1 > h_2 > h_3 > \cdots > 0 , \qquad h_0 < h_{-1} < h_{-2} < \cdots < 0 .$$

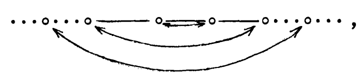
Then $h_{m_i} = -h_i$ $(i > 1)$ implies $m_i = -i + 1$ $(i > 1)$ and $\sigma e_{ii} = e_{-i+1,-i+1}$, and the action on the simple roots are $\sigma\gamma_{i,i+1} = \gamma_{-i,-i+1}$.

We can summarize this in the following.

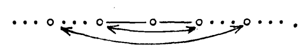
Theorem 2.2.2. *Every conjugacy class of L^*-automorphisms of L_A containing an involution has an element leaving the Cartan subalgebra $\tilde{H}$ invariant and inducing on iH one of the following involutive rotations*:

(i) $\sigma_0 = \mathrm{id}$,

(ii) $\sigma_1(\gamma_{i,i+1}) = \gamma_{-i-1,i}$



(iii) $\sigma_2(\gamma_{i,i+1}) = \gamma_{-1,i+1}$



Proof. According to Corollary 1.5.2, in each conjugacy class of L^*-automorphisms of L_A containing an involution we can select an element S leaving $\tilde{H}$ and a regular self-adjoint element in it fixed. If we denote by σ' the rotation induced by S, there exists a rotation σ, conjugate to σ', which is equal to either σ_0, σ_1 or σ_2. Since σ' and σ are conjugate, we can find a rotation θ in F such that $\sigma = \theta\sigma^1\theta^{-1}$. If T is an L^*-automorphism of L_A extending θ [11], then $S_1 = TST^{-1}$ is the required involution. q.e.d.

Now all that remains is to study the involutions which induce in iH those kinds of rotations.

2.3. L^*-automorphisms leaving $\tilde{H}$ pointwise fixed. The statement "if $T \in \mathrm{Aut}(\tilde{L})$ leaves a Cartan subalgebra $\tilde{H}$ pointwise fixed, then $T = e^{\mathrm{ad}(h)}$ for some $h \in H$" is not true for separable L^*-algebras as the following example shows.

Example. $h = \sum_j i\Pi_{2j,2j}$ is a diagonal bounded skew-hermitian operator on E. $T = e^{\mathrm{ad}(h)}$ (L^*-automorphism of L_A implemented by the unitary operator e^h) leaves $\tilde{H}$ invariant, and $Te_{i,i+1} = -e_{i,i+1}$. If an element in $\tilde{H}$ induces such an L^*-automorphism, then each component must be congruent to the corresponding component of h modulo $2\Pi i$, but then it cannot be in L_A.

We have instead the following:

Theorem 2.3.1. *If T is an L^*-automorphism of L_A leaving $\tilde{H}$ pointwise fixed, then $T = e^{\mathrm{ad}(h)}$ $(e^{\mathrm{ad}(h)}a = e^h a e^{-h})$ where h is a diagonal bounded skew-hermitian operator on E.*

Proof. Since $T|\tilde{H} = \mathrm{id}$, T leaves each one of the 1-dimensional spaces

$\{e_{ij}\}_C$ invariant (e_{ij} is a root vector of γ_{ij}). If $Te_{i,i+1} = \nu_i e_{i,i+1}$ ($i \in Z$), then the numbers $\mu_i = \operatorname{Log}(\nu_i)$ are purely imaginary complex numbers because $|\nu_i| = 1$. Set

$$h'_0 = 0 , \qquad h'_i = \mu_0 - \mu_1 - \cdots - \mu_{-1} ,$$
$$h'_{-i} = \mu_{-1} + \mu_{-2} + \mu_{-i} \qquad (i > 0) ,$$

reduce each one of them modulo $2\Pi i$, and call it h_i. Then $0 \leq |h_i| \leq 2\Pi$. Hence the element $h = \sum_i h_i e_{ii}$ is the required one, i.e.,

$$e^{\operatorname{ad}(h)} e_{i,i+1} = e^h e_{i,i+1} e^{-h} = e^{h_i - h_{i+1}} e_{i,i+1}$$
$$= e^{\mu_i \pm 2k\Pi i} e_{i,i+1} = \nu_i e_{i,i+1} .$$

T and $e^{\operatorname{ad}(h)}$ are two L^*-automorphisms of L_A, which coincide on $\tilde{H}$ and on the root-spaces corresponding to the elements of Π, so everywhere.

Corollary 2.3.1. *If R is an L^*-automorphism of L_A leaving $\tilde{H}$ invariant and inducing in iH a rotation implemented by an unitary operator U, then R itself is implemented by an unitary operator on E.*

Proof. Let T be the L^*-automorphism implemented by u, i.e., $Ta = uau^{-1}$ ($a \in L_A$). Then $T^{-1}R \mid \tilde{H} = \mathrm{id}$, and by the theorem, it is an L^*-automorphism implemented by an unitary of E, say v. Thus $Ra = (uv)a(uv)^{-1}$ for all $a \in L_A$. q.e.d.

Let S be an involutive L^*-automorphism of L_A leaving $\tilde{H}$ pointwise fixed. Then $S = e^{\operatorname{ad}(h)}$ where h is a diagonal bounded skew-hermitian operator of E. Set $h = \Pi i\phi$ ($\phi = \sum_i \phi_i e_{ii}$). Since all the components of ϕ are real numbers and S is involutive, we have

$$e^{\Pi i \operatorname{ad}(\phi)} e_{ij} = \pm e_{ij} , \qquad \phi_i - \phi_j \in Z \ (i, j \in Z) .$$

We are allowed to perform the following operations on the components of ϕ without changing the conjugacy class of S:

(i) Add or substract one and the same number to all the components of ϕ.

(ii) Reduce any components of ϕ modulo 2.

(iii) Permute the components of ϕ.

With the first two, we do not change $e^{\Pi i \operatorname{ad}(\phi)}$, and with the third, which is a rotation, we get an element conjugate to S. Thus ϕ can be reduced to the following normal forms:

$$AIII(0): \quad \phi = 0 .$$

$$AIII(n): \quad \phi = \sum_{i=1}^{n} e_{ii} \qquad (1 \leq i \leq \infty) .$$

$$AIII(\infty): \quad \phi = \sum_{i=1}^{\infty} e_{ii} .$$

Now we take each case separately, and compute the structure of the complexification of the characteristic subalgebra and the corresponding maximal abelian L^*-subalgebra in $\tilde{K}$.

Remark 2.3.2. For the rest of the paper, we use the following notation: L_A, L_B, L_C are simple L^*-algebras of types A, B, C respectively. A_n, B_n, C_n, D_n are simple n-dimensional Lie algebras of types A, B, C, D respectively. $\tilde{H}_A$, $\tilde{H}_C$, $\tilde{H}_{\mathrm{I}}$, $\tilde{H}_{\mathrm{II}}$ are Cartan subalgebras in L_A, L_C, L_B of types I, II, respectively. $\tilde{H}_{A_n}$, $\tilde{H}_{B_n}$, $\tilde{H}_{C_n}$, $\tilde{H}_{D_n}$ denote Cartan subalgebras in A_n, B_n, C_n, D_n respectively, and Π^1 a system of simple roots in $\Delta_1^1 \cup \Delta_3^1$ ($\Delta_1^1 = \Delta_1$).

