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CONVERGENCE OF THE YAMABE FLOW FOR ARBITRARY INITIAL ENERGY

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Abstract

We consider the Yamabe flow $\frac{\partial g}{\partial t} = -(R_g - r_g)\, g$, where g is a Riemannian metric on a compact manifold M, R_g denotes its scalar curvature, and r_g denotes the mean value of the scalar curvature. We prove convergence of the Yamabe flow if the dimension n satisfies $3 \leq n \leq 5$ or the initial metric is locally conformally flat.

1. Introduction

In this paper, we present a general convergence result for the Yamabe flow in conformal geometry. Let M be a compact manifold of dimension $n \geq 3$ without boundary and let g be a Riemannian metric on M. Along the Yamabe flow, the Riemannian metric is deformed according to

$$\frac{\partial}{\partial t} g = -(R_g - r_g)\, g, \tag{1}$$

where R_g is the scalar curvature of g and r_g is the mean value of R_g, i.e.,

$$r_g = \frac{\int_M R_g \, dvol_g}{\int_M dvol_g}. \tag{2}$$

The Yamabe energy of a Riemannian metric g on M is defined as

$$\frac{\int_M R_g \, dvol_g}{\left(\int_M dvol_g\right)^{\frac{n-2}{n}}}. \tag{3}$$

Moreover, the Yamabe constant of a Riemannian metric g_0 is defined as the infimum of the Yamabe energy among metrics conformally equivalent to g_0, i.e.,

$$Y(M, g_0) = \inf_{\substack{g = u^{\frac{4}{n-2}} g_0 \\ u \in C^\infty(M),\, u > 0}} \frac{\int_M R_g \, dvol_g}{\left(\int_M dvol_g\right)^{\frac{n-2}{n}}}. \tag{4}$$

By definition, $Y(M, g_0)$ depends only on the conformal class of g_0.

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Since the Yamabe flow preserves the conformal structure, we may write $g = u^{\frac{4}{n-2}}\, g_0$, where g_0 is a fixed background metric on M and u is a positive function. The scalar curvature of g is related to the scalar curvature of g_0 by

$$R_g = -u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2}\, \Delta_{g_0} u - R_{g_0}\, u \right). \tag{5}$$

Hence, the Yamabe flow reduces to the following evolution equation for the conformal factor:

$$\frac{\partial}{\partial t} u^{\frac{n+2}{n-2}} = \frac{n+2}{4} \left(\frac{4(n-1)}{n-2}\, \Delta_{g_0} u - R_{g_0}\, u + r_g\, u^{\frac{n+2}{n-2}} \right). \tag{6}$$

Moreover, the Yamabe constant of g_0 can be written as

$$Y(M, g_0) = \inf_{u \in C^\infty(M),\, u>0} \frac{\int_M \big(\frac{4(n-1)}{n-2}\, |du|_{g_0}^2 + R_{g_0}\, u^2\big)\, dvol_{g_0}}{\big(\int_M u^{\frac{2n}{n-2}}\, dvol_{g_0}\big)^{\frac{n-2}{n}}}. \tag{7}$$

In case $Y(M, g_0) \leq 0$, it is not difficult to show that the conformal factor is uniformly bounded above and below. Moreover, the flow converges to a metric of constant scalar curvature as $t \to \infty$.

The case $Y(M, g_0) > 0$ is more interesting. Chow [**6**] proved the convergence of the flow for locally conformally flat metrics with positive Ricci curvature. Ye [**20**] later extended the result to all locally conformally flat metrics.

Recently, Struwe and Schwetlick [**17**] proved convergence of the Yamabe flow in lower dimensions ($3 \leq n \leq 5$) under the assumption that the Yamabe energy of the initial metric is less than $(Y(M, g_0)^{\frac{n}{2}} + Y(S^n)^{\frac{n}{2}})^{\frac{2}{n}}$, where $Y(S^n)$ denotes the Yamabe energy of the standard sphere S^n. Under this assumption, it is shown that any singularity consists of at most one bubble, and the positive mass theorem precludes the formation of a singularity of this type.

In the present paper, we prove convergence of the flow for arbitrary initial energy.

Theorem 1.1. *Suppose that either $3 \leq n \leq 5$ or M is locally conformally flat. Moreover, assume that M is not conformally equivalent to the standard sphere S^n. Then, for every choice of the initial metric, the Yamabe flow exists for all time and converges to a metric with constant scalar curvature.*

We expect that the methods of this paper can be used to prove convergence of the Yamabe flow in dimension $n \geq 6$ under a technical condition on the Weyl tensor (compare [**1**], [**11**], Theorem B, and [**16**], Theorem 4.1 on p. 219).

2. Longtime existence

In light of the foregoing discussion, it suffices to consider the case $Y(M, g_0) > 0$. In this case, we can find a background metric g_0, which is conformally equivalent to the initial metric and has positive scalar curvature R_{g_0} (see [15], Lemma 1.1). Let $g(t) = u(t)^{\frac{4}{n-2}}\, g_0$ be a solution of the Yamabe flow

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - r_{g(t)})\, g(t).$$

Since the volume of M does not change under the evolution, we may assume that

$$\int_M dvol_{g(t)} = 1$$

for all $t \geq 0$. With this normalization, the mean value of the scalar curvature can be written as

$$r_{g(t)} = \int_M R_{g(t)}\, dvol_{g(t)}.$$

Using the identity

$$\frac{\partial}{\partial t} R_{g(t)} = (n-1)\, \Delta_{g(t)} R_{g(t)} + R_{g(t)}\, (R_{g(t)} - r_{g(t)}), \tag{8}$$

we obtain

$$\frac{d}{dt} r_{g(t)} = -\frac{n-2}{2} \int_M (R_{g(t)} - r_{g(t)})^2\, dvol_{g(t)}. \tag{9}$$

In particular, the function $t \mapsto r_{g(t)}$ is decreasing.

Proposition 2.1. *The scalar curvature of the metric $g(t)$ satisfies*

$$\inf_M R_{g(t)} \geq \min\Big\{\inf_M R_{g(0)}, 0\Big\} \tag{10}$$

for all $t \geq 0$.

Proof. The scalar curvature satisfies the evolution equation

$$\frac{\partial}{\partial t} R_{g(t)} = (n-1)\, \Delta_{g(t)} R_{g(t)} + R_{g(t)}\, (R_{g(t)} - r_{g(t)}).$$

Observe that $r_{g(t)} > 0$ since $Y(M, g_0) > 0$. Hence, the assertion follows from the maximum principle.

For abbreviation, let

$$\sigma = \max\Big\{\sup_M (1 - R_{g(0)}), 1\Big\},$$

so that $R_{g(t)} + \sigma \geq 1$ for all $t \geq 0$.

The following two results are similar to Lemma 3.3 in [17]. Our arguments mostly follow those of Struwe and Schwetlick, but we do not require that the scalar curvature is positive everywhere.

Lemma 2.2. *For every $p > 2$, we have*

(11)
$$\begin{aligned}
&\frac{d}{dt}\int_M (R_{g(t)} + \sigma)^{p-1}\, dvol_{g(t)} \\
&\quad = -\frac{4(n-1)(p-2)}{p-1}\int_M \Big| d(R_{g(t)} + \sigma)^{\frac{p-1}{2}}\Big|^2_{g(t)}\, dvol_{g(t)} \\
&\qquad - \Big(\frac{n+2}{2} - p\Big)\int_M \big((R_{g(t)}+\sigma)^{p-1} - (r_{g(t)}+\sigma)^{p-1}\big)(R_{g(t)} - r_{g(t)})\, dvol_{g(t)} \\
&\qquad - (p-1)\int_M \sigma\,\big((R_{g(t)} + \sigma)^{p-2} - (r_{g(t)} + \sigma)^{p-2}\big)\,(R_{g(t)} - r_{g(t)})\, dvol_{g(t)}.
\end{aligned}$$

Proof. This follows immediately from the evolution equation for the scalar curvature.

Lemma 2.3. *For every $p > \max\{\frac{n}{2}, 2\}$, we have*

(12)
$$\begin{aligned}
\frac{d}{dt}\int_M |R_{g(t)} - r_{g(t)}|^p\, dvol_{g(t)} &\leq C\,\Big(\int_M |R_{g(t)} - r_{g(t)}|^p\, dvol_{g(t)}\Big)^{\frac{2p-n+2}{2p-n}} \\
&\quad + C\int_M |R_{g(t)} - r_{g(t)}|^p\, dvol_{g(t)}
\end{aligned}$$

for some uniform constant C independent of t.

Proof. Using the evolution equation for the scalar curvature, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_M |R_{g(t)} - r_{g(t)}|^p\, dvol_{g(t)} \\
&\quad = -\frac{(n-2)(p-1)}{p} \\
&\qquad \cdot \int_M \Big(\frac{4(n-1)}{n-2}\,\big|d\,|R_{g(t)} - r_{g(t)}|^{\frac{p}{2}}\big|^2_{g(t)} + R_{g(t)}\,|R_{g(t)} - r_{g(t)}|^p\Big)\, dvol_{g(t)} \\
&\qquad + \Big(\frac{(n-2)(p-1)}{p} + p - \frac{n}{2}\Big)\int_M |R_{g(t)} - r_{g(t)}|^p\,(R_{g(t)} - r_{g(t)})\, dvol_{g(t)} \\
&\qquad + \Big(\frac{(n-2)(p-1)}{p} + p\Big)\int_M r_{g(t)}\,|R_{g(t)} - r_{g(t)}|^p\, dvol_{g(t)}
\end{aligned}$$

$$+ \frac{(n-2)p}{2} \int_M (R_{g(t)} - r_{g(t)})^2 dvol_{g(t)}$$
$$\cdot \int_M |R_{g(t)} - r_{g(t)}|^{p-2} (R_{g(t)} - r_{g(t)}) dvol_{g(t)},$$

hence

$$\begin{aligned}
&\frac{d}{dt} \int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} \\
&\quad \leq -\frac{(n-2)(p-1)}{p} \, Y(M, g_0) \left(\int_M |R_{g(t)} - r_{g(t)}|^{\frac{pn}{n-2}} \, dvol_{g(t)} \right)^{\frac{n-2}{n}} \\
&\qquad + \left(\frac{(n-2)(p-1)}{p} + p - \frac{n}{2} \right) \int_M |R_{g(t)} - r_{g(t)}|^{p+1} \, dvol_{g(t)} \\
&\qquad + \left(\frac{(n-2)(p-1)}{p} + p \right) \int_M r_{g(0)} \, |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} \\
&\qquad + \frac{(n-2)p}{2} \left(\int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} \right)^{\frac{p+1}{p}}.
\end{aligned}$$

Since $p > \frac{n}{2}$, this implies

$$\begin{aligned}
\frac{d}{dt} \int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} \leq C \left(\int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} \right)^{\frac{2p-n+2}{2p-n}} \\
+ C \int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)}
\end{aligned}$$

by Hölder's inequality. From this, the assertion follows.

Proposition 2.4. *Given any $T > 0$, we can find positive constants $C(T)$ and $c(T)$ such that*

$$\sup_M u(t) \leq C(T)$$

and

$$\inf_M u(t) \geq c(T)$$

for all $0 \leq t \leq T$.

Proof. The function $u(t)$ satisfies

$$\frac{\partial}{\partial t} u(t) = -\frac{n-2}{4} \, (R_{g(t)} - r_{g(t)}) \leq \frac{n-2}{4} \, (r_{g(0)} + \sigma).$$

Thus, we conclude that

$$\sup_M u(t) \leq C(T)$$

for all $0 \leq t \leq T$. Hence, if we define

$$P = R_{g_0} + \sigma \Big(\sup_{0 \leq t \leq T} \sup_M u(t) \Big)^{\frac{4}{n-2}},$$

then we obtain

$$\begin{aligned} &- \frac{4(n-1)}{n-2} \, \Delta_{g_0} u(t) + P \, u(t) \\ &\geq - \frac{4(n-1)}{n-2} \, \Delta_{g_0} u(t) + R_{g_0} \, u(t) + \sigma \, u(t)^{\frac{n+2}{n-2}} \\ &= (R_{g(t)} + \sigma) \, u(t)^{\frac{n+2}{n-2}} \geq 0 \end{aligned} \tag{13}$$

for all $0 \leq t \leq T$. By Corollary A.3, we can find a positive constant $c(T)$ such that

$$\inf_M u(t) \, \Big(\sup_M u(t) \Big)^{\frac{n+2}{n-2}} \geq c(T)$$

for all $0 \leq t \leq T$. Since $\sup_M u(t) \leq C(T)$, the assertion follows.

Lemma 2.5. *For every $T > 0$, there exists a constant $C(T)$ such that*

$$\int_M |R_{g(t)} - r_{g(t)}|^{\frac{n^2}{2(n-2)}} \, dvol_{g(t)} \leq C(T) \tag{14}$$

for all $0 \leq t \leq T$.

Proof. Using Lemma 2.2 with $p = \frac{n+2}{2} > 2$, we obtain

$$\sup_{0 \leq t \leq T} \int_M (R_{g(t)} + \sigma)^{\frac{n}{2}} \, dvol_{g_0} \leq C$$

and

$$\int_0^T \int_M \big| d(R_{g(t)} + \sigma)^{\frac{n}{4}} \big|_{g(t)}^2 \, dvol_{g(t)} \, dt \leq C.$$

Using Proposition 2.4 and Sobolev's inequality, we conclude that

$$\int_0^T \bigg(\int_M (R_{g(t)} + \sigma)^{\frac{n^2}{2(n-2)}} \, dvol_{g(t)} \bigg)^{\frac{n-2}{n}} \, dt \leq C(T).$$

From this, it follows that

$$\int_0^T \bigg(\int_M |R_{g(t)} - r_{g(t)}|^{\frac{n^2}{2(n-2)}} \, dvol_{g(t)} \bigg)^{\frac{n-2}{n}} \, dt \leq C(T).$$

We now apply Lemma 2.3 with $p = \frac{n^2}{2(n-2)} > \max\{\frac{n}{2}, 2\}$. This implies

$$\frac{d}{dt} \log \left(\int_M |R_{g(t)} - r_{g(t)}|^{\frac{n^2}{2(n-2)}} \, dvol_{g(t)} \right) \leq C \left(\int_M |R_{g(t)} - r_{g(t)}|^{\frac{n^2}{2(n-2)}} \, dvol_{g(t)} \right)^{\frac{n-2}{n}} + C.$$

From this, the assertion follows.

Proposition 2.6. *Let $0 < \alpha < \min\{\frac{4}{n}, 1\}$. Given any $T > 0$, there exists a constant $C(T)$ such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C(T) \left((t_1 - t_2)^{\frac{\alpha}{2}} + d(x_1, x_2)^\alpha \right) \tag{15}$$

for all $x_1, x_2 \in M$ and all $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$.

Proof. Let $\alpha = 2 - \frac{n}{p}$, where $\frac{n}{2} < p < \min\left\{\frac{n^2}{2(n-2)}, n\right\}$. Using Lemma 2.5 and Proposition 2.4, we obtain

$$\int_M \left| \frac{4(n-1)}{n-2} \, \Delta_{g_0} u(t) - R_{g_0} u(t) \right|^p dvol_{g_0} \leq C(T) \tag{16}$$

and

$$\int_M \left| \frac{\partial}{\partial t} u(t) \right|^p dvol_{g(t)} \leq C(T) \tag{17}$$

for all $t \in [0, T]$. The first inequality implies that

$$|u(x_1, t) - u(x_2, t)| \leq C(T) \, d(x_1, x_2)^\alpha$$

for all $x_1, x_2 \in M$ and $t \in [0, T]$. Using the second inequality, we obtain

$$\begin{aligned}
&|u(x, t_1) - u(x, t_2)| \\
&\leq C \, (t_1 - t_2)^{-\frac{n}{2}} \int_{B_{\sqrt{t_1 - t_2}}(x)} |u(x, t_1) - u(x, t_2)| \, dvol_{g_0} \\
&\leq C \, (t_1 - t_2)^{-\frac{n}{2}} \int_{B_{\sqrt{t_1 - t_2}}(x)} |u(t_1) - u(t_2)| \, dvol_{g_0} + C(T) \, (t_1 - t_2)^{\frac{\alpha}{2}} \\
&\leq C \, (t_1 - t_2)^{-\frac{n-2}{2}} \sup_{t_1 \geq t \geq t_2} \int_{B_{\sqrt{t_1 - t_2}}(x)} \left| \frac{\partial}{\partial t} u(t) \right| dvol_{g_0} + C(T) \, (t_1 - t_2)^{\frac{\alpha}{2}} \\
&\leq C \, (t_1 - t_2)^{\frac{\alpha}{2}} \sup_{t_1 \geq t \geq t_2} \left(\int_M \left| \frac{\partial}{\partial t} u(t) \right|^p dvol_{g_0} \right)^{\frac{1}{p}} + C(T) \, (t_1 - t_2)^{\frac{\alpha}{2}} \\
&\leq C(T) \, (t_1 - t_2)^{\frac{\alpha}{2}}
\end{aligned}$$

for all $x \in M$ and all $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$. This proves the assertion.

We can now use the standard regularity theory for parabolic equations (see [7], Theorem 5 on p. 64) to show that all higher order derivatives of u are uniformly bounded on every fixed time interval $[0,T]$. Therefore, the flow exists for all time.

3. Proof of the main result assuming Proposition 3.3

Proposition 3.1. *Fix* $\max\{\frac{n}{2}, 2\} < p < \frac{n+2}{2}$. *Then, we have*

$$\lim_{t\to\infty} \int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} = 0. \tag{18}$$

Proof. It follows from Lemma 2.2 that

$$\begin{aligned} &\frac{d}{dt} \int_M (R_{g(t)} + \sigma)^{p-1} \, dvol_{g(t)} \\ &\quad \leq -\Big(\frac{n+2}{2} - p\Big) \int_M \big((R_{g(t)} + \sigma)^{p-1} - (r_{g(t)} + \sigma)^{p-1}\big) \\ &\qquad \cdot (R_{g(t)} - r_{g(t)}) \, dvol_{g(t)}. \end{aligned}$$

Since $p > 2$, we have

$$\big((R_{g(t)} + \sigma)^{p-1} - (r_{g(t)} + \sigma)^{p-1}\big)\,(R_{g(t)} - r_{g(t)}) \geq c\,|R_{g(t)} - r_{g(t)}|^p$$

for a suitable constant $c > 0$. Since $p < \frac{n+2}{2}$, it follows that

$$\frac{d}{dt} \int_M (R_{g(t)} + \sigma)^{p-1} \, dvol_{g(t)} \leq -c \int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)}.$$

Thus, we conclude that

$$\int_0^\infty \int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} \, dt \leq C,$$

hence

$$\liminf_{t\to\infty} \int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} \, dt = 0.$$

On the other hand, since $p > \max\left\{\frac{n}{2}, 2\right\}$, we have

$$\begin{aligned} \frac{d}{dt} \int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} &\leq C \left(\int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)}\right)^{\frac{2p-n+2}{2p-n}} \\ &\quad + C \int_M |R_{g(t)} - r_{g(t)}|^p \, dvol_{g(t)} \end{aligned}$$

by Lemma 2.3. From this, the assertion follows.

Hence, if we define

$$r_\infty = \lim_{t\to\infty} r_{g(t)}, \tag{19}$$

then we obtain the following result:

Corollary 3.2. *For every* $1 < p < \frac{n+2}{2}$, *we have*

$$\lim_{t \to \infty} \int_M |R_{g(t)} - r_\infty|^p \, dvol_{g(t)} = 0. \tag{20}$$

The proof of Theorem 1.1 will be based on the following proposition. The proof of Proposition 3.3 will occupy Sections 4–7.

Proposition 3.3. *Let* $\{t_\nu : \nu \in \mathbb{N}\}$ *be a sequence of times such that* $t_\nu \to \infty$ *as* $\nu \to \infty$. *Then, we can find a real number* $0 < \gamma < 1$ *and a constant* C *such that, after passing to a subsequence, we have*

$$r_{g(t_\nu)} - r_\infty \leq C \bigg(\int_M u(t_\nu)^{\frac{2n}{n-2}} \, |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \bigg)^{\frac{n+2}{2n} (1+\gamma)} \tag{21}$$

for all integers ν *in that subsequence. Note that* γ *and* C *may depend on the sequence* $\{t_\nu : \nu \in \mathbb{N}\}$.

The following result is an immediate consequence of Proposition 3.3.

Proposition 3.4. *There exist real numbers* $0 < \gamma < 1$ *and* $t_0 > 0$ *such that*

$$r_{g(t)} - r_\infty \leq \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, |R_{g(t)} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \bigg)^{\frac{n+2}{2n} (1+\gamma)} \tag{22}$$

for all $t \geq t_0$.

