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# RANDOM CONSTRUCTION OF RIEMANN SURFACES

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## Abstract

We develop a new approach for the study of "typical" Riemann surfaces with high genus. The method that we use is the construction of random Riemann surfaces from oriented cubic graphs. This construction enables us to get a control over the global geometry properties of compact Riemann surfaces. We use the theory of random regular graphs to show that almost all such surfaces have large first eigenvalues and large Cheeger constants. Moreover a closer analysis of the probability space of oriented cubic graphs shows that on a typical surface there is a large embedded hyperbolic ball.

## 1. Introduction

In this paper, we address the following question: What does a typical compact Riemann surface of large genus look like geometrically?

By a Riemann surface, we mean an oriented surface with a complete, finite-area metric of constant curvature-1.

In the standard geometric picture of Riemann surfaces via Fenchel–Nielsen coordinates, it is difficult to keep track of global geometric quantities such as the first eigenvalue of the Laplacian, the injectivity radius, and the diameter. Here, we present a model for looking at Riemann surfaces based on 3-regular graphs, with which it is easier to control the global geometry.

The idea of using 3-regular graphs to study the first eigenvalue of Riemann surfaces originated in Buser [11], [12], who associated cubic graphs to Riemann surfaces as a tool for comparing the spectral geometry of surfaces with the spectral geometry of graphs. We introduce a somewhat different method, which associates to each 3-regular graph $\Gamma$

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with an orientation $\mathcal{O}$, to be defined below, a finite area Riemann surface $S^O(\Gamma, \mathcal{O})$, and we let $S^C(\Gamma, \mathcal{O})$ denote its conformal compactification.

Our main technical result, Theorem 2.1, shows that the geometry of $S^C(\Gamma, \mathcal{O})$ in its hyperbolic metric is controlled, with asymptotic probability 1 as the number of vertices in $\Gamma$ tend to infinity, by the geometry of $S^O(\Gamma, \mathcal{O})$ in its hyperbolic metric, which in turn is controlled by the geometry of $(\Gamma, \mathcal{O})$. This is then combined with results due to Bollobás [**4**] to give, in Theorem 2.2, a fairly complete picture of the spectral geometry (first eigenvalue, Cheeger constant, shortest geodesic, and diameter) of a typical surface constructed this way.

We then pursue in Sections 8–11 a closer study of the structure of orientations and how they affect the geometry of $S^C(\Gamma, \mathcal{O})$. Our main result here is an estimate for the expected value of the volume of the largest embedded ball of a surface $S^C(\Gamma, \mathcal{O})$. We find that the expected volume of this ball is proportional to the volume of the surface, with a constant of proportionality independent of the size of the graph. Thus, the geometry of $S^C(\Gamma, \mathcal{O})$ is in general dominated by one very large ball of injectivity radius.

## 2. Statement of results

If $\Gamma$ is a finite 3-regular graph, an *orientation* $\mathcal{O}$ on $\Gamma$ is a function which assigns to each vertex $v$ of $\Gamma$ a cyclic ordering of the edges emanating from $v$. In Section 4 , we will show how, given a pair $(\Gamma, \mathcal{O})$, we may associate to $(\Gamma, \mathcal{O})$ two Riemann surfaces $S^O(\Gamma, \mathcal{O})$ and $S^C(\Gamma, \mathcal{O})$. $S^O(\Gamma, \mathcal{O})$ is constructed by associating an ideal hyperbolic triangle to each vertex of $\Gamma$, and gluing sides together according to the edges of the graph $\Gamma$ and the orientation $\mathcal{O}$. It is a finite-area Riemann surface with cusps.

The surface $S^C(\Gamma, \mathcal{O})$ is then the conformal compactification of $S^O(\Gamma, \mathcal{O})$.

It follows from a theorem of Belyi [**3**] (see [**18**] for a discussion of Belyi's Theorem), that the surfaces $S^C(\Gamma, \mathcal{O})$ are dense in the space of all Riemann surfaces. Thus, the process of randomly selecting a Riemann surface can be modeled on the process of picking a finite 3-regular graph with orientation at random.

Since the pair $(\Gamma, \mathcal{O})$ gives a description of $S^O(\Gamma, \mathcal{O})$ as an orbifold covering of $\mathbb{H}^2/PSL(2, \mathbb{Z})$, one can give a qualitative description of the global geometry of $S^O(\Gamma, \mathbb{Z})$ by a corresponding description of the pair $(\Gamma, \mathcal{O})$. Thus, for example, the first eigenvalue of $S^O(\Gamma, \mathcal{O})$ will be large if and only if the first eigenvalue of $\Gamma$ is large [**6**] and [**7**].

It is our observation here, building on the work of [**9**], that the same will be true of $S^C(\Gamma, \mathcal{O})$, provided that $S^O(\Gamma, \mathcal{O})$ satisfies a "large cusps" condition, to be described in Section 3. This has a purely combinatorial interpretation in terms of the pair $(\Gamma, \mathcal{O})$, and so can be analyzed with relative ease.

For $n$ a positive integer, let $\mathcal{F}_n^*$ denote the finite set of pairs $(\Gamma, \mathcal{O})$, where $\Gamma$ is a 3-regular graph on $2n$ vertices. We will endow $\mathcal{F}_n^*$ with a probability measure introduced and studied by Bollobás [**4**], [**5**], which we review in Section 5.

If $Q$ is a property of 3-regular graphs with orientation, denote by $\mathrm{Prob}_n[Q]$ the probability that a pair $(\Gamma, \mathcal{O})$ picked from $\mathcal{F}_n^*$ has property $Q$.

Our main technical result, shown in Section 6.

**Theorem 2.1.** *As $n \to \infty$,*

$$\mathrm{Prob}_n[S^O(\Gamma, \mathcal{O}) \textit{ satisfies the large cusps condition}] \to 1.$$

We will use Theorem 2.1 in order to study geometric properties of the surfaces $S^C(\Gamma, \mathcal{O})$. To that end, we define the Cheeger constant $h(S)$ of a Riemann surface $S$ by the formula

$$h(S) = \inf_C \frac{\mathrm{length}\,(C)}{\min[\mathrm{area}\,(A), \mathrm{area}\,(B)]},$$

where $C$ runs over (possibly disconnected) closed curves on $S$ which divide $S$ into two parts $A$ and $B$. It will then follow from Theorem 2.1 that

**Theorem 2.2.** *There exist constants $C_1$, $C_2$, $C_3$, and $C_4$ such that, as $n \to \infty$:*

(a) *The first eigenvalue $\lambda_1(S^C(\Gamma, \mathcal{O}))$ satisfies*

$$\mathrm{Prob}_n[\lambda_1(S^C(\Gamma, \mathcal{O})) \geq C_1] \to 1.$$

(b) *The Cheeger constant $h(S^C(\Gamma, \mathcal{O}))$ satisfies*

$$\mathrm{Prob}_n[h(S^C(\Gamma, \mathcal{O})) \geq C_2] \to 1.$$

(c) *The shortest geodesic $\mathrm{syst}(S^C(\Gamma, \mathcal{O}))$ satisfies*

$$Prob_n[\mathrm{syst}(S^C(\Gamma, \mathcal{O})) \geq C_3] \to 1.$$

(d) *The diameter $\mathrm{diam}\,(S^C(\Gamma, \mathcal{O}))$ satisfies*

$$\mathrm{Prob}_n[\mathrm{diam}\,(S^C(\Gamma, \mathcal{O})) \leq C_4 \log(\mathrm{genus}\,(S^C(\Gamma, \mathcal{O})))] \to 1.$$

Of these properties, (a) follows from (b) by Cheeger's inequality [**13**], while (d) also follows from (b) and (c) and the following well-known argument: if $M$ is a manifold and $B(r_0)$ is the infimum over all points of $M$ of the volume of a ball of radius $r_0$, then

$$\operatorname{diam}(M) \leq 2\left[r_0 + \frac{1}{h(M)}\log\left(\frac{\operatorname{vol}(M)}{2B(r_0)}\right)\right].$$

Property (b) for the surfaces $S^O(\Gamma, \mathcal{O})$ will follow from the corresponding result on graphs [**4**] together with [**6**] and [**7**], while the passage from the surfaces $S^O(\Gamma, \mathcal{O})$ to the surfaces $S^C(\Gamma, \mathcal{O})$ will follow from Theorem 2.1. For properties (c) and (d), the translation from graphs to surfaces is not as simple, but the idea is similar.

The forms of (a), (b), and (d) are sharp, up to the constants. Regarding (a), it follows from Cheng's Theorem [**14**] that a Riemann surface $S$ must have $\lambda_1(S) \leq 1/4 + \varepsilon$ for some $\varepsilon \to 0$ as genus$(S) \to \infty$. The upper bound $h(S) \leq 1 + \varepsilon$, is well-known, and follows from a similar argument. The estimate $\operatorname{diam}(S) \geq (\text{const}) \log(\text{genus}(S))$ follows from area considerations and Gauss–Bonnet.

The estimate in (c) is certainly not optimal, as there are Riemann surfaces whose injectivity radius grows like (const)[log(genus(S))]. Indeed, this occurs for the Platonic surfaces [**9**], and also for congruence coverings of compact arithmetically defined surfaces. It follows from our analysis that, for a given constant $C_5$, there is a positive constant $C_6$ such that

$$\text{Prob}_n[\text{syst}(S^C(\Gamma, \mathcal{O})) \geq C_5] \to C_6.$$

Thus, the probability of selecting a surface having injectivity radius at least a given large number is asymptotically positive, but certainly not asymptotically 1.

In the language of [**8**], Theorem 2.2 shows that, with probability $\to 1$, a typical Riemann surface is short, with a large first eigenvalue, but not necessarily fat.

In Section 8, we begin a study of the distribution of the closed left-hand-turn paths (LHT paths for short), defined in Definition 4.1, of $(\Gamma, \mathcal{O})$. We investigate two aspects of this distribution – the number of closed LHT paths and the length of the longest LHT path.

The first of these determines the genus of the surface $S^C(\Gamma, \mathcal{O})$. We will show the following theorem.

**Theorem 2.3.** *There exist constants $C_1$ and $C_2$ such that the expected value of the genus* genus$(S^C(\Gamma, \mathcal{O}))$*, where $(\Gamma, \mathcal{O})$ is randomly* *selected among oriented* 3*-regular graphs on* $2n$ *vertices, satisfies*

$$1 + n/2 - \left[C_1 + \frac{3\log(n)}{4}\right] \le E(\text{genus}\,(S^C(\Gamma,\mathcal{O}))) \le 1 + n/2 - \left[C_2 + \frac{\log(n)}{2}\right].$$

In Section 10, we show that the length of the longest left-hand-turn path in $(\Gamma,\mathcal{O})$ gives a lower bound for the volume of the largest embedded ball in $S^C(\Gamma,\mathcal{O})$. Denoting this volume by $\mathrm{Emb}(S^C(\Gamma,\mathcal{O}))$, we will show the following theorem.

