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EXISTENCE OF DIFFUSION ORBITS IN *A PRIORI* UNSTABLE HAMILTONIAN SYSTEMS

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Abstract

Under open and dense conditions we show that Arnold diffusion orbits exist in *a priori* unstable and time-periodic Hamiltonian systems with two degrees of freedom.

1. Introduction and results

By the Kolmogorov, Arnold and Moser (KAM) theory we know that there are many invariant tori in nearly integrable Hamiltonian systems with arbitrary n degrees of freedom. These tori are of n dimensions and occupy a nearly full Lebesgue measure set in the phase space. As an important consequence, all orbits are stable in autonomous system with two degrees of freedom, or time-periodic system with one degree of freedom, in the sense that the actions do not change much along the orbits. However, the KAM theory does not guarantee such stability when the system has three or more degrees of freedom for the autonomous case or when it has two or more degrees of freedom for the time-periodic case, simply because the KAM torus cannot separate the phase space (or integral manifold) into two disconnected parts.

In his celebrated paper [**1**], Arnold constructed an example of nearly integrable Hamiltonian system, where some orbits are unstable. His example is a time periodic system with two degrees of freedom. In Arnold's example, the perturbations are chosen so specifically that all hyperbolic invariant tori preserve in the perturbed system. Hence, one can use the so called Melnikov method to construct transition chain along which the action has substantial variation. However, in a generic case, the perturbed systems do not possess such a good property; some resonant gaps between invariant tori break up the transition

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chain. Thus, it seems unclear whether one can apply Arnold's method to find diffusion orbits. Despite this technical difficulty, Arnold asked whether there is such a phenomenon for a "typical" small perturbation. After nearly four decades of study, some remarkable generalizations of Arnold's result have been announced [**8**],[**16**],[**20**]. A few years ago, Xia [**20**] announced that Arnold diffusion exists in generic *a priori* unstable systems. Recently, Mather announced [**16**] that under so-called cusp residual condition, Arnold diffusion exists in *a priori* stable systems with two degrees of freedom in time-periodic case, or with three degrees of freedom in autonomous case. They claim that diffusion orbits can be constructed by variational method. Using geometrical method, some demonstration was provided in [**9**] to show that diffusion orbits exist in some types of *a priori* unstable and time-periodic Hamiltonian systems with two degrees of freedom.

In this paper, we study generic perturbations of *a priori* unstable Hamiltonian systems which have two degrees of freedom and are time-periodic, and give a complete proof of the existence of diffusion orbits by using variational method. The approach of our proof is different from the approaches proposed by Mather and by Xia (cf. [**16**] and [**20**]). The starting point of our proof is based on the previous work of Mather ([**14**], [**15**]). With his deep insight, Mather developed a new variational method to study Hamiltonian dynamics in higher dimensions. In [**14**], Mather established the variational set-up of time-dependent positive definite Lagrangian systems and showed the existence of minimal measures. By exploiting the properties of barrier functions in [**15**], he introduced the idea of C-equivalence and pointed out a possible way to construct connecting orbits. However, the difficulty in applying this method to interesting problems in higher dimensions is that we do not know the structures of related c-minimal orbit sets. In this paper, we have succeeded in getting sufficient information about the topological structure of the relevant Mañé sets and in providing the proof of a theorem of connecting C-equivalent Mañé sets formulated by Mather in [**15**]. Consequently, we are able to construct the diffusion orbits crossing the gaps. However, it appears unclear whether such C-equivalence can be established at the place where uncountably many whiskered tori cluster together. Fortunately, this is the place where there is no big gap. Arnold's mechanism can be used here because a transition chain of whisker tori clearly exists in this case. Crucially relying on such geometric structure, we are able to establish local variational principle (cf.

[5], [6]), the local minimum corresponds to some diffusion orbits crossing these whisker tori. It is the variational version of Arnold's mechanism. Another step in our proof is to show that we can join the orbits constructed by C-equivalence smoothly with the orbits which realize the minimum of the local variational principle. In this way, we do find some diffusion orbits in generic systems.

Given a Hamiltonian function $H(p,q,t)$ the Hamiltonian equation has the form:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{1.1}$$

The Hamiltonian function studied here has the following form:

$$H(p,q,t) = f(p_1) + g(p_2,q_2) + P(p,q,t), \tag{1.2}$$

where $p = (p_1,p_2) \in \mathbb{R}^2$, $q = (q_1,q_2) \in \mathbb{T}^2$, $H \in C^r$ $(r \geq 3)$, P is a time-1-periodic small perturbation. We assume it satisfies the following conditions:

1. $f+g$ is a convex function in p i.e., the Hessian matrix $\partial_{pp}(f+g)$ is positive definite, finite everywhere and has superlinear growth in p, $(f+g)/\|p\| \to \infty$ as $\|p\| \to \infty$,
2. it is *a priori* unstable in the sense that g has non-degenerate saddle critical point, i.e., $\partial_{p_2q_2}g^2 - \partial_{p_2p_2}g\partial_{q_2q_2}g > 0$ at (p_2^*, q_2^*). The function $g(p_2^*, q_2) : \mathbb{T} \to \mathbb{R}$ attains its maximum at q_2^*: $g(p_2^*, q_2^*) = \max_{q_2} g(p_2^*, q_2)$. Without loss of generality, we assume $(p_2^*, q_2^*) = 0$.

Let $\mathcal{B}_{\epsilon,K}$ denote a ball in the function space $C^r(\{(p,q) \in \mathbb{T}^2 \times \mathbb{R}^2 : \|p\| \leq K\} \to \mathbb{R})$, centered at the origin with radius of ϵ. Now, we can state the theorem which was formulated by Arnold in [1].

Theorem 1.1. *Let $A < B$ be two arbitrarily given numbers and assume H satisfies the above two conditions. There exists a small number $\epsilon > 0$, a large number $K > 0$ and an open and dense set $\mathcal{S}_{\epsilon,K} \subset \mathcal{B}_{\epsilon,K}$ such that for each $P \in \mathcal{S}_{\epsilon,K}$ there exists an orbit of the Hamiltonian flow which connects the region with $p_1 < A$ to the region with $p_1 > B$.*

We shall use variational argument to complete the proof. In Section 2, by using Legendre transformation we follow Mather's work [15] and put this problem into the Lagrangian formalism. The diffusion orbits are found by searching for the minimal action of the Lagrangian. Some properties such as upper semi-continuity of some set-valued functions are also proved in this section. In Section 3, we investigate the topological structure of some relevant Mañé sets. Section 4 is devoted to the study of the barrier function when the Aubry set contains a codimension one torus. In Section 5, by making use of the semi-continuity property shown in Section 2, we obtain the proof of a theorem of connecting C-equivalent Mañé sets, formulated by Mather in [**15**]. Based on the understanding of the topological structure of the relevant Mañé sets shown in Section 3, we establish the C-equivalence among those relevant Mañé sets and use this C-equivalence to construct the diffusion orbits crossing resonant gaps. In virtue of the techniques developed in [**6**] and the analytic expression of the barrier function obtained in Section 4, we join the orbits constructed by C-equivalence smoothly with the orbits constructed via transition chain. Thus, we obtain the diffusion orbits. In Section 6, we show the open and dense property.

2. Variational set-up

Roughly speaking, the diffusion orbits are constructed by connecting different c-minimal orbit sets, along which the Lagrange action takes its minimum. Therefore, we shall study the Lagrangian equation equivalent to the Hamiltonian equation (1.1):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0, \tag{2.1}$$

where the Lagrangian function $L(\dot q, q, t)$ is obtained from the Hamiltonian function (1.1) by using Legendre transformation $\mathcal{L}$: $(p, q, t) \to (\dot q, q, t)$ such that

$$L(\dot q, q, t) = \max_p \{\langle p, \dot q\rangle - H(p, q, t)\}. \tag{2.2}$$

Here, $\dot q = \dot q(p, q, t)$ is implicitly determined by $\dot q = \frac{\partial H}{\partial p}$. Since we study a nearly integrable system, the Lagrangian has the form of

$$L = L_0(q_2, \dot q) + L_1(q, \dot q, t),$$

where L_0 corresponds to $f + g$ through the Legendre transformation.

Throughout this paper, we use ϕ^t to denote the Euler–Lagrange flow determined by L, use Φ^t to denote the Hamiltonian flow determined by H. To specify the Euler–Lagrange (Hamiltonian) flow determined by other functions, we add the subscript, e.g. $\phi^t_{L_0}$, Φ^t_{f+g}, etc.

Clearly, Equation (2.1) corresponds to the critical point of the functional

$$A(\gamma) = \int L(\gamma, \dot{\gamma}, t)dt.$$

We can think that L is a function defined on $TM \times \mathbb{T}$ where $M = \mathbb{T}^2$. As $f + g$ is an integrable system and H is its small perturbation, every solution of H is well defined for $t \in \mathbb{R}$. By the assumptions on H, we see that L satisfies the following conditions introduced by Mather [**14**].

Positive definiteness. For every $(q, t) \in M \times \mathbb{T}$, the Lagrangian function is strictly convex in velocity: the Hessian $L_{\dot{q}\dot{q}}$ is positive definite.

Superlinear growth. We suppose that L has fiber-wise superlinear growth: for every $(q, t) \in M \times \mathbb{T}$, we have $L/\|\dot{q}\| \to \infty$ as $\|\dot{q}\| \to \infty$.

Completeness. All solutions of the Lagrange equations are well defined for all $t \in \mathbb{R}$.

Under these conditions, Mather established the theory of c-minimal measure and c-minimal orbits [**14**, **15**]. To introduce some basic results of Mather, let us observe the fact that the functional $\int Ldt$ has the same critical point as $\int (L - \eta_c)dt$ does if η_c is a closed 1-form on $M \times \mathbb{T}$, whose first de Rham co-homology class is c, i.e., $[\eta_c] = c$, in other words, their Lagrange equations are the same.

Let $I = [a, b]$ be a compact interval of time. A curve $\gamma \in C^1(I, M)$ is called a c-minimizer or a c-minimal curve if it minimizes the action among all curves $\xi \in C^1(I, M)$ which satisfy the same boundary conditions:

$$A_c(\gamma) = \min_{\substack{\xi(a)=\gamma(a)\\ \xi(b)=\gamma(b)}} \int_a^b (L - \eta_c)(d\xi(t), t)dt. \tag{2.3}$$

As we have the condition of completeness, the minimizer must be a C^1-curve by Tonelli's theorem. Without the completeness, the minimizer can fail to be [**2**]. If J is a non-compact interval, the curve $\gamma \in C^1(J, M)$ is said a c-minimizer if $\gamma|_I$ is c-minimal for any compact interval $I \subset J$. An orbit $X(t)$ of ϕ^t is called c-minimizing if the curve $\pi \circ X$ is c-minimizing, where the operator π is the standard projection from tangent bundle to the underlying manifold along the fibers; a point $(z, s) \in TM \times \mathbb{R}$ is c-minimizing if its orbit $\phi^t(z, s)$ is c-minimizing. We use $\tilde{\mathcal{G}}_L(c) \subset TM \times \mathbb{R}$ to denote the set of minimal orbits of $L - \eta_c$ (the

c-minimal orbits of L). We shall drop the subscript L when it is clear which Lagrangian is under consideration. It is not necessary to assume the periodicity of L in t for the definition of $\tilde{\mathcal{G}}$. When it is periodic in t, $\tilde{\mathcal{G}}(c) \subset TM \times \mathbb{R}$ is a non-empty compact subset of $TM \times \mathbb{T}$, invariant for the Euler–Lagrange flow ϕ^t.

We can extend the definition of action along a C^1-curve to the action on a probability measure. Let $\mathfrak{M}$ be the set of Borel probability measures on $TM \times \mathbb{T}$. For each $\nu \in \mathfrak{M}$, the action $A_c(\nu)$ is defined as the following:

$$A_c(\nu) = \int (L - \eta_c) d\nu. \tag{2.4}$$

Mather has proved [**14**] that for each first de Rham cohomology class c, there is a probability measure μ which minimizes the actions over $\mathfrak{M}$

$$A_c(\mu) = \inf_{\nu \in \mathfrak{M}} \int (L - \eta_c) d\nu.$$

This μ is invariant to the Euler–Lagrange flow. We use $\tilde{\mathcal{M}}(c)$ to denote the closure of the union of the support of all such measures, use $-\alpha(c) = A_c(\mu)$ to denote the minimum c-action. It defines a function α: $H^1(M, \mathbb{R}) \to \mathbb{R}$, usually called α-function. Its Legendre transformation β: $H_1(M, \mathbb{R}) \to \mathbb{R}$ is usually called β-function. Both functions are convex, finite everywhere and have super-linear growth [**14**]. As $\tilde{\mathcal{M}}(c)$ is defined as the limit measure of c-minimal orbits, the following lemma is a straightforward result of topological dynamics:

Lemma 2.1. *For each co-homological class c and each positive number ϵ, there exists a positive number $T_0 = T_0(c, \epsilon)$, such that if $T \geq T_0$ and γ: $[0, T] \to M \times \mathbb{T}$ is a curve minimizing the action of $L - \eta_c$, $[\eta_c] = c$, then there is $t \in [0, T]$ such that*

$$d(d\gamma(t), \tilde{\mathcal{M}}(c)) \leq \epsilon.$$

Before starting the existence proof of diffusion orbits, we need to introduce some more concepts and investigate some relevant properties, which shall be made use of below for our purpose.

We have defined the sets $\tilde{\mathcal{M}}(c)$ and $\tilde{\mathcal{G}}(c)$. It is easy to see that $\tilde{\mathcal{M}}(c)$ is contained in the set $\tilde{\mathcal{G}}(c)$. Between the set $\tilde{\mathcal{G}}$ and set $\tilde{\mathcal{M}}$, we can also define so-called Aubry set $\tilde{\mathcal{A}}(c)$ and Mañé set $\tilde{\mathcal{N}}(c)$ as well as the limit point set $\tilde{\mathcal{L}}(c)$.

As all orbits are well defined on the whole $\mathbb{R}$, they have ω-limit sets and α-limit sets. Let $\tilde{\omega}(c)$ be the union of ω-limit points of c-minimal orbits $X(t):[0,\infty)\to TM\times\mathbb{T}$, let $\tilde{\alpha}(c)$ be the union of α-limit points of c-minimal orbits $X(t):(-\infty,0]\to TM\times\mathbb{T}$. We call $\tilde{L}(c)=\tilde{\omega}(c)\cup\tilde{\alpha}(c)$ the limit set.

To define the Aubry set and the Mañé set, let us define

$$(2.5)\quad h_c(x,x',t,t')=\min_{\substack{\gamma\in C^1([t,t'],M)\\ \gamma(t)=x,\gamma(t')=x'}}\int_t^{t'}(L-\eta_c)(d\gamma(s),s)ds+(t'-t)\alpha(c),$$

$$F_c(x,x',s,s')=\inf_{\substack{s=t \bmod 1\\ s'=t' \bmod 1\\ t'\geq t+1}} h_c(x,x',t,t').$$

$$(2.6)\qquad h_c(x,x')=h_c(x,x',0,1),\qquad F_c(x,x')=F_c(x,x',0,0).$$

Let

$$h_c^n(x,x')=\min\left\{\sum_{i=0}^{n-1}h_c(m_i,m_{i+1}):m_0=x,\ m_n=x'\right.$$
$$\left.\text{and } m_i\in M \text{ for } 0\leq i\leq n\right\}$$

and let

$$(2.7)\qquad h_c^\infty(x,x')=\liminf_{n\to\infty}h_c^n(x,x'),$$

$$(2.8)\qquad d_c(x,x')=h_c^\infty(x,x')+h_c^\infty(x',x).$$

Mather showed in [**15**] that d_c is a pseudo-metric on the set $\{x\in M:h_c^\infty(x,x)=0\}$. A curve $\gamma\in C^1(\mathbb{R},M)$ is called c-semi-static if

$$A_c(\gamma|_{[a,b]})+\alpha(c)(b-a)=F_c(\gamma(a),\gamma(b),a \bmod 1,b \bmod 1)$$

for each $[a,b]\subset\mathbb{R}$. A curve $\gamma\in C^1(\mathbb{R},M)$ is called c-static if, in addition

$$A_c(\gamma|_{[a,b]})+\alpha(c)(b-a)=-F_c(\gamma(b),\gamma(a),b \bmod 1,a \bmod 1)$$

for each $[a,b]\subset\mathbb{R}$. An orbit $X(t)=(d\gamma(t),t \bmod 1)$ is called static (semi-static) if γ is static (semi-static). We call the Mañé set $\tilde{\mathcal{N}}(c)$ the union of global c-semi-static orbits, the set $\tilde{\mathcal{A}}(c)$ is defined as the union of global c-static orbits, we call it Aubry set.

We use $\mathcal{M}(c)$, $\mathcal{L}(c)$, $\mathcal{A}(c)$, $\mathcal{N}(c)$ and $\mathcal{G}(c)$ to denote the standard projection of $\tilde{\mathcal{M}}(c)$, $\tilde{\mathcal{L}}(c)$, $\tilde{\mathcal{A}}(c)$, $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{G}}(c)$ from $TM\times\mathbb{T}$ to $M\times\mathbb{T}$ respectively. We have the following inclusions [**4**].

$$\tilde{\mathcal{M}}(c)\subseteq\tilde{\mathcal{L}}(c)\subseteq\tilde{\mathcal{A}}(c)\subseteq\tilde{\mathcal{N}}(c)\subseteq\tilde{\mathcal{G}}(c). \tag{2.9}$$

The set $\tilde{\mathcal{G}}(c)$ and $\tilde{\mathcal{N}}(c)$ have the good property of upper semi-continuity in c. Restricted on $\mathcal{A}(c)$, the map $\pi^{-1}:\mathcal{A}(c)\to\tilde{\mathcal{A}}(c)$ is Lipschitz. We use $\tilde{\mathcal{N}}_s(c)=\tilde{\mathcal{N}}(c)|_{t=s}$ to denote the time section, and so on.

When necessary, we use the symbols $\tilde{\mathcal{G}}_L(c)$, $\tilde{\mathcal{N}}_L(c)$, $\tilde{\mathcal{A}}_L(c)$ and $\tilde{\mathcal{M}}_L(c)$ to denote the minimal orbit set, Mañé sets, Aubry set and Mather set determined by some Lagrangian L respectively, omitting the subscript L when the Lagrangian is clearly defined.

To describe these minimal orbit sets, Mather introduced two kinds of barrier functions B_c and B_c^*, it is defined as follows

$$B_c(q)=h_c^\infty(q,q)$$

$$B_c^*(q)=\min\{h_c^\infty(\xi,q)+h_c^\infty(q,\eta)-h_c^\infty(\xi,\eta):\forall\ \xi,\eta\in\mathcal{M}_0(c)\}. \tag{2.10}$$

Clearly, we have $0\le B_c^*\le B_c$. When $d_c(\xi,\eta)=0$ for all $\xi,\eta\in\mathcal{M}_0(c)$, then $B_c=B_c^*$ [**15**]. It is not hard to see that $\mathcal{A}_0(c)=\{x\in M:B_c(x)=0\}$. The following lemma is a modified version of the Proposition 2.1 in [**4**].

Lemma 2.2. *Let M be a compact, connected Riemannian manifold. Assume $L\in C^r(TM\times\mathbb{R},\mathbb{R})$ $(r\ge 2)$ satisfies the positive definite, superlinear-growth and completeness conditions. Considered as the function of t, L is assumed periodic for $t\in(-\infty,0]$ and for $t\in[1,\infty)$. Then, the map $L\to\tilde{\mathcal{G}}_L\subset TM\times\mathbb{R}$ is upper semi-continuous. As an, immediate consequence, $\tilde{\mathcal{G}}(c)$ is a non-empty compact set in $TM\times\mathbb{T}$ and the map $c\to\tilde{\mathcal{G}}(c)$ is upper semi-continuous if L is periodic in t.*

We can consider t is defined on $(\mathbb{T}\vee[0,1]\vee\mathbb{T})/\sim$, where $\sim$ is defined by identifying $\{0\}\in[0,1]$ with some point on one circle, and identifying $\{1\}\in[0,1]$ with some point on another circle. Let $U_k=\{(\zeta,q,t):(q,t)\in M\times(\mathbb{T}\vee[0,1]\vee\mathbb{T})/\sim,\ \|\zeta\|\le k\}$, $\cup_{k=1}^\infty U_k=TM\times\mathbb{R}$. Let $L_i\in C^r(TM\times\mathbb{T},\mathbb{R})$. We say L_i converges to L if for each $\epsilon>0$ and each U_k, there exists i_0 such that $\|L-L_i\|_{U_k}\le\epsilon$ if $i\ge i_0$.

Proof. Since M is connected and compact, any two point x_1, $x_2\in M$ can be connected by a geodesic. Let $\ell(x_1,x_2)$ be the length of the shortest geodesic connecting these two points, there is an upper bound $K_1>0$ of $\ell(x_1,x_2)$ uniformly for all $x_1,\ x_2\in M$. Let

$$K=\max_{\substack{(q,t)\in\mathbb{T}^2\times(\mathbb{T}\vee[0,1]\vee\mathbb{T})/\sim\\ \|\zeta\|\le K_1}} L(q,\zeta,t).$$

Given time interval $[a,b]$ with $b-a\geq 1$, if we reparametrize the shortest geodesic $\gamma(s)$ by $\bar\gamma(t)=\gamma(\ell(x_1,x_2)(t-a)/(b-a))$, then $\bar\gamma(t)$ is a C^1-curve such that $\bar\gamma(a)=x_1,\ \bar\gamma(b)=x_2$. Clearly, the action of L along this curve is not bigger than $K(b-a)$. Obviously, there is an upper bound uniformly for all minimizing action of L' if they close to L on $\{\|\zeta\|\le K_1\}$, still denoted by $K(b-a)$.

Since the super-linear growth is assumed, there are two constant C and D such that $L'(q,\dot q,t)\ge C\|\dot q\|-D$ for all $(q,\dot q,t)\in TM\times[a,b]$ and for all L' close to L. It follows that

$$\frac{\mathrm{dist}(\gamma(a),\gamma(b))}{b-a}\le\frac{1}{b-a}\int_a^b\|d\gamma\|\le\frac{(K+D)}{C} \tag{2.11}$$

if γ is a minimizer. As (2.11) holds for any $b-a\ge1$, it implies that there must be some $\tau_i\in[a+i,a+i+1]$ $(i\in\mathbb{Z})$ such that $\|\dot\gamma(\tau_i)\|\le C^{-1}(K+D)$. By the compactness of $M\times(\mathbb{T}\vee[0,1]\vee\mathbb{T})/\sim$, we see that there exists $K_2>0$ such that $\cup_{s\in[0,1]}\phi^s(\{q,\xi,t:(q,t)\in M\times(\mathbb{T}\vee[0,1]\vee\mathbb{T})/\sim,\|\xi\|\le C^{-1}(K+D)\})\subset\{q,\xi,t:(q,t)\in M\times(\mathbb{T}\vee[0,1]\vee\mathbb{T})/\sim,\|\xi\|\le K_2\}$.

Let $L_i\in C^r(TM\times\mathbb{R},\mathbb{R})$ be a sequence converging to L, let γ_i: $[a,b]\to M$ be the minimizer of L_i with $b-a\ge1$. By the argument above, we see there exists some $U_k\supset\{\xi,q,t:(q,t)\in M\times\mathbb{R},\|\xi\|\le K_2\}$, so that $\|(L(z,t)-L_i(z,t)\|_{U_k}\le\epsilon_i$. Here, $\epsilon_i\to0$ as $i\to\infty$. Thus,

$$\int_a^b L(d\gamma_i(t),t)dt\le(K+\epsilon_i)(b-a). \tag{2.12}$$

As all γ_i is a C^1-curve and the actions of L on each γ_i are bounded by (2.12), the set $\{\gamma_i\}_{i\in\mathbb{Z}_+}$ is compact in the C^0-topology (cf [**14**]). Moreover, this set is compact in the $C^1([a,b],M)$-topology as we have $\|\dot\gamma_i\|\le K_2$ and as $\partial^2L/\partial\dot q^2$ is positive definite. So, we can write the Lagrange equations in the form of $\ddot q=f(q,\dot q,t)$, which implies γ_i is bounded in C^2-topology.

Let γ: $[a,b]\to M$ be one of the accumulation points of this set. Clearly, γ: $[a,b]\to M$ is the minimizer of L and we have

$$A_c(\gamma)=\lim_{i\to\infty}\int_a^b L_i(d\gamma_i(t),t)dt.$$

We let $I_i = [-T_i, T_i]$ and let $T_i \to \infty$, there is a sequence of minimizers of L_i, γ_i: $I_i \to M$. By diagonal extraction argument, we can find a subsequence of γ_i which converges C^1 uniformly on each compact set to a C^1-curve γ: $\mathbb{R} \to M$ which is the minimizer of L on any compact interval of $\mathbb{R}$. This proves the upper semi-continuity of $L \to \tilde{\mathcal{G}}_L$.