AIII. $S = \mathrm{id}$, and the corresponding real form is the unique compact real form of L_A, i.e., $U = \{a \in L_A : a^* = -a\}$.

$$
\begin{aligned}
AIII(n).\quad & \phi = \sum_{i=1}^{n} e_{ii} \qquad (1 \leq i < \infty)\ ,\\
& \Delta_1 = \{\alpha_{ij} : 1 \leq i, j \leq n ; i, j < 1 ; i, j > n\}\ ,\\
& \Delta_2 = \text{all others}\ , \qquad \Delta_3 = \emptyset\ ,\\
& \tilde{Z} = \{h \in \tilde{H} : \alpha_{ij}(h) = 0,\ \alpha_{ij} \in \Delta_1\} = \{\phi\}_C\ ,\\
& \Pi^1 = \{\alpha_{i,i+1} : 1 \leq i \leq n-1\} \cup \{\alpha_{i,i+1} : i < 0 ;\\
& \qquad\qquad \alpha_{0,n+1} ; \alpha_{i,i+1} : i > n\}\ ,\\
& \tilde{H}_1^1 = \tilde{H}_{A_{n-1}} + \tilde{H}_A\ , \qquad \tilde{K} = \tilde{Z} + A_{n-1} + L_A\ .\\
AIII(\infty).\quad & \phi = \sum_{i=1}^{\infty} e_{ii}\ , \qquad \Delta_1 = \{\alpha_{i,j} : i, j < 1\ ,\quad i, j \geq 1\}\ ,\\
& \Delta_2 = \text{all others}\ , \qquad \Delta_3 = \emptyset\ ,\\
& \tilde{Z} = \{0\}, \text{ because } \phi \text{ is not a Hilbert-Schmidt operator},\\
& \Pi^1 = \{\alpha_{i,i+1} : i < 0\} \cup \{\alpha_{i,i+1} : i \geq 1\}\ ,\\
& \tilde{H}_1^1 = \tilde{H}_A + \tilde{H}_A\ , \qquad \tilde{K} = L_A + L_A\ .
\end{aligned}
$$

2.4. Involutions of L_A leaving invariant $\tilde{H}$ and inducing the rotation σ_1. Let S be such an involution. Setting $Se_{i,i+1} = \nu_i e_{-i-1,-i}$ we have $|\nu_i| = 1$ and $\nu_{-i-1} = \nu_i$ because S is an involutive unitary operator of L_A. We can assume all coefficients $\nu_i = 1$, i.e., for $j > 0$, denote $\mu_j = \mathrm{Log}\, \nu_i$ and $h_j^1 = \mu_0 + \cdots + \mu_{j-1}$. Reducing h_j^1 modulo $2\Pi i$, we get an element h_j having absolute value less than 2Π. If $h = \sum_{j=1}^{\infty} h_j e_{jj}$ (diagonal skew-hermitian bounded operator of E), then the involution $e^{\mathrm{ad}(-h)} S e^{\mathrm{ad}(h)}$ satisfies our claim and is conjugate to S. So we have $Se_{i,i+1} = e_{-i-1,i}$, and there is only one conjugacy class in $\mathrm{Aut}\,(L_A)$ containing an involution leaving $\tilde{H}$ invariant and indcing the rotation σ_1.

$$
\begin{aligned}
AI.\quad & \Delta_1 = \emptyset\ , \quad \Delta_2 = \emptyset\ , \quad \Delta_3 = \Delta\ ,\\
& \Delta_3^1 = \{\xi_{ij}^1 : \xi_{ij} \in \Delta\} = \{\xi_{ij}^1 : i, j > 0\}\ ,\\
& \Pi^1 = \{\xi_{01}^1, \xi_{12}^1, \xi_{23}^1, \cdots\}\ , \qquad 2\,|\xi_{01}^1|^2 = |\xi_{i,i+1}^1|^2\ , \quad (i > 0)\ ,\\
& \tilde{H} = \tilde{H}_I\ , \qquad \tilde{K} = L_B\ .
\end{aligned}
$$

2.5. Involutions of L_A leaving $\tilde{H}$ invariant and inducing the rotation σ_2. Let S be such an involution. Then $Se_{01} = \pm e_{01}$ and $Se_{i,i+1} = \rho_i e_{-1+1}$ $(i \neq 0)$. As before, we can have $\rho_i = 1$ $(i \neq 0)$ by changing to an involution conjugate to S if necessary. We have two possibilities:

AII. $Se_{01} = e_{01}$, $\quad Se_{i,i+1} = e_{-i,-i+1}$ $(i > 1)$,

$$\Delta_1 = \{\alpha_{ij}: i + j = 1\}, \quad \Delta_2 = \emptyset, \quad \Delta_3 = \{\xi_{ij}: i + j \neq 1\},$$
$$\Pi^1 = \{\alpha_{01}^1, \xi_{12}^1, \xi_{22}^1, \cdots\}, \quad |\alpha_{01}^1|^2 = 2\,|\xi_{i,i+1}^1|^2, \quad (i > 1),$$
$$\tilde{H}_1^1 = \tilde{H}_C, \qquad \tilde{K} = L_C.$$

AI. $Se_{01} = e_{01}$ $\quad Se_{i,i+1} = e_{-i,-i+1}$, $(i > 1)$,

$$\Delta_1 = \emptyset, \quad \Delta_2 = \{\beta_{ij}: i + j = 1\}, \quad \Delta_3 = \{\xi_{ij}: i + j \neq 1\},$$
$$\Pi^1 = \{\theta^1, \xi_{12}^1, \xi_{23}^1, \cdots\}, \quad \theta = \beta_{01} + \xi_{12}, \quad (\theta^1, \xi_{12}^1) = 0,$$
$$|\theta^1|^2 = |\xi_{i,i+1}^1|^2, \qquad (i > 1),$$
$$\tilde{H}_1^1 = \tilde{H}_{II}, \qquad \tilde{K} = L_B.$$

Remark 2.5.1. The real forms denoted by *AI* are L^*-isomorphic, and we postpone the proof until we study case *B* (observe that in this case, where we have $\tilde{K} = L_B$ and we select a Cartan subalgebra, we have two possibilities).

3. Real forms in simple complex L^*-algebras of type C

3.1. Description of L_C. Let J be an anticonjugation of E, i.e., $J(\alpha x + \beta y) = \bar{\alpha} Jx + \bar{\beta} Jy)$, $(Jx, Jy) = (y_1 x)$, $J^2 = \text{id}$, for $\alpha, \beta \in C$, $x, y \in E$. Then $L_C = \{a \in L_A: a^* = JaJ\}$ is a simple complex L^*-algebra of type C.