**Theorem 2.4.** *There exists a constant* $C$ *such that the expected value*

$$E(\mathrm{Emb}(S^C(\Gamma,\mathcal{O})))$$

*satisfies*

$$E(\mathrm{Emb}(\Gamma,\mathcal{O})) \ge (C - \varepsilon(n))\text{area}\,(S^C(\Gamma,\mathcal{O})),$$

*where* $\varepsilon(n)$ *is a function which tends to* 0 *as* $n \to \infty$*.*

Our proof gives a value of $C$ of $1/\pi$. We do not believe that this result is sharp. More generally, we believe that a closer analysis of the probability theory of the distribution of LHT paths will show that this distribution will look very much like a Poisson–Dirichlet distribution, see [**2**], [**24**], and [**22**] for a discussion, but our methods do not establish this. Theorem 2.4 responds to a question which was posed to us by David Kazhdan.

The results that we use from [**9**] are qualitative rather than quantitative. However, they have been put in quantitative form in [**20**], [**21**]. In particular, it follows from [**20**], [**21**] that whenever in the following, the condition of "cusps of length $\ge L$" is used, we may take $L = 7$.

The results of Theorem 2.2 were announced in [**10**], under the weaker conclusion that properties (a)–(d) occur with positive probability, rather than probability $\to 1$ as $n \to \infty$.

A weaker version of Theorem 2.3 was proved by Gamburd and Makover in [**16**].

## 3. Compactification of Riemann surfaces

In this section, we review the connection between a finite-area Riemann surface and its conformal compactification.

Let $S^0$ be a Riemann surface with a complete finite area metric of curvature $-1$. Then, $S^0$ has finitely many cusps neighborhoods $C^1, \dots, C^k$, such that, for each $C^i$ there is an isometry

$$f_i : C^i \to C^{y_i}$$

for some $y_i$, where $C^{y_i}$ is the space

$$C^{y_i} = \left\{ z \in \mathbb{C} : \Im(z) \geq \frac{1}{y_i} \right\} \Big/ (z \sim z+1),$$

endowed with the hyperbolic metric

$$ds^2 = \frac{1}{y^2} \left[ dx^2 + dy^2 \right].$$

The curve $h_i = f_i^{-1}(z : \Im(z) = \frac{1}{y_i})$ on $S^O$ is a closed horocycle on $C^i$ whose length is $y_i$. The length of the largest simple closed horocycle on the cusp $C^i$ is a measure of how large the cusp is.

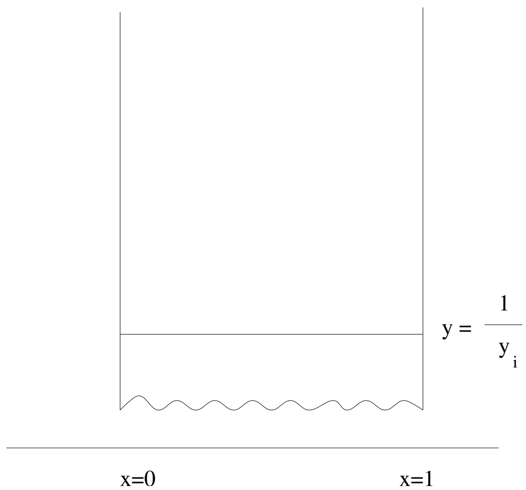


**Figure 1.** The length of a horocycle.

**Definition 3.1.** *$S^O$ has cusps of length $\geq L$ if we may choose all the $C^i$'s disjoint, with $y_i \geq L$ for all $i = 1, \ldots, k$.*

Given a finite-area Riemann surface $S^O$, there is a unique compact Riemann surface $S^C$ and finitely many points $p_1, \ldots, p_k$ of $S^C$ such that $S^O$ is conformally equivalent to $S^C - \{p_1, \ldots, p_k\}$. $S^C$ may be constructed from $S^O$ by observing that each cusp neighborhood $C^i$ is conformally equivalent to a punctured disk. One may fill in this puncture conformally and then reglue the disk to obtain the conformal structure on the closed surface $S^C$. By the Uniformization Theorem, there is a unique constant curvature metric which agrees with this conformal structure.

It is natural to raise the question of the relationship between the constant curvature metric on $S^O$ and the constant curvature metric on

$S^C$. In general, the relationship need not be close. For instance, the surface $S^C$ need not carry a hyperbolic metric even when $S^O$ carries one. However, it is shown in Theorem 2.1 of [9] that, if $S^O$ has cusps of length $\geq L$ for a suitably large $L$, then $S^C$ will carry a hyperbolic metric, and indeed this metric will be very closely related to the hyperbolic metric on $S^O$.

More precisely, we have the following theorem.

**Theorem 3.2** ([9])**.** *For every $\varepsilon$, there exists numbers $L, r$, and $y$ such that, if the cusps of $S^O$ have length $\geq L$, then, outside the union of cusp neighborhoods $\mathcal{U} = \cup_{i=1}^{k} f_i^{-1}(C^y) \subset S^O$ of the cusps $C^i$, and $\mathcal{V} = \cup_{i=1}^{k} B_r(p_i) \subset S^C$, the metrics $ds_C^2$ and $ds_O^2$ satisfy*

$$\frac{1}{(1+\varepsilon)} ds_O^2 \leq ds_C^2 \leq (1+\varepsilon) ds_O^2.$$

The proof of this theorem is based on the Ahlfors–Schwarz Lemma [1]. The idea of the proof is to build on the compact surface an intermediate metric $ds^2_{\text{int}}$ with curvature close to the curvature of the metric on the open surface, and to use the Ahlfors–Schwarz Lemma to compare this metric to the constant curvature metric. The large cusps condition enters precisely here, by giving the metric $ds^2_{\text{int}}$ sufficient time to evolve from the hyperbolic metric on the ball to the hyperbolic metric on the cusp, while keeping curvature close to constant.

For some explicit estimates of $L, r$, and $y$ in terms of $\varepsilon$, (see[21]).

It was shown in [9] that this result may be employed to show that, under the assumption of large cusps, the surfaces $S^O$ and $S^C$ share a number of global geometric properties.

**Theorem 3.3** ([9])**.** *For every $\varepsilon$, there exists an $L$ such that, if $S^O$ has cusps of length $\geq L$, then*

(a) *the Cheeger constants $h(S^O)$ and $h(S^C)$ satisfy*

$$\frac{1}{(1+\varepsilon)} h(S^O) \leq h(S^C) \leq (1+\varepsilon) h(S^O),$$

(b) *the shortest closed geodesics* $\operatorname{syst}(S^O)$ *and* $\operatorname{syst}(S^C)$ *satisfy*

$$\frac{1}{(1+\varepsilon)} \operatorname{syst}(S^O) \leq \operatorname{syst}(S^C).$$

We do not obtain an inequality of the form

$$\operatorname{syst}(S^C) \leq (\text{const}\,) \operatorname{syst}(S^O),$$

in (b), because it may happen that the shortest closed geodesic on $S^O$ becomes homotopically trivial on $S^C$.

## 4. Surfaces and 3-regular graphs

Let $\Gamma$ be a finite 3-regular graph. We will allow $\Gamma$ to contain loops and multiple edges.

An orientation $\mathcal{O}$ on the graph is the assignment, for each vertex $v$ of $\Gamma$, of a cyclic ordering of the three edges emanating from $v$. If $\Gamma$ has $2n$ vertices, then clearly there are $2^{2n}$ orientations on $\Gamma$.

We may think of an orientation on a 3-regular graph in the following way: Suppose one were to walk along the graph, then, when one approaches a vertex, the orientation allows one to distinguish between a left-hand-turn and a right-hand-turn at the vertex. Thus, any path on $\Gamma$ beginning at a vertex $v_0$ may be described by picking an initial direction and a series of $L$'s (signalling a left-hand-turn) and $R$'s (signalling a right-hand-turn).

To a pair $(\Gamma, \mathcal{O})$, we will associate two Riemann surfaces $S^O(\Gamma, \mathcal{O})$ and $S^C(\Gamma, \mathcal{O})$ as follows: We begin by considering the ideal hyperbolic triangle $T$ with vertices $0, 1$, and $\infty$ shown in Figure 2. The solid lines in Figure 2 are geodesics joining the points $i, i+1$, and $(i+1)/2$ with the point $(1+i\sqrt{3})/2$, while the dotted lines are horocycles of length 1 joining pairs of points from the set $\{i, i+1, (i+1)/2\}$. We may think of these points as "midpoints" of the corresponding sides, even though they are of infinite length. We may also think of the three solid lines as segments of a graph surrounding a vertex. We then give them the cyclic ordering $(i, i+1, (i+1)/2)$.

We may now construct the surface $S^O(\Gamma, \mathcal{O})$ from $(\Gamma, \mathcal{O})$ by placing on each vertex $v$ of $\Gamma$ a copy of $T$, so that the cyclic ordering of the segments in $T$ agrees with the orientation at the vertex $v$ in $\Gamma$. If two vertices of $\Gamma$ are joined by an edge, we glue the two copies of $T$ along the corresponding sides subject to the following two conditions:

(a) the midpoints of the two sides are glued together,
and
(b) the gluing preserves the orientation of the two copies of $T$.

The conditions (a) and (b) determine the gluing uniquely. It is easily seen that the surface $S^O(\Gamma, \mathcal{O})$ is a complete Riemann surface with finite area equal to $2\pi n$, where $2n$ is the number of vertices of $\Gamma$.

The surface $S^C(\Gamma, \mathcal{O})$ is then the conformal compactification of $S^O(\Gamma, \mathcal{O})$.

In the remainder of this section, we will describe how to read off many geometric properties of the surfaces $S^O(\Gamma, \mathcal{O})$ and $S^C(\Gamma, \mathcal{O})$ from the combinatorics of the pair $(\Gamma, \mathcal{O})$.

We begin with the observation that the topology of $S^O$ is easy to reconstruct from $(\Gamma, \mathcal{O})$. We will need the following definition:

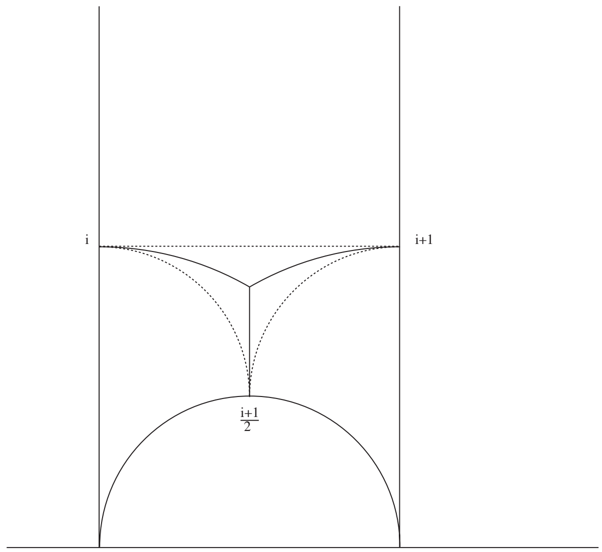


**Figure 2.** The marked ideal triangle $T$.

**Definition 4.1.** *A left-hand-turn path (for short, LHT path) on* $(\Gamma, \mathcal{O})$ *is a closed path on* $\Gamma$ *such that, at each vertex, the path turns left in the orientation* $\mathcal{O}$.

