# THE CLASSIFICATION OF DOUBLY PERIODIC MINIMAL TORI WITH PARALLEL ENDS

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## Abstract

Let $\mathcal{K}$ be the space of properly embedded minimal tori in quotients of $\mathbb{R}^3$ by two independent translations, with any fixed (even) number of parallel ends. After an appropriate normalization, we prove that $\mathcal{K}$ is a 3-dimensional real analytic manifold that reduces to the finite coverings of the examples defined by Karcher, Meeks and Rosenberg in [**9**, **10**, **15**]. The degenerate limits of surfaces in $\mathcal{K}$ are the catenoid, the helicoid and three 1-parameter families of surfaces: the simply and doubly periodic Scherk minimal surfaces and the Riemann minimal examples.

## 1. Introduction

In 1988, Karcher [**9**], defined a 1-parameter family of minimal tori in quotients of $\mathbb{R}^3$ by two independent translations. Each of these surfaces, called a *toroidal halfplane layer,* has four parallel Scherk-type ends in its smallest fundamental domain, is invariant by reflection symmetries in three orthogonal planes and contains four parallel straight lines through the ends that project orthogonally onto the corners of a planar rectangle contained in one of the reflective symmetry planes, and the edges of this rectangle are just the half-period vectors of the surface, see Figure 3 left. Using this richness of symmetries of the examples, he was able to give explicitly the Weierstrass representation of the toroidal halfplane layers in terms of elliptic functions on a 1-parameter family of rectangular tori (these examples will be denoted as $M_{\theta,0,0}$ with $0 < \theta < \frac{\pi}{2}$ in Section 3). Inside a brief remark in this paper [**9**], and later in another work [**10**], Karcher described two distinct 1-parameter deformations of each $M_{\theta,0,0}$ with smaller symmetry group (denoted by $M_{\theta,\alpha,0}, M_{\theta,0,\beta}$ in Section 3). In 1989, Meeks and Rosenberg [**15**], developed a general theory for doubly periodic minimal surfaces with finite topology in the quotient,

and used a completely different approach to find again the examples $M_{\theta,0,\beta}$ (although up to now, it was not clear that Meeks and Rosenberg's examples were the same as Karcher's). In fact, it is not difficult to produce a 3-parameter family of examples $M_{\theta,\alpha,\beta}$ containing all the above examples, see Section 3.

Hauswirth and Traizet [5], have extended previous ideas by Pérez and Ros [20] and Pérez [18], to study the moduli space of all properly embedded doubly periodic minimal surfaces with a given fixed finite topology in the quotient, proving that in the parallel ends case (resp. non-parallel) and after identifying by translations, homotheties and rotations, this moduli space is a real analytic manifold of dimension 3 (resp. 1) around a *non-degenerate* surface. The non-degeneracy condition concerns the space of infinitesimal deformations of a given surface; we will see in Section 3 that each example $M_{\theta,\alpha,\beta}$ is non-degenerate. All these facts point out to a possible global uniqueness of the four ended surfaces $M_{\theta,\alpha,\beta}$ and their $k$-sheeted coverings (here, $k$ is any positive integer), which in the sequel will be referred to as *standard examples,* among all properly embedded doubly periodic minimal surfaces with genus one and any number of parallel ends in the quotient. In this paper, we solve this uniqueness problem in the affirmative.

**Theorem 1.** *If $M$ is a properly embedded doubly periodic minimal surface with genus one in the quotient and parallel ends, then $M$ is a standard example.*

**Remark 1.** The theorem does not hold if we remove the hypothesis on the ends to be parallel, as demonstrated in the 4-ended tori discovered by Hoffman, Karcher and Wei in [6].

The analysis of the space of standard examples $\mathcal{K}$ with $4k$ parallel ends can be obviously reduced to the case $k = 1$ by taking $k$-sheeted coverings. In the four ended case, the space $\mathcal{K}_1 = \{M_{\theta,\alpha,\beta}\}_{\theta,\alpha,\beta}$ of standard examples is studied in detail in [23]. $\mathcal{K}_1$ is a 3-dimensional real analytic manifold and the degenerate limits of sequences in $\mathcal{K}_1$ are the catenoid, the helicoid, any singly or doubly periodic Scherk minimal surface and any Riemann minimal example (see for instance [14] for a description of these last surfaces). Furthermore, $\mathcal{K}_1$ is self-conjugate, in the sense that the conjugate surface (Two minimal surfaces $M_1, M_2 \subset \mathbb{R}^3$ are *conjugate* if the local coordinate functions of $M_2$ are harmonic conjugate to the coordinate functions of $M_1$.) of any element in $\mathcal{K}_1$ also belongs to $\mathcal{K}_1$. Since any standard example admits an orientation reversing involution without fixed points, Theorem 1 also gives a classification of all properly embedded minimal Klein bottles with parallel ends in doubly periodic quotients of $\mathbb{R}^3$.

The cyclic covering of any minimal surface satisfying the hypotheses of Theorem 1, associated to the homology class generated by a loop around one of its ends, produces a properly embedded minimal surface in the quotient of $\mathbb{R}^3$ by a translation, with genus zero and infinitely many Scherk-type ends (thus with infinite total curvature). Hence, Theorem 1 can be viewed as a first step toward the classification of all properly embedded minimal surfaces of genus zero in the complete flat 3-manifold $\mathbb{R}^2 \times \mathbb{S}^1$. Concerning this open problem and except for the flat examples (which reduce to the plane $\mathbb{R}^2 \times \{\theta\}$ and the cylinder $\mathbb{R} \times \mathbb{S}^1$), all the simple ends of any such surface are simultaneously asymptotic to ends of helicoids or to ends of Scherk minimal surfaces. In the finite topology case, the number of ends is necessarily even and the helicoid is known to be the unique genus zero example with a finite number of helicoidal ends (Pérez and Ros [19]), and it was conjectured that for any integer $k \geq 2$, the only examples with genus zero and $2k$ Scherk-type ends are the $(2k-3)$-parameter family of saddle towers having as building blocks the conjugate surface of a Jenkins–Serrin graph over the right triangle which, after symmetrization, generates a convex $2k$-gon with all edges of the same length (in particular, for $k = 2$ the only examples are the singly-periodic Scherk minimal surfaces). The validity of this conjecture is proved by Pérez and Traizet in [22].

The proof of Theorem 1 is a modified application of the machinery developed by Meeks, Pérez and Ros in their characterization of the Riemann minimal examples [14], hence, this reference could be a helpful source to the reader. For $k \in \mathbb{N}$ fixed, one considers the space $\mathcal{S}$ of properly embedded doubly periodic minimal surfaces of genus one in the quotient and $4k$ parallel ends. The goal is to prove that $\mathcal{S}$ reduces to the space $\mathcal{K}$ of standard examples (i.e. $k$-sheeted coverings to the surfaces $M_{\theta,\alpha,\beta} \in \mathcal{K}_1$). It is easy to check that $\mathcal{K}$ is open and closed in $\mathcal{S}$ and forms a connected component of $\mathcal{S}$. Thus, its remains to prove that $\mathcal{S}$ is connected. The argument is based on modeling $\mathcal{S}$ as an analytic subset in a complex manifold $\mathcal{W}$ of finite dimension (roughly, $\mathcal{W}$ consists of all admissible Weierstrass data for our problem). Then, the procedure has three steps:

- *Properness:* Uniform curvature estimates are proven for a sequence of surfaces in $\mathcal{S}$ constrained to certain natural normalizations in terms of the period vector at the ends and of the flux of these surfaces (this flux will be defined in Section 4).
- *Openness:* Any surface in $\mathcal{S} - \mathcal{K}$ can be minimally deformed by moving its period at the ends and its flux. This step depends on the properness part and both together imply, assuming $\mathcal{S} - \mathcal{K} \neq \emptyset$

(the proof of Theorem 1 is by contradiction), that any period at the ends and flux can be achieved by non-standard examples.

- *Uniqueness around a boundary point of $\mathcal{S}$:* Only standard examples can occur nearby a certain minimal surface outside of $\mathcal{S}$ which is obtained as a smooth limit of surfaces in $\mathcal{S}$.

The uniqueness property in the third point together with the last sentence in the openness point lead directly to the desired contradiction, thereby proving Theorem 1.

Although the above strategy is adapted from the one in [**14**], there are several major differences between that work and ours, among which we would like to emphasize two. The main tool in our proof is a detailed study of the map $C$ that associates to each $M \in \mathcal{S}$ two geometric invariants: its period at the ends and its flux along a non-trivial homology class with vanishing period vector. The first main difference between this work and [**14**] is that $C$ is not proper, in contrast with the properness of the flux map in [**14**]. Fortunately, $C|_{\mathcal{S}-\mathcal{K}}$ becomes proper (recall we assumed $\mathcal{S} - \mathcal{K} \neq \emptyset$). This restricted properness follows from curvature estimates as in the first step of the above procedure, together with a local uniqueness argument similar to the third step, performed around any singly periodic Scherk minimal surface, considered as a point in the boundary $\partial \mathcal{S}$. In fact, to describe the complete list of limits of sequences in $\mathcal{S}$, we need a new characterization of the singly periodic Scherk minimal surfaces among all properly embedded singly periodic genus zero minimal surfaces with Scherk-type ends, provided that all ends are parallel except two of them. This result is a special case of the main theorem in [**22**] and it will be used here.

Once we know that $C|_{\mathcal{S}-\mathcal{K}}$ is proper and open, we need a local uniqueness result at a point of $\partial \mathcal{S}$ other than a singly periodic Scherk minimal surface, to conclude the third step in our strategy. This boundary point will be the catenoid, and the local uniqueness follows from the Inverse Function theorem. The second main difference of this approach with that appearing in [**14**] is a technical difficulty: nearby the catenoidal limit, the ends of surfaces in $\mathcal{S}$ group in couples, each couple giving rise to a single end of the limit catenoid. Such a collapsing phenomenon makes certain residues blow-up, and a careful study of rates of degeneration is needed to rectify the mapping to which the Inverse Function theorem is applied to.

The paper is organized as follows. In Section 2, we recall the necessary background to tackle our problem. Section 3 is devoted to introduce briefly the 3-parameter family $\mathcal{K}$ of standard examples. The complex manifold of admissible Weierstrass data $\mathcal{W}$ and the natural mappings on it are studied in Section 4. In Section 5, we obtain the curvature

estimates needed for the first point of our strategy. Sections 6 and 7 deal with the local uniqueness around the singly periodic Scherk minimal surfaces and the catenoid, respectively. The second point of our above strategy (openness) is the goal of Section 8, and finally Section 9 contains the proof of Theorem 1.

## 2. Preliminaries

Let $\widetilde{M} \subset \mathbb{R}^3$ be a connected orientable (Unless otherwise stated, all surfaces in the paper are supposed to be orientable.) properly embedded minimal surface, invariant by a rank 2 lattice $\mathcal{P}$ generated by two linearly independent translations $T_1, T_2$ (we will call such an $\widetilde{M}$ a *doubly periodic minimal surface*). $\widetilde{M}$ induces a properly embedded minimal surface $M = \widetilde{M}/\mathcal{P}$ in the complete flat 3-manifold $\mathbb{R}^3/\mathcal{P} = \mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}$ is a 2-dimensional torus. Reciprocally, if $M \subset \mathbb{T} \times \mathbb{R}$ is a properly embedded non-flat minimal surface, then its lift $\widetilde{M} \subset \mathbb{R}^3$ is a connected doubly periodic minimal surface by the Strong Halfspace theorem of Hoffman and Meeks [7]. Existence and classification theorems for doubly periodic minimal surfaces are usually tackled by considering the quotient surfaces in $\mathbb{T} \times \mathbb{R}$. An important result by Meeks and Rosenberg [15], insures that a properly embedded minimal surface $M \subset \mathbb{T} \times \mathbb{R}$ has finite topology if and only if it has finite total curvature, and in this case, $M$ has an even number of ends, each one asymptotic to a flat annulus (Scherk-type end). Later, Meeks [13] proved that any properly embedded minimal surface in $\mathbb{T} \times \mathbb{R}$ has a finite number of ends, so the finiteness of its total curvature is equivalent to the finiteness of its genus. It also follows from [15] that such a surface is conformally a finitely punctured Riemann surface whose ends representatives can be considered to be punctured disks.

When normalized so that the lattice of periods $\mathcal{P}$ is horizontal, we distinguish two types of ends, depending on whether the well defined third coordinate function on $M$ tends to $\infty$ (*top* end) or to $-\infty$ (*bottom* end) at the corresponding end. By separation properties, there are an even number of top (resp. bottom) ends. Because of embeddedness, top (resp. bottom) ends are always parallel to each other. If the top ends are not parallel to the bottom ends, then there exists an algebraic obstruction on the period lattice, which must be *commensurable* as in the classical doubly periodic minimal surfaces defined by Scherk in [25] or in the 4-ended tori found by Hoffman, Karcher and Wei [6]. If the top ends are parallel to the bottom ends, then the number of elements of both families of ends coincide, therefore the total number of ends of $M$ is a multiple of four. For details, see [15].

We will focus on the parallel ends setting, where the simplest possible topology is a finitely punctured torus (properly embedded minimal planar domains in $\mathbb{T} \times \mathbb{R}$ must have non-parallel ends by Theorem 4 in [**15**]; in fact, Lazard–Holly and Meeks [**11**] proved that the doubly periodic Scherk minimal surfaces are the unique possible examples with genus zero). Our main goal is to give a complete classification of all examples with genus one and parallel ends. To do this, we will first normalize appropriately the surfaces under consideration.

Given a positive integer $k$, let $\mathcal{S}$ be the space of all properly embedded minimal tori in a quotient $\mathbb{R}^3/\mathcal{P} = \mathbb{T} \times \mathbb{R}$ modulo a rank 2 lattice $\mathcal{P}$ generated by two independent translations (which depend on the surface), one of them being in the direction of the $x_2$-axis, with $4k$ horizontal Scherk-type ends. Given $M \in \mathcal{S}$, we denote respectively by $P_\Gamma \in \mathcal{P}$ and $F_\Gamma$ the period and flux vectors of $M$ along an oriented closed curve $\Gamma \subset M$. By the Divergence theorem, $P_\Gamma$ and $F_\Gamma$ only depend on the homology class of $\Gamma$ in $M$. The period and flux vectors $H$, $F$ at an end of $M$ (considering the end to be a punctured disk, $H$ and $F$ are the period and flux vectors along a small loop around the puncture with respect to the inward pointing conormal vector) are related by the equation $F = H \wedge N_0$, where $N_0$ is the value of the Gauss map at the puncture. In our normalization, each of the period vectors at the ends of $M$ is of the form $H = \pm(0, \pi a, 0)$ with $a > 0$. The end is called a *left end* if $F = (-\pi a, 0, 0)$, and a *right end* if $F = (\pi a, 0, 0)$. As $M$ is embedded, each family of left or right sided ends is naturally cyclically ordered by limiting heights. In fact, the maximum principle at infinity [**16**] implies that consecutive left (resp. right) ends $E, E'$ are at positive distance. Furthermore, since $M$ separates $\mathbb{T} \times \mathbb{R}$ (see [**7**]), the limit normal vectors at $E, E'$ are opposite.

We will denote by $\widetilde{M} \subset \mathbb{R}^3$ the (connected) doubly periodic minimal surface obtained by lifting $M$. Since the Gauss map has a constant value at points in a $\mathcal{P}$-orbit of $\widetilde{M}$, the stereographically projected Gauss map $g : \widetilde{M} \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ descends to $M$. As $M$ has finite total curvature, $g$ extends meromorphically to the conformal torus $\mathbb{M}$ obtained after attaching the ends to $M$, with the values $0, \infty$ at the punctures. As the period lattice $\mathcal{P}$ is not horizontal, the third coordinate function $x_3$ of $\widetilde{M}$ is multivalued on $M$, but the *height differential* $dh = \frac{\partial x_3}{\partial z} dz$ defines a meromorphic differential on $M$ (here, $z$ is a holomorphic coordinate). Since $M$ has finite total curvature and horizontal ends, $dh$ extends to a holomorphic differential on $\mathbb{M}$. The next statement collects some elementary properties satisfied by any surface in $\mathcal{S}$. Given $v \in \mathcal{P} - \{0\}$,

$\widetilde{M}/v$ will stand for the singly periodic minimal surface obtained as the quotient of $\widetilde{M}$ by the translation of vector $v$.

**Proposition 1.** *Given $M \in \mathcal{S}$, the following properties hold:*

1. *$g : \mathbb{M} \to \overline{\mathbb{C}}$ has degree $2k$, total branching number $4k$, does not take vertical directions on $M$ and it is unbranched at the ends.*

2. *The period vectors at the ends coincide up to sign and we will denote them by $H = \pm(0, \pi a, 0)$, $a > 0$. The period lattice $\mathcal{P}$ of $\widetilde{M}$ is generated by $H$ and a non-horizontal vector $T \in \mathbb{R}^3$, which is the period vector along a closed curve $\gamma_1 \subset M$ such that $[\gamma_1] \neq 0$ in the homology group $H_1(\mathbb{M}, \mathbb{Z})$.*

3. *Let $\mathcal{E}$ be the set of Scherk-type ends of $\widetilde{M}/H$. Then, $(\widetilde{M}/H) \cup \mathcal{E}$ is conformally $\mathbb{C}^* = \mathbb{C} - \{0\}$, and the height differential can be expressed as $dh = c\frac{dz}{z}$ in $\mathbb{C}^*$, with $c \in \mathbb{R}^* = \mathbb{R} - \{0\}$.*

*Let $\Pi \subset \mathbb{R}^3$ be a horizontal plane.*

4. *If neither end of the annulus $\Pi/H$ is asymptotic to an end in $\mathcal{E}$, then $\Pi/H$ intersects $\widetilde{M}/H$ transversely in a simple closed curve, whose period vector either vanishes or equals $\pm H$.*

5. *We divide $\mathcal{E}$ into right ends and left ends, depending on whether the flux vector at the corresponding end (with the inward pointing conormal vector) is $(a, 0, 0)$ or $(-a, 0, 0)$, $a > 0$. If $\Pi/H$ is asymptotic to an end in $\mathcal{E}$, then $(\widetilde{M} \cap \Pi)/H$ consists of one or two properly embedded arcs. If $(\widetilde{M} \cap \Pi)/H$ is one arc $\Gamma$, then both ends of $\Gamma$ diverge to the same end in $\mathcal{E}$. In the two arcs intersection case, both arcs travel from one left end to one right end in $\mathcal{E}$.*

6. *There exists an embedded closed curve $\gamma_2 \subset M$ such that $\{[\gamma_1], [\gamma_2]\}$ is a basis of $H_1(\mathbb{M}, \mathbb{Z})$ and $P_{\gamma_2} = 0$. Up to orientation, $\gamma_2$ represents the unique non-trivial homology class in $H_1(\mathbb{M}, \mathbb{Z})$ with associated period zero and an embedded representative.*

7. *Let $[\gamma] \in H_1(M, \mathbb{Z})$ be a homology class with an embedded representative that generates the homology group of $(\widetilde{M}/H) \cup \mathcal{E}$. Then, the third component $(F_\gamma)_3$ of the flux of $M$ along any representative $\gamma \in [\gamma]$ neither vanishes nor depends on $[\gamma]$ (up to orientation).*

*Proof.* Since $dh$ has no poles on the torus $\mathbb{M}$, it cannot have zeros, which implies that the only zeros and poles of $g$ are at the ends. As at each puncture, one of the meromorphic differentials $g\,dh, \frac{dh}{g}$ has a simple pole (Meeks and Rosenberg [**15**]), we conclude that $g$ is unbranched at the ends. Since $M$ has $4k$ ends, $g$ must have degree $2k$ and the Riemann–Hurwitz formula implies that its total branching number is $4k$, so statement 1 is proved.

Denote by $T_1, T_2$ two generators of $\mathcal{P}$. As $M$ is properly embedded with horizontal ends, we can assume that $T_1$ is not horizontal and $T_2$ is in the direction of the $x_2$-axis. It follows that the period vector at an end $E$ of $M$ is $H = nT_2$ with $n \in \mathbb{Z} - \{0\}$, and that $T_1$ is the period vector along a closed curve $\gamma_1 \subset M$ with $[\gamma_1] \neq 0$ in $H_1(\mathbb{M}, \mathbb{Z})$. Part 2 of the proposition will be proved if we show that $n = \pm 1$, by taking $T = T_1$. Arguing by contradiction, assume $|n| \geq 2$. Let $\widetilde{E}$ be a lifting of $E$ to $\mathbb{R}^3$ and $P = \{x_1 = d\} \subset \mathbb{R}^3$ a vertical plane containing the period vector to $E$, where $d \in \mathbb{R}$. By taking $d$ large enough in absolute value, the flat cylinder $P/H$ intersects $\widetilde{E}/H$ transversely along an embedded closed curve $\Lambda$ that generates the homology group of $P/H$ (and $\Lambda$ can be taken arbitrarily close to the quotient modulo $H$ of a horizontal straight line in $P$). Since $M$ is invariant by $T_2$, $\Lambda + T_2$ is also contained in $(\widetilde{M} \cap P)/H$ and $\Lambda + T_2$ generates the homology of $P/H$. Since $\widetilde{M}/H$ is embedded, we only can have the following two possibilities.

1) $\Lambda, \Lambda + T_2$ are disjoint.
2) $\Lambda = \Lambda + T_2$.

Note that $\Lambda$ can be considered as the graph of a $|H|$-periodic smooth function $f = f(x) : \mathbb{R} \to \mathbb{R}$ so that $(d, x, f(x))$ parametrizes the lifting of $\Lambda$ to $P$. If case (1) occurs, then the graphs $x \mapsto (d, x, f(x))$ and $x \mapsto (d, x, f(x - |T_2|))$ are disjoint, say $f(x) < f(x - |T_2|)$ for each $x \in \mathbb{R}$. Then, one finds a contradiction by taking $x$ as a point where $f(x - |T_2|)$ attains its minimum value. Therefore, (1) cannot hold. If case (2) holds, then the Gauss map of $\widetilde{E}$ descends to the quotient by $T_2$, having a zero or pole at the corresponding puncture. Since $|n| \geq 2$, this implies that the Gauss map $g$ of $E = \widetilde{E}/H$ has branching order $|n| - 1 > 0$ at the puncture. This contradicts that $g$ is unbranched at the ends of $M$, thereby proving statement 2 of the proposition.

By the above arguments, the natural covering map from $\widetilde{M}/H$ onto $M$ extends in an unbranched way through the Scherk-type ends of $\widetilde{M}/H$. Since $M$ is a finitely punctured torus, $\widetilde{M}/H$ is a cylinder with infinitely many points removed. Note that the third coordinate function $x_3$ is a well defined harmonic function on $\widetilde{M}/H$ that extends smoothly through $\mathcal{E}$, giving rise to a proper harmonic function on $(\widetilde{M}/H) \cup \mathcal{E}$. Therefore, $(\widetilde{M}/H) \cup \mathcal{E}$ is conformally $\mathbb{C}^* = \mathbb{C} - \{0\}$ and $x_3$ can be expressed as $x_3(z) = c \ln |z| + c'$ with $c \in \mathbb{R}^*$, $c' \in \mathbb{R}$, which is statement 3 of the proposition. This description of $x_3$ implies that $(\widetilde{M} \cap \Pi)/H$ corresponds in the $\mathbb{C}^*$-model to a possible punctured circle $C_r = \{|z| = r\}$ for certain $r > 0$. The hypotheses in item 4 correspond to the case that $C_r$ does not contain ends in $\mathcal{E}$, so the conclusion of 4 is clear. Under the hypotheses

of item 5, $C_r$ contains at most two ends of $\mathcal{E}$ (one left end and/or one right end because ends of the same side are separated by heights), and 5 also holds easily.