We can find an o.n.b. $\{e_i: i \in Z\}$, which will be fixed throughout this chapter, such that $Je_i = -sg(i)e_{-i}$ for all i; considering the elements of L_C as matrices, the condition $a^* = JaJ$ reads $a_{ij} = -sg(i)s(j)a_{-j-i}$. The diagonal elements in L_C form a Cartan subalgebra. Conversely, given any Cartan subalgebra in L_C we can find an o.n.b. having the property mentioned above with respect to J, such that all the elements in the Cartan subalgebra are precisely the diagonal elements in L_C. Let $\tilde{H}$ be the Cartan subalgebra of all diagonal elements, i.e., $\tilde{H} = \{h \in L_C: h = \sum_{i=1}^{\infty} h_i(e_{ii} - e_{-i-i})\}$. We denote $e_{ii} - e_{-i-i}$ by f_i $(i > 1)$. The linear functional $\lambda_i: \tilde{H} \to C$ defined by $\lambda_i: H \to C$ defined by $\lambda_i(h) = h_i$ is bounded, and the system Δ of nonzero roots of L_C relative to $\tilde{H}$ is:

root		*root vector*
$\lambda_i - \lambda_j = \frac{1}{2}(f_i - f_j)$	$(i \neq j)$	$e_{ij} - e_{-j-i}$
$\lambda_i + \lambda_j = \frac{1}{2}(f_i + f_j)$	$(i < j)$	$e_{i,-j} - e_{j,-i}$
$-\lambda_i - \lambda_j = -\frac{1}{2}(f_i + f_j)$	$(i < j)$	$e_{-i,j} - e_{i,-j}$

$$2\lambda_i = f_j \qquad (i > 0) \qquad e_{i,-i}$$
$$-2\lambda_i = -f_i \qquad (i > 0) \qquad e_{-i,i}$$

A system of simple roots, which will be frequently used, is

$$\Pi = \{2\lambda_1, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \cdots, \lambda_{n+1} - \lambda_n, \cdots\} .$$

3.2. Rotations. Let σ be any rotation in iH; just because σ is an orthogonal map, it permutes the roots of the form $\pm 2\lambda_i$. Define U by:

$$Ue_i = e_{m_i} , \quad Ue_{-i} = e_{-m_i} \quad \text{if } \sigma(2\lambda_i) = 2\lambda_{m_i} ,$$
$$Ue_i = e_{-m_i} , \quad Ue_{-i} = e_{m_i} \quad \text{if } \sigma(2\lambda_i) = -2\lambda_{m_i} .$$

Then U can be extended to an unitary operator of E. Let T be the L^*-automorphism of L_A implemented by U. T leaves L_C invariant, and thus its restriction to L_C, which we denote again by T, is an L^*-automorphism leaving $\tilde{H}$ invariant.

$$Tf_i = +f_{m_i} \quad \text{if } \sigma(2\lambda) = 2\lambda_{m_i} , \quad \text{or} \quad Tf_i = f_{m_i} \quad \text{if } \sigma(2\lambda_i) = -2\lambda_{m_i} .$$

Hence $T(2\lambda_i) = \sigma(2\lambda_i)$ $(i > 0)$. Since $\{2\lambda_i\}$ is an orthogonal set which expands $\tilde{H}$, we have $T|iH = \sigma$. We summarize all of these in the following.

Theorem 3.2.1. *Let σ be any rotation. Then we can find a permutation of the positive integers $\{m_1, m_2, \cdots\}$ and an L^*-automorphism implemented by an unitary operator U of E such that $T|iH = \sigma$ and $Tf_i = \pm f_{m_i}$ $(i > 0)$.*

Remark 3.2.2. In particular the map of iH onto iH which changes the sign of one of the components of every element in iH is a rotation.

Let σ be an involutive rotation leaving a regular self-adjoint element h fixed. Since h is regular, we have $2h_i \neq 0$, $h_i - h_j \neq 0$, and $h_i + h_j \neq 0$. According to Theorem 3.2.1, $\sigma f_i = \pm f_{m_i}$ for some permutation of the positive integers. So $\sigma h = h$ becomes $\sum h_i f_{m_i} = \sum h_i f_i$ and $h_i = \pm h_{m_i}$ $(i > 0)$. Hence $m_i = i$ $(i > 0)$, and σ is the identity. All this consideration together with Corollary 1.5.3 makes up the following:

Theorem 3.2.3. *Every conjugacy class of L^*-automorphisms of L_C containing an involution also has an involutive L^*-automorphisms leaving $\tilde{H}$ pointwise fixed.*

Thus all it remains to do is to study the involutions of L_C leaving $\tilde{H}$ pointwise fixed.

Theorem 3.2.4. *Let T be an L^*-automorphism of L_C leaving $\tilde{H}$ pointwise fixed. Then we can find a bounded diagonal skew-hermitian operator $h = \sum h_i f_i$ on E such that $e^{\mathrm{ad}\, h} = T$.*

Proof. The proof is completely similar to that of Theorem 2.3.1, so we omit it.

Let S be an involution of L_C leaving $\tilde{H}$ pointwise fixed. Then according to

Theorem 3.2.4 there exists a bounded diagonal skew-hermitian operator $h = \sum_{i=1}^{\infty} (\Pi i\phi_i)f_i$ such that $e^{\mathrm{ad}\, h} = S$, and we have that $2\phi_1$ and $\phi_i - \phi_{i+1}$ $(i > 0)$ are integers.

We are allowed to perform the following operations on the components of $\phi = \sum_{i=1}^{\infty} \phi_i$ without changing the conjugacy class of the involution S.

(i) Add or substruct one and the same integer to all the components of ϕ.
(ii) Reduce each component of ϕ modulo 2.
(iii) Permute the components of ϕ.
(iv) Change the sign of the components of ϕ.

So the possibilities are:

$$CI: \quad \phi = \sum_{i=1}^{\infty} \tfrac{1}{2} f_i \,,$$

$$CII(0): \quad \phi = \mathrm{id} \,,$$

$$CII(n): \quad \phi = \sum_{i=1}^{n} f_i \qquad (1 < n < \infty) \,,$$

$$CII(\infty): \quad \phi = \sum_{i=1}^{\infty} f_{2i} \,.$$

Now we take each case separately, and compute the characteristic subalgebra $\tilde{K}$ and the corresponding maximal abelian L^*-subalgebra $\tilde{H}_1$ in $\tilde{K}$.

$$CI. \quad \phi = \sum_{i=1}^{\infty} \tfrac{1}{2} f_i \,, \quad \Delta_1 = \{\lambda_i - \lambda_j : i \neq j\} \,, \quad \Delta_2 = \text{all others}, \quad \Delta_3 = \emptyset \,.$$

$$\tilde{Z} = \{0\} \,, \quad \Pi^1 = \{\lambda_1 - \lambda_2, \lambda_1 - \lambda_2, \cdots\} \,, \quad \tilde{H}_1 = \tilde{H}_A \,, \quad \tilde{K} = L_A \,.$$

$$CII(0). \quad \phi = \mathrm{id} \,, \qquad \tilde{K} = \text{unique compact real form of } L_C \,.$$

$$CII(n). \quad \phi = \sum_{i=1}^{n} f_i \,,$$

$$\Delta_1 = \{\pm 2\lambda_i : \text{all } i;\ \lambda_i - \lambda_j,\ \pm(\lambda_i + \lambda_j) : 1 < i, j < n,\ n < i, j\} \,,$$

$$\Delta_2 = \text{all others} \,, \quad \Delta_3 = \emptyset \,, \quad \tilde{Z} = \{\phi\}_C \,,$$

$$\Pi^1 = \{2\lambda_1, \lambda_2 - \lambda_1, \cdots, \lambda_n - \lambda_{n-1}\} \cup \{2\lambda_{n+1}, \lambda_{n+2} - \lambda_{n+1}, \cdots\} \,,$$

$$\tilde{H}_1^1 = \tilde{H}_{Cn} + \tilde{H}_C \,, \qquad \tilde{K} = \tilde{Z} + C_n + L_C \,.$$

$$CII(\infty). \quad \phi = \sum_{i=1}^{\infty} f_{2i} \,,$$

$$\Delta_1 = \{\pm 2\lambda_i : \text{all } i;\ \pm\lambda_{2i} \pm \lambda_{2j},\ \pm\lambda_{2i+1} \pm \lambda_{2j+1} : \text{all } i, j\} \,,$$

$$\Delta_2 = \text{all others} \,, \quad \Delta_3 = \emptyset \,, \quad \tilde{Z} = \{0\} \,,$$

$$\Pi^1 = \{2\lambda_1, \lambda_3 - \lambda_1, \cdots\} \cup \{2\lambda_2, \lambda_4 - \lambda_2, \cdots\} \,,$$

$$\tilde{H}_1^1 = \tilde{H}_C + \tilde{H}_C \,, \qquad \tilde{K} = L_C + L_C \,.$$