Traveling on a path on $\Gamma$ which always turns left describes a path on $S^O(\Gamma, \mathcal{O})$ which travels around a cusp. Indeed, if we set $LHT = LHT(\Gamma, \mathcal{O})$ to be the number of disjoint left-hand-turn paths, then the topology of $S^O(\Gamma, \mathcal{O})$ is easily describable in terms of $LHT$ and $n$, where again $2n$ is the number of vertices in $\Gamma$. Indeed, the graph $\Gamma$ divides $S^O(\Gamma, \mathcal{O})$ into $LHT$ regions, each bordered by a left-hand-turn path and containing one cusp in its interior. From this, we can immediately read off the signature of $S^O(\Gamma, \mathcal{O})$ by the Euler characteristic. The genus of $S^O(\Gamma, \mathcal{O})$ is given by

$$\text{genus } = 1 + \frac{n - LHT}{2}$$

and the number of cusps is $LHT$.

Note that the usual orientation on the 3-regular graph which is the 1-skeleton of the cube contains six left-hand-turn paths, giving that the associated surface is a sphere with six punctures, while a choice of a different orientation on this graph can have either two, four, or six left-hand-turn paths, so that the associated surface can have genus 0, 1, or

2. Thus, the topology of $S^O(\Gamma, \mathcal{O})$ is heavily dependent on the choice of $\mathcal{O}$.

The geometry of the cusps can also be read off from $(\Gamma, \mathcal{O})$. To that end, we observe that the horocycles on the various copies of $T$ fit together to form a system of disjoint closed horocycles about the cusps of $S^O(\Gamma, \mathcal{O})$. We call this system of horocycles the *canonical horocycles* of $S^O(\Gamma, \mathcal{O})$. The length of each closed horocycle in this set is precisely the length of the corresponding left-hand-turn path, since the length of the horocycle joining $i$ to $i+1$ has length 1. Thus, $S^O(\Gamma, \mathcal{O})$ has cusps of length $\geq L$ if the length of any left-hand-turn path on $(\Gamma, \mathcal{O})$ is at least $L$.

The converse to this is not true, as it is possible to choose a system of horocycles other than the canonical horocycles such that the length of the shortest horocycle for the new system is larger than the length of the shortest canonical horocycle. We will return to this idea in Section 6.

The Cheeger constant $h(S^O(\Gamma, \mathcal{O}))$ can be estimated in terms of the graph $(\Gamma, \mathcal{O})$ as well. Recall that, by analogy with the Cheeger constant of a manifold, the Cheeger constant $h(\Gamma)$ of a graph $\Gamma$ is given by

$$h(\Gamma) = \inf_E \frac{\#(E)}{\min(\#(A), \#(B))},$$

where $E$ is a collection of edges such that $\Gamma - E$ disconnects into two components $A$ and $B$, and $\#(A)$ (resp. $\#(B)$) is the number of vertices in $A$ (resp. $B$).

Then, we have the following theorem.

**Theorem 4.2** ([**6**], [**7**]). *There are positive constants $C_1$ and $C_2$ such that*

$$C_1 h(\Gamma) \leq h(S^O(\Gamma, \mathcal{O})) \leq C_2 h(\Gamma)$$

*for all finite* 3*-regular graphs* $\Gamma$.

In effect, the pair $(\Gamma, \mathcal{O})$ describes $S^O(\Gamma, \mathcal{O})$ as an orbifold covering space of the orbifold $\mathbb{H}^2/PSL(2, \mathbb{Z})$. The behavior of the Cheeger constant of a finite covering of a compact manifold in terms of the graph of a covering is described in [**6**]. In the present case, the base manifold is not compact, but rather a finite-area Riemann surface (with singularities). The additional complication which this difficulty presents is solved in [**7**].

This gives Theorem 4.2.

Note, in particular, that the quantity $h(\Gamma)$ of Theorem 4.2 depends only on $\Gamma$ and not on $\mathcal{O}$.

The geodesics of $S^O(\Gamma, \mathcal{O})$ are also describable in terms of $(\Gamma, \mathcal{O})$. To explain this, let $\mathcal{L}$ and $\mathcal{R}$ denote the matrices

$$\mathcal{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

A closed path $\mathcal{P}$ of length $k$ on the graph may be described by starting at a midpoint of an edge, and then giving a sequence $(w_1, \ldots, w_k)$, where each $w_i$ is either $l$ or $r$, signifying a left or right turn at the upcoming vertex. We then consider the matrix

$$M_{\mathcal{P}} = W_1 \ldots W_k,$$

where $W_j = \mathcal{L}$ if $w_j = l$ and $W_j = \mathcal{R}$ if $w_j = r$. The closed path $\mathcal{P}$ on $\Gamma$ is then homotopic to a closed geodesic $\gamma(\mathcal{P})$ on $S^O(\Gamma, \mathcal{O})$ whose length $\text{length}\,(\gamma(\mathcal{P}))$ is given by

$$2\cosh\left(\frac{\text{length}\,(\gamma(\mathcal{P}))}{2}\right) = \text{tr}\,(M_{\mathcal{P}}).$$

Note that $\text{length}\,(\gamma(\mathcal{P}))$ depends very strongly on $\mathcal{O}$. Indeed, if the path $\mathcal{P}$ contains only left-hand-turns, then $\text{length}\,(\gamma(\mathcal{P})) = 0$, and if $\gamma(\mathcal{P})$ is a path of length $r$ containing precisely one right-hand-turn, then

$$\text{length}\,(\gamma(\mathcal{P})) = 2\log\left(\frac{(1+r) + \sqrt{(1+r)^2 - 4}}{2}\right) \sim 2\log(1+r)$$

and hence grows linearly in $\log(r)$. On the other hand, if the path $\mathcal{P}$ of length $r$ consists of alternating left- and right-hand-turns, then

$$\text{length}\,(\gamma(\mathcal{P})) = r\log\left(\frac{3+\sqrt{5}}{2}\right),$$

which is linear in $r$.

We now consider the description of $S^C(\Gamma, \mathcal{O})$ in terms of $(\Gamma, \mathcal{O})$. We will carry out this description under the assumption that the cusps of $S^O(\Gamma, \mathcal{O})$ are large, i.e., they satisfy the condition in Theorem 3.3.

**Theorem 4.3.** *Assume that the cusps of $S^O(\Gamma, \mathcal{O})$ have length $\geq L = L(\varepsilon)$. Then, there exist constants $C_1, C_2, C_3, C_4$, and $C_5$ depending only on $L$ such that:*

(a) *The Cheeger constant $h(S^C(\Gamma, \mathcal{O}))$ satisfies*

$$C_1 h(\Gamma) \leq h(S^C(\Gamma, \mathcal{O})) \leq C_2 h(\Gamma).$$

(b) *The shortest closed geodesic $\text{syst}(S^C(\Gamma, \mathcal{O}))$ satisfies*

$$\text{syst}(S^C(\Gamma, \mathcal{O})) \geq C_3 \log(1 + \text{syst}(\Gamma)) \geq C_4,$$

*where $\text{syst}(\Gamma)$ is the girth of the graph $\Gamma$.*

(c) *The genus of* $S^C(\Gamma, \mathcal{O})$ *satisfies*

$$\text{genus}\,(S^C(\Gamma, \mathcal{O})) \geq C_5 \#(\Gamma).$$

*Proof.*

(a) follows from Theorem 3.3 (a) and Theorem 4.2.

(b) follows from Theorem 3.3 (b) and the calculation of lengths of geodesics on $S^O(\Gamma, \mathcal{O})$.

(c) follows from the formula for the genus of $S^O(\Gamma, \mathcal{O})$, which is also the genus of $S^C(\Gamma, \mathcal{O})$, together with the simple observation that if each cusp in $S^O(\Gamma, \mathcal{O})$ is bounded by a horocycle of length at least $L$, then the number of such cusps is bounded by $\frac{1}{L}[\text{area}\,(S^O(\Gamma, \mathcal{O}))]$, since $L$ is the area inside a horocycle of length $L$.

We will sharpen (c) considerably in Section 8.

## 5. The Bollobás model

In this section, we discuss a model, due to Bollobás, for studying the process of randomly selecting a 3-regular graph.

The problem of putting a probability measure on the set of 3-regular graphs on $2n$ vertices would not appear at first sight to be difficult, since this is a finite set. It has, however, proven problematic to find a model which is amenable to meaningful calculation, and a number of different models have been proposed and studied, each with its own benefits and drawbacks. See Janson [**17**] for a discussion and comparison of the different models.

We will use a model introduced by Bollobás ([**4**], [**5**]). Bollobás considered the problem for $k$-regular graphs, $k$ arbitrary, but we will need only the case $k = 3$. This model has the advantage that calculations of an asymptotic character (as $n \to \infty$) can be carried out with relative ease.

For each $n$, let $\mathcal{F}_n$ denote the finite set of 3-regular graphs on $2n$ vertices. We put a probability measure on $\mathcal{F}_n$ in the following way: We consider a hat with $6n$ balls, labeled by the integers $1, 2, \ldots, (2n)$, with three copies of each number occurring. We begin with a graph consisting only of vertices labeled with the integers $1, \ldots, (2n)$.

We then add edges to the graph at random by selecting pairs of balls from the hat, without replacement. If at step $i$ the integers $l_i$ and $m_i$ are selected, we add to the graph an edge joining $l_i$ and $m_i$.

We modify this picture to handle orientations in the following manner: We distinguish between the three balls with the same number by adding one of the letters $a, b$, and $c$. Thus, the balls are labeled

$1a, 1b, 1c, \ldots, (2n)a, (2n)b,\ (2n)c$. We denote the set of draws from this collection by $\mathcal{F}_n^*$.

In this way, the three edges from vertex $i$ are labeled by one of the letters $\{a, b, c\}$. We may then put a cyclic ordering on these edges by the cyclic ordering $(a, b, c)$. Thus, the probability measure on $\mathcal{F}_n^*$ gives us a probability measure on the set of oriented 3-regular graphs $(\Gamma, \mathcal{O})$ on $2n$ vertices.

We will need two results of Bollobás concerning the model $\mathcal{F}_n$. In the statement of these results, it does not matter whether or not we include loops and multiple edges, as can be seen from Theorem 5.3.

The first result concerns the Cheeger constant of a graph.

**Theorem 5.1** ([4]). *There is a constant $C > 0$ such that the probability of a graph $\Gamma$ chosen randomly from $\mathcal{F}_n$ having Cheeger constant $h(\Gamma)$ greater than $C$ satisfies*

$$\mathrm{Prob}_n[h(\Gamma) > C] \to 1 \textit{ as } n \to \infty.$$

Bollobás gives numerical estimates showing that $C > 2/11$.

To state the second result, we recall the notion of an asymptotic Poisson distribution.