In order to see item 6, let $\beta$ be a compact horizontal level section of $M$. By the description above, $\beta$ generates the homology of the cylinder $(\widetilde{M}/H) \cup \mathcal{E}$. Since the period vector $T = P_{\gamma_1}$ is not horizontal, we conclude that $\{[\gamma_1], [\beta]\}$ is a basis of $H_1(\mathbb{M}, \mathbb{Z})$. If $P_\beta = 0$, then the first assertion in 6 is proved with $\gamma_2 = \beta$. If $P_\beta \neq 0$, then $P_\beta = \pm H$ by item 4. In this case, we choose as $\gamma_2$ an embedded closed curve in $M$ homologous to $\beta$ in $(\widetilde{M}/H) \cup \mathcal{E}$ such that $\beta \cup \gamma_2$ bounds just an end with period vector $-P_\beta$. Finally, suppose that $[\Gamma] \in H_1(\mathbb{M}, \mathbb{Z}) - \{0\}$ has an embedded representative $\Gamma$ (which can be assumed to lie in $M$) with $P_\Gamma = 0$. Since $\gamma_1, \gamma_2$ and small loops $\alpha_1, \ldots, \alpha_{4k}$ around the punctures generate $H_1(M, \mathbb{Z})$, we can write

$$(1) \qquad [\Gamma] = a_1[\gamma_1] + a_2[\gamma_2] + \sum_{i=1}^{4k} b_i[\alpha_i] \quad \text{in } H_1(M, \mathbb{Z})$$

for integers $a_1, a_2, b_1, \ldots, b_{4k}$. Taking periods in (1) and having in mind that $P_{\gamma_1} = T$ and that the periods at the ends are $\pm H$, we obtain $0 = P_\Gamma = a_1 T + bH$, where $b \in \mathbb{Z}$. As $T, H$ are linearly independent, it follows $a_1 = 0$. Now, (1) implies that $[\Gamma] = a_2[\gamma_2]$ in $H_1(\mathbb{M}, \mathbb{Z})$, so the embeddedness of $\Gamma$ forces $a_2$ to be $\pm 1$. This proves 6.

Finally, recall we have shown that any compact horizontal level section $\beta$ of $M$ is an embedded closed curve such that $\{[\gamma_1], [\beta]\}$ is a basis for $H_1(\mathbb{M}, \mathbb{Z})$. As the conormal vector to $M$ along $\beta$ has third coordinate with constant sign, we have $(F_\beta)_3 \neq 0$. Note that if $\gamma \subset M$ is an embedded closed curve homologous to $\beta$ in $(\widetilde{M}/H) \cup \mathcal{E}$, then $\gamma \cup \beta$ bounds a finite number of ends in $\mathcal{E}$, whose fluxes are all horizontal. Thus, $(F_\gamma)_3 = (F_\beta)_3$ and the proposition is proved. q.e.d.

**Remark 2.**

(i) We cannot expect the curve $\gamma_2$ in item 6 above to be a compact horizontal section in $\mathbb{R}^3$. For instance, all the horizontal level curves of the standard example $M_{\theta, \frac{\pi}{2}, 0}$ in Section 3 are open arcs when viewed in $\mathbb{R}^3$, see Figure 4 right.

(ii) With the notation in Proposition 1, the fact that all the fluxes at the ends of $M \in \mathcal{S}$ point to the $x_1$-axis implies that the component $(F_\gamma)_2$ of $F_\gamma$ in the direction of the $x_2$-axis is also independent (up to orientation) of the homology class $[\gamma] \in H_1(M, \mathbb{Z})$ satisfying the hypotheses of item 7.

There is a last natural normalization on the surfaces in $\mathcal{S}$, which we now explain. Given $M \in \mathcal{S}$, Proposition 1 gives a non-trivial homology class in $H_1(\mathbb{M}, \mathbb{Z})$ with an embedded representative $\gamma_2 \subset M$ such that $P_{\gamma_2} = 0$ and $(F_{\gamma_2})_3 > 0$. In the sequel, we will always normalize our surfaces so that $(F_{\gamma_2})_3 = 2\pi$, which can be achieved after an homothety. Note that this normalization is independent of the homology class of $\gamma_2$ in $H_1(M, \mathbb{Z})$ (up to orientation), see item 7 of Proposition 1.

We label by $\widetilde{\mathcal{S}}$ the set of *marked surfaces* $(M, p_1, \ldots, p_{2k}, q_1, \ldots, q_{2k}, [\gamma_2])$ where

1) $M$ is a surface in $\mathcal{S}$ whose period lattice is generated by $H, T \in \mathbb{R}^3$, where $H = (0, a, 0)$, $T = (T_1, T_2, T_3)$ and $a, T_3 > 0$;
2) $\{p_1, \ldots, p_{2k}\} = g^{-1}(0)$, $\{q_1, \ldots, q_{2k}\} = g^{-1}(\infty)$ and the cyclically ordered lists $(p_1, q_1, \ldots, p_k, q_k)$, $(p_{k+1}, q_{k+1}, \ldots, p_{2k}, q_{2k})$ are the two families of "sided" ends of $M$, both cyclically ordered by increasing heights in the quotient;
3) $[\gamma_2] \in H_1(M, \mathbb{Z})$ is the homology class of an embedded closed curve $\gamma_2 \subset M$ satisfying $P_{\gamma_2} = 0$, $(F_{\gamma_2})_3 = 2\pi$. We additionally impose that $\gamma_2$ lifts to a curve contained in a fundamental domain of the doubly periodic lifting of $M$ lying between two horizontal planes $\Pi, \Pi + T$.

We will identify in $\widetilde{\mathcal{S}}$ two marked surfaces that differ by a translation that preserves both orientation, the above "sided" ordering of their lists of ends and the associated homology classes. The same geometric surface in $\mathcal{S}$ can be viewed as different marked surfaces in $\widetilde{\mathcal{S}}$. We will simply denote as $M \in \widetilde{\mathcal{S}}$ the marked surfaces unless it leads to confusion. $\widetilde{\mathcal{S}}$ can be naturally endowed with a topology: a sequence of marked surfaces $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ converges to $M \in \widetilde{\mathcal{S}}$ if the associated sequence of minimal surfaces $\{M_n\}_n \subset \mathcal{S}$ converges smoothly to $M \in \mathcal{S}$ (in the uniform topology on compact sets), the ordered list of ends associated to $M_n$ converges to the corresponding one for $M$ and the homology classes $[\gamma_{2,n}] \in H_1(M_n, \mathbb{Z})$ in the last component of the marked surfaces $M_n$ have representatives converging uniformly to a representative of the last component of $M$. With this topology, a geometric minimal surface $M \in \mathcal{S}$ produces a discrete subset in $\widetilde{\mathcal{S}}$.

Consider $M \in \widetilde{\mathcal{S}}$ with Gauss map $g$ and height differential $dh$. An elementary calculation gives the periods $P_{p_j}, P_{q_j}$ and fluxes $F_{p_j}, F_{q_j}$ at the ends of $M$ as follows:

$$(2) \qquad \begin{aligned} P_{p_j} + i\, F_{p_j} &= \pi \mathrm{Res}_{p_j}\left(g^{-1}\, dh\right)(i, -1, 0), \\ P_{q_j} + i\, F_{q_j} &= -\pi \mathrm{Res}_{q_j}(g\, dh)(i, 1, 0), \end{aligned}$$

where $\mathrm{Res}_A$ denotes the residue of the corresponding meromorphic differential at a point $A$. The fact that $P_{p_j}, P_{q_j}$ point to the $x_2$-axis translates into $\mathrm{Res}_{p_j}\left(g^{-1}\,dh\right), \mathrm{Res}_{q_j}(g\,dh) \in \mathbb{R}$. By definition of the ordering of the ends of $M$ as a marked surface, we have that

$$(3) \quad \mathrm{Res}_{p_j}\left(g^{-1}\,dh\right) = -\mathrm{Res}_{q_j}(g\,dh) = \left\{ \begin{array}{ll} a & (1 \le j \le k) \\ -a & (k+1 \le j \le 2k), \end{array} \right.$$
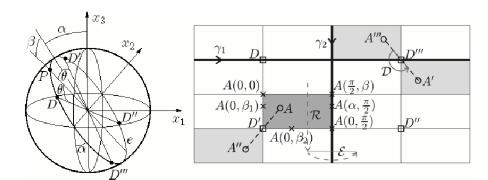
for certain $a \in \mathbb{R}^*$ (the case $a > 0$ corresponds to $p_1, \ldots, p_k, q_1, \ldots, q_k$ being right ends of $M$, see the definition of left and right ends before the statement of Proposition 1). Recall that $P_{\gamma_2} = 0$ and $(F_{\gamma_2})_3 = 2\pi$. Thus,

$$(4) \qquad \overline{\int_{\gamma_2} g^{-1}\,dh} = \int_{\gamma_2} g\,dh, \qquad \int_{\gamma_2} dh = 2\pi i.$$

## 3. Standard examples

We dedicate this Section to introduce briefly the 3-parameter family of standard examples $\mathcal{K} \subset \mathcal{S}$ to which the uniqueness Theorem 1 applies. Some of the properties of $\mathcal{K}$ in this Section are straightforward computations that can be found in detail in [**23**]. As the standard examples with $4k$-ends are nothing but $k$-sheeted coverings of 4-ended standard examples in $\mathcal{K}_1$, we will concentrate on these last ones. Each $M_{\theta,\alpha,\beta} \in \mathcal{K}_1$ is determined by the 4 branch values of its Gauss map, which consist of two pairs of antipodal points $D, D', D'' = -D, D''' = -D'$ in the sphere $\mathbb{S}^2$. Since the Gauss map of any surface in $\mathcal{S}$ is unbranched at the ends, $D, D', D'', D'''$ must be different from the North and South Poles. We also let $e$ be the equator in $\mathbb{S}^2$ that contains $D, D', D'', D'''$. Given a point $P \in e$, the branch values of the Gauss map can be determined by giving only one angle $\theta \in (0, \frac{\pi}{2})$, in such a way that the position vectors of $D, D'$ form an angle of $2\theta$ and the position vector of $P$ bisects such an angle. We will call a *spherical configuration* to any set $\{D, D', D'', D'''\}$ as above.

Given $(\theta, \alpha, \beta) \in (0, \frac{\pi}{2}) \times [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ with $(\alpha, \beta) \ne (0, \theta)$, we define the spherical configuration of the potential standard example $M_{\theta,\alpha,\beta}$ by means of the angle $\theta$, the equator $e$ and the point $P$ as follows:

1) Let $e_0$ be the inverse image of the imaginary axis in $\mathbb{C}$ through the stereographic projection from the North Pole of $\mathbb{S}^2$. Then, $e$ is the image of $e_0$ through the rotation by angle $\alpha$ around the $x_2$-axis.
2) $P$ is the image of the North Pole through the composition of a rotation by angle $\beta$ around the $x_1$-axis with a rotation by angle $\alpha$ around the $x_2$-axis, see Figure 1 left.

**Figure 1.** Left: Spherical configuration of $M_{\theta,\alpha,\beta}$. Right: Behavior of the pole $A(\alpha,\beta)$ of $g$ in the dark shaded rectangle $\mathcal{R}$ (here, $0 < \beta_1 < \theta < \beta_2 < \frac{\pi}{2}$, $\alpha \in (0, \frac{\pi}{2})$ and $\beta \in [0, \frac{\pi}{2}]$). The remaining ends of $M_{\theta,\alpha,\beta}$ move in the light shaded rectangles as $\alpha, \beta$ vary.

Projecting stereographically from the North Pole the branch points $D, D', D'', D'''$ for the spherical configuration associated to $(\theta, 0, 0)$, one just finds the four roots of the polynomial $(z^2 + \lambda^2)(z^2 + \lambda^{-2})$ where $\lambda = \lambda(\theta) = \cot\frac{\theta}{2}$. Therefore, the underlying conformal compactification of the potential surface $M_{\theta,0,0}$ is the rectangular torus

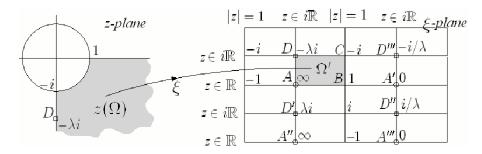$$\Sigma_\theta = \left\{(z,w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z^2 + \lambda^2)(z^2 + \lambda^{-2})\right\},$$

and its extended Gauss map is the $z$-projection $(z,w) \in \Sigma_\theta \mapsto z \in \overline{\mathbb{C}}$ on $\Sigma_\theta$. Note that the spherical configuration for angles $(\theta, \alpha, \beta)$ differs from the one associated to $(\theta, 0, 0)$ in a Möbius transformation $\varphi$. Thus, the compactification of any $M_{\theta,\alpha,\beta}$, which is the branched covering of $\mathbb{S}^2$ through its Gauss map, is $\Sigma_\theta$. Furthermore, the composition of the Gauss map of $M_{\theta,0,0}$ with $\varphi$ gives the Gauss map of the potential example $M_{\theta,\alpha,\beta}$:

$$g(z,w) = \frac{z\left(i\cos(\frac{\alpha-\beta}{2}) + \cos(\frac{\alpha+\beta}{2})\right) + \sin(\frac{\alpha-\beta}{2}) + i\sin(\frac{\alpha+\beta}{2})}{\cos(\frac{\alpha-\beta}{2}) + i\cos(\frac{\alpha+\beta}{2}) - z\left(i\sin(\frac{\alpha-\beta}{2}) + \sin(\frac{\alpha+\beta}{2})\right)},$$
$$(z,w) \in \Sigma_\theta,$$

and its ends are $\{A, A', A'', A'''\} = g^{-1}(\{0, \infty\})$. As the height differential $dh$ of $M_{\theta,\alpha,\beta}$ is a holomorphic 1-form on $\Sigma_\theta$, we have $dh = \mu\frac{dz}{w}$ for certain $\mu = \mu(\theta, \alpha, \beta) \in \mathbb{C}^*$.

It will be also useful to have a second representation of $\Sigma_\theta$ as a quotient of the $\xi$-plane $\mathbb{C}$ by two orthogonal translations. Let $\Omega \subset \Sigma_\theta$ be

**Figure 2.** The biholomorphism $\xi$ between the shaded regions. We have included some values of the $z$-projection in the $\xi$-plane model of $\Sigma_\theta$.

one of the two connected components of $\{(z,w) \in \Sigma_\theta \ / \ |z| > 1, -\frac{\pi}{2} < \arg(z) < 0\}$. $\Omega$ is topologically a disk and its boundary contains just one branch point $D_1 = (-\lambda i, w(-\lambda i))$ and one pole $A_1 = (\infty, \infty)$ of the $z$-projection. Let $\Omega'$ be an open rectangle of consecutive vertices $A, B, C, D \in \mathbb{C}$ with the segment $\overline{AB}$ being horizontal, such that there exists a biholomorphism $\xi : \{|z| > 1, -\frac{\pi}{2} < \arg(z) < 0\} \to \Omega'$ with boundary values $\xi(\infty) = A$, $\xi(1) = B$, $\xi(-i) = C$ and $\xi(-\lambda i) = D$. Then, the composition of the $z$-projection with $\xi$ defines a biholomorphism between $\Omega$ and $\Omega'$. After symmetric extension of this biholomorphism across the boundary curves of $\Omega, \Omega'$, we will get a biholomorphism from $\Sigma_\theta$ to the quotient of the $\xi$-plane modulo the translations given by four times the sides of the rectangle $\Omega'$, see Figure 2. Note that if $\alpha = \beta = 0$, then $g(z,w) = z$ and the equalities $\xi(\infty) = A$, $\xi(-\lambda i) = D$ justify to use the same symbols $A, D$ previously defined for two of the corners of the rectangle $\Omega'$.

The above identification allows us to see the ends $A, A', A'', A'''$ as functions of $(\alpha, \beta) \in [0, \frac{\pi}{2}]^2 - \{(0, \theta)\}$ valued in the $\xi$-plane model of $\Sigma_\theta$. $A(\alpha, \beta)$ moves on the dark shaded rectangle $\mathcal{R}$ in Figure 1 right. For $(\alpha, \beta) = (0,0)$, we choose $A(0,0)$ as the up-left corner of $\mathcal{R}$. As $\alpha$ increases in $[0, \frac{\pi}{2}]$, the end $A(\alpha, 0)$ of $M_{\theta, \alpha, 0}$ moves continuously and horizontally to the right, until reaching for $\alpha = \frac{\pi}{2}$ the up-right corner of $\mathcal{R}$. The curve $\beta \in [0, \theta) \mapsto A(0, \beta)$ parametrizes downwards the left vertical boundary edge of $\mathcal{R}$, until reaching the branch point $D'$. The curve $\beta \in (\theta, \frac{\pi}{2}] \mapsto A(0, \beta)$ parametrizes the lower horizontal edge of $\partial \mathcal{R}$ from $D'$ until the down-right corner of $\mathcal{R}$. The curve $\alpha \in [0, \frac{\pi}{2}] \mapsto A(\alpha, \frac{\pi}{2})$ parametrizes upwards the right vertical edge of $\partial \mathcal{R}$. Finally, the curve $\beta \in [0, \frac{\pi}{2}] \mapsto A(\frac{\pi}{2}, \beta)$ is constant at the up-right corner of $\mathcal{R}$. These boundary values of $A(\alpha, \beta)$ extend continuously and bijectively from $(0, \frac{\pi}{2})^2$ to the interior of $\mathcal{R}$. The behavior of the remaining three

ends of $M_{\theta,\alpha,\beta}$ on the $\xi$-plane model can be deduced from $A(\alpha,\beta)$ by using the isometry group $\mathrm{Iso}(\theta,\alpha,\beta)$ of the induced metric $ds^2 = \frac{1}{4}(|g| + |g|^{-1})^2|dh|^2$, which we now investigate.

First, note that the identity in $\mathbb{S}^2$ lifts via $g$ to two different isometries of $ds^2$, namely the identity in $\Sigma_\theta$ and the deck transformation $\mathcal{D}(z,w) = (z,-w)$, both restricted to $\Sigma_\theta - g^{-1}(\{0,\infty\})$. $\mathcal{D}$ corresponds in the $\xi$-plane to the 180°-rotation about any of the branch points of the $z$-projection. The antipodal map $\aleph$ on $\mathbb{S}^2$ also leaves invariant both the spherical configuration of $M_{\theta,\alpha,\beta}$ and the set $\{(0,0,\pm1)\}$, and the equality $\left[w\left(\frac{-1}{\overline{z}}\right)\right]^2 = \frac{[w(z)]^2}{\overline{z}^4}$ for any $(z,w) \in \Sigma_\theta$ implies that $\aleph$ lifts through $g$ to two isometries of $ds^2$, which we call $\mathcal{E}$ and $\mathcal{F} = \mathcal{D} \circ \mathcal{E}$. Both $\mathcal{E}, \mathcal{F}$ are antiholomorphic involutions of $\Sigma_\theta$ without fixed points. This property implies that one of these involutions, say $\mathcal{E}$, corresponds in the $\xi$-plane model to the composition of the translation by the vector $\overline{D, D'}$ with the symmetry with respect to the right vertical edge of $\partial\mathcal{R}$, see Figure 1 right. The remaining ends of $M_{\theta,\beta,\alpha}$ in terms of $A = A(\alpha,\beta)$ are (up to relabeling)

$$(5) \qquad A'' = \mathcal{D}(A), \qquad A''' = \mathcal{E}(A), \qquad A' = \mathcal{D}(A''').$$

We now study the period problem for $M_{\theta,\alpha,\beta}$. Recall that $dh = \mu\frac{dz}{w}$ on $\Sigma_\theta$ with $\mu \in \mathbb{C}^*$. From now on, take $\mu \in \mathbb{R}^*$. The period $P_A$ and flux $F_A$ of $M_{\theta,\alpha,\beta}$ at the end $A$ with $g(A) = \infty$ are given by

$$(6) \qquad P_A = \pi\mu\big(i\,E(\theta,\alpha,\beta),0\big), \qquad F_A = \pi\mu\big(E(\theta,\alpha,\beta),0\big),$$

where we have used the identification $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ by $(a,b,c) \equiv (a + ib,c)$, and $E(\theta,\alpha,\beta) = [\cos^2\alpha + \csc^2\theta(\sin\alpha\cos\beta - i\sin\beta)^2]^{-1/2}$ (we have chosen a branch of $w$ for computing (6), which only affects the result up to sign). The periods and fluxes at $A', A'', A'''$ can be easily obtained using (5), (6) and the pullbacks by $\mathcal{D}, \mathcal{E}$ of the Weierstrass form $\Phi = \big(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1\big)dh$ of $M_{\theta,\alpha,\beta}$:

$$(7) \qquad\qquad\qquad \mathcal{D}^*\Phi = -\Phi, \qquad \mathcal{E}^*\Phi = -\overline{\Phi}.$$

Concerning the period problem in homology, let $\gamma_1, \gamma_2$ be the simple closed curves in $\Sigma_\theta$ obtained respectively as quotients of the horizontal and vertical lines in the $\xi$-plane passing through $D, D'''$ and through the right vertical edge of $\partial\mathcal{R}$ (see Figure 1 right). Clearly $\{[\gamma_1], [\gamma_2]\}$ is a basis of $H_1(\Sigma_\theta, \mathbb{Z})$. Both $\gamma_1, \gamma_2$ miss the ends $A, A', A'', A'''$ except for certain extreme values of $(\alpha,\beta)$ (for instance, $A(\pi/2,0)$ lies in $\gamma_2$), but one can think of the homology classes $[\gamma_1], [\gamma_2] \in H_1(M_{\theta,\alpha,\beta}, \mathbb{Z})$ as being independent of $(\alpha,\beta)$. Furthermore,

$$(8)\;\; \mathcal{E}_*[\gamma_1] = -[\gamma_1] - [\gamma_A] - [\gamma_{A'}], \qquad \mathcal{E}_*[\gamma_2] = [\gamma_2] \quad \text{in } H_1(M_{\theta,\alpha,\beta}, \mathbb{Z}),$$

where $\gamma_A, \gamma_{A'}$ denote small loops around $A, A'$. Equations (5), (7) and the first equality in (8) imply $-\overline{\int_{\gamma_1} \Phi} = -\int_{\gamma_1} \Phi - \int_{\gamma_A} \Phi + \overline{\int_{\gamma_A} \Phi}$. Taking imaginary parts, we find

$$F_{\gamma_1} = -F_{\gamma_A} = -F_A. \tag{9}$$

Similarly, the second equalities in (7), (8) insure that $\int_{\gamma_2} \Phi = -\overline{\int_{\gamma_2} \Phi}$, whose real part gives

$$P_{\gamma_2} = 0. \tag{10}$$

Equation (10) implies that we can take $\gamma_2$ as the embedded closed curve appearing in item $6$ of Proposition 1 (except for the extreme values of $(\alpha, \beta)$ mentioned above, in which we deform $\gamma_2$ keeping $[\gamma_2] \in H_1(M_{\theta,\alpha,\beta}, \mathbb{Z})$ constant in $\alpha, \beta$). As $dh$ is holomorphic and non-trivial on $\Sigma_\theta$, (10) also implies that $\int_{\gamma_2} dh \in i\mathbb{R}^*$, so we must rescale to have $\int_{\gamma_2} dh = 2\pi i$ as part of the normalization of our surfaces in $\mathcal{S}$. This rescaling gives $\mu$:

$$\mu(\theta) = \frac{\pi}{2\displaystyle\int_{-\pi/2}^{0} \sqrt{2\cos(2t) + \lambda^2(\theta) + \frac{1}{\lambda^2(\theta)}}\, dt} = \frac{\pi \csc\theta}{\text{Elliptic } \mathrm{K}(\sin^2\theta)}, \tag{11}$$

where $\mathrm{EllipticK}(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2 u}\, du$ is a complete elliptic integral of the first kind. Since $\Re \int_{\gamma_2} dh = 0$ and $dh$ is holomorphic on $\Sigma_\theta$, we deduce that $P_{\gamma_1}$ has non-vanishing third component. In particular, $P_A$ and $P_{\gamma_1}$ are linearly independent. All these facts imply that $M_{\theta,\alpha,\beta}$ is a complete immersed minimal surface invariant by the rank 2 lattice generated by $P_A, P_{\gamma_1}$, with four horizontal embedded Scherk-type ends and genus one in the quotient.

We claim that $M_{\theta,\alpha,\beta}$ is embedded for any $(\theta, \alpha, \beta)$. First, note that $M_{\theta,0,0}$ is the toroidal halfplane layer defined by Karcher in [**9**], who proved that $M_{\theta,0,0}$ decomposes in 16 congruent disjoint pieces, each one being the conjugate surface of certain Jenkins–Serrin graph. In particular, $M_{\theta,0,0}$ is embedded for each $\theta \in (0, \frac{\pi}{2})$. Since for $\theta$ fixed the heights of the ends of $M_{\theta,\alpha,\beta}$ depend continuously on $(\alpha, \beta)$ in the connected set $[0, \frac{\pi}{2}]^2 - \{(0,\theta)\}$ (this is clear in the $\xi$-plane model), a standard application of the maximum principle insures that $M_{\theta,\alpha,\beta}$ is embedded for all values of $\theta, \alpha, \beta$.

**Remark 3.** Since $P_A$ does not necessarily point to the $x_2$-axis, we must possibly rotate $M_{\theta,\alpha,\beta}$ by a suitable angle around the $x_3$-axis in order to see it in $\mathcal{S}$.