4. Real forms in simple L^*-algebras of type B

In this chapter, we determine the real forms of a simple L^*-algebra L_B of type B, up to L^*-isomorphisms. The proof of the fact that the different classes obtained are not L^*-isomorphic involves not only the structure of the characteristic subalgebra $\tilde{K}$ but also the different choices of maximal abelian L^*-subalgebras in $\tilde{K}$. As we mention before, we have no conjugacy theorem for Cartan subalgebras of simple L^*-algebras of type B. Indeed, it is known [14] that there are two conjugacy classes of Cartan subalgebras. Elements of different classes are not conjugate under any L^*-automorphism of L_B whatsoever.

4.1. Description of L_B. Let E be a separable complex Hilbert space, and J a conjugation of E, i.e., $J(\alpha x + \beta y) = \bar{\alpha}Jx + \bar{\beta}Jy$, $(Jx, Jy) = (y, x)$, $J^2 = \mathrm{id}$ ($\alpha, \beta \in C$, and $x, y \in E$). Let L_A be the L^*-algebra of all Hilbert-Schmidt operators on E. Then

$$L_B = \{a \in L_A \colon a^* = -JaJ\}$$

is a simple complex L^*-algebra of type B.

The two conjugacy classes of Cartan subalgebras of L_B will be referred to as of types I, II respectively.

Cartan subalgebras of type I: We can find in E an o.n.b. $\{e_i \colon i \in Z\}$, which will be fixed throughout this section every time when we consider type I, such that $Je_i = e_{-i}$ $(i \neq 0)$ and $Je_0 = e_0$. With respect to this basis, the elements of L_B are matrices $a = (a_{ij}) = \sum_{ij} a_{ij}e_{ij}$. The condition $a^* = -JaJ$ becomes $-a_{i,j} = a_{-j,-i}$. Let $\tilde{H}_I$ denote the set of all diagonal matrices in L_B. Then $\tilde{H}_I$ is a Cartan subalgebra of type I. Conversely, given any Cartan subalgebra of type I, we can find an o.n.b. as above such that the Cartan subalgebra is precisely the set of diagonal matrices. An element $h \in \tilde{H}_I$ can be written as $h = \sum h_i f_i$, where $h_0 = 0$ and $f_i = e_{ii} - e_{-i-i}$ $(i > 0)$. The linear functional $\lambda_i \colon \tilde{H}_I \to C$ defined by $\lambda_i(h) = h_i$ is bounded, and the root system Δ_I of L_B with respect to $\tilde{H}_I$ is:

root		*root vector*
$\lambda_i - \lambda_j = \frac{1}{2}(f_i - f_j)$	$(i \neq j)$	$e_{ij} - e_{-j-i}$
$\lambda_i + \lambda_j = \frac{1}{2}(f_i + f_j)$	$(i < j)$	$e_{j,-i} - e_{i,-j}$
$-\lambda_i - \lambda_j = -\frac{1}{2}(f_i + f_j)$	$(i < j)$	$e_{-i,j} - e_{-j,i}$
$\lambda_i = \frac{1}{2}f_i$	$(i > 0)$	$e_{i,0} - e_{0,-i}$
$-\lambda_i = -\frac{1}{2}f_i$	$(i > 0)$	$e_{0,i} - e_{-i,0}$

A system of simple roots is:

$$\Pi_I = \{\lambda_1, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \cdots, \lambda_m - \lambda_{m-1}, \cdots\},$$
$$2|\lambda_1| = |\lambda_i - \lambda_{i-1}|^2, \qquad (i > 1).$$

Cartan subalgebras of type II: We can find in E an o.n.b. $\{e_i\colon i \neq 0, i \in Z\}$, which will be fixed throughout this chapter every time when we consider type II, such that $Je_i = e_{-i}$ (all i). With respect to this basis, the condition $a^* = -JaJ$ becomes $a_{ij} = -a_{-j,-i}$. Let $\tilde{H}_{II}$ denote the set of all diagonal matrices in L_B. Then $\tilde{H}_{II}$ is a Cartan subalgebra of type II. Conversely, any Cartan subalgebra of type II can be expressed in this form with respect to a suitable o.n.b. of E having the above property with respect to J. An element h in $\tilde{H}_{II}$ can be written as $h = \sum_{i>0} h_i f_i$ where $f_i = e_{ii} - e_{-i,-i}$ $(i > 0)$. The linear functional $\lambda_i\colon \tilde{H}_{II} \to C$ is bounded and the root Δ_{II} system of L_B with respect to $\tilde{H}_{II}$ is:

root		*root vector*
$\lambda_i - \lambda_j = \frac{1}{2}(f_i - f_j)$	$(i \neq j)$	$e_{ij} - e_{-j,-i}$
$\lambda_i + \lambda_j = \frac{1}{2}(f_i + f_j)$	$(i < j)$	$e_{j,-i} - e_{i,-j}$
$-\lambda_i - \lambda_j = -\frac{1}{2}(f_i + f_j)$	$(i < j)$	$e_{-i,j} - e_{-j,i}$

A system of simple roots is:

$$\Pi_{II} = \{\lambda_1 + \lambda_2, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \cdots, \lambda_m - \lambda_{m-1}, \cdots\} ,$$

where all the roots have ths same length.

4.2. Rotations. We consider two cases.

Rotations in Cartan subalgebras of type I. Let σ be a rotation in iH_I. Since σ is an orthogonal linear transformation, it interchanges the roots of the form $\pm\lambda_i$. Thus we can find a permutation of the positive integers $\{m_1, m_2, \cdots\}$ such that $\sigma\lambda_i = \pm\lambda_i$. Let U be the unitary operator of E defined by:

$$\begin{aligned} &Ue_i = e_{m_i} , \quad Ue_{-i} = e_{-m_i} \quad \text{if } \sigma\lambda_i = \lambda_{m_i} , \\ &Ue_i = e_{-m_i} , \quad Ue_{-i} = e_{m_i} \quad \text{if } \sigma\lambda_i = -\lambda_{m_i} , \\ &Ue_0 = e_0 . \end{aligned}$$

The L^*-automorphism T of L_A implemented by U leaves both L_B and $\tilde{H}_1$ invariant. We have

$$Tf_i = f_{m_i} \text{ if } \sigma\lambda_i = \lambda_{m_i} , \quad \text{or} \quad Tf_i = -f_{m_i} \text{ if } \sigma\lambda_i = -\lambda_{m_i} .$$

This amounts to say that $(T \mid iH_I)\lambda_j = \sigma\lambda_j$, and one has $T \mid iH_I = \sigma$ since the elements f_i generate H_I.