Proof. Suppose this is not true. Then, there exists a sequence of times $\{t_\nu : \nu \in \mathbb{N}\}$ such that $t_\nu \geq \nu$ and

$$r_{g(t_\nu)} - r_\infty \geq \bigg(\int_M u(t_\nu)^{\frac{2n}{n-2}} \, |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \bigg)^{\frac{n+2}{2n} (1+\frac{1}{\nu})}$$

for all $\nu \in \mathbb{N}$. We now apply Proposition 3.3 to this sequence $\{t_\nu : \nu \in \mathbb{N}\}$. Hence, there exists an infinite subset $I \subset \mathbb{N}$, a real number $0 < \gamma < 1$ and a real number C such that

$$r_{g(t_\nu)} - r_\infty \leq C \bigg(\int_M u(t_\nu)^{\frac{2n}{n-2}} \, |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \bigg)^{\frac{n+2}{2n} (1+\gamma)}$$

for all $\nu \in I$. Thus, we conclude that

$$1 \leq C \bigg(\int_M u(t_\nu)^{\frac{2n}{n-2}} \, |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \bigg)^{\frac{n+2}{2n} (\gamma-\frac{1}{\nu})}$$

for all $\nu \in I$. On the other hand, we have

$$\lim_{\nu \to \infty} \bigg(\int_M u(t_\nu)^{\frac{2n}{n-2}} \, |R_{g(t_\nu)} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \bigg)^{\frac{n+2}{2n} (\gamma-\frac{1}{\nu})} = 0$$

by Corollary 3.2. This is a contradiction.

Proposition 3.5. *We have*

$$\int_0^\infty \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, (R_{g(t)} - r_{g(t)})^2 \, dvol_{g_0} \bigg)^{\frac{1}{2}} \, dt \leq C. \tag{23}$$

Proof. It follows from Proposition 3.4 that

$$\begin{aligned} r_{g(t)} - r_\infty \leq C \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, |R_{g(t)} - r_{g(t)}|^{\frac{2n}{n+2}} \, dvol_{g_0} \bigg)^{\frac{n+2}{2n} \, (1+\gamma)} \\ + C \, (r_{g(t)} - r_\infty)^{1+\gamma}, \end{aligned}$$

hence,

$$r_{g(t)} - r_\infty \leq C \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, |R_{g(t)} - r_{g(t)}|^{\frac{2n}{n+2}} \, dvol_{g_0} \bigg)^{\frac{n+2}{2n} \, (1+\gamma)} \tag{24}$$

if t is sufficiently large. Therefore, we obtain

$$\begin{aligned} \frac{d}{dt} \, (r_{g(t)} - r_\infty) &= -\frac{n-2}{2} \int_M u(t)^{\frac{2n}{n-2}} \, (R_{g(t)} - r_{g(t)})^2 \, dvol_{g_0} \\ &\leq -\frac{n-2}{2} \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, |R_{g(t)} - r_{g(t)}|^{\frac{2n}{n+2}} \, dvol_{g_0} \bigg)^{\frac{n+2}{n}} \\ &\leq -c \, (r_{g(t)} - r_\infty)^{\frac{2}{1+\gamma}}, \end{aligned} \tag{25}$$

where c is a positive constant independent of t. This implies

$$\frac{d}{dt} \, (r_{g(t)} - r_\infty)^{-\frac{1-\gamma}{1+\gamma}} \geq c. \tag{26}$$

From this, it follows that

$$(r_{g(t)} - r_\infty)^{-\frac{1-\gamma}{1+\gamma}} \geq c \, t,$$

hence

$$r_{g(t)} - r_\infty \leq C \, t^{-\frac{1+\gamma}{1-\gamma}} \tag{27}$$

if t is sufficiently large. Using Hölder's inequality, we obtain

$$\begin{aligned} &\int_T^{2T} \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, (R_{g(t)} - r_{g(t)})^2 \, dvol_{g_0} \bigg)^{\frac{1}{2}} \, dt \\ &\leq \bigg(T \int_T^{2T} \int_M u(t)^{\frac{2n}{n-2}} \, (R_{g(t)} - r_{g(t)})^2 \, dvol_{g_0} \, dt \bigg)^{\frac{1}{2}} \\ &\leq \bigg(\frac{2}{n-2} \, T \, (r_{g(T)} - r_{g(2T)}) \bigg)^{\frac{1}{2}} \\ &\leq C \, T^{-\frac{\gamma}{1-\gamma}} \end{aligned} \tag{28}$$

if T is sufficiently large. Since $0 < \gamma < 1$, we conclude that

$$\begin{aligned}
(29) \quad & \int_0^\infty \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, (R_{g(t)} - r_{g(t)})^2 \, dvol_{g_0} \bigg)^{\frac{1}{2}} \, dt \\
&= \int_0^1 \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, (R_{g(t)} - r_{g(t)})^2 \, dvol_{g_0} \bigg)^{\frac{1}{2}} \, dt \\
&\quad + \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, (R_{g(t)} - r_{g(t)})^2 \, dvol_{g_0} \bigg)^{\frac{1}{2}} \, dt \\
&\leq C \sum_{k=0}^\infty 2^{-\frac{\gamma}{1-\gamma} \, k} \leq C.
\end{aligned}$$

This proves the assertion.

Proposition 3.6. *Given any $\eta_0 > 0$, we can find a real number $r > 0$ such that*

$$\int_{B_r(x)} u(t)^{\frac{2n}{n-2}} \, dvol_{g_0} \leq \eta_0$$

for all $x \in M$ and $t \geq 0$.

Proof. We can find a real number $T > 0$ such that

$$(30) \qquad \int_T^\infty \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, (R_{g(t)} - r_{g(t)})^2 \, dvol_{g_0} \bigg)^{\frac{1}{2}} \, dt \leq \frac{\eta_0}{n}.$$

We now choose a real number $r > 0$ such that

$$(31) \qquad \int_{B_r(x)} u(t)^{\frac{2n}{n-2}} \, dvol_{g_0} \leq \frac{\eta_0}{2}$$

for all $x \in M$ and $0 \leq t \leq T$. Then, we have

$$\begin{aligned}
(32) \quad & \int_{B_r(x)} u(t)^{\frac{2n}{n-2}} \, dvol_{g_0} \\
&\leq \int_{B_r(x)} u(T)^{\frac{2n}{n-2}} \, dvol_{g_0} \\
&\quad + \frac{n}{2} \int_T^\infty \bigg(\int_M u(t)^{\frac{2n}{n-2}} \, (R_{g(t)} - r_{g(t)})^2 \, dvol_{g_0} \bigg)^{\frac{1}{2}} \, dt \leq \eta_0
\end{aligned}$$

for all $x \in M$ and $t \geq T$. This proves the assertion.

Proposition 3.7. *The function $u(t)$ satisfies*

$$(33) \qquad \sup_M u(t) \leq C$$

and

$$\inf_M u(t) \geq c \tag{34}$$

for all $t \geq 0$. Here, C and c are positive constants independent of t.

Proof. Fix $\frac{n}{2} < q < p < \frac{n+2}{2}$. By Corollary 3.2, we have

$$\int_M |R_{g(t)}|^p \, dvol_{g(t)} \leq C \tag{35}$$

for some constant C independent of t. By Proposition 3.6, we can find a constant $r > 0$ independent of t such that

$$\int_{B_r(x)} dvol_{g(t)} \leq \eta_0 \tag{36}$$

for all $x \in M$ and $t \geq 0$. Using Hölder's inequality, we obtain

$$\int_{B_r(x)} |R_{g(t)}|^q dvol_{g(t)} \leq \left(\int_{B_r(x)} dvol_{g(t)} \right)^{\frac{p-q}{p}} \left(\int_{B_r(x)} |R_{g(t)}|^p dvol_{g(t)} \right)^{\frac{q}{p}}.$$

Hence, if we choose η_0 sufficiently small, then we have

$$\int_{B_r(x)} |R_{g(t)}|^q \, dvol_{g(t)} \leq \eta_1 \tag{37}$$

for all $x \in M$ and $t \geq 0$. Here, η_1 is the constant appearing in Proposition A.1. Using Proposition A.1, we conclude that $u(t)$ is uniformly bounded from above. Hence, if we define

$$P = R_{g_0} + \sigma \left(\sup_{t \geq 0} \sup_M u(t) \right)^{\frac{4}{n-2}},$$

then we obtain

$$\begin{aligned} &-\frac{4(n-1)}{n-2} \, \Delta_{g_0} u(t) + P \, u(t) \\ &\geq -\frac{4(n-1)}{n-2} \, \Delta_{g_0} u(t) + R_{g_0} \, u(t) + \sigma \, u(t)^{\frac{n+2}{n-2}} \\ &= (R_{g(t)} + \sigma) \, u(t)^{\frac{n+2}{n-2}} \geq 0. \end{aligned} \tag{38}$$

According to Corollary A.3, we can find a positive constant c such that

$$\inf_M u(t) \left(\sup_M u(t) \right)^{\frac{n+2}{n-2}} \geq c$$

for all $t \geq 0$. Since $u(t)$ is uniformly bounded from above, we conclude that $u(t)$ is uniformly bounded from below.

Proposition 3.8. *Let $0 < \alpha < \frac{4}{n+2}$. Then, the function u satisfies*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C\left((t_1 - t_2)^{\frac{\alpha}{2}} + d(x_1, x_2)^\alpha\right) \tag{39}$$

for all $x_1, x_2 \in M$ and $0 < t_1 - t_2 < 1$. Here, C is a positive constant independent of t_1 and t_2.

Proof. Let $\alpha = 2 - \frac{n}{p}$, where $\frac{n}{2} < p < \frac{n+2}{2}$. Using Corollary 3.2 and Proposition 3.7, we obtain

$$\int_M \Big|\frac{4(n-1)}{n-2}\,\Delta_{g_0} u(t) - R_{g_0} u(t)\Big|^p \, dvol_{g_0} \leq C \tag{40}$$

and

$$\int_M \Big|\frac{\partial}{\partial t} u(t)\Big|^p \, dvol_{g(t)} \leq C, \tag{41}$$

where C is a positive constant independent of t. The first inequality implies that

$$|u(x_1, t) - u(x_2, t)| \leq C\, d(x_1, x_2)^\alpha$$

for all $x_1, x_2 \in M$ and $t \geq 0$. Using the second inequality, we obtain

$$\begin{aligned}
&|u(x, t_1) - u(x, t_2)| \\
&\leq C\,(t_1 - t_2)^{-\frac{n}{2}} \int_{B_{\sqrt{t_1 - t_2}}(x)} |u(x, t_1) - u(x, t_2)|\, dvol_{g_0} \\
&\leq C\,(t_1 - t_2)^{-\frac{n}{2}} \int_{B_{\sqrt{t_1 - t_2}}(x)} |u(t_1) - u(t_2)|\, dvol_{g_0} + C\,(t_1 - t_2)^{\frac{\alpha}{2}} \\
&\leq C\,(t_1 - t_2)^{-\frac{n-2}{2}} \sup_{t_1 \geq t \geq t_2} \int_{B_{\sqrt{t_1 - t_2}}(x)} \Big|\frac{\partial}{\partial t} u(t)\Big|\, dvol_{g_0} + C\,(t_1 - t_2)^{\frac{\alpha}{2}} \\
&\leq C\,(t_1 - t_2)^{\frac{\alpha}{2}} \sup_{t_1 \geq t \geq t_2} \bigg(\int_M \Big|\frac{\partial}{\partial t} u(t)\Big|^p dvol_{g_0}\bigg)^{\frac{1}{p}} + C\,(t_1 - t_2)^{\frac{\alpha}{2}} \\
&\leq C\,(t_1 - t_2)^{\frac{\alpha}{2}}
\end{aligned}$$

for all $x \in M$ and $0 < t_1 - t_2 < 1$. This proves the assertion.

In view of Proposition 3.8, we may apply the standard regularity theory for parabolic equations (see [**7**], Theorem 5 on p. 64) to derive uniform estimates for all higher order derivatives of u. The uniqueness of the asymptotic limit follows from Proposition 3.5. This completes the proof of Theorem 1.1.

4. Blow-up analysis

The remaining part of this paper will be concerned with the proof of Proposition 3.3. Let $\{t_\nu : \nu \in \mathbb{N}\}$ be a sequence of times such that $t_\nu \to \infty$ as $\nu \to \infty$. For abbreviation, let $u_\nu = u(t_\nu)$ and $g_\nu = g(t_\nu) = u(t_\nu)^{\frac{4}{n-2}}\, g_0 = u_\nu^{\frac{4}{n-2}}\, g_0$. The normalization condition implies that

$$\int_M dvol_{g_\nu} = 1,$$

hence

$$\int_M u_\nu^{\frac{2n}{n-2}}\, dvol_{g_0} = 1 \tag{42}$$

for all $\nu \in \mathbb{N}$. Moreover, it follows from Corollary 3.2 that

$$\int_M |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}}\, dvol_{g_\nu} \to 0,$$

hence

$$\int_M \Big| \frac{4(n-1)}{n-2}\, \Delta_{g_0} u_\nu - R_{g_0}\, u_\nu + r_\infty\, u_\nu^{\frac{n+2}{n-2}} \Big|^{\frac{2n}{n+2}}\, dvol_{g_0} \to 0 \tag{43}$$

as $\nu \to \infty$.

At this point, we may apply the following compactness result due to Struwe [**19**]. A similar result for the harmonic map heat flow can be found in [**12**].

Proposition 4.1. *Let $\{u_\nu : \nu \in \mathbb{N}\}$ be a sequence of positive functions satisfying (42) and (43). After passing to a subsequence if necessary, we can find a non-negative integer m, a non-negative smooth function u_∞ and a sequence of m-tuplets $(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)_{1 \leq k \leq m}$ with the following properties:*

(i) *The function u_∞ satisfies the equation*

$$\frac{4(n-1)}{n-2}\, \Delta_{g_0} u_\infty - R_{g_0}\, u_\infty + r_\infty\, u_\infty^{\frac{n+2}{n-2}} = 0. \tag{44}$$

(ii) *For all $i \neq j$, we have*

$$\frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{d(x_{i,\nu}^*, x_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^*\, \varepsilon_{j,\nu}^*} \to \infty \tag{45}$$

as $\nu \to \infty$.

(iii) *We have*

$$\Big\| u_\nu - u_\infty - \sum_{k=1}^m \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \Big\|_{H^1(M)} \to 0 \tag{46}$$

as $\nu \to \infty$. Here, the functions $\overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)}$ are the standard test functions constructed in Appendix B.

Proof. The first and the third statement follow from results of Struwe [19]. The second statement is due to Bahri and Coron [3].

Proposition 4.2. *If u_∞ vanishes at one point in M, then u_∞ vanishes everywhere.*

Proof. By Proposition 4.1, the function u_∞ satisfies

$$-\frac{4(n-1)}{n-2}\,\Delta_{g_0} u_\infty + R_{g_0}\, u_\infty = r_\infty\, u_\infty^{\frac{n+2}{n-2}} \geq 0.$$

By assumption, the background metric g_0 has positive scalar curvature. Hence, if u_∞ attains a non-positive minimum, then u_∞ is constant by the strong maximum principle (see [8], Theorem 8.19 on p. 198).

The cases $u_\infty \equiv 0$ and $u_\infty > 0$ need to be discussed separately. The case $u_\infty \equiv 0$ will be studied in Section 5. In Section 6, we deal with the case $u_\infty > 0$. The proof of Proposition 3.3 will be completed in Section 7.

For convenience, we define two functionals $E(u)$ and $F(u)$ by

$$E(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2}\,|du|_{g_0}^2 + R_{g_0}\, u^2\right) dvol_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}}\, dvol_{g_0}\right)^{\frac{n-2}{n}}} \tag{47}$$

and

$$F(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2}\,|du|_{g_0}^2 + R_{g_0}\, u^2\right) dvol_{g_0}}{\int_M u^{\frac{2n}{n-2}}\, dvol_{g_0}}. \tag{48}$$

Then, we have

$$\begin{aligned} 1 &= \lim_{\nu\to\infty} \int_M u_\nu^{\frac{2n}{n-2}}\, dvol_{g_0} \\ &= \lim_{\nu\to\infty} \left(\int_M u_\infty^{\frac{2n}{n-2}}\, dvol_{g_0} + \sum_{k=1}^m \int_M \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)}^{\frac{2n}{n-2}}\, dvol_{g_0}\right) \\ &= \left(\frac{E(u_\infty)}{r_\infty}\right)^{\frac{n}{2}} + m\left(\frac{Y(S^n)}{r_\infty}\right)^{\frac{n}{2}}, \end{aligned}$$

hence,

$$r_\infty = (E(u_\infty)^{\frac{n}{2}} + m\, Y(S^n)^{\frac{n}{2}})^{\frac{2}{n}} \tag{49}$$

(compare [17], Lemma 3.4).

5. The case $u_\infty \equiv 0$

Throughout this section, we will assume that $u_\infty \equiv 0$. For every $\nu \in \mathbb{N}$, we denote by $\mathcal{A}_\nu$ the set of all m-tuplets $(x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m} \in (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m$ such that

$$d(x_k, x_{k,\nu}^*) \leq \varepsilon_{k,\nu}^*, \quad \frac{1}{2} \leq \frac{\varepsilon_k}{\varepsilon_{k,\nu}^*} \leq 2, \quad \frac{1}{2} \leq \alpha_k \leq 2 \tag{50}$$

for all $1 \leq k \leq m$. Moreover, we can find an m-tuplet $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m} \in \mathcal{A}_\nu$ such that

$$\begin{aligned} &\int_M \bigg(\frac{4(n-1)}{n-2} \bigg| d\Big(u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \Big) \bigg|_{g_0}^2 \\ &\qquad + R_{g_0} \Big(u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \Big)^2 \bigg) \, dvol_{g_0} \\ &\leq \int_M \bigg(\frac{4(n-1)}{n-2} \bigg| d\Big(u_\nu - \sum_{k=1}^m \alpha_k \, \overline{u}_{(x_k, \varepsilon_k)} \Big) \bigg|_{g_0}^2 \\ &\qquad + R_{g_0} \Big(u_\nu - \sum_{k=1}^m \alpha_k \, \overline{u}_{(x_k, \varepsilon_k)} \Big)^2 \bigg) \, dvol_{g_0} \end{aligned} \tag{51}$$

for all $(x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m} \in \mathcal{A}_\nu$.

Proposition 5.1.

(i) *For all $i \neq j$, we have*

$$\frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \, \varepsilon_{j,\nu}} \to \infty \tag{52}$$

as $\nu \to \infty$.

(ii) *We have*

$$\Big\| u_\nu - \sum_{k=1}^m \alpha_{k,\nu} \, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \Big\|_{H^1(M)} \to 0 \tag{53}$$

as $\nu \to \infty$.

Proof. (i) Since $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m} \in \mathcal{A}_\nu$, we have

$$\begin{aligned} &32 \, \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + 32 \, \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + 8 \, \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \, \varepsilon_{j,\nu}} \\ &\quad \geq 8 \, \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 8 \, \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + 2 \, \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}^* \, \varepsilon_{j,\nu}^*} \end{aligned}$$

$$\geq 4\,\frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 4\,\frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{(d(x_{i,\nu}, x_{j,\nu}) + \varepsilon_{i,\nu}^* + \varepsilon_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^*\,\varepsilon_{j,\nu}^*}$$
$$\geq 4\,\frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 4\,\frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{d(x_{i,\nu}^*, x_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^*\,\varepsilon_{j,\nu}^*},$$

and the expression on the right-hand side tends to ∞ as $\nu \to \infty$.

(ii) By definition of $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}$, we have

$$\int_M \bigg(\frac{4(n-1)}{n-2} \Big| d\Big(u_\nu - \sum_{k=1}^m \alpha_{k,\nu}\,\overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}\Big)\Big|_{g_0}^2 + R_{g_0}\Big(u_\nu - \sum_{k=1}^m \alpha_{k,\nu}\,\overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}\Big)^2\bigg)\,dvol_{g_0}$$
$$\leq \int_M \bigg(\frac{4(n-1)}{n-2} \Big| d\Big(u_\nu - \sum_{k=1}^m \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)}\Big)\Big|_{g_0}^2 + R_{g_0}\Big(u_\nu - \sum_{k=1}^m \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)}\Big)^2\bigg)\,dvol_{g_0}.$$

By Proposition 4.1, the expression on the right-hand side tends to 0 as $\nu \to \infty$. This proves the assertion.

Proposition 5.2. *We have*

$$d(x_{k,\nu}, x_{k,\nu}^*) \leq o(1)\,\varepsilon_{k,\nu}^*, \qquad \frac{\varepsilon_{k,\nu}}{\varepsilon_{k,\nu}^*} = 1 + o(1), \qquad \alpha_{k,\nu} = 1 + o(1) \tag{54}$$

for all $1 \leq k \leq m$. *In particular,* $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}$ *is an interior point of* $\mathcal{A}_\nu$ *if* ν *is sufficiently large.*

Proof. Observe that

$$\Big\| \sum_{k=1}^m \alpha_{k,\nu}\,\overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} - \sum_{k=1}^m \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)}\Big\|_{H^1(M)}$$
$$\leq \Big\| u_\nu - \sum_{k=1}^m \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)}\Big\|_{H^1(M)} + \Big\| u_\nu - \sum_{k=1}^m \alpha_{k,\nu}\,\overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}\Big\|_{H^1(M)}$$
$$= o(1)$$

by Propositions 4.1 and 5.1. From this, the assertion follows.