Next, we study the conjugate surface $M^*_{\theta,\alpha,\beta}$ of $M_{\theta,\alpha,\beta}$. Since the flux (resp. the period) of the conjugate surface along a given curve in the parameter domain equals the period (resp. the opposite of the flux) of the original surface along the same curve, we deduce from (6), (9) and (10) that the period of $M^*_{\theta,\alpha,\beta}$ at its ends is $\pm\pi\mu(E(\theta,\alpha,\beta),0) \in \mathbb{C} \times \mathbb{R}$, its period along $\gamma_1$ is $\pi\mu(E(\theta,\alpha,\beta),0)$ and the third component of the period of $M^*_{\theta,\alpha,\beta}$ along $\gamma_2$ is $-2\pi$. Therefore, $M^*_{\theta,\alpha,\beta}$ is a complete immersed doubly periodic torus with four horizontal embedded Scherk-type ends, whose period lattice is generated by $\pi\mu(E(\theta,\alpha,0),0)$ and $(F(\gamma_2),2\pi)$, where $F(\gamma_2) \in \mathbb{C}$ denotes the horizontal part of the flux of $M_{\theta,\alpha,\beta}$ along $\gamma_2$. Again any $M^*_{\theta,\alpha,\beta}$ is embedded by the maximum principle and the embeddedness of $M^*_{\theta,0,0}$ (which decomposes in 16 congruent Jenkins–Serrin graphs). Also note that, in order to see $M^*_{\theta,\alpha,\beta}$ inside the normalized space $\mathcal{S}$, we must rotate this surface around the $x_3$-axis and rescale it suitably. After this identification, the curve with period zero in the sense of item *6* of Proposition 1 can be taken as $\gamma^*_2 = \gamma_1 + \gamma_A \subset M^*_{\theta,\alpha,\beta}$.

We have defined two families of examples $\mathcal{A} = \{M_{\theta,\alpha,\beta}\}$, $\mathcal{A}^* = \{M^*_{\theta,\alpha,\beta}\}$ inside $\mathcal{S}$, with $(\theta,\alpha,\beta)$ varying in

$$\mathcal{I} = \left\{(\theta,\alpha,\beta) \in (0,\frac{\pi}{2}) \times [0,\frac{\pi}{2}]^2 \mid (\alpha,\beta) \neq (0,\theta)\right\}.$$

Clearly, this definition can be extended to larger ranges in $(\theta,\alpha,\beta)$, but such an extension only produces symmetric images of these surfaces with respect to certain planes orthogonal to the $x_1, x_2$ or $x_3$-axes. Nevertheless, some of these geometrically equivalent surfaces are considered as distinct points in the space $\widetilde{\mathcal{S}}$ defined in Section 2. Another interesting property is that $\mathcal{A}^* = \mathcal{A}$ (modulo the aforementioned identifications), as the following lemma shows. In this sense, we can assure that the space of standard examples is *self-conjugate*.

**Lemma 1.** *Given $(\theta,\alpha,\beta) \in \mathcal{I}$, the surface $M^*_{\theta,\alpha,\beta}$ coincides with $M_{\frac{\pi}{2}-\theta,\alpha,\frac{\pi}{2}-\beta}$ up to a symmetry in a plane orthogonal to the $x_2$-axis.*

*Proof.* Fix $(\theta,\alpha,\beta) \in \mathcal{I}$. By direct calculation, $\Sigma_{\frac{\pi}{2}-\theta} = \{(\widetilde{z},\widetilde{w}) \mid \widetilde{w}^2 = (\widetilde{z}^2 - 1)^2 + 4\widetilde{z}^2 \sec^2\theta\}$. Since the Möbius transformation $\varphi(z) = \frac{1-iz}{-i+z}$ applies the set of branch points of the $z$-projection of $\Sigma_\theta$ bijectively on the set of branch points of the $\widetilde{z}$-projection of $\Sigma_{\frac{\pi}{2}-\theta}$, it follows that $\Delta(z,w) = (\varphi(z), \widetilde{w}(\varphi(z)))$ is a biholomorphism between $\Sigma_\theta$ and $\Sigma_{\frac{\pi}{2}-\theta}$. A simple thought of the role of the angle $\beta$ gives that if we extend the definition of the standard examples to $\beta' \in [-\frac{\pi}{2},0)$, then $M_{\theta,\alpha,\beta'}$ is nothing but the reflected image of $M_{\theta,\alpha,-\beta'}$ with respect to the $(x_1,x_3)$-plane

(up to a translation). Furthermore, it is straightforward to check that $g_{\theta,\alpha,-\beta} = g_{\frac{\pi}{2}-\theta,\alpha,\frac{\pi}{2}-\beta} \circ \Delta$, where the subindex for $g$ means the parameter angles of the standard example whose Gauss map is $g$. Denoting by $dh_\theta$ its height differential (which only depends on $\theta$), a direct computation gives $\Delta^* dh_{\frac{\pi}{2}-\theta} = \mu(\frac{\pi}{2}-\theta)\frac{\varphi'(z)\,dz}{\overline{w}(\varphi(z))} = i\frac{\mu(\frac{\pi}{2}-\theta)}{\tan\theta}\frac{dz}{w} = i\frac{\mu(\frac{\pi}{2}-\theta)}{\mu(\theta)\tan\theta}\,dh_\theta$. Now, the lemma is proved.                                                q.e.d.

One could ask for which angles $(\theta,\alpha,\beta)$ the surface $M_{\theta,\alpha,\beta}$ is self-conjugate (i.e. when it is congruent to its conjugate surface). First, note that for any $\theta \in (0,\frac{\pi}{2})$, $M_{\theta,\frac{\pi}{2},\beta}$ does not depend on $\beta \in [0,\frac{\pi}{2}]$ since its spherical configuration is just the rotation image of the one of $M_{\theta,\frac{\pi}{2},0}$ around the $x_3$-axis by angle $\beta$. Having this in mind and using Lemma 1, we have that the range of angles for which $M_{\theta,\alpha,\beta}$ is self-conjugate is $\left(\{\frac{\pi}{4}\} \times (0,\frac{\pi}{2}] \times \{\frac{\pi}{4}\}\right) \cup \left(\{\frac{\pi}{4}\} \times \{\frac{\pi}{2}\} \times [0,\frac{\pi}{2}]\right)$.

As the branch values of the Gauss map of any $M_{\theta,\alpha,\beta}$ lie on a spherical equator, a result by Montiel and Ros [17] insures that the space of bounded Jacobi functions on $M_{\theta,\alpha,\beta}$ is 3-dimensional (they reduce to the linear functions of the Gauss map), a condition usually referred in literature as the *non-degeneracy* of $M_{\theta,\alpha,\beta}$. This non-degeneracy can be interpreted by means of an Implicit Function theorem argument to obtain that around $M_{\theta,\alpha,\beta}$, the space $\mathcal{S}$ is a 3-dimensional real analytic manifold (Hauswirth and Traizet [5]); in particular, the only elements in $\mathcal{S}$ around a standard example are themselves standard. This local uniqueness result will be extended in the large by Theorem 1 in this paper.

We finish this section by summarizing some additional properties of the standard examples, that can be checked using their Weierstrass representation. For details, see [23].

**Proposition 2.**

1) *Given $\theta \in (0,\frac{\pi}{2})$, $M_{\theta,0,0}$ has three reflection symmetries $S_1,S_2,S_3$ in orthogonal planes and contains four straight lines parallel to the $x_1$-axis, each of which induce the same $180°$-rotation symmetry $R_D$. The isometry group $\mathrm{Iso}(\theta,0,0)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, with generators $S_1,S_2,S_3,R_D$ (see Figure 3 left).*

2) *For any $(\theta,\alpha) \in (0,\frac{\pi}{2})^2$, $M_{\theta,\alpha,0}$ is invariant by a reflection $S_2$ in a plane orthogonal to the $x_2$-axis and by a $180°$-rotation $R_2$ around a line parallel to the $x_2$-axis that cuts the surface orthogonally (Figure 4 left). $\mathrm{Iso}(\theta,\alpha,0)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with generators $S_2,R_2,\mathcal{D}$. Furthermore, $\mathrm{Iso}(\theta,\frac{\pi}{2},0) = \mathrm{Iso}(\theta,0,0)$ (Figure 4 right).*

3) *For any $(\theta,\beta) \in (0,\frac{\pi}{2})^2 - \{(\theta,\theta)\}$, $M_{\theta,0,\beta}$ is invariant by a reflection $S_1$ in a plane orthogonal to the $x_1$-axis, by a $180°$-rotation*
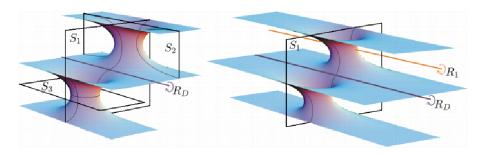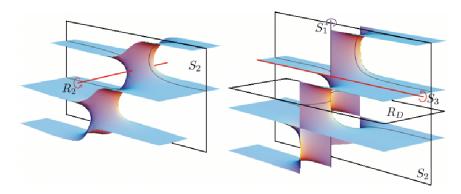
**Figure 3.** Left: The toroidal halfplane layer $M_{\pi/4,0,0}$.
Right: The surface $M_{\pi/4,0,\pi/8}$.

symmetry $R_D$ around a straight line parallel to the $x_1$-axis con-
tained in the surface, and by a $180°$-rotation $R_1$ around another
line parallel to the $x_1$-axis that cuts the surface orthogonally. The
group $\mathrm{Iso}(\theta,0,\beta)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with generators $S_1$,
$R_D$, $R_1$ (Figure 3 right). Furthermore, $\mathrm{Iso}(\theta,0,\frac{\pi}{2}) =\mathrm{Iso}(\theta,0,0)$.

4) For any $(\theta,\alpha,\beta) \in (0,\frac{\pi}{2})^3$, $\mathrm{Iso}(\theta,\alpha,\beta)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$
with generators $\mathcal{D},\mathcal{E}$.

5) For any $(\theta,\alpha) \in (0,\frac{\pi}{2})^2$, $M_{\theta,\alpha,\frac{\pi}{2}}$ is invariant by a $180°$-rotation $S_3$
around a straight line parallel to the $x_2$-axis contained in the sur-
face, and by the composition $R_3$ of a reflection symmetry across a
plane orthogonal to the $x_2$-axis with a translation by half a hori-
zontal period. In this case, $\mathrm{Iso}(\theta,\alpha,\frac{\pi}{2})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$,
with generators $S_3, R_3, \mathcal{D}$.

6) When $(\theta,\alpha,\beta) \to (0,0,0)$, $M_{\theta,\alpha,\beta}$ converges smoothly to two ver-
tical catenoids, both with flux $(0,0,2\pi)$.

7) Let $\theta_0 \in (0,\frac{\pi}{2})$. When $(\theta,\alpha,\beta) \to (\theta_0,0,\theta_0)$, $M_{\theta,\alpha,\beta}$ converges
to a Riemann minimal example with two horizontal ends, ver-
tical part of its flux $2\pi$ and branch values of its Gauss map at
$0, \infty, i\tan\theta_0, -i\cot\theta_0 \in \overline{\mathbb{C}}$.

8) When $(\theta,\alpha,\beta) \to (\frac{\pi}{2},0,\frac{\pi}{2})$, $M_{\theta,\alpha,\beta}$ converges (after blowing up) to
two vertical helicoids.

9) Let $(\alpha_0,\beta_0) \in [\theta,\frac{\pi}{2}]^2 - \{(0,0)\}$. When $(\theta,\alpha,\beta) \to (0,\alpha_0,\beta_0)$,
$M_{\theta,\alpha,0}$ converges to two singly periodic Scherk minimal surfaces,
each one with two horizontal ends and two ends forming angle
$\arccos(\cos\alpha_0\cos\beta_0)$ with the horizontal.

10) Let $(\alpha_0,\beta_0) \in [\theta,\frac{\pi}{2}]^2 - \{(0,\frac{\pi}{2})\}$. When $(\theta,\alpha,\beta) \to (\frac{\pi}{2},\alpha_0,\beta_0)$,
$M_{\theta,\alpha,0}$ converges (after blowing up) to two doubly periodic Scherk
minimal surfaces, each one with two horizontal ends and two ends
forming angle $\arccos(\cos\alpha_0\sin\beta_0)$ with the horizontal.

**Figure 4.** Left: The surface $M_{\pi/4,\pi/4,0}$. Right: $M_{\pi/4,\pi/2,0}$, all whose horizontal level curves are open arcs in $\mathbb{R}^3$.

## 4. The moduli space $\mathcal{W}$ of Weierstrass representations

Any surface in the hypotheses of Theorem 1 can be seen as an element of a finite dimensional complex manifold by means of the *Hurwitz schemes*, a process which endows families of meromorphic functions of prescribed degree with structures of finite dimensional complex manifolds, see for instance [**3**]. This general construction is simpler in our setting, as we now explain.

For $k \in \mathbb{N}$ fixed, we will call $\mathcal{W}$ to the space of lists $(\mathbb{M}, g, p_1, \ldots, p_{2k}, q_1, \ldots, q_{2k}, [\gamma])$, where $g : \mathbb{M} \to \overline{\mathbb{C}}$ is a meromorphic degree $2k$ function defined on a torus $\mathbb{M}$, which is unbranched at its zeros $\{p_1, \ldots, p_{2k}\} = g^{-1}(0)$ and poles $\{q_1, \ldots, q_{2k}\} = g^{-1}(\infty)$, and $[\gamma]$ is a homology class in $g^{-1}(\mathbb{C}^*)$ with $[\gamma] \neq 0$ in $H_1(\mathbb{M}, \mathbb{Z})$. Note that the same map $g$ can be viewed as an infinite subset of $\mathcal{W}$ by considering different orderings on the points $p_j, q_j$ and different homology classes $[\gamma]$. This infinite subset associated to the same $g$ will be discrete with the topology to be defined below. We will simply denote by $g$ the elements of $\mathcal{W}$, which will be referred to as *marked meromorphic maps*.

Next, we endow $\mathcal{W}$ with a topology. Given $g \in \mathcal{W}$, let $b_1, \ldots, b_l \in \mathbb{C}^*$, $l \leq 4k$, be the distinct branch values of $g$ (by the Riemann–Hurwitz formula, $g$ has $4k$ branch points counting multiplicity). Take $l$ small pairwise disjoint disks $D_i$ in $\mathbb{C}^*$ centered at the points $b_i$ and let $\Omega = \overline{\mathbb{C}} - \cup_{i=1}^l D_i$. By taking the radii of these disks small enough, we can assume that both $\Omega$ and $g^{-1}(\Omega)$ are connected and each component of $g^{-1}(D_i)$ is a disk $(1 \leq i \leq l)$. Therefore, $g : g^{-1}(\Omega) \to \Omega$ is an unbranched $2k$-sheeted covering with connected total space, and each component of $g^{-1}(D_i)$ contains at most one branch point of $g$, possibly of high

multiplicity. With these data $g, D_1, \ldots, D_l$, we define a neighborhood $U(g)$ of $g$ in $\mathcal{W}$ as the set of marked meromorphic maps $f \in \mathcal{W}$ such that:

1) $f|_{f^{-1}(\Omega)}$ *is unbranched and the restrictions* $f : f^{-1}(\Omega) \to \Omega$, $g : g^{-1}(\Omega) \to \Omega$ *are isomorphic as covering maps.* This allows us to identify conformally $f^{-1}(\Omega)$ and $g^{-1}(\Omega)$, so that zeros (resp. poles) of $g$ identify with zeros (resp. poles) of $f$.

2) *Each component of* $f^{-1}(D_i)$ *is a disk in the torus* $f^{-1}(\overline{\mathbb{C}})$. These disks are in bijective correspondence with those of $g$ by means of the identification in item 1), and the total branching order of $g, f$ at corresponding disks necessarily coincides.

3) The identification in item 1) gives a bijection between the zeros (resp. poles) of $g$ and those of $f$. *We impose that the ordering in the set of zeros and poles of the marked meromorphic map* $f$ *coincides with the corresponding one for* $g$ *through this bijection.*

4) The homology class $[\gamma] \in H_1(g^{-1}(\mathbb{C}^*), \mathbb{Z})$ can be represented by an embedded cycle $\gamma \in g^{-1}(\Omega)$, which by item 1) can be also seen as an embedded cycle in $f^{-1}(\Omega)$. Such a cycle induces an homology class in $f^{-1}(\mathbb{C}^*)$, which we denote $[\gamma]$ as well. It is easy to show that $[\gamma] \neq 0$ in $H_1(f^{-1}(\mathbb{C}^*), \mathbb{Z})$. *We impose that the homology class associated to the marked meromorphic map* $f$ *coincides with* $[\gamma]$.

Next, we endow $\mathcal{W}$ with a structure of $4k$-dimensional complex manifold. The topology defined above implies that if $g \in \mathcal{W}$ has $4k$ distinct branch values, then every $f \in \mathcal{W}$ close enough to $g$ has also $4k$ distinct branch values. In this case, the map that applies this $f$ to its list of branch values (choosing an ordering) is a local chart for $\mathcal{W}$ around $g$. Around a generic $g \in \mathcal{W}$, a local chart can be obtained by exchanging the branch values of $g$ by the list $(\sigma_1(g), \ldots, \sigma_{4k}(g))$, where $\sigma_i(g)$ is the value of the symmetric elementary polynomial of degree $i$ on the unordered list of $4k$ (not necessarily distinct) branch values of $g$, $1 \leq i \leq 4k$. These symmetric elementary polynomials can be considered as globally defined holomorphic functions $\sigma_i : \mathcal{W} \to \mathbb{C}$, $1 \leq i \leq 4k$. Also, the map $(\sigma_1, \ldots, \sigma_{4k}) : \mathcal{W} \to \mathbb{C}^{4k}$ is a local diffeomorphism, hence, $\mathcal{W}$ can be seen as an open submanifold of $\mathbb{C}^{4k}$.

The following result deals with compact analytic subvarieties of $\mathcal{W}$. A subset $V$ of a complex manifold $N$ is said to be an *analytic subvariety* if for any $p \in N$ there exists a neighborhood $U$ of $p$ in $N$ and a finite number of holomorphic functions $f_1, \ldots, f_r$ on $U$ such that $U \cap V = \{q \in U \mid f_i(q) = 0, \ 1 \leq i \leq r\}$. The proof of the following lemma can be found in [**14**], Lemma 4 (although the definition of $\mathcal{W}$ in [**14**] is different, the proof still works in our case).

**Lemma 2.** *The only compact analytic subvarieties of $\mathcal{W}$ are finite subsets.*

**4.1. The height differential associated to a marked meromorphic map.** Let $g = (\mathbb{M}, g, p_1, \ldots, p_{2k}, q_1, \ldots, q_{2k}, [\gamma]) \in \mathcal{W}$ be a marked meromorphic map. Since the complex space of holomorphic differentials on $\mathbb{M}$ is 1-dimensional, there exists a unique holomorphic 1-form $\phi = \phi(g)$ on $\mathbb{M}$ such that

$$(12) \qquad \int_\gamma \phi = 2\pi i.$$

The pair $(g, \phi)$ must be seen as the Weierstrass data of a potential surface in the setting of Theorem 1, defined on $M = g^{-1}(\mathbb{C}^*)$, with Gauss map $g$ and height differential $\phi$. Equation (12) means that the period of $(g, \phi)$ along $\gamma$ is horizontal and its flux along $\gamma$ has third coordinate $2\pi$. We will say that $g \in \mathcal{W}$ *closes periods* when there exists $a \in \mathbb{R}^*$ such that (3) and the first equation in (4) hold with $dh = \phi$ and $\gamma_2 = \gamma$ (note that the second equation in (4) holds by definition of $\phi$).

**Lemma 3.** *If $g \in \mathcal{W}$ closes periods, then $(g, \phi)$ is the Weierstrass pair of a properly immersed minimal surface $M \subset \mathbb{T} \times \mathbb{R}$ for a certain 2-dimensional flat torus $\mathbb{T}$, with total curvature $8k\pi$ and $4k$ horizontal Scherk-type ends. Furthermore, the fluxes at the ends $p_1, q_1, \ldots, p_k, q_k$ are equal to $(\pi a, 0, 0)$ and opposite to the fluxes at $p_{k+1}, q_{k+1}, \ldots, p_{2k}, q_{2k}$ (here $a \in \mathbb{R}^*$ comes from equation (3)).*

*Proof.* Since $g$ closes periods, it follows that the period vectors of $(g, \phi)$ at the ends are $\pm(0, \pi a, 0)$. Hence, it suffices to check that if $\alpha \subset g^{-1}(\mathbb{C}^*)$ is a closed curve with $\{[\alpha], [\gamma]\}$ forming a basis of $H_1(g^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$, then the period of $(g, \phi)$ along $\alpha$ is linearly independent of $(0, \pi a, 0)$. This holds because $\phi$ is a non-zero holomorphic differential on the torus $g^{-1}(\overline{\mathbb{C}})$ with purely imaginary integral along $\gamma$, hence $\Re \int_\alpha \phi \neq 0$. q.e.d.

Equation (3) gives $4k$ residue equations on a marked meromorphic map $g \in \mathcal{W}$, which are necessary for $g$ to close periods with $dh = \phi$. The fact that the sum of the residues of a meromorphic differential on a compact Riemann surface equals zero implies that it suffices to impose (3) for $1 \leq j \leq 2k - 1$, so we end up with $4k - 2$ equations. Provided that the first equation in (4) also holds for $g$ with $\gamma_2 = \gamma$, the horizontal component of the flux of the corresponding immersed minimal surface $M$ in Lemma 3, along $\gamma$ is given by

$$(13) \qquad F(\gamma) = i \int_\gamma g\phi \in \mathbb{C} \equiv \mathbb{R}^2.$$

**4.2. The ligature map.** We define the *ligature map* $L : \mathcal{W} \to \mathbb{C}^{4k}$ as the map that associates to each marked meromorphic map $g \in \mathcal{W}$ the $4k$-tuple

$$
\begin{aligned}
L(g) \;=\; & \Big( \mathrm{Res}_{p_1} \left( g^{-1} \phi \right), \ldots, \mathrm{Res}_{p_{2k-1}} \left( g^{-1} \phi \right), \\
& \mathrm{Res}_{q_1}(g\phi), \ldots, \mathrm{Res}_{q_{2k-1}}(g\phi), \int_\gamma g^{-1} \phi, \int_\gamma g\phi \Big).
\end{aligned}
$$

As $\phi$ depends holomorphically on $g$ and all the components of $L$ can be computed as integrals along curves contained in $g^{-1}(\Omega)$ (see the definition of the topology of $\mathcal{W}$), we conclude that $L$ is holomorphic. We consider the subset of marked meromorphic maps that close periods, $\mathcal{M} = \{ g \in \mathcal{W} \mid L(g) = L_{(a,b)}, \text{ with } a \in \mathbb{R}^*, \; b \in \mathbb{C} \}$, where

$$
L_{(a,b)} = \Big( \underbrace{a, \ldots, a}_{1 \leq j \leq k}, \underbrace{-a, \ldots, -a}_{k+1 \leq j \leq 2k-1}, \underbrace{-a, \ldots, -a}_{2k \leq j \leq 3k-1}, \underbrace{a, \ldots, a}_{3k \leq j \leq 4k-2}, b, \overline{b} \Big) \in \mathbb{C}^{4k}.
$$

The following map $J$ defines a canonical injection from the topological space $\widetilde{\mathcal{S}}$ of marked surfaces into $\mathcal{M}$,

$$
(M, p_1, \ldots, p_{2k}, q_1, \ldots, q_{2k}, [\gamma_2])
$$
$$
\overset{J}{\mapsto} \left( g^{-1}(\overline{\mathbb{C}}), g, p_1, \ldots, p_{2k}, q_1, \ldots, q_{2k}, [\gamma_2] \right),
$$

with $g$ being the Gauss map of $M$.

**Lemma 4.** $J : \widetilde{\mathcal{S}} \to \mathcal{M}$ *is an embedding, where $\mathcal{M}$ has the restricted topology from the one of $\mathcal{W}$. Moreover, if we identify $\widetilde{\mathcal{S}}$ with $J(\widetilde{\mathcal{S}})$, then $\widetilde{\mathcal{S}}$ is open and closed in $\mathcal{M}$.*

*Proof.* First, consider a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ converging to a marked surface $M \in \widetilde{\mathcal{S}}$. It is a standard fact that the Gauss map $g_n$ of $M_n$ converges uniformly as $n \to \infty$ to the Gauss map $g$ of $M$ on the compactified torus. From here, it is not difficult to check that $J(M_n)$ lies in an arbitrarily small neighborhood of $J(M)$ in the topology of $\mathcal{W}$ for $n$ large enough, so $J$ is continuous. Both the continuity of the inverse map $J^{-1}$ and the openness and closeness of $\widetilde{\mathcal{S}}$ in $\mathcal{M}$ can be deduced from an argument similar to the one in the proof of Lemma 6 of [14] (in fact, our setting is easier than [14] because the Gauss map of any surface in $\mathcal{S}$ is unbranched at the ends). q.e.d.