We have thus proved the following

Theorem 4.2.1. *Let σ be any rotation in iH_I. Then there exist a permutation $\{m_1, m_2, \cdots\}$ of the positive integers and an L^*-automorphism T, implemented by an unitary operator of E, such that*

$$T \mid iH_I = \sigma , \quad Tf_i = \pm fm_i \qquad (i > 0) .$$

Suppose h is a regular element in iH_I. Then $\gamma(h) \neq 0$ for all $\gamma \in \Delta_I$, and the components of h are all different and different from zero. If $\sigma h = h$, then $\sum h_i f_i = \sum \pm h_i f_{m_i}$ and $h_i = \pm h_{m_i}$ $(i > 0)$. Thus $h_i = h_{m_i}$ and $i = m_i$ $(i > 0)$, and we have proved the following

Theorem 4.2.2. *A rotation in iH_I, which leaves a regular element fixed, is the identity.*

Rotations in Cartan subalgebras of type II: A root of the system Δ_{II} is expressed as a function of two linear functionals λ_i, λ_j as shown above. Thus we can denote any such a root as γ_{ij}. When it is necessary to distinguish between the two types of roots appearing in the list, we use μ_{ij} and ν_{ij}, where $\mu_{ij} = \lambda_i - \lambda_j$, $\nu_{ij} = \lambda_i + \lambda_j$. With this notation, the system of simple roots is written as:

$$\Pi_{II} = \{\nu_{12}, \mu_{21}, \mu_{32}, \cdots, \mu_{i+1,i}, \cdots\} .$$

Let σ be a rotation in iH_{II}. Like in the previous case, we shall define a permutation $\{m_1, m_2, \cdots\}$ and a unitary operator U. We consider two "consecutive" roots in Π_{II}, say $\mu_{i+1,i}$ and $\mu_{i+2,i+1}$, and let $\sigma(\mu_{i+1,i}) = \gamma_{m,n}$ and $\sigma(\mu_{i+2,i+1}) = \gamma_{p,q}$. We claim that the pairs (m, n) and (p, q) have one and only one common entry. In fact, since $(\mu_{i+1,i}, \mu_{i+2,i+1}) = -1$ and the map σ is orthogonal, we must have $(\gamma_{m,n}, \gamma_{p,q}) = -1$. Hence they have at least one common entry. One the other hand, if the set $\{m, n\} = \{p, q\}$ it can be easily checked that $(\gamma_{m,n}, \gamma_{p,q})$ is either 0 or ± 2. Hence there is only one common entry. Similarly, we can check that if μ_{12} is mapped into $\gamma_{m,n}$, then ν_{12} is mapped into some $\gamma^1_{m,n}$ (same subindexes). Thus it follows that σ maps the system Π_{II} onto the system of simple roots:

$$\Pi^1 = \{\gamma^1_{m_1,m_2}, \gamma_{m_2,m_1}, \gamma_{m_1,m_2}, \cdots\} .$$

Lemma 4.2.3. *$\{m_1, m_2, \cdots\}$ is a permutation of the positive integers.*

Proof. The above considerations show that the mapping $i \to m_i$ is well defined. Since Π^1 is again a system of simple roots, the map is onto. Again the above considerations show that m_i, m_{i+1}, m_{i+2} are all different. Since two nonconsecutive roots are orthogonal, m_{i+3} is also different from all of them. Proceeding by an easy induction, we can see that $m_i \neq m_j$ of $i \neq j$, i.e., the mapping is one-to-one. q.e.d.

Let U be the unitary operator defined on the basis $\{e_i\}$ as follows, according to the different forms of γ_{m_{i+1},m_i}: if γ_{m_{i+1},m_i} is equal to:
$\mu_{m_{i+1},m_{i+1}}$ then $Ue_i = e_{m_i}$, $Ue_{i+1} = e_{m_{i+1}}$, $Ue_{-i} = e_{-m_i}$, $Ue_{-i-1} = e_{-m_{i+1}}$;
$-\mu_{m_{i+1},m_i}$ then $Ue_i = e_{-m_i}$, $Ue_{i+1} = e_{-m_{i+1}}$, $Ue_{-i} = e_{m_i}$, $Ue_{-i-1} = e_{m_{i+1}}$;
ν_{m_{i+1},m_i} then $Ue_i = e_{-m_i}$, $Ue_{i+1} = e_{m_{i+1}}$, $Ue_{-i} = e_{m_i}$, $Ue_{-i-1} = e_{-m_{i+1}}$;
$-\nu_{m_{i+1},m_i}$ then $Ue_i = e_{m_i}$, $Ue_{i+1} = e_{-m_{i+1}}$, $Ue_{-i} = e_{-m_i}$, $Ue_{-i-1} = e_{m_{i+1}}$.

The L^*-automorphism T of L_A implemented by U leaves L_B and $\tilde{H}_{II}$ invariant, since $Tf_i = \pm f_{m_i}$. Next, we show that $T \mid iH_{II} = \sigma$. Suppose, for instance, that $\sigma\mu_{i+1,i} = -\nu_{m_{i+1},m_i}$. Then we have

$$Ue_i = e_{m_i}, \quad Ue_{i+1} = e_{-m_{i+1}}, \quad Ue_{-i} = e_{-m_i}, \quad Ue_{-i-1} = e_{m_{i+1}},$$

and so

$$Tf_i = f_{m_i}, \qquad Tf_{i+1} = -f_{m_{i+1}}.$$

Hence

$$T\mu_{i+1,i} = T(\tfrac{1}{2}(f_{i+1} - f_i)) = -\tfrac{1}{2}(f_{m_{i+1}} + f_{m_i}) = -\nu_{m_{i+1},m_i}.$$

Similarly, we can check the result in the other cases. We have thus proved the following

Theorem 4.2.4. *Let σ be any rotation in iH_I. Then there exist a permutation $\{m_1, m_2, \cdots\}$ of the positive integers and an L^*-automorphism T, implemented by an unitary operator of E, such that*

$$T \mid iH_{II} = \sigma, \quad Tf_i = \pm f_{m_i} \qquad (i > 0).$$

Now let $h \in iH_{II}$ be a regular element such that $\sigma h = h$. By the regularity we have $\gamma(h) \neq 0$ for all $\gamma \in \Delta_{II}$, i.e., at most one of the components is equal to 0. We consider two cases:

(a) $h_i \neq 0$ for all i. Then $\sigma h = h$ implies $\sum \pm h_i f_{m_i} = \sum h_i f_i$ and $\pm h_{m_i} = h_i$ for all i. Hence $h_i = h_{m_i}$ and $i = m_i$ for all i.

(b) Some $h_i = 0$. Since a permutation of the f_i's is a rotation, we may assume that $h_1 = 0$. Then $\pm h_{m_i} = h_i$ for all i implies $\sigma f_1 = \pm f_1$ and $\sigma f_i = f_i$ $(i > 1)$.

Thus we have the following

Theorem 4.2.5. *A rotation in iH_{II}, which leaves a regular element fixed, either is the identity or permutes μ_{21} and ν_{12} leaving the rest of the roots in Π_{II} fixed.*

4.3. L^*-automorphisms of L_B leaving a Cartan subalgebra invariant. According to Corollary 1.5.3., all which remains to be done is to study the involutions leaving H_I pointwise fixed and the involutions leaving $\tilde{H}_{II}$ invariant and inducing one of the rotations mentioned in Theorem 4.2.5.

Theorem 4.3.1. *Let T be an L^*-automorphism of L_B leaving $\tilde{H}_I$ or $\tilde{H}_{II}$ pointwise fixed. Then we can find a bounded diagonal skew-hermitian operator $h = \sum h_i f_i$ of E such that $e^{\mathrm{ad}(h)} = T$.*

Proof. The proof is completely similar to that of Theorem 2.3.1, so we omit it.