In the sequel, we assume that

$$\varepsilon_{i,\nu} \leq \varepsilon_{j,\nu} \quad \text{for} \quad i \leq j. \tag{55}$$

We now decompose the function u_ν as

$$u_\nu = v_\nu + w_\nu, \tag{56}$$

where

$$v_\nu = \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}, \tag{57}$$

and

$$w_\nu = u_\nu - \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}. \tag{58}$$

Note that the function w_ν satisfies

$$\int_M \Big(\frac{4(n-1)}{n-2}\, |dw_\nu|_{g_0}^2 + R_{g_0}\, w_\nu^2\Big)\, dvol_{g_0} = o(1) \tag{59}$$

by Proposition 5.1.

Proposition 5.3.

(i) *For every $1 \leq k \leq m$, we have*

$$\bigg|\int_M \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}\, w_\nu\, dvol_{g_0}\bigg| \leq o(1) \bigg(\int_M |w_\nu|^{\frac{2n}{n-2}}\, dvol_{g_0}\bigg)^{\frac{n-2}{2n}}. \tag{60}$$

(ii) *For every $1 \leq k \leq m$, we have*

$$\begin{aligned} &\bigg|\int_M \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}\, \frac{\varepsilon_{k,\nu}^2 - d(x_{k,\nu},x)^2}{\varepsilon_{k,\nu}^2 + d(x_{k,\nu},x)^2}\, w_\nu\, dvol_{g_0}\bigg| \\ &\leq o(1) \bigg(\int_M |w_\nu|^{\frac{2n}{n-2}}\, dvol_{g_0}\bigg)^{\frac{n-2}{2n}}. \end{aligned} \tag{61}$$

(iii) *For all $1 \leq k \leq m$, we have*

$$\begin{aligned} &\bigg|\int_M \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}\, \frac{\varepsilon_{k,\nu}\, \exp_{x_{k,\nu}}^{-1}(x)}{\varepsilon_{k,\nu}^2 + d(x_{k,\nu},x)^2}\, w_\nu\, dvol_{g_0}\bigg| \\ &\leq o(1) \bigg(\int_M |w_\nu|^{\frac{2n}{n-2}}\, dvol_{g_0}\bigg)^{\frac{n-2}{2n}}. \end{aligned} \tag{62}$$

Proof. By definition of $(x_{k,\nu},\varepsilon_{k,\nu},\alpha_{k,\nu})_{1\leq k\leq m}$, we have

$$\int_M \Big(\frac{4(n-1)}{n-2}\, \langle d\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}, dw_\nu\rangle_{g_0} + R_{g_0}\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\, w_\nu\Big)\, dvol_{g_0} = 0,$$

hence,

$$\int_M \Big(\frac{4(n-1)}{n-2}\, \Delta_{g_0}\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} - R_{g_0}\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\Big)\, w_\nu\, dvol_{g_0} = 0$$

for all $1 \leq k \leq m$. Using the estimate

$$\Big\| \frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} - R_{g_0} \, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} + r_\infty \, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{2n}{n+2}}(M)} = o(1),$$

we conclude that

$$\bigg| \int_M \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{n+2}{n-2}} \, w_\nu \, dvol_{g_0} \bigg| \leq o(1) \, \|w_\nu\|_{L^{\frac{2n}{n-2}}(M)}$$

for all $1 \leq k \leq m$. This proves (i).

The remaining statements follow similarly.

In the next step, we prove uniform estimates for the second variation operator of the Yamabe functional at v_ν. A similar estimate was derived by Bahri (see [4], Proposition 3.1 on p. 64).

Proposition 5.4. *If ν is sufficiently large, then we have*

$$\begin{aligned} (63) \qquad & \frac{n+2}{n-2} \, r_\infty \int_M \sum_{j=1}^m \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{4}{n-2}} \, w_\nu^2 \, dvol_{g_0} \\ & \leq (1-c) \int_M \Big(\frac{4(n-1)}{n-2} \, |dw_\nu|_{g_0}^2 + R_{g_0} \, w_\nu^2 \Big) \, dvol_{g_0} \end{aligned}$$

for some positive constant c independent of ν.

Proof. Suppose this is not true. Upon rescaling, we obtain a sequence of functions $\{\widetilde{w}_\nu : \nu \in \mathbb{N}\}$ such that

$$(64) \qquad \int_M \Big(\frac{4(n-1)}{n-2} \, |d\widetilde{w}_\nu|_{g_0}^2 + R_{g_0} \, \widetilde{w}_\nu^2 \Big) \, dvol_{g_0} = 1$$

and

$$(65) \qquad \lim_{\nu \to \infty} \frac{n+2}{n-2} \, r_\infty \int_M \sum_{j=1}^m \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{4}{n-2}} \, \widetilde{w}_\nu^2 \, dvol_{g_0} \geq 1.$$

Note that

$$(66) \qquad \int_M |\widetilde{w}_\nu|^{\frac{2n}{n-2}} \, dvol_{g_0} \leq Y(M,g_0)^{-\frac{n}{n-2}}$$

by (64). In view of Proposition 5.1, we can find a sequence $\{N_\nu : \nu \in \mathbb{N}\}$ such that $N_\nu \to \infty$, $N_\nu \varepsilon_{j,\nu} \to 0$ for all $1 \leq j \leq m$, and

$$(67) \qquad \frac{1}{N_\nu} \, \frac{\varepsilon_{j,\nu} + d(x_{i,\nu}, x_{j,\nu})}{\varepsilon_{i,\nu}} \to \infty$$

for all $i < j$. Let

$$(68) \qquad \Omega_{j,\nu} = B_{N_\nu \varepsilon_{j,\nu}}(x_{j,\nu}) \setminus \bigcup_{i=1}^{j-1} B_{N_\nu \varepsilon_{i,\nu}}(x_{i,\nu})$$

for every $1 \leq j \leq m$. In view of (64) and (65), we can find an integer $1 \leq j \leq m$ such that

$$\lim_{\nu \to \infty} \int_M \overline{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{4}{n-2}} \, \widetilde{w}_\nu^2 \, dvol_{g_0} > 0 \tag{69}$$

and

$$\begin{aligned} &\lim_{\nu \to \infty} \int_{\Omega_{j,\nu}} \left(\frac{4(n-1)}{n-2} \, |d\widetilde{w}_\nu|_{g_0}^2 + R_{g_0} \, \widetilde{w}_\nu^2 \right) dvol_{g_0} \\ &\leq \lim_{\nu \to \infty} \frac{n+2}{n-2} \, r_\infty \int_M \overline{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{4}{n-2}} \, \widetilde{w}_\nu^2 \, dvol_{g_0}. \end{aligned} \tag{70}$$

We now define a sequence of functions $\hat{w}_\nu : TM_{x_{j,\nu}} \to \mathbb{R}$ by

$$\hat{w}_\nu(\xi) = \varepsilon_{j,\nu}^{\frac{n-2}{2}} \, \widetilde{w}_\nu(\exp_{x_{j,\nu}}(\varepsilon_{j,\nu} \, \xi))$$

for $\xi \in TM_{x_{j,\nu}}$. The sequence $\{\hat{w}_\nu : \nu \in \mathbb{N}\}$ satisfies

$$\lim_{\nu \to \infty} \int_{\{\xi \in TM_{x_{j,\nu}} : |\xi| \leq N_\nu\}} \frac{4(n-1)}{n-2} |d\hat{w}_\nu(\xi)|^2 \, d\xi \leq 1$$

and

$$\lim_{\nu \to \infty} \int_{\{\xi \in TM_{xj,\nu} : |\xi| \leq N_\nu\}} |\hat{w}_\nu(\xi)|^{\frac{2n}{n-2}} d\xi \leq Y(M, g_0)^{-\frac{n}{n-2}}.$$

Hence, if we take the weak limit as $\nu \to \infty$, then we obtain a function $\hat{w} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^2 \hat{w}(\xi)^2 \, d\xi > 0$$

and

$$\int_{\mathbb{R}^n} |d\hat{w}(\xi)|^2 \, d\xi \leq n(n+2) \int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^2 \hat{w}(\xi)^2 \, d\xi.$$

Moreover, it follows from Proposition 5.3 that

$$\int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \hat{w}(\xi) \, d\xi = 0$$

$$\int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \frac{1 - |\xi|^2}{1 + |\xi|^2} \, \hat{w}(\xi) \, d\xi = 0$$

$$\int_{\mathbb{R}^n} \left(\frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \frac{\xi}{1 + |\xi|^2} \, \hat{w}(\xi) \, d\xi = 0.$$

Using a result of Rey, we conclude that $\hat{w}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ (see [**13**], Appendix D, pp. 49–51). This is a contradiction.

Corollary 5.5. *If ν is sufficiently large, then we have*

$$\frac{n+2}{n-2}\, r_\infty \int_M v_\nu^{\frac{4}{n-2}}\, w_\nu^2\, dvol_{g_0} \leq (1-c) \int_M \left(\frac{4(n-1)}{n-2}\, |dw_\nu|_{g_0}^2 + R_{g_0}\, w_\nu^2 \right) dvol_{g_0} \tag{71}$$

for some positive constant c independent of ν.

Proof. By definition of v_ν, we have

$$\int_M \left| v_\nu^{\frac{4}{n-2}} - \sum_{j=1}^m \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{4}{n-2}} \right|^{\frac{n}{2}} dvol_{g_0} = o(1).$$

Therefore, the assertion follows from Proposition 5.4.

Proposition 5.6. *The Yamabe energy of v_ν satisfies the estimate*

$$E(v_\nu) \leq \left(\sum_{k=1}^m E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}} \right)^{\frac{2}{n}} \tag{72}$$

if ν is sufficiently large.

Proof. Using the identity

$$\begin{aligned} &\int_M \left(\frac{4(n-1)}{n-2}\, |dv_\nu|_{g_0}^2 + R_{g_0}\, v_\nu^2 \right) dvol_{g_0} \\ &= \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 \left(\frac{4(n-1)}{n-2}\, |d\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}|_{g_0}^2 + R_{g_0}\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^2 \right) dvol_{g_0} \\ &\quad + 2 \int_M \sum_{i<j} \alpha_{i,\nu}\, \alpha_{j,\nu} \left(\frac{4(n-1)}{n-2}\, \langle d\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}, d\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \rangle_{g_0} \right. \\ &\qquad \left. + R_{g_0}\, \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \right) dvol_{g_0}, \end{aligned} \tag{73}$$

we obtain

$$\begin{aligned} &E(v_\nu) \left(\int_M v_\nu^{\frac{2n}{n-2}}\, dvol_{g_0} \right)^{\frac{n-2}{n}} \\ &= \int_M \sum_{k=1}^m \alpha_{k,\nu}^2\, F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}}\, dvol_{g_0} \\ &\quad - 2 \int_M \sum_{i<j} \alpha_{i,\nu}\, \alpha_{j,\nu}\, \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \end{aligned} \tag{74}$$

$$\cdot \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} - R_{g_0} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \Big) \, dvol_{g_0}.$$

Moreover, we have

$$\begin{aligned}
(75) \quad & \Big(\sum_{k=1}^m E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}} \Big)^{\frac{2}{n}} \Big(\int_M v_\nu^{\frac{2n}{n-2}} \, dvol_{g_0} \Big)^{\frac{n-2}{n}} \\
&= \Big(\int_M \Big(\sum_{k=1}^m F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}} \, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \Big) \, dvol_{g_0} \Big)^{\frac{2}{n}} \\
&\quad \cdot \Big(\int_M v_\nu^{\frac{2n}{n-2}} \, dvol_{g_0} \Big)^{\frac{n-2}{n}} \\
&\geq \int_M \Big(\sum_{k=1}^m F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}} \, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \Big)^{\frac{2}{n}} \, v_\nu^2 \, dvol_{g_0} \\
&\geq \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 \, F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}) \, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \, dvol_{g_0} \\
&\quad + 2 \int_M \sum_{i<j} \alpha_{i,\nu} \, \alpha_{j,\nu} \, \Big(F(\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})})^{\frac{n}{2}} \, \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}^{\frac{2n}{n-2}} \\
&\quad\quad + F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})^{\frac{n}{2}} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{2n}{n-2}} \Big)^{\frac{2}{n}} \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \, dvol_{g_0}
\end{aligned}$$

by Hölder's inequality. Consider a pair $i < j$. We can find positive constants c and C independent of ν such that

$$\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}(x)^{\frac{n+2}{n-2}} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}(x) \geq c \Big(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}\varepsilon_{j,\nu}} \Big)^{-\frac{n-2}{2}} \, \varepsilon_{i,\nu}^{-n}$$

and

$$\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}(x) \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}(x)^{\frac{n+2}{n-2}} \leq C \Big(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}\varepsilon_{j,\nu}} \Big)^{-\frac{n+2}{2}} \, \varepsilon_{i,\nu}^{-n}$$

if $d(x_{i,\nu}, x) \leq \varepsilon_{i,\nu}$ and ν is sufficiently large. From this, it follows that

$$\begin{aligned}
& \Big(F(\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})})^{\frac{n}{2}} \, \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}^{\frac{2n}{n-2}} + F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})^{\frac{n}{2}} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{2n}{n-2}} \Big)^{\frac{2}{n}} \\
&\quad \cdot \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \\
&\quad \geq F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}) \, \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \\
&\quad\quad + c \Big(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}\varepsilon_{j,\nu}} \Big)^{-\frac{n-2}{2}} \, \varepsilon_{i,\nu}^{-n} \, 1_{\{d(x_{i,\nu},x) \leq \varepsilon_{i,\nu}\}}
\end{aligned}$$

for ν sufficiently large. Integration over M yields

$$\begin{aligned}(76)\qquad &\int_M \Big(F(\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}^{\frac{2n}{n-2}} + F(\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}} \\ &\quad \cdot \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}\,dvol_{g_0} \\ &\quad \geq \int_M F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}}\,dvol_{g_0} \\ &\qquad + c\left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu},x_{j,\nu})^2}{\varepsilon_{i,\nu}\,\varepsilon_{j,\nu}}\right)^{-\frac{n-2}{2}}\end{aligned}$$

if ν is sufficiently large. From this, it follows that

$$\begin{aligned}(77)\quad &\Big(\sum_{k=1}^m E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big(\int_M v_\nu^{\frac{2n}{n-2}}\,dvol_{g_0}\Big)^{\frac{n-2}{n}} \\ &\geq \int_M \sum_{k=1}^m \alpha_{k,\nu}^2\,F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}}\,dvol_{g_0} \\ &\quad + 2\int_M \sum_{i<j} \alpha_{i,\nu}\,\alpha_{j,\nu}\,F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}}\,dvol_{g_0} \\ &\quad + c\sum_{i<j}\left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu},x_{j,\nu})^2}{\varepsilon_{i,\nu}\,\varepsilon_{j,\nu}}\right)^{-\frac{n-2}{2}}.\end{aligned}$$

Putting these facts together, we obtain

$$\begin{aligned}(78)\qquad &E(v_\nu)\Big(\int_M v_\nu^{\frac{2n}{n-2}}\,dvol_{g_0}\Big)^{\frac{n-2}{n}} \\ &\quad \leq \Big(\sum_{k=1}^m E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big(\int_M v_\nu^{\frac{2n}{n-2}}\,dvol_{g_0}\Big)^{\frac{n-2}{n}} \\ &\qquad - 2\int_M \sum_{i<j}\alpha_{i,\nu}\,\alpha_{j,\nu}\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0}\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \\ &\qquad\quad - R_{g_0}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} + F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}}\Big)\,dvol_{g_0} \\ &\qquad - c\sum_{i<j}\left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu},x_{j,\nu})^2}{\varepsilon_{i,\nu}\,\varepsilon_{j,\nu}}\right)^{-\frac{n-2}{2}}.\end{aligned}$$

Since $F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}) = r_\infty + o(1)$, it follows from Lemmas B.4 and B.5 that

(79)

$$\begin{aligned}
&\int_M \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \Big| \frac{4(n-1)}{n-2} \Delta_{g_0} \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \\
&\qquad - R_{g_0} \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} + F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}) \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \Big| \, dvol_{g_0} \\
&\leq \int_M \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \Big| \frac{4(n-1)}{n-2} \Delta_{g_0} \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \\
&\qquad - R_{g_0} \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} + r_\infty \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \Big| \, dvol_{g_0} \\
&\quad + |F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}) - r_\infty| \int_M \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \, dvol_{g_0} \\
&\leq C \Big(\delta^4 + \delta^{n-2} + \frac{\varepsilon_{j,\nu}^2}{\delta^2} \Big) \Big(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \Big)^{-\frac{n-2}{2}} \\
&\quad + o(1) \Big(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu} \varepsilon_{j,\nu}} \Big)^{-\frac{n-2}{2}}
\end{aligned}$$

for $i < j$. Hence, if we choose δ sufficiently small, the assertion follows.

Corollary 5.7. *If ν is sufficiently large, then the Yamabe energy of v_ν satisfies the estimate*

$$E(v_\nu) \leq (m \, Y(S^n)^{\frac{n}{2}})^{\frac{2}{n}}. \tag{80}$$

Proof. Using Proposition B.3, we obtain

$$E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}) \leq Y(S^n)$$

for all $1 \leq k \leq m$. Hence, the assertion follows from Proposition 5.6.

6. The case $u_\infty > 0$

We next discuss the case $u_\infty > 0$.

Proposition 6.1. *There exists a sequence of smooth functions $\{\psi_a : a \in \mathbb{N}\}$ and a sequence of positive real numbers $\{\lambda_a : a \in \mathbb{N}\}$ with the following properties:*

(i) *For every $a \in \mathbb{N}$, the function ψ_a satisfies the equation*

$$\frac{4(n-1)}{n-2} \Delta_{g_0} \psi_a - R_{g_0} \psi_a + \lambda_a \, u_\infty^{\frac{4}{n-2}} \psi_a = 0. \tag{81}$$

(ii) *For all* $a, b \in \mathbb{N}$, *we have*

$$\int_M u_\infty^{\frac{4}{n-2}} \, \psi_a \, \psi_b \, dvol_{g_0} = \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq b \end{cases}. \tag{82}$$

(iii) *The span of* $\{\psi_a : a \in \mathbb{N}\}$ *is dense in* $L^2(M)$.

(iv) $\lambda_a \to \infty$ *as* $a \to \infty$.

Proof. Consider the linear operator

$$\psi \mapsto u_\infty^{-\frac{4}{n-2}} \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \psi - R_{g_0} \, \psi \Big).$$

This operator is symmetric with respect to the inner product

$$(\psi_1, \psi_2) \mapsto \int_M u_\infty^{\frac{4}{n-2}} \, \psi_1 \, \psi_2 \, dvol_{g_0}$$

on $L^2(M)$. Hence, the assertion follows from the spectral theorem.

Let A be a finite subset of $\mathbb{N}$ such that $\lambda_a > \frac{n+2}{n-2} \, r_\infty$ for all $a \notin A$. We denote by Π the projection operator

$$\begin{aligned} \Pi f &= \sum_{a \notin A} \Big(\int_M \psi_a \, f \, dvol_{g_0} \Big) \, u_\infty^{\frac{4}{n-2}} \, \psi_a \\ &= f - \sum_{a \in A} \Big(\int_M \psi_a \, f \, dvol_{g_0} \Big) \, u_\infty^{\frac{4}{n-2}} \, \psi_a. \end{aligned} \tag{83}$$

Lemma 6.2. *For every* $1 \leq p < \infty$, *we can find a constant* C *such that*

$$\begin{aligned} \|f\|_{L^p(M)} \leq C \, \Big\| \frac{4(n-1)}{n-2} \, \Delta_{g_0} f - R_{g_0} \, f + \frac{n+2}{n-2} \, r_\infty \, u_\infty^{\frac{4}{n-2}} \, f \Big\|_{L^p(M)} \\ + C \sup_{a \in A} \Big| \int_M u_\infty^{\frac{4}{n-2}} \, \psi_a \, f \, dvol_{g_0} \Big|. \end{aligned} \tag{84}$$

Proof. Assume that this is not true. By compactness, we can find a function $f \in L^p(M)$ satisfying $\|f\|_{L^p(M)} = 1$,

$$\int_M u_\infty^{\frac{4}{n-2}} \, \psi_a \, f \, dvol_{g_0} = 0 \tag{85}$$

for all $a \in A$ and

$$\frac{4(n-1)}{n-2} \, \Delta_{g_0} f - R_{g_0} \, f + \frac{n+2}{n-2} \, r_\infty \, u_\infty^{\frac{4}{n-2}} \, f = 0 \tag{86}$$

in the sense of distributions. Hence, if we use the function ψ_a as a test function, then we obtain

$$\Big(\lambda_a - \frac{n+2}{n-2} \, r_\infty \Big) \int_M u_\infty^{\frac{4}{n-2}} \, \psi_a \, f \, dvol_{g_0} = 0$$

for all $a \in \mathbb{N}$. In particular, we have

$$\int_M u_\infty^{\frac{4}{n-2}}\, \psi_a\, f\, dvol_{g_0} = 0 \tag{87}$$

for all $a \notin A$. Thus, we conclude that $f = 0$. This is a contradiction.

Lemma 6.3.