Given L periodic in t, we let $L_c = L - \eta_c$ where η_c is a closed one form such that $[\eta_c] = c$. η_c is a linear function in $\dot{q}$. If $c_i \to c$, we can choose a sequence of closed 1-form η_{c_i} such that $[\eta_{c_i}] = c_i$ and $|\eta_{c_i} - \eta_c|_{\|\dot{q}\| \le K_1} \to 0$. In this case, $L_{c_i} \to L_c$ implies $c_i \to c$. Since the c-minimal orbits are independent of the choice of η_i, applying the argument above, we obtain the upper semi-continuity $c \to \tilde{\mathcal{G}}(c)$. q.e.d.

In the application, the set $\tilde{\mathcal{G}}(c)$ seems too big to be used for the construction of connecting orbits in interesting problems. Mañé sets seem good candidates. In the time-periodic case, Mañé set can be a proper subset of $\tilde{\mathcal{G}}(c)$, $\tilde{\mathcal{N}}(c) \subsetneq \tilde{\mathcal{G}}(c)$. It is closely related to the problem whether the Lax–Oleinik semi-group converges or not, some example can be found in [**10**]. To establish the connection between two Mañé sets, we consider a modified Lagrangian

$$L_{\eta,\mu} = L - \eta - \mu,$$

where η is a closed 1-form on M such that $[\eta] = c$, μ is a 1-form depending on t in the way that the restriction of μ on $\{t \le 0\}$ is 0, the restriction on $\{t \ge 1\}$ is a closed 1-form $\bar{\mu}$ on M with $[\bar{\mu}] = c' - c$. Let $m_0, m_1 \in M$, we define

$$(2.13) \qquad h_{\eta,\mu}^{T_0,T_1}(m_0, m_1) = \inf_{\substack{\gamma(-T_0)=m_0 \\ \gamma(T_1)=m_1}} \int_{-T_0}^{T_1} (L - \eta - \mu)(d\gamma(t), t)dt + T_0\alpha(c) + T_1\alpha(c').$$

Clearly, $\exists\, m^* \in M$ and some constants C_μ, $C_{\eta,\mu}$, independent of T_0, T_1, such that

$$\begin{aligned} h_{\eta,\mu}^{T_0,T_1}(m_0, m_1) &\le h_c^{T_0}(m_0, m^*) + h_{c'}^{T_2}(m^*, m_2) + C_\mu \\ &\le C_{\eta,\mu}. \end{aligned}$$

Thus, its limit infimum is bounded

$$(2.14) \qquad \begin{aligned} h_{\eta,\mu}^{\infty}(m_0, m_1) &= \liminf_{T_0,T_1 \to \infty} h_{\eta,\mu}^{T_0,T_1}(m_0, m_1) \\ &\le C_{\eta,\mu}. \end{aligned}$$

Let $\{T_0^i\}_{i\in\mathbb{Z}_+}$ and $\{T_1^i\}_{i\in\mathbb{Z}_+}$ be the sequence of positive integers such that $T_j^i \to \infty$ $(j = 0, 1)$ as $i \to \infty$ and the following limit exists

$$\lim_{i\to\infty} h_{\eta,\mu}^{T_0^i,T_1^i}(m_0, m_1) = h_{\eta,\mu}^{\infty}(m_0, m_1).$$

Let $\gamma_i(t, m_0, m_1)$: $[-T_0^i, T_1^i] \to M$ be a minimizer connecting m_0 and m_1

$$h_{\eta,\mu}^{T_0^i,T_1^i}(m_0, m_1) = \int_{-T_0^i}^{T_1^i} (L - \eta - \mu)(d\gamma_i(t), t)dt + T_0^i\alpha(c) + T_1^i\alpha(c').$$

From the proof of Lemma 2.2, we can see that for any compact interval $[a, b]$, there is some $I \in \mathbb{Z}_+$ such that the set $\{\gamma_i\}_{i\ge I}$ is pre-compact in $C^1([a, b], M)$.

Lemma 2.3. *Let γ: $\mathbb{R} \to M$ be an accumulation point of $\{\gamma_i\}$. If $s \ge 1$, then*

$$A_{L_{\eta,\mu}}(\gamma|[s, \tau]) = \inf_{\substack{\tau_1-\tau\in\mathbb{Z},\tau_1>s\\ \gamma^*(s)=\gamma(s)\\ \gamma^*(\tau_1)=\gamma(\tau)}} \int_s^{\tau_1} (L - \eta - \mu)(d\gamma^*(t), t)dt + (\tau_1 - \tau)\alpha(c'), \tag{2.15a}$$

if $\tau \le 0$, then

$$A_{L_{\eta,\mu}}(\gamma|[s, \tau]) = \inf_{\substack{s_1-s\in\mathbb{Z},s_1<\tau\\ \gamma^*(s_1)=\gamma(s)\\ \gamma^*(\tau)=\gamma(\tau)}} \int_{s_1}^{\tau} (L - \eta - \mu)(d\gamma^*(t), t)dt - (s_1 - s)\alpha(c), \tag{2.15b}$$

if $s \le 0$ and $\tau \ge 1$, then

$$A_{L_{\eta,\mu}}(\gamma|[s, \tau]) = \inf_{\substack{s_1-s\in\mathbb{Z},\tau_1-\tau\in\mathbb{Z}\\ s_1\le 0,\tau_1\ge 1\\ \gamma^*(s_1)=\gamma(s)\\ \gamma^*(\tau_1)=\gamma(\tau)}} \int_{s_1}^{\tau_1} (L - \eta - \mu)(d\gamma^*(t), t)dt - (s_1 - s)\alpha(c) - (\tau_1 - \tau)\alpha(c'). \tag{2.15c}$$

Proof. To show that, let us suppose the contrary, for instance, (2.15b) does not hold. Thus, there would exist $\Delta > 0$, $s < \tau \le 0$, $s_1 < \tau \le 0$, $s_1 - s \in \mathbb{Z}$ and a curve γ^*: $[s_1, \tau] \to M$ with $\gamma^*(s_1) = \gamma(s)$, $\gamma^*(\tau) = \gamma(\tau)$ such that

$$A_{L_{\eta,\mu}}(\gamma|[s, \tau]) \ge \int_{s_1}^{\tau} (L - \eta - \mu)(d\gamma^*(t), t)dt - (s_1 - s)\alpha(c) + \Delta.$$

Let $\epsilon = \frac{1}{4}\Delta$. By the definition of limit infimum, there exist $T_0^{i_0} > 0$ and $T_1^{i_0} > 0$ such that

$$h_{\eta,\mu}^{T_0,T_1}(m_0,m_1) \geq h_{\eta,\mu}^{\infty}(m_0,m_1) - \epsilon \quad \forall \;\; T_0 \geq T_0^{i_0},\; T_1 \geq T_1^{i_0}, \tag{2.16}$$

there exist subsequences $T_j^{i_k}$ $(j = 0, 1,\ k = 0, 1, 2, \dots)$ such that for all $k > 0$

$$T_0^{i_k} - T_0^{i_0} \geq s - s_1, \tag{2.17}$$

$$|h_{\eta,\mu}^{T_0^{i_k},T_1^{i_k}}(m_0,m_1) - h_{\eta,\mu}^{\infty}(m_0,m_1)| < \epsilon. \tag{2.18}$$

By taking a further subsequence, we can assume $\gamma_{i_k} \to \gamma$. In this case, we can choose sufficiently large k such that $\gamma_{i_k}(s)$ and $\gamma_{i_k}(\tau)$ are so close to $\gamma(s)$ and $\gamma(\tau)$, respectively that we can construct a curve $\gamma_{i_k}^*\colon [s_1,\tau] \to M$ which has the same endpoints as γ_{i_k}: $\gamma_i^*(s_1) = \gamma_i(s)$, $\gamma_i^*(\tau) = \gamma_i(\tau)$ and satisfies the following

(2.19)
$$A_{L_{\eta,\mu}}(\gamma_{i_k}|[s,\tau]) \geq \int_{s_1}^{\tau} (L - \eta - \mu)(d\gamma_{i_k}^*(t), t)dt - (s_1 - s)\alpha(c) + \frac{3}{4}\Delta.$$

Let $T_0' = T_0^{i_k} + (s - s_1)$, if we extend $\gamma_{i_k}^*$ to $\mathbb{R} \to M$ such that

$$\gamma_{i_k}^* = \begin{cases} \gamma_{i_k}(t - s_1 + s), & t \leq s_1, \\ \gamma_{i_k}^*(t), & s_1 \leq t \leq \tau, \\ \gamma_{i_k}(t), & t \geq \tau, \end{cases}$$

then, we obtain from (2.18) and (2.19) that

$$\begin{aligned} h_{\eta,\mu}^{T_0',T_1^{i_k}}(m_0,m_1) &\leq A_{L_{\eta,\mu}}(\gamma_{i_k}^*|[-T_0', T_1^{i_k}]) - T_1^{i_k}\alpha(c') - T_0'\alpha(c) \\ &\leq A_{L_{\eta,\mu}}(\gamma_{i_k}|[-T_0^{i_k}, T_1^{i_k}]) - T_1^{i_k}\alpha(c') - T_0^{i_k}\alpha(c) - \frac{3}{4}\Delta \\ &\leq h_{\eta,\mu}^{\infty}(m_0,m_1) - 2\epsilon. \end{aligned}$$

but this contradicts (2.16) since $T_0' \geq T_0^{i_0}$ and $T_1^{i_k} \geq T_1^{i_0}$, guaranteed by (2.17). (2.15a) and (2.15c) can be proved in the same way. q.e.d.

We define

$$\tilde{\mathcal{N}}_{\eta,\mu} = \{d\gamma \in \tilde{\mathcal{G}}_{L_{\eta,\mu}} : \quad \text{(2.15a), (2.15b) and (2.15c) hold}\}.$$

This definition is similar to the definition of a Mañé set, but L is replaced by $L_{\eta,\mu}$.

Lemma 2.4. *The map* $(\eta,\mu)\to\tilde{\mathcal{N}}_{\eta,\mu}$ *is upper semi-continuous.* $\tilde{\mathcal{N}}_{\eta,0}=\tilde{\mathcal{N}}(c)$ *if* $[\eta]=c$. *Consequently, the map* $c\to\tilde{\mathcal{N}}(c)$ *is upper semi-continuous.*

Proof. Let $\eta_i\to\eta$ and $\mu_i\to\mu$, let $\gamma_i\in\tilde{\mathcal{N}}_{\eta_i,\mu_i}$ and let γ be an accumulation point of the set $\{\gamma_i\in\tilde{\mathcal{N}}_{\eta_i,\mu_i}\}_{i\in\mathbb{Z}^+}$. Clearly, $\gamma\in\tilde{\mathcal{N}}_{\eta,\mu}$. If $\gamma\notin\tilde{\mathcal{N}}_{\eta,\mu}$, there would be two points $\gamma(s)$,$\gamma(\tau)\in M$ such that one of the following three possible cases takes place. Either $\gamma(s)$ and $\gamma(\tau)\in M$ can be connected by another curve γ^*: $[s+n,\tau]\to M$ with smaller action

$$A_{\eta,\mu}(\gamma|[s,\tau])<A_{\eta,\mu}(\gamma^*|[s+n,\tau])-n\alpha(c),$$

in the case $\tau<0$; or there would a curve γ^*: $[s,\tau+n]\to M$ such that

$$A_{\eta,\mu}(\gamma|[s,\tau])<A_{\eta,\mu}(\gamma^*|[s,\tau+n])-n\alpha(c'),$$

in the case $s\geq 1$, or when $s\leq 0$ and $\tau\geq 1$, there would be a curve γ^*: $[s+n_1,\tau+n_2]\to M$ such that

$$A_{\eta,\mu}(\gamma|[s,\tau])<A_{\eta,\mu}(\gamma^*|[s+n_1,\tau+n_2])-n_1\alpha(c)-n_2\alpha(c'),$$

where $s+n_1\leq 0$, $\tau+n_2\geq 1$. Since γ is an accumulation point of γ_i, for any small $\epsilon>0$, there would be sufficiently large i such that $\|\gamma-\gamma_i\|_{C^1[s,t]}<\epsilon$, it follows that $\gamma_i\notin\tilde{\mathcal{N}}_{\eta_i,\mu_i}$, but that is absurd.

Let us consider the case that $\mu=0$. In this case, $L-\eta$ is periodic in t. If some orbit $\gamma\in\tilde{\mathcal{N}}_{\eta,0}$: $\mathbb{R}\to M$ is not semi-static, then there exist $s<\tau\in\mathbb{R}$, $n\in\mathbb{Z}$, $\Delta>0$ and a curve γ^*: $[s,\tau+n]\to M$ such that $\gamma^*(s)=\gamma(s)$, $\gamma^*(\tau+n)=\gamma(\tau)$ and

$$A_{\eta,0}(\gamma|[s,\tau])\geq A_{\eta,0}(\gamma^*|[s,\tau+n])-n\alpha(c)+\Delta.$$

We can extend γ^* to $[s_1,\tau_1+n]\to M$ such that $s_1\leq\min\{s,0\}$, $\min\{\tau_1,\tau_1+n\}\geq 1$, $\tau_1\geq\tau$ and

$$\gamma^*=\begin{cases}\gamma(t), & s_1\leq t\leq s,\\ \gamma^*(t), & s\leq t\leq\tau+n,\\ \gamma(t-n), & \tau+n\leq t\leq\tau_1+n.\end{cases}$$

Since $L-\eta$ is periodic in t, we would have

$$A_{\eta,0}(\gamma|[s_1,\tau_1])\geq A_{\eta,0}(\tau^*\gamma|[s_1,\tau_1+n])-n\alpha(c)+\Delta.$$

but this contradicts to (2.15c). q.e.d.

The upper semi-continuity of $c \to \tilde{\mathcal{N}}(c)$ will be fully exploited to build the C-equivalence among some $\tilde{\mathcal{N}}(c)$, the construction of diffusion orbits in this paper depends crucially on this property. Towards that, we shall also make use of the Lipschitz property of the Aubry sets. Let π: $TM \times \mathbb{T} \to M \times \mathbb{T}$ be the projection along the fibers. Mather discovered the following (cf. [**14**, **15**]).

Lemma 2.5. *π: $\tilde{\mathcal{A}}(c) \to M \times \mathbb{T}$ is injective. Its inverse (considered as a map from $\mathcal{A}(c) = \pi\tilde{\mathcal{A}}(c)$ to $\tilde{\mathcal{A}}(c)$) is Lipschitz, i.e., $\exists$ a constant C_L such that for any $x, y \in \mathcal{A}(c)$, we have*

$$\mathrm{dist}(\pi^{-1}(x), \pi^{-1}(y)) \leq C_L \mathrm{dist}(x, y).$$

The concept of regular Lagrangian is useful for us in this paper. L is said to be c-regular if the following limit exists for all (x, x', s, s')

$$h_c^{\infty}(x, x', s, s') = \lim_{k \to \infty} h_c^k(x, x', s, s'). \tag{2.20}$$

Lemma 2.6 ([**4**])**.** *If $\tilde{\mathcal{M}}(c)$ is minimal in the sense of topological dynamics and if there exists a sequence γ_n of n-periodic curves such that $A_c(\gamma_n) \to 0$ as $n \to \infty$, then L_c is regular, hence $\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c) = \tilde{\mathcal{G}}(c)$.*

For the completeness sake, we shall present his proof in the appendix. Applying this lemma to the area-preserving twist map, we have the following:

Corollary 2.7. *Let $\omega \in \mathbb{R}\backslash\mathbb{Q}$ be the rotation number and $c = \beta'(\omega)$, then L_c is regular and $\tilde{\mathcal{G}}(c) = \tilde{\mathcal{A}}(c)$.*

3. Structure of some c-minimal orbit sets

Our construction of connecting orbits between different c-minimal orbit sets exploits fully the upper-semi continuity of the set-valued function $c \to \tilde{\mathcal{N}}(c)$, and the structure of the relevant Mañé sets.

Let us consider the Hamiltonian flow Φ^t which is a small perturbation of Φ^t_{f+g}. Let Φ and Φ_{f+g} be their time-1-maps. As the cylinder $\mathbb{T} \times \mathbb{R} \times \{(q_2, p_2) = (0, 0)\} = \Sigma_0$ is the normally hyperbolic invariant manifold for Φ_{f+g} and the *a priori* unstable condition is assumed, it follows from the fundamental theorem of normally hyperbolic invariant manifold (cf. [**11**]) that there is $\epsilon = \epsilon(A, B) > 0$ such that if $\|P\|_{C^r} \leq \epsilon$ on the region $\{|p| \leq \max(|A|, |B|)+1\}$ the map Φ^{s+k} ($k \in \mathbb{Z}$) also has a C^{r-1} invariant manifold $\Sigma(s) \subset \mathbb{R}^2 \times \mathbb{T}^2$, provided that $r \geq 2$. This manifold is a small deformation of the manifold $\Sigma_0|_{\{|p_1|\le \max(|A|,|B|)+1\}}$, and is also normally hyperbolic and time-1-periodic. Let $\Sigma = \Sigma(0)$, it can be considered as the image of a map ψ: $\Sigma_0 \to \mathbb{R}^2 \times \mathbb{T}^2$, $\Sigma = \{p_1, q_1, p_2(p_1, q_1), q_2(p_1, q_1)\}$. This map induces a 2-form $\psi^*\omega$ on Σ_0

$$\psi^*\omega = \left(1 + \frac{\partial(p_2, q_2)}{\partial(p_1, q_1)}\right) dp_1 \wedge dq_1.$$

Since the second de Rham co-homology group of Σ_0 is trivial, by using Moser's argument on the isotopy of symplectic forms [**17**], we find that there exists a diffeomorphism ψ_1 on $\Sigma_0|_{\{|p_1|\le \max(|A|,|B|)+1\}}$ such that

$$(\psi \circ \psi_1)^*\omega = dp_1 \wedge dq_1. \tag{3.1}$$

Since Σ is invariant for Φ and $\Phi^*\omega = \omega$, we have

$$\Big((\psi \circ \psi_1)^{-1} \circ \Phi \circ (\psi \circ \psi_1)\Big)^* dp_1 \wedge dq_1 = dp_1 \wedge dq_1$$

i.e., $(\psi \circ \psi_1)^{-1} \circ \Phi \circ (\psi \circ \psi_1)$ preserves the standard area. Clearly, it is exact and twist since it is a small perturbation of Φ_f. In this sense, we say that the restriction of Φ on Σ is obviously area-preserving and twist. If $r > 4$, there are many invariant homotopically non-trivial curves, including many KAM curves. As it still remains open whether the invariant curves of irrational rotation number must be differentiable, we can only assume all these curves are Lipschitz. Given $\rho \in \mathbb{R}$, there is an Aubry–Mather set with rotation number ρ, which is either an invariant circle, or a Denjoy set if $\rho \in \mathbb{R}\backslash\mathbb{Q}$, or periodic orbits if $\rho \in \mathbb{Q}$. Under the generic condition, we can assume there are no homotopically non-trivial invariant curves with rational rotation number for $\Phi|_\Sigma$, and there is only one minimal periodic orbit on Σ for each rational rotation number.

Let us consider the Legendre transformation $\mathcal{L}$. By abuse of terminology, we continue to denote $\Sigma(s)$ and its image under the Legendre transformation by the same symbol. Let

$$\tilde{\Sigma} = \bigcup_{s\in\mathbb{T}} (\Sigma(s), s),$$

which has the normal hyperbolicity as well. Under the Legendre transformation, those Aubry–Mather sets (invariant curves, Denjoy sets or minimal periodic orbits) on Σ correspond to the support of some c-minimal measures. Recall $H^1(M, \mathbb{R}) = \mathbb{R}^2$. We claim that each of these sets corresponds to an interval or a rectangle in $H^1(M, \mathbb{R})$. In other words, for all c in this interval (rectangle), the time-1-section of the support of the c-minimal measure is exactly this Aubry–Mather set.

Towards that goal, we introduce the coordinate transformation

$$(p_1, q_2, p_2, q_2) \to (p_1, q_2, p_2 + \zeta(q_2), q_2),$$

where ζ is defined in the way such that

$$\frac{\partial g}{\partial p_2}(\zeta(q_2), q_2) = 0 \tag{3.2}$$

and let $g'(p_2, q_2) = g(p_2 + \zeta(q_2), q_2)$. By the assumption on g, we now have

$$\frac{\partial^2 g'}{\partial q_2^2}(0,0) < 0, \quad \frac{\partial^2 g'}{\partial p_2 \partial q_2}(0, q_2) = 0.$$

To simplify the notation, we still use g to denote the function g'. Let L_0 be the Lagrangian obtained from $f + g$ by Legendre transformation. It has the form

$$L_0(q_2, \dot{q}) = \ell_1(\dot{q}_1) + \ell_2(q_2, \dot{q}_2),$$

where ℓ_1 and ℓ_2 are the Legendre transformation of f and g, respectively. As g is a convex function in p_2, $\dot{q}_2 = \dot{q}_2(p_2, q_2) = \partial_{p_2} g(p_2, q_2)$, we find from (3.2) and the convexity of g that $\dot{q}_2(0, q_2) = 0$ and $\partial \dot{q}_2/\partial p_2 > 0$. Thus, ℓ_2 can be written in the form

$$\ell_2(q_2, \dot{q}_2) = V(q_2) + U(q_2, \dot{q}_2),$$

where $V(q_2) = -g(0, q_2)$, $U \geq 0$ is a convex function in $\dot{q}_2$ with super-linear growth, attains its minimum at $\dot{q}_2 = 0$ ($\forall q_2 \in \mathbb{T}$). By the assumption, V has a global minimum at $q_2 = 0$ which is non-degenerate.

Now, let us consider the β function of L_0. Under the flow $\phi^t_{L_0}$, an invariant circle on Σ with irrational rotation number ρ is the support of a unique minimal measure $\mu_{(\rho,0)}$ whose rotation vector is $(\rho, 0)$. There exist $c_1 \in \mathbb{R}$ and $-\infty < c_2^- < 0 < c_2^+ < \infty$ such that $\mu_{(\rho,0)}$ is c-minimal for $c \in \{c_1\} \times [c_2^-, c_2^+]$. We have $c_2^- < c_2^+$ since the β function of the twist map has corner at rational numbers. β is differentiable at some rational number p/q if and only if there exists a homotopically non-trivial invariant curve of rotation number p/q, and consists entirely of periodic orbits of period q [**3**], [**13**]. From the property that both α and β functions are finite everywhere and has super-linear growth we find that $-\infty < c_2^-$ and $c_2^+ < \infty$.

Next, let us consider the α function of L. We use $c = (c_1, c_2) \in \mathbb{R}^2$ to denote a first de Rham cohomology class of M. For each $c \in \mathbb{R} \times (c_2^-, c_2^+)$, the action variable on each c-minimal orbit of L_0 takes value $(p_1, 0)$ which is independent of t. Let A^*, B^* be such numbers that for each $c \in [A^*, B^*] \times (c_2^-, c_2^+)$ the corresponding p_1 satisfies the condition

$$A - 1 \le p_1 \le B + 1.$$

Lemma 3.1. *There exists $\epsilon_0 > 0$, if $\|P\|_{C^2} \le \epsilon_0$ on the region $\{|p| \le \max(|A|, |B|) + 1\}$, there is a strip $C = [A^*, B^*] \times [-c_2^*, c_2^*] \subset H^1(M, \mathbb{R})$ $(c_2^* > 0)$, such that for each $c \in C$, the c-minimal orbit set $\tilde{\mathcal{G}}(c) \subset \tilde{\Sigma}$.*

Proof. Note the Lagrangian flow of L_0 is integrable and is decoupled between two phase sub-space $(q_1, \dot{q}_1)$ and $(q_2, \dot{q}_2)$. The second component of the flow $\phi^t_{L_0}, \phi^t_{\ell_2}$ has two homoclinic loops Γ^+ and Γ^-, which can be thought as the graph of the functions $G^\pm(q_2)$, i.e., $\Gamma^\pm = \{q_2, G^\pm(q_2)\}$. The orbit dq_2^+ on Γ^+ encircles the cylinder $\mathbb{T} \times \mathbb{R}$ in counter clockwise direction $(\dot{q}_2 > 0)$, the orbit dq_2^- on Γ^- encircles the cylinder in clockwise direction $(\dot{q}_2 < 0)$. Clearly, we have some positive numbers $C_A^\pm > 0$ such that

$$\int_{-\infty}^{\infty} \ell_2(q_2^\pm(t), \dot{q}_2^\pm(t))dt = C_A^\pm.$$

Let

$$c_2^+ = \frac{1}{2\pi} C_A^+, \qquad c_2^- = \frac{1}{2\pi} C_A^-.$$

It is obvious that for each $c \in \mathbb{R} \times (-c_2^-, c_2^+)$, $\tilde{\mathcal{G}}_{L_0}$ is contained in $\tilde{\Sigma}$. By the upper semi-continuity of the set function $(c, L) \to \tilde{\mathcal{G}}_L(c)$, there exist $\epsilon = \epsilon(A, B) > 0$ and $c_2^* > 0$ such that if $c \in [A^*, B^*] \times [-c_2^*, c_2^*]$ and if $\|L_1\|_{C^2} \le \epsilon$, then $\tilde{\mathcal{G}}(c)$ is contained in a small neighborhood of $\tilde{\mathcal{G}}_{L_0}(c)$. Here, $\| \cdot \|_{C^2}$ is the norm in the function space $C^2(\{(\dot{q}, q) \in \mathbb{R}^2 \times \mathbb{T}^2 : \|\dot{q}\| \le K\}, \mathbb{R})$, $K > 0$ is a sufficiently large number. Since $\tilde{\mathcal{G}}(c)$ is invariant, by the normal hyperbolicity of the invariant cylinder, $\tilde{\mathcal{L}} \subset \tilde{\Sigma}$. q.e.d.

