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A CONDITION FOR PARACOMPACTNESS OF A MANIFOLD

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1. Introduction

It is known that if a differential manifold M is paracompact, then it can be made into a Riemannian manifold with a unique torsion-free Levi-Civita connection. In discussing the structure of Minkowski spaces (see [5]), the author came across a condition for paracompactness of a manifold. This condition is stated and proved as Theorem 1, which is the main result of this paper. We begin by introducing some geometrical preliminaries.

2. Geometrical preliminaries

By a differentiable n-dimensional manifold of class C^r, we mean a Hausdorff connected locally Euclidean topological space with a fixed C^r atlas. We assume r to be large enough to ensure the smoothness of the operations involved. By a pseudo-Riemannian manifold, we mean a manifold with fundamental tensor of arbitrary signature (definite or indefinite). Let L denote the principal fibre bundle of linear frames on M with structure group $G = GL(n, R)$, and let H be the closed subgroup of G which leaves a given nondegenerate quadratic form on $\boldsymbol{R}^n$ invariant. Expressing $x \in \boldsymbol{R}^n$ in terms of its natural basis, we can write the quadratic form $Q: R^n \to R$ as

$$Q(x) = a_{ij}x^i x^j ,$$

where $x = (x^1, \cdots, x^n) \in \boldsymbol{R}^n$, $a_{ij} \in R$, and summation convention is used. Consider the action of G on $L \times G/H$, given by

$$a \cdot (l, \xi) = (a \cdot l, \xi \cdot a^{-1}) \in L \times G/H$$

for $a \in G$ and $(l, \xi) \in L \times G/H$, where a acts on the frame l by acting on each vector in the frame and G/H is regarded as a right coset space.

The quotient space of $L \times G/H$ under this action of G is denoted by $E(M, G/H, G, L)$ or E for short. The map $L \times G/H \to L \to M$ induces the map $\pi_E : E \to M$, and a differential structure is introduced in E in a natural manner by using π_E (see [4]). The surjective map $(l, \xi) \mapsto \xi \cdot l$ of $L \times G/H$ onto L/H

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factors through E, and allows us to identify E with L/H. Consequently L can be regarded as a fibre bundle over E with structure group H. Let $\gamma: L \to E = L/H$ be the natural projection. We are now in a position to state the main result as

Theorem 1. *Let L be the principal fibre bundle of linear frames over an n-dimensional real differentiable manifold with structure group G, H be the closed subgroup of G which leaves invariant a given nondegenerate quadratic form on R, and $E(M, G/H, G, L)$ be the associated bundle of L with fibre G/H. Then M is paracompact if E admits a cross-section.*

3. Proof of Theorem 1

The proof is divided into several lemmas. We omit the proofs of Lemmas 1 and 2 as they follow easily from the standard constructions (see, for example, [4]).

Lemma 1. *Let $\sigma: M \to E$ be a cross-section of E. Then there exists a unique (depending on σ) reduced subbundle P of L with H as its structure group.*

Lemma 2. *There exists a unique torsion-free connection in the bundle P which makes M into a pseudo-Riemannian space with fundamental tensor induced by the quadratic form Q.*

Lemma 3. *L can be made into a Riemannian manifold and hence is paracompact.*

Proof. Using the pseudo-Riemannian structure on M and its Levi-Civita connection, we obtain the Cartan differential forms on L denoted by θ_i, W_{ij} where $i, j = 1, \cdots, n$. These forms are linearly independent and make L globally parallelizable. Using classical notation we can make L into a Riemannian manifold with "metric" given by

$$ds^2 = \sum_i \theta_i^2 + \sum_{ij} W_{ij}^2 .$$

Thus L is a metric space and hence paracompact by A. H. Stone's theorem.

Lemma 4. *M is paracompact.*

Proof. Since M is connected, L has at most two connected components, open and closed in L, and therefore it is sufficient to restrict our considerations to a component of L, say L'. Clearly L' is locally compact and paracompact, and hence can be written as a countable union of compact sets K_n such that K_n is contained in the interior of K_{n+1} (see, for example, [1, Chapter I, § 5, Theorem 5, p. 107]). Also, each K_n is metrizable, and therefore $\pi(K_n)$ is also metrizable, where π is the restriction to L' of the projection of L onto M. (For a proof, see, for example, [2, Chapter IX, § 2, Proposition 17, p. 44). Since π is an open mapping, $\pi(K_n)$ is contained in the interior of $\pi(K_{n+1})$; this implies that M, which is the union of $\pi(K_n)$, is metrizable (see [3, (12.4.7), p. 13]) and hence paracompact.

Lemmas 3 and 4 lead to the following corollaries which characterize the paracompactness of M.

Corollary 1. *M is paracompact if and only if L admits a connection.*

Proof. Cartan forms can be constructed when a connection on L is given, and the remaining parts of Lemma 3 and Lemma 4 now go through.

As a special case of Corollary 1 we have the following:

Corollary 2. *M is paracompact if and only if it admits a pseudo-Riemannian structure.*

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