**Definition 5.2.**

(a) *A random variable $X$ which takes values in the natural numbers $\mathbb{Z}^+$ is a Poisson distribution with mean $\mu$ if*

$$\mathrm{Prob}(X = k) = e^{-\mu} \frac{\mu^k}{k!}.$$

*The mean $\mu$ is the expected value of $X$.*

(b) *Let $\{X^n\}$ be a family of random variables on the probability spaces $\{P_n\}$. The $\{X^n\}$ are asymptotic Poisson distributions as $n \to \infty$ if there exists $\mu$ such that*

$$\lim_{n\to\infty} \mathrm{Prob}(X^n = k) = e^{-\mu} \frac{\mu^k}{k!}$$

*for all $k$.*

(c) *The families $\{X_i^n\}$ are asymptotically independent Poisson distributions if, for each $i$, the random variables $X_i^n$ tend to a Poisson distribution $X_i$ as $n \to \infty$, and if the variables $X_i$ are independent.*

A well-known example of an asymptotic Poisson distribution is given by the hatcheck lady who returns hats in a random fashion to the $n$ guests at a party. The random variable $X^n$ which is the number of guests who receive the correct hat is asymptotically Poisson with mean 1 as $n \to \infty$. Bollobás proves:

**Theorem 5.3** ([**5**])**.** *Let $X_i$ denote the number of closed paths in $\Gamma$ of length $i$. Then, the random variables $X_i$ on $\mathcal{F}_n$ are asymptotically independent Poisson distributions with means*

$$\lambda_i = \frac{2^i}{2i}.$$

In our case, we have an additional structure on the graph- "the orientation". We are distinguishing between short paths that agree with the orientation and those that do not. To do so, we look at all the possible orientations on a given closed path of length $i$. There are $2^i$ possible orientations, but only two yield a left-hand path. Therefore, we get the following corollary.

**Corollary 5.4.** *Let $Y_i$ be the random variable on $\mathcal{F}_n^*$ which associates to $(\Gamma, \mathcal{O})$ the number of left-hand-turn paths of length $i$. Then, the $Y_i$ are asymptotically independent Poisson distributions with means*

$$\mu_i = \frac{1}{i}.$$

Theorem 5.3 and Corollary 5.4 imply that short geodesics and small cusps will occur with positive probability in the surfaces $S^O(\Gamma, \mathcal{O})$, asymptotically as $n \to \infty$. One would expect on the grounds of asymptotic independence that as $n \to \infty$, these phenomena appear far apart. The following elementary lemma makes this expectation precise (compare [**5**], Theorem 32).

**Lemma 5.5.** *For fixed numbers $l_1, l_2$, and $d$, let $Q_n(l_1, l_2, d)$ denote the probability that a graph picked from $\mathcal{F}_n$ (resp. $\mathcal{F}_n^*$) has closed paths $\gamma_1$ and $\gamma_2$ of length $l_1$ and $l_2$ respectively, which are a distance $d$ apart. Then*

$$Q_n(l_1, l_2, d) \to 0 \text{ as } n \to \infty.$$

*Proof.* We first observe that, since the statement is independent of the orientation $\mathcal{O}$, we may restrict our attention to picking from $\mathcal{F}_n$. We will show that, for every $\varepsilon$, for $n$ sufficiently large, we have that

$$Q_n(l_1, l_2, d) < \varepsilon.$$

Since the number of closed paths of length $l_1$ are asymptotically Poisson distributed by Theorem 5.3, given $\varepsilon_1$, we may find $N(\varepsilon_1)$ such that with probability $> 1 - \varepsilon_1$, the number of closed paths of length $l_1$ is less than $N(\varepsilon_1)$.

Now, let $\gamma$ be a closed path in $\Gamma$ of length $l_1$. We consider the $l_1 \cdot 2^{d+[l_2/2]-1}$ vertices which are at distance at most $d+[l_2/2]$ from $\gamma$. When $n$ is large compared to $l_1 \cdot 2^{d+[l_2/2]}$, with probability $\to 1$ as $n \to \infty$, no vertex in this set will have been selected twice. This implies that there will be no closed path of length $l_2$ at distance $d$ from $\gamma$.

Applying this estimate to each of the $< N(\varepsilon_1)$ closed paths of length $l_1$, then gives the lemma.

## 6. Large cusps

In this section, we complete the proof of Theorem 2.1. We will reformulate it in the following way.

**Theorem 2.1A.** *Given $L$, as $n \to \infty$, we have*

$$\mathrm{Prob}_n[S^O(\Gamma, \mathcal{O}) \textit{ has cusps of length } \geq L] \to 1.$$

*Proof.* We begin the proof by calculating the probability that the canonical horocycles of $S^O(\Gamma, \mathcal{O})$ all have length $\geq L$. This is precisely the probability that all the random variables $Y_i$ of Corollary 5.4 have the value 0, for $0 < i < L$. By Corollary 5.4, this is asymptotically

$$e^{-\sum_{i=0}^{L-1} \frac{1}{i}} \sim e^{-\gamma}(L-1)^{-1},$$

where $\gamma$ is Euler's constant.

Hence, the theorem is proved if we replace the conclusion "probability $\to 1$" with the weaker conclusion "probability $\geq$ (const ) $> 0$ as $n \to \infty$." This is a sufficiently strong version of the theorem to obtain the results announced in [**10**].

We now show how to obtain the stronger results of Theorem 2.1A. To that end, suppose that the cusp $C_0$ of the surface $S^O(\Gamma, \mathcal{O})$ has a canonical horocycle of length $< L$. We would like to choose a larger horocycle about this cusp.

There are two obstructions to choosing such a larger horocycle. The first obstruction is that as we increase the length of the horocycle about $C_0$, it may cease to be injective. This will happen if there is a short closed geodesic in the neighborhood of $C_0$.

The second obstruction is that, as we increase the length of the canonical horocycle about $C_0$, we must decrease the lengths of the horocycles of nearby cusps in order to keep the interiors of the horocycles disjoint. When we decrease the length of a nearby horocycle, it may then cease to have length $\geq L$.

Both of these considerations are handled by Lemma 5.5. Indeed, both of these obstructions arise from the possibility that in the graph $\Gamma$, there may be a short closed path close to the left-hand-turn path corresponding to $C_0$. According to Lemma 5.5, the probability of this occurring is asymptotically 0.

This argument is illustrated in Figure 3. The cusp in question lies between the vertical lines $x = 0$ and $x = 2$, and the canonical horocycle, the line $y = 1$, has length 2. We increase its length to 8 by lowering the horocycle to the line $y = 1/4$. This will be possible if none of the points $x + iy$ with $0 < x < 2, y > 1/4$ are identified in the surface (the first obstruction), and if all the horocycles which meet the line $y = 1/4$ have images in the surface which are sufficiently long (the second obstruction).

This concludes the proof of the theorem.

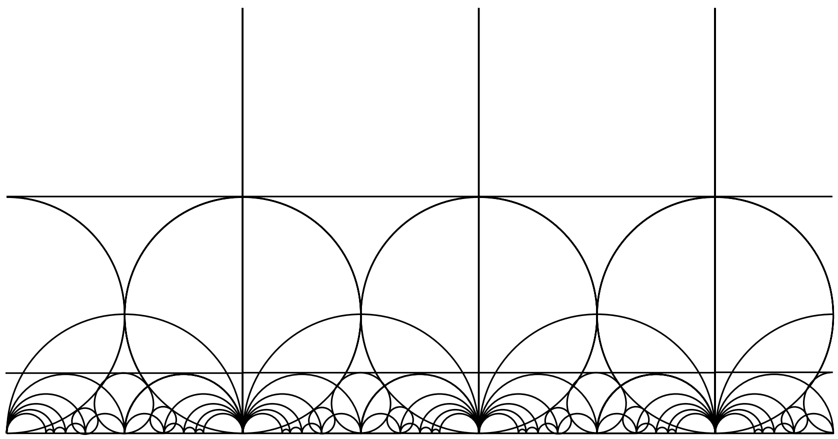

**Figure 3.** Increasing the size of horocycle.

## 7. Proof of Theorem 2.2

We now derive Theorem 2.2 from Theorem 2.1.

For a given $\varepsilon$, we set $L = L(\varepsilon)$ as in Theorem 3.2. Using Theorem 2.1, we have that, as $n \to \infty$,

$$\mathrm{Prob}_n(S^O(\Gamma, \mathcal{O}) \text{ has cusps of length } \geq L) \to 1.$$

Theorem 2.2 (b) and (c) now follow immediately from this and Theorem 4.3.

Theorem 2.2 (a) follows from Theorem 2.2 (b) and Cheeger's inequality [**13**].

It remains to establish (d).

We remark that a standard argument (see [**19**]) gives an upper bound for diameter in terms of volume and lower bounds for the Cheeger constant and injectivity radius. Thus, (d) is a formal consequence of (b) and (c).

We will give a different argument which better reflects the geometry of $S^C(\Gamma, \mathcal{O})$, and which also has the advantage that it gives better constants when the injectivity radius is small. We will show the following Lemma.

**Lemma 7.1.** *Suppose that $S^O(\Gamma, \mathcal{O})$ has cusps of length $\geq L$. Let $D_0$ denote the diameter of $\Gamma$, and $L_0$ the length of the largest canonical horocycle of $S^O(\Gamma, \mathcal{O})$.*

*Then, there exist constants $C_1$ and $C_2$, depending only on $L$, such that*

$$\operatorname{diam}(S^C(\Gamma, \mathcal{O})) \leq 2[C_1 \log(L_0) + C_2] + C_3 D_0.$$

*Proof.* Recall from [**9**] that the metric $ds^2_{\text{int}}$ on $S^C(\Gamma, \mathcal{O})$ is obtained by multiplying the hyperbolic metric $ds^2_O$ by a conformal factor in each cusp, so that in each cusp the metric is changed to a metric which is close to that of the hyperbolic ball whose perimeter is the length of the corresponding horocycle.

By Theorem 3.2, the metrics $ds^2_C$ and $ds^2_{\text{int}}$ satisfy

$$\frac{1}{(1+\varepsilon)} ds^2_C \leq ds^2_{\text{int}} \leq (1+\varepsilon) ds^2_C,$$

so that, up to a constant factor uniform in $L$, we may carry out calculations in the metric $ds^2_{\text{int}}$.

We remark that the construction of large horocycles of Theorem 2.1 involves enlarging the small canonical horocycles by at most a fixed amount while shrinking the large horocycles by at most a fixed amount. It follows that if $X$ is any point of $S^C(\Gamma, \mathcal{O})$, its distance to a point $Y$ which is a copy of the point $(1 + i\sqrt{3}/2)$ is at most $C_1 \log(L_0) + C_2$.

A path between two copies $Y_1$ and $Y_2$ of the point $(1 + i\sqrt{3}/2)$ is then given by a path along the graph, from which we conclude that the distance from $Y_1$ to $Y_2$ is bounded by $C_3 D_0$.

The lemma now follows.

To conclude the proof of Theorem 2.2 (c), we observe that the argument of [**19**] gives the upper bound

$$D_0 \leq (\text{const}) \log(\#(\Gamma)) = (\text{const}) \log(2n),$$

while trivially $L_0 \leq 4n$.

Using the upper bound

$$\operatorname{genus}(S^C(\Gamma, \mathcal{O})) \geq (\text{const}) 2n$$

from Theorem 4.3 completes the proof.