Although the structure of minimal measures is unclear in general case, we know very well the structures of those $\tilde{\mathcal{M}}(c) \subset \tilde{\Sigma}$ since the time-1-map Φ is an area-preserving twist map when it is restricted to Σ. Under the projection from $TM \times \mathbb{T}$ to $TM \times \{t = 0\}$, the support of those c-minimal measures are the image of those Aubry–Mather sets under the Legendre transformation $\mathcal{L}$, they are homotopically non-trivial invariant curves, Denjoy sets or minimal periodic orbits on Σ. We use Γ to denote those Aubry–Mather sets on Σ in the Hamiltonian formalism, let $\Gamma(t) = \Phi^t_H(\Gamma) \subset \Sigma(t)$, $\tilde{\Gamma} = \cup_{t\in\mathbb{T}}(\mathcal{L}(\Gamma(t)), t)$.

Before going onto the study of some c-minimal measures, let us note a fact as follows:

Proposition 3.2. *Let $c', c^* \in H^1(M,\mathbb{R})$, μ' and μ^* be the corresponding minimal measures respectively. If $\langle c'-c^*, \rho(\mu')\rangle = \langle c'-c^*, \rho(\mu^*)\rangle = 0$, then $\alpha(c') = \alpha(c^*)$.*

Proof. By the definition of the α function we find that

$$\begin{aligned}-\alpha(c') &= \inf_{\nu\in\mathfrak{M}} \int (L-\eta_{c'})d\nu = \int (L-\eta_{c'})d\mu' \\ &= \int (L-\eta_{c^*})d\mu' + \langle c^*-c', \rho(\mu')\rangle \\ &\geq -\alpha(c^*).\end{aligned}$$

In the same way, we find that $\alpha(c^*) \leq \alpha(c')$. q.e.d.

Lemma 3.3. *Assume $\tilde{\Gamma} \in \tilde{\mathcal{M}}(\bar{c})$ for some $\bar{c} = (\bar{c}_1, \bar{c}_2) \in [A^*, B^*] \times [-c_2^*, c_2^*]$. There is an interval $I = I(\bar{c}_1) = \{(c_1, c_2) \in H^1(M,\mathbb{R}) : c_1 = \bar{c}_1, a(c_1) \leq c_2 \leq b(c_1)\}$ with $-\infty < a(c_1) < 0 < b(c_1) < \infty$, such that $\tilde{\mathcal{M}}(c) = \tilde{\Gamma}$ for all $c \in \mathrm{Int}I$, $\tilde{\mathcal{M}}(c) \supseteq \tilde{\Gamma}$ for $c \in \partial I$. If there is an invariant curve containing Γ, we have further $\tilde{\mathcal{M}}(c) = \tilde{\Gamma}$ for all $c \in I$.*

Proof. Let $\bar{\mu}$ be a $\bar{c}$-minimal measure. We have shown in the lemma 3.1 that the support of $\bar{\mu}$ must be contained in $\tilde{\Sigma}$. Note the time-1-map is an area-preserving twist map when it is restricted on the cylinder, $\mathrm{supp}(\bar{\mu})|_{t=0}$ is exactly an Aubry–Mather set. When the rotation number is irrational, it follows from the theory for twist map that $\bar{\mu}$ is uniquely ergodic; if the rotation number is rational, we have assumed that there is only one minimal periodic orbit. Thus, the minimal measure of consideration here is always uniquely ergodic, i.e., $\mathrm{supp}(\bar{\mu}) = \tilde{\Gamma}$. Let $\phi^t(z,\theta) \in TM \times \mathbb{T}$ be the Lagrangian flow, z_t be the TM component,

$\hat{\eta} = dq_2$. For any invariant measure μ, if $\text{supp}(\mu) \subset \tilde{\Sigma}$, we have

$$\begin{aligned} \int \hat{\eta} d\mu &= \frac{1}{T}\int_0^T ds \int (\hat{\eta}\circ\phi^s) d\mu \\ &= \frac{1}{T}\int_0^T ds \int \langle \hat{\eta}, z_s\rangle d\mu(z) \\ &\le \frac{1}{T}\int \left|\int_0^T \langle \hat{\eta}, z_s\rangle ds\right| d\mu(z) \\ &\le \frac{2\pi}{T} \to 0 \end{aligned} \tag{3.3}$$

as $T \to \infty$. Since $\int \hat{\eta} d\mu$ is independent of T, $\int \hat{\eta} d\mu = 0$. Therefore, it follows from the Proposition 3.2 that $\alpha(\bar{c}) = \alpha(\hat{c})$ if both $\bar{c}$- and $\hat{c}$-minimal measures are on $\tilde{\Sigma}$ with $\bar{c} - \hat{c} = (0, c_2)$. As the β function for a twist map is strictly convex, $\tilde{\mathcal{M}}(\bar{c}) = \tilde{\mathcal{M}}(\hat{c})$. Let $I(\bar{c}_1) = \{c \in H^1(M,\mathbb{R}) : c_1 = \bar{c}_1,\ \tilde{\mathcal{M}}(c) \supseteq \tilde{\Gamma}\}$. As the α function is convex and has super-linear growth, I is connected and $-\infty < a < 0 < b < \infty$. What remains to show is that I is closed. If not, there was a sequence $c_k \to c$ such that $\tilde{\Gamma} \subset \tilde{\mathcal{M}}(c_k)$ and $\tilde{\Gamma} \not\subseteq \tilde{\mathcal{M}}(c)$, consequently, there would exist μ such that $A_c(\mu_{\tilde{\Gamma}}) > A_c(\mu)$, where $\mu_{\tilde{\Gamma}}$ is the invariant measure on $\tilde{\Gamma}$. Let k be sufficiently large so that c_k is sufficiently close to c, then

$$A_{c_k}(\mu) = \int L d\mu - \langle \rho(\mu), c_k\rangle = A_c(\mu) - \langle \rho(\mu), c_k - c\rangle < A_c(\mu_{\tilde{\Gamma}}).$$

On the other hand, it follows from $\langle c - c_k, \rho(\mu_{\tilde{\Gamma}})\rangle = 0$ that $A_c(\mu_{\tilde{\Gamma}}) = A_{c_k}(\mu_{\tilde{\Gamma}})$. Thus, we have $A_{c_k}(\mu_{\tilde{\Gamma}}) > A_{c_k}(\mu)$, but it contradicts the fact that $\mu_{\tilde{\Gamma}}$ is c_k-minimal measure.

If there is another measure μ which can also minimize the c-action of L when $a(c_1) < c_2 < b(c_1)$, then $\langle dq_2, \mu\rangle = 0$. Indeed, for all $a(c_1) < c_2' < b(c_1)$, we have

$$\begin{aligned} &\int \Big(L - c_1\dot{q}_1 - c_2'\dot{q}_2\Big) d\mu_\Gamma \\ &= \int \Big(L - c_1\dot{q}_1 - c_2\dot{q}_2\Big) d\mu_\Gamma \\ &= \int \Big(L - c_1\dot{q}_1 - c_2\dot{q}_2\Big) d\mu \qquad \text{(by assumption)} \\ &= \int \Big(L - c_1\dot{q}_1 - c_2'\dot{q}_2\Big) d\mu + (c_2' - c_2)\langle dq_2, \mu\rangle. \end{aligned}$$

Thus, we can choose c_2' in the way that $(c_2'-c_2)\langle dq_2,\mu\rangle > 0$ if $\langle dq_2,\mu\rangle \neq 0$, but this contradicts to the minimality of μ_Γ. Consequently, we always have

$$\int (L - c_1\dot{q}_1)d\mu = \int (L - c_1\dot{q}_1)d\mu_{\tilde{\Gamma}},$$

which is independent of the value c_2 takes between $a(c_1)$ and $b(c_1)$, it implies that $\mu = \mu_{\tilde{\Gamma}}$ since $\mu_{\tilde{\Gamma}}$ is the only c-minimal measure when $|c_2| \leq c_2^*$.

Let us consider the case when Γ is contained in an invariant curve and $c_2 \in \partial I$. Recall that there exists an invariant curve if and only if the Peierls' barrier function is identically equal to zero, the Aubry set $\tilde{\mathcal{A}}(c)$ contains a co-dimensional one torus in this case. Let π be the projection $TM \times \mathbb{T} \to M \times \mathbb{T}$. Because the inverse map π^{-1} defined the Aubry sets is Lipschitz and $\pi\tilde{\Gamma}$ contains a codimension 1 torus, any c-minimal curve $\gamma \subset \mathcal{A}(c)$ cannot cross the 2-torus $\pi\tilde{\Gamma} \subset T^2 \times \mathbb{T}$. Thus, there exist $\delta > 0$ such that for any $T > 0$

$$-\delta \leq \left| \int_{-T}^{T} \dot{\gamma}_2(t)dt \right| \leq 2\pi + \delta.$$

So, if μ is also a c-minimal measure and $c' = (c_1, 0)$, then

$$\begin{aligned} A_{c'}(\mu_{\tilde{\Gamma}}) &= \int (L - c_1\dot{q}_1)d\mu_{\tilde{\Gamma}} \\ &= \int (L - c_1\dot{q}_1 - c_2\dot{q}_2)d\mu \quad \text{(by condition)} \\ &= \int (L - c_1\dot{q}_1)d\mu \quad \text{(by (3.3))} \\ &= A_{c'}(\mu) \end{aligned}$$

it implies that the only minimal measure is $\mu_{\tilde{\Gamma}}$. q.e.d.

It follows from the lemma 3.3 that there is a strip $\mathcal{S} = \{(c_1, c_2) \in H^1(M,\mathbb{R}) : c_1 \in \mathbb{R},\, a(c_1) \leq c_2 \leq b(c_1),\, A^* < c_1 < B^*,\, -\infty < a(c_1) < 0 < b(c_1) < \infty\}$, such that if $c \in \text{int}\mathcal{S}$, the c-minimal measure is on $\tilde{\Sigma}$ and is uniquely ergodic. If $c \in \partial\mathcal{S} \cap \{A^* < c_1 < B^*\}$ and $\Gamma \subset \tilde{\mathcal{M}}(c)$ is contained in an invariant curve, the c-minimal measure is also uniquely ergodic. In these cases, we have $\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c)$.

In the following, we use $I(c_1) = \{c = (c_1, c_2) : a(c_1) \leq c_2 \leq b(c_1)\}$ to denote the maximal interval in the following sense: for each $c' = (c_1, c_2')$ with $a(c_1) < c_2' < b(c_1)$, the c'-minimal measure has some $\tilde{\Gamma} \subset \tilde{\Sigma}$ as its support, this $\tilde{\Gamma}$ is not contained in the support of any c^*-minimal measure where $c^* = (c_1, c_2^*)$ with either $c_c^* < a(c_1)$ or $c_2^* > b(c_1)$.

Lemma 3.4. *Let $\tilde{\Gamma} \subset \tilde{\Sigma}$ be the support of some minimal measure for $\bar{c} \in I(c_1)$, we assume that it has dense orbit. Then, $\tilde{\mathcal{N}}(c) \subset \tilde{\Sigma}$ for each $c \in \mathrm{int}I(c_1) = \{(c_1, c_2) : a(c_1) < c_2 < b(c_1)\}$. If Γ is an invariant curve or a Denjoy set contained in an invariant curve, and if $c \in \partial I = \{(c_1, c_2) : c_2 = a(c_1)$ or $c_2 = b(c_1)\}$, we have further that $\tilde{\mathcal{N}}(c)$ consists of $\tilde{\Gamma}$ and the c-minimal orbits homoclinic to $\tilde{\Gamma}$.*

Proof. Let us consider a c-minimal orbit $d\gamma$ with $c \in \mathrm{int}I(c_1)$ ($c \in I(c_1)$ if Γ is an invariant curve). If this orbit is not contained in $\tilde{\mathcal{M}}(c) = \tilde{\Gamma}$, then $d\gamma$ is semi-asymptotic to $\tilde{\Gamma}$ as $t \to \pm\infty$. We say an orbit is semi-asymptotic to an invariant set Γ as $t \to \infty$ if every invariant subset of its ω-limit set that is minimal in Birkhoff sense is contained in Γ. We use the argument in [7] to show it. Let N be a minimal (in Birkhoff sense) invariant subset of the ω-limit set of $d\gamma$, there exists a sequence $t_k \to \infty$ such that $\mathrm{dist}(d\gamma(t_k), N) \to 0$. We claim that there is a sequence $T_k \to \infty$ such that

$$\lim_{k\to\infty} \sup\{\mathrm{dist}(d\gamma(t), N) : t_k \le t \le t_k + T_k\} \to 0. \tag{3.4}$$

If not, there exist $d > 0$, $T > 0$ and a subsequence t_j of the sequence t_k such that $\mathrm{dist}(d\gamma(t), N) \ge d$ for every j and some $s_j \in [t_j, t_j + T]$. As $\gamma(t)$ is a c-minimal curve, $d\gamma$ lies in a bounded region of $TM \times \mathbb{T}$, the closure of the orbit is compact. Thus, for some subsequence t_i of the sequence t_j, the sequence $d\gamma(t_i)$ and $d\gamma(s_i)$ are convergent to some points $x \in N$ and $y \in TM \times \mathbb{T}$ respectively, where $\mathrm{dist}(y, N) \ge d$. Consequently, $\phi^{t_0}(x) = y$ for some $0 < t_0 \le T$. This contradicts the invariance of N to the Euler–Lagrange flow.

Let μ_n be the probability measure evenly distributed along $d\gamma[t_k, t_k + T_k]$, μ be an accumulation point of $\{\mu_n\}$. As $d\gamma$ is a c-minimal orbit of the Lagrange system μ is a c-minimal measure, i.e., $\mu = \mu_{\tilde{\Gamma}}$. From (3.4), we see $\mathrm{dist}(N, \tilde{\Gamma}) = 0$. As $\tilde{\Gamma}$ has dense orbit, $N = \tilde{\Gamma}$, i.e., the ω-limit set of $d\gamma$ has only one minimal invariant subset $\tilde{\Gamma}$ (in Birkhoff sense). In the same way, we can show that the α-limit set of $d\gamma$ has only one minimal invariant subset $\tilde{\Gamma}$ also.

Let $c \in \mathrm{int}I$ and $d\gamma \in \tilde{\mathcal{N}}(c)$. Note $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c)$ in this case. For each $\xi \in \pi(\Gamma)$, if $k_{ij} \to \infty$ ($i = 1, 2$) as $j \to \infty$ are the two sequences such that $d\gamma(-k_{1j}), d\gamma(k_{2j}) \to \pi^{-1}(\xi)$, then we claim that

$$\lim_{j\to\infty}\int_{-k_{1j}}^{k_{2j}} \dot\gamma_2(t)dt = 0. \tag{3.5}$$

In fact, for any $\xi \in \pi(\Gamma)$, there exist two sequences $k_{ij} \to \infty$ as $j \to \infty$ $(i = 1, 2)$ such that $d\gamma(-k_{ij}) \to \pi^{-1}(\xi)$ and $d\gamma(k_{2j}) \to \pi^{-1}(\xi)$ as $j \to \infty$. It follows from the fact that γ is c-static that

$$h_c^{k_{1j}}(\gamma(-k_{1j}), \gamma(0)) + h_c^{k_{2j}}(\gamma(0), \gamma(k_{2j})) \to 0.$$

If (3.5) does not hold, by choosing a subsequence again (we use the same symbol), we would have

$$\left|\lim_{j\to\infty}\int_{-k_{1}j}^{k_{2}j} \dot\gamma_2(t)dt\right| \geq 2\pi > 0.$$

In this case, let us consider the barrier function $B^*_{c'}$ where $c' = (c_1, c_2')$. Since $c - c' = (0, c_2 - c_2')$, we obtain from Proposition 3.2 that $\alpha(c') = \alpha(c)$, so

$$\begin{aligned}
B_{c'}(\gamma(0)) &\leq \liminf_{j\to\infty}\int_{-k_{1j}}^{k_{2j}} \big(L(d\gamma(t), t) - c_1\dot\gamma_1(t) - c_2'\dot\gamma_2(t) - \alpha(c')\big)\, dt \\
&\leq \liminf_{j\to\infty}\int_{-k_{1j}}^{k_{2j}} \left(L(d\gamma(t), t) - c_1\dot\gamma_1(t) - c_2\dot\gamma_2(t) - \alpha(c)\right) dt \\
&\quad + (c_2 - c_2')\lim_{j\to\infty}\int_{-k_{1j}}^{k_{2j}} \dot\gamma_2(t)dt \\
&\leq -2|c_2 - c_2'|\pi < 0
\end{aligned}$$

as we can choose $c_2' > c_2$ or $c_2' < c_2$ accordingly. But this is absurd since barrier function is non-negative.

Now, let us derive from (3.5) that there is no c-semi-static orbit that is not contained in $\tilde\Sigma$. In fact, we find that $d\gamma \in \tilde{\mathcal{N}}((c_1, 0))$. To see that, we obtain from (3.5) that the term $c_2\dot\gamma_2$ has no contribution to the action along the curve $\gamma|_{[-k_{1j}, k_{2j}]}$:

$$\int_{-k_{1j}}^{k_{2j}} (L - c_1\dot\gamma_1 - c_2\dot\gamma_2)dt \to \int_{-k_{1j}}^{k_{2j}} (L - c_1\dot\gamma_1)dt \quad \text{as } j \to \infty. \tag{3.6}$$

Note $k_{ij} \to \infty$ as $j \to \infty$ $(i = 1, 2)$. If $d\gamma \notin \tilde{\mathcal{N}}((c_1, 0))$, there would exist $j' \in \mathbb{Z}^+$, $k' \in \mathbb{Z}$, $E > 0$ and a curve ζ: $[-k_{1j}, k_{2j} + k'] \to M$ such that $\zeta(-k_{1j'})=\gamma(-k_{1j'})$, $\zeta(k_{2j}+k')=\gamma(k_{2j'})$

$$\begin{aligned}(3.7)\qquad &\int_{-k_{1j'}}^{k_{2j'}}(L(d\gamma(t),t)-c_1\dot\gamma_1+\alpha((c_1,0)))dt\\ &\geq\int_{-k_{1j'}}^{k_{2j'}+k'}(L(d\zeta(t),t)-c_1\dot\zeta_1+\alpha((c_1,0)))dt+E\\ &\geq F_{(c_1,0)}(\gamma(-k_{1j'}),\gamma(k_{2j}))+E\end{aligned}$$

and

$$(3.8)\qquad \left|\int_{-k_{1j'}}^{k_{2j'}+k'}\dot\zeta_2 dt\right|\to 0.$$

The second condition (3.8) follows from the facts that $\tilde{\mathcal{N}}((c_1,0))\subset\tilde\Sigma$ and that $\gamma(-k_{ij})\to\xi\in\mathcal{M}_0((c_1,0))=\mathcal{M}_0(c)$. Let $j-j'$ be sufficiently large, we construct a curve ζ': $[-k_{1j},k_{2j}+k']\to M$ such that

$$\zeta'(t)=\begin{cases}\gamma(t), & t\in[-k_{1j},-k_{1j'}],\\ \zeta(t), & t\in[-k_{1j'},k_{2j'}+k'],\\ \gamma(t-k'), & t\in[k_{2j'}+k',k_{2j}+k'].\end{cases}$$

It follows from (3.5–3.8) that

$$\begin{aligned}&\int_{-k_{1j}}^{k_{2j}+k'}(L(d\zeta'(t),t)-\langle c,\dot\zeta'\rangle)dt\\ &\quad<\int_{-k_{1j}}^{k_{2j}}(L(d\gamma(t)-c_1\dot\gamma_1)dt-E\\ &\quad\leq\int_{-k_{1j}}^{k_{2j}}(L(d\gamma(t),t)-\langle c,\dot\gamma\rangle)dt-\frac{E}{2},\end{aligned}$$

but this contradicts the property that $d\gamma\in\tilde{\mathcal{N}}(c)$.

Finally, let us consider the case when $c\in\partial I$ and there is an invariant circle containing Γ. In this case, we obtain from the Lemma 3.3 that $\mu_{\tilde\Gamma}$ is the only minimal measure still. According to the upper semi-continuity of the set-valued function $c\to\tilde{\mathcal{N}}(c)$ that $\tilde{\mathcal{N}}(c')$ should be in a small neighborhood of $\tilde{\mathcal{N}}(c)$ if c' is close to c. It implies that $\tilde{\mathcal{N}}(c)$ should contain some orbits outside of $\tilde\Sigma$. If this is not true, $\tilde{\mathcal{N}}(c')$ would be in a small neighborhood of $\tilde\Sigma$ for some $c'=(c_1,c_2')$ with $c_2<a(c_1)$ or $c_2>b(c_1)$. As we have normally hyperbolic structure in the neighborhood of $\tilde{\Sigma}$, any invariant set should be on $\tilde{\Sigma}$, consequently, we would have $\tilde{\mathcal{M}}(c') = \tilde{\Gamma}$ as the map induced by the Euler–Lagrange flow on this manifold corresponds to a twist area-preserving map on Σ. But this contradicts the definition of $I(c_1)$.

At the beginning of the proof, we have shown that any c-minimal orbits must be semi-asymptotic to the support of the minimal measure if it is uniquely ergodic. What remains to be shown is that such orbit is homoclinic to the invariant circle in this case. As Γ is contained in an invariant circle, denoted by Γ^*, the Aubry set contains a codimension 1 torus $\tilde{\Gamma}^* = \cup_{t\in[0,1]}(\phi^t(\mathcal{L}(\Gamma^*)), t)$, because $P_\omega(q) = B_c(q)$ for all $q \in \pi(\Gamma^*)$ when $\omega = \partial_1\alpha(c)$ is irrational, and because the necessary and sufficient condition for the existence of invariant circle is the Peierls' barrier function is identically equal to zero. Due to the Lipschitz property of the Aubry set, any c-minimal curve can not cross $\pi(\tilde{\Gamma}^*)$, so

$$\int_{-k}^{k} \dot{\gamma}_2(t)dt \le 2\pi + \mathcal{O}(\|P\|) \quad \forall\ k \in \mathbb{Z}^+.$$

As $d\gamma$ is semi-asymptotic to $\tilde{\Gamma}$, $d\gamma$ enters the small neighborhood of $\tilde{\Sigma}$. If $d\gamma$ does not fall either on the stable manifold or on the unstable manifold, then it will go outside of the neighborhood again. It implies that $d\gamma$ is a multi-bump solution of the Lagrange equation. As we did in the proof of the Lemma 3.1, we can construct a curve ζ by cutting off all other bumps and leave only one bump. In this case, the c-action of ζ is smaller than that of γ, but this is absurd. Thus, $d\gamma(t) \in W^s_{loc}(\tilde{\Gamma}^*)\cup W^u_{loc}(\tilde{\Gamma}^*)\backslash\{\tilde{\Gamma}^*\}$ when $d\gamma(t)$ is in a small neighborhood of $\tilde{\Sigma}$. q.e.d.