In the sequel, we will identify $J$ with the inclusion map and see $\widetilde{\mathcal{S}}$ as a subset of $\mathcal{M}$.

**Proposition 3.** *Let $H \in \mathbb{R}^3 - \{0\}$ be a vector parallel to the $x_2$-axis and $F \in \mathbb{C}$. Then, the set of marked surfaces $M \in \widetilde{\mathcal{S}}$ whose periods at the ends are $\pm H$ and whose flux vector along the homology class in the last component of $M$ equals $(F, 2\pi) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3$ is an analytic subvariety of $\mathcal{W}$.*

*Proof.* Fix $a \in \mathbb{R}^*$ and $b \in \mathbb{C}$. It follows from (2) and (13) that the set $\mathcal{M}(a, b) = \{g \in \mathcal{W} \mid L(g) = L_{(a,b)}\}$ coincides with the set of immersed minimal surfaces $M \in \mathcal{M}$ whose periods and fluxes at the ends are $P_{p_j} = -P_{p_{k+j}} = -P_{q_j} = P_{q_{k+j}} = -(0, \pi a, 0)$, $F_{p_j} = -F_{p_{k+j}} = F_{q_j} = -F_{q_{k+j}} = (\pi a, 0, 0)$, for $j = 1, \ldots, k$, and whose flux along the last component $[\gamma]$ of the marked meromorphic map $g$ is $F_\gamma = (i\overline{b}, 2\pi)$. As $L$ is holomorphic, $\mathcal{M}(a, b)$ is an analytic subvariety of $\mathcal{W}$. As a simultaneously open and closed subset of an analytic subvariety is also an analytic subvariety, Lemma 4 implies that $\widetilde{\mathcal{S}} \cap \mathcal{M}(a, b)$ is an analytic subvariety of $\mathcal{W}$, which proves the proposition. q.e.d.

**Definition 1.** The value of the ligature map $L$ at a marked surface $M \in \widetilde{\mathcal{S}}$ is determined by two numbers $a \in \mathbb{R}^*, b \in \mathbb{C}$ so that $\mathrm{Res}_{p_1}(g^{-1} dh) = a$ and $F_{\gamma_2} = (i\overline{b}, 2\pi)$. We define the *classifying map* $C : \widetilde{\mathcal{S}} \to \mathbb{R}^* \times \mathbb{C}$ by $C(M) = (a, b)$.

**Remark 4.** Let $M \in \mathcal{S}$ be a geometric surface, seen as two marked surfaces $M_1, M_2 \in \widetilde{\mathcal{S}}$ with associated homology classes $[\gamma_2(M_1)], [\gamma_2(M_2)] \in H_1(M, \mathbb{Z})$ such that $[\gamma_2(M_1)] = [\gamma_2(M_2)]$ in $H_1(\mathbb{M}, \mathbb{Z})$ (here, $\mathbb{M}$ is the compactification of $M$). Then, $\gamma_2(M_1) \cup \gamma_2(M_2)$ bounds an even number of ends whose periods add up to zero, and the components of $C$ at $M_1, M_2$ satisfy $a(M_1) = \pm a(M_2) \in \mathbb{R}^*$, $b(M_1) = b(M_2) + m\pi a(M_1)$ with $m \in \mathbb{Z}$ even.

## 5. Properness

In the sequel, we will denote by $K_\Sigma$ the Gaussian curvature function of any surface $\Sigma$.

**Lemma 5.** *Let $\mathcal{P}$ be a rank 2 lattice in $\mathbb{R}^3$ and $\Sigma \subset \mathbb{R}^3/\mathcal{P}$ a properly embedded non-flat (orientable) minimal surface with finite topology and $4k$ horizontal Scherk-type ends. Denote by $H, T$ two generators of $\mathcal{P}$ and assume that $H$ points to the $x_2$-axis. Suppose also that $|K_\Sigma| \leq c$ where $c > 0$. Then,*

(i) *Both $\|H\|$ and the vertical distance between consecutive left (resp. right) ends of $\Sigma$ are not less than $2/\sqrt{c}$, and the third coordinate of $T$ satisfies $|T_3| \geq 4k/\sqrt{c}$.*

(ii) *The injectivity radius of $\mathbb{R}^3/\mathcal{P}$ is bounded by below by $1/\sqrt{c}$ and $\Sigma$ admits a regular open neighborhood of radius $1/\sqrt{c}$ in $\mathbb{R}^3/\mathcal{P}$.*

*Proof.* Let $\widetilde{\Sigma} \subset \mathbb{R}^3$ be the connected properly embedded doubly periodic minimal surface obtained by lifting $\Sigma$. As $|K_{\widetilde{\Sigma}}| \leq c$ and the ends of $\widetilde{\Sigma}$ are asymptotic to horizontal halfplanes, a standard application of the maximum principle shows that $\widetilde{\Sigma}$ has a regular neighborhood $\widetilde{\Sigma}(\varepsilon)$ of positive radius $\varepsilon = 1/\sqrt{c}$ (see [**16**, **24**] for similar arguments). In particular, the vertical separation between two consecutive left ends of $\widetilde{\Sigma}$ in the same fundamental domain is greater than or equal to $2\varepsilon$. As $\Sigma$ has $2k$ left ends, it follows that $|T_3| \geq 4k\varepsilon$. Now, take a point $p \in \widetilde{\Sigma}$ where the normal vector to $\widetilde{\Sigma}$ points to $(0, 1, 0)$. $\widetilde{\Sigma}$ can be expressed locally around $p$ as a graph $\mathcal{G}$ over a disk in the tangent space at $p$ and the same holds around the translated point $p + H$. Reasoning as above with $\mathcal{G}, \mathcal{G} + H$ instead of consecutive left ends of $\widetilde{\Sigma}$, we have that $\|H\| \geq 2\varepsilon$. This proves (i).

Let $r$ be the injectivity radius of $\mathbb{R}^3/\mathcal{P}$. Since $\mathbb{R}^3/\mathcal{P}$ is flat, there are no conjugate points on geodesics in $\mathbb{R}^3/\mathcal{P}$ and thus, $r$ is half of the minimum length of a closed geodesic in $\mathbb{R}^3/\mathcal{P}$. Therefore,

$$2r = \min_{v \in \mathcal{P}-\{0\}} \|v\| \geq \min(\|H\|, |T_3|) \geq \min\left(\frac{2}{\sqrt{c}}, \frac{4k}{\sqrt{c}}\right) = \frac{2}{\sqrt{c}}.$$

Finally, any two distinct points $p, q \in \mathbb{R}^3$ translated by a vector in $\mathcal{P}$ are separated by a distance greater than or equal to $2r$. It follows that $\widetilde{\Sigma}(\varepsilon)/\mathcal{P}$ is a regular neighborhood of $\Sigma$ of radius $1/\sqrt{c}$.          q.e.d.

Next, we describe all possible limits of sequences $\{M_n\}_n$ of surfaces under the hypotheses of Theorem 1, with the additional assumption of having uniform curvature bounds. For such a surface $M_n$, Proposition 1 insures that its period lattice $\mathcal{P}_n$ is generated by the horizontal period vector $H_n = (0, \pi a_n, 0)$ at the ends of $M_n$ and by a non-horizontal vector $T_n = P_{\gamma_1(n)} \neq 0$, where $a_n > 0$, $\gamma_1(n) \subset M_n$ is a closed curve so that $[\gamma_1(n)] \neq 0$ in $H_1(\mathbb{M}_n, \mathbb{Z})$ and $\mathbb{M}_n$ is the compactification of $M_n$. We will use this notation throughout the proof of the following proposition.

**Proposition 4.** *Let $\{\widetilde{M}_n\}_n$ be a sequence of doubly periodic minimal surfaces, each $\widetilde{M}_n$ invariant by a rank 2 lattice $\mathcal{P}_n$ as above. Suppose that for all $n$, $M_n = \widetilde{M}_n/\mathcal{P}_n$ has genus 1 and $4k$ horizontal Scherk-type ends, $\widetilde{M}_n$ passes through the origin of $\mathbb{R}^3$ and $|K_{\widetilde{M}_n}(0)| = 1$ is a maximum value of $|K_{\widetilde{M}_n}|$. Then (after passing to a subsequence), $\widetilde{M}_n$ converges uniformly on compact subsets of $\mathbb{R}^3$ with multiplicity 1 to a*

*properly embedded minimal surface* $\widetilde{M}_\infty \subset \mathbb{R}^3$ *which lies in one of the following cases:*

(i) $\widetilde{M}_\infty$ *is a vertical catenoid with flux* $(0, 0, 2\pi)$. *In this case, both* $\{H_n\}_n, \{T_n\}_n$ *are unbounded for any choice of* $T_n$ *as above.*

(ii) $\widetilde{M}_\infty$ *is a vertical helicoid with period vector* $(0, 0, 2\pi m)$ *for some* $m \in \mathbb{N}$. *Now,* $\{H_n\}_n$ *is unbounded and there exists a choice of* $T_n$ *for which* $\{T_n\}_n \to (0, 0, 2\pi m)$ *as* $n \to \infty$.

(iii) $\widetilde{M}_\infty$ *is a Riemann minimal example with horizontal ends. Moreover,* $\{H_n\}_n$ *is unbounded and certain choices of the* $T_n$ *converge to the period vector of* $\widetilde{M}_\infty$ *as* $n \to \infty$.

(iv) $\widetilde{M}_\infty$ *is a singly periodic Scherk minimal surface, two of whose ends are horizontal. Furthermore, any choice of* $\{T_n\}_n$ *is unbounded,* $\{H_n\}_n$ *converges to the period vector* $H_\infty = (0, a, 0)$ *of* $\widetilde{M}_\infty$ *(with* $a > 0$), *and* $\widetilde{M}_\infty/H_\infty$ *has genus zero.*

(v) $\widetilde{M}_\infty$ *is a doubly periodic Scherk minimal surface. In this case,* $\{H_n\}_n, \{T_n\}_n$ *converge respectively to period vectors* $H_\infty, T_\infty$ *of* $\widetilde{M}_\infty$, *and* $\widetilde{M}_\infty/\{H_\infty, T_\infty\}$ *has genus zero with at least two horizontal ends and exactly two non-horizontal ends.*

(vi) $\widetilde{M}_\infty$ *is a doubly periodic minimal surface invariant by a rank* 2 *lattice* $\mathcal{P}_\infty$, $M_\infty = \widetilde{M}_\infty/\mathcal{P}_\infty$ *has genus one and* $4k$ *horizontal Scherk-type ends, and* $\{H_n\}_n \to H_\infty$, $\{T_n\}_n \to T_\infty$, *where* $H_\infty, T_\infty$ *are defined by Proposition 1 applied to* $M_\infty$.

*Proof.* By Lemma 5, $\{\widetilde{M}_n\}_n$ has local uniform area bounds. As $\{|K_{\widetilde{M}_n}|\}_n$ is uniformly bounded and the origin is an accumulation point of $\{\widetilde{M}_n\}_n$, after choosing a subsequence $\{\widetilde{M}_n\}_n$ converges uniformly on compact subsets of $\mathbb{R}^3$ to a properly embedded minimal surface $\widetilde{M}_\infty$ with $0 \in \widetilde{M}_\infty$ and $|K_{\widetilde{M}_\infty}(0)| = 1$. Since $\widetilde{M}_\infty$ is complete, orientable and not flat, it is not stable (see e.g. do Carmo and Peng [1] or Fischer–Colbrie and Schoen [2]). This implies that the multiplicity of the limit $\{\widetilde{M}_n\}_n \to \widetilde{M}_\infty$ is one, see for instance [21]. We now discuss on the boundedness of the sequences $\{H_n\}_n, \{T_n\}_n$.

**Suppose that both $\{H_n\}_n$ and any choice of $\{T_n\}_n$ are unbounded.** After passing to a subsequence, we can assume that the minimum length of a non-zero vector in $\mathcal{P}_n$ diverges to $\infty$ as $n \to \infty$. Thus, $\widetilde{M}_\infty$ can be viewed as limit of surfaces with boundary $\widetilde{\Sigma}_n$ contained in fundamental domains of $\widetilde{M}_n$, so $\widetilde{M}_\infty$ has finite total curvature. As each $\widetilde{\Sigma}_n$ has genus zero, the same holds for $\widetilde{M}_\infty$. Therefore, $\widetilde{M}_\infty$ is a catenoid (López and Ros [12]). As the Gauss map of $\widetilde{M}_n$ omits the

vertical directions for all $n$, the same holds for $\widetilde{M}_\infty$ by the Open Mapping theorem. Hence, $\widetilde{M}_\infty$ is a vertical catenoid. Since $|K_{\widetilde{M}_\infty}|$ reaches a maximum at the origin with value 1, the flux of $\widetilde{M}_\infty$ must be $(0, 0, 2\pi)$.

**Suppose that $\{H_n\}_n$ is unbounded and there is a bounded choice of $\{T_n\}_n$.** We do not loss generality assuming that $\{H_n\}_n \to \infty$ and $\{T_n\}_n \to T_\infty \in \mathbb{R}^3$. By Lemma 5, $|(T_n)_3| \geq 4k$ for all $n$, thus $|(T_\infty)_3| \geq 4k$. In particular, $T_\infty \neq 0$ and $\widetilde{M}_\infty$ is singly periodic, invariant by the translation by $T_\infty$. As the Gauss map of $\widetilde{M}_\infty/T_\infty$ has degree at most $2k$ and this surface is properly embedded in $\mathbb{R}^3/T_\infty$, it must have finite topology, and all its ends are simultaneously asymptotic to ends of planes, helicoids or Scherk minimal surfaces. Note that $\widetilde{M}_\infty/T_\infty$ cannot have Scherk-type ends, because otherwise $\widetilde{M}_\infty/T_\infty$ would have at least two horizontal Scherk-type ends (the Gauss map of $\widetilde{M}_\infty/T_\infty$ omits the vertical directions), so $T_\infty$ would be horizontal, a contradiction.

Next, we prove that if the ends of $\widetilde{M}_\infty/T_\infty$ are helicoidal, then $\widetilde{M}_\infty$ is a vertical helicoid. As the Gauss map $g_\infty$ of $\widetilde{M}_\infty/T_\infty$ misses $0, \infty$, we deduce that the height differential $dh_\infty$ of $\widetilde{M}_\infty/T_\infty$ has no zeros and all the ends of $\widetilde{M}_\infty/T_\infty$ are asymptotic to vertical helicoids, where $dh_\infty$ has simple poles. Calling $P$ to the number of ends of $\widetilde{M}_\infty/T_\infty$ and $G$ to the genus of its compactification, it follows that $P + 2(G-1) = 0$ from where $P = 2$ and $G = 0$. In this situation, it is well known that $\widetilde{M}_\infty$ is a vertical helicoid (Toubiana [**26**] or Pérez and Ros [**19**]). Furthermore, its period vector is of the form $(0, 0, 2\pi m)$ with $m \in \mathbb{N}$ because the maximum absolute curvature of $\widetilde{M}_\infty/T_\infty$ is one. Thus (ii) holds.

Now, assume that the ends of $\widetilde{M}_\infty/T_\infty$ are planar. A direct lifting argument and the maximum principle insure that $\widetilde{M}_\infty/T_\infty$ cannot have genus zero, thus, it is a properly embedded minimal torus in $\mathbb{R}^3/T_\infty$ with a finite number of planar ends. By a theorem of Meeks, Pérez and Ros [**14**], $\widetilde{M}_\infty/T_\infty$ is a Riemann minimal example. As its Gauss map misses the vertical directions, the ends of $\widetilde{M}_\infty/T_\infty$ are horizontal and we have (iii).

**Suppose that $\{H_n\}_n$ is bounded and any choice of $\{T_n\}_n$ is unbounded.** After extracting a subsequence, we can assume $\{H_n\}_n \to H_\infty = (0, a, 0)$ with $a \in [0, \infty)$ and $\{T_n\}_n \to \infty$. By Lemma 5, $a \geq 2$ and $\widetilde{M}_\infty$ is singly periodic, invariant by translation by $H_\infty$. The well defined Gauss map $g_\infty$ of $\widetilde{M}_\infty/H_\infty$ has degree less than or equal to $2k$, hence, $\widetilde{M}_\infty/H_\infty$ has finite total curvature. As $\widetilde{M}_\infty/H_\infty$ is embedded and proper, all its ends are again simultaneously planar, helicoidal or

of Scherk-type. Moreover, $\widetilde{M}_\infty/H_\infty$ has at least two ends with vertical limit normal vector because $g_\infty$ omits $0, \infty$. As $H_\infty$ is horizontal, properness of $\widetilde{M}_\infty/H_\infty$ prevents its ends to be planar. Also, its ends are not helicoidal because in such case the period vectors at the ends would be vertical. Hence, $\widetilde{M}_\infty/H_\infty$ has Scherk-type ends (at least two of them horizontal) and genus zero or one.

**Assertion 1.** In the above situation, the well defined height differential $dh_\infty$ of $\widetilde{M}_\infty/H_\infty$ cannot have a zero at any horizontal end (equivalently, $g_\infty$ is unbranched at the horizontal ends of $\widetilde{M}_\infty/H_\infty$).

To see that the assertion holds, suppose on the contrary that $g_\infty$ is branched at a puncture $p_\infty$ corresponding to a horizontal Scherk-type end of $\widetilde{M}_\infty/H_\infty$, say with value zero. Since the Gauss map $g_n$ of $\widetilde{M}_n/H_n$ is unbranched at its zeros, there exists an integer $l \geq 2$ such that $l$ distinct zeros $p_1(n), \ldots, p_l(n)$ of $g_n$ converge to $p_\infty$ (we can think of pieces of the compactifications of $\widetilde{M}_\infty/T_\infty, \widetilde{M}_n/T_n$ defined on a common disk $D$ centered at $p_\infty$ so that $p_1(n), \ldots, p_l(n) \in D$ for $n$ large). Moreover, the heights of the ends of $\widetilde{M}_n/H_n$ corresponding to $p_1(n), \ldots, p_l(n)$ converge to the height of the end $p_\infty$ of $\widetilde{M}_\infty/H_\infty$. By Lemma 5, we deduce that $l = 2$ and $p_1(n), p_2(n)$ are a left end and a right end, and so their fluxes are opposite and the same holds for their periods. Therefore, both the period and flux of $\widetilde{M}_\infty/H_\infty$ at $p_\infty$ vanish, a contradiction. This proves Assertion 1.

Note that $dh_\infty$ has a pole at each non-horizontal Scherk-type end of $\widetilde{M}_\infty/H_\infty$ (and there are at least two of these ends by the maximum principle). Using Assertion 1 and the fact that $g_\infty$ misses $0, \infty$ on $\widetilde{M}_\infty/H_\infty$, we conclude that $dh_\infty$ has no zeros at the compactification of $\widetilde{M}_\infty/H_\infty$. Thus, $\widetilde{M}_\infty/H_\infty$ has genus zero and $dh_\infty$ has exactly two poles or equivalently, $\widetilde{M}_\infty/H_\infty$ has exactly two non-horizontal Scherk-type ends. In this situation, Pérez and Traizet [**22**] have proved that $\widetilde{M}_\infty/H_\infty$ is a singly periodic Scherk minimal surface of genus zero. This shows (iv).

**Suppose that $\{H_n\}_n$ is bounded and there exists a bounded choice of $\{T_n\}_n$.** As before, it can be supposed that $\{H_n\}_n \to H_\infty = (0, a, 0)$ with $a \geq 2$ and $\{T_n\}_n \to T_\infty \in \mathbb{R}^3$ with $|(T_\infty)_3| \geq 4k$. Thus, $\widetilde{M}_\infty$ is doubly periodic, invariant by the rank 2 lattice $\mathcal{P}_\infty$ generated by $H_\infty, T_\infty$ and $\widetilde{M}_\infty/\mathcal{P}_\infty$ has genus zero or one with a finite number of Scherk-type ends.

Since the Gauss map $g_\infty$ of $\widetilde{M}_\infty/\mathcal{P}_\infty$ misses $0, \infty$, this surface has at least two horizontal ends. A suitable modification of Assertion 1 gives

that the height differential $dh_\infty$ of $\widetilde{M}_\infty/\mathcal{P}_\infty$ (which is also well defined) does not vanish at any of the horizontal ends of this surface. Therefore, either $\widetilde{M}_\infty/\mathcal{P}_\infty$ has genus zero with exactly two non-horizontal Scherk-type ends, or it has genus one and all its ends are horizontal. In the first case, a theorem by Wei [**28**] (see also Lazard–Holly and Meeks [**11**]) insures that $\widetilde{M}_\infty$ is a doubly periodic Scherk minimal surface, so (v) is proved. Finally, assume that $\widetilde{M}_\infty/\mathcal{P}_\infty$ has genus one. Since the total curvature of $M_n$ is $8k\pi$, $\widetilde{M}_\infty/\mathcal{P}_\infty$ must have total curvature at most $8k\pi$. Using the Meeks–Rosenberg formula [**15**] and the fact that $\widetilde{M}_\infty/\mathcal{P}_\infty$ is properly embedded with genus one and Scherk-type ends, we deduce that $\widetilde{M}_\infty/\mathcal{P}_\infty$ has at most $4k$ ends. On the other hand, the spacing between left (resp. right) ends of $M_n$ is bounded away from zero by Lemma 5, which implies that the $2k$ heights corresponding to the left (resp. right) ends of $M_n$ converge to $2k$ distinct heights in $\mathbb{R}^3/H_\infty$. Since non-compact horizontal level sets of the $\widetilde{M}_n/H_n$ tend to non-compact level sets of $\widetilde{M}_\infty/H_\infty$, we conclude that $\widetilde{M}_\infty/\mathcal{P}_\infty$ has exactly $4k$ horizontal ends. Now, the proof is complete.                q.e.d.

The following result is a crucial curvature estimate in terms of the classifying map $C$.

**Proposition 5.** *Let $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ be a sequence of marked surfaces. Suppose that $C(M_n) = (a_n, b_n) \in \mathbb{R}^* \times \mathbb{C}$ satisfies*

1) *$\{a_n\}_n$ is bounded away from zero.*
2) *$\{|b_n|\}_n$ is bounded from above.*

*Then, the sequence of Gaussian curvatures $\{K_{M_n}\}_n$ is uniformly bounded.*

*Proof.* The proof is based on the one of Theorem 4 in [**14**], so we will only go into the details of what is new in this setting. Arguing by contradiction, assume that $\lambda_n := \max_{M_n} \sqrt{|K_{M_n}|} \to \infty$ as $n \to \infty$. Let $\Sigma_n = \lambda_n M_n \subset \mathbb{R}^3/\lambda_n \mathcal{P}_n$, where $\mathcal{P}_n = \mathrm{Span}\{H_n, T_n\}$ is the rank 2 lattice associated to $M_n$ and $H_n$ is the period vector at the ends of $M_n$ (up to sign). Let us also denote by $\widehat{\Sigma}_n, \widetilde{\Sigma}_n$, the respective liftings of $\Sigma_n$ to $\mathbb{R}^3/\lambda_n T_n$ and to $\mathbb{R}^3$.

After translation of $\widetilde{\Sigma}_n$ to have maximum absolute Gauss curvature one at the origin, Proposition 4 implies that (after passing to a subsequence) $\{\widetilde{\Sigma}_n\}_n$ converges smoothly to a properly embedded minimal surface $\mathcal{H}_1 \subset \mathbb{R}^3$, which must lie in one of the six possibilities in Proposition 4. Since $a_n$ is bounded away from zero and $\lambda_n \to \infty$, the period vectors $\lambda_n H_n$ at the ends of $\Sigma_n$ diverge to $\infty$. Therefore, $\mathcal{H}_1$ must be

a vertical catenoid, a vertical helicoid or a Riemann minimal example with horizontal ends. If $\mathcal{H}_1$ were a catenoid or a Riemann minimal example, then the vertical part of the flux of $\Sigma_n$ along a compact horizontal section, which is $2\pi\lambda_n$ by item 7 of Proposition 1, would converge to the vertical part of the flux of $\mathcal{H}_1$, which is finite. This contradicts that $\lambda_n \to \infty$ and so, $\mathcal{H}_1$ is a vertical helicoid with period vector $T = (0, 0, 2\pi m)$ for certain $m \in \mathbb{N}$ (furthermore, we can choose the period vector $T_n$ of $M_n$ with $\{\lambda_n T_n\}_n \to T$ as $n \to \infty$, see Proposition 4).