(i) *There exists a constant C such that*

$$\begin{aligned} \|f\|_{L^{\frac{n+2}{n-2}}(M)} \leq C\, \Big\| \Pi\Big(\frac{4(n-1)}{n-2}\, \Delta_{g_0} f - R_{g_0}\, f \\ + \frac{n+2}{n-2}\, r_\infty\, u_\infty^{\frac{4}{n-2}}\, f\Big)\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ + C \sup_{a \in A} \Big| \int_M u_\infty^{\frac{4}{n-2}}\, \psi_a\, f\, dvol_{g_0} \Big|. \end{aligned} \tag{88}$$

(ii) *There exists a constant C such that*

$$\begin{aligned} \|f\|_{L^1(M)} \leq C\, \Big\| \Pi\Big(\frac{4(n-1)}{n-2}\, \Delta_{g_0} f - R_{g_0}\, f + \frac{n+2}{n-2}\, r_\infty\, u_\infty^{\frac{4}{n-2}}\, f\Big)\Big\|_{L^1(M)} \\ + C \sup_{a \in A} \Big| \int_M u_\infty^{\frac{4}{n-2}}\, \psi_a\, f\, dvol_{g_0} \Big|. \end{aligned} \tag{89}$$

Proof. (i) It follows from standard elliptic regularity theory that

$$\begin{aligned} \|f\|_{L^{\frac{n+2}{n-2}}(M)} \leq C\, \Big\| \frac{4(n-1)}{n-2}\, \Delta_{g_0} f - R_{g_0}\, f + \frac{n+2}{n-2}\, r_\infty\, u_\infty^{\frac{4}{n-2}}\, f\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ + C\, \|f\|_{L^{\frac{n(n+2)}{n^2+4}}(M)}. \end{aligned} \tag{90}$$

Using Lemma 6.2, we obtain

$$\begin{aligned} \|f\|_{L^{\frac{n+2}{n-2}}(M)} \leq C\, \Big\| \frac{4(n-1)}{n-2}\, \Delta_{g_0} f - R_{g_0}\, f + \frac{n+2}{n-2}\, r_\infty\, u_\infty^{\frac{4}{n-2}}\, f\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ + C \sup_{a \in A} \Big| \int_M u_\infty^{\frac{4}{n-2}}\, \psi_a\, f\, dvol_{g_0} \Big|. \end{aligned} \tag{91}$$

By definition of Π, we have

$$\begin{aligned}(92)\quad &\frac{4(n-1)}{n-2}\,\Delta_{g_0} f - R_{g_0}\,f + \frac{n+2}{n-2}\,r_\infty\,u_\infty^{\frac{4}{n-2}}\,f \\ &= \Pi\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0} f - R_{g_0}\,f + r_\infty\,u_\infty^{\frac{4}{n-2}}\,f\Big) \\ &\quad - \sum_{a \in A} \Big(\lambda_a - \frac{n+2}{n-2}\,r_\infty\Big) \left(\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,f\,dvol_{g_0}\right) u_\infty^{\frac{4}{n-2}}\,\psi_a.\end{aligned}$$

This implies

$$\begin{aligned}(93)\quad &\Big\|\frac{4(n-1)}{n-2}\,\Delta_{g_0} f - R_{g_0}\,f + \frac{n+2}{n-2}\,r_\infty\,u_\infty^{\frac{4}{n-2}}\,f\Big\|_{L^q(M)} \\ &\leq \Big\|\Pi\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0} f - R_{g_0}\,f + r_\infty\,u_\infty^{\frac{4}{n-2}}\,f\Big)\Big\|_{L^q(M)} \\ &\quad + C\,\sup_{a \in A} \left|\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,f\,dvol_{g_0}\right|.\end{aligned}$$

Putting these facts together, the assertion follows.

(ii) It follows from Lemma 6.2 that

$$\begin{aligned}(94)\quad \|f\|_{L^1(M)} &\leq C\,\Big\|\frac{4(n-1)}{n-2}\,\Delta_{g_0} f - R_{g_0}\,f + \frac{n+2}{n-2}\,r_\infty\,u_\infty^{\frac{4}{n-2}}\,f\Big\|_{L^1(M)} \\ &\quad + C\,\sup_{a \in A} \left|\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,f\,dvol_{g_0}\right|.\end{aligned}$$

By definition of Π, we have

$$\begin{aligned}(95)\quad &\frac{4(n-1)}{n-2}\,\Delta_{g_0} f - R_{g_0}\,f + \frac{n+2}{n-2}\,r_\infty\,u_\infty^{\frac{4}{n-2}}\,f \\ &= \Pi\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0} f - R_{g_0}\,f + r_\infty\,u_\infty^{\frac{4}{n-2}}\,f\Big) \\ &\quad - \sum_{a \in A} \Big(\lambda_a - \frac{n+2}{n-2}\,r_\infty\Big) \left(\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,f\,dvol_{g_0}\right) u_\infty^{\frac{4}{n-2}}\,\psi_a.\end{aligned}$$

This implies

$$\begin{aligned}(96)\quad &\Big\|\frac{4(n-1)}{n-2}\,\Delta_{g_0} f - R_{g_0}\,f + \frac{n+2}{n-2}\,r_\infty\,u_\infty^{\frac{4}{n-2}}\,f\Big\|_{L^1(M)} \\ &\leq \Big\|\Pi\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0} f - R_{g_0}\,f + r_\infty\,u_\infty^{\frac{4}{n-2}}\,f\Big)\Big\|_{L^1(M)} \\ &\quad + C\,\sup_{a \in A} \left|\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,f\,dvol_{g_0}\right|.\end{aligned}$$

From this, the assertion follows.

Lemma 6.4. *There exists a positive real number* ζ *with the following significance: for every vector* $z \in \mathbb{R}^A$ *with* $|z| \leq \zeta$*, there exists a smooth function* $\overline{u}_z$ *such that*

$$\int_M u_\infty^{\frac{4}{n-2}} \, (\overline{u}_z - u_\infty) \, \psi_a \, dvol_{g_0} = z_a \tag{97}$$

for all $a \in A$ *and*

$$\Pi\Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_z - R_{g_0} \, \overline{u}_z + r_\infty \, \overline{u}_z^{\frac{n+2}{n-2}}\Big) = 0. \tag{98}$$

Furthermore, the map $z \mapsto \overline{u}_z$ *is real analytic.*

Proof. This is a consequence of the implicite function theorem.

Lemma 6.5. *There exists a real number* $0 < \gamma < 1$ *such that*

(99)

$$\begin{aligned} &E(\overline{u}_z) - E(u_\infty) \\ &\quad \leq C \sup_{a \in A} \bigg| \int_M \psi_a \, \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_z - R_{g_0} \, \overline{u}_z + r_\infty \, \overline{u}_z^{\frac{n+2}{n-2}}\Big) \, dvol_{g_0} \bigg|^{1+\gamma} \end{aligned}$$

if z *is sufficiently small.*

Proof. Note that the function $z \mapsto E(\overline{u}_z)$ is real analytic. According to results of Lojasiewicz (see [**18**], equation (2.4) on p. 538), there exists a real number $0 < \gamma < 1$ such that

$$|E(\overline{u}_z) - E(u_\infty)| \leq \sup_{a \in A} \Big| \frac{\partial}{\partial z_a} E(\overline{u}_z) \Big|^{1+\gamma} \tag{100}$$

if z is sufficiently small. The partial derivatives of the function $z \mapsto E(\overline{u}_z)$ are given by

(101)

$$\begin{aligned} \frac{\partial}{\partial z_a} E(\overline{u}_z) &= -2 \, \frac{\int_M \widetilde{\psi}_{a,z} \, \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_z - R_{g_0} \, \overline{u}_z + r_\infty \, \overline{u}_z^{\frac{n+2}{n-2}}\Big) \, dvol_{g_0}}{\Big(\int_M \overline{u}_z^{\frac{2n}{n-2}} \, dvol_{g_0}\Big)^{\frac{n-2}{n}}} \\ &\quad - 2 \, (F(\overline{u}_z) - r_\infty) \, \frac{\int_M \overline{u}_z^{\frac{n+2}{n-2}} \, \widetilde{\psi}_{a,z} \, dvol_{g_0}}{\Big(\int_M \overline{u}_z^{\frac{2n}{n-2}} \, dvol_{g_0}\Big)^{\frac{n-2}{n}}}, \end{aligned}$$

where $\widetilde{\psi}_{a,z} = \frac{\partial}{\partial z_a} \overline{u}_z$ for $a \in A$. The function $\widetilde{\psi}_{a,z}$ satisfies

$$\int_M u_\infty^{\frac{4}{n-2}} \, \widetilde{\psi}_{a,z} \, \psi_b \, dvol_{g_0} = \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq b \end{cases} \tag{102}$$

for all $b \in A$ and

$$\Pi\bigg(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \widetilde{\psi}_{a,z} - R_{g_0} \, \widetilde{\psi}_{a,z} + \frac{n+2}{n-2} \, r_\infty \, \overline{u}_z^{\frac{4}{n-2}} \, \widetilde{\psi}_{a,z}\bigg) = 0. \tag{103}$$

Using the identity

$$\Pi\bigg(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_z - R_{g_0} \, \overline{u}_z + r_\infty \, \overline{u}_z^{\frac{n+2}{n-2}}\bigg) = 0,$$

we obtain

(104)

$$\begin{aligned} \frac{\partial}{\partial z_a} E(\overline{u}_z) = {} & -2 \, \frac{\int_M \psi_a \, \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_z - R_{g_0} \, \overline{u}_z + r_\infty \, \overline{u}_z^{\frac{n+2}{n-2}}\Big) \, dvol_{g_0}}{\Big(\int_M \overline{u}_z^{\frac{2n}{n-2}} \, dvol_{g_0}\Big)^{\frac{n-2}{n}}} \\ & + 2 \sum_{b \in A} \frac{\int_M \overline{u}_z^{\frac{n+2}{n-2}} \, \widetilde{\psi}_{a,z} \, dvol_{g_0} \, \int_M u_\infty^{\frac{4}{n-2}} \, \overline{u}_z \, \psi_b \, dvol_{g_0}}{\int_M \overline{u}_z^{\frac{2n}{n-2}} \, dvol_{g_0}} \\ & \quad \cdot \frac{\int_M \psi_b \, \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_z - R_{g_0} \, \overline{u}_z + r_\infty \, \overline{u}_z^{\frac{n+2}{n-2}}\Big) \, dvol_{g_0}}{\Big(\int_M \overline{u}_z^{\frac{2n}{n-2}} \, dvol_{g_0}\Big)^{\frac{n-2}{n}}} \end{aligned}$$

for all $a \in A$. Thus, we conclude that

(105)

$$\begin{aligned} & \sup_{a \in A} \Big| \frac{\partial}{\partial z_a} E(\overline{u}_z) \Big| \\ & \quad \leq C \sup_{a \in A} \bigg| \int_M \psi_a \, \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_z - R_{g_0} \, \overline{u}_z + r_\infty \, \overline{u}_z^{\frac{n+2}{n-2}}\Big) \, dvol_{g_0} \bigg|. \end{aligned}$$

From this, the assertion follows.

For every $\nu \in \mathbb{N}$, we denote by $\mathcal{A}_\nu$ the set of all pairs $\big(z, (x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m}\big) \in \mathbb{R}^A \times (M \times \mathbb{R}_+ \times \mathbb{R}_+)^m$ such that

$$|z| \leq \zeta \tag{106}$$

and

$$d(x_k, x_{k,\nu}^*) \leq \varepsilon_{k,\nu}^*, \quad \frac{1}{2} \leq \frac{\varepsilon_k}{\varepsilon_{k,\nu}^*} \leq 2, \quad \frac{1}{2} \leq \alpha_k \leq 2 \tag{107}$$

for all $1 \leq k \leq m$. Moreover, we can find a pair

$$\big(z_\nu, (x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}\big) \in \mathcal{A}_\nu$$

such that

$$\begin{aligned}
(108) \qquad & \int_M \bigg(\frac{4(n-1)}{n-2} \bigg| d\Big(u_\nu - \overline{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}\Big) \bigg|_{g_0}^2 \\
&\quad + R_{g_0} \Big(u_\nu - \overline{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \Big)^2 \bigg) \, dvol_{g_0} \\
&\leq \int_M \bigg(\frac{4(n-1)}{n-2} \bigg| d\Big(u_\nu - \overline{u}_z - \sum_{k=1}^m \alpha_k\, \overline{u}_{(x_k, \varepsilon_k)}\Big) \bigg|_{g_0}^2 \\
&\quad + R_{g_0} \Big(u_\nu - \overline{u}_z - \sum_{k=1}^m \alpha_k\, \overline{u}_{(x_k, \varepsilon_k)} \Big)^2 \bigg) \, dvol_{g_0}
\end{aligned}$$

for all $\big(z, (x_k, \varepsilon_k, \alpha_k)_{1 \leq k \leq m}\big) \in \mathcal{A}_\nu$.

Proposition 6.6.

(i) *For all $i \neq j$, we have*

$$(109) \qquad \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}\, \varepsilon_{j,\nu}} \to \infty$$

as $\nu \to \infty$.

(ii) *We have*

$$(110) \qquad \Big\| u_\nu - \overline{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \Big\|_{H^1(M)} \to 0$$

as $\nu \to \infty$.

Proof. (i) Since $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m} \in \mathcal{A}_\nu$, we have

$$\begin{aligned}
& 32\, \frac{\varepsilon_{i,\nu}}{\varepsilon_{j,\nu}} + 32\, \frac{\varepsilon_{j,\nu}}{\varepsilon_{i,\nu}} + 8\, \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}\, \varepsilon_{j,\nu}} \\
&\geq 8\, \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 8\, \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + 2\, \frac{d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}^*\, \varepsilon_{j,\nu}^*} \\
&\geq 4\, \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 4\, \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{(d(x_{i,\nu}, x_{j,\nu}) + \varepsilon_{i,\nu}^* + \varepsilon_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^*\, \varepsilon_{j,\nu}^*} \\
&\geq 4\, \frac{\varepsilon_{i,\nu}^*}{\varepsilon_{j,\nu}^*} + 4\, \frac{\varepsilon_{j,\nu}^*}{\varepsilon_{i,\nu}^*} + \frac{d(x_{i,\nu}^*, x_{j,\nu}^*)^2}{\varepsilon_{i,\nu}^*\, \varepsilon_{j,\nu}^*},
\end{aligned}$$

and the expression on the right-hand side tends to ∞ as $\nu \to \infty$.

(ii) By definition of $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}$, we have

$$\begin{aligned}
&\int_M \bigg(\frac{4(n-1)}{n-2} \bigg| d\bigg(u_\nu - \overline{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \bigg) \bigg|_{g_0}^2 \\
&\qquad\qquad + R_{g_0} \bigg(u_\nu - \overline{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \bigg)^2 \bigg)\, dvol_{g_0} \\
&\leq \int_M \bigg(\frac{4(n-1)}{n-2} \bigg| d\bigg(u_\nu - u_\infty - \sum_{k=1}^m \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \bigg) \bigg|_{g_0}^2 \\
&\qquad\qquad + R_{g_0} \bigg(u_\nu - u_\infty - \sum_{k=1}^m \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \bigg)^2 \bigg)\, dvol_{g_0}.
\end{aligned}$$

By Proposition 4.1, the expression on the right-hand side tends to 0 as $\nu \to \infty$. This proves the assertion.

Proposition 6.7. *We have*

$$|z_\nu| = o(1) \tag{111}$$

and

$$d(x_{k,\nu}, x_{k,\nu}^*) \leq o(1)\, \varepsilon_{k,\nu}^*, \quad \frac{\varepsilon_{k,\nu}}{\varepsilon_{k,\nu}^*} = 1 + o(1), \quad \alpha_{k,\nu} = 1 + o(1) \tag{112}$$

for all $1 \leq k \leq m$. *In particular,* $\big(z_\nu, (x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}\big)$ *is an interior point of* $\mathcal{A}_\nu$ *if* ν *is sufficiently large.*

Proof. Observe that

$$\begin{aligned}
&\bigg\| \overline{u}_{z_\nu} + \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} - u_\infty - \sum_{k=1}^m \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \bigg\|_{H^1(M)} \\
&\leq \bigg\| u_\nu - u_\infty - \sum_{k=1}^m \overline{u}_{(x_{k,\nu}^*, \varepsilon_{k,\nu}^*)} \bigg\|_{H^1(M)} \\
&\quad + \bigg\| u_\nu - \overline{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})} \bigg\|_{H^1(M)} \\
&= o(1)
\end{aligned}$$

by Propositions 4.1 and 6.6. From this, the assertion follows.

As above, we assume that

$$\varepsilon_{i,\nu} \leq \varepsilon_{j,\nu} \quad \text{for} \quad i \leq j. \tag{113}$$

We now decompose the function u_ν as

$$u_\nu = v_\nu + w_\nu, \tag{114}$$

where

$$v_\nu = \overline{u}_{z_\nu} + \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}, \tag{115}$$

and

$$w_\nu = u_\nu - \overline{u}_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}. \tag{116}$$

Note that the function w_ν satisfies

$$\int_M \Big(\frac{4(n-1)}{n-2}\, |dw_\nu|_{g_0}^2 + R_{g_0}\, w_\nu^2 \Big)\, dvol_{g_0} = o(1) \tag{117}$$

by Proposition 6.6.

Proposition 6.8.

(i) *For every $a \in A$, we have*

$$\bigg| \int_M u_\infty^{\frac{4}{n-2}}\, \psi_a\, w_\nu\, dvol_{g_0} \bigg| \leq o(1) \int_M |w_\nu|\, dvol_{g_0}. \tag{118}$$

(ii) *For every $1 \leq k \leq m$, we have*

$$\bigg| \int_M \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}\, w_\nu\, dvol_{g_0} \bigg| \leq o(1) \bigg(\int_M |w_\nu|^{\frac{2n}{n-2}}\, dvol_{g_0} \bigg)^{\frac{n-2}{2n}}. \tag{119}$$

(iii) *For every $1 \leq k \leq m$, we have*

$$\begin{aligned} &\bigg| \int_M \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}\, \frac{\varepsilon_{k,\nu}^2 - d(x_{k,\nu}, x)^2}{\varepsilon_{k,\nu}^2 + d(x_{k,\nu}, x)^2}\, w_\nu\, dvol_{g_0} \bigg| \\ &\leq o(1) \bigg(\int_M |w_\nu|^{\frac{2n}{n-2}}\, dvol_{g_0} \bigg)^{\frac{n-2}{2n}}. \end{aligned} \tag{120}$$

(iv) *For all $1 \leq k \leq m$, we have*

$$\begin{aligned} &\bigg| \int_M \overline{u}_{(x_{k,\nu}, \varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}\, \frac{\varepsilon_{k,\nu}\, \exp_{x_{k,\nu}}^{-1}(x)}{\varepsilon_{k,\nu}^2 + d(x_{k,\nu}, x)^2}\, w_\nu\, dvol_{g_0} \bigg| \\ &\leq o(1) \bigg(\int_M |w_\nu|^{\frac{2n}{n-2}}\, dvol_{g_0} \bigg)^{\frac{n-2}{2n}}. \end{aligned} \tag{121}$$

Proof. (i) As above, let $\widetilde{\psi}_{a,z} = \frac{\partial}{\partial z_a} \overline{u}_z$. By definition of

$$\big(z_\nu, (x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m} \big),$$

we have

$$\int_M \Big(\frac{4(n-1)}{n-2}\, \langle d\widetilde{\psi}_{a,z_\nu}, dw_\nu \rangle_{g_0} + R_{g_0}\, \widetilde{\psi}_{a,z_\nu}\, w_\nu \Big)\, dvol_{g_0} = 0.$$

This implies

$$\begin{aligned}
&\lambda_a \int_M u_\infty^{\frac{4}{n-2}}\, \psi_a\, w_\nu\, dvol_{g_0} \\
&= -\int_M \Big(\frac{4(n-1)}{n-2}\, \Delta_{g_0}\psi_a - R_{g_0}\,\psi_a\Big)\, w_\nu\, dvol_{g_0} \\
&= \int_M \Big(\frac{4(n-1)}{n-2}\, \Delta_{g_0}(\widetilde{\psi}_{a,z_\nu} - \psi_a) - R_{g_0}\,(\widetilde{\psi}_{a,z_\nu} - \psi_a)\Big)\, w_\nu\, dvol_{g_0}.
\end{aligned}$$

Since $\lambda_a > 0$, we conclude that

$$\Big| \int_M u_\infty^{\frac{4}{n-2}}\, \psi_a\, w_\nu\, dvol_{g_0} \Big| \leq o(1)\, \|w_\nu\|_{L^1(M)}$$

for all $a \in A$.