To each orbit $d\gamma$ homoclinic to $\tilde{\Gamma}$, we can associate an element $[\gamma] \in H_1(M\times\mathbb{T}, \tilde{U}, \mathbb{Z}) = \mathbb{Z}$ where $\tilde{U}$ is a small neighborhood of $\pi(\tilde{\Gamma}^*) \subset M\times T$ when Γ is contained in an invariant circle Γ^*. We can see from this lemma that the necessary condition for a homoclinic orbit $\{d\gamma\} \subset \tilde{\mathcal{N}}(c)$ is $[\gamma] = \pm 1$. In general, the time-1-section $\mathcal{N}_0(c)\backslash\pi(\mathcal{L}(\Gamma))$ is homotopically trivial. By definition, we mean that there exists an open neighborhood $U = \cup_{i=1}^m U_i$ of $\mathcal{N}_0(c)$ such that $U_i \cap U_j = \varnothing$ if $i \ne j$, U_0 is an open neighborhood of $\mathcal{L}(\Gamma)$ and each U_i $(i \ne 0)$ is contractible to one point. In this case, we have

$$i_*H_1(U, \mathbb{R}) \subset \text{span}([\zeta]),$$

where i is the standard inclusion map, $\zeta = (\zeta_1, 0) : [0,1] \to M$ with $\zeta_1(0) = \zeta_1(1)$. By the Lipschitz property of $\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c)$ in this case, we may choose bounded, mutually disjoint open sets $\tilde{U}_i$ in TM such that $\pi\tilde{U}_i = U_i$ and $\cup\tilde{U}_i \supset \tilde{\mathcal{N}}_0(c)$. Under this assumption, we have

Lemma 3.5. *Assume $c = (c_1, b(c_1))$, $\tilde{\mathcal{M}}(c) = \tilde{\Gamma}$ and $\mathcal{N}_0(c)\backslash\pi(\tilde{\Gamma})$ is homotopically trivial. Let $c' = (c_1, c_2')$ with $c_2' - b(c_1) > 0$ being sufficiently small. If $\tilde{\mathcal{M}}(c')$ is uniquely ergodic, then there exists a neighborhood $N_{c'}$ of $\mathcal{N}_0(c')$ such that $i_*H(N_{c'}, \mathbb{R}) = 0$.*

Proof. By assumption, we can choose $\tilde{U} = \cup_{i=0}^m \tilde{U}_i$, a neighborhood of $\tilde{\mathcal{N}}(c)$ such that $\pi(\tilde{U}_i) \cap \pi(\tilde{U}_j) = \varnothing$ if $i \neq j$, $\tilde{U}_0$ is an open neighborhood of $\mathcal{L}(\Gamma)$ and each U_i $(i \neq 0)$ is contractible to one point. By the upper-semi continuity of $c \to \tilde{\mathcal{N}}(c)$, $\tilde{\mathcal{N}}(c') \subset \tilde{U}$ if $c_2' - b(c_1)$ sufficiently small. We claim that for each $z \in \tilde{U}_0 \cap \tilde{\mathcal{N}}(c')$, $\exists$ an integer $k(z) \in \mathbb{Z}_+$ such that $\phi^{k(z)}(z) \notin \tilde{U}_0$ and there is an uniform upper bound $K \in \mathbb{Z}_+$ for all these $k(z)$. If this is not true, for any $k > 0$, $k \in \mathbb{Z}$ there is $z_k \in \tilde{U}_0$ such that $\phi^l(z_k) \in \tilde{U}_0$, $\forall 0 \leq l \leq k$. Let ν_k be a probability measure distributed evenly on $\phi^t(z)$ $(0 \leq t \leq k)$ and let $k \to \infty$, we find there is an accumulation point ν, $\mathrm{supp}(\nu) \subset \tilde{U}_0$. Obviously, $\nu \in \tilde{\mathcal{M}}(c)$. As there is a normally hyperbolic structure on $\tilde{\Sigma}$, the invariant set in $\tilde{U}_0$ must be on $\tilde{\Sigma}$, it follows that $\tilde{\mathcal{M}}(c) \subset \tilde{\Sigma}$, but it contradicts the definition of $I(c_1)$.

By the upper semi-continuity of $c \to \tilde{\mathcal{N}}(c)$ and the assumption on the intersection of the stable and unstable manifolds, we see that $\tilde{\mathcal{N}}_0(c')\backslash\tilde{U}_0$ can be covered by finite mutually disjoint open sets, each of them is homotopic to a point. As each point in $\tilde{U}_0$ shall go outside under the time-1-map ϕ^1, the whole $\tilde{\mathcal{N}}_0(c')$ can be covered by finite mutually disjoint, homotopically trivial open sets. Because $\tilde{\mathcal{M}}(c')$ is assumed uniquely ergodic, we obtain from Lemma 2.5 that $\tilde{\mathcal{N}}(c') = \tilde{\mathcal{A}}(c')$. The Lipschitz property of $\mathcal{A}(c')$ guarantees that $\mathcal{N}_0(c') = \mathcal{A}_0(c')$ is also homotopically trivial. q.e.d.

4. Some Barrier functions

In this section we consider a co-homology class $c = (c_1, b(c_1))$ such that $\mathcal{A}(c)$ contains a 2-torus in $\mathbb{T}^2 \times \mathbb{T}$, i.e., its time-1-sections have an invariant circle on the cylinder, and study the relevant barrier functions introduced in [15]. The study for $c=(c_1,a(c_1))$ is the same. According to our assumptions, the rotation number of this circle is irrational. To go further with our proof, let us turn back to the Hamiltonian formalism temporarily to look at something.

Let $\Phi_H=\Phi_H^1$ be the time-1-map of the Hamiltonian flow Φ_H^t. It has an invariant cylinder Σ. Restricted to the cylinder Σ, this map is clearly twist and area-preserving, thus the invariant circle Γ is Lipschitz. When $P=0$, we have the cylinder $\mathbb{T}\times\mathbb{R}\times\{q_2=p_2=0\}$ as the normally hyperbolic manifold for Φ_{f+g}. Each orbit on this manifold lies in an invariant circle and has zero Lyapunov exponent only. Both the stable and unstable manifolds have two branches. Each of them has an invariant fibration $\{q_1=p_1=\text{constant}, p_2=\tilde G^{\pm}(q_2)\}$ if we use $\{q_2,\tilde G^{\pm}(q_2)\}$ to denote the homoclinic loops of Φ_g in the space of (q_2,p_2). Under a small perturbation, the invariant circle on Σ is the graph of a small function, i.e., $\Gamma=\{q_1\in\mathbb{T}, p=p_\Gamma(q_1), q_2=q_{2\Gamma}(q_1)\}$. From the theory of normally hyperbolic manifolds, we know that the fibration has C^{r-2}-smoothness on the base points. As Γ is an invariant circle, all stable (unstable) fibers with base points on Γ constitute the local stable (unstable) manifold $W_H^{s,u}(\Gamma)$ of Γ. Both the stable and the unstable manifolds have two branches corresponding to $(c_1,b(c_1))$ and $(c_1,a(c_1))$ respectively. Let us consider the branch corresponding to $(c_1,b(c_1))$. In the covering space $T(\mathbb{T}\times\mathbb{R})$, one lift of a unstable manifold originates from $\{p=p_\Gamma(q_1), q_2=q_{2\Gamma}(q_1)\}$ and extends to right, one lift of stable manifold originates from $\{p=p_\Gamma(q_1), q_2=q_{2\Gamma}(q_1)+2\pi\}$ and extends to left. When $P=0$, these two manifolds coincide with each other and are graphs above $0\le q_2\le 2\pi$. Thus, for suitably small $a>0$, there exists $\epsilon>0$ such that if $\|P\|\le\epsilon$, the unstable manifold is a graph above the region $\{q_{2\Gamma}(q_1)\le q_2\le 2\pi-a\}$ and the stable manifold keeps horizontal in the region $\{a\le q_2\le q_{2\Gamma}(q_1)+2\pi\}$, i.e., they are the graphs of some functions in the relevant regions,

$$
\begin{aligned}
(4.1)\qquad W^u(\Gamma)&=\{q,p^u(q): q_1\in\mathbb{T},\ q_{2\Gamma}(q_1)\le q_2\le 2\pi-a\},\\
W^s(\Gamma)&=\{q,p^s(q): q_1\in\mathbb{T},\ a\le q_2\le q_{2\Gamma}(q_1)+2\pi\}.
\end{aligned}
$$

Although each stable (unstable) fiber has C^{r-2}-smoothness, the base points of these fibers fall on a circle for which we can only assume Lipschitz smoothness, these manifolds are at least Lipschitz, i.e., $p^{s,u}(q)$ in (4.1) are at least Lipschitz. We choose suitably small $a>0$ such that the time for any $d\gamma_2$ to cross the strip $\{a\le q_2\le 2\pi-a\}$ is longer than

1. Such assumption is feasible as Φ_H^t is a small perturbation of Φ_{f+g}^t for which this assumption is clearly true.

If there is another invariant circle Γ_1 very close to Γ, by the smoothness of the invariant fibration, we see that $W_H^{s,u}(\Gamma_1)$ are also graphs above the relevant region. Let $\Gamma(A)$ be the highest circle on Σ where $p_1 \leq A$, let $\Gamma(B)$ be the lowest circle where $p_1 \geq B$. As all invariant circles on Σ make up a closed set, it is reasonable to assert that we have some $\epsilon > 0$ such that if $\|P\| \leq \epsilon$, the stable and unstable manifolds of all Γ between $\Gamma(A)$ and $\Gamma(B)$ can keep horizontal in the region $\{a \leq q_2 \leq q_{2\Gamma}(q_1) + 2\pi\}$ and $\{q_{2\Gamma}(q_1) \leq q_2 \leq 2\pi - a\}$, respectively.

As the Hamiltonian system under study has standard symplectic structure, each horizontal Lagrangian sub-manifold is a graph of some closed 1-form defined on M. We know that the stable (unstable) manifold of some smooth isotropic manifold is a Lagrangian manifold, therefore, if we use $(q, p(q))$ $(p(q) \in C^1)$ to denote such a smooth manifold, then

$$\frac{\partial p_1}{\partial q_2} = \frac{\partial p_2}{\partial q_1}, \tag{4.2}$$

it follows that there exists a C^2-function $S(q)$ and constant vector $c \in \mathbb{R}^2$ such that

$$\frac{\partial S}{\partial q_1} + c_1 = p_1, \qquad \frac{\partial S}{\partial q_2} + c_2 = p_2. \tag{4.3}$$

If we consider the manifold as the graph of some closed 1-form, $c \in H^1(M, \mathbb{R})$ is the cohomology class of this closed 1-form. Since a Lipschitz function is differentiable almost everywhere, we claim that there exists a $C^{1,1}$-function S so that (4.3) holds. Here, we use $C^{k,\alpha}$ to denote those functions whose kth derivative is α-Hölder.

Lemma 4.1. *Let Γ be an invariant circle on the cylinder Σ, let $W^{s,u}(\Gamma)$ be its stable (unstable) manifold, which is a graph over a connected open set $U \subset M$ with $\pi(\Gamma) \in U$, then there exists $C^{1,1}$ functions $S^{s,u}$: $U \to \mathbb{R}$ and a constant vector $c \in \mathbb{R}^2$ such that $\{W_H^{s,u} : q \in U\} = \{(q, dS^{s,u}(q)) + c : \, q \in U\}$.*

Proof. Let us consider the case of a stable manifold. By the condition that $W^s(\Gamma)$ is a graph, there is a Lipschitz function $p = (p_1, p_2)$: $U \to R^2$ such that $W^s(\Gamma) = \{(q, p^s(q)) : \, q \in U\}$. Let γ be a closed curve which is the boundary of some topological disk σ on W^s. Since γ is on the stable manifold, $\Phi_H^k(\gamma)$ approaches uniformly to Γ, it implies that $\Phi_H^k(\gamma)$

is such a closed curve going from a point to another point and returning back along almost the same path when k is sufficiently large. As Φ_H is a symplectic diffeomorphism, k can be arbitrary large, we have

$$\iint_\sigma dp \wedge dq = \oint_\gamma pdq = \oint_{\Phi_H^k(\gamma)} pdq = 0. \tag{4.4}$$

Note p is Lipschitz, by the theorem of Rademacher [**18**], p is differentiable almost everywhere in U. As γ is arbitrarily chosen, (4.2) holds for almost all $q \in U$. Consequently, there exists a $C^{1,1}$-function S^s and $c \in \mathbb{R}^2$ such that $p^s = dS^s + c$. In the same way, we obtain a $C^{1,1}$-function S^u and $c' \in \mathbb{R}^2$ such that $p^u = dS^u + c'$. As W_H^s intersects W_H^u at the whole Γ, $c' = c$. q.e.d.

In fact, for almost all initial points $(q, p^s(q))$, p is differentiable at all $\Phi_H^k(q, p^s(q))$ $(\forall k \in \mathbb{Z}^+)$. To see that, let O be an open set in U. For each $k \in \mathbb{Z}^+$, there is a full Lebesgue measure set $O_k \subset \pi(\Phi_H^k\{O, p(O)\})$ where p is differentiable. Since Φ is a diffeomorphism, the set

$$O^* = \bigcap_{k=0}^{\infty} \pi\Big(\Phi_H^{-k}\{O_k, p^s(O_k)\}\Big)$$

is a full Lebesgue measure subset of O. For any point $q \in O^*$, p is differentiable at the points $\pi(\Phi_H^k(q, p^s(q)))$ for all $k \in \mathbb{Z}^+$.

Let us consider the Hamiltonian flow. If the locally horizontal stable (unstable) manifold has the form

$$W_H^{s,u} = \{(q, p^{s,u}(q,t), t) : (q,t) \in U \times \mathbb{T}\}$$

and if we call the 2-form $\Omega = \sum dp_i \wedge dq_i - dH \wedge dt$, then $(p^{s,u}, t)^*\Omega = 0$. In the covering space $\mathbb{R}^2 \times \mathbb{R}$, we find that there exists $\bar{S}^{s,u}(q,t)$ such that $d\bar{S}^{s,u} = p^{s,u}(q,t)dq - H(p^{s,u}(q,t), q, t)dt$. By applying the standard argument (see, for instance, the appendix 2 in [**14**]), we find that

$$L^{s,u} = L - \langle \partial_q \bar{S}^{s,u}, \dot{q}\rangle - \partial_t \bar{S}^{s,u} \tag{4.5}$$

attains its minimum at $\partial_q \bar{S}^{s,u}$ as the function $\dot{q}$. Note $L_{\dot{q}}^{s,u} = L_{\dot{q}} - \partial_q \bar{S}^{s,u}$ is Lipschitz, $dL_{\dot{q}}^{s,u}/dt$ and $L_q^{s,u}$ exist almost everywhere. Since $W^{s,u}$ is a manifold consisting of the trajectories of the Euler–Lagrange flow, it follows from the Euler–Lagrange equation $dL_{\dot{q}}/dt = L_q$ and (4.2) that $L_q^{s,u} = 0$ almost everywhere. The absolute continuity of L implies that $L^{s,u}$ is a function of t alone. Therefore, by adding some function of t to $\bar{S}^{s,u}$, we can make $L^{s,u} = 0$. Note the local stable (unstable) manifold can be thought as the graph of some function defined on $\{(q,t)\in\mathbb{T}^2\times\mathbb{T}: a\le q_2\le q_{2\Gamma}(q_1,t)+2\pi\}$ ($\{(q,t)\in\mathbb{T}^2\times\mathbb{T}: q_{2\Gamma}(q_1,t)\le q_2\le 2\pi-a\}$), where $q_{2\Gamma}(q_1,t)$ is such a function that $\pi(\tilde\Gamma)=\{(q,t): q_2=q_{2\Gamma}(q_1,t)\}$, $q_{2\Gamma}(q_1)=q_{2\Gamma}(q_1,0)$. The first co-homology group is $\mathbb{R}\times\{0\}\times\mathbb{R}$. Thus, there exists a function $S^u(q,t)$: $\{\{(q,t)\in\mathbb{T}^2\times\mathbb{T}: q_{2\Gamma}(q_1,t)\le q_2\le 2\pi-a\}\to\mathbb{R}$, $S^s(q,t)$: $\{(q,t)\in\mathbb{T}^2\times\mathbb{T}: a\le q_2\le q_{2\Gamma}(q_1,t)+2\pi\}\to\mathbb{R}$ and $(c_1^*,0,\alpha^*)$ such that $\bar S^{s,u}(q,t)=S^{s,u}(q,t)+c_1^*q_1+\alpha^*t$, where we have used the fact that both the stable and the unstable manifolds coincide with each other at $\tilde\Gamma$. In this case, we obtain from (4.5) that

$$L^{s,u}=L-\langle(c_1^*,0),\dot q\rangle-\langle\partial_q S^{s,u},\dot q\rangle-\partial_t S^{s,u}$$

attains its minimum at $W^{s,u}$ as the function of $\dot q$ with $L^{s,u}|_{W^{s,u}}=\alpha^*$. Thus, for all $d\gamma$ on $\tilde\Gamma$ we have

$$\int_{-\infty}^{\infty}\Big(L(d\gamma(t),t)-\langle(c_1^*,0),\dot\gamma\rangle-\langle\partial_q S^{s,u}(\gamma(t),t),\dot\gamma\rangle - \partial_t S^{s,u}(\gamma(t),t)-\alpha^*\Big)dt=0. \tag{4.6}$$

We have mentioned before that the Euler–Lagrange equation for $L-\eta_c$ is the same as that for L if η_c is a closed 1-form. In local coordinates, we can write $\eta_c=\langle c(q),\dot q\rangle$. If we use $H_{\eta_c}(p,q,t)$ to denote the Legendre transformation

$$H_{\eta_c}(p,q,t)=\max_p\Big\{\langle p,\dot q\rangle-\Big(L-\langle c(q),\dot q\rangle\Big)\Big\},$$

then we obtain

$$p+c(q)=\frac{\partial L}{\partial\dot q}.$$

It implies that $H_{\eta_c}(p,q,t)=H(p+c(q),q,t)$. As η_c is closed, the coordinate translation $(p,q)\to(p+c(q),q)$ is symplectic. Under such a coordinate translation the horizontal stable (unstable) manifold is the graph of the function $p^{s,u}(q)-c(q)$.

We know that $\tilde\Gamma$ is contained in some Aubry set $\mathcal{A}(c)=\{B_c=0\}$ where $c=(c_1,c_2')$ with $a(c_1)\le c_2'\le b(c_1)$. From the above arguments and the Proposition 3.2, we can see that $c_1=c_1^*$ and $\alpha^*=\alpha(c)$.

To study the barrier function B_c^*, we consider the covering of $\mathbb{T}^2$ given by $\mathbb{T}\times\mathbb{R}$, let $\tilde\Gamma_k$ be the lift of $\tilde\Gamma$ which is close to $\mathbb{T}\times\{2k\pi\}\times\{p_1=\text{const.},p_2=0\}\times\mathbb{T}$. Without lose of generality, we single out one lift of the unstable manifold W^u that extends from $\tilde\Gamma_0$ and keep horizontal over $\{(q,t)\in\mathbb{T}^2\times\mathbb{T}: q_{2\Gamma}(q_1,t)\le q_2\le 2\pi-a\}$ and single out one lift of the stable manifold W^s that extends from $\tilde{\Gamma}_1$ and keep horizontal over $\{(q,t)\in\mathbb{T}^2\times\mathbb{T}: a\le q_2\le q_{2\Gamma}(q_1,t)+2\pi\}$.

Recall $c=(c_1,b(c_1))$. Since $L^{s,u}$ attains its minimum on the local horizontal stable (unstable) manifold, for $q\in\mathbb{T}\times(a,2\pi-a)$, we claim that there exists only one c-minimal orbit $d\gamma_c^s$: $\mathbb{R}^+\to TM$ as well as only one c-minimal orbit $d\gamma_c^u$: $\mathbb{R}^-\to TM$ such that $\gamma^{s,u}(0)=q$. In fact, such an orbit $d\gamma_c^{s,u}$ lies on the local stable (unstable) manifold.

There are two steps to verify our claim. The first step is to show that $\gamma^{s,u}$ does not cross the codimension one torus $\tilde{\Gamma}\subset\mathbb{T}^2\times\mathbb{T}$. It follows immediately from Lemma 4.2. To state this lemma, we define the set of forward and backward semi-static curves:

$$\tilde{\mathcal{N}}^+(c)=\{(z,s)\in TM\times\mathbb{T}:\ \pi\circ\phi_L^t(z,s)|_{[0,+\infty)} \text{ is } c\text{-semi-static}\},$$

$$\tilde{\mathcal{N}}^-(c)=\{(z,s)\in TM\times\mathbb{T}:\ \pi\circ\phi_L^t(z,s)|_{(-\infty,0]} \text{ is } c\text{-semi-static}\}.$$

Lemma 4.2. *If $\mathcal{M}(c)$ is uniquely ergodic, $u\in\mathcal{A}_0(c)$, then there exists a unique $v\in T_uM$ such that $(u,v)\in\tilde{\mathcal{N}}_0^+(c)$ (or $\tilde{\mathcal{N}}_0^-(c)$). Moreover, $(u,v)\in\tilde{\mathcal{A}}_0(c)$.*

Proof. Let us suppose the contrary. Then, there would exist $(u,v)\in\tilde{\mathcal{A}}_0(c)$ and a forward c-semi-static curve $\gamma_+(t)$ with $\gamma_+(0)=u$ and $\dot\gamma_+(0)\neq v$. In this case, for any $u_1\in\mathcal{M}_0(c)$, there exist two sequences $k_i,k_i'\to\infty$ such that

$$\pi\circ\phi_L^{k_i}(u,v)\to u_1,\qquad \gamma_+(k_i')\to u_1$$

and

$$\begin{aligned}&\lim_{k_i\to\infty}\int_0^{k_i}(L-\eta_c)(\phi_L^t(u,v),t)dt+k_i\alpha(c)\\&=\lim_{k_i'\to\infty}\int_0^{k_i'}(L-\eta_c)(d\gamma_+(t),t)dt+k_i\alpha(c)\\&=h_c^\infty(u,u_1).\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}&h_c^\infty(\pi\circ\phi_L^{-1}(u,v),u_1)\\&\quad=F_c(\pi\circ\phi_L^{-1}(u,v),u)+h_c^\infty(u,u_1)\\&\quad=F_c(\pi\circ\phi_L^{-1}(u,v),u)+\lim_{k_i'\to\infty}\int_0^{k_i'}(L-\eta_c)(d\gamma_+(t),t)dt\\&\quad>h_c^\infty(\pi\circ\phi_L^{-1}(u,v),u_1),\end{aligned}$$

where the last inequality follows from the facts that $\dot{\gamma}_+(0) \neq v$ and the minimizer must be a C^1-curve. But this is absurd. q.e.d.

For the second step of the proof, we consider the problem in the covering space $\mathbb{T} \times \mathbb{R}$ and single out a lift of the stable (unstable) manifold of the invariant circle. The stable (unstable) manifold has two branches: $W^{s,u}_{l,r}$

$$W^{s,u}_r = W^{s,u} \cap \{q_{2\Gamma}(q_1,t) \leq q_2 \leq 2\pi - a\},$$
$$W^{s,u}_l = W^{s,u} \cap \{-2\pi + a \leq q_2 \leq q_{2\Gamma}(q_1,t)\}.$$

These two branches of the manifold joined together smoothly at the invariant torus. Let us consider the unstable manifold. There is a smooth function S^u: $\{-2\pi + a \leq q_2 \leq 2\pi - a\} \to \mathbb{R}$ such that $\mathrm{graph}(dS^u) = W^u_l \cup W^u_r$. Note $W^u_l|_{(q_1,t)=\text{constant}}$ is below the zero section of the cotangent bundle while $W^u_r|_{(q_1,t)=\text{constant}}$ is above the zero section if we restrict them in the sub-cotangent bundle $T^*\mathbb{T}$. If L_1 is sufficiently small, then there exist some $c'_2 > 0$ and a periodic function $q_2 = q_2(q_1,t)$ such that $q_2(q_1,t) \leq a$, $|q_2(q_1,t) - a|$ very small and

$$S^u(q_1, 2\pi - q_2(q_1,t), t) - S^u(q_1, -q_2(q_1,t), t) - 2\pi c'_2 = 0.$$

Thus, we can extend $S^u - c'_2 q_2$ periodically so that $S^u - c'_2 q_2$ is a continuous function defined on $\mathbb{T}^2 \times \mathbb{T}$. Note that this function is not differentiable at the 2-dimensional torus $\{(q,t) \in \mathbb{T}^3 : q_2 = q_2(q_1,t)\}$. Since $L^u + \alpha(c) = 0\}$ when it is restricted on $W^u \cap \{-q_2(q_1) \leq q_2 \leq 2\pi - q_2(q_1)\}$ and strictly positive elsewhere, the backward c-semi static orbits must lie on W^u_r if it approaches $\tilde{\Gamma}$ from the right-hand side.

There might be another possibility that the backward c-semi-static orbits approaches $\tilde{\Gamma}$ from the left-hand side. Similarly, there exist $\tilde{c}_2 < 0$ and a periodic function $\tilde{q}_2 = \tilde{q}_2(q_1,t)$ with $|\tilde{q}_2(q_1,t) - a|$ very small such that

$$S^u(q_1, \tilde{q}_2(q_1,t), t) - S^u(q_1, -2\pi + \tilde{q}_2(q_1,t), t) - 2\pi\tilde{c}_2 = 0.$$

In this case, we can also extend $S^u - \tilde{c}_2 q_2$ periodically so that $S^u - \tilde{c}_2 q_2$ is a continuous function defined on $\mathbb{T}^2 \times \mathbb{T}$. Because $\gamma^u(0) \in \{a < q_2 < 2\pi - a\}$, and $c = (c_1, b(c_1))$, it is clear that the c-action along the orbit lying on W^u_l is bigger than the c-action along the orbit lying on W^u_r. This asserts our claim.