(I) Case of Cartan subalgebras of type I. Let S be an involutive L^*-automorphism of L_B leaving $\tilde{H}_I$ pointwise fixed. Then $S = e^{\Pi i\, ad(\phi)}$ where $\phi = \sum \phi_i f_i$ is a bounded diagonal symmetric operator of E. Since S is involutive and all the components of ϕ are real numbers, we have that ϕ_1 and $\phi_{i+1} - \phi_i$ $(i > 1)$ are integers.

We are allowed to perform the following operations on the components of ϕ without changing the conjugacy class of the involution S.

(i) Change the sign of one of the components of ϕ.

(ii) Permute the components of ϕ.

(iii) Reduce each component of ϕ modulo Z.

Thus ϕ can be reduced to the following normal forms:

$$BI(0): \quad \phi = 0 .$$

$$BI(n): \quad \phi = \sum_{i=1}^{n} f_i \qquad (n = 1, 2, \cdots) .$$

$$BI(\infty): \quad \phi = \sum_{i=1}^{\infty} f_{2i} .$$

$$DI(n): \quad \phi = \sum_{i=n}^{\infty} f_i \qquad (n = 1, 2, \cdots) .$$

Now we take each case separately, and compute the characteristic subalgebra and the corresponding maximal abelian subalgebra in $\tilde{K}$. (See Remark 2.5.2 for notation.)

$BI(0)$. $S = \mathrm{id}$, and the corresponding real form is the unique compact real form in L_B.

$$BI(n). \quad \phi = \sum_{i=1}^{n} f_i ,$$

$$\Delta_1 = \{\lambda_i \colon i > n ;\ \pm\lambda_i \pm \lambda_j \colon 1 \leq i, j \leq n \text{ or } n < i, j\} ,$$

$$\Delta_2 = \text{all others} , \qquad \Delta_3 = \emptyset .$$

For $n = 1$:

$$\tilde{Z} = \{f_1\}_C ,$$

$$\Pi^1 = \{\lambda_2, \lambda_3 - \lambda_2, \cdots\} , \quad \tilde{H}_1^1 = \tilde{H}_1 , \quad \tilde{K} = \tilde{Z} + L_B .$$

For $n > 1$:

$$\tilde{Z} = \{0\} ,$$

$$\Pi^1 = \{\lambda_1 + \lambda_2, \lambda_2 - \lambda_1, \cdots, \lambda_n - \lambda_{n-1}\} \cup \{\lambda_{n+1}, \lambda_{n+2} - \lambda_{n+1}, \cdots\} ,$$

$$\tilde{H}_1^1 = \tilde{H}_{D_n} + \tilde{H}_I , \qquad \tilde{K} = D_n + L_B .$$

$$BI(\infty). \quad \phi = \sum_{i=1}^{\infty} f_{2i} ,$$

$$\Delta_1 = \{\lambda_{2i+1}, \pm\lambda_{2i} \pm \lambda_{2j}, \pm\lambda_{2i+1} \pm \lambda_{2j+1} \colon \text{all } i, j\} \cup \{\lambda_{2i+1} \colon i > 0\} ,$$

$$\Delta_2 = \text{all others} , \quad \Delta_3 = \emptyset , \quad \tilde{Z} = \{0\} ,$$

$$\Pi^1 = \{\lambda_1, \lambda_3 - \lambda_1, \cdots\} \cup \{\lambda_2 + \lambda_4, \lambda_4 - \lambda_2, \cdots\} ,$$

$$\tilde{H}_1^1 = \tilde{H}_I + \tilde{H}_{II} , \qquad \tilde{K} = L_B + L_B .$$

$$DI(n). \quad \phi = \sum_{i=n}^{\infty} f_i \qquad (n = 1, 2, \cdots),$$

$$\Delta_1 = \{\lambda_i : i < n; \pm\lambda_i \pm \lambda_j : 1 < i, j < n \text{ or } n < i, j\},$$

$$\Delta_2 = \text{all others}, \quad \Delta_3 = \emptyset, \quad \tilde{Z} = \{0\},$$

$$\Pi^1 = \{\lambda_1, \lambda_2 - \lambda_1, \cdots, \lambda_{n-1} - \lambda_{n-2}\} \cup \{\lambda_n + \lambda_{n+1}, \lambda_{n+1} - \lambda_n, \cdots\},$$

$$\tilde{H}_1^1 = \tilde{H}_{B_{n-1}} + \tilde{H}_{II}, \qquad \tilde{K} = B_{n-1} + L_B.$$

(IIa) Let S be an involution leaving $\tilde{H}_{II}$ pointwise fixed. Then $S = e^{\Pi i \operatorname{ad}(\phi)}$ and, as before, $\phi_2 + \phi_1, \phi_2 - \phi_1, \phi_3 - \phi_2, \cdots$ in Z. We are allowed to perform the following operations on the components of ϕ changing the conjugacy class of the involution S.

(i) Add or substract one and the same integer to every component of ϕ.

(ii) Reduce each component of ϕ modulo z.

(iii) Permute the components of ϕ.

(iv) Change the sign of any component ϕ.

Thus the possibilities are:

$$BI(0): \quad \phi = 0,$$

$$BI(n): \quad \phi = \sum_{i=1}^{n} f_i,$$

$$BI(\infty): \quad \phi = \sum_{i=1}^{\infty} f_{2i} + 1,$$

$$DII: \quad \phi = \sum_{i=1}^{\infty} \tfrac{1}{2} f_i.$$

Now we take each case separately.

$BI(0)$. $S = \text{id}$, and the corresponding real form is the unique compact real form in L_B.

$$BI(n). \quad \phi = \sum_{i=1}^{n} f_i \qquad (n = 1, 2, \cdots),$$

$$\Delta_1 = \{\pm\lambda_i \pm \lambda_j : 1 \leq i, j \leq m \text{ or } i, j > m\},$$

$$\Delta_2 = \text{all others}, \qquad \Delta_3 = \emptyset.$$

For $n = 1$:

$$\tilde{Z} = \{f_1\}_C, \qquad \Pi^1 = \{\lambda_2 + \lambda_3, \lambda_3 - \lambda_2, \cdots\},$$

$$\tilde{H}_1^1 = \tilde{H}_{II}, \qquad \tilde{K} = \tilde{Z} + L_B.$$

For $n > 1$:

$$\tilde{Z} = \{0\},$$

$$\Pi^1 = \{\lambda_1 + \lambda_2, \lambda_2 - \lambda_1, \cdots, \lambda_n - \lambda_{n-1}\}$$
$$\cup \{\lambda_{n+1}\lambda_{n+2}, \lambda_{n+2} - \lambda_{n+1}, \cdots\} ,$$
$$\tilde{H}_1^1 = \tilde{H}_{D_n} + \tilde{H}_{II} , \qquad \tilde{K} = D_n + L_B .$$

$BI(\infty)$. $\phi = \sum_{i=1}^{\infty} f_{2i+1}$, $\Delta_1 = \{\pm\lambda_{2i} \pm \lambda_{2j}, \pm\lambda_{2i+1} \pm \lambda_{2j+1}\text{: all } i, j\}$,

$$\Delta_2 = \text{all others} , \qquad \Delta_3 = \emptyset ,$$
$$\Pi^1 = \{\lambda_1 + \lambda_3, \lambda_3 - \lambda_1, \lambda_5 - \lambda_3, \cdots\}$$
$$\cup \{\lambda_2 + \lambda_4, \lambda_4 - \lambda_2, \lambda_6 - \lambda_4, \cdots\},$$
$$\tilde{H}_1^1 = \tilde{H}_{II} + \tilde{H}_{II} , \qquad \tilde{K} = L_B + L_B .$$