## 8. The expected genus

In this section, we will estimate the expected value

$$E_n(\text{genus}\,(S^C(\Gamma,\mathcal{O})))$$

of the surface $S^C(\Gamma,\mathcal{O}))$, where $(\Gamma,\mathcal{O})$ is randomly picked from the set $\mathcal{F}_n^*$.

We will introduce the following notation: In what follows, we will describe a random process $A$ by a sequence of choices. At each step of the process, the choice $(A)_i$ will be conditioned on the choices made up to step $(i-1)$. If $Y_i = Y(A_i)$ denotes a random variable of the $i$-th step of the process, we will denote by $E(Y_i)$ and $\mathrm{Var}\,(Y_i)$ the expectation and variance of $Y_i$, and by $E_n(Y)$ and $\mathrm{Var}\,_n(Y)$ the expected value and variance of the variable $Y = \sum_i (Y_i)$ over the entire process. Thus, we have in particular that

$$E_n(Y) = \sum_i E(Y_i).$$

We will show the following theorem.

**Theorem 2.3.** *There exist constants $C_1$ and $C_2$ such that*

$$1 + n/2 - \left[C_1 + \frac{3\log(n)}{4}\right] \leq E_n(\text{genus}\,) \leq 1 + n/2 + - \left[C_2 + \frac{\log(n)}{2}\right].$$

Recalling that $LHT((\Gamma,\mathcal{O}))$ denotes the number of closed left-hand-turn paths on $(\Gamma,\mathcal{O})$, we have from the formula

$$\text{genus}\, = 1 + \frac{n - LHT}{2},$$

this formula is equivalent to the existence of constants $C_1'$ and $C_2'$ such that

$$C_1' + \log(n) \leq E_n(LHT) \leq C_2' + (3/2)\log(n).$$

We begin our calculation by presenting the Bollobás model in a somewhat different way. We begin by considering $2n$ vertices in the plane, each with three edges of length 1/2, as shown in Figure 4. The free endpoints of the edges will be called *ends*.

We add to this picture the $2n$ curves of length 1 corresponding to the left-hand and right-hand paths leading from each end (see Figure 5). We can view these lines as the dotted horolines in the ideal hyperbolic triangle Figure 2.

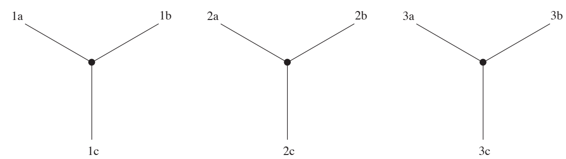


**Figure 4.** The Bollobás model revisited.

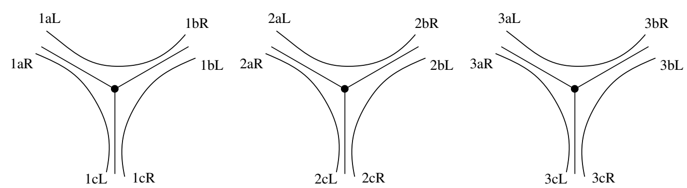


**Figure 5.** Left-hand and right-hand-turn path segments.

We now describe a process $W$, which models the drawing process for $\mathcal{F}_n^*$.

At each step $i$, we pick at random two ends not previously chosen, and glue them together as shown in Figure 6, so that the left-hand path segment from one end is glued to the right-hand path segment from the other.

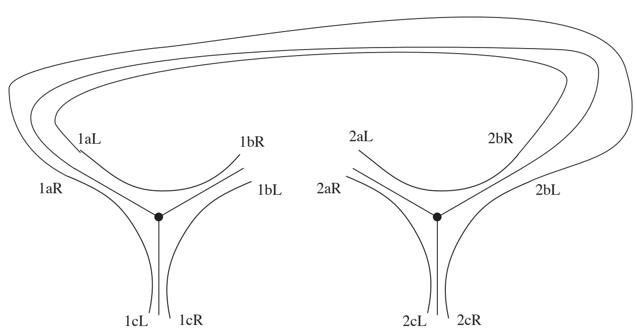


**Figure 6.** End $1a$ glued to end $2b$.

Let us calculate the expected number of closed left-hand-turn paths formed at the $i$-th step.

Suppose that the first end selected has the property that the LHT path leading from this end does not coincide with the right-hand-turn path leading from this end. Then, there are two possible choices of ends which will lead to a closed LHT path, namely the other ends of the left-hand path and right-hand path leading from the same end. These two ends may coincide, in which case, two closed LHT paths will be formed by this gluing.

Thus, we have that, in this case, the expected number of LHT paths created at the $i$-th step is $2/(6n-2i+1)$.

We will say that an end is a *bottleneck* if the left-hand-turn path leading from the end coincides with the right-hand-turn path leading from it, in other words following the LHT path from the starting end will lead us back to the same end.

If the first-chosen end is a bottleneck, then a closed LHT path will be formed from the gluing if and only if the second chosen end is also a bottleneck.

It would thus appear that the expected number $(LHT)_i$ of closed LHT paths formed at step $i$ would then depend on how many bottlenecks are present after step $i$-1. To get around this difficulty, let $X_i$ be the random variable which is the number of closed LHT paths formed at step $i$ which are not formed by gluing two bottlenecks together, and let $B_i$ be the random variable which counts the number of bottlenecks formed at step $i$.

To calculate $B_i$, let us label the first chosen end $e_1$. Following the left-hand-turn path from $e_1$, we arrive at an end $e_2$, and following the LHT path from $e_2$, we arrive at an end $e_3$. Similarly, we may follow the right-hand-turn path from $e_1$ to arrive at an end $e_2'$, and following the right-hand-turn path from $e_2'$, we arrive at $e_3'$.

If we now glue $e_1$ to $e_3$, we produce a bottleneck at $e_2$, and if we glue $e_1$ to $e_3'$, we produce a bottleneck at $e_2'$.

This process is illustrated in Figure 7, where we see that gluing end $1a$ to end $3a$ will create a bottleneck at end $2b$, while gluing end $1a$ to end $3b$ will create a bottleneck at end $1c$.

It may happen that both $e_2$ and $e_2'$ agree with $e_1$, in which case, $e_1$ is a bottleneck. In this case, gluing end $e_1$ to any end will not produce a bottleneck.

It may happen that $e_3 = e_3'$. In this case, gluing $e_1$ to $e_3$ will produce two bottlenecks, at $e_2$ and $e_2'$, respectively.

Finally, it may happen that $e_2 = e_2'$, in which case $e_3 = e_3' = e_1$, and gluing $e_1$ to any end will not create a new bottleneck. If $e_1$ is glued to a bottleneck, then this gluing extends the bottleneck, but we do not consider this as producing a new bottleneck.

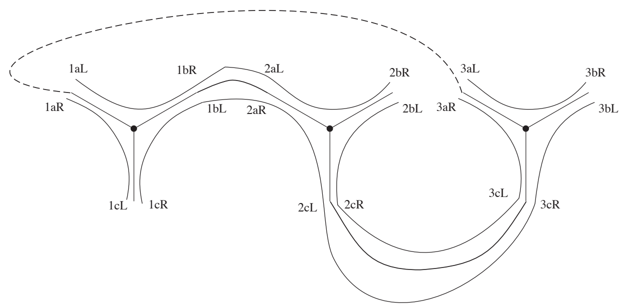


**Figure 7.** Creating a bottleneck.

Thus, in various situations, $B_i$ may take the values 0 for all choices of ends, 1 for two choices of ends, or 2 for one choice of end.

We now consider the situation where the first-chosen end is a bottleneck. Then, $X_i$ will necessarily be 0, since the only possibility for forming a closed LHT path will arise from joining two bottlenecks together. It is also easily checked that $B_i$ will also be 0.

Summarizing, we have:

(i) If the first-chosen end is not a bottleneck, then

$$E(X_i) = \frac{2}{6n - 2i + 1} \geq E(B_i).$$

(ii) If the first-chosen end is a bottleneck, then

$$E(X_i) = E(B_i) = 0.$$

(iii) In either event, we have the estimates

$$E(X_i) \leq \frac{2}{6n - 2i + 1},$$
$$E(B_i) \leq \frac{2}{6n - 2i + 1}.$$

Clearly,

$$LHT \leq \sum_i X_i + (1/2) \sum_i B_i,$$

where the inequality arises because a bottleneck created at one step may be destroyed at a later step.

We thus have that

$$E_n(LHT) \leq \sum_{i=1}^{3n} \frac{2}{6n-2i+1} + (1/2)\left(\sum_{i=1}^{3n} \frac{2}{6n-2i+1}\right)$$
$$= (3/2)\left(\sum_i \frac{1}{(1/2+(3n-i))}\right).$$

Using the upper estimate

$$\frac{1}{1/2+x} \leq \log(x+1) - \log(x), \quad x \geq 1,$$

we see that

$$E_n(LHT) \leq (3/2)[2+\log(3n)].$$

To obtain the lower bound for $E_n(LHT)$, we will define another process $W'$ as follows: We first begin by lexicographically ordering the balls $1a < 1b < 1c < 2a < \cdots < (2n)c$. At step $i$, we choose as the first end the end which has not yet been glued, which is not a bottleneck, and whose label is the lowest in the ordering with these two properties. If all the remaining ends are bottlenecks, we choose the free end which is lowest in the ordering. We then choose the remaining end randomly from among the remaining free ends, and glue them together.

We remark that the distribution of $(\Gamma, \mathcal{O})$ constructed by process $W'$ is exactly the same as in the process $W$. This can be seen as follows: Given a draw from $\mathcal{F}_n^*$, we regard this draw as an unordered matching of the balls (i.e., we disregard the order in which the pairs are drawn). Then, this unordered matching occurs precisely one way among the draws satisfying the conditions of $W'$.

If we denote by $(LHT)'_i$, the number of closed LHT paths created at the $i$-th step, and $(LHT)' = \sum_i (LHT)'_i$, it follows that

$$E(LHT)' = E(LHT(W)),$$

since the number of $LHT$ paths in the graph $(\Gamma, \mathcal{O})$ does not depend on the order in which the ends are glued. It will be the case that the expected number of bottleneck will differ, as this does depend on the order of gluing.

We now estimate $E(LHT)'$ and $E(LHT)'_i$ from below. We see that

$$E((LHT)'_i) = \frac{2}{6n-2i+1}$$

if there exist non-bottlenecks at the $i$-th step, and

$$E((LHT)'_i) = 1$$

otherwise.

Note that if, at the last step, there are no bottlenecks, then $(LHT)'_{3n}$ is 2, while if at the last step there are two bottlenecks, then $(LHT)'_{3n} = 1$. We thus have in any event the lower estimate

$$\begin{aligned} E((LHT)_i)' &\geq \frac{2}{6n-2i+1} \qquad i \neq 3n \\ &\geq 1, \qquad\qquad\qquad i = 3n. \end{aligned}$$

Thus,

$$\begin{aligned} E_n((LHT)') &= \sum_i E((LHT)'_i) \\ &\geq 1 + \sum_{i=1}^{3n-1} \frac{1}{1/2 + (3n-i)} \\ &\geq 1 - \gamma' + \log(3n), \end{aligned}$$

where $\gamma'$ is the "modified Euler's constant"

$$\gamma' = \lim_{n\to\infty} \left[ \log(3n) - \sum_{i=1}^{3n-1} \frac{1}{(1/2+i)} \right].$$

Note that it is standard that the limit exists, and the bracketed term is monotonically increasing in $n$.