Since $L^{s,u}+\alpha(c)=0$ on $W^{s,u}$, for arbitrary $T>0$, we have

$$
\begin{aligned}
(4.7)\quad &\int_{-T}^{0}\Big(L(d\gamma_c^u(t),t)-\langle c,\dot\gamma_c^u(t)\rangle-\alpha(c)\Big)dt\\
&\quad = S^u(\gamma_c^u(0),0)-S^u(\gamma_c^u(-T),-T)-b(c_1)(\bar\gamma_{c2}^u(0)-\bar\gamma_{c2}^u(-T)),\\
&\int_{0}^{T}\Big(L(d\gamma_c^s(t),t)-\langle c,\dot\gamma_c^s(t)\rangle-\alpha(c)\Big)dt\\
&\quad = S^s(\gamma_c^s(T),T)-S^s(\gamma_c^s(0),0)-b(c_1)(\bar\gamma_{c2}^s(T)-\bar\gamma_{c2}^s(0)).
\end{aligned}
$$

Since Φ is an area-preserving twist map when it is restricted on the cylinder, from the Lemma 2.6 and the Corollary 2.7, we see that L_c is regular. Therefore, for any $\varepsilon>0$, $0\le s<1$, $0\le t<1$ and $q',q^*\in M$, there exists $K_0\in\mathbb{Z}^+$ such that

$$|h_c^\infty(q',q^*,s,t)-h_c^K(q',q^*,s,t)|\le\varepsilon\quad\forall K_0\le K\in\mathbb{Z}.$$

Since $\mathcal{M}(c)$ is uniquely ergodic in this case, for any $\delta>0$, $0\le t<1$, $\gamma^s\colon\mathbb{R}^+\to M$ with $\gamma^s(0)=q\in\mathbb{T}\times(a,2\pi-a)$ and $q^*\in\mathcal{M}_t(c)$ there exists a sequence of $\{K_i\}_{i=1}^\infty$ ($K_i\in\mathbb{Z}^+$) such that

$$d(\gamma^s(t+K_i),q^*)\le\delta.$$

It is easy to construct an absolutely continuous curve $\zeta\colon[s,K_i+t]\to M$ such that $\zeta(t)=\gamma^s(t)$ as $s\le t\le K_i+t-2$, $d(d\zeta(t),d\gamma^s(t))\le\delta$ as $K_i+t-2\le t\le K_i+t$ and $\zeta(K_i+t)=q^*$. As $\bar L^s$ attains its minimum at W^s for each $(q,t)\in U$, it follows from the convexity of L in $\dot q$ and (4.7) that

$$
\begin{aligned}
0\le&\int_s^{K_i+t}\Big(L_c(d\zeta(t),t))-\alpha(c)\Big)dt\\
&-S^s(q^*+(0,2\pi),t)+S^s(q',s)-b(c_1)(q_2^*-q_2)\\
\le&\, o(\delta),
\end{aligned}
$$

where $L_c = L - \langle c, \dot q\rangle$. If $\gamma^s_{K_i}\colon [s, K_i + t] \to M$ is the minimizer of $h_c^{K_i}(q, q^*, s, t)$, then

$$\begin{aligned}
0 \le& \int_s^{K_i+t} \Big(L_c(d\gamma^s_{K_i}(t), t)) - \alpha(c)\Big) dt \\
&- S^s(q^* + (0, 2\pi), t) + S^s(q, s) + b(c_1) \int_s^{K_i+t} \dot\gamma^s_{K_i 2}(t) dt \\
\le& \int_s^{K_i+t} \Big(L_c(d\zeta(t), t)) - \alpha(c)\Big) dt \\
&- S^s(q^* + (0, 2\pi), t) + S^s(q, s) + b(c_1)(q_2^* - q_2) \\
\le& \, o(\delta).
\end{aligned}$$

It is easy to see that $d\gamma^s_{K_i}(t)$ keeps close to the branch of the stable manifold which corresponds to the cohomology class $c = (c_1, b(c_1))$ if K_i is sufficiently large. Thus, we have

$$\int_s^{K_i+t} \dot\gamma^s_{K_i 2}(t) dt = q_2^* + 2\pi - q_2.$$

Therefore, we assert that for all $q \in \mathbb{T} \times (a, 2\pi - a)$, $q^* \in \pi(\tilde\Gamma_t(c))$ and $s, t \in \mathbb{T}$

(4.8)
$$\begin{aligned}
h_c^\infty(q, q^*, s, t) &= S^s(q^* + (0, 2\pi), t) - S^s(q, s) - b(c_1)(q_2^* + 2\pi - q_2), \\
h_c^\infty(q^*, q, s, t) &= S^u(q, s) - S^u(q^*, t) - b(c_1)(q_2 - q_2^*).
\end{aligned}$$

In fact, we have seen that (4.8) holds for $q^* \in \mathcal{M}_t(c)$, $q \in \mathbb{T} \times (a, 2\pi - a)$ or $q \in \pi(\tilde\Gamma|_s)$. As there exists an invariant circle on which the rotation number is irrational, we see that $B_c(q) = P_\omega(q) \equiv 0$ for all $q \in \pi(\Gamma)$, thus $d_c(\hat q, q^*) = 0$ for all $q^* \in \mathcal{M}(c)$ and $\hat q \in \pi(\Gamma)$, where $\omega = \partial_1 \alpha(c)$, P_ω is the Peierls' barrier function. Consequently, we have $h_c^\infty(q, \hat q) = h_c^\infty(q, q^*) + h_c^\infty(q^*, \hat q)$. Therefore, we obtain (4.8) for any $q \in \mathbb{T} \times (a, 2\pi - a)$ and any $q^* \in \pi(\tilde\Gamma_t)$. As $dS^s|_{\pi(\Gamma)} = dS^u|_{\pi(\Gamma)}$, by adding a constant, we can assume that $S^s(q + (0, 2\pi), t) = S^u(q, t)$ if $(q, t) \in \pi(\tilde\Gamma)$. Since the c-minimal measure is uniquely ergodic, we have the following:

Lemma 4.3. *Let $q \in \mathbb{T} \times (a, 2\pi - a)$, then*

(4.9)
$$B_c^*(q) = S^u(q, 0) - S^s(q, 0) - 2\pi b(c_1).$$

Proof. Since $\tilde{\mathcal{M}}(c)$ is uniquely ergodic, by definition of B_c^*, the property $S^s(q+(0,2\pi),t) = S^u(q,t)$ if $(q,t) \in \pi(\tilde{\Gamma})$ and (4.8), we have

$$\begin{aligned}
B_c^*(q) &= \min_{\xi,\eta} \Big\{ h_c^\infty(\xi,q) + h_c^\infty(q,\eta) - h_c^\infty(\xi,\eta) : \xi,\eta \in \mathcal{M}(c) \Big\} \\
&= \min_{\xi} \Big\{ h_c^\infty(\xi,q) + h_c^\infty(q,\xi) : \xi \in \mathcal{M}(c) \Big\} \\
&= S^u(q,0) - S^s(q,0) - 2\pi b(c_1).
\end{aligned}$$

q.e.d.

Next, we consider the stable (unstable) manifold of all invariant circles. Different invariant circle determines different stable and unstable manifold, so we have a family of these manifolds. We claim that this family of stable (unstable) manifolds can be parameterized by some parameter σ so that both $p_1^{s,u}$ and $p_2^{s,u}$ have $\frac{1}{2}$-Hölder continuity in σ. Indeed, we arbitrarily choose one circle Γ_0 and parameterize another circle Γ_σ by the algebraic area between Γ_σ and Γ_0,

$$\sigma = \int_0^1 (\Gamma_\sigma(q_1) - \Gamma_0(q_1))dq_1. \tag{4.10}$$

This integration is in the sense that we pull it back to the standard cylinder by $\psi \circ \psi_1 \in \text{diff}(\Sigma_0, \Sigma)$ (cf. (3.1)). In this way, we obtain one-parameter family curves Γ: $\mathbb{T} \times \mathbb{S} \to \Sigma$ in which $\mathbb{S} \subset [A', B']$ is a closed set. Usually, $\mathbb{S}$ is a Cantor with positive Lebesgue measure, A' and B' correspond to the curves where the action $p_1 \le A$ and $p_1 \ge B$ respectively. Clearly, for each $\sigma \in \mathbb{S}$, there is only one $c_1 = c_1(\sigma)$ such that $\Gamma_\sigma = \tilde{\mathcal{M}}_0(c)$ for all $c \in I(c_1(\sigma))$ as the rotation number is irrational. We can think Γ_σ as a map to function space C^0 equipped with supremum norm Γ: $\mathbb{S} \to C^0(\mathbb{T}, \mathbb{R})$,

$$\|\Gamma_{\sigma_1} - \Gamma_{\sigma_2}\| = \max_{q_1 \in \mathbb{T}} |\Gamma(q_1,\sigma_1) - \Gamma(q_1,\sigma_2)|.$$

Direct calculation shows

$$|\sigma_1 - \sigma_2| \ge \frac{1}{2C_h} \left(\max_{q_1 \in \mathbb{T}} |\Gamma(q_1,\sigma_1) - \Gamma(q_1,\sigma_2)| \right)^2,$$

where C_h is the Lipschitz constant for the twist map, it follows that

$$\|\Gamma_{\sigma_1} - \Gamma_{\sigma_2}\| \le C_s |\sigma_1 - \sigma_2|^{\frac{1}{2}}, \tag{4.11}$$

where $C_s = \sqrt{2C_h}$. Since the stable (unstable) fibers have C^{r-2}-smoothness on their base points on Σ, $p_\sigma^{s,u}$ is also $\frac{1}{2}$-Hölder continuous in

σ. Thus, there exist two families of $C^{1,1}$ functions $S^u_\sigma(q,t)$: $\{(q,t) : q_{2\Gamma_\sigma}(q_1,t) \le q_2 \le 2\pi - a\} \to M$ and $S^s_\sigma(q,t)$: $\{(q,t) : a \le q_2 \le q_{2\Gamma_\sigma}(q_1,t)+2\pi\} \to M$, which are also $\frac{1}{2}$-Hölder continuous in σ. Remember for each $\sigma \in \mathbb{S}$, $B^*_{c(\sigma)}(q)$ can always take the value zero as its minimum in the region $\{a \le q_2 \le 2\pi - a\}$, it follows from the $\frac{1}{2}$-Hölder continuity of $S^{s,u}_{c(\sigma)}$ and the expression of $B^*_{c(\sigma)}$ given by (4.9) that $b(c_1(\sigma))$ also has $\frac{1}{2}$-Hölder continuity in σ. For $z \in \mathbb{T}$, there is unique $z_\sigma(t) \in \pi(\tilde{\Gamma}_{\sigma t})$ such that $z_\sigma(t) = (z, q_{2\Gamma_\sigma}(z,t))$. Let $c(\sigma) = (c_1(\sigma), b(c_1(\sigma)))$, we have:

Lemma 4.4. *For all $q \in \mathbb{T} \times (a, 2\pi - a)$, $z \in \mathbb{T}$ and $s,t \in \mathbb{T}$ the functions $S^{s,u}_\sigma(q)$, $h^\infty_{c(\sigma)}(q, z_\sigma(t), s, t)$, $h^\infty_{c(\sigma)}(z_\sigma(t), q, s, t)$ and $B^*_{c(\sigma)}(q)$ are $\frac{1}{2}$-Hölder continuous in $\sigma \in \mathbb{S}$.*

Different from B^*_c, h^∞_c depends on the choice of the closed 1-form η_c (cf. [**15**]). To guarantee the Hölder continuity, we choose $\eta_c = \langle c(\sigma), \dot{q}\rangle$ in above lemma.

5. Construction of connecting orbits

Throughout this section, we shall make the following hypotheses, their verification shall be postponed to Section 6.

(H1): For each $\sigma \in \mathbb{S} \subset [A', B']$, the set $\{B^*_{c(\sigma)} = 0\} \cap \{a \le q_2 \le 2\pi - a\}$ is totally disconnected.

Remark. By the choice of a, the set $\{B^*_{c(\sigma)} = 0\} \cap \{a \le q_2 \le 2\pi - a\}$ is not empty since $d\gamma_2$ cannot cross the strip $\{a \le q_2 \le 2\pi - a\}$ under one step of the map ϕ, there must be some points on time-1-section of the minimal orbits whose projection fall into the strip. By the definition of $\mathbb{S}$, for each $\sigma \in \mathbb{S}$, $\tilde{\mathcal{A}}_0(c(\sigma))$ contains an invariant circle on the cylinder. In this case, we have an explicit expression of $B^*_c(q)$ in the strip. The hypothesis (H1) implies the minimal critical point set of $S^s_{c(\sigma)} - S^u_{c(\sigma)}$ consists of discrete points, and there must be some minimal points in the interior of this strip.

(H2): If the rotation number of Γ is rational, then the associated c-minimal measure has its support only at a periodic orbit. The set of minimal homoclinic orbits in Σ to this periodic orbit is topologically trivial.

Before making the third hypothesis, let us note that the union of all invariant circles on the cylinder forms a closed set. These circles do not intersect each other, so the complementary set consists of countably many invariant annulus.

(H3): Let Γ be an invariant circle on Σ, associated with co-homology class c. If this circle is on the boundary of a gap, then for small $\delta > 0$, there exists $c' = (c_1, c'_2)$ with either $0 < c'_2 - b(c_1) < \delta$ or $-\delta < c'_2 - a(c_1) < 0$ such that $\mathcal{M}(c')$ is uniquely ergodic.

According to the study in the last section, we know that $\tilde{\mathcal{N}}_0(c')$ is homotopically trivial, but this does not guarantee that $\mathcal{N}_0(c')$ is also homotopically trivial on M, since the projection from $\tilde{\mathcal{N}}(c') \to \mathcal{N}(c')$ is not necessarily injective. If $\tilde{\mathcal{M}}(c')$ is uniquely ergodic, then $\tilde{\mathcal{N}}(c') = \tilde{\mathcal{A}}(c')$. The Lipschitz property of $\mathcal{A}(c')$ implies that $\mathcal{N}_0(c')$ is homotopically trivial in this case. Given arbitrary small $d > 0$, there are only finitely many invariant circles which are the boundary of some annulus with width not smaller than d. Actually, we require the third hypothesis only for these tori.

The first task in this section is to build a C-equivalent sequence $\{c^{(i)}\}_{i=1}^m$, where $c_1^{(1)} = c_1(\sigma')$, $a(c_1^{(1)}) \leq c_2^{(1)} \leq b(c_1^{(1)})$, $c_1^{(m)} = c_1(\sigma^*)$, $a(c_1^{(m)}) \leq c_2^{(m)} \leq b(c_1^{(m)})$ and $\sigma' < \sigma^*$ correspond to two invariant circles which make up the whole boundary of a gap. Thus, a theorem of connecting C-equivalent Mañé sets is used to construct the diffusion orbits crossing this gap. This kind of theorem was discovered by Mather in [**15**] where the proof was sketched. To make use of this theorem, we shall give a complete proof first. A theorem of connecting different $\mathcal{G}(c)$ was proved by Bernard recently [**4**].

To any subset A of M, we associate a subspace of $H_1(M, \mathbb{R})$

$$V(A) = \bigcap \Big\{ i_{U*} H_1(U, \mathbb{R}) : U \text{ is an open neighborhood of } A \Big\}, \tag{5.1}$$

where i_{U*}: $H_1(U, \mathbb{R}) \to H_1(M, \mathbb{R})$ is the map induced by the inclusion. Clearly, there exists an open neighborhood U of A such that $V(A) = i_{U*} H_1(U)$. Let $V^{\perp}(A)$ be the annihilator of $V(A)$. In other words, if $c \in H^1(M, \mathbb{R})$, then $c \in V^{\perp}$ if and only if $\langle c, h \rangle = 0$ for all $h \in V(A)$. Given $c \in H^1(M, \mathbb{R})$, we define

$$R(c) = \sum_{t \in \mathbb{T}} (V(\mathcal{N}_t(c)))^{\perp}. \tag{5.2}$$

In [**4**], $R(c)$ is defined by using $\mathcal{G}(c)$ instead of using $\mathcal{N}(c)$.

We say a continuous curve Γ: $\mathbb{R} \to H^1(M, \mathbb{R})$ is admissible if for each $t \in \mathbb{R}$ there exists $\delta > 0$ such that $\Gamma(t) - \Gamma(t_0) \in R(\Gamma(t_0))$ for all

$t \in [t_0 - \delta, t_0 + \delta]$. We say c, $c' \in H^1(M, \mathbb{R})$ are C-equivalent if there is an admissible curve Γ: $[0, 1] \to M$ such that $\Gamma(0) = c$ and $\Gamma(1) = c'$.

Let U be an open subset of $M \times \mathbb{T}$, we can think it as the open subset in $M \times \mathbb{R}$ of points (q, t) such that $(q, t \bmod 1) \in U$. The 1-form μ on $M \times \mathbb{R}$ is called a U-step form if there is a closed form $\bar{\mu}$ on $M \times \mathbb{T}$, also considered as a periodic 1-form on $M \times \mathbb{R}$, such that the restriction of μ to $t \leq 0$ is 0, the restriction of μ to $t \geq 1$ is $\bar{\mu}$, and such that the restriction of μ to the set $U \cup \{t \leq 0\} \cup \{t \geq 1\}$ is closed. In the application in this paper, $\bar{\mu}$ is chosen as a closed form on M.

If the first de Rham cohomology class $d \in R(c)$, then there exists an open neighborhood U of $\mathcal{N}(c)$ and a U-step form μ such that $[\bar{\mu}] = d$. Such a neighborhood U will be called an adapted neighborhood. Indeed, similar to the arguments in [**4**], let us fix a time $t \in [0, 1]$ and a cohomology class $d \in V(\mathcal{N}_t(c))^{\perp}$. There exist an open neighborhood Ω of $\mathcal{N}_t(c)$ and a $\delta > 0$ such that $V(\Omega) = V(\mathcal{N}_t(c))$ and such that $\mathcal{N}_s(c) \subset \Omega$ for all $s \in [t - \delta, t + \delta]$. As $d \in R(c)$, we can take a closed form $\bar{\mu}$ on M whose support is disjoint from Ω and such that $[\bar{\mu}] = d$. Let ρ: $\mathbb{R} \to \mathbb{R}$ be a smooth function such that $\rho = 0$ on $(-\infty, t - \delta]$, $\rho = 1$ on $[t + \delta, \infty)$ and $0 \leq \rho \leq 1$ for all $t \in \mathbb{R}$ and let $U = M \times ((0, t - \delta) \cup (t + \delta, 1)) \cup \Omega \times [t - \delta, t + \delta]$. Obviously, the form

$$\mu = \rho(t)\bar{\mu}$$

is an U-step form satisfying the required conditions.

Let Γ: $[0, 1] \to H^1(M, \mathbb{R})$ be an admissible curve such that $\Gamma(0) = c$ and $\Gamma(1) = c'$. For each $t \in [0, 1]$ and an adapted neighborhood $U(t)$, let $\eta(t)$ be a closed 1-form on M such that $[\eta(t)] = \Gamma(t)$. There exists $\delta(t) > 0$ such that $\Gamma(s) - \Gamma(t) \in R(\Gamma(t))$ and a U-step form $\mu(s)$ with $[\bar{\mu}(s)] = \Gamma(s) - \Gamma(t)$ if $s \in (t - \delta, t + \delta)$. According to the upper semi-continuity $(\eta, \mu) \to \tilde{\mathcal{N}}_{\eta,\mu}$ proved in Lemma 2.4, we can assume that

$$\pi(\tilde{\mathcal{N}}_{\eta(t),\mu(s)}) + \epsilon(t) \subset U(t) \tag{5.3}$$

if we take suitably small $\delta(t)$. In this paper, we use $U + a$ to denote the set $\{x \in M : \operatorname{dist}(x, U) \leq a\}$. Clearly, there is a finite increasing sequence $\{t_i\}_{0 \leq i \leq N}$ such that

$$\begin{gathered} \bigcup_{i=0}^{N} (t_i - \delta(t_i), t_i + \delta(t_i)) \supset [0, 1], \\ t_{i-1} > t_i - \delta(t_i), \qquad t_{i+1} < t_i + \delta(t_i) \end{gathered} \tag{5.4}$$

and (5.3) holds for each t_i, and each $s \in (t_i - \delta(t_i), t_i + \delta(t_i))$. In the following, we shall use ϵ_i, δ_i, U_i, η_i and μ_i to denote $\epsilon(t_i)$, $\delta(t_i)$, $U(t_i)$, $\eta(t_i)$ and $\mu(t_i)$ respectively. Thus, we have

$$\eta_i = \eta_0 + \sum_{j=0}^{i-1} \bar{\mu}_j. \tag{5.5}$$

Let us fix some $0 \le i \le N$ and consider the function $h^{T_0,T_1}_{\eta_i,\mu_i}(m_0, m_1)$ defined in (2.13). For each small $\epsilon_i^* > 0$ and $(m_0, m_1) \in M \times M$ there exists $(\breve{T}_0^i, \breve{T}_1^i) = (\breve{T}_0^i, \breve{T}_1^i)(\epsilon_i^*, m_0, m_1) \in \mathbb{Z}^+$ such that

$$h^{T_0,T_1}_{\eta_i,\mu_i}(m_0, m_1) \ge h^{\infty}_{\eta_i,\mu_i}(m_0, m_1) - \epsilon_i^* \quad \forall T_j \ge \breve{T}_j^i, \quad j = 0, 1. \tag{5.6}$$

Obviously, there are infinitely many $T_j \ge \breve{T}_j^i$ $(j = 0, 1)$ such that

$$|h^{T_0,T_1}_{\eta_i,\mu_i}(m_0, m_1) - h^{\infty}_{\eta_i,\mu_i}(m_0, m_1)| \le \epsilon_i^*. \tag{5.7}$$

Let $\gamma_i\,(t,\, m_0,\, m_1,\, T_0,\, T_1) : [-T_0,\, T_1] \to M$ be the minimizer of $h^{T_0,T_1}_{\eta_i,\mu_i}$ (m_0, m_1), it follows from Lemma 2.3 that if $\epsilon_i^* > 0$ is sufficiently small, $\breve{T}_j^i$ $(j = 0, 1)$ are sufficiently large, and T_0, T_1 are chosen so that (5.7) holds, then

$$d\gamma_i(t, m_0, m_1, T_0, T_1) \in \tilde{\mathcal{N}}_{\eta_i,\mu_i}(t) + \epsilon_i \quad \forall 0 \le t \le 1. \tag{5.8}$$

From the Lipschitz property of $h^{T_0,T_1}_{\eta_i,\mu_i}(m_0, m_1)$ in (m_0, m_1) and the compactness of M, we see that there are $\breve{T}_j^i = \breve{T}_j^i(\epsilon_i)$ $(j = 0, 1)$, independent of (m_0, m_1), so that (5.6) holds for all $T_j \ge \breve{T}_j^i$. We can see also that there exist $\hat{T}_j^i(\epsilon_i) > \breve{T}_j^i(\epsilon_i)$ $(j = 0, 1)$ so that for any $(m_0, m_1) \in M \times M$, there are $T_j = T_j(m_0, m_1)$ with $\breve{T}_j^i \le T_j \le \hat{T}_j^i$ $(j = 0, 1)$ such that (5.7) and consequently (5.8) hold. Note that for different (m_0, m_1), we may need different $T_j \ge \breve{T}_j^i$.