Let $\pi_n : \mathbb{R}^3/\lambda_n\mathcal{P}_n \to \{x_3 = 0\}/\lambda_n\mathcal{P}_n$ be the linear projection in the direction of $T_n$, $\Pi_n^H : \mathbb{R}^3/\lambda_n T_n \to \mathbb{R}^3/\lambda_n\mathcal{P}_n$ the quotient projection modulo $\lambda_n H_n$ and $\mathbb{D}$ the unit disk centered at the origin in $\{x_3 = 0\}$. From now on we will only consider $n$ large so that $\mathcal{H}_1(n) = \Sigma_n \cap \pi_n^{-1}(\mathbb{D}/\lambda_n\mathcal{P}_n)$ is connected and extremely close to a piece of $\mathcal{H}_1$ containing its axis, $\lambda_n T_n$ being close to $T$. Let $\overline{\Sigma}_n$ be the torus obtained by attaching to $\Sigma_n$ its $4k$ ends. Note that $\mathcal{H}_1(n)$ does not separate $\overline{\Sigma}_n$, hence, $\mathcal{F}_n = \overline{\Sigma}_n - \mathcal{H}_1(n)$ is a compact annulus that contains the $4k$ ends of $\Sigma_n$.

Let $N_n$ be the Gauss map of $\Sigma_n$. We claim that for all $n$ large, $N_n$ takes horizontal values on $\mathcal{F}_n$. To see this, consider the compact intersection $\Gamma_n$ of $\Sigma_n$ with a totally geodesic horizontal cylinder in $\mathbb{R}^3/\lambda_n\mathcal{P}_n$ not asymptotic to the ends of $\Sigma_n$. Viewed in $\overline{\Sigma}_n$, $\Gamma_n$ is a non-nulhomotopic embedded closed curve whose intersections with $\mathcal{H}_1(n), \mathcal{F}_n$ are two open arcs with common end points $A_n, B_n$. As $\mathcal{H}_1(n)$ is very close to a piece of a vertical helicoid containing its axis, we deduce that $N_n(A_n), N_n(B_n)$ are in different hemispheres of $\mathbb{S}^2$ with respect to the vertical direction. Now, our claim follows by continuity.

Let $\mathbb{S}^1 \subset \mathbb{S}^2$ be the horizontal equator in the sphere. We next show that for all $\theta \in \mathbb{S}^1$ and $n \in \mathbb{N}$ large enough, $\mathcal{F}_n \cap N_n^{-1}(\theta)$ is not empty. Since $\mathcal{H}_1(n)$ is almost a piece of a vertical helicoid containing its axis, $(N_n|_{\mathcal{H}_1(n)})^{-1}(\mathbb{S}^1)$ must be a simple closed curve in $\Sigma_n$ (viewed in $\mathbb{R}^3$, $(N_n|_{\mathcal{H}_1(n)})^{-1}(\mathbb{S}^1)$ is an open embedded arc whose end points differ in $\lambda_n T_n$) that covers $\mathbb{S}^1$ with finite multiplicity through $N_n$. As $N_n$ is horizontal somewhere in $\mathcal{F}_n$, we deduce that the number of sheets of this last covering is less than $2k$, which implies that for all $\theta \in \mathbb{S}^1$, $\mathcal{F}_n \cap N_n^{-1}(\theta)$ is not empty as desired.
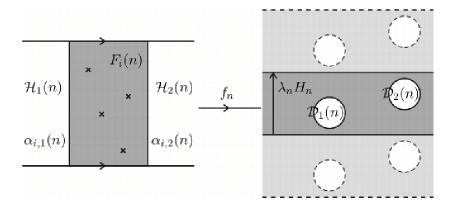
Next, we produce a second helicoid as a limit of different translations of the $\widetilde{\Sigma}_n$. Since the total branching number of $N_n$ is $4k$, there exists a spherical disk $D_{\mathbb{S}^2}(\theta_n, \varepsilon) \subset \mathbb{S}^2$ centered at some $\theta_n \in \mathbb{S}^1$ with uniform radius $\varepsilon > 0$ which is free of branch values of $N_n$. By the last paragraph, for $n$ large, we can find points $p_n \in \widetilde{\Sigma}_n$ with $N_n(p_n) = \theta_n$ and whose projections into $\mathbb{R}^3/\lambda_n\mathcal{P}_n$ lie inside $\mathcal{F}_n$. The sequence of translated surfaces $\{\widetilde{\Sigma}_n - p_n\}_n$ has uniform curvature and area bounds and

all the $\widetilde{\Sigma}_n - p_n$ have normal vector $\theta_n$ at the origin. After passing to a subsequence, $\widetilde{\Sigma}_n - p_n$ converges uniformly on compact subsets of $\mathbb{R}^3$ to a properly embedded minimal surface $\mathcal{H}_2$ having vertical tangent plane at the origin. $\mathcal{H}_2$ is not flat, because otherwise one could construct an arc $\widetilde{\alpha}_n$ contained in $\{x \in \widetilde{\Sigma}_n \mid d_{\widetilde{\Sigma}_n}(x, p_n) < \frac{3}{2}\|T\|\}$, $\widetilde{\alpha}_n$ giving rise to the period vector $\lambda_n T_n$. By construction, $N_n(\widetilde{\alpha}_n)$ would lie in $D_{\mathbb{S}^2}(\theta_n, \varepsilon)$ for $n$ large enough, but this last disk is free of branch values of $N_n$ hence $N_n^{-1}(D_{\mathbb{S}^2}(\theta_n, \varepsilon))$ consists of disjoint disks inside $\Sigma_n$, one of which would contain $\alpha_n = \widetilde{\alpha}_n / \lambda_n \mathcal{P}_n$ in contradiction with the fact that $\alpha_n$ is homotopically non-trivial on $\Sigma_n$. Hence, $\mathcal{H}_2$ is not flat. Applying Proposition 4 (suitably modified so that the absolute Gaussian curvature of the surfaces in the sequence is not 1 at the origin, but it is uniformly bounded by above and that the limit surface is not a plane) and our previous arguments to eliminate all limits other than a helicoid, we conclude that $\mathcal{H}_2$ is another vertical helicoid. Since both $\mathcal{H}_1, \mathcal{H}_2$ are limits of translations of the $\widetilde{\Sigma}_n$, the period vector of $\mathcal{H}_2$ is again $T = \lim_n \lambda_n T_n$.

Consider two disjoint round disks $\mathcal{D}_1(n), \mathcal{D}_2(n)$ in the totally geodesic cylinder $\{x_3 = 0\}/\lambda_n \mathcal{P}_n \subset \mathbb{R}^3/\lambda_n \mathcal{P}_n$, with common radius $r_n$ such that the annular component $\mathcal{H}_i(n) = \Sigma_n \cap \pi_n^{-1}(\mathcal{D}_i(n))$ is arbitrarily close to a translated copy of the forming helicoid $\mathcal{H}_i/T$ minus neighborhoods of its ends, $i = 1, 2$. After passing to a subsequence, we can also choose $r_n$ so that

1) $r_n \to \infty$ as $n \to \infty$.
2) $\frac{r_n}{\lambda_n} \to 0$ as $n \to \infty$.
3) The normal direction to $\Sigma_n$ along the helix-type curves in the boundary of $\mathcal{H}_i(n)$ makes an angle less than $1/n$ with the vertical, $i = 1, 2$.

(Note that we have exchanged the former $\mathcal{H}_1(n) = \Sigma_n \cap \pi_n^{-1}(\mathbb{D})$ by a bigger one $\mathcal{H}_1(n) = \Sigma_n \cap \pi_n^{-1}(D_1(n))$ but all the preceding arguments remain valid now.) We claim that the extended Gauss map $N_n$ applies $\overline{\Sigma}_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$ in the spherical disks centered at the North and South Poles of $\mathbb{S}^2$ with radius $1/n$. Since $N_n$ is an open map, the condition 3 above shows that it suffices to check that $\Sigma_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$ has no points with horizontal normal vector. If horizontal normal vectors occurred in $\Sigma_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$, then our arguments above would give further vertical helicoids $\mathcal{H}_3, \ldots, \mathcal{H}_s$ as limits of translations of subsequences of the $\widetilde{\Sigma}_n$ (finitely many limits because each one consumes at least $-4\pi$ of total curvature, and the total curvature of the $\Sigma_n$ is fixed $-8k\pi$). Therefore, we have $s$ pairwise disjoint annuli $\mathcal{H}_j(n) = \Sigma_n \cap \pi_n^{-1}(\mathcal{D}_j(n))$, each one arbitrarily close to the corresponding vertical helicoid $\mathcal{H}_j$ minus a neighborhood of its two

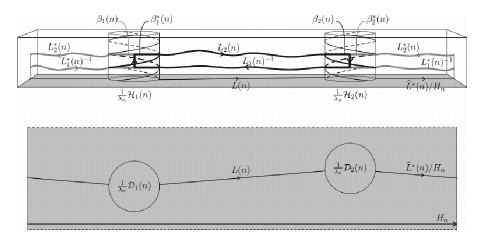**Figure 5.** In this case, the covering map $f_n$ has two sheets.

ends, where each $\mathcal{D}_j(n) \subset \{x_3 = 0\}/\lambda_n \mathcal{P}_n$ is a round disk as before, $j = 1, \ldots, s$. In this setting, the complement of $\mathcal{H}_1(n) \cup \ldots \cup \mathcal{H}_s(n)$ in $\overline{\Sigma}_n$ would consist of $s$ closed annuli $\mathcal{F}_1(n), \ldots, \mathcal{F}_s(n)$ labelled so that $\mathcal{H}_i(n)$ is consecutive to $\mathcal{F}_i(n)$ and all $\mathcal{H}_i(n), \mathcal{F}_i(n)$ are disposed cyclically for $i = 1, \ldots, s$. Moreover, $N_n(\mathcal{F}_1(n) \cup \ldots \cup \mathcal{F}_s(n))$ omits $\mathbb{S}^1$ hence, $N_n(\mathcal{F}_1(n) \cup \ldots \cup \mathcal{F}_s(n))$ is contained in a small neighborhood of the vertical directions in $\mathbb{S}^2$. When restricted to the surface $\mathcal{F}_i(n)$, the projection $\pi_n$ extends smoothly through the ends producing a map

$$(14) \qquad f_n := \pi_n|_{\mathcal{F}_i(n)} : \mathcal{F}_i(n) \to \frac{\{x_3 = 0\} \cup \{\infty\}}{\lambda_n \mathcal{P}_n} - \cup_{j=1}^s \mathcal{D}_j(n).$$

$f_n$ is a proper local diffeomorphism, hence, a finite sheeted covering map. Since $\partial \mathcal{F}_i(n)$ has two components it follows that $s = 2$, a contradiction. Thus, $N_n[\overline{\Sigma}_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))] \subset D_{\mathbb{S}^2}(*, 1/n)$ where $* = 0, \infty$.

So far, we have proved that the two components $\mathcal{F}_1(n), \mathcal{F}_2(n)$ of $\overline{\Sigma}_n - (\mathcal{H}_1(n) \cup \mathcal{H}_2(n))$ are closed annuli, each one is non-compact when viewed in $\Sigma_n$. The number of ends of $\Sigma_n$ in $\mathcal{F}_i(n)$ is twice the number of sheets $\#_i$ of the covering map $f_n$ in (14). Moreover, the boundary components $\alpha_{i,1}(n), \alpha_{i,2}(n)$ of $\mathcal{F}_i(n)$ apply by $f_n$ respectively on the circumferences $\partial \mathcal{D}_1(n), \partial \mathcal{D}_2(n)$, both with multiplicity $\#_i$. Since $\alpha_{1,j}(n), \alpha_{2,j}(n)$ are the helix-type boundary curves of the forming helicoid $\mathcal{H}_j(n)$, we deduce that $\#_1 = \#_2$. Since there are the same number of left and right ends of $\Sigma_n$ in $\mathcal{F}_i(n)$, it follows that $\#_1 = k$, see Figure 5.

In this setting and coming back to the original scale, one can follow the argument in pages 117–118 of [**14**] to construct a closed embedded curve $\Gamma_n \subset M_n$ formed by four consecutive arcs $L_1(n)^{-1} * \beta_1(n) * L_2(n) * \beta_2(n)$, where $L_1(n), L_2(n)$ are liftings of the distance minimizing horizontal segment $L(n)$ from $\frac{1}{\lambda_n} \partial \mathcal{D}_1(n)$ to $\frac{1}{\lambda_n} \partial \mathcal{D}_2(n)$, lying in consecutive

**Figure 6.** Front and top views of two helicoids forming with the direction of $L(n)$ tending to the direction of the $x_2$-axis. In the front view, the sided ends are placed ahead and behind the box that contains both helicoids.

sheets by the covering

$$f_n : \tfrac{1}{\lambda_n}[\mathcal{F}_1(n) \cup \mathcal{F}_2(n)] \to \frac{\{x_3=0\}\cup\{\infty\}}{\mathcal{P}_n} - \tfrac{1}{\lambda_n}[\mathcal{D}_1(n) \cup \mathcal{D}_2(n)]$$

and $\beta_1(n), \beta_2(n)$ are small arcs contained in $\tfrac{1}{\lambda_n}\mathcal{H}_1(n), \tfrac{1}{\lambda_n}\mathcal{H}_2(n)$ respectively (we abuse of notation keeping the label $f_n$ in the original scale of $M_n$, although it has been defined in the scale of $\Sigma_n = \lambda_n M_n$). For later uses, we will describe $\beta_1(n), \beta_2(n)$ more precisely. $\beta_1(n)$ consists of the union of three consecutive arcs $l_1, l_2, l_3$, where $l_1, l_3$ are at almost constant height and $l_2 \subset N_n^{-1}(\mathbb{S}^1)$, and $\beta_2(n)$ is similarly defined, see Figure 6.

Next, we study the relation between the homology classes $[\Gamma_n], [\gamma_2(n)]$ $\in H_1(M_n, \mathbb{Z})$, where $[\gamma_2(n)]$ is the last component of the marked surface $M_n \in \widetilde{\mathcal{S}}$ (recall that $C(M_n) = (a_n, b_n)$ where $F_{\gamma_2(n)} = (F(\gamma_2(n)), 2\pi) = (i\overline{b_n}, 2\pi)$ is the flux vector of $M_n$ along $\gamma_2(n)$). Let $g_n$ be the complex Gauss map of $M_n$. Since $\Gamma_n$ is embedded, non-trivial in $H_1(g_n^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$ and has period zero, Proposition 1, implies that $\Gamma_n$ can be oriented so that $[\Gamma_n] = [\gamma_2(n)]$ in $H_1(g_n^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$. Viewed in $H_1(M_n, \mathbb{Z})$, the classes $[\Gamma_n], [\gamma_2(n)]$ differ by a finite sum of loops around ends, from where
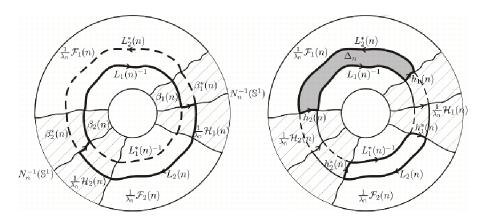
(15) $$F(\Gamma_n) = i\overline{b_n} + t(n)\pi a_n,$$

where $t(n) \in \mathbb{Z}$. Since both $\Gamma_n, \gamma_2(n)$ can be chosen in the same fundamental domain of the doubly periodic lifting $\widetilde{M_n}$ of $M_n$ lying between

two horizontal planes $\Pi, \Pi + T_n$, the embeddedness of both curves insures that $\{t(n)\}_n$ is bounded. Passing to a subsequence, we can suppose that $t = t(n)$ does not depend on $n$. Also, note that $t$ is even since the periods of $M_n$ along both $\Gamma_n, \gamma_2(n)$ are zero. By item 7 of Proposition 1, $(F_{\Gamma_n})_3 = (F_{\gamma_2(n)})_3 = 2\pi$. This property implies that the lengths of $L_1(n), L_2(n)$ diverge to $\infty$ as $n \to \infty$, see [14] page 118. Those arguments in [14] also show that the limit as $n \to \infty$ of the horizontal flux $F(\Gamma_n)$ can be computed as the limit of the horizontal fluxes along $L_1(n) \cup L_2(n)$, and that the horizontal flux along $L_1(n) \cup L_2(n)$ divided by the length of $L(n)$ converges to a complex number of modulus 2 as $n \to \infty$, so $F(\Gamma_n) \to \infty$. Dividing equation (15) by $|F(\Gamma_n)|$ and using that $b_n$ is bounded, we have (after extracting a subsequence) that both $|F(\Gamma_n)|^{-1}F(\Gamma_n)$, $t|F(\Gamma_n)|^{-1}\pi a_n$ converge to the same limit $e^{i\theta}$, $\theta \in [0, 2\pi)$, from where $t \neq 0$ and $a_n \to \infty$. Since $a_n \in \mathbb{R}^*$, we also have $\theta = 0$ or $\pi$. In particular, the direction of the segment $L(n)$ tends to the direction of the $x_2$-axis as $n \to \infty$.

Consider respective liftings $\widehat{L}(n), \widehat{\mathcal{D}}_1(n), \widehat{\mathcal{D}}_2(n)$ of $L(n), \mathcal{D}_1(n), \mathcal{D}_2(n)$ such that $\widehat{L}(n) \cup \frac{1}{\lambda_n}\left[\widehat{\mathcal{D}}_1(n), \widehat{\mathcal{D}}_2(n)\right]$ lies in the same fundamental domain of $\widetilde{M}_n/T_n$. Let $\widehat{L}^*(n)$ be the length minimizing horizontal segment from $\frac{1}{\lambda_n}\partial\widehat{\mathcal{D}}_2(n)$ to $\frac{1}{\lambda_n}\partial\widehat{\mathcal{D}}_1(n) + H_n$. Let $\Gamma_n^* \subset M_n$ be another embedded closed curve constructed in a similar way as $\Gamma_n$, i.e. $\Gamma_n^* = L_1^*(n)^{-1} * \beta_1^*(n) * L_2^*(n) * \beta_2^*(n)$ where $L_1^*(n), L_2^*(n)$ are liftings in consecutive sheets of $\widehat{L}^*(n)/H_n$ and $\beta_1^*(n), \beta_2^*(n)$ are small arcs inside $\frac{1}{\lambda_n}\mathcal{H}_1(n), \frac{1}{\lambda_n}\mathcal{H}_2(n)$ respectively. As before, each of the $\beta_i^*(n)$ consists of the consecutive union of three arcs, two at almost constant height joined by a central one which we choose as $\beta_i(n) \cap N_n^{-1}(\mathbb{S}^1)$, $i = 1, 2$. We orient $\Gamma_n^*$ in such a way that $\Gamma_n, \Gamma_n^*$ share their orientations along the arcs $\beta_i(n) \cap N_n^{-1}(\mathbb{S}^1)$, $i = 1, 2$. Viewed in $H_1(g_n^{-1}(\overline{\mathbb{C}}), \mathbb{Z})$, it holds $[\Gamma_n] = -[\Gamma_n^*]$, see Figure 7 left. As above, we have that after passing to a subsequence, $F(\Gamma_n^*) = -i\overline{b_n} + t^*\pi a_n$ for certain non-zero even integer $t^*$.

Let $\zeta_n \subset M_n$ be the embedded closed curve defined by $\zeta_n = L_1(n)^{-1} * h_1(n) * L_2^*(n) * h_2(n)$, where $h_1(n) \subset \frac{1}{\lambda_n}\partial\mathcal{H}_1(n)$ is a helix-type curve joining the end point of $L_1(n)^{-1}$ with the starting point of $L_2^*(n)$ (so the covering map $f_n$ restricts to $h_1(n)$ as a diffeomorphism onto an arc in $\frac{1}{\lambda_n}\partial\mathcal{D}_1(n)$ arbitrarily close to a halfcircumference), and $h_2(n) \subset \frac{1}{\lambda_n}\partial\mathcal{H}_2(n)$ is similarly defined. Note that we can endow $\zeta_n$ with a natural orientation by restricting the orientation of either $\Gamma_n$ or $\Gamma_n^*$ to their common arcs with $\zeta_n$ (both restrictions give rise to the same orientation). $\zeta_n$ is the boundary of a disk $\Delta_n$ inside $\frac{1}{\lambda_n}[\mathcal{F}_1(n) \cup \mathcal{F}_2(n)]$. We do not lose generality by supposing $\Delta_n \subset \frac{1}{\lambda_n}\mathcal{F}_1(n)$. Since $f_n$ restricts

**Figure 7.** Left: The curves $\Gamma_n, \Gamma_n^*$. Right: The curve $\zeta_n$.

to $\zeta_n$ as a diffeomorphism onto a curve that represents the period vector $H_n$, we deduce that $\Delta_n$ has just one end of $M_n$. In particular, $F(\zeta_n) = \varepsilon \pi a_n$, with $\varepsilon = \pm 1$. Now, consider for $i = 1, 2$ the helix-type curve $h_i^*(n) \subset \frac{1}{\lambda_n} \partial \mathcal{H}_i(n)$ that joins the corresponding extreme points of $L_1^*(n)^{-1}$ and of $L_2(n)$. $h_i^*(n)$ is almost opposite to $h_i(n)$ with respect to the almost axis of $\frac{1}{\lambda_n} \partial \mathcal{H}_i(n)$, and let $\zeta_n^*$ be the closed curve $L_1^*(n)^{-1} * h_1^*(n) * L_2(n) * h_2^*(n)$ endowed with the orientation induced by the ones of either $\Gamma_n$ or $\Gamma_n^*$ along their common arcs with $\zeta_n^*$. The same argument as before gives $F(\zeta_n^*) = \pm \pi a_n$. Since the unit conormal vector along $L_1(n)^{-1}$ projects horizontally on the same side as the unit conormal vector along $L_2(n)$, we conclude that $F(\zeta_n^*) = F(\zeta_n)$. A similar argument with the unit conormal vector implies that $\frac{F(\Gamma_n)}{F(\Gamma_n^*)}$ converges to a positive real number, from where $\frac{t}{t^*} > 0$.

As we are assuming $\frac{r_n}{\lambda_n} \to 0$, then the lengths of $\beta_i(n), \beta_i^*(n), h_i(n)$, $h_i^*(n)$ go to 0 as $n \to \infty$, $i = 1, 2$. Thus, the limit of $F(\Gamma_n) + F(\Gamma_n^*)$ coincide with the limit of $F(\zeta_n) + F(\zeta_n^*) = 2\varepsilon \pi a_n$ as $n \to \infty$, and

$$2 = \lim_{n \to \infty} \frac{(F(\Gamma_n) + F(\Gamma_n^*))}{\varepsilon \pi a_n} = \lim_{n \to \infty} \frac{-2i\overline{b_n} + (t + t^*)\pi a_n}{\varepsilon \pi a_n} = \varepsilon(t + t^*),$$

which contradicts that both $t, t^*$ are non-zero even integers with the same sign, thereby proving Proposition 5.                    q.e.d.

**Remark 5.** If we remove the hypothesis 2) in Proposition 5, then it is possible to find two highly sheeted vertical helicoids forming inside surfaces in $\mathcal{S}$, which makes their curvatures to blow-up. For instance, the standard examples $M_{\theta, 0, \pi/2}$ with $\theta \nearrow \pi/2$ contain two helicoids

forming with axes joined horizontally by a line parallel to the period vector at the ends. The surfaces $M_{\theta,0,\beta}$ with $\beta < \theta < \pi/2$ and $\beta \nearrow \pi/2$ have also two forming helicoids, but now their axes join by a horizontal line orthogonal to the period vector at the ends.

## 6. Uniqueness of examples around the singly periodic Scherk surfaces

In this Section, we will prove that if $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ degenerates in a singly periodic Scherk minimal surface (case (iv) of Proposition 4), then $L(M_n)$ tends to a tuple in $\mathbb{C}^{4k}$. In particular, the classifying map $C : \widetilde{\mathcal{S}} \to \mathbb{R}^* \times \mathbb{C}$ cannot be proper. In order to overcome this lack of properness, we will prove that only standard examples can occur in $\widetilde{\mathcal{S}}$ nearby the singly periodic Scherk limit. This will be essential when proving that the restriction of $C$ to the space of non-standard examples is proper (Theorem 5).