Theorem 2.3 now follows from this and $E_n(LHT(W)) = E_n((LHT)')$.

We now give an alternate argument for the lower bound which does not involve the process $W'$. It does, however, produce slightly weaker constants.

Assume that at the $i$-th step in process $W$, there are $B_i$ bottlenecks. Then, the probability of choosing a bottleneck for the first end is $B_i/(6n-2i+2)$, and for then choosing a bottleneck for the second end is $(B_i - 1)/(6n-2i+1)$. Thus, the probability of producing a closed $LHT$ path at step $i$ is

$$(LHT)_i = \frac{6n-2i+2-B_i}{6n-2i+2}\frac{2}{6n-2i+1} + \frac{B_i}{6n-2i+2}\frac{B_i-1}{6n-2i+1}.$$

When $B_i = 0$ or is $> 2$, this is

$$\geq \frac{2}{6n-2i+1}.$$

When $B_i = 1$ or 2, this is

$$\frac{6n-2i+1}{6n-2i+2}\frac{2}{6n-2i+1}.$$

Now, choose $C$ less than 1. Then, we will have

$$(LHT)_i \geq C\frac{2}{6n-2i+1}$$

for all $i$ such that

$$(6n-2i) > \frac{2C-1}{1-C}.$$

This will involve only finitely many terms, independent of $n$, so we will have

$$LHT \geq C\log(3n) + C'$$

for some $C'$ independent of $n$, by the same calculation as before. We note that as $C \to 1$, we have $C' \to \infty$.

## 9. Some combinatorial considerations

It is clear from the preceeding section that the probability theory of random 3-regular oriented graphs is complicated by the appearance of special types of LHT paths such as bottlenecks. In this section, we study the ways in which such configurations occur. The results of this section will then be used in section 10.

To facilitate the discussion, we need to introduce some terminology.

**Definition 9.1.** *Let $\gamma$ be an LHT segment. Then the* circuit *passing through $\gamma$ is the sequence of LHT segments such that the first segment is $\gamma$, and every two adjacent segments share a common end.*

*A circuit is called a $k$-*circuit *if it contains precisely $k$ ends.*

Intuitively, we can describe a circuit as follows: We walk along a LHT path, until we reach an open end. At the open end, we will leap to the next LHT segment on the other side of the edge. And we continue this walk until we return to our starting point. A circuit is the combinatorial configuration of such LHT paths segments. The number $k$ represents the number of open ends in the configuration.

Thus, a 0-circuit is a closed LHT path, and a 1-circuit is a bottleneck. We will call a 2-circuit a *pipe*. A good example of a 3-circuit is one of the pieces of Figure 5.

It is easy to see that when we glue two ends from two different circuits of length $k$ and $l$ respectively, we create one new circuit of length $k+l-2$. When we glue together two ends of the same circuit of length $k$, we create two new circuits of length $l_1$ and $l_2$, where $l_1 + l_2 = k-2$.

In particular, there are only two ways to create a pipe – either a bottleneck is joined to a 3-circuit, or a $k$-circuit is joined to itself, for $k \geq 4$, to create a pipe and a $(k-4)$-circuit. We have seen above that there is only one way to create a bottleneck, namely by joining a $k$-circuit to itself, for $k \geq 3$ to create a bottleneck and a $k-3$-circuit.

We remark that, in contrast to the notion of a closed LHT path, the notion of a bottleneck or pipe is not intrinsic to the graph $(\Gamma, \mathcal{O})$ itself, but rather depends on the order in which $(\Gamma, \mathcal{O})$ is put together.

We now return to the process $W$ defined in the previous section, and recall that $B_i$ is the random variable which counts the number of bottlenecks at step $i$ of $W$.

We may similarly define $P_i$ to be the number of pipe ends (that is, twice the number of pipes) created at step $i$ of the process. As was the case with $B_i$, when a pipe is joined to another pipe, we do not count this as forming a new pipe, but rather as extending a previous pipe. Let $B = \sum_i B_i$ and $P = \sum_i P_i$.

We will show the following Lemma.

**Lemma 9.2.** *There exists constants $C_1$ and $C_2$ such that with asymptotic probability tending to 1 as $n \to \infty$,*

(i) $B \leq C_1 \log(n)$
*and*
(ii) $P \leq C_2 \log(n)$.

*Proof.* We will use the following standard consequence of Chebycheff's inequality: If $X$ is a random variable, then

$$\mathrm{Prob}\,(|X - E(X)| > \alpha E(X)) \leq \mathrm{Var}\,(X)[\alpha^2 E(X)^2]^{-1}.$$

In order to apply this inequality, we will introduce the following device, which will produce constants larger than necessary, but which has the advantage of being simple. Recall from the discussion of Theorem 2.3 that if the first-chosen end does not lie on a bottleneck, pipe, or 4-circuit, then there are two choices of ends which will give $B_i$ a value of 1, and for the remaining choices will give a value of 0. If the first-chosen end lies on a bottleneck or pipe end, then $B_i = 0$, while if the first-chosen end lies on a 4-circuit, there will be one choice of end for which $B_i$ will have the value 2.

We introduce the random variable $\widetilde{B}_i$ as follows: If the first-chosen end is not on a 1-,2-, or 4-circuit, we set

$$\widetilde{B}_i = 2B_i.$$

If the first-chosen end lies on a 4-circuit, we choose one end $x_i$ different from the end which will create a bottleneck, and set

$$\begin{aligned} \widetilde{B}_i &= 2 \quad \text{if the second-chosen end is } x_i \\ &= 2B_i \quad \text{otherwise .} \end{aligned}$$

If the first-chosen end is a pipe or bottleneck, we arbitrarily choose two ends $x_i, y_i$, and set

$$\begin{aligned} \widetilde{B}_i &= 2 \quad \text{if the second end is } x_i \text{ or } y_i \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Thus, in all cases, there are two choices of ends which will gives $\widetilde{B}_i$ the value 2, and for the remaining choices we have $\widetilde{B}_i = 0$.

Clearly, $B_i \le \widetilde{B}_i$, and hence $B \le \widetilde{B}$, since $\widetilde{B}_i$ overcounts some bottlenecks and also counts choices which do not produce bottlenecks. We have

$$E(\widetilde{B}_i) = \frac{4}{6n-2i+1},$$

and so

$$E(\widetilde{B}) = \sum_i E(\widetilde{B}_i) \sim C\log(n).$$

But $\mathrm{Var}\,(\widetilde{B}_i)$ can also be calculated easily. We will use

$$\begin{aligned}\mathrm{Var}\,(\widetilde{B}_i) &= E(\widetilde{B}_i^2) - (E(\widetilde{B}_i))^2\\ &\le E(\widetilde{B}_i^2)\\ &\le 2E(B_i),\end{aligned}$$

since the maximum value of $\widetilde{B}_i$ is 2.

But the $\widetilde{B}_i$'s are now independent, so that

$$\mathrm{Var}\,(\widetilde{B}) = \sum_i \mathrm{Var}\,(\widetilde{B}_i) \le 2\sum_i E(\widetilde{B}_i) \sim 2C\log(n).$$

From Chebycheff's inequality, we see that, for some $C'$,

$$\mathrm{Prob}\,(\widetilde{B} > C'\log(n)) < \frac{\mathrm{const}}{\log(n)} \to 0 \text{ as } n \to \infty.$$

Since $B \le \widetilde{B}$, this establishes (i).

We may proceed similarly to establish (ii). Let $P^A$ denote the number of pipe ends created by joining a bottleneck with a 3-circuit, and $P^B$ the number of pipe ends created by joining a $k$-circuit to itself, as described above.

Since a bottleneck is involved in the first method, we have the upper estimate $P^A \le B$. To estimate $P^B$, we may use the same device as before: If the first-chosen end lies on a $k$-circuit, $k > 6$, then there are two choices of ends which will produce a pipe, while for $k \le 6$, we may arbitrarily choose ends to define a variable $\widetilde{P}_i$ so that

$$P_i^B \le \widetilde{P}_i$$

and

$$\begin{aligned}\widetilde{P}_i &= 4 \quad \text{for two choices of ends}\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

From this, we conclude as before that

$$E(\widetilde{P}) \sim C^{(2)} \log(n)$$
$$\mathrm{Var}\,(\widetilde{P}) \sim C^{(3)} \log(n)$$
$$P^B \leq \widetilde{P}$$

and hence

$$\mathrm{Prob}\,(P^B > C^{(4)} \log(n)) < \frac{C^{(5)}}{\log(n)}$$

by Chebycheff's inequality.

This establishes (ii).

## 10. The largest embedded ball

If $S$ is a compact Riemann surface, denote by $\mathrm{Emb}(S)$ the area of the largest embedded ball in $S$. This is the maximum over $p \in S$ of the area of the ball at $p$ whose radius is the injectivity radius at $p$.

In this section and the next, we will show the following theorem.

**Theorem 2.4.** *There exists a constant $C > 0$ such that the expected value $E(\mathrm{Emb}\,(S^C(\Gamma, \mathcal{O})))$ satisfies*

$$E(\mathrm{Emb}\,(S^C(\Gamma, \mathcal{O}))) \geq (C - \varepsilon(n))\mathrm{area}\,(S^C(\Gamma, \mathcal{O})),$$

*where $\varepsilon(n)$ is a function which goes to $0$ as $n \to \infty$.*

We will present two calculations of $C$. The first one will yield a value of $C$ of $3/4\pi$. The second one, which is somewhat more technical, will give a value of $C$ of $1/\pi$. Neither estimate is sharp, and we believe that the correct value for $C$ is substantially larger.

In this section, we will first present the geometric considerations which reduce Theorem 2.4 to a statement about the probability theory of random oriented 3-regular graphs. We then present an outline of how we solve this probabilistic problem, by estimating it by a model problem. The calculations necessary to solve this model problem are then carried out in Section 11.

As a first step towards Theorem 2.4, we note the estimate

$$\mathrm{area}\,(S^C(\Gamma, \mathcal{O})) \leq 2\pi n$$

for $(\Gamma, \mathcal{O})$ chosen randomly from oriented 3-regular graphs on $2n$ vertices.

For $(\Gamma, \mathcal{O})$ an oriented 3-regular graph, let $\mathcal{L}(\Gamma.\mathcal{O})$ be the length of the longest left-hand-turn path in $(\Gamma, \mathcal{O})$. Then, this left-hand-turn path goes over to a closed horocycle of length $\mathcal{L}(\Gamma, \mathcal{O})$ in $S^O(\Gamma, \mathcal{O})$, which bounds a cusp neighborhood with area $\mathcal{L}(\Gamma, \mathcal{O})$.