We are now ready to construct a connecting orbit joining $\mathcal{N}(c_0)$ and $\mathcal{N}(c_N)$. We consider τ_i as the time translation $(q, t) \to (q, t + \tau_i)$ on $M \times \mathbb{R}$, and define the modified Lagrangian

$$\tilde{L} = L - \eta_0 - \sum_{i=0}^{N-1} (-\tau_i)^* \mu_i. \tag{5.9}$$

For each $\vec{\tau} = (\tau_0, \tau_1, \dots, \tau_{N-1})$, the following variational problem

$$
\begin{aligned}
&h_{\tilde{L}}^{T_0,T_N}(m, m', \vec{\tau}) \\
&\quad = \inf_{\substack{\gamma(-T_0)=m \\ \gamma(T_N+\tau_{N-1})=m'}} \int_{-T_0}^{T_N+\tau_{N-1}} \left(L - \eta_0 - \sum_{i=0}^{N-1} (-\tau_i)^* \mu_i \right) (d\gamma(t), t) dt \\
&\qquad - \sum_{i=1}^{N-1} (\tau_i - \tau_{i-1}) \alpha(c_i) - T_0 \alpha(c_0) - T_N \alpha(c_N)
\end{aligned}
$$

has a C^1-minimizer $\gamma(t, m, m', \vec{\tau}, T_0, T_N)$ which is clearly the solution of the Euler–Lagrangian equation determined by $\tilde{L}$. We need to show that it can be the extremal of L if we suitably choose $\vec{\tau}$, T_0 and T_N. We define

$$
\begin{aligned}
\Lambda = \Big\{ \vec{\tau} \in \mathbb{Z}^N : \max\{\breve{T}_0^i, \breve{T}_1^{i-1} + 1\} \le \tau_i - \tau_{i-1} \le \max\{\hat{T}_0^i, \hat{T}_1^{i-1} + 1\} \\
\forall 1 \le i \le N-1, \tau_0 = 0 \Big\}
\end{aligned}
$$

and take the minimum of $h_{\tilde{L}}^{T_0,T_N}(m, m', \vec{\tau})$ over Λ

$$
F_{\tilde{L}}(m, m', T_0, T_N) = \min_{\vec{\tau} \in \Lambda} h_{\tilde{L}}^{T_0,T_N}(m, m', \vec{\tau}). \tag{5.10}
$$

Let $\vec{\tau}^*(T_0, T_N)$ be the minimal point about $\vec{\tau}$. If $\gamma(t, m, m', T_0, T_N)$ is the minimizer of $F_{\tilde{L}}(m, m', T_0, T_N)$, we claim that for $t \in [\tau_i, \tau_i + 1]$ and $0 < i < N-1$

$$
d\gamma(t, m, m', T_0, T_N) \in (-\tau_i)^* \left(\tilde{\mathcal{N}}_{\eta_{i-1}, \mu_i}|_t \right) + \epsilon_i. \tag{5.11}
$$

In fact, let us to choose $m_i = \gamma(\tau_{i-1}+1)$, $m_i' = \gamma(\tau_{i+1})$ for $0 < i < N-1$. Since $\gamma(t, m, m', T_0, T_N)$ is the minimizer of $F_{\tilde{L}}(m, m', T_0, T_N)$, thus

(5.12)

$$
\begin{aligned}
A_{\eta_{i-1}, \mu_i}((-\tau_i)^* \gamma|_{\tau_{i-1}+1}^{\tau_{i+1}}) = &\inf_{\substack{\gamma^*(-T_0)=m_i \\ \gamma^*(T_1)=m_i' \\ \breve{T}_0^i \le T_0 \le \hat{T}_0^i \\ \breve{T}_1^i \le T_1 \le \hat{T}_1^i}} \int_{-T_0}^{T_1} (L - \eta_i - \mu_i)(d\gamma^*(t), t) dt \\
&- T_0 \alpha(c_i) - T_1 \alpha(c_{i+1}).
\end{aligned}
$$

So, we obtain (5.11) from (5.6–5.8), (5.12) and the choice of $\breve{T}_j^i$ as well as $\hat{T}_j^i$ $(j = 0, 1)$. We define the infimum limit of $F_{\tilde{L}}(m, m', T_0, T_N)$

$$h_{\tilde{L}}^{\infty}(m, m') = \liminf_{T_0, T_1 \to \infty} F_{\tilde{L}}(m, m', T_0, T_N). \tag{5.13}$$

Let T_j^k $(j = 0, N)$ be the subsequences such that $T_j^k \to \infty$ as $k \to \infty$

$$|F_{\tilde{L}}(m, m', T_0^k, T_N^k) - h_{\tilde{L}}^{\infty}(m, m')| \le \min\{\epsilon_0^*, \epsilon_N^*\} \quad \forall\ k,$$

as well as

$$\lim_{k \to \infty} F_{\tilde{L}}(m, m', T_0^k, T_N^k) = h_{\tilde{L}}^{\infty}(m, m')$$

and let $\gamma_k(t, m, m') = \gamma(t, m, m', T_0^k, T_N^k)$ be the minimizer of $F_{\tilde{L}}(m, m', T_0^k, T_N^k)$. It is easy to see that (5.11) holds also for $i = 0, N$. From (5.3), (5.12) and the definition of U_i, we obtain that $d\gamma_k(t)$ is the extremal of L with the boundary condition $\gamma_k(-T_0^k) = m$, $\gamma_k(T_N^k + \tau_{N-1}^*) = m'$. Clearly, for any compact interval $[a, b]$ the set $\{\gamma_k\}_{k \ge \bar{k}}$ is pre-compact in the $C^1([a, b], M)$ topology if $\bar{k}$ is suitably large. Let γ: $\mathbb{R} \to M$ be the accumulation point of $\{\gamma_k\}$, then $d\gamma$ is the solution of the Euler–Lagrange equation determined by L and

$$\alpha(d\gamma) \subseteq \tilde{\mathcal{A}}(c_0), \qquad \omega(d\gamma) \subseteq \tilde{\mathcal{A}}(c_N).$$

Consider a bi-infinite sequence $(\ldots, c_i, \ldots)$ of C-equivalent cohomology classes and a sequence $(\ldots, \varepsilon_i, \ldots)$ of small positive numbers. Let $\{\tau_i\}_{-\infty}^{\infty}$ be a monotone sequence of integers such $\tau_0 = 0$, $\tau_i \to \pm\infty$ as $i \to \pm\infty$. Let

$$\nu_N = \sum_{i=-N}^{N} (-\tau_i)^* \mu_i.$$

For each $\vec{\tau}_N = (\tau_{-N}, \ldots, \tau_{N-1})$, we consider the following variational problem

$$\begin{aligned} h_{\tilde{L}}^{T_{-N}, T_N}(m, m', \vec{\tau}_N) &= \inf_{\substack{\gamma(-T_{-N} - \tau_{-N}) = m \\ \gamma(T_N + \tau_{N-1}) = m'}} \int_{-T_{-N} - \tau_{-N}}^{T_N + \tau_{N-1}} \Big(L - \eta_0 - \nu_N\Big)(d\gamma(t), t)dt \\ &\quad - \sum_{i=-N+1}^{N-1} (\tau_i - \tau_{i-1})\alpha(c_i) - T_{-N}\alpha(c_{-N}) - T_N\alpha(c_N). \end{aligned}$$

Let Λ_N be the set of $2N$ dimensional integer vectors defined in the same way as for Λ with the subscripts ranging over $(-N,\dots,N-1)$ instead of $(0,\dots,N-1)$. Let $\gamma_N(t,m,m',T_{-N},T_N)$ be the minimizer of

$$F_{\tilde L}(m,m',T_{-N},T_N)=\min_{\vec\tau\in\Lambda_N} h_{\tilde L}^{T_{-N},T_N}(m,m',\vec\tau_N).$$

With the same arguments above, we can make $\gamma_N(t,m,m',T_{-N},T_N)$ be the extremal of L by choosing suitably large T_{-N}, and T_N. From (5.3) and (5.11), we can see that $d\gamma_N$ passes within a distance of ε_i of each $\tilde{\mathcal L}(c_i)$ for $-N\le i\le N$ if we set $\hat T^i_j$ suitably large for each $j=0,1$ and each $-N\le i\le N$. Let γ: $\mathbb R\to M$ be an accumulation point of the set $\{\Gamma_N\}_{N\ge N_0}^\infty$, $d\gamma$ clearly determines a trajectory of the Euler–Lagrange flow of L which passes within a distance of ε_i of each $\tilde{\mathcal A}(c_i)$ for all $i\in\mathbb Z$. Therefore, we have proved the theorem:

Theorem 5.1 ([**15**])**.** *Suppose c_0 and c_N are C-equivalent classes. Then, there is a trajectory of the Euler–Lagrange flow of L whose α-limit set lies in $\tilde{\mathcal A}(c_0)$ and whose ω-limit set lies in $\tilde{\mathcal A}(c_N)$.*

Consider a bi-infinite sequence $(\dots,c_i,\dots)$ of C-equivalent cohomology classes and a sequence $(\dots,\varepsilon_i,\dots)$ of small positive numbers. Then, there is a trajectory of the Euler–Lagrange flow of L which passes within a distance of ε_i of each $\tilde{\mathcal A}(c_i)$ in turn.

The next step is to establish C-equivalence among some Mañé sets of the special L given by (2.2). Let us consider the first de Rham cohomology class $c\in H^1(M,\mathbb R)$ such that the support of c-minimal measure uniquely sits on $\tilde\Gamma\subset\tilde\Sigma$. First, we consider the case that Γ is a Denjoy set and there is no invariant circle containing Γ. The rotation number of Γ is irrational. By the well-known knowledge, we see that the β-function for the twist map is differentiable at the point of irrational number, it implies that there is only one c_1 such that $\tilde\Gamma$ is the support of c-minimal measure if $c\in \mathrm{int}I(c_1)$. We see from the Lemma 3.4 that $\tilde{\mathcal N}(c)=\tilde{\mathcal M}(c)$ when $a(c_1)<c_2<b(c_1)$. By the upper semi-continuity of $c\to\tilde{\mathcal N}(c)$, we find that there exists $\delta>0$, if $c'\in J=((c_1-\delta,c_1+\delta)\times(a(c_1)+\delta,b(c_1)-\delta)$, then $\mathcal N(c')$ is in a small neighborhood of $\pi(\tilde\Gamma)$. Thus, each of such $\mathcal N_0(c)$ is homotopically trivial. Therefore, all $c'\in J$ are C-equivalent.

Next, let us consider the case when Γ consists of single periodic orbit. Since the β-function of the twist map has a corner at the rational rotation number, there is a flat piece of the α-function of the twist map, over the interval $[c_1^-, c_1^+]$. Consequently, there is a rectangle $(c_1^-, c_1^+) \times (a(c_1), b(c_1)) \in H^1(M, \mathbb{R})$ such that all c-minimal measures have their support on $\tilde{\Gamma}$ if c is in this rectangle. When $c_1^- < c_1 < c_1^+$, $a(c_1) < c_2 < b(c_1)$, $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{M}}(c)$. When $a(c_1) < c_2 < b(c_1)$ and $c_1 = c_1^-$ or $c_1 = c_1^+$, $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{M}}(c) \cup \{$minimal homoclinic orbit in $\tilde{\Sigma}\}$. Due to the upper semi-continuity of $c \to \tilde{\mathcal{N}}(c)$ and the hypothesis of (H2), we find that there exists $\delta > 0$, if $c' \in J = ((c_1^- - \delta, c_1^+ + \delta) \times (a(c_1) + \delta, b(c_1) - \delta)$, then $\mathcal{N}_0(c)$ is homotopically trivial. Thus, all $c' \in J$ are C-equivalent.

Finally, we consider the case when Γ is contained in an invariant circle on the boundary of a gap. In this case, $\tilde{\Gamma}$ is the support of that c-minimal measure with $c \in I(\bar{c}_1) = \{(\bar{c}_1, c_2) : a(\bar{c}_1) \le c_2 \le b(\bar{c}_1)\}$. Because of the hypotheses (H1), (H3) and in virtue of the Lemma 3.3–3.5, we have $\mathcal{N}_0(c) \subset U = \cup_{i=0}^m U_i$, where $U_i \cap U_j = \varnothing$ if $i \ne j$, U_0 is an open neighborhood of Γ, all other U_i $(i \ne 0)$ are open set contractible to one point. Let $J = (\bar{c}_1 - \delta, \bar{c}_1 + \delta) \times (a(\bar{c}_1) - \delta, b(\bar{c}_1) + \delta)$. Due to the upper semi-continuity of $c \to \tilde{\mathcal{N}}(c)$, we can see that $\mathcal{N}_0(c) \subset U$ for all $c \in J$ if $\delta > 0$ is sufficiently small. To establish the C-equivalent relationship between any two $c, c' \in J$, let us consider first the special case when $c, c' \in J$ and $c - c' = (0, c_2 - c_2')$. Let $\Gamma(s) = (c_1, sc_2 + (1-s))c_2'$ for $0 \le s \le 1$, obviously, $[dq_2]$ is the annihilator of $V_{\Gamma(s)}(t)$ $\forall s \in [0,1]$, $t \in \mathbb{T}$. Thus, $\Gamma(s)$ is an admissible curve. Second, let us consider the case when $c = (c_1, c_2) \in J$, but $c_2 > b(\bar{c}_1)$ or $c_2 < a(\bar{c}_1)$. Under the hypotheses (H1) and (H3), for any $\delta > 0$, there exists $c = (\bar{c}_1, c_2)$ with $b(\bar{c}_1) < c_2 < b(\bar{c}_1) + \delta$ or $a(\bar{c}_1) - \delta < c_2 < a(\bar{c}_1)$ such that $\mathcal{N}_t(c)$ is homopotically trivial for any $t \in \mathbb{T}$. Therefore, $\exists\, \delta' > 0$ such that for all $c' \in B_{\delta'}(c)$, $\mathcal{N}_0(c')$ is homotopically trivial. Replacing δ with δ' in the definition of J, we find that all $c \in J$ are C-equivalent. In fact, given any two c, $c' \in J$, we can construct the admissible curve as follows. Let Γ: $[0,3] \to H^1(M, \mathbb{R})$,

$$\Gamma(s) = \begin{cases} sc + (1-s)\tilde{c}, & 0 \le s \le 1, \\ (s-1)\tilde{c} + (2-s)\tilde{c}', & 1 \le s \le 2, \\ (s-2)\tilde{c}' + (3-s)c', & 2 \le s \le 3 \end{cases}$$

in which $\tilde{c}$ and $\tilde{c}' \in J$ are defined in the way $\tilde{c}_2 = \tilde{c}_2' > b(\bar{c}_1)$ or $\tilde{c}_2 = \tilde{c}_2' < a(\bar{c}_1)$, $\tilde{c}_1 = c_1$ and $\tilde{c}_1' = c_1'$, both $\mathcal{N}_t(\tilde{c})$ and $\mathcal{N}_t(\tilde{c}')$ are homotopically trivial.

Lemma 5.2. *We assume the hypotheses* (H1–H3). *Let* $\hat{c} = (c_1(\sigma'), 0)$ *and* $\bar{c} = (c_1(\sigma^*), 0)$ *be two co-homology classes such that* $\tilde{\mathcal{N}}_0(\hat{c})$ *and* $\tilde{\mathcal{N}}_0(\bar{c})$ *make up the whole boundary of some given gap with* $\sigma' < \sigma^*$. *Then,* $\hat{c}$ *and* $\bar{c}$ *are* C*-equivalent.*

Proof. By assumption, there is no other invariant circle between $\mathcal{N}_0(\hat{c})$ and $\mathcal{N}_0(\bar{c})$. In this case, we have shown that for any $c_1(\sigma') \le c_1 \le c_1(\sigma^*)$, there is an open rectangle $J(c_1) \subset H^1(M, \mathbb{R})$ containing $(c_1, 0)$ such that all $c \in J(c_1)$ are C-equivalent. By the compactness of the interval $[c_1(\sigma'), c_1(\sigma^*)]$, there is a sequence $\{c_1^{(i)}\}_{i=0}^m$ such that $\cup_{i=0}^m J(c_1^{(i)}) \supset [c_1(\sigma'), c_1(\sigma^*)] \times \{0\}$. Obviously, the C-equivalence has transitivity. q.e.d.

This C-equivalence establishes the existence of the diffusion orbits crossing gaps as we have the Theorem 5.1.

To go further, we need to know more details of U-step forms. Let η_j be any given closed 1-form such that $[\eta_j] = c^{(j)}$ for $j = 1, k$. A natural question is whether there exists such kind of $\mu(t)$ so that $\mu(t) = \eta_1$ for $t \le 0$ and $\mu(t) = \eta_k$ for $t \ge \tau_k + 1$ even though $c^{(1)}$ is equivalent to $c^{(k)}$? In general, we do not know whether it is true or not, but in our case, the answer is yes.

Lemma 5.3. *Let* $c^{(1)} = (c_1^{(1)}, c_2^{(1)})$, $c^{(k)} = (c_1^{(k)}, c_2^{(k)})$ *be two cohomology classes connected by an admissible curve* Γ, *where* $a(c_1^{(1)}) \le c_2^{(1)} \le b(c_1^{(1)})$, $a(c_1^{(k)}) \le c_2^{(k)} \le b(c_1^{(k)})$, *and* $\tilde{\mathcal{M}}(c^{(1)})$, $\tilde{\mathcal{M}}(c^{(k)}) \subset \tilde{\Sigma}$. *Let* η_1, η_k *be two closed one forms such that* $[\eta_1] = c^{(1)}$, $[\eta_k] = c^{(k)}$. *Then, there exists a composition of finite* U*-step forms* $\mu(t)$ *such that* $\mu(t) = \eta_1$ *for* $t \le 0$ *and* $\mu(t) = \eta_k$ *for* $t \ge \tau_k + 1$.

Proof. Since Φ is an area-preserving twist map when it is restricted on the cylinder, by the hypothesis (H2), there is some c with $c_1^{(1)} < c_1 < c_1^{(k)}$, $c_2 = 0$ such that its semi-static minimal orbit set consists of single m-periodic orbit with $m > 1$. Thus, for each $s \in \mathbb{T}$, $\mathcal{N}_s$ consists of several points, $\mathcal{N}_s(c) = \cup\{q_i(s)\}$. Consequently, there exist $\delta > 0$, and $0 < s_1 < s_2 < 1$ such that

$$\Big(\cup\{q_i(s_1)\} + 3\delta\Big) \cap \Big(\cup\{q_i(s_2)\} + 3\delta\Big) = \varnothing.$$

There also exists $\epsilon > 0$ such that $0 < s_1 - \epsilon < s_1 + \epsilon < s_2 - \epsilon < s_2 + \epsilon < 1$ and $\cup\{q_i(s)\} \subset \cup\{q_i(s_j)\} + \frac{1}{2}\delta$ when $|s - s_j| < \epsilon$ for $j = 1, 2$.

Let η be an any exact 1-form, we claim there exists a U-step form ν such that $\nu(t) = 0$ for $t \leq 0$ and $\nu(t) = \eta$ for $t \geq 1$, where U is a neighborhood of $\mathcal{N}(c) = \cup_{s\in\mathbb{T}} \cup_i \{q_i(s)\}$. Let F: $M \to \mathbb{R}$ be the function such that $\eta = dF$. Let $\lambda_\delta(q)$: $M \to \mathbb{R}$ be a smooth function $\lambda_\delta = 1$ when $\|q\| \leq \delta$, $0 < \lambda_\delta < 1$ when $\delta < \|q\| < 2\delta$ and $\lambda_\delta = 0$ when $\|q\| \geq 2\delta$. Let

$$F^* = \left(1 - \sum_{i=1}^{k} \lambda_\delta(q - q_i(s_1))\right) F, \qquad \tilde{F} = \left(\sum_{i=1}^{k} \lambda_\delta(q - q_i(s_1))\right) F,$$

obviously, $\operatorname{supp}(dF^*) \cap (\cup\{q_i(s_1)\} + 2\delta) = \varnothing$, $\operatorname{supp}(d\tilde{F}) \cap (\cup\{q_i(s_2)\} + 2\delta) = \varnothing$. If we choose

$$\nu = \rho(t - s_1 + \epsilon)dF^* + \rho(t - s_2 + \epsilon)d\tilde{F},$$

where $\rho = 0$ for $t \leq 0$, $0 < \rho < 1$ for $0 < t < 2\epsilon$ and $\rho = 1$ for all $t \geq 2\epsilon$, then $\nu(t) = 0$ for $t \leq s_1 - \epsilon$ and $\nu(t) = dF$ for $t \geq s_2 + \epsilon$. Let $U = \cup_{j=1,2}((\cup\{q_i(s_j)\} + \delta) \times [s_j - \epsilon, s_j + \epsilon]) \cup M \times ([0, s_1 - \epsilon] \cup [s_1 + \epsilon, s_2 - \epsilon] \cup [s_2 + \epsilon, 1])$, then $d\nu|_U = 0$.

Since both $[\eta_1]$ and $[\eta_2]$ are C-equivalent to c, there are two compositions of U-step forms ν_1, ν_2 such that

$$\begin{aligned} &\eta_1 + \nu_1(t) = \langle c, dq\rangle + dF_1, && t \geq \tau_1;\\ &\nu_2(t) = 0, && t \leq \tau_1 + 1,\\ &\langle c, dq\rangle + \nu_2(t) = \eta_2 + dF_2, && t \geq \tau_2. \end{aligned}$$

By the demonstration above, there is a U-step form ν such that $\nu(t) = -d(F_1 + F_2)$ when $t \geq 1$. Clearly, the 1-form $\mu = (-\tau_1)^*\nu + \nu_1 + \nu_2$ is what we are looking for. q.e.d.

The remaining work in this section is to join the orbit crossing the gaps smoothly with the orbit constructed via Arnold's mechanism. We shall make use of some ideas developed in [**5**] and in [**6**], it is showed that the diffusion orbits in several examples, constructed by transition chains, are actually the orbits which locally minimize the Lagrangian action.

Let us consider the barrier function of those cohomology classes corresponding to an invariant circle Γ_c on the cylinder. In this case,

$\mathcal{M}_0(c) \subseteq \Gamma_c$ and $d_c(\xi, \xi') = 0$ for all $\xi, \xi' \in \pi(\Gamma_c)$. Thus

$$B_c^*(q) = \min_{\xi,\eta \in \mathcal{M}(c)} \{h_c^\infty(\xi, q) + h_c^\infty(q, \eta) - h_c^\infty(\xi, \eta)\} \\ = h_c^\infty(\xi, q) + h_c^\infty(q, \xi) \quad \forall\, \xi \in \pi(\Gamma_c). \tag{5.14}$$

Under the hypothesis (H1), the set $\{B^*_{c(\sigma)} = 0\} \cap \mathbb{T} \times (a, 2\pi - a)$ is totally disconnected for all $\sigma \in \mathbb{S}$. Thus, for any given $\sigma \in \mathbb{S}$ and any $\epsilon > 0$, there are finite and mutual disjoint balls $\mathcal{B}_\epsilon(q_i)$ and $\delta = \delta(\sigma, \epsilon) > 0$ such that $\cup \mathcal{B}_\epsilon(q_i) \supset \{B^*_{c(\sigma)} = 0\} \cap \mathbb{T} \times (a, 2\pi - a)$ and

$$\min\{B^*_{c(\sigma)}(q) : q \in \partial \mathcal{B}_\epsilon(q_i), \forall i\} \geq 2\delta, \quad B^*_{c(\sigma)}(q_i) = 0.$$

In other words, as a function of q. $B^*_{c(\sigma)}$ reaches its minimum in $\{a \leq q_2 \leq 2\pi - a\}$ away from the boundary

$$\min_{q \in \partial \mathcal{B}_\epsilon(q_i)} B^*_{c(\sigma)}(q) - \min_{q \in \mathcal{B}_\epsilon(q_i)} B^*_{c(\sigma)}(q) \geq 2\delta. \tag{5.15}$$

Recall for each $z \in \mathbb{T}$, there is unique $z_\sigma \in \pi(\Gamma_\sigma)$ such that $z_\sigma = (z, q_{2\Gamma_\sigma}(z))$. From (5.14), (5.15) and the Hölder continuity guaranteed by Lemma 4.4, we find that for each $z \in \mathbb{T}$

$$\min_{q \in \partial \mathcal{B}_\epsilon(q_i)} h^\infty_{c(\sigma)}(z_\sigma, q) + h^\infty_{c(\sigma')}(q, z_{\sigma'}) \\ - \min_{q \in \mathcal{B}_\epsilon(q_i)} h^\infty_{c(\sigma)}(z_\sigma, q) + h^\infty_{c(\sigma')}(q, z_{\sigma'}) \geq \frac{3}{2}\delta, \tag{5.16}$$

provided that σ' is sufficiently close to σ. As these functions depend on the choice of closed 1-form η_c, to obtain (5.16) we choose $\eta_c = \langle c(\sigma), \dot{q} \rangle$. In general, $h^\infty_{c(\sigma)}(z_\sigma, q) + h^\infty_{c(\sigma')}(q, z_{\sigma'})$ is also the function of z, but its variation over $z \in \mathbb{T}$ is very small if σ' is sufficiently close to σ, because $q_{2\Gamma_\sigma}(z)$ has $\frac{1}{2}$-Hölder continuity in σ. Since $\mathbb{S}$ is compact, there exist $\delta = \delta(\epsilon)$ and $\epsilon_1 = \epsilon_1(\epsilon, \delta)$, independent of σ, such that (5.15) and (5.16) hold if $|\sigma - \sigma'| \leq \epsilon_1$.

We say σ_j is linked with σ_{j+1} by transition torus with some persistency if $\sigma_{j+1} \in \mathbb{S}$ is so close to σ_j such that

$$|c_1(\sigma_j) - c_1(\sigma_{j+1})| \leq \frac{1}{4}\delta \tag{5.17}$$

and (5.16) holds where we replace σ and σ' by σ_j and σ_{j+1}, respectively. We say σ_j is linked with σ_k by transition chain with some persistency if there there is a sequence $\sigma_j, \sigma_{j+1}, \ldots, \sigma_{k-1}, \sigma_k$ in $\mathbb{S}$ such that for each $j \leq i < k$ σ_i is linked with σ_{i+1} by transition torus with some persistency. To be brief, we shall say in the following that they are linked by transition torus (chain). Note that $\mathbb{S} \subset [A', B']$ is compact, we can find finitely many $\sigma_k \in \mathbb{S}$ $(0 \le k \le K)$ such that we have one of the following alternatives for each $k < K$: either σ_k is linked with σ_{k+1} by transition chain, or Γ_{σ_k} and $\Gamma_{\sigma_{k+1}}$ make up the boundary of an annulus of Birkhoff instability, i.e., there is no other invariant circle between Γ_{σ_k} and $\Gamma_{\sigma_{k+1}}$. In the following, we shall use Γ_i to denote Γ_{σ_i} and use z_i to denote z_{σ_i}.