DII. $\phi = \sum_{i=1}^{\infty} \frac{1}{2} f_i$, $\Delta_1 = \{\lambda_i - \lambda_j\text{: all } i, j\}$, $\Delta_2 = \text{all others}$,

$$\Delta_3 = \emptyset , \quad \tilde{Z} = \{0\} , \quad \Pi^1 = \{\lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \cdots\} ,$$
$$\tilde{H}_1^1 = \tilde{H}_A , \quad \tilde{K} = L_A .$$

(IIb) Let σ be a rotation leaving the system $\Pi_{II} = \{\rho_1, \rho_2, \cdots\}(\rho_1 = \nu_{12}, \rho_2 = \mu_{21}, \cdots)$ invariant and defined by $\sigma_1 = \rho_2$, $\sigma\rho_2 = \rho_1$, $\sigma\rho_i = \rho_i$ $(i > 2)$. Let e_{ρ_i} be a root vector corresponding to the root ρ_i, and denote by S_ρ the involution of L_B defined by $S_\sigma e_{\rho_i} = e_{\sigma\rho_i}$ (all i) and $S_\sigma | iH_{II} = \sigma$, [11]. Let S be any involution of L_B leaving $\tilde{H}_{II}$ invariant and $S | H_{II} = \sigma$. Then $Se_{\rho_i} = \nu_i e_{\sigma\rho_i}$. Since $\sigma\rho_1 = \rho_2$, we can assume (as in § 2.4) that $\nu_1 = \nu_2 = 1$. Now $SS_\sigma = S_\sigma S$ is an involution leaving $\tilde{H}_{II}$ pointwise fixed; hence

$$SS_\sigma = e^{\Pi i \,\mathrm{ad}(\phi)} ,$$

where we can assume $h_1 = h_2 = 0$ and $h_i = 0$ or 1 for $i > 2$.

Thus the possibilities are:

$BI(\infty)$. $\phi = \sum_{i=1}^{\infty} f_{2i+1}$,

$DI(n)$. $\phi = \sum_{n+1}^{\infty} f_i$ $\qquad (n = 2, 3, \cdots)$,

$DI(1)$. $\phi = 0$.

Now we take each case separately.

$BI(\infty)$. $\phi = \sum_{i=1}^{\infty} f_{2i+1}$, $\Delta_1 = \{\pm\lambda_{2i} \pm \lambda_{2j}, \pm\lambda_{2i+1} \pm \lambda_{2i+1}\text{: } i, j \geq 1\}$,

$$\Delta_2 = \{\pm\lambda_{2i} \pm \lambda_{2j+1}\text{: } i, j \geq 1\} , \qquad \Delta_3 = \{\pm\lambda_i \pm \lambda_i\text{: } i > 1\} ,$$
$$\Delta_3^1 = \{\pm\lambda_i\text{: } i > 1\} , \quad \Pi^1 = \{\lambda_2, \lambda_4 - \lambda_2, \cdots\} \cup \{\lambda_3, \lambda_5 - \lambda_3, \cdots\} ,$$
$$\tilde{H}_1^1 = \tilde{H}_I + \tilde{H}_I , \qquad \tilde{K} = L_B + L_B .$$

$$DI(n).\quad \phi = \sum_{n+1}^{\infty} f_i \qquad (n = 2, 3, \cdots)\,,$$
$$\Delta_1 = \{\pm\lambda_i \pm \lambda_j : 2 \le i, j \le n \text{ or } n+1 \le i, j\}\,,$$
$$\Delta_2 = \{\pm\lambda_i \pm \lambda_j : 1 \le i \le n+1 \text{ and } n+1 \le j\}\,,$$
$$\Delta_3 = \{\pm\lambda_1 \pm \lambda_i : i > 1\}\,, \qquad \Delta_3^1 = \{\pm\lambda_i : i > 1\}\,,$$
$$\Pi^1 = \{\lambda_2, \lambda_3 - \lambda_2, \cdots, \lambda_n - \lambda_{n-1}\} \cup \{\lambda_{n+1}, \lambda_{n+2} - \lambda_{n+1}, \cdots\}\,,$$
$$\tilde{H}_1^1 = \tilde{H}_{B_{n-1}} + \tilde{H}_I\,, \qquad \tilde{K} = B_{n-1} + L_B\,.$$
$$DI(1).\quad \phi = 0 \quad (S = S_\sigma)\,, \quad \Delta_1 = \{\pm\lambda_i \pm \lambda_j : i, j > 1\}\,, \quad \Delta_2 = \emptyset\,,$$
$$\Delta_3 = \{\pm\lambda_1 \pm \lambda_i : i > 1\}\,, \qquad \Delta_3^1 = \{\pm\lambda_i : i > 1\}\,,$$
$$\Pi^1 = \{\lambda_2, \lambda_3 - \lambda_2, \cdots\}\,, \quad \tilde{H}_1^1 = \tilde{H}_I\,, \quad \tilde{K} = L_B\,.$$

Remark 4.3.2. In (I), (IIa), (IIb) (§ 4.3) and in § 2.5 we have used the same notation (e.g., $BI(n)$, $DI(1)$, $\cdots$) to denote certain real forms which are obtained in a different manner. We shall now prove that they are actually L^*-isomorphic to each other. For instance, let us show that the real forms of type $BI(\infty)$ given in (I), (IIa) and (IIb) are L^*-isomorphic. In general, if S is an involution of L_B and $\tilde{H}_1$ is a maximal abelian L^*-subalgebra of $\tilde{K}$ (1-eigenspace of S in L_B), then we can find an L^*-automorphism T such that $TST^{-1} = S^1$ is one of the involutions listed in (I), (IIa) and (IIb), and $T\tilde{H}_1$ is the corresponding maximal abelian L^*-algebra to S^1:

Consider a real form L of L_B such that $L = K + M$ and $\tilde{K} + L_B + L_B$. By taking a Cartan subalgebra in each simple component of $\tilde{K}$, we can select $\tilde{H}_1$ to be one of the following three non-conjugate Cartan subalgebras:

$$\text{(i)}\quad \tilde{H}_I + \tilde{H}_I\,, \qquad \text{(ii)}\quad \tilde{H}_I + \tilde{H}_{II}\,, \qquad \text{(iii)}\quad \tilde{H}_I + \tilde{H}_{II}\,.$$

Let S be the involutive L^*-automorphism of L_B associated to L, and let us take $\tilde{H}_1$ to be of type (i). It is impossible to the L^*-automorphism mentioned above such that TST^{-1} is either one of the involutions in (I) or (IIa), because in either case we get $T\tilde{H}_I = \tilde{H}_{II}$ which is a contradiction. So there is only one case left, and L is L^*-isomorphic to the real form of type $BI(\infty)$ in II(b).

Similarly, taking $\tilde{H}_1$ to be of type (ii) we can show that L is L^*-isomorphic to the real form of type $BI(\infty)$ on (I); and if $\tilde{H}_1$ is taking to be of type (iii), then L is L^*-isomorphic to the real form of type $BI(\infty)$ in II(a).