(ii) By definition of $(x_{k,\nu}, \varepsilon_{k,\nu}, \alpha_{k,\nu})_{1 \leq k \leq m}$, we have

$$\int_M \Big(\frac{4(n-1)}{n-2}\, \langle d\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}, dw_\nu\rangle + R_{g_0}\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\, w_\nu\Big)\, dvol_{g_0} = 0,$$

hence

$$\int_M \Big(\frac{4(n-1)}{n-2}\, \Delta_{g_0}\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} - R_{g_0}\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\Big)\, w_\nu\, dvol_{g_0} = 0$$

for all $1 \leq k \leq m$. Using the estimate

$$\Big\| \frac{4(n-1)}{n-2}\, \Delta_{g_0}\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} - R_{g_0}\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} + r_\infty\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{2n}{n+2}}(M)} = o(1),$$

we conclude that

$$\Big| \int_M \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{n+2}{n-2}}\, w_\nu\, dvol_{g_0} \Big| \leq o(1)\, \|w_\nu\|_{L^{\frac{2n}{n-2}}(M)}$$

for all $1 \leq k \leq m$.

The remaining statements follow similarly.

As above, we need an estimate for the second variation operator of the Yamabe functional at v_ν.

Proposition 6.9. *If ν is sufficiently large, then we have*

$$\begin{aligned}
\text{(122)} \qquad &\frac{n+2}{n-2}\, r_\infty \int_M \Big(u_\infty^{\frac{4}{n-2}} + \sum_{j=1}^m \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{4}{n-2}}\Big)\, w_\nu^2\, dvol_{g_0} \\
&\leq (1-c) \int_M \Big(\frac{4(n-1)}{n-2}\, |dw_\nu|_{g_0}^2 + R_{g_0}\, w_\nu^2\Big)\, dvol_{g_0}
\end{aligned}$$

for some positive constant c independent of ν.

Proof. Suppose this is not true. Upon rescaling, we obtain a sequence of functions $\{\widetilde{w}_\nu : \nu \in \mathbb{N}\}$ such that

$$\int_M \Big(\frac{4(n-1)}{n-2}\,|d\widetilde{w}_\nu|_{g_0}^2 + R_{g_0}\,\widetilde{w}_\nu^2\Big)\,dvol_{g_0} = 1 \tag{123}$$

and

$$\lim_{\nu\to\infty} \frac{n+2}{n-2}\,r_\infty \int_M \bigg(u_\infty^{\frac{4}{n-2}} + \sum_{j=1}^m \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{4}{n-2}}\bigg)\,\widetilde{w}_\nu^2\,dvol_{g_0} \geq 1. \tag{124}$$

Observe that

$$\int_M |\widetilde{w}_\nu|^{\frac{2n}{n-2}}\,dvol_{g_0} \leq Y(M,g_0)^{-\frac{n}{n-2}} \tag{125}$$

by (123). In view of Proposition 6.6, we can find a sequence $\{N_\nu : \nu \in \mathbb{N}\}$ such that $N_\nu \to \infty$, $N_\nu \varepsilon_{j,\nu} \to 0$ for all $1 \leq j \leq m$, and

$$\frac{1}{N_\nu}\,\frac{\varepsilon_{j,\nu} + d(x_{i,\nu}, x_{j,\nu})}{\varepsilon_{i,\nu}} \to \infty \tag{126}$$

for all $i < j$. Let

$$\Omega_{j,\nu} = B_{N_\nu \varepsilon_{j,\nu}}(x_{j,\nu}) \setminus \bigcup_{i=1}^{j-1} B_{N_\nu \varepsilon_{i,\nu}}(x_{i,\nu}) \tag{127}$$

for every $1 \leq j \leq m$. In view of (123) and (124), there are only two possibilities:

Case 1. Suppose that

$$\lim_{\nu\to\infty} \int_M u_\infty^{\frac{4}{n-2}}\,\widetilde{w}_\nu^2\,dvol_{g_0} > 0 \tag{128}$$

and

$$\begin{aligned} &\lim_{\nu\to\infty} \int_{M\setminus\bigcup_{j=1}^m \Omega_{j,\nu}} \Big(\frac{4(n-1)}{n-2}\,|d\widetilde{w}_\nu|_{g_0}^2 + R_{g_0}\,\widetilde{w}_\nu^2\Big)\,dvol_{g_0} \\ &\quad\leq \lim_{\nu\to\infty} \frac{n+2}{n-2}\,r_\infty \int_M u_\infty^{\frac{4}{n-2}}\,\widetilde{w}_\nu^2\,dvol_{g_0}. \end{aligned} \tag{129}$$

Let $\widetilde{w}$ be the weak limit of the sequence $\{\widetilde{w}_\nu : \nu \in \mathbb{N}\}$. Then, the function $\widetilde{w}$ satisfies

$$\int_M u_\infty^{\frac{4}{n-2}}\,\widetilde{w}^2\,dvol_{g_0} > 0$$

and

$$\int_M \Big(\frac{4(n-1)}{n-2}\,|d\widetilde{w}|_{g_0}^2 + R_{g_0}\,\widetilde{w}^2\Big)dvol_{g_0} \\ \leq \frac{n+2}{n-2}\,r_\infty \int_M u_\infty^{\frac{4}{n-2}}\,\widetilde{w}^2\,dvol_{g_0}.$$

This implies

$$\sum_{a\in\mathbb{N}} \lambda_a \left(\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,\widetilde{w}\,dvol_{g_0}\right)^2 \\ \leq \sum_{a\in\mathbb{N}} \frac{n+2}{n-2}\,r_\infty \left(\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,\widetilde{w}\,dvol_{g_0}\right)^2.$$

Using Proposition 6.8, we obtain

$$\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,\widetilde{w}\,dvol_{g_0} = 0$$

for all $a \in A$. Thus, we conclude that $\widetilde{w}(x) = 0$ for all $x \in M$. This is a contradiction.

Case 2. Suppose that there exists an integer $1 \leq j \leq m$ such that

$$\lim_{\nu\to\infty} \int_M \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{4}{n-2}}\,\widetilde{w}_\nu^2\,dvol_{g_0} > 0 \tag{130}$$

and

$$\lim_{\nu\to\infty} \int_{\Omega_{j,\nu}} \Big(\frac{4(n-1)}{n-2}\,|d\widetilde{w}_\nu|_{g_0}^2 + R_{g_0}\,\widetilde{w}_\nu^2\Big)\,dvol_{g_0} \\ \leq \lim_{\nu\to\infty} \frac{n+2}{n-2}\,r_\infty \int_M \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{4}{n-2}}\,\widetilde{w}_\nu^2\,dvol_{g_0}. \tag{131}$$

We now define a sequence of functions $\hat{w}_\nu : TM_{x_{j,\nu}} \to \mathbb{R}$ by

$$\hat{w}_\nu(\xi) = \varepsilon_{j,\nu}^{\frac{n-2}{2}}\,\widetilde{w}_\nu(\exp_{x_{j,\nu}}(\varepsilon_{j,\nu}\,\xi))$$

for $\xi \in TM_{x_{j,\nu}}$. The sequence $\{\hat{w}_\nu : \nu \in \mathbb{N}\}$ satisfies

$$\lim_{\nu\to\infty} \int_{\{\xi\in TM_{x_{j,\nu}}:|\xi|\leq N_\nu\}} \frac{4(n-1)}{n-2}|d\hat{w}_\nu(\xi)|^2\,d\xi \leq 1$$

and

$$\lim_{\nu\to\infty} \int_{\{\xi\in TM_{x_{j,\nu}}:|\xi|\leq N_\nu\}} |\hat{w}_\nu(\xi)|^{\frac{2n}{n-2}}d\xi \leq Y(M,g_0)^{-\frac{n}{n-2}}.$$

Hence, if we take the weak limit as $\nu \to \infty$, then we obtain a function $\hat{w} : \mathbb{R}^n \to \mathbb{R}$ such that
$$\int_{\mathbb{R}^n} \Big(\frac{1}{1+|\xi|^2}\Big)^2 \, \hat{w}(\xi)^2 \, d\xi > 0$$
and
$$\int_{\mathbb{R}^n} |d\hat{w}(\xi)|^2 \, d\xi \leq n(n+2) \int_{\mathbb{R}^n} \Big(\frac{1}{1+|\xi|^2}\Big)^2 \, \hat{w}(\xi)^2 \, d\xi.$$
Moreover, it follows from Proposition 5.3 that
$$\int_{\mathbb{R}^n} \Big(\frac{1}{1+|\xi|^2}\Big)^{\frac{n+2}{2}} \, \hat{w}(\xi) \, d\xi = 0$$
$$\int_{\mathbb{R}^n} \Big(\frac{1}{1+|\xi|^2}\Big)^{\frac{n+2}{2}} \, \frac{1-|\xi|^2}{1+|\xi|^2} \, \hat{w}(\xi) \, d\xi = 0$$
$$\int_{\mathbb{R}^n} \Big(\frac{1}{1+|\xi|^2}\Big)^{\frac{n+2}{2}} \, \frac{\xi}{1+|\xi|^2} \, \hat{w}(\xi) \, d\xi = 0.$$
Using a result of Rey, we conclude that $\hat{w}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ (see [**13**], Appendix D, pp. 49–51). This is a contradiction.

Corollary 6.10. *If ν is sufficiently large, then we have*

$$\begin{aligned} &\frac{n+2}{n-2} \, r_\infty \int_M v_\nu^{\frac{4}{n-2}} \, w_\nu^2 \, dvol_{g_0} \\ &\leq (1-c) \int_M \Big(\frac{4(n-1)}{n-2} \, |dw_\nu|_{g_0}^2 + R_{g_0} \, w_\nu^2\Big) \, dvol_{g_0} \end{aligned} \tag{132}$$

for some positive constant c independent of ν.

Proof. By definition of v_ν, we have
$$\int_M \Big| v_\nu^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}} - \sum_{j=1}^m \overline{u}_{(x_{j,\nu}, \varepsilon_{j,\nu})}^{\frac{4}{n-2}} \Big|^{\frac{n}{2}} \, dvol_{g_0} = o(1).$$
Hence, the assertion follows from Proposition 6.9.

Lemma 6.11. *The difference $u_\nu - \overline{u}_{z_\nu}$ satisfies the estimate*

$$\|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}} \leq C \, \|u_\nu^{\frac{n+2}{n-2}} \, (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} \tag{133}$$

if ν is sufficiently large.

Proof. Using the identities
$$\frac{4(n-1)}{n-2} \, \Delta_{g_0} u_\nu - R_{g_0} \, u_\nu + r_\infty \, u_\nu^{\frac{n+2}{n-2}} = -u_\nu^{\frac{n+2}{n-2}} \, (R_{g_\nu} - r_\infty)$$

and

$$\Pi\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0}\overline{u}_{z_\nu} - R_{g_0}\,\overline{u}_{z_\nu} + r_\infty\,\overline{u}_{z_\nu}^{\frac{n+2}{n-2}}\Big) = 0,$$

we obtain

(134)

$$\begin{aligned}
&\Pi\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0}(u_\nu - \overline{u}_{z_\nu}) - R_{g_0}\,(u_\nu - \overline{u}_{z_\nu}) \\
&\qquad + \frac{n+2}{n-2}\,r_\infty\,u_\infty^{\frac{4}{n-2}}\,(u_\nu - \overline{u}_{z_\nu})\Big) \\
&\quad = \Pi\bigg(- u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu} - r_\infty) - \frac{n+2}{n-2}\,r_\infty\,(\overline{u}_{z_\nu}^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}})\,(u_\nu - \overline{u}_{z_\nu}) \\
&\qquad + r_\infty\,\Big(\overline{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\,\overline{u}_{z_\nu}^{\frac{4}{n-2}}\,(u_\nu - \overline{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}}\Big)\bigg).
\end{aligned}$$

Using the inequality

$$\begin{aligned}
&\|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)} \\
&\qquad \leq C\,\Big\|\Pi\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0}(u_\nu - \overline{u}_{z_\nu}) - R_{g_0}\,(u_\nu - \overline{u}_{z_\nu}) \\
&\qquad\qquad + \frac{n+2}{n-2}\,r_\infty\,u_\infty^{\frac{4}{n-2}}\,(u_\nu - \overline{u}_{z_\nu})\Big)\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\
&\qquad\quad + C\,\sup_{a\in A}\Big|\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,(u_\nu - \overline{u}_{z_\nu})\,dvol_{g_0}\Big|,
\end{aligned} \tag{135}$$

we conclude that

$$\begin{aligned}
&\|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)} \\
&\quad \leq C\,\|u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu} - r_\infty)\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\
&\qquad + C\,\big\|(\overline{u}_{z_\nu}^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}})\,(u_\nu - \overline{u}_{z_\nu})\big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\
&\qquad + C\,\Big\|\overline{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\,\overline{u}_{z_\nu}^{\frac{4}{n-2}}\,(u_\nu - \overline{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\
&\qquad + C\,\sup_{a\in A}\Big|\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,(u_\nu - \overline{u}_{z_\nu})\,dvol_{g_0}\Big|.
\end{aligned} \tag{136}$$

Using the pointwise estimate

$$\begin{aligned}
&\Big|\overline{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\,\overline{u}_{z_\nu}^{\frac{4}{n-2}}\,(u_\nu - \overline{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}}\Big| \\
&\quad \leq C\,\overline{u}_{z_\nu}^{\max\{0,\frac{4}{n-2}-1\}}\,|u_\nu - \overline{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2},2\}} + C\,|u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}},
\end{aligned} \tag{137}$$

we obtain

$$\Big\| \overline{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\, \overline{u}_{z_\nu}^{\frac{4}{n-2}}\, (u_\nu - \overline{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \leq C\, \Big\| |u_\nu - \overline{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2},2\}} + |u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)}. \tag{138}$$

Note that

$$\begin{aligned} &\Big\| |u_\nu - \overline{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2},2\}} + |u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \\ &\leq \Big\| |u_\nu - \overline{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2},2\}} + |u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} \\ &\quad + \Big\| |u_\nu - \overline{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2},2\}} + |u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M \setminus \bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} \\ &\leq C \sum_{k=1}^m (N\varepsilon_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}} \Big\| |u_\nu - \overline{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2},2\}} + |u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{2n}{n+2}}(M)} \\ &\quad + C\, \Big\| |u_\nu - \overline{u}_{z_\nu}|^{\min\{\frac{4}{n-2},1\}} + |u_\nu - \overline{u}_{z_\nu}|^{\frac{4}{n-2}} \Big\|_{L^{\frac{n}{2}}(M \setminus \bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} \\ &\qquad \cdot \|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)} \end{aligned} \tag{139}$$

by Hölder's inequality. Since

$$\begin{aligned} \|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{2n}{n-2}}(M \setminus \bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} &= \Big\| \sum_{k=1}^m \alpha_{k,\nu}\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} + w_\nu \Big\|_{L^{\frac{2n}{n-2}}(M \setminus \bigcup_{k=1}^m B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} \\ &\leq \sum_{k=1}^m \alpha_{k,\nu}\, \|\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\|_{L^{\frac{2n}{n-2}}(M \setminus B_{N\varepsilon_{k,\nu}}(x_{k,\nu}))} + \|w_\nu\|_{L^{\frac{2n}{n-2}}(M)} \\ &\leq C\, N^{-\frac{n-2}{2}} + o(1), \end{aligned} \tag{140}$$

it follows that

$$\Big\| \overline{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\, \overline{u}_{z_\nu}^{\frac{4}{n-2}}\, (u_\nu - \overline{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{n(n+2)}{n^2+4}}(M)} \leq C \sum_{k=1}^m (N\varepsilon_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}} + (C\, N^{-2} + o(1))\, \|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}. \tag{141}$$

Moreover, we have

$$\begin{aligned}(142)\qquad &\sup_{a\in A}\left|\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,(u_\nu-\overline{u}_{z_\nu})\,dvol_{g_0}\right| \\ &\quad= \sup_{a\in A}\left|\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\left(\sum_{k=1}^m \alpha_{k,\nu}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}+w_\nu\right)dvol_{g_0}\right| \\ &\quad\leq C\sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}+o(1)\,\|w_\nu\|_{L^1(M)} \\ &\quad\leq C\sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}+o(1)\left\|u_\nu-\overline{u}_{z_\nu}-\sum_{k=1}^m \alpha_{k,\nu}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\right\|_{L^1(M)} \\ &\quad\leq C\sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}+o(1)\,\|u_\nu-\overline{u}_{z_\nu}\|_{L^1(M)}.\end{aligned}$$

Putting these facts together, we conclude that

$$\begin{aligned}(143)\qquad \|u_\nu-\overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)} &\leq C\,\|u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu}-r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} \\ &\quad+C\sum_{k=1}^m (N\varepsilon_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}}+C\sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} \\ &\quad+(C\,N^{-2}+o(1))\,\|u_\nu-\overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}.\end{aligned}$$

Hence, if we choose N sufficiently large, then we obtain
(144)

$$\|u_\nu-\overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}\leq C\,\|u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu}-r_\infty)\|_{L^{\frac{2n}{n+2}}(M)}+C\sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{(n-2)^2}{2(n+2)}}.$$

From this, the assertion follows.

Lemma 6.12. *The difference* $u_\nu-\overline{u}_{z_\nu}$ *satisfies the estimate*

$$(145)\qquad \|u_\nu-\overline{u}_{z_\nu}\|_{L^1(M)}\leq C\,\|u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu}-r_\infty)\|_{L^{\frac{2n}{n+2}}(M)}+C\sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}$$

if ν *is sufficiently large.*

Proof. Using the inequality

$$\begin{aligned} &\|u_\nu - \overline{u}_{z_\nu}\|_{L^1(M)} \\ &\leq C\,\Big\|\Pi\Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0}(u_\nu - \overline{u}_{z_\nu}) - R_{g_0}\,(u_\nu - \overline{u}_{z_\nu}) \\ &\quad + \frac{n+2}{n-2}\,r_\infty\,u_\infty^{\frac{4}{n-2}}\,(u_\nu - \overline{u}_{z_\nu})\Big)\Big\|_{L^1(M)} \\ &\quad + C\,\sup_{a\in A}\Big|\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,(u_\nu - \overline{u}_{z_\nu})\,dvol_{g_0}\Big| \end{aligned} \tag{146}$$

and (134) we conclude that

$$\begin{aligned} &\|u_\nu - \overline{u}_{z_\nu}\|_{L^1(M)} \\ &\leq C\,\|u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu} - r_\infty)\|_{L^1(M)} \\ &\quad + C\,\big\|(\overline{u}_{z_\nu}^{\frac{4}{n-2}} - u_\infty^{\frac{4}{n-2}})\,(u_\nu - \overline{u}_{z_\nu})\big\|_{L^1(M)} \\ &\quad + C\,\Big\|\overline{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\,\overline{u}_{z_\nu}^{\frac{4}{n-2}}\,(u_\nu - \overline{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}}\Big\|_{L^1(M)} \\ &\quad + C\,\sup_{a\in A}\Big|\int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,(u_\nu - \overline{u}_{z_\nu})\,dvol_{g_0}\Big|. \end{aligned} \tag{147}$$

Using the pointwise estimate

$$\begin{aligned} &\Big|\overline{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\,\overline{u}_{z_\nu}^{\frac{4}{n-2}}\,(u_\nu - \overline{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}}\Big| \\ &\leq C\,\overline{u}_{z_\nu}^{\max\{0,\frac{4}{n-2}-1\}}\,|u_\nu - \overline{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2},2\}} + C\,|u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}}, \end{aligned} \tag{148}$$

we obtain

$$\begin{aligned} &\Big\|\overline{u}_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\,\overline{u}_{z_\nu}^{\frac{4}{n-2}}\,(u_\nu - \overline{u}_{z_\nu}) - u_\nu^{\frac{n+2}{n-2}}\Big\|_{L^1(M)} \\ &\leq C\,\Big\||u_\nu - \overline{u}_{z_\nu}|^{\min\{\frac{n+2}{n-2},2\}} + C\,|u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}}\Big\|_{L^1(M)} \\ &\leq C\,\big\||u_\nu - \overline{u}_{z_\nu}|\big\|_{L^1(M)}^{\max\{0,1-\frac{n-2}{4}\}}\,\big\||u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}}\big\|_{L^1(M)}^{\min\{1,\frac{n-2}{4}\}} \\ &\quad + C\,\big\||u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}}\big\|_{L^1(M)} \\ &\leq C\,\big\||u_\nu - \overline{u}_{z_\nu}|\big\|_{L^1(M)}^{\max\{0,1-\frac{n-2}{4}\}}\,\|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}\,\min\{1,\frac{n-2}{4}\}} \\ &\quad + C\,\|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}} \end{aligned} \tag{149}$$

by Hölder's inequality. Moreover, we have

$$\sup_{a \in A} \left| \int_M u_\infty^{\frac{4}{n-2}} \, \psi_a \, (u_\nu - \overline{u}_{z_\nu}) \, dvol_{g_0} \right| \leq C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} + o(1) \, \|u_\nu - \overline{u}_{z_\nu}\|_{L^1(M)}. \tag{150}$$

Putting these facts together, we conclude that

$$\begin{aligned} \|u_\nu - \overline{u}_{z_\nu}\|_{L^1(M)} &\leq C \, \|u_\nu^{\frac{n+2}{n-2}} \, (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} \\ &\quad + C \, \|u_\nu - \overline{u}_{z_\nu}\|_{L^1(M)}^{\max\{0, 1 - \frac{n-2}{4}\}} \, \|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2} \min\{1, \frac{n-2}{4}\}} \\ &\quad + C \, \|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} \\ &\quad + o(1) \, \|u_\nu - \overline{u}_{z_\nu}\|_{L^1(M)}. \end{aligned} \tag{151}$$

Since $\max\{0, 1 - \frac{n-2}{4}\} < 1$, this implies

$$\begin{aligned} \|u_\nu - \overline{u}_{z_\nu}\|_{L^1(M)} &\leq C \, \|u_\nu^{\frac{n+2}{n-2}} \, (R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} \\ &\quad + C \, \|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}. \end{aligned} \tag{152}$$

The assertion follows now from Lemma 6.11.