Let us consider a sequence of invariant circles Γ_i $(i = 0, 1, \dots, \ell, \ell+1)$ on the cylinder Σ such that Γ_1 is linked with Γ_ℓ through the transition chain $\Gamma_2, \dots, \Gamma_{\ell-1}$, and there are two annuli of Birkhoff instability, one has Γ_0 and Γ_1 as its boundary, another one has Γ_ℓ and $\Gamma_{\ell+1}$ as its boundary. By the construction of this transition chain, we know that for each $1 \le i < \ell$ there is $x_i \in \{B^*_{c(\sigma_i)} = 0\} \cap (a, 2\pi - a)$ such that for any $z \in \mathbb{T}$

$$\begin{aligned}(5.16\text{i}) \qquad &\min_{q \in \partial \mathcal{B}_\epsilon(x_i)} h^\infty_{c(\sigma_i)}(z_i, q) + h^\infty_{c(\sigma_{i+1})}(q, z_{i+1}) \\ &\qquad - \min_{q \in \mathcal{B}_\epsilon(x_i)} h^\infty_{c(\sigma_i)}(z_i, q) + h^\infty_{c(\sigma_{i+1})}(q, z_{i+1}) \ge \frac{3}{2}\delta.\end{aligned}$$

As in [**6**], let us consider the covering of $\mathbb{T}^2$ given by $\bar{M} = \mathbb{T} \times \mathbb{R}$. For each x_i, we identify it with its lift in the region $\mathbb{T} \times (0, 2\pi)$ and single out a point on its lift, $\bar{x}_i = x_i + (0, 2i\pi)$. We also identify each z_i with its lift $z_i + (0, 2i\pi)$. For $i \in (1, 2, \dots, \ell - 1)$, we introduce a smooth function Ψ_i: $\mathbb{T} \times \mathbb{R} \to \mathbb{R}$ which vanishes outside $\{q : |q - \bar{x}_i| \le 2\epsilon\}$ and such that

$$(5.18) \qquad \nabla \Psi_i(q) = c(\sigma_{i+1}) - c(\sigma_i) \qquad \forall q : \ |q - \bar{x}_i| \le \epsilon.$$

If we set

$$(5.19) \qquad \bar{c}_i(q) = c(\sigma_i) + \nabla \Psi_i(q),$$

then

$$h^\infty_{\bar{c}_i}(z, q) + h^\infty_{\bar{c}_{i+1}}(q, z) = h^\infty_{c(\sigma_i)}(z, q) + h^\infty_{c(\sigma_{i+1})}(q, z) + \Psi_{i+1}(q) - \Psi_i(q).$$

Note $\Psi_{i+1}(q) = 0$ as $q \in \mathcal{B}_\epsilon(q_i)$. If we require further that σ_{i+1} is so close to σ_i that (5.17) holds, we obtain from (5.16i) and (5.18) that

$$\begin{aligned}(5.20) \qquad &\min_{q \in \partial \mathcal{B}_\epsilon(x_i)} h^\infty_{\bar{c}_i}(z_i, q) + h^\infty_{\bar{c}_{i+1}}(q, z_{i+1}) \\ &\qquad - \min_{q \in \mathcal{B}_\epsilon(x_i)} h^\infty_{\bar{c}_i}(z_i, q) + h^\infty_{\bar{c}_{i+1}}(q, z_{i+1}) \ge \delta.\end{aligned}$$

Let $\mathbb{B} = \mathcal{B}_\epsilon(x_1) \times \mathcal{B}_\epsilon(x_2) \times \cdots \times \mathcal{B}_\epsilon(x_{\ell-1})$, $Q = (q_1, \ldots, q_{\ell-1})$, $\vec{n} = (n_0, n_1, \ldots, n_\ell) \in \mathbb{Z}^{\ell+1}$ and define

$$\mathfrak{h}(Q, z_1, z_\ell, \vec{n}) = \sum_{i=1}^{\ell-2} h_{\bar{c}_{i+1}}^{n_{i+1}-n_i}(q_i, q_{i+1}) + h_{\bar{c}_1}^{n_1-n_0}(z_1, q_1) + h_{\bar{c}_\ell}^{n_\ell-n_{\ell-1}}(q_{\ell-1}, z_\ell). \tag{5.21}$$

We see that $\mathfrak{h}$, as the function of Q, takes its local minimum in the interior of $\mathbb{B}$ if $n_{i+1} - n_i$ is sufficiently large for all $1 \le i \le \ell - 1$. In fact, let x_i^* be the point where the function of q $h_{\bar{c}_i}^{\infty}(z_i, q) + h_{\bar{c}_{i+1}}^{\infty}(q, z_{i+1})$ attains its local minimum in $\mathcal{B}_\epsilon(x_i)$, we find that the function of Q

$$\sum_{i=1}^{\ell-1} h_{\bar{c}_i}^{\infty}(z_i, q_i) + h_{\bar{c}_{i+1}}^{\infty}(q_i, z_{i+1})$$

takes its local minimum at the point $(x_1^*, x_2^*, \ldots, x_{\ell-1}^*)$ which is obviously in the interior of $\mathbb{B}$. Thus, the local minimum of $\mathfrak{h}$ is in the interior of $\mathbb{B}$ if the following holds

$$\lim_{n_{i+1}-n_i \to \infty} h_{\bar{c}_{i+1}}^{n_{i+1}-n_i}(q_i, q_{i+1}) = h_{\bar{c}_{i+1}}^{\infty}(q_i, z_{i+1}) + h_{\bar{c}_{i+1}}^{\infty}(z_{i+1}, q_{i+1}). \tag{5.22}$$

To show this, let us state a lemma.

Lemma 5.4. *Assume $\tilde{\mathcal{M}}(c)$ has a dense orbit. For any $m_0, m_1 \in M$, let γ: $[0, K] \to M$ be c-minimal curve connecting m_0 and m_1, $\gamma(0) = m_0$, $\gamma(K) = m_1$. For any $\delta > 0$, any $K_1 \in \mathbb{Z}^+$ and any $z \in \mathcal{M}_0(c)$, $\exists$ $K_0 \in \mathbb{Z}^+$, if $K \ge K_0$, then there exists $T \in \mathbb{Z}^+$ such that $\gamma(T) \in B_\delta(z)$, $T \ge K_1$ and $K_0 - T \ge K_1$.*

Proof. For any $\delta^* > 0$, there is $K_0 \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^+$ such that $d\gamma(k) \in \tilde{\mathcal{M}}_0(c) + \delta^*$ if $K \ge K_0$, otherwise there would be another c-minimal measure. For any $z \in \mathcal{M}_0(c)$, by choosing sufficiently small δ^* and sufficiently large K_0, there is some $T \in \mathbb{Z}^+$ so that $\gamma(T)$ is in δ-neighborhood of z. Clearly, for any $K_1 \in \mathbb{Z}^+$, there exists such T so that $T \ge K_1$ and $K_0 - T \ge K_1$, otherwise there would be another c-minimal measure also. q.e.d.

Applying the lemma to this problem, we find that for each $m \in \mathcal{M}(c_i)$, each small $\delta > 0$ and each large $K > 0$, there exist $n^* \in \mathbb{Z}^+$ such that if $n \ge n^*$, then there exists $z_n \in \mathcal{B}_\delta(m)$ such that

$$h_{c_i}^{n}(q, q') = h_{c_i}^{n_1}(q, z_n) + h_{c_i}^{n_2}(z_n, q'),$$

where $n = n_1 + n_2$ with $n_1, n_2 \geq K$. Using the Lipschitz property of $h^n_{c_i}(m, m')$ in (m, m') we find that for each small $\epsilon > 0$ the following holds

$$\left|h^n_{c_i}(q, q') - h^{n_1}_{c_i}(q, m) - h^{n_2}_{c_i}(m, q')\right| < \epsilon$$

if δ is sufficiently small and n^* is sufficiently large. Since we consider the $\mathcal{M}(c_i)$ which is on the cylinder with irrational number, thanks to the corollary 2.7, we know that $L - \langle \bar{c}_i(q), \dot{q}\rangle$ is regular for each $0 \leq i \leq \ell + 1$, (5.22) follows from the property that $d_{c_i}(m, m') = 0$ for all $m, m' \in \pi(\Gamma_i)$. Denote the corresponding minimizer by γ: $[n_0, n_\ell] \to M$, we use $\gamma_i(t)$ to denote its restriction on the time interval $[n_i, n_{i+1}]$. Once $\gamma(t)$ reaches its local minimum in the interior of $\mathbb{B}$, standard argument shows that

$$\frac{\partial L_{\bar{c}_i}}{\partial \dot{q}}(d\gamma_i(t), t) = \frac{\partial L_{\bar{c}_{i+1}}}{\partial \dot{q}}(d\gamma_{i+1}(t), t)$$

holds at $t = n_{i+1}$. Note that $L_{\bar{c}_i} = L_{\bar{c}_{i+1}}$ in the neighborhood of $\mathcal{B}_\epsilon(\bar{x}_i)$ by the definition of $\Psi_i(q)$, we get

$$\dot{\gamma}_i(n_{i+1}) = \dot{\gamma}_{i+1}(n_{i+1}),$$

thus, $\gamma(t)$ is a solution of the Euler–Lagrange equation over the time interval $[n_0, n_\ell]$.

In fact, we can remove the restriction on z_1 and z_ℓ that there is $z \in \mathbb{T}$ so that $z_j = (z, q_{2\Gamma_j}(z))$ for $j = 1, \ell$. We can replace z_j by any point $z^*_j \in \mathcal{M}(c_j)$ simply because $d_{c_j}(m, m') = 0$ for all $m, m' \in \pi(\Gamma_j)$. Thus, the function of $Q = (q_1, \ldots, q_{\ell-1})$

$$\begin{aligned}\sum_{i=2}^{\ell-2} & h^\infty_{\bar{c}_i}(z_i, q_i) + h^\infty_{\bar{c}_{i+1}}(q_i, z_{i+1}) + h^\infty_{\bar{c}_1}(z^*_1, q_1) + h^\infty_{\bar{c}_\ell}(q_\ell, z^*_\ell) \\ &= \sum_{i=1}^{\ell-1} h^\infty_{\bar{c}_i}(z_i, q_i) + h^\infty_{\bar{c}_{i+1}}(q_i, z_{i+1}) + h^\infty_{\bar{c}_1}(z^*_1, z_1) + h^\infty_{\bar{c}_\ell}(z_\ell, z^*_\ell)\end{aligned}$$

also reaches its local minimum at the point $(x^*_1, x^*_2, \ldots, x^*_{\ell-1})$. So, the Lipschitz property of h^n_c enable us to assert that there exist large $(\Delta n_1, \Delta n_2, \ldots, \Delta n_\ell)$ and small $\delta^* > 0$, if $n_i - n_{i-1} \geq \Delta n_i$, $z_j \in \mathcal{B}_{\delta^*}(z^*_j)$ for $j = 1, \ell$, then as the function of Q, $\mathfrak{h}(Q, z_1, z_\ell, \vec{n})$ reaches its local minimum in the interior of $\mathbb{B}$.

Now, we are ready to construct an orbit γ: $\mathbb{R} \to M$ such that $\alpha(d\gamma) \supset \Gamma_0$ and $\omega(d\gamma) \supset \Gamma_{\ell+1}$.

By the condition, Γ_i and Γ_{i+1} make up the boundary of the resonant zone Z_i for $i=0,\ell$. For $i=0,\ell+1$, we let $c^{(i)}$ be a co-homology class such that $\{B^*_{c^{(i)}}=0\}=\Gamma_i$. For $i=1,\ell$, we let $c^{(i)}$ be a co-homology class such that $c^{(i)}=(c_1^{(i)},b(c_1^{(i)}))$, in this case, $\{B^*_{c^{(i)}}=0\}=\Gamma_i\cup\{\text{its } c^{(i)}\text{-minimal homoclinic orbits}\}$. Since the C-equivalence between $c^{(i)}$ and $c^{(i+1)}$has been established for $i=0,\ell$, in analogy to the proof of Theorem 5.1, we can find the composition of finite U-step forms ν_j

$$\nu_j=\sum_{i=0}^{N_j}(-\tau_i^j)^*\mu_i^j,\quad j=1,2$$

such that their cohomology classes are $[\nu_1(t)|_{t\le 0}]=0$, $[\nu_1(t)|_{t\ge\tau^1_{N_1}+1}]=c^{(1)}-c^{(0)}$, $[\nu_2(t)|_{t\le 0}]=0$ and $[\nu_2(t)|_{t\ge\tau^2_{N_2}+1}]=c^{(\ell+1)}-c^{(\ell)}$, where τ_i^j is the time translation $(q,t)\to(q,t+\tau_i^j)$. Moreover, by Lemma 5.3, we can choose those ν_j such that $\nu_1(t)|_{t\le 0}=0$, $\nu_1(t)|_{t\ge\tau^1_{N_1}+1}=\langle\bar c_1(q)-c(\sigma_0),dq\rangle$, $\nu_2(t)|_{t\le 0}=0$ (see (5.18) for the definition of $\bar c_i(q)$) and $\nu_2(t)|_{t\ge\tau^2_{N_2}+1}=\langle c(\sigma_{\ell+1})-c(\sigma_\ell),dq\rangle$. Let $\eta_0^1=\langle c(\sigma_0),dq\rangle$, $\eta_0^2=\langle\bar c_\ell(q),dq\rangle$, $\eta_i^j=\eta_0^j+\sum_{k=0}^{i-1}\bar\mu_k^j$ and $c_i^j=[\eta_i^j]$, then $\eta^1_{N_1+1}=\langle\bar c_1(q),dq\rangle$, $\eta^2_{N_2+1}=\langle c(\sigma_{\ell+1}),dq\rangle$. Based on the proof of Theorem 5.1, we can choose each μ_i^j, the adapted neighborhood U_i^j and $\epsilon_i^j>0$ so that

$$\pi(\tilde{\mathcal N}_{\eta_i^j,\mu_i^j})+\epsilon_i^j\subset U_i^j\quad\forall\ j=1,2,\ \ 0\le i\le N_j. \tag{5.6ij}$$

For each $\epsilon_i^{j*}>0$, there exist $\hat T^j_{ki},\breve T^j_{ki}\in\mathbb Z_+$ with $\hat T^j_{ki}>\breve T^j_{ki}$, $(k=0,1)$ such that

$$h^{T_0,T_1}_{\eta_i^j,\mu_i^j}(m_0,m_1)\ge h^{\infty}_{\eta_i^j,\mu_i^j}(m_0,m_1)-\epsilon_i^{j*}\qquad\forall\ T_k\ge\breve T^j_{ki},(k=0,1),$$
$$\forall\ (m_0,m_1)\in M\times M$$

for any given $(m_0,m_1)\in M\times M$, there exists $T_k=T_k(m_0,m_1)$ with $\breve T^j_{ki}\le T_k\le\hat T^j_{ki}$ such that

$$\left|h^{T_0,T_1}_{\eta_i^j,\mu_i^j}(m_0,m_1)-h^{\infty}_{\eta_i^j,\mu_i^j}(m_0,m_1)\right|\le\epsilon_i^{j*}. \tag{5.7ij}$$

Let $\gamma_i^j\,(t,\,m_0,\,m_1,\,T_0,\,T_1):[-T_0,\,T_1]\to M$ be the minimizer of $h^{T_0,T_1}_{\eta_i^j,\mu_i^j}(m_0,m_1)$. Let $\breve T^j_{ki}$ be set so large and $\epsilon_i^{j*}>0$ be set so small such that if (5.7ij) holds, then

$$d\gamma_i^j(t, m_0, m_1, T_0, T_1) \in \tilde{\mathcal{N}}_{\eta_i^j, \mu_i^j} + \epsilon_i^j \quad \forall\, 0 \le t \le 1. \tag{5.8ij}$$

We define the index set for $\vec{\tau}^j = (\tau_0^j, \tau_1^j, \ldots, \tau_{N_j}^j)$

$$\Lambda^j = \Big\{ \vec{\tau}^j \in \mathbb{Z}^{N_j} : \max\{\breve{T}^j_{0(i-1)}, \breve{T}^j_{1i} + 1\} \\ \le \tau_i^j - \tau_{i-1}^j \le \max\{\hat{T}^j_{0(i-1)}, \hat{T}^j_{1i} + 1\}, \forall 1 \le i \le N_j, \tau_0^j = 0 \Big\}$$

and introduce a modified Lagrangian depending on the parameters $\vec{\tau}^j$ $(j = 1, 2)$ and $\vec{n}$

$$\tilde{L} = \begin{cases} L - \langle c(\sigma_0), \dot{q}\rangle - (\tau^1_{N_1} + 1)^* \nu_1, & t \le n_1, \\ L - \langle c(\sigma_j) + \varrho_j(t) \nabla \Psi_j(q), \dot{q}\rangle, & n_{j-1} \le t \le n_j, \\ & 2 \le j \le \ell - 1, \\ L - \langle c(\sigma_\ell), \dot{q}\rangle + (n_\ell + \tau^2_{N_2} + 1)^* \nu_2, & t \ge n_{\ell-1}, \end{cases}$$

where ϱ_j is a smooth function such that $\varrho_j(t) = 0$ for $t \le \frac{1}{2}(n_{j+1} + n_j)$, $0 < \varrho_j < 1$ when $\frac{1}{2}(n_{j+1} + n_j) < t < \frac{1}{2}(n_{j+1} + n_j) + 1$ and $\varrho_j = 1$ when $t \ge \frac{1}{2}(n_{j+1} + n_j) + 1$, this function is well defined if $n_j - n_{j-1} \ge 4$. Clearly, $\tilde{L}$ is smooth in

$$(\dot{q}, q, t) \in TM \times \left\{ \mathbb{R} \backslash \bigcup_{i=1}^{\ell-1} \{n_i\} \right\} \cup \bigcup_{i=1}^{\ell-1} T\mathcal{B}_\epsilon(x_i) \times \Big(n_i - 1, n_i + 1 \Big)$$

For each $(m, m') \in M \times M$, $Q = (q_1, \ldots, q_{\ell-1}) \in \mathbb{B}$ let

$$\begin{aligned} & h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m', Q, \vec{\tau}^1, \vec{\tau}^2, \vec{n}) \\ & \quad = \inf_{\substack{\gamma(-T_0^*) = m \\ \gamma(T^*_{\ell+1}) = m' \\ \gamma(n_j) = q_j \\ j = 1, \ldots, \ell-1}} \int_{-T_0^*}^{T^*_{\ell+1}} \tilde{L}(d\gamma(t), t)\, dt \\ & \qquad + \sum_{\substack{1 \le i \le N_j \\ j = 1,2}} (\tau_i^j - \tau_{i-1}^j) \alpha(c_i^j) + n_0 \alpha(c_1) \\ & \qquad + (n_\ell - n_{\ell-1}) \alpha(c_\ell) + T_0 \alpha(c_0) + T_{\ell+1} \alpha(c_{\ell+1}) \end{aligned}$$

where $T_0^* = T_0 + \tau^1_{N_1} + 1$, $T^*_{\ell+1} = T_{\ell+1} + n_\ell + \tau^2_{N_2} + 1$ and $T_0, T_{\ell+1} > 0$. In virtue of the Lemma 5.4, we can take sufficiently large n_1' so that any $c(\sigma_1)$-minimal curve γ_1: $[0, n_1] \to M$ with $n_1 \ge n_1'$ has a point

$\gamma_1(n_0) \in B_{\delta^*}(z_1^*)$ with $n_1 - n_0 \geq \Delta n_1$. Similarly, we can take sufficiently large $\Delta n'_\ell$ so that any $c(\sigma_\ell)$-minimal curve γ_ℓ: $[n_{\ell-1}, n'_\ell] \to M$ with $n'_\ell - n_{\ell-1} \geq \Delta n'_\ell$ has a point $\gamma_\ell(n_\ell) \in B_{\delta^*}(z_{\ell+1})$ with $n'_\ell - n_{\ell-1} > n_\ell - n_{\ell-1} \geq \Delta n_\ell$. We can also take suitably large n_i ($i = 2, 3, \ldots, \ell-1$) so that $n_{i+1} - n_i \geq \Delta n_i$ for each $1 \leq i \leq \ell - 1$. Under these conditions, we take the minimum of $h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m', Q, \vec{\tau}^1, \vec{\tau}^2, \vec{n})$ over $\mathbb{B}$

$$h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m', \vec{\tau}^1, \vec{\tau}^2, \vec{n}) = \min_{Q \in \mathbb{B}} h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m', Q, \vec{\tau}^1, \vec{\tau}^2, \vec{n}).$$

Let $\gamma(t) = \gamma(t, m, m', \vec{\tau}^1, \vec{\tau}^2, \vec{n})$ be the minimizer of $h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m', \vec{\tau}^1, \vec{\tau}^2, \vec{n})$. Recall that the support of Ψ_i is a small ball. For each cohomology class under our consideration here, the support of the minimal measure is on the cylinder, the hyperbolicity of the cylinder. Let us see that $\gamma(t)$ is outside of the support of $\nabla \Psi_i$ if both $t - n_i$ and $n_{i+1} - t$ are suitably large. In other words, for $t \in [\frac{1}{2}(n_{i+1} + n_i), \frac{1}{2}(n_{i+1} + n_i) + 1]$, $\gamma(t)$ falls into the area where $\langle \varrho_i(t) \nabla \Psi_i(q), dq \rangle$ is exact. Thus, $d\gamma$ solves the Euler–Lagrange equation of L for $t \in [n_0, n_\ell]$ if we repeat the argument for the function $\mathfrak{h}(Q, z_1, z_\ell, \vec{n})$.

Next, by choosing sufficiently large values for $\breve{T}^1_{1N_1}$, $\hat{T}^1_{1N_1}$, $\breve{T}^2_{00}$ and $\hat{T}^2_{00}$, we can assume $\breve{T}^1_{1N_1} \geq n'_1$ and $\breve{T}^2_{00} \geq n_{\ell-1} + \Delta n'_\ell$. In this case, let us consider the minimum of $h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m', \vec{\tau}^1, \vec{\tau}^2, \vec{n})$ over $\Lambda^1 \times \Lambda^2 \times \{\breve{T}^1_{1N_1} \leq n_1 \leq \hat{T}^1_{1N_1}\} \times \{\breve{T}^2_{00} + n_{\ell-1} \leq n_\ell \leq \hat{T}^2_{00} + n_{\ell-1}\}$

$$h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m') = \min_{\substack{\vec{\tau}^1 \in \Lambda^1, \vec{\tau}^2 \in \Lambda^2 \\ \breve{T}^1_{1N_1} \leq n_1 \leq \hat{T}^1_{1N_1} \\ \breve{T}^2_{00} + n_{\ell-1} \leq n_\ell \leq \hat{T}^2_{00} + n_{\ell-1}}} h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m', \vec{\tau}^1, \vec{\tau}^2, \vec{n}).$$

Denote by $\vec{\tau}^{j*}$, n_1^* and n_ℓ^* where the the minimum is reached. Let $\gamma(t) = \gamma(t, m, m', T_0, T_{\ell+1})$ be the minimizer of $h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m')$. Let $\tau_{1j} = \tau^1_j - \tau^1_{N_1} - 1$, $\tau_{2j} = \tau^2_j + \tau^2_{N_2} + 1 + n_\ell$. From the proof of Theorem 5.1, we can see that (5.6ij), (5.7ij) and (5.8ij) hold for $(-\tau_{ij})^* \gamma$ at $j = 1, 2$, $0 \leq i \leq N_j$ except for $(i, j) = (0, 1), (N_2, 2)$.

As the third step, we consider the limit infimum

$$h_{\tilde{L}}^{\infty}(m, m') = \liminf_{\substack{T_0 \to \infty \\ T_{\ell+1} \to \infty}} h_{\tilde{L}}^{T_0, T_{\ell+1}}(m, m').$$

Let T_0^k, $T_{\ell+1}^k$ be the subsequence so that $T_0^k \to \infty$, $T_{\ell+1}^k \to \infty$ as $k \to \infty$,

$$|h_{\tilde{L}}^{T_0^k, T_{\ell+1}^k}(m, m') - h_{\tilde{L}}^{\infty}(m, m')| \leq \min\{\epsilon_0^{1*}, \epsilon_{N_2}^{2*}\} \quad \forall\ k,$$

and

$$\lim_{k\to\infty} h_{\tilde{L}}^{T_0^k, T_{\ell+1}^k}(m, m') = h_{\tilde{L}}^{\infty}(m, m')$$

and let γ_k: $[-T_0^{k*}, T_{\ell+1}^{k*}] \to M$ be the minimizer of $h_{\tilde{L}}^{T_0^k, T_{\ell+1}^k}(m, m')$, where $T_0^{k*} = T_0^k + \tau_{N1}^{1*} + 1$, $T_{\ell+1}^{k*} = T_{\ell+1}^k + n_\ell^* + \tau_{N_2}^{2*} + 1$. By the similar argument to prove Theorem 5.1, we can see (5.6ij), (5.7ij), (5.8ij) hold for $(-\tau_{ij})^*\gamma_k$ at $(i,j) = (0,1), (N_2, 2)$ also. In this case, $d\gamma_k$ is a solution of the Euler–Lagrange equation induced by L. For each small δ, $d\gamma_k$ connects $\tilde{\Gamma}_0 + \delta$ with $\tilde{\Gamma}_{\ell+1} + \delta$ if k is sufficiently large. Let γ: $\mathbb{R} \to M$ be the accumulation point of $\{\gamma_k\}_{k\in\mathbb{Z}^+}$, then $\alpha(d\gamma) = \Gamma_0$ and $\omega(d\gamma) = \Gamma_{\ell+1}$ since $\tilde{\mathcal{A}}(c_i) = \Gamma_i$ for $i = 0, \ell+1$.