If $C_L$ denotes the corresponding cusp, let $C_L^*$ denote the point in $S^C(\Gamma, \mathcal{O})$ which is the image of $C_L$ under the compactification $S^O(\Gamma, \mathcal{O}) \to S^C(\Gamma, \mathcal{O})$.

According to [**9**], for any $\varepsilon$, there exists a constant $L_\varepsilon$ such that if the cusps of $S^O(\Gamma, \mathcal{O})$ are of length at least $L_\varepsilon$, then $C_L^*$ will be the center of an embedded ball of volume at least $1/(1+\varepsilon)$ times the area of the horocycle neighborhood. Thus, if $S^O(\Gamma, \mathcal{O})$ has all cusps of length at least $L_\varepsilon$, then

$$\mathrm{Emb}(S^C(\Gamma, \mathcal{O})) \geq \frac{1}{1+\varepsilon} \mathcal{L}(\Gamma, \mathcal{O}).$$

Since by Theorem 2.1, with asymptotic probability 1 $S^O(\Gamma, \mathcal{O})$ will have all cusps of length at least $L_\varepsilon$, Theorem 2.4 will be a consequence of following theorem.

**Theorem 10.1.** *There is a constant $C_2$ such that the length $\mathcal{L}(\Gamma, \mathcal{O})$ of the longest left-hand-turn path of $(\Gamma, \mathcal{O})$ drawn from 3-regular graphs on $2n$ vertices satisfies*

$$E(\mathcal{L}(\Gamma, \mathcal{O})) \geq (C_2 - \varepsilon(n))n,$$

*where $\varepsilon(n) \to 0$ as $n \to \infty$.*

The constant $C$ of Theorem 2.4 and the constant $C_2$ of Theorem 10.1 are related by

$$C = C_2/(2\pi).$$

We will not in fact estimate $E(\mathcal{L}(\Gamma, \mathcal{O}))$ directly. What we will in fact estimate is the expected length of the left-hand-turn path starting from a given end. While there is no reason to expect that this LHT path is the longest, one does expect that a randomly chosen end will lie on a long LHT path rather than a shorter one, so it is not surprising that this calculation can yield an answer of the same order of magnitude as the expected value of the longest LHT path. On the other hand, it is considerably easier to calculate.

Without loss of generality, we may take as our starting end the end $1a$.

Our general strategy is to describe a process of gluing whereby at each step of this process, we extend the LHT path leading from $1a$. We will estimate the expected number of steps before the LHT path from $1a$ closes up, stopping the process. Since we add at least length 1 to the length of the LHT path from $1a$, this estimated stopping time will also be a lower estimate for the length of the closed LHT path leading from $1a$. This is in fact a great underestimate, because we expect that, after a large number of step, the length of the pieces we are adding to this LHT path will grow.

At each step of the process, we will describe below, we will call the free end at the left-hand side of the segment leading from $1a$ the *initial end*, and the free end at the right-hand side of this segment the *current end*. At the beginning of the process, the initial end is $1a$, but it can change under rare circumstances, which will be described below. The current end will change at each step of the process.

The idea is now to glue the current end to a randomly chosen end. The path will close up if this randomly chosen end is the initial end. As in Section 8, we expect complications if the current end is a bottleneck, or in other words, if the current end coincides with the initial end. In this case, the LHT path will close up on the next step if it is glued to another bottleneck, and so the probability of the process terminating grows with the number of bottlenecks. We will, therefore, have to be careful when the current end is a bottleneck.

This can occur when the current end is glued to the other end of the right-hand-turn path leading from $1a$ (which will be $1c$ at the beginning of the process).

There is an additional complication which did not arise in Section 8. If the current end is an end of a pipe, whose other end must then be the initial end, then gluing this end to a bottleneck will produce a bottleneck at the initial end. We illustrate this state of affairs in Figure 8. In this figure, the current end is $1c$, and gluing at either of the bottlenecks $4c$ or $6c$ will produce a bottleneck at the initial end $1a$.

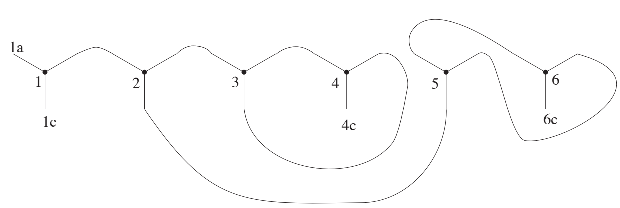


**Figure 8.** The additional complication.

To handle these problems, we will construct a process $Z$, to be defined below, such that these problems will only interfere if the number of pipes and bottlenecks is large compared to the number of remaining ends. We will then use Lemma 9.2 to say that, with probability asymptotically approaching 1 as $n \to \infty$, this will happen only very late in the process, when the length of the LHT path from $1a$ is already very large.

We will make use of the fact that if one has two processes $A$ and $B$ such that the stopping probabilities of $A$ are always less than that of $B$, then the expected stopping time of $A$ must be larger than that of $B$. With this understood, we may model the stopping time of $Z$ on that of a process $Z_{n,\delta}$ whose stopping times can be calculated. This calculation will then be carried out in Section 11.

We now describe the process $Z$. At the $i$-th step, if case (iia) below did not arise in the previous step, we pick an end at random and glue it to the current end. There are three possibilities to consider:

(i) The randomly chosen end is the initial end. In this case, performing this gluing creates a closed $LHT$ path containing the initial end, and the process stops.

(ii) A bottleneck is created at the initial end. This will occur if the randomly chosen end is the end which is joined to the initial end by the right-hand-turn path leading from the initial end. If this occurs, we draw another end. There are two possibilities:

(iia) This end is neither a bottleneck nor a pipe. In this case, the process continues at step $i + 1$ by performing this gluing.

(iib) This end is a bottleneck or a pipe. Then, the process stops. When one performs this gluing, if the randomly-chosen end is a bottleneck, then a LHT path is created which contains the initial end. If the randomly-chosen end is a pipe, then a new bottleneck is created which includes the initial end.

(iii) Neither case (i) or case (ii) above occurs. Then, we proceed to step $i + 1$ of the process.

We now calculate the probability that the process will come to an end at step $i$.

If at the end of step $i - 1$, the process ended in case (iia), then the probability of the process ending at step $i$ is 0.

If at the end of step $i - 1$, the current end is not part of a pipe, then with probability $1/(6n - 2i + 1)$, case (i) occurs, and with probability $1/(6n - 2i + 1)$, case (ii) occurs. If we let $(B + P)_i$ denote the number of bottlenecks and pipe ends after the first gluing, then the probability of case (iib) occurring at the second choice is $(B + P)_i/(6n - 2i - 1)$. Hence, the probability of the process stopping at the $i$-th step is

$$\frac{1}{6n - 2i + 1} + \left(\frac{1}{6n - 2i + 1}\right)\left(\frac{(B + P)_i}{6n - 2i - 1}\right).$$

If at the end of step $i-1$, the current end is part of a pipe, then case (ii) will occur when the randomly-chosen end is a bottleneck. This will occur with probability $(B + P)_i/(6n - 2i + 1)$. The probability that case (iib) occurs is then

$$\frac{(B+P)_i[(B+P)_i-1]}{(6n-2i+1)(6n-2i-1)}.$$

Thus, the probability of the process ending at the $i$-th step is at most

$$\frac{1}{(6n-2i+1)}+\frac{(B+P)_i^2}{6n-2i-1}.$$

Hence, in all cases, the probability of the process ending at the $i$-th step is at most

$$\frac{1}{(6n-2i+1)}\left[1+\frac{(B+P)_i^2}{6n-2i-1}\right]. \tag{1}$$

For a given $\delta$, we would like to know when the expression in equation (1) is at most

$$(Z_{n,\delta})_i=\frac{1+\delta}{(6n-2i+1)}.$$

This will happen when

$$\frac{(B+P)_i^2}{\delta}<6n-2i,$$

or, alternatively, when

$$i<3n-\frac{(B+P)_i^2}{2\delta}. \tag{2}$$

We now use the overestimate

$$(B+P)_i\le(B+P).$$

By Lemma 9.2, the estimate

$$(B+P)<(\text{const})\log(n)$$

will hold asymptotically almost surely, so that in calculating the expected stopping time of $Z$, we may use this estimate, after multiplying by a factor which goes to 1 as $n\to\infty$.

We may then rewrite Equation (2) as

$$i<(3-\varepsilon)n \quad \text{for } \varepsilon=\frac{(\text{const})^2\log^2(n)}{2n\delta}, \tag{3}$$

where the last expression clearly goes to 0 as $n\to\infty$ for fixed $\delta$.

In Section 11, we will compute the expected stopping time of the process $Z_{n,\delta}$ as well as the contribution to this expected stopping time after $(3-\varepsilon)n$ steps. This will then give a lower estimate for the stopping time of the process $Z$, completing the proof of Theorems 10.1 and 2.4.

## 11. Expected stopping times

Let $Z_{n,\delta}$ be a process whose probability of stopping at step $i$, conditioned on arriving at step $i-1$, is

$$(Z_{n,\delta})_i = \frac{1+\delta}{6n-2i+1}, \quad i \leq 3n.$$

We would first like to estimate the stopping time of $Z_{n,\delta}$. It is given by

$$\begin{aligned} E_{n,\delta} = \frac{1+\delta}{6n-1} &+ \frac{(6n-2-\delta)}{6n-1}\frac{(1+\delta)}{6n-3}\cdot 2 \\ &+ \frac{(6n-2-\delta)(6n-4-\delta)}{(6n-1)(6n-3)}\frac{1+\delta}{6n-5}\cdot 3 + \cdots \end{aligned}$$

When $\delta = 1$, the terms in the products collapse, so that

$$\begin{aligned} E_{n,1} &= \frac{2}{6n-1} + \frac{2}{6n-1}\cdot 2 + \cdots \\ &= \frac{2}{6n-1}[1+2+\cdots+3n] \\ &= \frac{2}{6n-1}\left[\frac{(3n)(3n+1)}{2}\right] \\ &\sim \frac{3n}{2} \quad \text{for } n \text{ large.} \end{aligned}$$

When $\delta = 0$, the evaluation of $E_{n,0}$ is less simple, but still elementary. We will need the following lemma.

**Lemma 11.1.**

$$\frac{1}{6n-1} + \frac{6n-2}{6n-1}\frac{1}{6n-3} + \cdots + \frac{(6n-2)(6n-4)\dots 1}{(6n-1)(6n-3)\dots 1} = 1.$$

*Proof.* This can be seen either by inductively adding the last two remaining terms

$$\frac{4}{5}\frac{1}{3} + \frac{4}{5}\frac{2}{3} = \frac{4}{5}$$

$$\frac{6}{7}\frac{1}{5} + \frac{6}{7}\frac{4}{5} = \frac{6}{7} \quad \text{etc.}$$

or by observing that this is the probability of the process stopping after $3n$ steps, which is certainty.