Lemma 6.13. *We have*

$$\begin{aligned} &\sup_{a \in A} \left| \int_M \psi_a \left(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_{z_\nu} - R_{g_0} \, \overline{u}_{z_\nu} + r_\infty \, \overline{u}_{z_\nu}^{\frac{n+2}{n-2}} \right) dvol_{g_0} \right| \\ &\leq C \left(\int_M u_\nu^{\frac{2n}{n-2}} \, |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \right)^{\frac{n+2}{2n}} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} \end{aligned} \tag{153}$$

if ν is sufficiently large.

Proof. Integration by parts yields

$$\begin{aligned} &\int_M \psi_a \left(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_{z_\nu} - R_{g_0} \, \overline{u}_{z_\nu} + r_\infty \, \overline{u}_{z_\nu}^{\frac{n+2}{n-2}} \right) dvol_{g_0} \\ &\quad = \int_M \psi_a \left(\frac{4(n-1)}{n-2} \, \Delta_{g_0} u_\nu - R_{g_0} \, u_\nu + r_\infty \, u_\nu^{\frac{n+2}{n-2}} \right) dvol_{g_0} \\ &\qquad + \lambda_a \int_M u_\infty^{\frac{4}{n-2}} \, \psi_a \, (u_\nu - \overline{u}_{z_\nu}) \, dvol_{g_0} - r_\infty \int_M \psi_a \, (u_\nu^{\frac{n+2}{n-2}} - \overline{u}_{z_\nu}^{\frac{n+2}{n-2}}) \, dvol_{g_0}. \end{aligned} \tag{154}$$

Using the identity

$$\frac{4(n-1)}{n-2}\,\Delta_{g_0} u_\nu - R_{g_0}\,u_\nu + r_\infty\,u_\nu^{\frac{n+2}{n-2}} = -u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu} - r_\infty),$$

we obtain

(155)

$$\begin{aligned}
&\int_M \psi_a \Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0}\overline{u}_{z_\nu} - R_{g_0}\,\overline{u}_{z_\nu} + r_\infty\,\overline{u}_{z_\nu}^{\frac{n+2}{n-2}}\Big)\,dvol_{g_0} \\
&\quad = -\int_M \psi_a\,u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu} - r_\infty)\,dvol_{g_0} \\
&\qquad + \lambda_a \int_M u_\infty^{\frac{4}{n-2}}\,\psi_a\,(u_\nu - \overline{u}_{z_\nu})\,dvol_{g_0} - r_\infty \int_M \psi_a\,(u_\nu^{\frac{n+2}{n-2}} - \overline{u}_{z_\nu}^{\frac{n+2}{n-2}})\,dvol_{g_0}.
\end{aligned}$$

Using the pointwise estimate

$$|u_\nu^{\frac{n+2}{n-2}} - \overline{u}_{z_\nu}^{\frac{n+2}{n-2}}| \leq C\,\overline{u}_{z_\nu}^{\frac{4}{n-2}}\,|u_\nu - \overline{u}_{z_\nu}| + C\,|u_\nu - \overline{u}_{z_\nu}|^{\frac{n+2}{n-2}}, \tag{156}$$

we conclude that

$$\begin{aligned}
&\sup_{a\in A}\Big|\int_M \psi_a \Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0}\overline{u}_{z_\nu} - R_{g_0}\,\overline{u}_{z_\nu} + r_\infty\,\overline{u}_{z_\nu}^{\frac{n+2}{n-2}}\Big)\,dvol_{g_0}\Big| \\
&\qquad \leq C\,\|u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} \\
&\qquad\quad + C\,\|u_\nu - \overline{u}_{z_\nu}\|_{L^1(M)} + C\,\|u_\nu - \overline{u}_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(M)}^{\frac{n+2}{n-2}}.
\end{aligned} \tag{157}$$

Hence, it follows from Lemma 6.11 and Lemma 6.12 that

$$\begin{aligned}
&\sup_{a\in A}\Big|\int_M \psi_a \Big(\frac{4(n-1)}{n-2}\,\Delta_{g_0}\overline{u}_{z_\nu} - R_{g_0}\,\overline{u}_{z_\nu} + r_\infty\,\overline{u}_{z_\nu}^{\frac{n+2}{n-2}}\Big)\,dvol_{g_0}\Big| \\
&\qquad \leq C\,\|u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu} - r_\infty)\|_{L^{\frac{2n}{n+2}}(M)} + C\sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}}.
\end{aligned} \tag{158}$$

This proves the assertion.

Proposition 6.14. *The Yamabe energy of $\overline{u}_{z_\nu}$ satisfies the estimate*

(159)

$$\begin{aligned}
&E(\overline{u}_{z_\nu}) - E(u_\infty) \\
&\quad \leq C\Big(\int_M u_\nu^{\frac{2n}{n-2}}\,|R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}}\,dvol_{g_0}\Big)^{\frac{n+2}{2n}\,(1+\gamma)} + C\sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}\,(1+\gamma)}
\end{aligned}$$

if ν is sufficiently large.

Proof. This follows immediately from Lemmas 6.5 and 6.13.

Proposition 6.15. *The Yamabe energy of v_ν satisfies the estimate*

$$E(v_\nu) \leq \Big(E(\overline{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^m E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}} \Big)^{\frac{2}{n}} - c \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} \tag{160}$$

if ν is sufficiently large.

Proof. Using the identity

(161)

$$\begin{aligned}
&\int_M \Big(\frac{4(n-1)}{n-2} \, |dv_\nu|_{g_0}^2 + R_{g_0} \, v_\nu^2 \Big) \, dvol_{g_0} \\
&\quad = \int_M \Big(\frac{4(n-1)}{n-2} \, |d\overline{u}_{z_\nu}|_{g_0}^2 + R_{g_0} \, \overline{u}_{z_\nu}^2 \Big) \, dvol_{g_0} \\
&\qquad + \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 \, \Big(\frac{4(n-1)}{n-2} \, |d\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}|_{g_0}^2 + R_{g_0} \, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^2 \Big) \, dvol_{g_0} \\
&\qquad + 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} \, \Big(\frac{4(n-1)}{n-2} \, \langle d\overline{u}_{z_\nu}, d\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} \rangle_{g_0} \\
&\qquad\quad + R_{g_0} \, \overline{u}_{z_\nu} \, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} \Big) \, dvol_{g_0} \\
&\qquad + 2 \int_M \sum_{i<j} \alpha_{i,\nu} \, \alpha_{j,\nu} \, \Big(\frac{4(n-1)}{n-2} \, \langle d\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}, d\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \rangle_{g_0} \\
&\qquad\quad + R_{g_0} \, \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \Big) \, dvol_{g_0},
\end{aligned}$$

we obtain

(162)

$$\begin{aligned}
&E(v_\nu) \, \Big(\int_M v_\nu^{\frac{2n}{n-2}} \, dvol_{g_0} \Big)^{\frac{n-2}{n}} \\
&\quad = \int_M F(\overline{u}_{z_\nu}) \, \overline{u}_{z_\nu}^{\frac{2n}{n-2}} \, dvol_{g_0} \\
&\qquad + \int_M \sum_{k=1}^m \alpha_{k,\nu}^2 \, F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}) \, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}} \, dvol_{g_0} \\
&\qquad - 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} \, \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_{z_\nu} - R_{g_0} \, \overline{u}_{z_\nu} \Big) \, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} \, dvol_{g_0} \\
&\qquad - 2 \int_M \sum_{i<j} \alpha_{i,\nu} \, \alpha_{j,\nu} \, \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \\
&\qquad\quad \cdot \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} - R_{g_0} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \Big) \, dvol_{g_0}.
\end{aligned}$$

Moreover, we have

(163)

$$\begin{aligned}
&\bigg(E(\overline{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^m E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}}\bigg)^{\frac{2}{n}} \bigg(\int_M v_\nu^{\frac{2n}{n-2}}\,dvol_{g_0}\bigg)^{\frac{n-2}{n}} \\
&\quad= \bigg(\int_M \Big(F(\overline{u}_{z_\nu})^{\frac{n}{2}}\,\overline{u}_{z_\nu}^{\frac{2n}{n-2}} + \sum_{k=1}^m F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}}\Big)\,dvol_{g_0}\bigg)^{\frac{2}{n}} \\
&\qquad\quad \cdot \bigg(\int_M v_\nu^{\frac{2n}{n-2}}\,dvol_{g_0}\bigg)^{\frac{n-2}{n}} \\
&\quad\geq \int_M \bigg(F(\overline{u}_{z_\nu})^{\frac{n}{2}}\,\overline{u}_{z_\nu}^{\frac{2n}{n-2}} + \sum_{k=1}^m F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}}\bigg)^{\frac{2}{n}} v_\nu^2\,dvol_{g_0} \\
&\quad\geq \int_M F(\overline{u}_{z_\nu})\,\overline{u}_{z_\nu}^{\frac{2n}{n-2}}\,dvol_{g_0} \\
&\qquad + \int_M \sum_{k=1}^m \alpha_{k,\nu}^2\,F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}}\,dvol_{g_0} \\
&\qquad + 2\int_M \sum_{k=1}^m \alpha_{k,\nu}\Big(F(\overline{u}_{z_\nu})^{\frac{n}{2}}\,\overline{u}_{z_\nu}^{\frac{2n}{n-2}} + F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}} \\
&\qquad\quad \cdot \overline{u}_{z_\nu}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\,dvol_{g_0} \\
&\qquad + 2\int_M \sum_{i<j} \alpha_{i,\nu}\,\alpha_{j,\nu}\Big(F(\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}^{\frac{2n}{n-2}} \\
&\qquad\quad + F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}}\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}\,dvol_{g_0}
\end{aligned}$$

by Hölder's inequality. Let $1 \leq k \leq m$. Using the inequality

$$\begin{aligned}
&\Big(F(\overline{u}_{z_\nu})^{\frac{n}{2}}\,\overline{u}_{z_\nu}^{\frac{2n}{n-2}} + F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}}\,\overline{u}_{z_\nu}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} \\
&\quad\geq F(\overline{u}_{z_\nu})\,\overline{u}_{z_\nu}^{\frac{n+2}{n-2}}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})} + c\,\varepsilon_{k,\nu}^{-\frac{n+2}{2}}\,1_{\{d(x_{k,\nu},x)\leq\varepsilon_{k,\nu}\}},
\end{aligned}$$

we obtain

(164)

$$\begin{aligned}
&\int_M \Big(F(\overline{u}_{z_\nu})^{\frac{n}{2}}\,\overline{u}_{z_\nu}^{\frac{2n}{n-2}} + F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}}\,\overline{u}_{z_\nu}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\,dvol_{g_0} \\
&\quad\geq \int_M F(\overline{u}_{z_\nu})\,\overline{u}_{z_\nu}^{\frac{n+2}{n-2}}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\,dvol_{g_0} + c\,\varepsilon_{k,\nu}^{\frac{n-2}{2}}
\end{aligned}$$

if ν is sufficiently large. We next consider a pair $i < j$. We can find positive constants c and C independent of ν such that

$$\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}(x)^{\frac{n+2}{n-2}}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}(x) \geq c\left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu},x_{j,\nu})^2}{\varepsilon_{i,\nu}\varepsilon_{j,\nu}}\right)^{-\frac{n-2}{2}}\varepsilon_{i,\nu}^{-n}$$

and

$$\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}(x)\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}(x)^{\frac{n+2}{n-2}} \leq C\left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu},x_{j,\nu})^2}{\varepsilon_{i,\nu}\varepsilon_{j,\nu}}\right)^{-\frac{n+2}{2}}\varepsilon_{i,\nu}^{-n}$$

if $d(x_{i,\nu},x) \leq \varepsilon_{i,\nu}$ and ν is sufficiently large. From this, it follows that

$$\begin{aligned}
&\Big(F(\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}^{\frac{2n}{n-2}} + F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}} \\
&\qquad\cdot\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \\
&\quad\geq F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \\
&\qquad + c\left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu},x_{j,\nu})^2}{\varepsilon_{i,\nu}\varepsilon_{j,\nu}}\right)^{-\frac{n-2}{2}}\varepsilon_{i,\nu}^{-n}\,1_{\{d(x_{i,\nu},x)\leq\varepsilon_{i,\nu}\}}
\end{aligned}$$

for ν sufficiently large. Integration over M yields

$$\begin{aligned}
(165)\qquad &\int_M \Big(F(\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}^{\frac{2n}{n-2}} + F(\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})})^{\frac{n}{2}}\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}^{\frac{2n}{n-2}}\Big)^{\frac{2}{n}} \\
&\qquad\cdot\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}\,dvol_{g_0} \\
&\quad\geq \int_M F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})\,\overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\,\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}}\,dvol_{g_0} \\
&\qquad + c\left(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu},x_{j,\nu})^2}{\varepsilon_{i,\nu}\,\varepsilon_{j,\nu}}\right)^{-\frac{n-2}{2}}
\end{aligned}$$

if ν is sufficiently large. From this, it follows that

$$\begin{aligned}
(166)\qquad &\Big(E(\overline{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^m E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}}\Big)^{\frac{2}{n}}\left(\int_M v_\nu^{\frac{2n}{n-2}}\,dvol_{g_0}\right)^{\frac{n-2}{n}} \\
&\geq \int_M F(\overline{u}_{z_\nu})\,\overline{u}_{z_\nu}^{\frac{2n}{n-2}}\,dvol_{g_0} \\
&\quad + \int_M \sum_{k=1}^m \alpha_{k,\nu}^2\,F(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}^{\frac{2n}{n-2}}\,dvol_{g_0} \\
&\quad + 2\int_M \sum_{k=1}^m \alpha_{k,\nu}\,F(\overline{u}_{z_\nu})\,\overline{u}_{z_\nu}^{\frac{n+2}{n-2}}\,\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\,dvol_{g_0}
\end{aligned}$$

$$+ 2 \int_M \sum_{i<j} \alpha_{i,\nu}\, \alpha_{j,\nu}\, F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})\, \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})}\, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}}\, dvol_{g_0}$$
$$+ c \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} + c \sum_{i<j} \Big(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}\, \varepsilon_{j,\nu}} \Big)^{-\frac{n-2}{2}}.$$

Putting these facts together, we obtain

$$\begin{aligned} (167)\quad & E(v_\nu) \Big(\int_M v_\nu^{\frac{2n}{n-2}}\, dvol_{g_0} \Big)^{\frac{n-2}{n}} \\ &\leq \Big(E(\overline{u}_{z_\nu})^{\frac{n}{2}} + \sum_{k=1}^m E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})})^{\frac{n}{2}} \Big)^{\frac{2}{n}} \Big(\int_M v_\nu^{\frac{2n}{n-2}}\, dvol_{g_0} \Big)^{\frac{n-2}{n}} \\ &\quad - 2 \int_M \sum_{k=1}^m \alpha_{k,\nu} \Big(\frac{4(n-1)}{n-2}\, \Delta_{g_0} \overline{u}_{z_\nu} \\ &\quad - R_{g_0}\, \overline{u}_{z_\nu} + F(\overline{u}_{z_\nu})\, \overline{u}_{z_\nu}^{\frac{n+2}{n-2}} \Big)\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\, dvol_{g_0} \\ &\quad - 2 \int_M \sum_{i<j} \alpha_{i,\nu}\, \alpha_{j,\nu}\, \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \Big(\frac{4(n-1)}{n-2}\, \Delta_{g_0} \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \\ &\quad - R_{g_0}\, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} + F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})\, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \Big)\, dvol_{g_0} \\ &\quad - c \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}} - c \sum_{i<j} \Big(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}\, \varepsilon_{j,\nu}} \Big)^{-\frac{n-2}{2}}. \end{aligned}$$

Note that

$$\begin{aligned} (168)\quad & \int_M \Big| \frac{4(n-1)}{n-2}\, \Delta_{g_0} \overline{u}_{z_\nu} - R_{g_0}\, \overline{u}_{z_\nu} + F(\overline{u}_{z_\nu})\, \overline{u}_{z_\nu}^{\frac{n+2}{n-2}} \Big|\, \overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}\, dvol_{g_0} \\ &\leq o(1)\, \varepsilon_{k,\nu}^{\frac{n-2}{2}}. \end{aligned}$$

Moreover, since $F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}) = r_\infty + o(1)$, it follows from Lemmas B.4 and B.5 that

$$\begin{aligned} (169)\quad & \int_M \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \Big| \frac{4(n-1)}{n-2}\, \Delta_{g_0} \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \\ &\qquad - R_{g_0}\, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} + F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})})\, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \Big|\, dvol_{g_0} \\ &\leq \int_M \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \Big| \frac{4(n-1)}{n-2}\, \Delta_{g_0} \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} \\ &\qquad - R_{g_0}\, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})} + r_\infty\, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \Big|\, dvol_{g_0} \end{aligned}$$

$$+ |F(\overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}) - r_\infty| \int_M \overline{u}_{(x_{i,\nu},\varepsilon_{i,\nu})} \, \overline{u}_{(x_{j,\nu},\varepsilon_{j,\nu})}^{\frac{n+2}{n-2}} \, dvol_{g_0}$$
$$\leq C \Big(\delta^4 + \delta^{n-2} + \frac{\varepsilon_{j,\nu}^2}{\delta^2}\Big) \Big(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}\varepsilon_{j,\nu}}\Big)^{-\frac{n-2}{2}}$$
$$+ o(1) \Big(\frac{\varepsilon_{j,\nu}^2 + d(x_{i,\nu}, x_{j,\nu})^2}{\varepsilon_{i,\nu}\varepsilon_{j,\nu}}\Big)^{-\frac{n-2}{2}}$$

for $i < j$. Hence, if we choose δ sufficiently small, the assertion follows.

Corollary 6.16. *If ν is sufficiently large, then the Yamabe energy of v_ν satisfies the estimate*

$$E(v_\nu) \leq (E(u_\infty)^{\frac{n}{2}} + m \, Y(S^n)^{\frac{n}{2}})^{\frac{2}{n}} + C \Big(\int_M u_\nu^{\frac{2n}{n-2}} \, |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0}\Big)^{\frac{n+2}{2n}(1+\gamma)}. \tag{170}$$

Proof. Using Proposition 6.14 and Proposition B.3, we obtain

$$E(\overline{u}_{z_\nu}) \leq E(u_\infty) + C \Big(\int_M u_\nu^{\frac{2n}{n-2}} \, |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0}\Big)^{\frac{n+2}{2n}(1+\gamma)} + C \sum_{k=1}^m \varepsilon_{k,\nu}^{\frac{n-2}{2}(1+\gamma)}$$

and

$$E(\overline{u}_{(x_{k,\nu},\varepsilon_{k,\nu})}) \leq Y(S^n)$$

for all $1 \leq k \leq m$. Hence, the assertion follows from Proposition 6.15.