**Proposition 6.** *Let* $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ *be a sequence of marked surfaces with* $\{H_n\}_n \to H_\infty = (0, \pi a, 0)$ $(a \in \mathbb{R}^*)$, $\{T_n\}_n \to \infty$ (*for any choice of* $T_n$ *as in Proposition* 4) *and* $\{C(M_n)\}_n \to (a, b)$ *where* $b \in \mathbb{C}$. *Then, for* $n$ *large, the geometric surface* $M_n$ *is close to* $2k$ *translated images of arbitrarily large compact regions of a singly periodic Scherk minimal surface of genus zero with two horizontal ends, together with* $2k$ *annular regions* $C_n(1), \ldots, C_n(2k)$ *each of which has two distinct simple branch points of the Gauss map* $g_n : M_n \to \overline{\mathbb{C}}$. *Moreover, there exists a non-horizontal plane* $\Pi \subset \mathbb{R}^3$ *such that any annulus* $C_n(j)$ *is a graph over the intersection of* $\Pi/H_n$ *with a certain horizontal slab,* $j = 1, \ldots, 2k$.

*Proof.* Note that it suffices to show that there exists a subsequence of $\{M_n\}_n$ verifying the conclusions of the proposition. Since the total branching number of the Gauss map $N_n$ of $M_n$ is fixed $4k$, we can find a small $\varepsilon > 0$ such that for every $n$, there exists a $\theta_n \in \mathbb{S}^2 \cap \{x_2 = 0\}$ with the disk $D_{\mathbb{S}^2}(\theta_n, \varepsilon)$ of radius $\varepsilon$ in $\mathbb{S}^2$ around $\theta_n$ disjoint from the branch locus of $N_n$ and from the North and South Poles. In particular, $N_n^{-1}(\theta_n)$ consists of $2k$ distinct points $p_n(1), \ldots, p_n(2k) \in M_n$. We label by $\widetilde{p}_n(i)$ the lift of $p_n(i)$ to a fundamental domain of the doubly periodic surface $\widetilde{M}_n \subset \mathbb{R}^3$ whose quotient is $M_n$, $i = 1, \ldots, 2k$. Since $\{C(M_n)\}_n$ converges in $\mathbb{R}^* \times \mathbb{C}$, Proposition 5 and Lemma 5 imply that the sequence $\{\widetilde{M}_n - \widetilde{p}_n(1)\}_n$ has uniform curvature and local area bounds. After passing to a subsequence, $\widetilde{M}_n - \widetilde{p}_n(1)$ will converge to a (not necessarily connected) properly embedded minimal surface $\widetilde{M}_\infty(1) \subset \mathbb{R}^3$ uniformly on compact subsets of $\mathbb{R}^3$. Since $\theta_n \in \mathbb{S}^2 \cap \{x_2 = 0\}$, any plane orthogonal to the limit of $\{\theta_n\}_n$ contains the vector $H_\infty$. Using this fact,

minor modifications in the arguments inside the proof of Proposition 5 (when we proved that $\mathcal{H}_2$ is not flat) show that $\widetilde{M}_\infty(1)$ is not flat. Applying Proposition 4 with the modified hypothesis of the limit surface not being flat instead of the normalization of the absolute curvature to have a maximum at the origin, and using that $H_n$ converges while any choice of $T_n$ diverges, we deduce that $\widetilde{M}_\infty(1)$ is a singly periodic Scherk minimal surface of genus zero, two of whose ends are horizontal.

Consider an infinite closed horizontal cylinder $\mathcal{C} \subset \mathbb{R}^3$ around the $x_2$-axis with radius large enough such that if we define $\Omega(1) = \big(\widetilde{M}_\infty(1) \cap \mathcal{C}\big)/H_\infty$, then $\big(\widetilde{M}_\infty(1)/H_\infty\big) - \Omega(1)$ consists of four extremely flat annular Scherk-type ends. Since the region $\widetilde{\Omega}_n(1) = [\widetilde{M}_n - \widetilde{p}_n(1)] \cap \mathcal{C}$ satisfies that $\widetilde{\Omega}_n(1)/H_n$ converges uniformly to $\Omega(1)$ as $n \to \infty$ and $\Omega(1)$ has injective Gauss map, we deduce that $\widetilde{\Omega}_n(1)/H_n$ has also injective Gauss map for $n$ large. This implies that the points $\widetilde{p}_n(i) - \widetilde{p}_n(1)$, $2 \le i \le 2k$, are outside $\widetilde{\Omega}_n(1)$. Using the same arguments, we conclude that each of the sequences $\{\widetilde{M}_n - \widetilde{p}_n(2)\}_n, \ldots, \{\widetilde{M}_n - \widetilde{p}_n(2k)\}_n$ (after passing to a common subsequence) converges to respective singly periodic Scherk minimal surfaces $\widetilde{M}_\infty(2), \ldots, \widetilde{M}_\infty(2k)$, each one with two horizontal ends. Note that, these singly periodic Scherk limits have the same period vector $H_\infty$. Taking the radius of $\mathcal{C}$ large enough, we can assume that $\big(\widetilde{M}_\infty(i)/H_\infty\big) - \Omega(i)$ consists of four extremely flat annular Scherk-type ends, where $\Omega(i) = \big(\widetilde{M}_\infty(i) \cap \mathcal{C}\big)/H_\infty$ for each $i = 1, \ldots, 2k$. As before, if we define $\widetilde{\Omega}_n(i) = [\widetilde{M}_n - \widetilde{p}_n(i)] \cap \mathcal{C}$ then $\widetilde{\Omega}_n(i)/H_n$ converges uniformly to $\Omega(i)$ as $n \to \infty$. Furthermore, $\big[\big(\widetilde{\Omega}_n(1) + \widetilde{p}_n(1)\big) \cup \ldots \cup \big(\widetilde{\Omega}_n(2k) + \widetilde{p}_n(2k)\big)\big]/H_n$ embeds into $M_n$. Since any compact horizontal level section of $M_n$ is connected and any compact horizontal level section of $\widetilde{\Omega}_n(i)/H_n$ is a closed curve, we conclude that the minimum closed horizontal slab $S_n(i) \subset \mathbb{R}^3/H_n$ containing to $[\widetilde{\Omega}_n(i) + \widetilde{p}_n(i)]/H_n$ satisfies $S_n(i) \cap S_n(j) = \emptyset$ whenever $i \ne j$. Since the radius of the cylinder $\mathcal{C}$ can be made arbitrarily large, we also conclude that both the width of the slabs $S_n(i)$ and $(T_n)_3$ diverge to $\infty$ as $n$ increases.

We claim that all the limits $\widetilde{M}_\infty(1), \ldots, \widetilde{M}_\infty(2k)$ are in fact the same singly periodic Scherk minimal surface. Clearly, it suffices to check that the angle between the non-horizontal ends of these surfaces and the horizontal does not depend on $i = 1 \ldots, 2k$. Without loss of generality, we can assume that $\widetilde{M}_\infty(1), \ldots, \widetilde{M}_\infty(2k)$ are ordered increasingly in heights, in the sense that for all $n$, $S_n(i + 1)$ lies above $S_n(i)$, $i = 1, \ldots, 2k - 1$. Since the absolute total curvature of $M_n$ is $8k\pi$, the absolute total curvature of a Scherk minimal surface with genus zero is

$4\pi$ and we have $2k$ of these limit surfaces, we conclude that the Gauss map $N_n$ restricted to the complement of $M_n \cap \left( \cup_{i=1}^{2k} (S_n(i)/T_n) \right)$ covers a set in $\mathbb{S}^2$ of arbitrarily small area. Let $C_n(j)$ be the component of $M_n - \left( \cup_{i=1}^{2k} (S_n(i)/T_n) \right)$ that glues to $\left[ \widetilde{\Omega}_n(j) + \widetilde{p}_n(j) \right]/\mathcal{P}_n, \left[ \widetilde{\Omega}_n(j+1) + \widetilde{p}_n(j+1) \right]/\mathcal{P}_n$ (here, we identify cyclically $2k+1$ with 1). Since $C_n(j)$ is a compact annulus in $M_n$, the Divergence theorem shows that the flux vectors of $C_n(j)$ along its boundary curves are opposite; but such flux vectors converge as $n \to \infty$ to the fluxes of the limit Scherk surfaces $\widetilde{M}_\infty(j)/H_\infty, \widetilde{M}_\infty(j+1)/H_\infty$ around their respective upward and downward pointing non-horizontal ends. Thus, $\widetilde{M}_\infty(1) = \ldots = \widetilde{M}_\infty(2k)$ (and the Gauss map of $\widetilde{M}_\infty(j)$ is opposite to the one of $\widetilde{M}_\infty(j+1)$).

Since the Gauss map of a singly periodic Scherk surface of genus zero is unbranched, the $4k$ branch values (counting multiplicity) of the Gauss map $g_n$ of $M_n$ are located in $C_n(1) \cup \ldots \cup C_n(2k)$. Given $j = 1, \ldots, 2k$, $g_n$ restricts to each of the two boundary components of the annulus $C_n(j)$ as a bijection onto the boundary of a small spherical disk centered at a point $N(j)$, this one being the limit normal vector of $\widetilde{M}_\infty(j)$ at its upward pointing non-horizontal Scherk-type end or equivalently, the limit normal vector of $\widetilde{M}_\infty(j+1)$ at its downward pointing non-horizontal Scherk-type end. By gluing two suitable disks $D_1, D_2$ to $C_n(j)$ along its boundary components, one can construct a meromorphic degree 2 map $G : C_n(j) \cup D_1 \cup D_2 \to \overline{\mathbb{C}}$. Since $C_n(j) \cup D_1 \cup D_2$ is a sphere, Riemann–Hurwitz formula gives that $G$ has total branching number 2 and so, $G$ has exactly two distinct simple branch points which lie necessarily in $C_n(j)$. Finally, let $\Pi \subset \mathbb{R}^3$ be a plane parallel to the non-horizontal ends of $\widetilde{M}_\infty(1)$. Since for $n$ large $g_n|_{C_n(j)}$ is contained in an arbitrarily small spherical disk centered in $N(j)$, we conclude that $C_n(j)$ can be expressed as the graph of a function $u_n(j) : (\Pi/H_n) \cap S_n'(j) \to \mathbb{R}$, where $S_n'(j)$ is the horizontal slab (quotiented by $H_n$) between $S_n(j)$ and $S_n(j+1)$. This finishes the proof. q.e.d.

Let $S_\rho$ be the singly periodic Scherk minimal surface that appears as a limit in Proposition 6, with period vector $H_\infty$ pointing to the $x_2$-axis and normal vectors at its ends (stereographically projected) $0, \infty, \rho, -1/\rho$ ($0 < \rho \leq 1$). Hence, we can parametrize $S_\rho$ by the Weierstrass data

$$(16) \quad g(z) = z, \quad dh = c\frac{dz}{(z - \rho)(\rho z + 1)}, \quad z \in \overline{\mathbb{C}} - \{0, \infty, \rho, -1/\rho\},$$

where $c \in \mathbb{R}^*$. To determine $c$, note that the intersection of $S_\rho$ with the quotient by $H_\infty$ of a horizontal plane at large positive height consists of a compact embedded curve $\Gamma$ with period $H_\infty$ (up to sign). $\Gamma$ is

the uniform limit of compact horizontal level sections $\Gamma_n$ of the surfaces $M_n$ (suitably translated and with the notation of Proposition 6). Since the vertical parts of the period and flux vectors of $M_n$ along $\Gamma_n$ are respectively $0, 2\pi$ for all $n$, it follows that

$$2\pi i = c \int_\Gamma \frac{dz}{(z-\rho)(\rho z+1)} = 2\pi i c \operatorname{Res}_\rho \frac{dz}{(z-\rho)(\rho z+1)} = \frac{2\pi i c}{\rho^2+1},$$

thus, $c = \rho^2 + 1$ (we have assumed that the limit normal vector of $S_\rho$ at its upward pointing non-horizontal end is $\rho$ and oriented $\Gamma$ so that the second equality above holds; these choices determine the sign of $c$).

**6.1. Weierstrass data.** Next, we give a local chart for $\mathcal{W}$ around the boundary point described in Proposition 6. Fix $\rho \in (0,1)$ and let $D(*,\varepsilon) \subset \mathbb{C}$ be a small disk of radius $\varepsilon > 0$ centered at $* = \rho, -1/\rho$. Given $k$ unordered couples of points $a_{2i-1}, b_{2i-1} \in D(\rho,\varepsilon)$ with $a_{2i-1} \neq b_{2i-1}$ and another $k$ couples $a_{2i}, b_{2i} \in D(-1/\rho,\varepsilon)$ with $a_{2i} \neq b_{2i}$, $1 \le i \le k$, we can construct a marked meromorphic map $g \in \mathcal{W}$ associated to these couples as follows. Consider $2k$ copies $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2, \ldots, \overline{\mathbb{C}}_{2k}$ of $\overline{\mathbb{C}}$. Cut $\overline{\mathbb{C}}_1$ along small disjoint arcs $\beta_1, \beta_2$ so that $\beta_1$ joins $a_1$ with $b_1$ and $\beta_2$ joins $a_2$ with $b_2$. Cut $\overline{\mathbb{C}}_2$ along a copy of $\beta_2$ and also along an arc $\beta_3$ joining $a_3$ with $b_3$, and glue $\overline{\mathbb{C}}_1$ with $\overline{\mathbb{C}}_2$ along the common cut $\beta_2$ in the usual way. Repeat the process so that $\overline{\mathbb{C}}_{2k}$ glues with $\overline{\mathbb{C}}_1$ along the common cut $\beta_1$. This surgery produces a torus $\mathbb{M}$ and the natural $z$-map on each copy of $\overline{\mathbb{C}}$ gives a well defined degree $2k$ meromorphic map $g : \mathbb{M} \to \overline{\mathbb{C}}$ with branch values $\{a_1, \ldots, b_{2k}\}$. The ordered list of zeros and poles of the marked meromorphic map to be defined will be $(0_1, \ldots, 0_{2k}, \infty_1, \ldots, \infty_{2k})$, where the subindexes refer to the copy of $\overline{\mathbb{C}}$ which the zero or pole of $z$ belongs to (we do not lose generality by assuming that $0, \infty$ do not lie in none of the $\beta_j$-curves). Finally, the non-trivial homology class $[\gamma] \in H_1(\mathbb{M} - \{\text{zeros, poles}\}, \mathbb{Z})$ is defined to be the class of $\{|z| = 1\}$ in $\overline{\mathbb{C}}_1$ with the anticlockwise orientation.

**Remark 6.** For any $\rho \in (0,1)$, the circle $\{|z| = 1\}$ is a closed embedded curve in $S_\rho$ with period zero. When $\rho = 1$, we must perturb slightly $\{|z| = 1\}$ to keep the vanishing period condition true. This fact justifies the above choice of the homology class $[\gamma]$.

With the above procedure, the map $(a_1, b_1, \ldots, a_{2k}, b_{2k}) \mapsto g \in \mathcal{W}$ is not injective, as one can exchange $a_i$ by $b_i$ obtaining the same $g$. Rather than parametrizing these marked meromorphic maps by the lists of its branch values, one can use the symmetric elementary polynomials in two variables for each couple $a_i, b_i$. Note that when $a_i, b_i$ are close to $\rho$ (resp. $-1/\rho$) then $(a_i + b_i, a_i b_i)$ is close to $(2\rho, \rho^2)$ (resp. $(-2/\rho, 1/\rho^2)$).

The computations that follow simplify if we use the arithmetic and geometric means instead of the elementary symmetric polynomials, which is possible since the map $(u, v) \mapsto (u/2, \sqrt{v})$ is a local diffeomorphism outside $(0, 0)$. We define

$$x_i = \frac{1}{2}(a_i + b_i), \qquad y_i = \sqrt{a_i b_i},$$

thus $(x_i, y_i)$ lies in a neighborhood of $(\rho, \rho)$ or of $(-1/\rho, 1/\rho)$. Given $\varepsilon' > 0$, we label $\mathcal{U}(\varepsilon') = \left[ D(\rho, \varepsilon') \times D(\rho, \varepsilon') \times D(\frac{-1}{\rho}, \varepsilon') \times D(\frac{1}{\rho}, \varepsilon') \right]^k$. As $x_i^2 - y_i^2 = \frac{1}{4}(a_i - b_i)^2$, the condition $a_i \neq b_i$ necessary for the above construction is equivalent to $x_i^2 \neq y_i^2$. Let

$$\mathcal{A} = \{(x_1, y_1, \ldots, x_{2k}, y_{2k}) \mid x_i^2 = y_i^2 \text{ for some } i = 1 \ldots, 2k\}.$$

Clearly, $\mathcal{A}$ is an analytic subvariety of $\mathbb{C}^{4k}$. It can be shown that for $\varepsilon' > 0$ small, the correspondence $\mathbf{z} = (x_1, y_1, \ldots, x_{2k}, y_{2k}) \in \mathcal{U}(\varepsilon') - \mathcal{A} \mapsto \aleph(\mathbf{z}) = g \in \mathcal{W}$ defines a local chart for $\mathcal{W}$.

**Remark 7.**

(i) If a marked meromorphic map $g = \aleph(\mathbf{z})$ produces a marked surface $M$, then the ordered list $(0_1, \ldots, 0_{2k}, \infty_1, \ldots, \infty_{2k})$ does not necessarily coincide with the ordering on the ends of $M \in \widetilde{\mathcal{S}}$. This is only a matter of notation and will not affect to the arguments that follow.

(ii) Consider a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ with $\{C(M_n) = (a_n, b_n)\}_n \to (a, b) \in \mathbb{R}^* \times \mathbb{C}$ and $\{T_n\}_n \to \infty$ as $n \to \infty$ (for any choice of $T_n$). By Proposition 6, the sequence of geometric surfaces $\{M_n\}_n$ converges uniformly to $2k$-copies of a singly periodic Scherk minimal surface $S_\rho$ parametrized as in (16), for certain $\rho \in (0, 1]$. Let $\Gamma$ be the curve $\{|z| = 1\}$ viewed in one of the copies of $S_\rho$. Clearly, $\Gamma$ is the uniform limit as $n \to \infty$ of a sequence of closed curves $\Gamma_n \subset M_n$ with $P_{\Gamma_n} = 0$. After exchanging the homology class of the marked surface $M_n$ by $[\Gamma_n] \in H_1(M_n, \mathbb{Z})$, we can see the same geometric surface $M_n$ as a new marked surface $M_n'$ inside the domain of the chart $\aleph$ for $n$ large enough (also note that the second component of $C(M_n')$ differs from $b_n$ in a fixed even multiple of $i\pi a_n$).

**6.2. Holomorphic extension.** When $\mathbf{z} \in \mathcal{A}$, the continuous extension of the above cut-and-paste process gives a Riemann surface with nodes (see [**8**] page 245 for the definition of a Riemann surface with nodes), each node occurring between copies $\overline{\mathbb{C}}_{j-1}, \overline{\mathbb{C}}_j$ such that $a_j = b_j$. The corresponding differential $\phi$ also extends through $\mathbf{z}$, in the following manner.

**Proposition 7.** *Each* $\mathbf{z} \in \mathcal{A}$ *produces* $l$ *spheres* $S_1, \ldots, S_l$ *joined by* $l$ *node points* $P_i, Q_i \in S_i$ *(here* $Q_i = P_{i+1}$ *and the subindexes are cyclic),* $g$ *degenerates in* $l$ *non-constant meromorphic maps* $g(i) : S_i \to \overline{\mathbb{C}}$ *with the degrees of* $g(1), \ldots, g(l)$ *adding up to* $2k$ *and* $g(i)$ *takes the values* $\rho$ *and/or* $-1/\rho$ *at* $P_i, Q_i$. $\phi$ *degenerates in the* $l$ *unique meromorphic differentials* $\phi(i)$ *on* $S_i$, *such that* $\phi(i)$ *has exactly two simple poles at* $P_i, Q_i$ *with residues* 1 *at* $P_i$ *and* $-1$ *at* $Q_i$ *(these residues are determined by the equation* $\int_{|z|=1} \phi = 2\pi i$). *Finally, both* $g$ *and* $\phi$ *depend holomorphically on all parameters (including at points of* $\mathcal{A}$).

*Proof.* See Lemma 8 of [**14**] for a similar situation as ours (tori degenerating in spheres); see also Section 3.4 of [**27**] for arbitrary genus. q.e.d.

**Lemma 6.** *The ligature map* $L$ *extends holomorphically to* $\mathcal{U}(\varepsilon')$.

*Proof.* Since $\mathcal{U}(\varepsilon') \cap \mathcal{A}$ is an analytic subvariety of $\mathcal{U}(\varepsilon')$, it suffices to check that $L$ is bounded by the Riemann Extension theorem for several variables (see for instance [**4**] page 9). Let $g_n = \aleph(\mathbf{z}(n))$ be a sequence with $\mathbf{z}(n) \in \mathcal{U}(\varepsilon') - \mathcal{A}$ converging to $\mathbf{z} \in \mathcal{U}(\varepsilon') \cap \mathcal{A}$. Note that each component of $L(g_n)$ can be written as an integral along a curve $\alpha$ independent of $n$ which lies in one of the twice punctured spheres $S_j - \{P_j, Q_j\}$ of Proposition 7, of a holomorphic differential $\varphi(n)$, and that $\{\varphi(n)\}_n$ converge uniformly on $\alpha$ as $n \to \infty$ to some holomorphic differential $\varphi$ on $S_j - \{P_j, Q_j\}$. From here, we directly deduce that $L(g_n)$ is bounded as desired. q.e.d.

### 6.3. Partial derivatives and Inverse Function theorem.

**Theorem 2.** *There exists* $\varepsilon' > 0$ *small such that* $L|_{\mathcal{U}(\varepsilon')}$ *is a biholomorphism.*

*Proof.* Given a list $\mathbf{z} = (x_1, y_1, \ldots, x_{2k}, y_{2k}) \in \mathcal{U}(\varepsilon') - \mathcal{A}$, we will denote by $g = \aleph(\mathbf{z})$ the associated marked meromorphic map. Recall that $0_j, \infty_j$ belong to the copy $\overline{\mathbb{C}}_j$ of $\overline{\mathbb{C}}$. For $j = 1, \ldots, k$, we let $\Gamma_{2j-1}$ denote the closed curve in $g^{-1}(\overline{\mathbb{C}})$ that corresponds to the loop $\{|z| = 1\}$ on $\overline{\mathbb{C}}_{2j-1}$ (if $\rho = 1$, then, we modify slightly $\{|z| = 1\}$ as mentioned in Remark 6). We orient $\Gamma_1$ to coincide with the last component of $g$ and the remaining $\Gamma_{2j-1}$ to be homologous to $\Gamma_1$ in $g^{-1}(\overline{\mathbb{C}})$. Hence, $\text{Res}_{0_{2j}}(\frac{\phi}{g}) + \text{Res}_{0_{2j+1}}(\frac{\phi}{g}) = \frac{1}{2\pi i}\left(\int_{\Gamma_{2j+1}} \frac{\phi}{g} - \int_{\Gamma_{2j-1}} \frac{\phi}{g}\right)$ and a similar formula holds for poles of $g$. Thus, the composition of $L$ with a certain regular

linear transformation in $\mathbb{C}^{4k}$ can be written as $\widehat{L} : \mathcal{U}(\varepsilon') \to \mathbb{C}^{4k}$, where

$$\widehat{L}(\mathbf{z}) = \Big( \underbrace{\mathrm{Res}_{0_{2j-1}}\Big(\frac{\phi}{g}\Big), \mathrm{Res}_{\infty_{2j-1}}(g\phi)}_{1 \leq j \leq k}, \underbrace{\int_{\Gamma_{2j-1}} \frac{\phi}{g}, \int_{\Gamma_{2j-1}} g\phi}_{1 \leq j \leq k} \Big).$$

By the Inverse Function theorem, it suffices to prove that the differential of $\widehat{L}$ at the point $(\rho, \rho, \frac{-1}{\rho}, \frac{1}{\rho})^k \in \mathbb{C}^{4k}$ corresponding to $(2k$ copies of) the singly periodic Scherk minimal surface $S_\rho$, is an automorphism of $\mathbb{C}^{4k}$. The computations that follow are similar to those in the proof of Lemma 9 in [**14**], so we will only explain them briefly.