We then have that

$$E_{n,0} = 1 + \frac{6n-2}{6n-1} + \frac{(6n-2)(6n-4)}{(6n-1)(6n-3)} + \cdots + \frac{(6n-2)\dots 2}{(6n-1)\dots 3}.$$

Again, adding the right-hand terms inductively, using

$$\frac{2}{3}+1=\frac{5}{3}, \quad \frac{4}{3}+1=\frac{7}{3}, \quad \text{etc.}$$

we see that

$$E_{n,0}=1+\frac{6n-2}{3}=\frac{6n+1}{3}\sim 2n.$$

It is evident that $E_{n,\delta}$ increases as $\delta$ decreases from 1 to 0. We do not have a closed formula for $E_{n,\delta}$ in general, but we will not need it in what follows.

We now consider what happens when we stop the process after $(3-\varepsilon)$ steps. In terms of the problem at hand, this amounts to estimating the stopping time of a process $Z_{n,\delta,\varepsilon}$ for which the estimate $(1+\delta)/(6n-2i+1)$ becomes unreliable after $(3-\varepsilon)$ steps. We would like to say that the contributions from the last $\varepsilon n$ steps is essentially negligible.

Denoting this by $E_{n,\delta,\varepsilon}$, we have that $E_{n,\delta,\varepsilon}$ is clearly greater than the contribution to $E_{n,\delta}$ after the first $(3-\varepsilon)n$ terms.

When $\delta=1$, we obtain that this is at least

$$\begin{aligned}&\frac{2}{6n-1}[1+\cdots+(3-\varepsilon)n]\\ &\quad=\left(\frac{2}{6n-1}\right)\frac{(3-\varepsilon)n[(3-\varepsilon)n+1]}{2}\\ &\quad\sim\frac{(3-2\varepsilon)}{2}n.\end{aligned}$$

When $\delta=0$, we will proceed as follows: We will estimate the contribution from the last $\varepsilon n$ terms, and then subtract this from the total. We will overestimate the contribution from the last $\varepsilon n$ terms by replacing the multiplication by $i$ with multiplication by $3n$.

We, thus, have an overestimate of the last $\varepsilon n$ terms by

$$\begin{aligned}&(3n)\left(\frac{6n-2}{6n-1}\right)\cdots\left(\frac{6n-2((3-\varepsilon)n)+2}{6n-2((3-\varepsilon)n+3)}\right)\\ &\quad=(3n)\left[\left(\frac{6n-2}{6n-1}\right)\cdots\left(\frac{2\varepsilon n+2}{2\varepsilon n+3}\right)\right].\end{aligned}$$

It will be convenient to set $l = 3n - 1$. We then have that

$$\left(\frac{6n-2}{6n-1}\right)\dots\left(\frac{2\varepsilon n+2}{2\varepsilon n+3}\right) = \frac{(2l)(2l-2)\dots(2\varepsilon n+2)}{(2l+1)\dots(2\varepsilon n+3)}$$
$$= \frac{[(2l)\dots(2\varepsilon n+2)]^2}{(2l+1)(2l)\dots(2\varepsilon n+2)}$$
$$= \left[\frac{2^{2l}(l!)^2}{(2l+1)!}\right]\left[\frac{2^{2\varepsilon n}((\varepsilon n)!)^2}{(2\varepsilon n+1)!}\right]^{-1}. \tag{4}$$

We may now estimate this last expression by Stirling's formula, which we will use in the following form.

**Lemma 11.2** (Stirling's formula [**15**], [**23**])**.**

$$\sqrt{2\pi}n^{n+1/2}e^{-n+\frac{1}{(12n+1)}} < n! < \sqrt{2\pi}n^{n+1/2}e^{-n+\frac{1}{12n}}.$$

Substituting in our expression, and noting that the terms involving $e^{1/12n}$ and $e^{1/(12n+1)}$ contribute constants which go to 1 as $n \to \infty$, we get

$$\frac{2^{2l}(l!)^2}{(2l+1)!} \sim \frac{1}{2l+1}\left[\frac{2^{2l}2\pi l^{2l+1}e^{-2l}}{\sqrt{2\pi}(2l)^{2l+1/2}e^{-2l}}\right]$$
$$= \frac{1}{(2l+1)}\sqrt{\pi}l^{1/2},$$

and similarly for the terms involving $\varepsilon n$, so that the expression (4) is

$$\sim \frac{l^{1/2}}{(\varepsilon n)^{1/2}}\frac{(\varepsilon n+1)}{(2l+1)} \sim (\text{const}\,)\sqrt{\varepsilon}.$$

Hence, the error term is bounded by

$$(\text{const}\,)\sqrt{\varepsilon}n.$$

We then have that

$$E_{n,0,\varepsilon} > (2 - \text{const}\,\sqrt{\varepsilon})n.$$

We now observe that we may estimate the last $\varepsilon n$ terms for arbitrary $\delta$ by estimating the last $\varepsilon n$ terms when $\delta = 0$, so that in general

$$E_{n,\delta,\varepsilon} > E_{n,\delta} - (\text{const}\,)\sqrt{\varepsilon}n.$$

We may now establish Theorem 10.1 with the constant $C' = 2$, by letting $n$ get large for fixed $\delta$, and then letting $\delta \to 0$.

We remark that we may evaluate the limit distribution, as $n \to \infty$, of the processes $E_{n,\delta}$, for $0 \le \delta \le 1$. Applying Stirling's Formula to the terms

$$\begin{aligned}(1+\delta)&\frac{(6n-2-\delta)(6n-4-\delta)\dots(6n-2i-\delta+2)}{(6n-1)(6n-3)\dots(6n-2i+1)}\\ &=\frac{1+\delta}{6n-1}\left[\frac{(3n-1-\delta/2)\dots(3n-i-\delta/2+1)}{(3n-1/2-1)\dots(3n-i+1/2)}\right]\\ &=\frac{1+\delta}{6n-1}\left[\frac{\Gamma(3n-\delta/2)\Gamma(3n+1/2-i)}{\Gamma(3n-i-\delta/2+1)\Gamma(3n-1/2)}\right],\end{aligned}$$

we see that the limit distribution is given by the function

$$f_\delta(x)=(1+\delta)(1-2x)^{(\delta/2-1/2)},\quad 0\le x\le 1/2,$$

from which we may easily compute the limit expected value and variance as

$$E(f_\delta)=\frac{1}{\delta+3}$$

and

$$\mathrm{Var}\,(f_\delta)=\frac{\delta+1}{(\delta+3)^2(\delta+5)}.$$

Indeed, for $\delta<1$, the distribution is weighted toward the higher end of $[0,1/2]$, so that for $\delta=0$, the median is $3/8$, and with probability $1/3$ the value of $f_0$ is at least $4/9$.

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After writing this paper the first author Robert Brooks passed away tragically. I dedicate this paper to him. It was a great privilege to have him as a mentor and friend. I tremendously miss his advice and enthusiasm.

## References

[1] L.V. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **43**(**3**) (1938) 359–364, MR 1501949, Zbl 0018.41002.

[2] R. Arratia, A.D. Barbour, & S. Tavaré, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2003, MR 2032426.

[3] G.V. Belyĭ, *Galois extensions of a maximal cyclotomic field*, Izv. Akad. Nauk SSSR Ser. Mat. **43**(**2**) (1979) 267–276, 479, MR 0534593, Zbl 0409.12012.

[4] B. Bollobás, *The isoperimetric number of random regular graphs*, European J. Combin. **9**(**3**) (1988) 241–244, MR 0947025, Zbl 0673.05086.

[5] B. Bollobás, *Random Graphs*, second ed., Cambridge University Press, Cambridge, 2001, MR 1864966, Zbl 0979.05003.

[6] R. Brooks, *The spectral geometry of a tower of coverings*, J. Differential Geom. **23**(**1**) (1986) 97–107, MR 0840402, Zbl 0576.58033.

[7] R. Brooks, *Some remarks on volume and diameter of Riemannian manifolds*, J. Differential Geom. **27**(**1**) (1988) 81–86, MR 0918458, Zbl 0637.53059.

[8] R. Brooks, *Some relations between spectral geometry and number theory*, Topology '90 (Columbus, OH, 1990) (Berlin), de Gruyter, Berlin, 1992, 61–75, MR 1184403, Zbl 0764.58031.

[9] R. Brooks, *Platonic surfaces*, Comment. Math. Helv. **74**(**1**) (1999) 156–170, MR 1677565, Zbl 0920.30037.

[10] R. Brooks & E. Makover, *The first eigenvalue of a Riemann surface*, Electron. Res. Announc. Amer. Math. Soc. **5** (1999) 76–81 (electronic), MR 1696823, Zbl 0948.58003.

[11] P. Buser, *Cubic graphs and the first eigenvalue of a Riemann surface*, Math. Z. **162**(**1**) (1978) 87–99, MR 0505920, Zbl 0379.53024.

[12] P. Buser, *On the bipartition of graphs*, Discrete Appl. Math. **9**(**1**) (1984) 105–109, MR 0754431, Zbl 0544.05038.

[13] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis (Papers dedicated to Salomon Bochner, 1969) (Princeton, N. J.), Princeton University Press, Princeton, NJ, 1970, 195–199, MR 0402831, Zbl 0212.44903.

[14] S.Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. **143**(**3**) (1975) 289–297, MR 0378001, Zbl 0329.53035.

[15] W. Feller, *An introduction to probability theory and its applications*, Vol. I, John Wiley & Sons Inc., New York, 1968, MR 0228020, Zbl 0155.23101.

[16] A. Gamburd & E. Makover, *On the genus of a random Riemann surface*, Complex manifolds and hyperbolic geometry (Guanajuato, 2001) (Providence, RI), Contemp. Math., **311**, Amer. Math. Soc., Providence, RI, 2002, 133–140, MR 1940168.

[17] S. Janson, T. Łuczak, & A. Rucinski, *Random graphs*, Wiley-Interscience, New York, 2000, MR 1782847, Zbl 0968.05003.

[18] G. Jones & D. Singerman, *Belyĭ functions, hypermaps and Galois groups*, Bull. London Math. Soc. **28**(**6**) (1996) 561–590, MR 1405488, Zbl 0853.14017.

[19] A. Lubotzky, *Discrete Groups, Expanding Graphs, and Invariant Measures*, Birkhäuser Verlag, Basel, 1994, with an appendix by Jonathan D. Rogawski, MR 1308046, Zbl 0826.22012.

[20] D. Mangoubi, *Conformal extension of metrics of negative curvature*, J. Anal. Math. **91** (2003) 193–209, MR 2037407.

[21] D. Mangoubi, *Riemann surfaces and* 3*-regular graphs*, Master's thesis, Technion, February 2001.

[22] E. Mayer-Wolf, O. Zeitouni, & M. Zerner, *Asymptotics of certain coagulation-fragmentation processes and invariant Poisson-Dirichlet measures*, Elec. J. Prob. **7** (2002) 1–25, MR 1902841, Zbl 1007.60100.

[23] E.W. Weisstein, *Stirling's approximation*, http://mathworld.wolfram.com/StirlingsApproximation.html.

[24] N.V. Tsilevich, *Stationary random partitions of a natural series*, Teor. Veroyatnost. i Primenen. **44**(**1**) (1999) 55–73, MR 1751188, Zbl 0960.60012.

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