7. Proof of Proposition 3.3

Using the identity

$$R_{g_\nu} = -u_\nu^{-\frac{n+2}{n-2}} \Big(\frac{4(n-1)}{n-2} \, \Delta_{g_0} u_\nu - R_{g_0} \, u_\nu\Big),$$

we obtain

$$\begin{aligned} r_{g_\nu} &= \int_M \Big(\frac{4(n-1)}{n-2} \, |du_\nu|_{g_0}^2 + R_{g_0} \, u_\nu^2\Big) \, dvol_{g_0} \\ &= \int_M \Big(\frac{4(n-1)}{n-2} \, |dv_\nu|_{g_0}^2 + R_{g_0} \, v_\nu^2\Big) \, dvol_{g_0} \\ &\quad + 2 \int_M u_\nu^{\frac{n+2}{n-2}} \, R_{g_\nu} \, w_\nu \, dvol_{g_0} \\ &\quad - \int_M \Big(\frac{4(n-1)}{n-2} \, |dw_\nu|_{g_0}^2 + R_{g_0} \, w_\nu^2\Big) \, dvol_{g_0}. \end{aligned} \tag{171}$$

This implies

(172)

$$\begin{aligned} r_{g_\nu} &= E(v_\nu) \bigg(\int_M v_\nu^{\frac{2n}{n-2}} \, dvol_{g_0} \bigg)^{\frac{n-2}{n}} \\ &\quad + 2 \int_M u_\nu^{\frac{n+2}{n-2}} \, (R_{g_\nu} - r_\infty) \, w_\nu \, dvol_{g_0} \\ &\quad - \int_M \Big(\frac{4(n-1)}{n-2} \, |dw_\nu|_{g_0}^2 + R_{g_0} \, w_\nu^2 - \frac{n+2}{n-2} \, r_\infty \, v_\nu^{\frac{4}{n-2}} \, w_\nu^2 \Big) \, dvol_{g_0} \\ &\quad + r_\infty \int_M \Big(- \frac{n+2}{n-2} \, v_\nu^{\frac{4}{n-2}} \, w_\nu^2 + 2 \, (v_\nu + w_\nu)^{\frac{n+2}{n-2}} \, w_\nu \Big) \, dvol_{g_0}. \end{aligned}$$

In view of the volume normalization, we have

$$\int_M (v_\nu + w_\nu)^{\frac{2n}{n-2}} \, dvol_{g_0} = 1. \tag{173}$$

Furthermore, it is not difficult to show that

$$\begin{aligned} &\bigg(\int_M v_\nu^{\frac{2n}{n-2}} \, dvol_{g_0} \bigg)^{\frac{n-2}{n}} - 1 \\ &\qquad \leq \frac{n-2}{n} \bigg(\int_M v_\nu^{\frac{2n}{n-2}} \, dvol_{g_0} \bigg) - \frac{n-2}{n}, \end{aligned} \tag{174}$$

hence

$$\begin{aligned} &\bigg(\int_M v_\nu^{\frac{2n}{n-2}} \, dvol_{g_0} \bigg)^{\frac{n-2}{n}} - 1 \\ &\qquad \leq \int_M \Big(\frac{n-2}{n} \, v_\nu^{\frac{2n}{n-2}} - \frac{n-2}{n} \, (v_\nu + w_\nu)^{\frac{2n}{n-2}} \Big) \, dvol_{g_0}. \end{aligned} \tag{175}$$

From this, it follows that

(176)

$$\begin{aligned} r_{g_\nu} &\leq r_\infty + (E(v_\nu) - r_\infty) \bigg(\int_M v_\nu^{\frac{2n}{n-2}} \, dvol_{g_0} \bigg)^{\frac{n-2}{n}} \\ &\quad + 2 \int_M u_\nu^{\frac{n+2}{n-2}} \, (R_{g_\nu} - r_\infty) \, w \, dvol_{g_0} \\ &\quad - \int_M \Big(\frac{4(n-1)}{n-2} \, |dw_\nu|_{g_0}^2 + R_{g_0} \, w_\nu^2 - \frac{n+2}{n-2} \, r_\infty \, v_\nu^{\frac{4}{n-2}} \, w_\nu^2 \Big) \, dvol_{g_0} \\ &\quad + r_\infty \int_M \Big(\frac{n-2}{n} \, v_\nu^{\frac{2n}{n-2}} - \frac{n+2}{n-2} \, v_\nu^{\frac{4}{n-2}} \, w_\nu^2 \\ &\qquad + 2 \, (v_\nu + w_\nu)^{\frac{n+2}{n-2}} \, w_\nu - \frac{n-2}{n} \, (v_\nu + w_\nu)^{\frac{2n}{n-2}} \Big) \, dvol_{g_0}. \end{aligned}$$

Using Hölder's inequality, we obtain

(177)
$$\begin{aligned}&\int_M u_\nu^{\frac{n+2}{n-2}}\,(R_{g_\nu}-r_\infty)\,w_\nu\,dvol_{g_0}\\&\quad\leq\bigg(\int_M u_\nu^{\frac{2n}{n-2}}\,|R_{g_\nu}-r_\infty|^{\frac{2n}{n+2}}\,dvol_{g_0}\bigg)^{\frac{n+2}{2n}}\,\bigg(\int_M |w_\nu|^{\frac{2n}{n-2}}\,dvol_{g_0}\bigg)^{\frac{n-2}{2n}}.\end{aligned}$$

Moreover, it follows from Corollarys 5.5 and 6.10 that

$$\begin{aligned}&\int_M\Big(\frac{4(n-1)}{n-2}\,|dw_\nu|_{g_0}^2+R_{g_0}\,w_\nu^2-\frac{n+2}{n-2}\,r_\infty\,v_\nu^{\frac{4}{n-2}}\,w_\nu^2\Big)\,dvol_{g_0}\\&\qquad\geq c\int_M\Big(\frac{4(n-1)}{n-2}\,|dw_\nu|_{g_0}^2+R_{g_0}\,w_\nu^2\Big)\,dvol_{g_0},\end{aligned}\tag{178}$$

hence

$$\begin{aligned}&\int_M\Big(\frac{4(n-1)}{n-2}\,|dw_\nu|_{g_0}^2+R_{g_0}\,w_\nu^2-\frac{n+2}{n-2}\,r_\infty\,v_\nu^{\frac{4}{n-2}}\,w_\nu^2\Big)\,dvol_{g_0}\\&\qquad\geq c\bigg(\int_M|w_\nu|^{\frac{2n}{n-2}}\,dvol_{g_0}\bigg)^{\frac{n-2}{n}}.\end{aligned}\tag{179}$$

Finally, it follows from the pointwise estimate

$$\begin{aligned}&\Big|\frac{n-2}{n}\,v_\nu^{\frac{2n}{n-2}}-\frac{n+2}{n-2}\,v_\nu^{\frac{4}{n-2}}\,w_\nu^2\\&\qquad+2\,(v_\nu+w_\nu)^{\frac{n+2}{n-2}}\,w_\nu-\frac{n-2}{n}\,(v_\nu+w_\nu)^{\frac{2n}{n-2}}\Big|\\&\qquad\leq C\,v_\nu^{\max\{0,\frac{4}{n-2}-1\}}\,|w_\nu|^{\min\{\frac{2n}{n-2},3\}}+C\,|w_\nu|^{\frac{2n}{n-2}}\end{aligned}\tag{180}$$

that

(181)
$$\begin{aligned}&\int_M\Big|\frac{n-2}{n}\,v_\nu^{\frac{2n}{n-2}}-\frac{n+2}{n-2}\,v_\nu^{\frac{4}{n-2}}\,w_\nu^2\\&\quad+2\,(v_\nu+w_\nu)^{\frac{n+2}{n-2}}\,w_\nu-\frac{n-2}{n}\,(v_\nu+w_\nu)^{\frac{2n}{n-2}}\Big|\,dvol_{g_0}\\&\quad\leq C\int_M v_\nu^{\max\{0,\frac{2n}{n-2}-3\}}\,|w_\nu|^{\min\{\frac{2n}{n-2},3\}}\,dvol_{g_0}+C\int_M|w_\nu|^{\frac{2n}{n-2}}\,dvol_{g_0}\\&\quad\leq C\bigg(\int_M|w_\nu|^{\frac{2n}{n-2}}\,dvol_{g_0}\bigg)^{\frac{n-2}{n}\,\min\{\frac{n}{n-2},\frac{3}{2}\}}.\end{aligned}$$

Thus, we conclude that

$$r_{g_\nu} \leq r_\infty + (E(v_\nu) - r_\infty) \left(\int_M v_\nu^{\frac{2n}{n-2}} \, dvol_{g_0} \right)^{\frac{n-2}{n}} + C \left(\int_M u^{\frac{2n}{n-2}} \, |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \right)^{\frac{n+2}{n}}. \tag{182}$$

Since

$$E(v_\nu) - r_\infty \leq C \left(\int_M u_\nu^{\frac{2n}{n-2}} \, |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \right)^{\frac{n+2}{2n}(1+\gamma)},$$

we obtain

$$r_{g_\nu} \leq r_\infty + C \left(\int_M u_\nu^{\frac{2n}{n-2}} \, |R_{g_\nu} - r_\infty|^{\frac{2n}{n+2}} \, dvol_{g_0} \right)^{\frac{n+2}{2n}(1+\gamma)}. \tag{183}$$

This completes the proof of Proposition 3.3.

Appendix A. The interior regularity theorem and the Harnack inequality

Proposition A.1. *Let $q > \frac{n}{2}$. Then, we can find positive constants η_1 and C with the following significance: if $g = u^{\frac{4}{n-2}} g_0$ is a conformal metric such that*

$$\int_{B_r(x)} dvol_g \leq 1 \tag{184}$$

and

$$\int_{B_r(x)} |R_g|^q \, dvol_g \leq \eta_1, \tag{185}$$

then we have

$$u(x) \leq C \, r^{-\frac{n-2}{2}} \left(\int_{B_r(x)} dvol_g \right)^{\frac{n-2}{2n}}. \tag{186}$$

Proof. Let r_0 be a real number such that $r_0 < r$ and

$$(r-s)^{\frac{n-2}{2}} \sup_{B_s(x)} u \leq (r - r_0)^{\frac{n-2}{2}} \sup_{B_{r_0}(x)} u$$

for all $s < r$. Moreover, we choose a point $x_0 \in B_{r_0}(x)$ such that

$$\sup_{B_{r_0}(x)} u = u(x_0).$$

Using a standard interior estimate for linear elliptic equations (see [8], Theorem 8.17 on p. 194), we obtain

$$\begin{aligned}(187)\quad & s^{\frac{n-2}{2}}\,u(x_0) \\ &\leq C\left(\int_{B_s(x_0)} u^{\frac{2n}{n-2}}\,dvol_{g_0}\right)^{\frac{n-2}{2n}} \\ &\quad + C\,s^{\frac{n+2}{2}-\frac{n}{q}}\left(\int_{B_s(x_0)}\Big|\frac{4(n-1)}{n-2}\,\Delta_{g_0}u - R_{g_0}\,u\Big|^q\,dvol_{g_0}\right)^{\frac{1}{q}}\end{aligned}$$

for $s \leq \frac{r-r_0}{2}$. From this, it follows that

$$\begin{aligned}(188)\quad s^{\frac{n-2}{2}}\,u(x_0) &\leq C\left(\int_{B_s(x_0)} dvol_g\right)^{\frac{n-2}{2n}} \\ &\quad + C\,s^{\frac{n+2}{2}-\frac{n}{q}}\left(\int_{B_r(x_0)} u^{\frac{n+2}{n-2}\,q-\frac{2n}{n-2}}\,|R_g|^q\,dvol_g\right)^{\frac{1}{q}}\end{aligned}$$

for $s \leq \frac{r-r_0}{2}$. By definition of r_0 and x_0, we have

$$(189)\qquad \sup_{B_{\frac{r-r_0}{2}}(x_0)} u \leq \sup_{B_{\frac{r+r_0}{2}}(x)} u \leq 2^{\frac{n-2}{2}} \sup_{B_{r_0}(x)} u = 2^{\frac{n-2}{2}}\,u(x_0).$$

Hence, we can find a fixed constant K such that

$$\begin{aligned}(190)\quad s^{\frac{n-2}{2}}\,u(x_0) &\leq K\left(\int_{B_s(x_0)} dvol_g\right)^{\frac{n-2}{2n}} \\ &\quad + K\,(s^{\frac{n-2}{2}}\,u(x_0))^{\frac{n+2}{n-2}-\frac{2n}{n-2}\,\frac{1}{q}}\left(\int_{B_s(x_0)} |R_g|^q\,dvol_g\right)^{\frac{1}{q}}\end{aligned}$$

for all $s \leq \frac{r-r_0}{2}$. We now choose $\eta_1 > 0$ such that

$$(2K)^{\frac{n+2}{n-2}-\frac{2n}{n-2}\,\frac{1}{q}}\,\eta_1^{\frac{1}{q}} \leq \frac{1}{2}.$$

We claim that $(\frac{r-r_0}{2})^{\frac{n-2}{2}}\,u(x_0) \leq 2K$. Indeed, if $2K \leq (\frac{r-r_0}{2})^{\frac{n-2}{2}}\,u(x_0)$, then we may apply inequality (190) with $s = (\frac{2K}{u(x_0)})^{\frac{2}{n-2}} \leq \frac{r-r_0}{2}$. This yields

$$\begin{aligned}2K &\leq K\left(\int_{B_r(x)} dvol_g\right)^{\frac{n-2}{2n}} \\ &\quad + K\,(2K)^{\frac{n+2}{n-2}-\frac{2n}{n-2}\,\frac{1}{q}}\left(\int_{B_r(x)} |R_g|^q\,dvol_g\right)^{\frac{1}{q}},\end{aligned}$$

hence

$$2K \leq K + K\,(2K)^{\frac{n+2}{n-2} - \frac{2n}{n-2}\,\frac{1}{q}}\,\eta_1^{\frac{1}{q}}.$$

This contradicts the choice of η_1. Thus, we conclude that

$$\Big(\frac{r - r_0}{2}\Big)^{\frac{n-2}{2}}\,u(x_0) \leq 2K. \tag{191}$$

Using (190) with $s = \frac{r-r_0}{2}$, we obtain

$$\begin{aligned}\Big(\frac{r - r_0}{2}\Big)^{\frac{n-2}{2}}\,u(x_0) \leq K\,\bigg(\int_{B_r(x)} dvol_g\bigg)^{\frac{n-2}{2n}} \\ + K\,(2K)^{\frac{4}{n-2} - \frac{2n}{n-2}\,\frac{1}{q}}\,\bigg(\int_{B_r(x)} |R_g|^q\,dvol_g\bigg)^{\frac{1}{q}} \\ \cdot \Big(\frac{r - r_0}{2}\Big)^{\frac{n-2}{2}}\,u(x_0).\end{aligned} \tag{192}$$

This implies

$$\begin{aligned}\Big(\frac{r - r_0}{2}\Big)^{\frac{n-2}{2}}\,u(x_0) \leq K\,\bigg(\int_{B_r(x)} dvol_g\bigg)^{\frac{n-2}{2n}} \\ + \frac{1}{2}\,(2K)^{\frac{n+2}{n-2} - \frac{2n}{n-2}\,\frac{1}{q}}\,\eta_1^{\frac{1}{q}}\,\Big(\frac{r - r_0}{2}\Big)^{\frac{n-2}{2}}\,u(x_0),\end{aligned} \tag{193}$$

hence

$$\Big(\frac{r - r_0}{2}\Big)^{\frac{n-2}{2}}\,u(x_0) \leq 2K\,\bigg(\int_{B_r(x)} dvol_g\bigg)^{\frac{n-2}{2n}}. \tag{194}$$

Thus, we conclude that

$$r^{\frac{n-2}{2}}\,u(x) \leq (r - r_0)^{\frac{n-2}{2}}\,u(x_0) \leq 2^{\frac{n}{2}}\,K\,\bigg(\int_{B_r(x)} dvol_g\bigg)^{\frac{n-2}{2n}}. \tag{195}$$

This proves the assertion.

Proposition A.2. *Let P be a smooth function on M. Moreover, assume that u is a positive function on M such that*

$$-\frac{4(n-1)}{n-2}\,\Delta_{g_0} u + P\,u \geq 0.$$

Then, there exists a constant C, depending only on g_0 and P, such that

$$\int_M u\,dvol_{g_0} \leq C\,\inf_M u. \tag{196}$$

Proof. Fix $r > 0$ sufficiently small. Using the weak Harnack inequality for linear elliptic equations (see [8], Theorem 8.18 on p. 194), we obtain

$$\int_{B_{2r}(x)} u\, dvol_{g_0} \leq L_0 \inf_{B_r(x)} u$$

for some constant L_0. In particular, we have

$$\int_{B_{2r}(x)} u\, dvol_{g_0} \leq L_1 \int_{B_r(x)} u\, dvol_{g_0}$$

for some constant L_1. We claim that

$$\int_{B_r(x)} u\, dvol_{g_0} \leq L_0\, L_1^{m-1}\, u(y)$$

whenever $d(x, y) \leq mr$. Indeed, if $d(x, y) \leq mr$, then we can find a sequence of points $\{x_k : 1 \leq k \leq m\}$ such that $x_1 = x$, $d(x_m, y) \leq r$, and $d(x_k, x_{k+1}) \leq r$ for all $1 \leq k \leq m - 1$. From this, it follows that

$$\int_{B_{2r}(x_m)} u\, dvol_{g_0} \leq L_0 \inf_{B_r(x_m)} u \leq L_0\, u(y).$$

Moreover, we have

$$\int_{B_r(x_k)} u\, dvol_{g_0} \leq \int_{B_{2r}(x_{k+1})} u\, dvol_{g_0} \leq L_1 \int_{B_r(x_{k+1})} u\, dvol_{g_0}$$

for all $1 \leq k \leq m - 1$. Therefore, we obtain

$$\int_{B_r(x)} u\, dvol_{g_0} \leq L_0\, L_1^{m-1}\, u(y)$$

as claimed. Hence, we can find a constant C such that

$$\int_{B_r(x)} u\, dvol_{g_0} \leq C u(y)$$

for all $x, y \in M$. From this, the assertion follows.

Corollary A.3. *Let P be a smooth function on M. Moreover, assume that u is a positive function on M such that*

$$-\frac{4(n-1)}{n-2}\, \Delta_{g_0} u + P\, u \geq 0.$$

Then, there exists a constant C, depending only on g_0 and P, such that

$$\int_M u^{\frac{2n}{n-2}}\, dvol_{g_0} \leq C \inf_M u \left(\sup_M u \right)^{\frac{n+2}{n-2}}. \tag{197}$$

Appendix B. Estimates for the functions $\overline{u}_{(x_k,\varepsilon_k)}$

We first review the definition of conformal normal coordinates (see [11], Theorem 5.1 or [16], Theorem 3.1 on p. 208). Given any point $x \in M$, we can find a conformal metric $h_x = \varphi_x^{\frac{4}{n-2}} g_0$ and a map $\Phi_x : TM_x \to M$ with the following properties:

(i) The function φ_x satisfies $\varphi_x(x) = 1$ and $2^{-\frac{n-2}{2}} \leq \varphi_x(y) \leq 2^{\frac{n-2}{2}}$ for all $y \in M$.

(ii) The map $\Phi_x : TM_x \to M$ is the exponential map relative to the metric h_x. This means that $\Phi_x(0) = x$ and, for every vector $\xi \in TM_x$, the curve $\{\Phi_x(t\xi) : t \in \mathbb{R}\}$ is a geodesic relative to the metric h_x.

(iii) Let $D\Phi_x(\xi)$ be the differential of the map Φ_x at $\xi \in TM_x$. Then, the determinant of $D\Phi_x(\xi)$ relative to the metric h_x satisfies $\det D\Phi_x(\xi) = 1 + O(|\xi|^N)$. Here, N is a fixed positive integer which can be chosen arbitrary large.

(iv) The scalar curvature of h_x satisfies $|R_{h_x}(y)| \leq C\,\rho_x(y)^2$, where $\rho_x(y)$ denotes the Riemannian distance from x to y relative to the metric h_x.

For every point $x \in M$, we denote by G_x the Green's function of the conformal Laplacian relative to the metric h_x with pole at x. This implies

$$\frac{4(n-1)}{n-2}\,\Delta_{h_x} G_x(y) - R_{h_x}\,G_x(y) = 0 \tag{198}$$

for $y \neq x$. Moreover, the Green's function satisfies the estimates

$$\big|G_x(y) - \rho_x(y)^{2-n} - A_x\big| \leq C\,\rho_x(y) \tag{199}$$

and

$$\big|d\big(G_x(y) - \rho_x(y)^{2-n}\big)\big|_{h_x} \leq C, \tag{200}$$

where A_x is a constant depending on $x \in M$ (see [11], Lemma 6.4 or [16], Theorem 3.5 on p. 213). Observe that $A_x > 0$ for all $x \in M$ by the positive mass theorem. According to results of Habermann and Jost, the function $x \mapsto A_x$ is smooth (see [9], Proposition I.1.3 and [10], Proposition 3.5). In particular, this implies

$$\inf_{x \in M} A_x > 0. \tag{201}$$

Suppose that we are given a set of pairs $(x_k, \varepsilon_k)_{1 \leq k \leq m}$. For every $1 \leq k \leq m$, we define a function $\overline{u}_{(x_k,\varepsilon_k)}$ by

$$\overline{u}_{(x_k,\varepsilon_k)}(y) = \varphi_{x_k}(y)\,\overline{U}_{(x_k,\varepsilon_k)}(y), \tag{202}$$

where

(203)

$$\overline{U}_{(x_k,\varepsilon_k)}(y) = \Big(\frac{4n(n-1)}{r_\infty}\Big)^{\frac{n-2}{4}} \varepsilon_k^{\frac{n-2}{2}} \cdot \Big(\chi_\delta(\rho_{x_k}(y))\,(\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n-2}{2}} + (1-\chi_\delta(\rho_{x_k}(y)))\,G_{x_k}(y)\Big).$$

Here, the function $\chi_\delta : \mathbb{R} \to [0,1]$ is defined by $\chi_\delta(s) = \chi(s/\delta)$, where $\chi : \mathbb{R} \to [0,1]$ is a cut-off function satisfying $\chi(s) = 1$ for $s \leq 1$ and $\chi(s) = 0$ for $s \geq 2$. Moreover, δ is a positive real number such that $\varepsilon_k \ll \delta$ for all $1 \leq k \leq m$.