Fix $j = 1, \ldots, k$. To compute the partial derivative $\frac{\partial \widehat{L}}{\partial x_{2j-1}}(S_\rho)$, we differentiate at $x = \rho$ the composition of $\widehat{L}$ with the curve $x \in D(\rho, \varepsilon') \mapsto (\rho, \rho, \frac{-1}{\rho}, \frac{1}{\rho}, \ldots, \frac{1}{\rho}, x, \rho, \frac{-1}{\rho}, \frac{1}{\rho}, \ldots, \rho, \rho, \frac{-1}{\rho}, \frac{1}{\rho}) \in \mathcal{U}(\varepsilon')$, where $x$ is placed at the $(4j - 3)$-th component of the $4k$-tuple. This curve produces $2k - 1$ spheres $S_1, S_2, \ldots, S_{2j-2}, S_{2j}, \ldots, S_{2k}$ and meromorphic maps $g_1, g_2, \ldots, g_{2j-2}, g_{2j}, \ldots, g_{2k}$ on them such that

1) For each $m = 1, \ldots, 2k$ with $m \neq 2j - 2, 2j - 1$, $g_m : S_m \to \overline{\mathbb{C}}$ is a biholomorphism. Thus, we can parametrize $S_m$ by $\overline{\mathbb{C}}$ with $g_m(z) = z$. With this parameter, the node points correspond to $\rho$, $\frac{-1}{\rho}$, and $\phi = c \frac{dz}{(z-\rho)(z+\rho^{-1})}$ where $c \in \mathbb{C}^*$ is determined by the equation $\int_{|z|=1} \phi = 2\pi i$ (in particular, $c$ does not depend on $x$).

2) $g_{2j-2} : S_{2j-2} \to \overline{\mathbb{C}}$ has degree two. We can parametrize $S_{2j-2}$ by $\{(z, w) \in \overline{\mathbb{C}}^2 \mid w^2 = z^2 - 2xz + \rho^2\}$, hence, $w = \sqrt{z^2 - 2xz + \rho^2}$ is well defined on $S_{2j-2}$, (we fix the sign of the square root so that $w \sim z - \rho$ in $\overline{\mathbb{C}}_{2j-1}$ and $w \sim -(z - \rho)$ in $\overline{\mathbb{C}}_{2j-2}$). Let $Q$ denote the point $z = -1/\rho$ in $\overline{\mathbb{C}}_{2j-1}$. Then, $g(z, w) = z$ and $\phi = c(x) \frac{dz}{(z+\rho^{-1})w}$, where

(17)  $$-1 = \mathrm{Res}_Q \phi = \mathrm{Res}_Q \frac{c(x)\,dz}{(z+\rho^{-1})w} = \frac{c(x)}{w(Q)}.$$

As the components of $\widehat{L}$ computed on $S_m$ with $m \neq 2j-2, 2j-1$ do not depend on $x$, the corresponding derivative with respect to $x$ vanishes. From (17) one has $c(\rho) = -w(Q)|_{x=\rho} = \frac{\rho^2+1}{\rho}$, $c'(\rho) = -\frac{d}{dx}\big|_{x=\rho} w(Q) =$

$\frac{1}{\rho^2+1}$. An elementary calculation gives

$$\frac{d}{dx}\bigg|_{x=\rho}\mathrm{Res}_{0_{2j-1}}\left(\frac{\phi}{g}\right) = \frac{d}{dx}\bigg|_{x=\rho}\mathrm{Res}_{(0,-\rho)}\frac{c(x)\,dz}{z(z+\rho^{-1})w} = -c'(\rho),$$

$$\frac{d}{dx}\bigg|_{x=\rho}\mathrm{Res}_{\infty_{2j-1}}(g\phi) = -c'(\rho),$$

$$\frac{d}{dx}\bigg|_{x=\rho}\int_{\Gamma_{2j-1}}\frac{\phi}{g} = \frac{d}{dx}\bigg|_{x=\rho}\int_{|z|=1}\frac{c(x)\,dz}{z(z+\rho^{-1})w} = 0,$$

$$\frac{d}{dx}\bigg|_{x=\rho}\int_{\Gamma_{2j-1}}g\phi = \frac{2\pi i}{\rho^2+1},$$

where $\Gamma_{2j-1} \subset S_{2j-2}$ is the connected lift of $\{|z|=1\}$ to $\overline{\mathbb{C}}_{2j-1}$ through the $z$-map. Thus,

$$\frac{\partial\widehat{L}}{\partial x_{2j-1}}(S_\rho) = \tfrac{1}{\rho^2+1}\left(\mathbf{0}_{2j-2}, \overset{(2j-1)}{-1}, \overset{(2j)}{-1}, \mathbf{0}_{2k-2}, \overset{(2k+2j-1)}{0}, \overset{(2k+2j)}{2\pi i}, \mathbf{0}_{2k-2j}\right),$$

where $\mathbf{0}_n$ denotes the $n$-tuple $(0,\dots,0) \in \mathbb{C}^n$ for each positive integer $n$. Analogous derivation of the composition of $\widehat{L}$ with similar curves gives

$$\frac{\partial\widehat{L}}{\partial y_{2j-1}}(S_\rho) = \tfrac{\rho^2}{\rho^2+1}\left(\mathbf{0}_{2j-2}, \overset{(2j-1)}{\tfrac{-2\rho^2-1}{\rho^4}}, \overset{(2j)}{-1}, \mathbf{0}_{2k-2}, \overset{(2k+2j-1)}{0}, \overset{(2k+2j)}{2\pi i}, \mathbf{0}_{2k-2j}\right);$$

$$\frac{\partial\widehat{L}}{\partial x_{2j}}(S_\rho) = \tfrac{\rho^2}{\rho^2+1}\left(\mathbf{0}_{2j-2}, 1, 1, \mathbf{0}_{2k-2}, 2\pi i, 0, \mathbf{0}_{2k-2j}\right);$$

$$\frac{\partial\widehat{L}}{\partial y_{2j}}(S_\rho) = \tfrac{\rho^2(\rho^2+2)}{\rho^2+1}\left(\mathbf{0}_{2j-2}, 1, \tfrac{-1}{\rho^2(\rho^2+2)}, \mathbf{0}_{2k-2}, 2\pi i, 0, \mathbf{0}_{2k-2j}\right).$$

The absolute Jacobian of $\widehat{L}$ at the point $S_\rho$ is the absolute value of the determinant of the matrix whose rows are the above vectors, which turns out to be $(2\pi)^{2k}$, and the theorem follows.         q.e.d.

## 7. Uniqueness of examples around the catenoid

When a sequence $\{M_n\}_n \in \widetilde{\mathcal{S}}$ degenerates in a vertical catenoid (case (i) of Proposition 4), the residues in the ligature map $L$ diverge to $\infty$. In this Section, we will modify $L$ to have a well defined locally invertible extension through this boundary point of $\mathcal{W}$.

**Proposition 8.** *Let $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ be a sequence with $\{C(M_n) = (a_n, b_n)\}_n \to (\infty, 0)$. Then for $n$ large, the geometric surface $M_n$ is close to $2k$ translated images of arbitrarily large compact regions of a catenoid with flux $(0, 0, 2\pi)$, together with $2k$ regions $C_n(1), \dots, C_n(2k)$.*

*Each $C_n(j)$ is a twice punctured annulus with one left end, one right end of $M_n$ and two distinct simple branch points of its Gauss map. Furthermore, $C_n(j)$ is a graph over its horizontal projection on $\{x_3 = 0\}/H_n$.*

*Proof.* As in Proposition 6, it suffices to check that there exists a subsequence of $\{M_n\}_n$ verifying our assertions. Given $n \in \mathbb{N}$, let $p_n(1), \ldots, p_n(2k) \in M_n$ be the $2k$ distinct points applied by the Gauss map $N_n$ of $M_n$ on a prescribed regular value $\theta_n \in \mathbb{S}^2 \cap \{x_2 = 0\}$ such that a spherical disk $D_{\mathbb{S}^2}(\theta_n, \varepsilon)$ of sufficiently small uniform radius $\varepsilon$ centered at $\theta_n$ does not contain neither branch values of $N_n$ nor the North and South Poles. Let $\widetilde{M}_n \subset \mathbb{R}^3$ be the doubly periodic surface obtained by lifting $M_n$ and $\widetilde{p}_n(i)$ the point that corresponds to $p_n(i)$ in a fixed fundamental domain of $\widetilde{M}_n$, $i = 1 \ldots, 2k$. By Proposition 5 and Lemma 5, the sequence $\{\widetilde{M}_n - \widetilde{p}_n(1)\}_n$ has uniform curvature and local area bounds. After extracting a subsequence, $\widetilde{M}_n - \widetilde{p}_n(1)$ converges uniformly on compact subsets of $\mathbb{R}^3$ to a properly embedded minimal surface $\widetilde{M}_\infty(1) \subset \mathbb{R}^3$. The same argument as in the proof of Proposition 6 proves that $\widetilde{M}_\infty(1)$ is not flat, so it lies in one of the six cases in Proposition 4. Using that $a_n \to \infty$ as $n \to \infty$, we discard the cases (iv), (v), (vi) of that proposition. Since the vertical part of the flux of all the $M_n$ along a compact horizontal section is $2\pi$ and the vertical part of the conormal vector to $M_n$ along such a horizontal section cannot vanish, we conclude that $\widetilde{M}_\infty(1)$ is not a vertical helicoid. Now, suppose that $\widetilde{M}_\infty(1)$ is a Riemann minimal example, and let $F \in \mathbb{C}^*$ be the horizontal part of its flux along a compact horizontal section. Then, for $n$ large there exists a closed curve $\Gamma_n \subset \widetilde{M}_n - \widetilde{p}_n(1)$ such that $F(\Gamma_n)$ tends to $F$ as $n \to \infty$. But $F(\Gamma_n) - i\overline{b_n} = t(n)\pi a_n$ for certain even integer $t(n)$, which gives a contradiction after taking limits. Hence, $\widetilde{M}_\infty(1)$ is a vertical catenoid with flux $(0, 0, 2\pi)$.

Reasoning as in the proof of Proposition 6, we conclude that for $n$ large, the points $\widetilde{p}_n(i) - \widetilde{p}_n(1)$, $2 \leq i \leq 2k$, are outside a large compact domain of $\widetilde{M}_n - \widetilde{p}_n(1)$ arbitrarily close to a vertical catenoid, and that after passing to a subsequence, $\{\widetilde{M}_n - \widetilde{p}_n(2)\}_n, \ldots, \{\widetilde{M}_n - \widetilde{p}_n(2k)\}_n$ converge to different translations of $\widetilde{M}_\infty(1)$. Since an arbitrarily large compact region $\Omega(i)$ of $\widetilde{M}_\infty(1)$ can be uniformly approximated by compact regions $\widetilde{\Omega}_n(i) \subset \widetilde{M}_n - \widetilde{p}_n(i)$, it follows that $(T_n)_3 \to \infty$ as $n \to \infty$. Also note that the regions $[\widetilde{\Omega}_n(i) + \widetilde{p}_n(i)]/H_n$ can be chosen as the intersection of $\widetilde{M}_n/H_n$ with disjoint horizontal slabs $S_n(i) \subset \mathbb{R}^3/H_n$ whose widths go to $\infty$ as $n \to \infty$.

For $j = 1, \ldots, 2k$, let $C_n(j)$ be the component of $M_n - \left( \cup_{i=1}^{2k}(S_n(i)/T_n) \right)$ that glues to $[\widetilde{\Omega}_n(j) + \widetilde{p}_n(j)]/\mathcal{P}_n$, $[\widetilde{\Omega}_n(j+1) + \widetilde{p}_n(j+1)]/\mathcal{P}_n$ (we can

assume that $S_n(j+1)$ is directly above $S_n(j)$). A straightforward modification of the argument in Proposition 6 using the injectivity of the Gauss map of a catenoid shows that each $C_n(j)$ can be compactified by adding two ends $e_1(n), e_2(n)$ of $M_n$ to obtain a compact annulus with two single branch points of the Gauss map $N_n$ of $M_n$, and that $N_n$ applies bijectively each boundary curve of $C_n(j)$ on the boundary of a small spherical disk centered at $(0, 0, \pm 1)$. As the fluxes around $e_1(n), e_2(n)$ cancel (by the Divergence theorem and because the flux of $M_n$ along each boundary curve of $C_n(j)$ tends to finite vertical but the flux at a Scherk-type end of $M_n$ is horizontal and arbitrarily large), we deduce that $e_1(n)$ is a right end of $M_n$ and $e_2(n)$ a left end (or vice versa), both with the same limit normal vector. Finally, the horizontal projection of $C_n(j)$ onto its image in $\{x_3 = 0\}/H_n$ extends smoothly across $e_1(n), e_2(n)$ giving rise to a proper local diffeomorphism hence, a finite covering map, whose degree is one because $C_n(j)$ has one left end and one right end. This gives the graphical property of $C_n(j)$ and finishes the proof of the proposition.                                q.e.d.

**7.1. Weierstrass data.** Following the line of arguments in Section 6, we next show a local chart for $\mathcal{W}$ around the boundary point that appears in Proposition 8. Given $i = 1, \ldots, k$, choose points $a_{2i-1}, b_{2i-1}$ (resp. $a_{2i}, b_{2i}$) in a small punctured neighborhood of 0 (resp. of $\infty$) in $\overline{\mathbb{C}}$, such that $a_j \neq b_j$ for any $j$. These unordered couples can be considered as the branch values of a meromorphic map $g$ of degree $2k$, and a cut-and-paste construction analogous to the one in Section 6 gives rise to a marked meromorphic map $g \in \mathcal{W}$. Since the roles of $a_j$ and $b_j$ are symmetric, their elementary symmetric functions are the right parameters in this setting. We introduce the parameters

$$x_j = \tfrac{1}{2}(a_j + b_j), \quad y_j = a_j b_j \quad \text{if } j \text{ is odd}$$
$$x_j = \tfrac{1}{2}\left(\tfrac{1}{a_j} + \tfrac{1}{b_j}\right), \quad y_j = \tfrac{1}{a_j b_j} \quad \text{if } j \text{ is even}$$

so all parameters $x_j, y_j$, $1 \leq j \leq 2k$, are close to 0. Also the conditions on $a_j$, $b_j$ translate into $y_j \neq x_j^2$ and $y_j \neq 0$. In what follows, we abbreviate $\mathbf{x} = (x_1, \ldots, x_{2k})$, $\mathbf{y} = (y_1, \ldots, y_{2k})$. Given $\varepsilon > 0$, we let

$$D(0, \varepsilon)^{4k} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{4k} \mid |x_j|, |y_j| < \varepsilon \text{ for all } j = 1 \ldots, 2k\},$$
$$\mathcal{B} = \{(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k} \mid x_j^2 = y_j \text{ for some } j\},$$
$$\widehat{\mathcal{B}} = \{(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k} \mid y_j = 0 \text{ for some } j\}.$$

$\mathcal{B} \cup \widehat{\mathcal{B}}$ is an analytic subvariety of the polydisk $D(0, \varepsilon)^{4k}$ and the map $(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k} - (\mathcal{B} \cup \widehat{\mathcal{B}}) \mapsto \chi(\mathbf{x}, \mathbf{y}) = g \in \mathcal{W}$ is a local chart for $\mathcal{W}$. We also let $(\mathbf{0}, \mathbf{0})$ be the tuple $(\mathbf{x}, \mathbf{y}) = (0, \ldots, 0) \in D(0, \varepsilon)^{4k}$.
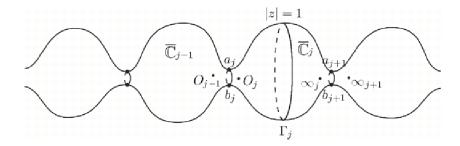
**Figure 8.** In this case, $j$ is odd.

**Remark 8.** An argument as in Remark 7-*(ii)* shows that given a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}}$ with $C(M_n) \to (\infty, 0)$, there exists another sequence of marked surfaces $\{M_n'\}_n$ inside the image of the chart $\chi$ such that for each $n$, $M_n, M_n'$ only differ in the homology class in the last component of the marked surface.

Next, we find the equations to solve in order to produce an immersed minimal surface. Let $\overline{\mathbb{C}}_j$ be the $j^{\text{th}}$ copy of $\overline{\mathbb{C}}$ and let $\Gamma_j$ be the circle defined by $|z| = 1$ in $\overline{\mathbb{C}}_j$, with the positive orientation if $j$ is odd and the negative orientation if $j$ is even. All these curves are homologous in $g^{-1}(\overline{\mathbb{C}})$, and $[\Gamma_1]$ is the last component of the marked meromorphic map $g$. We also write $0_j, \infty_j$ for the points $z = 0$ and $z = \infty$ in $\overline{\mathbb{C}}_j$, see Figure 8 for the case $j$ odd.

Recall that $\phi$ is defined as the unique holomorphic 1-form with $\int_\gamma \phi = 2\pi i$. Define for $1 \leq j \leq 2k$

$$A_j = \begin{cases} \int_{\Gamma_j} g^{-1}\phi & (j \text{ odd}) \\ \int_{\Gamma_{j+1}} g\phi & (j \text{ even}) \end{cases}$$

$$B_j = \begin{cases} \operatorname{Res}_{0_{j-1}}\left(g^{-1}\phi\right) \cdot \operatorname{Res}_{0_j}\left(g^{-1}\phi\right) & (j \text{ odd}) \\ \operatorname{Res}_{\infty_{j-1}}(g\phi) \cdot \operatorname{Res}_{\infty_j}(g\phi) & (j \text{ even}) \end{cases}$$

In this definition and in the sequel, we will adopt a cyclic convention on the subindexes, so when $j = 1$, $j - 1$ must be understood as $2k$. By the Residue theorem, we have for odd $j$

$$A_j - A_{j-2} = 2\pi i \left( \operatorname{Res}_{0_j} g^{-1}\phi + \operatorname{Res}_{0_{j-1}} g^{-1}\phi \right),$$

$$A_{j+1} - A_{j-1} = 2\pi i \left( \operatorname{Res}_{\infty_{j+1}} g\phi + \operatorname{Res}_{\infty_j} g\phi \right).$$

It follows easily that $g$ closes periods if and only if there exist $a \in \mathbb{R}$, $b \in \mathbb{C}$ such that

(18) $$\left.\begin{array}{ll} A_{2i-1} = b, A_{2i} = \overline{b} & \text{for all } i = 1, \ldots, k, \\ B_j = -a^2 & \text{for all } j = 1, \ldots, 2k, \end{array}\right\}$$

which are the equations, we have to solve.

**Remark 9.** The definition of $A_j$ and $B_j$ are motivated by the following fact: Fix $j$, say odd. There is no natural way to distinguish between the two zeros of $g$ which are close to the branch points $a_j$, $b_j$. We call them $0_{j-1}$ and $0_j$ for convenience, but this is a little bit artificial. Indeed, the fact that one zero belongs to $\overline{\mathbb{C}}_j$ or $\overline{\mathbb{C}}_{j-1}$ depends on the choice of the common cut $\beta_j$ connecting the points $a_j$ and $b_j$, but there is no way to choose $\beta_j$ depending continuously on the parameters (this is a homotopy issue). In other words, the residue of $g^{-1}\phi$ at $0_j$ is not a well defined function of the parameters. On the other hand, the unordered pair $\{0_{j-1}, 0_j\}$ depends continuously on the parameters. For this reason, we should consider the elementary symmetric functions of the residues at $0_{j-1}$ and $0_j$. This is essentially what we do in the definition of $A_j$ and $B_j$.

**7.2. Holomorphic extension.** When $a_j = b_j$ (which corresponds to $y_j = x_j^2$) for a given $j$, the definition of $g$ gives a Riemann surface with a node between $\overline{\mathbb{C}}_{j-1}$ and $\overline{\mathbb{C}}_j$. More precisely, each $(\mathbf{x}, \mathbf{y}) \in \mathcal{B}$ gives rise to a Riemann surface with nodes which consists of $l$ spheres $S_j$ joined by node points $P_j, Q_j$ so that $P_j = Q_{j+1}$, $l$ non-constant meromorphic maps $g(j) : S_j \to \overline{\mathbb{C}}$ with $\sum_j \deg(g(j)) = 2k$ and $g(j)(\{P_j, Q_j\}) \subset \{0, \infty\}$, and $l$ meromorphic differentials $\phi(j)$ on $S_j$ with just two simple poles at $P_j, Q_j$ and residues 1 at $P_j$, $-1$ at $Q_j$.

If $a_j = 0$ and $b_j \neq 0$ (or vice versa) for $j$ odd, then the conformal structure between the copies $\overline{\mathbb{C}}_{j-1}$ and $\overline{\mathbb{C}}_j$ does not degenerate, but the corresponding $z$-map has a double zero. For $j$ even, we have a similar behavior exchanging zero by pole. Thus, each $(\mathbf{x}, \mathbf{y}) \in \widehat{\mathcal{B}} - \mathcal{B}$ produces a conformal torus $\mathbb{M}$ and a single meromorphic degree $2k$ map $g : \mathbb{M} \to \overline{\mathbb{C}}$ with at least a double zero or pole. $\phi$ extends to the (unique) holomorphic differential on $\mathbb{M}$ with $\int_{\Gamma_1} \phi = 2\pi i$.

**Remark 10.** Although we will not use it, the points of the form $(\mathbf{x}, \mathbf{0}) \in \widehat{\mathcal{B}} - \mathcal{B}$ with $\mathbf{x} = (\lambda, -\lambda, \dots, \lambda, -\lambda)$ for $\lambda > 0$ small, represent boundary points of $\mathcal{W}$ corresponding to Riemann minimal examples close to a stack of vertical catenoids.

**Proposition 9.** *Both $g$ and $\phi$ depend holomorphically on all parameters $(\mathbf{x}, \mathbf{y})$ in a neighborhood of $(\mathbf{0}, \mathbf{0})$ (including at points of $\mathcal{B} \cup \widehat{\mathcal{B}}$).*

*Proof.* Same as Proposition 7.                                q.e.d.

**Proposition 10.** *For $1 \leq j \leq 2k$, the functions $A_j$ and $\widetilde{B}_j = y_j B_j$ extend holomorphically through $\mathcal{B} \cup \widehat{\mathcal{B}}$.*

*Proof.* The extendability of $A_j$ through a point $(\mathbf{x}_0, \mathbf{y}_0) \in \mathcal{B} \cup \widehat{\mathcal{B}}$ is a consequence of Proposition 9 since the curves $\Gamma_h$ stay in the limit Riemann surface minus its nodes (if any). For $\widetilde{B}_j$, we cannot apply Proposition 9 directly because some of the points $0_{j-1}, 0_j$ or $\infty_{j-1}, \infty_j$ where we compute the residues may collapse either into node points (when $(\mathbf{x}_0, \mathbf{y}_0) \in \mathcal{B}$) or into branch points of $g$ (when $(\mathbf{x}_0, \mathbf{y}_0) \in \widehat{\mathcal{B}} - \mathcal{B}$). Rather, we estimate the rate at which these residues blow-up.

We prove the extendability of $\widetilde{B}_j$ when $j$ is odd, the proof for $j$ even is similar. Consider a tuple $(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k} - (\mathcal{B} \cup \widehat{\mathcal{B}})$ close to $(\mathbf{0}, \mathbf{0})$, and let $\Omega \subset g^{-1}(\overline{\mathbb{C}})$ be the annulus bounded by $\Gamma_{j-1}$ and $\Gamma_j$, where $g = \aleph(\mathbf{x}, \mathbf{y})$. We shall first make a conformal representation of this domain into a standard annulus. $\Omega$ contains the branch points $a = a_j$, $b = b_j$. Recall that $a$, $b$ are close to 0. We introduce the functions

$$u = \sqrt{\frac{z - a}{z - b}}, \qquad v = \frac{1 + u}{1 - u}$$

and fix the sign of the square root by asking that $u \sim 1$ on $\Gamma_j$ and consequently $u \sim -1$ on $\Gamma_{j-1}$. Then both $u$ and $v$ are conformal representations of $\Omega$. On $\Gamma_j$ we may write $z = e^{i\theta}$

$$u = \sqrt{1 + \frac{(b - a)e^{-i\theta}}{1 - be^{-i\theta}}} \sim 1 + \frac{1}{2}(b - a)e^{-i\theta}$$

when $a, b$ are close to 0. Consequently, $u(\Omega)$ is close to $\overline{\mathbb{C}}$ minus the two disks $D(\pm 1, r/2)$ where $r = |b - a|$, and $v(\Omega)$ is close to the annulus $D(0, 4/r) - D(0, r/4)$. So we may write the Laurent series of $\phi$ with respect to the variable $v$ in the annulus $D(0, 1/r) - D(0, r)$ :

$$\phi = \sum_{n \in \mathbb{Z}} c_n v^n dv$$

where $c_n = \frac{1}{2\pi i} \int_{|v|=1} \frac{\phi}{v^{n+1}}$ depends holomorphically on all parameters. Since $\phi$ extends holomorphically, the integral of $|\phi|$ on each unit circle $|z| = 1$ extends analytically. Hence, it is bounded by some constant $C > 0$ (for the remainder of the proof, we shall denote by $C$ different positive constants). On $\Gamma_j$, we have $|v| \sim 4/r$, hence $|v| \geq 1/r$. This gives if $n + 1 \geq 0$

$$|c_n| = \frac{1}{2\pi} \left| \int_{\Gamma_j} \frac{\phi}{v^{n+1}} \right| \leq \frac{1}{2\pi} \int_{\Gamma_j} \frac{|\phi|}{|v|^{n+1}} \leq Cr^{n+1}.$$

On $\Gamma_{j-1}$, we have $|v| \sim r/4$, hence $|v| \le r$. This gives if $n + 1 \le 0$

$$|c_n| \le \frac{1}{2\pi} \int_{\Gamma_{j-1}} |\phi||v|^{|n+1|} \le Cr^{|n+1|}.$$

In summary,

(19)                $|c_n| \le Cr^{|n+1|}$        for all $n \in \mathbb{Z}$.