The construction of diffusion orbits can be done in the same way when there are finitely many resonant gaps.

6. Generic property

The construction of diffusion orbits is under the hypotheses (H1)–(H3). The task here is to show these hypotheses are dense properties in C^r-topology for $r \geq 3$. Since we are interested in the diffusion from $\{p_1 < A\}$ to $\{p_1 > B\}$, a compact domain for $\{\|p\| \leq K\} \times \mathbb{T}^2$ satisfies such an requirement if $K > 0$ is sufficiently large. The C^r-topology is endowed in the usual sense for functions $\{\|p\| \leq K\} \times \mathbb{T}^2 \to \mathbb{R}$.

The hypothesis (H1) is made only for those co-homology classes $c = (c_1, b(c_1))$, such that $\tilde{\mathcal{M}}_0(c)$ is contained in an invariant circle on the cylinder. Its Mañé set $\tilde{\mathcal{N}}(c)$ consists of the invariant circle and its minimal homoclinic orbits, i.e., $\{B_c^* = 0\}$. Let us look at this issue from the Hamiltonian dynamics point of view.

Since the system is positive definite in p, it has a generating function $G(q, q')$

$$G(q, q') = \inf_{\substack{\gamma\in C^1([0,1],\bar{M})\\ \gamma(0)=q,\gamma(1)=q'}} \int_0^1 L(\gamma(s), \dot{\gamma}(s), s)ds, \tag{6.1}$$

where (q, q') is in the covering space $\bar{M} = \mathbb{R}^2 \times \mathbb{R}^2$. Clearly, $G(q + 2m\pi, q' + 2m\pi) = G(q, q')$ for all $m \in \mathbb{Z}^2$. The map Φ_H: $(p, q) \to (p', q')$

is given by

$$p' = \partial_{q'} G(q, q'), \qquad p = -\partial_q G(q, q'). \tag{6.2}$$

Let π_1 be the standard projection from $\mathbb{R}^2$ to $\mathbb{T}^2$, let $c \in \mathbb{R}^2$ and

$$G_c(q, q') = G(q, q') - \langle c, q' - q \rangle$$

then

$$h_c(x, x') = \min_{\substack{\pi_1(q)=x \\ \pi_1(q')=x'}} G_c(q, q') - \alpha(c) \quad \text{(see (2.6))} \tag{6.3}$$

As the system is nearly integrable, the matrix $\partial^2_{q'q} G$ is non-degenerate everywhere. Thus we can solve the second equation in (6.2) and obtain somehow more explicit form of the map (6.2)

$$p' = \frac{\partial G}{\partial q'}(q, q'(p, q)), \qquad q' = q'(p, q). \tag{6.4}$$

Let us consider a small perturbation $G(q, q') + \kappa(q - q')G_1(q')$ of the generating function in which $0 \le \kappa(q-q') \le 1$, $\kappa(q-q') = 1$ if $|q-q'| \le K$ and $\kappa(q - q') = 0$ if $|q - q'| \ge K + 1$. We choose sufficiently large K so that $\{\|p\| \le \max(|A|, |B|)+1\}$ is contained in the set where $|q-q'| \le K$. In this set, the map will have the form

$$p' = \frac{\partial G}{\partial q'}(q, q'(p, q)) + \frac{\partial G_1}{\partial q'}(q'(p, q)), \quad q' = q'(p, q). \tag{6.5}$$

Note that both stable and unstable manifolds of Γ keep horizontal over the strip $U = \{a \le q_2 \le 2\pi - a\}$, restricting Φ to W^s and to W^u where they keep horizontal, and projecting it to the underline manifold M along the fibers, we obtain two maps f^s and f^u on M such that $\pi \circ \Phi = f^{s,u} \circ \pi$. We choose $G_1 \in C^r$ satisfying its support $supp(G_1) = \mathcal{B}_b(q^*)$ mod $2\pi m \subset U$ mod $2\pi m$ where $m \in \mathbb{Z}^2$. We see that $(f^u)^{-1}(\mathcal{B}_b(q^*)) \cap \mathcal{B}_b(q^*) = \varnothing$ and $f^s(\mathcal{B}_b(q^*)) \cap \mathcal{B}_b(q^*) = \varnothing$ if $b > 0$ is chosen suitably small. Let us consider the problem in the covering space $\mathbb{T} \times \mathbb{R}$ and assume one lift of the unstable manifold starting from $q_2 = 0$ to the right, one lift of the stable manifold starting from $q_2 = 2\pi$ to the left. From (6.5), we can see that the local stable manifold is not deformed $W^s|[q_2^* - b \le q_2 \le 2\pi + q_{2\Gamma}(q_1)] = \{q, dS^s + c(\sigma) : q_2^* - b \le q_2 \le 2\pi + q_{2\Gamma}(q_1)\}$, but the unstable manifold undergoes slight deformation, $W^u|[q_{2\Gamma}(q_1) \le q_2 \le q_2^* + b] = \{q, dS^u + dG_1 + c(\sigma) : q_{2\Gamma}(q_1) \le q_2 \le q_2^* + b\}$. It is easy to see that the barrier function has the form:

$$B^*_{c(\sigma)}(q) = S^u_{c(\sigma)}(q) - S^s_{c(\sigma)}(q) - G_1(q) + 2\pi b(c_1(\sigma)) \quad \text{if } q \in \mathcal{B}_b(q^*). \tag{6.6}$$

We should note the total action of the minimal orbit may be changed because of the perturbation, in other words, the associated cohomology class may be subjected to a small perturbation $(c_1, b(c_1)) \to (c_1, b(c_1) \pm \varepsilon)$.

Let $R_d = \{q \in M : |q_1 - q_1^*| \le d, |q_2 - q_2^*| \le d\} \subset \mathcal{B}_b(q^*)$, let $S_\sigma = S^u_{c(\sigma)} - S^s_{c(\sigma)} - G_1$ we define

$$Z(\sigma) = \left\{ q \in R_d : \, S_\sigma(q) = \min_{q \in R_d} S_\sigma \right\}.$$

We say a connected set V is non-trivial for R_d if either $\Pi_1(V \cap R_d) = \{q_1^* - d \le q_1 \le q_1^* + d\}$ or $\Pi_2(V \cap R_d) = \{q_2^* - d \le q_2 \le q_2^* + d\}$, where Π_i is the standard projection from $\mathbb{T}^2$ to its i-th component (i=1,2). Let $M_{d,q^*}(S) = \{q : S(q) = \min_{q \in R_d(q^*)} S\}$, we define a set in the function space $\mathfrak{F}(d, q^*) = C^0(R_d(q^*), \mathbb{R})$,

$$\begin{aligned} \mathfrak{Z}(d, q^*) = \Big\{ S \in \mathfrak{F}(d, q^*) : \, & M_{d,q^*}(S) \\ & \text{contains a set non-trivial for } R_d(q^*) \Big\}. \end{aligned}$$

Let

$$\mathfrak{Z}_1 = \left\{ S \in \mathfrak{Z}(d, q^*) : \, \Pi_1(M_{d,q^*}(S)) = \{q_1^* - d \le q_1 \le q_1^* + d\} \right\},$$
$$\mathfrak{Z}_2 = \left\{ S \in \mathfrak{Z}(d, q^*) : \, \Pi_2(M_{d,q^*}(S)) = \{q_2^* - d \le q_2 \le q_2^* + d\} \right\},$$

then

$$\mathfrak{Z}(d, q^*) = \mathfrak{Z}_1 \cup \mathfrak{Z}_2.$$

Our first task is to show for each generating function $G \in C^r(M \times M, \mathbb{R})$ and each $\epsilon > 0$, there is an open and dense set $\mathfrak{H}(d, q^*)$ of $\mathcal{B}_\epsilon(0) \subset C^r(R_d(q^*), \mathbb{R})$, for each $G_1 \in \mathfrak{H}(d, q^*)$, the image of S_σ from $[A', B']$ to $\mathfrak{F}$ has no intersection with the set $\mathfrak{Z}_i$.

Obviously, the set $\mathfrak{Z}_1$ is a closed set and has infinite co-dimensions in the following sense, there exists $\mathfrak{N}$, an infinite dimension subspace of $\mathfrak{F}$, such that $(S + F) \notin \mathfrak{Z}$ for all $S \in \mathfrak{Z}_1$ and $F \in \mathfrak{N} \backslash \{0\}$. In fact, for each non-constant function $F(q_1) \in C^0([q_1^* - d, q_1^* + d], \mathbb{R})$ with $F(q_1^*) = 0$ and each $S \in \mathfrak{Z}_1$, we have $S + F \notin \mathfrak{Z}_1$. Thus, we can choose $\mathfrak{N} = C^0([q_1^* - d, q_1^* + d], \mathbb{R})/\mathbb{R}$, which we think as the subspace of $C^0(R_d(q^*), \mathbb{R})$ consisting of those continuous functions independent of q_2.

On the other hand, as S_σ: $[A', B'] \to \mathfrak{F}$ has $\frac{1}{2}$-Hölder continuity, the image is compact and its box dimension is not bigger than 2

$$D_B(\mathfrak{F}_\sigma) \le 2. \tag{6.7}$$

where $\mathfrak{F}_\sigma = \{S_\sigma : \sigma \in [A', B']\}$. Clearly, this set is determined by the generating function G.

Lemma 6.1. *There is an open and dense set $\mathfrak{N}^* \subset \mathfrak{N}$ such that for all $F \in \mathfrak{N}^*$*

$$(\mathfrak{F}_\sigma + F) \cap \mathfrak{Z} = \varnothing. \tag{6.8}$$

Proof. The open property is obvious. If there were no density property, there would be n-dimensional ε-ball $\mathcal{B}_\varepsilon \subset \mathfrak{N}$ for some $\varepsilon > 0$, such that for each $F \in \mathcal{B}_\varepsilon$, there exists $S \in \mathfrak{F}_\sigma$ such that $F + S \in \mathfrak{Z}_1$ or $F + S \in \mathfrak{Z}_2$. For each $S \in \mathfrak{F}_\sigma$, there is at most one $F \in \mathcal{B}_\varepsilon$ so that $S+F \in \mathfrak{Z}_1$, for, otherwise, there would be $F' \neq F$ such that $F'+S \in \mathfrak{Z}_1$, but we can write $F' + S = F' - F + F + S$ where $F + S \in \mathfrak{Z}_1$ and $F' - F \in \mathfrak{N}\backslash\{0\}$, it contradicts the definition of $\mathfrak{N}$. Given $F \in \mathcal{B}_\varepsilon$, there might be more than one element in $\mathfrak{S}_F = \mathfrak{S}_F = \{S \in \mathfrak{F}_\sigma : S+F \in \mathfrak{Z}_1\}$. Given any two $F_1, F_2 \in \mathcal{B}_\varepsilon$, for any $S_1 \in \mathfrak{S}_{F_1}$ and any $S_2 \in \mathfrak{S}_{F_2}$, we have

$$\begin{aligned} d(S_1, S_2) &= \max_{q \in R_d(q^*)} |S_1(q) - S_2(q)| \\ &\ge \max_{|q_1 - q_1^*| \le d} \left| \min_{|q_2 - q_2^*| \le d} S_1(q_1, q_2) - \min_{|q_2 - q_2^*| \le d} S_2(q_1, q_2) \right| \\ &\ge \frac{1}{2} \mathrm{var}_{|q_1 - q_1^*| \le d} |F_1(q_1) - F_2(q_1)| \\ &\ge \frac{1}{2} d(F_1, F_2), \end{aligned} \tag{6.9}$$

where $d(\cdot, \cdot)$ is the C^0-metric. It follows from (6.9) and the definition of box dimension that

$$D_B(\mathfrak{F}_\sigma) \ge D_B(\mathcal{B}_\varepsilon) = n,$$

but this is absurd if we choose $n > 2$. The same argument can be applied to the set $\mathfrak{Z}_2$. q.e.d.

As C^r is dense in C^0, an open and dense set $\mathfrak{H}(d, q^*)$ of $\mathcal{B}_\epsilon \subset C^r(R_d(q^*), \mathbb{R})$ clearly exists such that for each perturbation of generating function $G_1 \in \mathfrak{H}(d, q^*)$, we have

$$\mathfrak{F}_\sigma \cap \mathfrak{Z}(d, q^*) = \varnothing \quad \forall\, \sigma \in \mathbb{S},$$

where, by abuse of terminology, we continue to denote S_σ and its restriction $R_d(q^*)$ by the same symbol.

Recall we have defined the set $U = \mathbb{T}\times[a, 2\pi - a]$ before. Let $M_U(S) = \{q : S(q) = \min_{q\in U} S\}$ and

$$\mathfrak{Z} = \Big\{S \in C^0(U, \mathbb{R}) :\, M_U(S) \text{ is totally disconnected}\Big\}.$$

Given $d_i > 0$, there are finite q_{ij} such that $\cup_j R_{d_i}(q_{ij}) \supset U$. Thus, there exists a sequence $d_i \to 0$ and a countable set $\{q_{ij}\}$ such that

$$\left(\bigcap_{i=1,j=1}^{\infty} \mathfrak{H}(d_i, q_{ij})\right) \bigcap \mathfrak{Z} = \varnothing.$$

Therefore, there is a generic set in $\mathcal{B}_\epsilon \subset C^r(U, \mathbb{R})$, the hypothesis (H1) holds for each G_1 in this generic set. Note U is an annulus, we can write $G_1 = G_1' + G_1^*$ so that both G_1' and G_1^* have simply connected support.

The perturbation to the generating function G can be achieved by perturbing the Hamiltonian function $H \to H' = H + \delta H$. Let Φ' be the map determined by the generating function $G + \kappa G_1'$, the symplectic diffeomorphism $\Psi = \Phi' \circ \Phi^{-1}$ is close to identity. We choose a smooth function $\rho(s)$ with $\rho(0) = 0$ and $\rho(1) = 1$, let Φ_s' be the symplectic map determined by $G + \rho(s)\kappa G_1'$ and let $\Psi_s = \Phi_s' \circ \Phi^{-1}$. Clearly, Ψ_s defines a symplectic isotopy between identity map and Ψ. Thus, there is a unique family of symplectic vector fields X_s: $T^*M \to TT^*M$ such that

$$\frac{d}{ds}\Psi_s = X_s \circ \Psi_s.$$

By the choice of perturbation, there is a simply connected and compact domain D_K such that $\Psi_s|_{T^*M\setminus D_K} = id$. It follows that there is a Hamiltonian $H_1(p, q, s)$ such that $dH_1(Y) = dp \wedge dq(X_s, Y)$ holds for any vector field Y. Re-parametrizing s by t, we can make H_1 smoothly and periodically depend on t. To see that dH_1 is also small, let us make use of a theorem of Weinstein [**19**]. A neighborhood of the identity in the symplectic diffeomorphism group of a compact symplectic manifold **M** can be identified with a neighborhood of the zero in the vector space of closed 1-forms on $\mathbf{M}$. Since Hamiltomorphism is a subgroup of symplectic diffeomorphism, there is a function H', sufficiently close to H, such that $\Phi_{H_1} \circ \Phi_H = \Phi^t_{H'}|_{t=1}$.

Thus, the density of (H1) is proved.

For the hypothesis (H2), let us consider the twist map on the cylinder. In this case, each co-homology class corresponds to a unique rotation number. Given any rational number $p/q \in \mathbb{Q}$, it is obvious that there is an open dense set in the space of area-preserving twist map such that there is only one minimal (p,q)-periodic orbit without homoclinic loop. Taking the intersection of countably open dense set, we obtain that (H2) is a generic property.

To verify the (H3), let us consider an invariant circle Γ_σ on Σ. There is an interval $I(c_1) = \{c = (c_1, c_2) \in \mathbb{R}^2 : a(c_1) \leq c_2 \leq b(c_1)\}$ such that $\text{supp}(\mathcal{M}_0(c)) \subseteq \Gamma_\sigma$ iff $c \in I(c_1)$. Let U be a small neighborhood of $\pi(\Gamma_\sigma)$. Under the hypothesis (H1), the set $\{B^*_c = 0\}\backslash U$ is homotopically trivial for $c = (c_1, a_1(c_1))$ and for $c = (c_1, b_1(c_1))$. By the upper semi-continuity of Mañé sets $c \to \tilde{\mathcal{N}}(c)$, the set $\mathcal{N}_0(c')$ is in a small neighborhood of $\{B^*_c = 0\}$ if $c' = (c_1, b(c_1) + \delta)$ with $\delta > 0$ sufficiently small. Let us consider such a minimal measure $\tilde{\mathcal{M}}(c')$. Let μ be an ergodic component of $\tilde{\mathcal{M}}(c')$, there exists $\epsilon^* > 0$ such that $\text{dist}(\text{supp}\mu, \tilde{\Gamma}_\sigma) \geq 3\epsilon^*$ for all $\sigma \in \mathbb{S}$. For any $\epsilon > 0$ with $\epsilon \leq \epsilon^*$, we can define a C^r-smooth function $L^\sigma_{k,\epsilon}$: $TM \times \mathbb{T} \to \mathbb{R}$ so that $L^\sigma_{k,\epsilon}(z,t) = 0$ if $(z,t) \in \text{supp}\mu + 2^{-k-1}\epsilon$, $L^\sigma_{k,\epsilon} = 2^{-k}\epsilon^{r+1}$ if $(z,t) \notin \text{supp}\mu + 2^{-k}\epsilon$ and $L^\sigma_{k,\epsilon}$ takes the value between $2^{-k}\epsilon^{r+1}$ and 0 elsewhere. Obviously, μ is the unique ergodic component of c'-minimal measure of the Lagrangian

$$L^\sigma_\epsilon = L + \frac{1}{r!}\sum_{k=1}^{\infty} L^\sigma_{k,\epsilon}$$

and $\|L^\sigma_\epsilon - L\|_{C^r} \leq \epsilon$. Since (H3) is required only for countable $\sigma \in \mathbb{S}$, we can choose even smaller ϵ_σ so that the supports of these $L^\sigma_{\epsilon_\sigma} - L$ have no intersection.

Note the perturbation we introduced for (H1) has compact support which has no intersection with the cylinder, the perturbation we introduced for (H3) does not touch the set $\{B^*_{c(\sigma)} = 0\}$ for all $\sigma \in \mathbb{S}$, and there is a dense set for P such that (H1)–(H3) hold. Thus, we obtain the density of the perturbation. Since the time for each orbit drifts from $p_1 < A$ to $p_1 > B$ is finite, the smooth dependence of solutions of ODE's on parameter guarantees the openness.

Therefore, the proof of the Theorem 1.1 is completed.

Appendix

In this appendix, we present the proof of the Lemma 2.6, given by Bernard [**4**], for the completeness sake.

Lemma 2.6. *If $\tilde{\mathcal{M}}(c)$ is minimal in the sense of topological dynamics and if there exists a sequence γ_n of n-periodic curves such that $A_c(\gamma_n) \to 0$ as $n \to \infty$, then L_c is regular, hence $\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c) = \tilde{\mathcal{G}}(c)$.*

Proof. As the first step, we show that the following limit exists for all $(x, t) \in \mathcal{M} \times \mathbb{T}$:

$$\lim_{n\to\infty} F_c(x, x, t, t+n) = 0. \tag{A.1}$$

By the condition, we can suppose these n-periodic curves γ_n are minimizers, their n-periodic orbits $X_n(t) = (d\gamma_n(t), t)$ is a compact subset of $TM \times \mathbb{T}$. Each subsequence of X_n has a convergent subsequence in the sense of Hausdorff topology. The limit set of such a sequence is obviously an invariant subset of $\tilde{\mathcal{M}}(c)$. Since $\tilde{\mathcal{M}}(c)$ is minimal, this limit set has to be $\tilde{\mathcal{M}}(c)$ itself. Therefore, the sequence of subsets X_n converges to $\tilde{\mathcal{M}}(c)$ in the Hausdorff topology. It follows that each point $(x, s) \in \tilde{\mathcal{M}}(c)$ is the limit of a sequence $(\gamma(t_n), s)$ with $t_n = s \bmod 1$ for each n. As F_c is of Lipschitz, we have

$$\begin{aligned}\limsup_{n\to\infty} F_c(x, x, t, t+n) &= \limsup_{n\to\infty} F_c(\gamma_n(t_n), \gamma_n(t_n), t, t+n) \\ &= \limsup_{n\to\infty} A_c(\gamma_n) \\ &= 0,\end{aligned}$$

which implies (A.1).

Next, we claim that (A.1) implies that $L - \eta_c$ is regular, i.e., for any (x, s), $(x', s') \in M \times \mathbb{T}$, $\epsilon > 0$, there exists T such that

$$F_c(x, x', t, t') \le h_c(x, x', t, t') + \epsilon$$

if t and t' satisfy $t = s \bmod 1$, $t' = s' \bmod 1$ and $t' \ge t + T$. Indeed, let K be the common Lipschitz constant of all functions $F_c(\cdot, \cdot, t, t')$ with $t' \ge t + 1$, let $t_0 = s \bmod 1$, $t_0' = s' \bmod 1$, let γ: $[t_0, t_0'] \to M$ be a minimizer with $\gamma(t_0) = x$ and $\gamma(t_0') = x'$, i.e., $A_c(\gamma) = F_c(x, x', t_0, t_0')$. We can make $t_0' - t_0$ is sufficiently large so that $\exists\, t_1 \in [t_0, t_0']$ such that

$\mathrm{dist}(\gamma(t_1), y) \le \epsilon/4K$ for some $y \in \mathcal{M}(c)|_{t=t_1}$, in virtue of standard argument of topological dynamics. Since $h_c(x, x', s, s') = \liminf F_c(x, x', t, t')$, we can suppose in addition that

$$F_c(x, x', t_0, t_0') \le h_c(x, x', s, s') + \frac{\epsilon}{2}.$$

Let $x_1 = \gamma(t_1)$, we have

$$F_c(x, x', t_0, t_0') = F_c(x, x_1, t_0, t_1) + F_c(x_1, x', t_1, t_0').$$

It follows that:

$$|F_c(x, x', t_0, t_0') - F_c(x, y, t_0, t_1) - F_c(y, x', t_1, t_0')| \le \frac{\epsilon}{2},$$

thus

$$F_c(x, y, t_0, t_1) + F_c(y, x', t_1, t_0') \le h_c(x, x', s, s') + \epsilon.$$

By the choice of t and t', we know that $\exists\ n \in \mathbb{N}$ such that $t' - t = t_0' - t_0 + n$. So, we have

$$\begin{aligned} F_c(x, x', t, t') &= F_c(x, x', t_0, t_0 + n) \\ &\le F_c(x, y, t_0, t_1) + F_c(y, y, t_1, t_1 + n) \\ &\quad + F_c(y, x', t_1 + n, t_0' + n). \end{aligned}$$

Let $n \to \infty$, thanks to (A.1), we obtain

$$\limsup F_c(x, x', t, t') \le h_c(x, x', s, s') + \epsilon.$$

As this holds for arbitrary $\epsilon > 0$, we see that L is regular.

As the third step, we claim that L is regular implies that $\tilde{\mathcal{G}} = \tilde{\mathcal{N}}$. Let $\gamma \in C^1(\mathbb{R}, M)$ be a minimizing curve, let $t_k \to -\infty$ be a sequence such that $s = t_k \bmod 1$ for all $k \in \mathbb{Z}$ and such that $\alpha = \lim \gamma(t_k)$, let $t_k' \to \infty$ be a sequence such that $s' = t_k' \bmod 1$ and such that $\omega = \lim \gamma(t_k')$. In this case

$$A(\gamma|_{[t_k, t_k']}) = F(\gamma(t_k), \gamma(t_k'), t_k, t_k') \to h(\alpha, \omega, s, s').$$

Let us consider a compact interval of times $[a, b]$, where $s' = a \bmod 1$ and $s = b \bmod 1$. For k sufficiently large, we have

$$A_c(\gamma|_{[a,b]}) = A_c(\gamma|_{[t_k, t_k']}) - A_c(\gamma|_{[t_k, a]}) - A_c(\gamma|_{[b, t_k']}).$$

Taking the limit, we obtain

$$A_c(\gamma|_{[a,b]}) = h_c(\alpha, \omega, s, s') - h_c(\alpha, \gamma(a), s, s) - h_c(\gamma(b), \omega, s', s').$$

On the other hand, we observe that if L is regular, then

$$h_c(\alpha, \omega, s, s') \le h_c(\alpha, \gamma(a), s, s) + \Phi_c(\gamma(a), \gamma(b), s, s') + h_c(\gamma(b), \omega, s', s')$$

it follows that

$$A(\gamma|_{[a,b]}) \le \Phi_c(\gamma(a), \gamma(b), s, s'),$$

hence γ is semi-static. It has been shown in [**15**] that $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c)$.
q.e.d.

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