Proposition B.1. *We have*

(204)

$$\begin{aligned} &\Big|\frac{4(n-1)}{n-2}\,\Delta_{h_{x_k}}\overline{U}_{(x_k,\varepsilon_k)}(y) - R_{h_{x_k}}\,\overline{U}_{(x_k,\varepsilon_k)}(y) + r_\infty\,\overline{U}_{(x_k,\varepsilon_k)}(y)^{\frac{n+2}{n-2}} \\ &\quad + \Big(\frac{4n(n-1)}{r_\infty}\Big)^{\frac{n-2}{4}} \varepsilon_k^{\frac{n-2}{2}} A_{x_k} \cdot \frac{4(n-1)}{n-2}\,\Delta_{h_{x_k}}\chi_\delta(\rho_{x_k}(y))\Big| \\ &\leq C\,\Big(\frac{\varepsilon_k}{\varepsilon_k^2 + \rho_{x_k}(y)^2}\Big)^{\frac{n-2}{2}} \rho_{x_k}(y)^2\,1_{\{\rho_{x_k}(y) \leq 2\delta\}} \\ &\quad + C\,\frac{\varepsilon_k^{\frac{n-2}{2}}}{\delta}\,1_{\{\delta \leq \rho_{x_k}(y) \leq 2\delta\}} + C\,\Big(\frac{\varepsilon_k}{\varepsilon_k^2 + \rho_{x_k}(y)^2}\Big)^{\frac{n+2}{2}} 1_{\{\rho_{x_k}(y) \geq \delta\}}. \end{aligned}$$

Proof. By definition of $\overline{U}_{(x_k,\varepsilon_k)}$, we have

$$\begin{aligned} &\frac{4(n-1)}{n-2}\,\Delta_{h_{x_k}}\overline{U}_{(x_k,\varepsilon_k)}(y) - R_{h_{x_k}}(y)\,\overline{U}_{(x_k,\varepsilon_k)}(y) + r_\infty\,\overline{U}_{(x_k,\varepsilon_k)}(y)^{\frac{n+2}{n-2}} \\ &\quad + \Big(\frac{4n(n-1)}{r_\infty}\Big)^{\frac{n-2}{4}} \varepsilon_k^{\frac{n-2}{2}} A_{x_k} \cdot \frac{4(n-1)}{n-2}\,\Delta_{h_{x_k}}\chi_\delta(\rho_{x_k}(y)) \\ &= \Big(\frac{4n(n-1)}{r_\infty}\Big)^{\frac{n-2}{4}} \varepsilon_k^{\frac{n-2}{2}} (I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7), \end{aligned}$$

where

$$\begin{aligned} I_1 &= \chi_\delta(\rho_{x_k}(y))\,\Big(\frac{4(n-1)}{n-2}\,\Delta_{h_{x_k}}(\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n-2}{2}} \\ &\qquad + 4n(n-1)\,\varepsilon_k^2\,(\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n+2}{2}}\Big) \\ I_2 &= -\chi_\delta(\rho_{x_k}(y))\,R_{h_{x_k}}(y)\,(\varepsilon_k^2 + \rho_{x_k}(y)^2)^{-\frac{n-2}{2}} \\ I_3 &= -\frac{4(n-1)}{n-2}\,\Delta_{h_{x_k}}\chi_\delta(\rho_{x_k}(y))\,\big(G_{x_k}(y) - \rho_{x_k}(y)^{2-n} - A_{x_k}\big) \end{aligned}$$

$$I_4 = \frac{4(n-1)}{n-2}\,\Delta_{h_{x_k}}\chi_\delta(\rho_{x_k}(y))\,\Big((\varepsilon_k^2+\rho_{x_k}(y)^2)^{-\frac{n-2}{2}} - \rho_{x_k}(y)^{2-n}\Big)$$
$$I_5 = -\frac{8(n-1)}{n-2}\,\Big\langle d\chi_\delta(\rho_{x_k}(y)), d\big(G_{x_k}(y) - \rho_{x_k}(y)^{2-n}\big)\Big\rangle_{h_{x_k}}$$
$$I_6 = \frac{8(n-1)}{n-2}\,\Big\langle d\chi_\delta(\rho_{x_k}(y)), d\Big((\varepsilon_k^2+\rho_{x_k}(y)^2)^{-\frac{n-2}{2}} - \rho_{x_k}(y)^{2-n}\Big)\Big\rangle_{h_{x_k}}$$
$$\begin{aligned} I_7 = 4n(n-1)\,\varepsilon_k^2\,\bigg[\Big(&\chi_\delta(\rho_{x_k}(y))\,(\varepsilon_k^2+\rho_{x_k}(y)^2)^{-\frac{n-2}{2}} \\ &+ (1-\chi_\delta(\rho_{x_k}(y)))\,G_{x_k}(y)\Big)^{\frac{n+2}{n-2}} \\ &- \chi_\delta(\rho_{x_k}(y))\,(\varepsilon_k^2+\rho_{x_k}(y)^2)^{-\frac{n+2}{2}}\bigg]. \end{aligned}$$

From this, the assertion follows.

Corollary B.2. *We have*

(205)
$$\begin{aligned} &\bigg|\frac{4(n-1)}{n-2}\,\Delta_{h_{x_k}}\overline{U}_{(x_k,\varepsilon_k)}(y) - R_{h_{x_k}}\,\overline{U}_{(x_k,\varepsilon_k)}(y) + r_\infty\,\overline{U}_{(x_k,\varepsilon_k)}(y)^{\frac{n+2}{n-2}}\bigg| \\ &\leq C\,\Big(\frac{\varepsilon_k}{\varepsilon_k^2+\rho_{x_k}(y)^2}\Big)^{\frac{n-2}{2}}\,\rho_{x_k}(y)^2\,1_{\{\rho_{x_k}(y)\leq 2\delta\}} \\ &\quad + C\,\frac{\varepsilon_k^{\frac{n-2}{2}}}{\delta^2}\,1_{\{\delta\leq\rho_{x_k}(y)\leq 2\delta\}} + C\,\Big(\frac{\varepsilon_k}{\varepsilon_k^2+\rho_{x_k}(y)^2}\Big)^{\frac{n+2}{2}}\,1_{\{\rho_{x_k}(y)\geq\delta\}}. \end{aligned}$$

We next estimate the Yamabe energy of the test function $\overline{u}_{(x_k,\varepsilon_k)}$.

Proposition B.3. *Suppose that either $3 \leq n \leq 5$ or M is locally conformally flat. Moreover, assume that M is not conformally equivalent to the standard sphere S^n. If δ is sufficiently small, then the Yamabe energy of the test function $\overline{u}_{(x_k,\varepsilon_k)}$ satisfies the estimate*

(206)
$$E(\overline{u}_{(x_k,\varepsilon_k)}) \leq Y(S^n) - c\,\varepsilon_k^{n-2} + C\,\delta\,\varepsilon_k^{n-2} + C\,\delta^{-n}\,\varepsilon_k^n$$

for some constant $c > 0$.

Proof. We adapt an argument due to Schoen (see [**14**] or [**16**], Theorem 4.1 on p. 219). We only consider the case $3 \leq n \leq 5$. Using Proposition B.1, we obtain

$$\begin{aligned} &\int_M \Big(\frac{4(n-1)}{n-2}\,|d\overline{U}_{(x_k,\varepsilon_k)}|_{h_{x_k}}^2 + R_{h_{x_k}}\,\overline{U}_{(x_k,\varepsilon_k)}^2 - r_\infty\,\overline{U}_{(x_k,\varepsilon_k)}^{\frac{2n}{n-2}}\Big)\,dvol_{h_{x_k}} \\ &\quad = -\int_M \Big(\frac{4(n-1)}{n-2}\,\Delta_{h_{x_k}}\overline{U}_{(x_k,\varepsilon_k)} - R_{h_{x_k}}\,\overline{U}_{(x_k,\varepsilon_k)} \end{aligned}$$

$$
\begin{aligned}
&\quad + r_\infty \, \overline{U}_{(x_k,\varepsilon_k)}^{\frac{n+2}{n-2}}\Big) \, \overline{U}_{(x_k,\varepsilon_k)} \, dvol_{h_{x_k}} \\
&\leq \Big(\frac{4n(n-1)}{r_\infty}\Big)^{\frac{n-2}{4}} \, \varepsilon_k^{\frac{n-2}{2}} \\
&\quad \cdot A_{x_k} \int_M \frac{4(n-1)}{n-2} \, \Delta_{h_{x_k}} \chi_\delta(\rho_{x_k}(y)) \, \overline{U}_{(x_k,\varepsilon_k)}(y) \, dvol_{h_{x_k}} \\
&\quad + C \, \delta^{6-n} \, \varepsilon_k^{n-2} + C \, \delta \, \varepsilon_k^{n-2} + C \, \delta^{-n} \, \varepsilon_k^n \\
&\leq \Big(\frac{4n(n-1)}{r_\infty}\Big)^{\frac{n-2}{2}} \, \varepsilon_k^{n-2} \\
&\quad \cdot A_{x_k} \int_M \frac{4(n-1)}{n-2} \, \Delta_{h_{x_k}} \chi_\delta(\rho_{x_k}(y)) \, \rho_{x_k}(y)^{2-n} \, dvol_{h_{x_k}} \\
&\quad + C \, \delta^{6-n} \, \varepsilon_k^{n-2} + C \, \delta \, \varepsilon_k^{n-2} + C \, \delta^{-n} \, \varepsilon_k^n \\
&\leq -\Big(\frac{4n(n-1)}{r_\infty}\Big)^{\frac{n-2}{2}} \cdot 4(n-1) \, \omega_{n-1} \, \varepsilon_k^{n-2} \, A_{x_k} \\
&\quad + C \, \delta^{6-n} \, \varepsilon_k^{n-2} + C \, \delta \, \varepsilon_k^{n-2} + C \, \delta^{-n} \, \varepsilon_k^n,
\end{aligned}
$$

where ω_{n-1} denotes the volume of the $(n-1)$-dimensional unit sphere S^{n-1}. This implies

$$
\begin{aligned}
&\int_M \Big(\frac{4(n-1)}{n-2} \, |d\overline{U}_{(x_k,\varepsilon_k)}|_{h_{x_k}}^2 + R_{h_{x_k}} \, \overline{U}_{(x_k,\varepsilon_k)}^2\Big) \, dvol_{h_{x_k}} \\
&\quad \leq r_\infty \int_M \overline{U}_{(x_k,\varepsilon_k)}^{\frac{2n}{n-2}} \, dvol_{h_{x_k}} \\
&\qquad - \Big(\frac{4n(n-1)}{r_\infty}\Big)^{\frac{n-2}{2}} \cdot 4(n-1) \, \omega_{n-1} \, \varepsilon_k^{n-2} \, A_{x_k} \\
&\qquad + C \, \delta^{6-n} \, \varepsilon_k^{n-2} + C \, \delta \, \varepsilon_k^{n-2} + C \, \delta^{-n} \, \varepsilon_k^n.
\end{aligned}
$$

Moreover, we have

$$
\int_M \overline{U}_{(x_k,\varepsilon_k)}^{\frac{2n}{n-2}} \, dvol_{h_{x_k}} \leq \Big(\frac{Y(S^n)}{r_\infty}\Big)^{\frac{n}{2}} + C \, \delta^{-n} \, \varepsilon_k^n.
$$

Putting these facts together, we conclude that

$$
\begin{aligned}
&\int_M \Big(\frac{4(n-1)}{n-2} \, |d\overline{U}_{(x_k,\varepsilon_k)}|_{h_{x_k}}^2 + R_{h_{x_k}} \, \overline{U}_{(x_k,\varepsilon_k)}^2\Big) \, dvol_{h_{x_k}} \\
&\quad \leq Y(S^n) \, \bigg(\int_M \overline{U}_{(x_k,\varepsilon_k)}^{\frac{2n}{n-2}} \, dvol_{h_{x_k}}\bigg)^{\frac{n-2}{n}}
\end{aligned}
$$

$$- \left(\frac{4n(n-1)}{r_\infty} \right)^{\frac{n-2}{2}} \cdot 4(n-1)\, \omega_{n-1}\, \varepsilon_k^{n-2}\, A_{x_k}$$
$$+ C\, \delta^{6-n}\, \varepsilon_k^{n-2} + C\, \delta\, \varepsilon_k^{n-2} + C\, \delta^{-n}\, \varepsilon_k^n.$$

Since $3 \leq n \leq 5$, the assertion follows.

Lemma B.4. *We have the estimate*

$$\int_M \overline{u}_{(x_i,\varepsilon_i)}\, \overline{u}_{(x_j,\varepsilon_j)}^{\frac{n+2}{n-2}}\, dvol_{g_0} \leq C \left(\frac{\varepsilon_j^2 + d(x_i,x_j)^2}{\varepsilon_i \varepsilon_j} \right)^{-\frac{n-2}{2}}. \tag{207}$$

Proof. On the set $\{2d(x_i,y) \leq \varepsilon_j + d(x_i,x_j)\}$, we have

$$\varepsilon_j + d(y,x_j) \geq \varepsilon_j + d(x_i,x_j) - d(x_i,y) \geq \frac{1}{2}\, (\varepsilon_j + d(x_i,x_j)).$$

Therefore, we obtain

$$\begin{aligned}
&\int_M \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i,y)^2)^{\frac{n-2}{2}}}\, \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y,x_j)^2)^{\frac{n+2}{2}}}\, dvol_{g_0} \\
&\quad \leq \int_{\{2d(x_i,y) \leq \varepsilon_j + d(x_i,x_j)\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i,y)^2)^{\frac{n-2}{2}}}\, \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y,x_j)^2)^{\frac{n+2}{2}}}\, dvol_{g_0} \\
&\qquad + \int_{\{2d(x_i,y) \geq \varepsilon_j + d(x_i,x_j)\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i,y)^2)^{\frac{n-2}{2}}}\, \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y,x_j)^2)^{\frac{n+2}{2}}}\, dvol_{g_0} \\
&\quad \leq C \int_{\{2d(x_i,y) \leq \varepsilon_j + d(x_i,x_j)\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i,y)^2)^{\frac{n-2}{2}}}\, \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i,x_j)^2)^{\frac{n}{2}}}\, dvol_{g_0} \\
&\qquad + C \int_M \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i,x_j)^2)^{\frac{n-2}{2}}}\, \frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2 + d(y,x_j)^2)^{\frac{n+2}{2}}}\, dvol_{g_0} \\
&\quad \leq C\, \frac{\varepsilon_i^{\frac{n-2}{2}}\, \varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i,x_j)^2)^{\frac{n-2}{2}}}.
\end{aligned}$$

This proves the assertion.

Lemma B.5. *We have the estimate*

$$\begin{aligned}
&\int_M \overline{u}_{(x_i,\varepsilon_i)} \left| \frac{4(n-1)}{n-2}\, \Delta_{g_0} \overline{u}_{(x_j,\varepsilon_j)} - R_{g_0}\, \overline{u}_{(x_j,\varepsilon_j)} + r_\infty\, \overline{u}_{(x_j,\varepsilon_j)}^{\frac{n+2}{n-2}} \right| dvol_{g_0} \\
&\quad \leq C \left(\delta^4 + \delta^{n-2} + \frac{\varepsilon_j^2}{\delta^2} \right) \left(\frac{\varepsilon_j^2 + d(x_i,x_j)^2}{\varepsilon_i \varepsilon_j} \right)^{-\frac{n-2}{2}}.
\end{aligned} \tag{208}$$

Proof. The inequality $2^{-\frac{n-2}{2}} \leq \varphi_x \leq 2^{\frac{n-2}{2}}$ implies $\frac{1}{2}\, d(x,y) \leq \rho_x(y) \leq 2d(x,y)$ for all $x,y \in M$. With the aid of Corollary B.2, we obtain

$$\begin{aligned}
&\Big| \frac{4(n-1)}{n-2}\, \Delta_{g_0} \overline{u}_{(x_j,\varepsilon_j)} - R_{g_0}\, \overline{u}_{(x_j,\varepsilon_j)} + r_\infty\, \overline{u}_{(x_j,\varepsilon_j)}^{\frac{n+2}{n-2}} \Big| \\
&\leq C\, (\delta^2 + \delta^{n-4}) \Big(\frac{\varepsilon_j}{\varepsilon_j^2 + d(y,x_j)^2} \Big)^{\frac{n-2}{2}}\, 1_{\{d(y,x_j) \leq 4\delta\}} \\
&\quad + C \Big(\frac{\varepsilon_j}{\varepsilon_j^2 + d(y,x_j)^2} \Big)^{\frac{n+2}{2}}\, 1_{\{d(y,x_j) \geq \frac{\delta}{2}\}}.
\end{aligned}$$

On the set $\{2d(x_i,y) \leq \varepsilon_j + d(x_i,x_j)\} \cap \{d(y,x_j) \leq 4\delta\}$, we have

$$\varepsilon_j + d(y,x_j) \geq \varepsilon_j + d(x_i,x_j) - d(x_i,y) \geq \frac{1}{2}\, (\varepsilon_j + d(x_i,x_j)),$$

hence

$$d(x_i,y) \leq \frac{1}{2}\, (\varepsilon_j + d(x_i,x_j)) \leq \varepsilon_j + d(y,x_j) \leq 8\delta.$$

From this, it follows that

$$\begin{aligned}
&\int_{\{d(y,x_j) \leq 4\delta\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i,y)^2)^{\frac{n-2}{2}}}\, \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(y,x_j)^2)^{\frac{n-2}{2}}}\, dvol_{g_0} \\
&\leq \int_{\substack{\{2d(x_i,y) \leq \varepsilon_j + d(x_i,x_j)\} \\ \cap \{d(y,x_j) \leq 4\delta\}}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i,y)^2)^{\frac{n-2}{2}}} \\
&\qquad \cdot \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(y,x_j)^2)^{\frac{n-2}{2}}}\, dvol_{g_0} \\
&\quad + \int_{\substack{\{2d(x_i,y) \geq \varepsilon_j + d(x_i,x_j)\} \\ \cap \{d(y,x_j) \leq 4\delta\}}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i,y)^2)^{\frac{n-2}{2}}} \\
&\qquad \cdot \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(y,x_j)^2)^{\frac{n-2}{2}}}\, dvol_{g_0} \\
&\leq C \int_{\{d(x_i,y) \leq 8\delta\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2 + d(x_i,y)^2)^{\frac{n-2}{2}}}\, \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i,x_j)^2)^{\frac{n-2}{2}}}\, dvol_{g_0} \\
&\quad + C \int_{\{d(y,x_j) \leq 4\delta\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(x_i,x_j)^2)^{\frac{n-2}{2}}}\, \frac{\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2 + d(y,x_j)^2)^{\frac{n-2}{2}}}\, dvol_{g_0}
\end{aligned}$$

$$\leq C\,\delta^2\,\frac{\varepsilon_i^{\frac{n-2}{2}}\,\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2+d(x_i,x_j)^2)^{\frac{n-2}{2}}}.$$

Similarly, on the set $\{2d(x_i,y) \leq \varepsilon_j + d(x_i,x_j)\} \cap \{d(y,x_j) \geq \frac{\delta}{2}\}$, we have

$$\varepsilon_j + d(y,x_j) \geq \varepsilon_j + d(x_i,x_j) - d(x_i,y) \geq \frac{1}{2}\,(\varepsilon_j + d(x_i,x_j)).$$

This implies

$$\begin{aligned}
&\int_{\{d(y,x_j)\geq\frac{\delta}{2}\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2+d(x_i,y)^2)^{\frac{n-2}{2}}}\,\frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2+d(y,x_j)^2)^{\frac{n+2}{2}}}\,dvol_{g_0} \\
&\leq \int_{\substack{\{2d(x_i,y)\leq\varepsilon_j+d(x_i,x_j)\}\\ \cap\{d(y,x_j)\geq\frac{\delta}{2}\}}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2+d(x_i,y)^2)^{\frac{n-2}{2}}} \\
&\qquad\cdot\frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2+d(y,x_j)^2)^{\frac{n+2}{2}}}\,dvol_{g_0} \\
&\quad+\int_{\substack{\{2d(x_i,y)\geq\varepsilon_j+d(x_i,x_j)\}\\ \cap\{d(y,x_j)\geq\frac{\delta}{2}\}}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2+d(x_i,y)^2)^{\frac{n-2}{2}}} \\
&\qquad\cdot\frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2+d(y,x_j)^2)^{\frac{n+2}{2}}}\,dvol_{g_0} \\
&\leq C\int_{\{2d(x_i,y)\leq\varepsilon_j+d(x_i,x_j)\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_i^2+d(x_i,y)^2)^{\frac{n-2}{2}}} \\
&\qquad\cdot\frac{\varepsilon_j^{\frac{n+2}{2}}}{\delta^2\,(\varepsilon_j^2+d(x_i,x_j)^2)^{\frac{n}{2}}}\,dvol_{g_0} \\
&\quad+C\int_{\{d(y,x_j)\geq\frac{\delta}{2}\}} \frac{\varepsilon_i^{\frac{n-2}{2}}}{(\varepsilon_j^2+d(x_i,x_j)^2)^{\frac{n-2}{2}}}\,\frac{\varepsilon_j^{\frac{n+2}{2}}}{(\varepsilon_j^2+d(y,x_j)^2)^{\frac{n+2}{2}}}\,dvol_{g_0} \\
&\leq C\,\frac{\varepsilon_j^2}{\delta^2}\,\frac{\varepsilon_i^{\frac{n-2}{2}}\,\varepsilon_j^{\frac{n-2}{2}}}{(\varepsilon_j^2+d(x_i,x_j)^2)^{\frac{n-2}{2}}}.
\end{aligned}$$

From this, the assertion follows.

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