We now compute the residues appearing in $B_j$. First note that $z = 0$ is equivalent to $v = \alpha$ or $v = 1/\alpha$, where

$$\alpha = \frac{1 + \sqrt{a/b}}{1 - \sqrt{a/b}}.$$

Furthermore, $g = z = \frac{a-b}{4v}(v - \alpha)(v - 1/\alpha)$ hence

$$\mathrm{Res}_{v=\alpha}\left(g^{-1}\phi\right) = \frac{4}{(a-b)(\alpha - 1/\alpha)} \sum_{n\in\mathbb{Z}} c_n \alpha^{n+1} = -\frac{1}{\sqrt{ab}} \sum_{n\in\mathbb{Z}} c_n \alpha^{n+1}.$$

(Observe here that this residue is indeed a multivalued function as was explained in Remark 9). From the equality $(b-a)\alpha = (\sqrt{a}+\sqrt{b})^2$, we see that $|\alpha| \ll 1/r$ and $|\alpha|^{-1} \ll 1/r$. Using this information, the estimate (19) and the convergence of the series $\sum_{n\ge 1} t^n$ for any $t \in (0,1)$, it is straightforward to check that $\sqrt{y_j}\,\mathrm{Res}\,(g^{-1}\phi)$ is bounded, where the residue is computed at $0_j$ or $0_{j-1}$. Hence, $y_j B_j$ is bounded on the set $y_j \ne x_j^2$, $y_j \ne 0$. Since $y_j B_j$ is a well defined holomorphic function, it extends holomorphically by the Riemann Extension theorem.        q.e.d.


## 7.3. Partial derivatives and Inverse Function theorem.

**Proposition 11.** *For each $j = 1, \ldots, 2k$ it holds*

$$A_j(\mathbf{0}, \mathbf{0}) = 0, \qquad \widetilde{B}_j(\mathbf{0}, \mathbf{0}) = -1, \qquad \frac{\partial A_{j-1}}{\partial x_j}(\mathbf{0}, \mathbf{0}) = 2\pi i$$

*(with $A_0$ understood as $A_{2k}$), and all remaining partial derivatives of the $A_h$ are zero. We will not need the partial derivatives of the $\widetilde{B}_h$.*

*Proof.* Let $(\mathbf{x}, \mathbf{y})_j = (0, \ldots, 0, x_j, 0, \ldots, 0, y_j, 0, \ldots, 0)$ with $y_j \ne x_j^2$ and $y_j \ne 0$. The Riemann surface associated to $(\mathbf{x}, \mathbf{y})_j$ has $2k - 1$ nodes which disconnect it into $2k - 1$ genus zero components. On $2k - 2$ of these components (which we will call *simple spheres*), the corresponding meromorphic map $g_m$ is the usual degree one $z$-map if we see the simple sphere as a copy of $\overline{\mathbb{C}}$, and the height differential $\phi$ becomes $\frac{dz}{z}$. The remaining component $S$ is obtained from the copies $\overline{\mathbb{C}}_{j-1}, \overline{\mathbb{C}}_j$ (so we will call $S$ a *double sphere*).

First, consider the case $j$ odd. Then, $S$ can be parametrized by $\{(z,w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z-a)(z-b)\}$ where $a+b = 2x_j$, $ab = y_j$, and the corresponding meromorphic map is $g(z,w) = z$. Hence, $w = \sqrt{(z-a)(z-b)}$ is well defined on $S$, where we convine that the sign of the square root is determined by $w \sim z$ in $\overline{\mathbb{C}}_j$ and $w \sim -z$ in $\overline{\mathbb{C}}_{j-1}$. Since $\phi$ has simple poles at the nodes $\infty_{j-1}$ and $\infty_j$ with respective residues $1$ and $-1$, and

$$\operatorname{Res}_{\infty_j} \frac{dz}{w} = \operatorname{Res}_{z=\infty} \frac{dz}{z\sqrt{1 - 2x_j/z + y_j/z^2}} = -1,$$

we conclude that $\phi = dz/w$. Thus,

$$(20) \qquad A_j((\mathbf{x}, \mathbf{y})_j) = \int_{\Gamma_j} \frac{dz}{zw} = -2\pi i \operatorname{Res}_{\infty_j} \frac{dz}{zw} = 0$$

and
(21)
$$A_{j-1}((\mathbf{x}, \mathbf{y})_j) = \int_{\Gamma_j} \frac{z\,dz}{w} = -2\pi i \operatorname{Res}_{z=\infty} \frac{dz}{\sqrt{1 - 2x_j/z + y_j/z^2}} = 2\pi i x_j.$$

These two equations imply that $A_j(\mathbf{0}, \mathbf{0}) = 0$ for every $j = 1, \ldots, 2k$ (not necessarily odd). Furthermore, given $h = 1, \ldots, 2k$ it holds

$$\frac{\partial A_j}{\partial x_h}(\mathbf{0}, \mathbf{0}) = \lim_{y_h \to 0} \frac{d}{dx_h}\bigg|_{x_h=0} A_j((\mathbf{x}, \mathbf{y})_h).$$

If $h \neq j-1, j$ then the integral in $A_j((\mathbf{x}, \mathbf{y})_h)$ can be computed in a simple sphere which does not depend on $x_h$, hence $\frac{\partial A_j}{\partial x_h}(\mathbf{0}, \mathbf{0}) = 0$. If $h = j$, then (20) implies $\frac{\partial A_j}{\partial x_j}(\mathbf{0}, \mathbf{0}) = 0$. If $h = j-1$, then (21) gives $\frac{\partial A_{j-1}}{\partial x_j}(\mathbf{0}, \mathbf{0}) = 2\pi i$.

Concerning $\widetilde{B}_j$, we write

$$\operatorname{Res}_{0_j}\left(g^{-1}\phi\right) = \frac{1}{\sqrt{y_j}}, \qquad \operatorname{Res}_{0_{j-1}}\left(g^{-1}\phi\right) = \frac{-1}{\sqrt{y_j}},$$

hence, $\widetilde{B}_j((\mathbf{x}, \mathbf{y})_j) = -1$ for all $j$ odd. The computations in the case $j$ even are similar, with the following modifications:

$$w = \sqrt{\left(\frac{1}{z} - \frac{1}{a}\right)\left(\frac{1}{z} - \frac{1}{b}\right)} = \sqrt{\frac{1}{z^2} - \frac{2x_j}{z} + y_j}, \qquad \phi = \frac{-dz}{z^2 w},$$

$$A_j((\mathbf{x}, \mathbf{y})_j) = \int_{\Gamma_{j+1}} \frac{-dz}{zw} = -2\pi i \operatorname{Res}_{0_{j+1}} \frac{dz}{zw} = 0,$$

$$A_{j-1}((\mathbf{x}, \mathbf{y})_j) = \int_{\Gamma_{j-1}} \frac{-dz}{z^3 w} = 2\pi i \operatorname{Res}_{0_{j-1}} \frac{-dz}{-z^2\sqrt{1 - 2x_j z + y_j z^2}} = 2\pi i x_j,$$

$$\operatorname{Res}_{\infty_j}(g\phi) = \frac{1}{\sqrt{y_j}}, \qquad \operatorname{Res}_{\infty_{j-1}}(g\phi) = \frac{-1}{\sqrt{y_j}},$$

from where one obtains similar conclusions to the case $j$ odd. The derivatives of the $A_h$ with respect to the variables $y_j$ clearly all vanish, and the proof is complete.                                    q.e.d.

Recall that given $j = 1, \ldots, 2j$, we have $1/B_j = y_j/\widetilde{B}_j$. Since $\widetilde{B}_j$ extends holomorphically to the polydisk $D(0, \varepsilon)^{4k}$ (Proposition 10) and $\widetilde{B}_j(\mathbf{0}, \mathbf{0}) = -1$ (Proposition 11), we deduce that $1/B_j$ extends holomorphically to $D(0, \varepsilon)^{4k}$ for $\varepsilon > 0$ small enough, and $(1/B_j)(\mathbf{0}, \mathbf{0}) = 0$. Hence, the map $\Theta : D(0, \varepsilon)^{4k} \to \mathbb{C}^{4k}$ given by

(22) $$\Theta = (A_1, \ldots, A_{2k}, 1/B_1, \ldots, 1/B_{2k})$$

is holomorphic and $\Theta(\mathbf{0}, \mathbf{0}) = (\mathbf{0}, \mathbf{0})$.

**Theorem 3.** *There exists $\varepsilon > 0$ small such that $\Theta$ restricts to $D(0, \varepsilon)^{4k}$ as a biholomorphism onto its image.*

*Proof.* It follows from Proposition 10 that

$$\frac{\partial(1/B_j)}{\partial x_i}(\mathbf{0}, \mathbf{0}) = 0, \qquad \frac{\partial(1/B_j)}{\partial y_i}(\mathbf{0}, \mathbf{0}) = -\delta_{ij},$$

Using these equations and the values of the partial derivatives of the functions $A_h$ at $(\mathbf{0}, \mathbf{0})$ given by Proposition 11, it is straightforward to check that the Jacobian matrix of $\Theta$ at $(\mathbf{0}, \mathbf{0})$ is invertible. Now, the theorem follows from the Inverse Function theorem.                    q.e.d.

**Remark 11.** As a consequence of Theorem 3, for $t < 0$ and $b \in \mathbb{C}$ close to $(\mathbf{0}, \mathbf{0})$ there exists a unique $(\mathbf{x}, \mathbf{y}) \in D(0, \varepsilon)^{4k}$ such that $\Theta(\mathbf{x}, \mathbf{y}) = (b, \overline{b}, \cdots, b, \overline{b}, t, \cdots, t)$. This uniqueness result and equation (18) imply that the space of immersed minimal surfaces sufficiently close to the boundary point of $\mathcal{W}$ given by the $2k$ catenoids (see Proposition 8) has three real freedom parameters.

## 8. Openness

Recall that $\mathcal{K} \subset \mathcal{S}$ represents the space of standard examples with $4k$ ends. A direct consequence of its construction is that $\mathcal{K}$ is closed in $\mathcal{S}$. We saw in Section 3, that $\mathcal{K}$ is open in $\mathcal{S}$, by the non-degeneracy of any standard example. Both closeness and openness remain valid for the space $\widetilde{\mathcal{K}}$ of marked standard examples inside $\widetilde{\mathcal{S}}$. By Lemma 4, $\widetilde{\mathcal{K}}$ is closed and open in $\mathcal{M}$. Our Theorem 1 reduces to prove that $\widetilde{\mathcal{K}} = \widetilde{\mathcal{S}}$, which will be proved by contradiction in Section 9. For this reason, in what follows we will assume $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \neq \emptyset$.

**Theorem 4.** *The classifying map* $C : \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \to \mathbb{R}^* \times \mathbb{C}$ *is open.*

*Proof.* Fix a marked surface $M \in \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$. It suffices to see that $C$ is open in a neighborhood of $M$ in $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$. Let $(a, b) = C(M) \in \mathbb{R}^* \times \mathbb{C}$ and $\mathcal{M}(a, b) = L^{-1}(L_{(a,b)}) \subset \mathcal{M}$ (with the notation of Subsection 4.2). Since $\widetilde{\mathcal{K}}$ is open and closed in $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}(a, b) = \widetilde{\mathcal{S}} \cap \mathcal{M}(a, b)$ is an analytic subvariety of $\mathcal{W}$ by Proposition 3, we conclude that $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b) = (\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}) \cap \widetilde{\mathcal{S}}(a, b)$ is an analytic subvariety of $\mathcal{W}$.

**Assertion 2.** $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b)$ is compact.

To prove Assertion 2, take a sequence $\{M_n\}_n \subset (\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a, b)$. By Proposition 5, the sequence of geometric surfaces $\{M_n\}_n$ has uniformly bounded Gaussian curvature. Similarly as in the proof of Proposition 6, we can find $\delta > 0$ such that for every $n$, there exists a point $p_n \in M_n$ where the normal vector $N_n(p_n)$ lies in $\mathbb{S}^2 \cap \{x_2 = 0\}$ and stays at spherical distance at least $\delta$ from the branch locus of the Gauss map $N_n$ of $M_n$. The usual limit process shows that, after passing to a subsequence, suitable liftings of $M_n - p_n$ converge smoothly as $n \to \infty$ to a properly embedded non-flat minimal surface $\widetilde{M}_\infty \subset \mathbb{R}^3$ in one of the six cases listed in Proposition 4. As $\widetilde{\mathcal{K}}$ is open in $\widetilde{\mathcal{S}}$, if $\widetilde{M}_\infty$ were in case (vi) of Proposition 4, then its quotient would actually be in $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$, hence it only remains to discard the first five possibilities of Proposition 4 for $\widetilde{M}_\infty$.

As the period vector $H_n$ at the ends of $M_n$ is $(0, \pi a, 0)$ for all $n$, the cases (i), (ii), (iii) of Proposition 4 are not possible for $\widetilde{M}_\infty$. If $\widetilde{M}_\infty$ is a singly periodic Scherk minimal surface $S_\rho$ with two horizontal ends, then any choice of the non-horizontal period vector $T_n$ of $M_n$ necessarily diverges to $\infty$ (case (iv) of Proposition 4). By Proposition 6 and Remark 7-(ii), for all $n$ large, we can find a new marked surface $M_n'$ which can be viewed inside the open subset $\mathcal{U}(\varepsilon')$ of $\mathbb{C}^{4k}$ appearing in Theorem 2, such that $M_n, M_n'$ only differ in the homology class of the last component of the marked surfaces. Since $L|_{\mathcal{U}(\varepsilon')}$ is a biholomorphism (Theorem 2), the space of tuples in $\mathcal{U}(\varepsilon')$ producing immersed minimal surfaces has three real freedom parameters. Since $\widetilde{\mathcal{K}}$ has real dimension three and $C|_{\widetilde{\mathcal{K}}}$ takes values arbitrarily close to $(a, b)$, we deduce that if $M' \in \widetilde{\mathcal{S}}$ lies in $\mathcal{U}(\varepsilon')$, then $C(M')$ coincides with the value of $C$ at a certain standard example $M_0 \in \widetilde{\mathcal{K}}$. In particular, $L(M') = L(M_0)$ and as $L|_{\mathcal{U}(\varepsilon')}$ is injective, we have $M' = M_0$. As this last equality cannot hold for $M'$ being $M_n' \in \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$, we conclude that $\widetilde{M}_\infty$ is not a singly periodic Scherk minimal surface.

Now, assume that $\widetilde{M}_\infty$ lies in case *(v)* of Proposition 4. Let $\Gamma$ be a component of the intersection of $\widetilde{M}_\infty$ with a horizontal plane $\{x_3 = c\}$ whose height does not coincide with the heights of the horizontal ends of $\widetilde{M}_\infty$. Since $\widetilde{M}_\infty$ has exactly two non-horizontal ends, $\Gamma$ is an embedded $U$-shaped curve with two almost parallel divergent ends, and if we denote by $\Pi \subset \mathbb{R}^3$ the plane passing through the origin parallel to the non-horizontal ends of $\widetilde{M}_\infty$, then the conormal vector to $\widetilde{M}_\infty$ along each of the divergent branch of $\Gamma$ becomes arbitrarily close to the upward pointing unit vector $\eta \in \Pi$ such that $\eta$ is orthogonal to $\Pi \cap \{x_3 = c\}$. Since translated liftings of the $M_n$ converge smoothly to $\widetilde{M}_\infty$, we deduce that $M_n$ contains arbitrarily large arcs at constant height along which the conormal vector $\eta_n$ is arbitrarily close to $\eta$. In particular, the integral of the third component of $\eta_n$ along such arcs becomes arbitrarily large. As the conormal vector of $M_n$ along any compact horizontal section misses the horizontal values (the Gauss map of $M_n$ is never vertical), it follows that the vertical component of the flux of $M_n$ along a compact horizontal section diverges to $\infty$ as $n \to \infty$. This contradicts our normalization on the surfaces of $\mathcal{S}$, and proves Assertion 2.

We now finish the proof of Theorem 4. By Assertion 2 and Lemma 2, $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a,b)$ is a finite subset, hence we can find an open set $\mathcal{U}$ of $\mathcal{W}$ containing $M$ such that $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})(a,b) \cap \mathcal{U} = \mathcal{M}(a,b) \cap \mathcal{U} = \{M\}$. In terms of the ligature map $L : \mathcal{W} \to \mathbb{C}^{4k}$, the last equality can be written as $L^{-1}(L_{(a,b)}) \cap \mathcal{U} = \{M\}$. Since $L$ is holomorphic, we can apply the Openness theorem for finite holomorphic maps (see [**4**] page 667) to conclude that $L|_\mathcal{U}$ is an open map. Finally, the relationship between the ligature map $L$ and the map $C$ gives the existence of a neighborhood of $M$ in $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$ where the restriction of $C$ is open.                q.e.d.

The same argument in the proof of Assertion 2 remains valid under the weaker hypothesis on $C(M_n)$ to converge to some $(a,b) \in \mathbb{R}^* \times \mathbb{C}$ instead of being constant on a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$. This proves the validity of the following statement.

**Theorem 5.** *The classifying map* $C : \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \to \mathbb{R}^* \times \mathbb{C}$ *is proper.*

## 9. The proof of Theorem 1

Recall that we were assuming $\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \neq \emptyset$. By Theorems 4 and 5, $C : \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}} \to \mathbb{R}^* \times \mathbb{C}$ is an open and proper map. Thus, $C(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})$ is an open and closed subset of $\mathbb{R}^* \times \mathbb{C}$. Since $C(\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}})$ has points in both connected components of $\mathbb{R}^* \times \mathbb{C}$, we deduce that $C|_{\widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}}$ is surjective. In particular, we can find a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}} - \widetilde{\mathcal{K}}$ such that $\{C(M_n)\}_n$

tends to $(\infty, 0)$ as $n$ goes to infinity. Now, the argument is similar to the one in the proof of Assertion 2 when we discarded the singly periodic Scherk limit, using Proposition 8, Remark 8 and Theorem 3 instead of Proposition 6, Remark 7-(ii) and Theorem 2.

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## References

[1] M. do Carmo & C.K. Peng, *Stable Minimal Surfaces in* $\mathbb{R}^3$ *Are Planes*, Bulletin of the AMS **1** (1979) 903–906, MR 0546314, Zbl 442.53013.

[2] D. Fischer-Colbrie & R. Schoen, *The Structure of Complete Stable Minimal Surfaces in* 3-*Manifolds of Non-Negative Scalar Curvature*, Comm. on Pure and Appl. Math. **33** (1980) 199–211, MR 0562550, Zbl 439.53060.

[3] W. Fulton, *Hurwitz schemes and irreducibility of moduli of algebraic curves*, Ann. of Math. (2) **90** (1969) 542–575, MR 0260752, Zbl 194.21901.

[4] P. Griffiths & J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics, Wiley-Interscience, 1978, MR 0507725, Zbl 408.14001.

[5] L. Hauswirth and M. Traizet, *The space of embedded doubly-periodic minimal surfaces*, Indiana Univ. Math. J. **51**(**5**) (2002) 1041–1079, MR 1947868, Zbl 1039.53012.

[6] D. Hoffman, H. Karcher & F. Wei, *The genus one helicoid and the minimal surfaces that led to its discovery*, Global Analysis and Modern Mathematics (K. Uhlenbeck, editor), Publish or Perish Press, 1993, 119–170, MR 1278754, Zbl 990.54130.

[7] D. Hoffman & W.H. Meeks III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101** (1990) 373–377, MR 1062966, Zbl 722.53054.

[8] Y. Imayoshi & M. Taniguchi (eds.), *An introduction to Teichmüller spaces*, Springer-Verlag, 1992, MR 1215481, Zbl 754.30001.

[9] H. Karcher, *Embedded minimal surfaces derived from Scherk's examples*, Manuscripta Math. **62** (1988) 83–114, MR 0958255, Zbl 658.53006.

[10] H. Karcher, *Construction of Minimal Surfaces*, Surveys in Geometry, 1989, 1–96, University of Tokyo, 1989, and Lecture Notes, **12**, SFB256, Bonn, 1989.

[11] H. Lazard-Holly & W.H. Meeks III, *The Classification of embedded doubly–periodic minimal surfaces of genus zero*, Invent. Math. **143** (2001) 1–27, MR 1802791, Zbl 992.18190.

[12] F.J. López & A. Ros, *On Embedded Complete Minimal Surfaces of Genus Zero*, J. of Differential Geometry **33**(**1**) 293–300, MR 1085145, Zbl 719.53004.

[13] W.H. Meeks III, *The geometry, topology, and existence of periodic minimal surfaces*, Part I, Proceedings of Symposia in Pure Math. **54** (1993) 333–374, MR 1216594, Zbl 812.49030.

[14] W.H. Meeks III, J. Pérez & A. Ros, *Uniqueness of the Riemann minimal examples*, Invent. Math. **131** (1998) 107–132, MR 1626477, Zbl 916.53004.

[15] W.H. Meeks III & H. Rosenberg, *The Global Theory of Doubly Periodic Minimal Surfaces*, Invent. Math. **97** (1989) 351–379, MR 1001845, Zbl 676.53068.

[16] W.H. Meeks III & H. Rosenberg, *The Maximum Principle at Infinity for Minimal Surfaces in Flat Three-Manifolds*, Comment. Math. Helvetici **65** (1990) 255–270, MR 1057243, Zbl 713.53008.

[17] S. Montiel & A. Ros, *Schrödinger operators associated to a holomorphic map*, in 'Global Differential Geometry and Global Analysis' (Berlin, 1990), Lecture Notes in Mathematics, **1481**, Springer-Verlag, 1991, 147–174, MR 1178529, Zbl 744.58007.

[18] J. Pérez, *On singly-periodic minimal surfaces with planar ends*, Trans. of the AMS **6** (1997) 2371–2389, MR 1407709, Zbl 882.53007.

[19] J. Pérez & A. Ros, *Some uniqueness and nonexistence theorems for embedded minimal surfaces*, Math. Ann. **295**(**3**) (1993) 513-525, MR 1204835, Zbl 789.53004.

[20] J. Pérez & A. Ros, *The space of properly embedded minimal surfaces with finite total curvature.*, Indiana Univ. Math. J. **45**(**1**) (1996) 177–204, MR 1406689, Zbl 864.53008.

[21] J. Pérez & A. Ros, *Properly embedded minimal surfaces with finite total curvature*, in 'The Global Theory of Minimal Surfaces in Flat Spaces' (G.P. Pirola, editor), 2002, 15–66, Lecture Notes in Math., **1775**, Springer-Verlag, MR 1901613, Zbl 1028.53005.

[22] J. Pérez & M. Traizet, *The classification of singly periodic minimal surfaces with genus zero and Scherk type ends*, to appear in Transactions of the A.M.S.

[23] M.M. Rodríguez, *Doubly Periodic Minimal Tori with Parallel Ends: Standard examples*, preprint, arXiv: math.DG/0501241.

[24] A. Ros, *Embedded minimal surfaces: forces, topology and symmetries*, Calc. Var. **4** (1996) 469–496, MR 1402733, Zbl 861.53008.

[25] H.F. Scherk, *Bemerkungen über die kleinste Fläche Innerhalb gegebener Grenzen*, J. R. Angew. Math. **13** (1835) 185–208.

[26] E. Toubiana, *On the Uniqueness of the Helicoid*, Annales de L'Institute Fourier **38**(**4**) (1988) 121–132, MR 0978243, Zbl 645.53032.

[27] M. Traizet, *An embedded minimal surface with no symmetries*, J. Differential Geometry **60**(**1**) (2002) 103–153, MR 1924593, Zbl 1054.53014.

[28] F. Wei, *Some existence and uniqueness theorems for doubly periodic minimal surfaces*, Invent. Math. **109** (1992) 113–136, MR 1168368, Zbl 773.53005.

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