As a result of the above considerations, we obtain the following

Theorem 4.3.3. *Two real forms of a simple complex L^*-algebra are L^*-isomorphic if and only if the corresponding characteristic subalgebras are L^*-isomorphic.*

5. Summary of the results

Let $\underline{E}$ be a separable Hilbert space, and $\Phi = \{e_i\}$ be an o.n.b., which we are going to reorder in different ways according to the case under consideration. gl $(\infty, \boldsymbol{C})_2$, the set of all Hilbert-Schmidt operators of E, is a simple complex L^*-algebra of type A. o $(\infty, \boldsymbol{C})_2 = \{a \in \mathrm{gl}\,(\infty, \boldsymbol{C})_2 \colon {}^t a = -a\}$ is a simple complex L^*-algebra of type B. Let $\Phi = \{e_{-1}, e_{-2}, \cdots, e_1, e_2, \cdots\}$ and $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, i.e., J is the bounded operator of E defined by $Je_{-i} = -e_i$ and $Je_i = e_{-i}$. Then sp $(\infty, \boldsymbol{C})_2 = \{a \in \mathrm{gl}\,(\infty, \boldsymbol{C})_2 \colon {}^t aJ + Ja = 0\}$ is a simple complex L^*-algebra of type C. We note that in this case we can turn E into a right vector space over K ($K = \{1, i, j, ij\}_K$, the algebra of quaternions) by defining the action of j by $xj = J\bar{x}$ for all $x \in E$; an o.n.b. of E over K is $\{e_1, e_2, \cdots\}$. An element $a \in \mathrm{gl}\,(\infty, \boldsymbol{C})_2$ is K-linear if and only if $J\bar{a} = aJ$, i.e., if a is of the form $\begin{bmatrix} a_1 & a_2 \\ -\bar{a}_2 & \bar{a}_1 \end{bmatrix}$, and when this is so, we shall use the matrix expression of a given by $a_1 + a_2 j$, in other words, as a linear operator of E over K. We denote by gl $(\infty, K)_2$ the set of all K-linear operators in gl $(\infty, \boldsymbol{C})_2$.

The *simple separable real L^*-algebras having a complex structure* are the real L^*-algebras obtained from gl $(\infty, \boldsymbol{C})_2$, o $(\infty, \boldsymbol{C})_2$ and sp $(\infty, \boldsymbol{C})_2$ by restriction of scalars.

The *compact simple separable real L^*-algebras* are

$$u(\infty, C)_2 = \{a \in \mathrm{gl}\,(\infty, C)_2 \colon a^* = -a\}\ ,$$
$$\mathrm{o}\,(\infty, R)_2 = \{a \in \mathrm{o}\,(\infty, C)_2 \colon a^* = -a\}\ ,$$
$$u(\infty, K)_2 = \{a \in \mathrm{gl}\,(\infty, K)_2 \colon {}^t\bar{a} + a = 0\}\ ,$$

where $\bar{x} = x_0 - x_1 i - x_2 j - x_3 ij$, if $x = x_0 + x_1 i + x_2 j + x_3 ij$ in K.

In the following, $\tilde{L}$ will denote a simple complex L^*-algebra, S an involutive L^*-automorphism of $\tilde{L}$, and L the real form of $\tilde{L}$ associated to S or a real form of $\tilde{L}$ conjugate to L.

The *noncompact simple separable real L^*-algebras* are

(a) $\Phi = \{e_1, e_2, \cdots, e_n, \cdots\}$, $K_n = \begin{bmatrix} -I_n & 0 \\ 0 & I \end{bmatrix}$.

$\underline{AI}$. $\tilde{L} = \mathrm{gl}\,(\infty, \boldsymbol{C})_2\ , \qquad Sa = -{}^t a\ ,$

$L = \mathrm{gl}\,(\infty, R)_2 =$ all real matrices in gl $(\infty, C)_2$.

$\underline{AIII(n)}$. $\tilde{L} = \mathrm{gl}\,(\infty, C)_2\ , \qquad Sa = K_n a K_n^{-1}\ ,$

$L = u(n, \infty)_2 = \{a \in \mathrm{gl}\,(\infty, C)_2 \colon {}^t\bar{a}K_n + K_n a = 0\}$.

$\underline{BDI(n)}$. $\tilde{L} = \mathrm{o}\,(\infty, C)_2\ , \qquad Sa = K_n a K_n^{-1}\ ,$

$L = \mathrm{o}\,(n, \infty)_2 = \{a \in \mathrm{gl}\,(\infty, R)_2 \colon {}^t a K_n + K_n a = 0\}$.

(b) $\Phi = \{e_{-1}, e_{-2}, \cdots, e_1 e_2, \cdots\}$, $K_\infty = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$.

$\underline{AIII(\infty)}$. $\tilde{L} = \mathrm{gl}\,(\infty, C)_2$, $Sa = K_\infty a K_\infty$, $L = u(\infty, \infty)_2$.

$\underline{BDI(\infty)}$. $\tilde{L} = \mathrm{o}\,(\infty, C)_2$, $Sa = K_\infty a K_\infty^{-1}$, $L = \mathrm{o}\,(\infty, \infty)_2$.

(c) $\Phi = \{e_{-1}, e_{-2}, \cdots, e_1, e_2, \cdots\}$, $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

$\underline{AII}$. $\tilde{L} = \mathrm{gl}\,(\infty, C)_2$, $Sa = -J{}^t a J^{-1}$, $\tilde{L} = \mathrm{gl}\,(\infty, K)_2$,

$\underline{CI}$. $\tilde{L} = \mathrm{sp}\,(\infty, C)_2$ $Sa = \bar{a}$,

$L = \mathrm{sp}\,(\infty, R)_2 =$ all real matrices in $\mathrm{sp}\,(\infty, C)_2$.

(d) $\Phi = \{e_{-1}, e_{-2}, \cdots, e_1, e_2, \cdots\}$, $K_{n,n} = \begin{bmatrix} K_n & 0 \\ 0 & K_n \end{bmatrix}$.

$\underline{CII(n)}$. $\tilde{L} = \mathrm{sp}\,(\infty, C)_2$, $Sa = K_{n,n} a K_{n,n}^{-1}$,

$L = u(n, \infty, K) = \{a \in \mathrm{gl}\,(\infty, K) \colon {}^t a K_n + K_n a = 0\}$,

where K_n is the operator of E over K defined by $K_n e_i = -e_i$ $(1 \le i \le n)$ and $K_n e_i = e_i$ $(i > n)$.

(e) $\Phi = \{e_{-1}, e_{-3}, \cdots, e_{-2}, e_{-4}, \cdots, e_1, e_3, \cdots, e_2, e_4, \cdots\}$,

$K_{\infty,\infty} = \begin{bmatrix} K_\infty & 0 \\ 0 & K_\infty \end{bmatrix}$.

$\underline{DIII}$. $\tilde{L} = \mathrm{o}\,(\infty, C)_2$, $Sa = JaJ^{-1}$,

$L = \mathrm{o}\,(\infty, K)_2 = \{a \in \mathrm{gl}\,(\infty, K)_2 \colon {}^t\tilde{a} + a = 0\}$,

where $\tilde{x} = x_0 + x_1 i - x_2 j + x_3 ij$, if $x = x_0 + x_1 i + x_2 j + x_3 ij$ in K.

$\underline{CII(\infty)}$. $\tilde{L} = \mathrm{sp}\,(\infty, C)_2$, $Sa = K_{\infty,\infty} a K_{\infty,\infty}^{-1}$, $L = u(\infty, \infty, K)_2$ (see $CII(n)$).

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