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POSITIVELY CURVED MANIFOLDS WITH LOW FIXED POINT COHOMOGENEITY

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Abstract

Positively curved manifolds of fixed point cohomogeneity one are classified up to equivariant diffeomorphism.

Introduction

The general program about classifying positively curved manifolds with large isometry groups (cf. [**10**]) has recently enjoyed considerable progress (see [**28**, **27**] and also [**20**, **7**]). Here, we will provide a solution to one particular aspect of this program.

If G is a (connected) compact Lie group of isometries on a (closed) riemannian manifold M, the dimension of the orbit space M/G, also called the *cohomogeneity*, $\mathrm{cohom}(M,G)$, of the action, provides a natural coarse measurement for the size of G. In particular, $\mathrm{cohom}(M,G)=0$ means that G acts transitively on M, i.e., $M=G/H$ is homogeneous. In this case, the *fixed point set* M^G of G is clearly empty. When $M^G \neq \emptyset$, the cohomogeneity is of course constrained by the dimension of $M^G \subset M$, since also $M^G \subset M/G$. Because of this, it is natural to view

$$\mathrm{cohomfix}(M,G) = \dim M/G - \dim M^G - 1$$

as an alternative measurement for the size of G, when fixed points are present. Note that, with this definition, $\mathrm{cohomfix}(M,G)=\mathrm{cohom}(M,G)$ when $M^G=\emptyset$, and $\mathrm{cohomfix}(M,G)=\mathrm{cohom}(S,G)$, where S is any normal sphere to a component $F \subset M^G$ of maximal dimension, when $M^G \neq \emptyset$. Clearly, $\mathrm{cohomfix}(M,G)=0$ simply means that G acts "as transitively as it possibly can" whether or not G has fixed points. When

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$M^G \neq \emptyset$ and $\operatorname{cohomfix}(M,G) = 0$, we, therefore, say as in [11] that M is *fixed point homogeneous* (we have chosen to subtract 1 from the definition of $\operatorname{cohomfix}(\cdot,\cdot)$ given in [11] since it seems to cause less confusion). When $\operatorname{cohomfix}(M,G) = k$, we will also say that (M,G), or simply M has *fixed point cohomogeneity* k. Although this formally includes all cohomogeneity k manifolds, we will throughout this paper use the notion of fixed point cohomogeneity only when the fixed point set is non-empty.

Fixed point homogeneous manifolds of positive curvature were classified up to equivariant diffeomorphism in [11]. Here, we will present a similar classification of positively curved fixed point cohomogeneity one manifolds. As a consequence, we get, in particular, the following

Theorem A. *Any positively curved simply connected manifold of fixed point cohomogeneity one is equivariantly diffeomorphic to an isometric action on a compact rank one symmetric space.*

With the exception of the Cayley plane, each of these spaces have many such structures. For a complete description including the group actions, and the cases where M is not simply connected, we refer to Section 2, and in particular, to our main result Theorem 2.10.

Note that our result in a sense is optimal, since the simply connected positively curved normal homogeneous Aloff–Wallach space $W_{1,1} = \mathrm{SU}(3)/\mathrm{S}^1_{1,1} = (\mathrm{SU}(3)\,\mathrm{SO}(3))/\mathrm{U}^\bullet(2)$ (cf. [26]) has an isometric circle action with fixed point cohomogeneity two and non-empty fixed point set.

We point out that Bredon in [2] investigated actions of fixed point cohomogeneity one in a general curvature free setting.

Results of the above type are often useful in other aspects of the symmetry program since actions of these types often occur as sub-actions of other large group actions (see e.g. [11] and [28]). Here are two extreme situations stemming from a general action $G \times M \to M$. If H is the principal isotropy group, then clearly $\operatorname{cohomfix}(M,H) = \operatorname{cohomfix}(G/H,H)$ is the fixed point cohomogeneity of the isotropy representation. If an isotropy group $K \subset G$ contains a normal subgroup $L \triangleleft G$, then $\operatorname{cohomfix}(M,L) = \operatorname{cohomfix}(S^\perp,L)$ is the fixed point cohomogeneity of L on the spherical slice representation. This yields a usefully general *reduction theorem* (see (1.4)) which extends the classification result of [11].

An important common feature among manifolds of fixed point cohomogeneity at most one, is that their orbit spaces are Alexandrov spaces with non-empty boundary. Since in our case these orbit spaces also have positive curvature, the Cheeger–Gromoll–Meyer soul theorem adapted to orbit spaces (cf. (1.2) and (1.3)) plays a pivotal role in determining their structure. In the first section, we recall the basic general prerequisites in this context, and prove the reduction theorem alluded to above. In Section 2, we briefly present the facts we need about cohomogeneity one manifolds, and exhibit a collection of examples of fixed point cohomogeneity one manifolds with positive curvature. In addition, extending of a result of Straume [**25**], we prove the following equivariant analogue of the Smale–Hatcher theorem [**23**, **14**] of independent interest.

Theorem B. *Suppose G is a compact connected Lie group which acts isometrically (hence linearly) by cohomogeneity one on a euclidean sphere S. Then, the inclusion*

$$\mathrm{O}^G(S) \to \mathrm{Diff}^G(S)$$

of the group of G-equivariant linear maps into the group of G-equivariant diffeomorphisms of S is homotopy equivalence.

This result is crucial for achieving our classification up to equivariant diffeomorphism rather than just equivariant homeomorphism. We point out that Morse theory applied to a suitable subspace of equivariant homeomorphisms of S provides a natural deformation retract onto $\mathrm{O}^G(S)$. Section 3 is devoted to a description of all possible geometric structures of orbit spaces of positively curved fixed point cohomogeneity one manifolds. There are all together four different types. This is then used in the final three sections to show that the examples exhibited in section two provide a complete classification.

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1. Basic setup and reduction

Throughout, M will denote a compact connected Riemannian manifold, and beginning with (1.2), we also assume that M has positive sectional curvature. Unless otherwise stated, G will be compact (connected) Lie group acting (almost) effectively and isometrically on M. Recall that each component of the *fixed point set* M^G of G is a totally geodesic submanifold of M. For $x \in M$, we let $G_x \subset G$ denote the *isotropy* (stabilizer) subgroup of G at x, and $Gx \simeq G/G_x$ the orbit through x. As is customary, we refer to the induced action of G_x on the tangent space T_xGx ($T_x^\top$ for short) to the orbit Gx, and on its orthogonal complement $T_x^\perp Gx$ ($T_x^\perp$ for short) as the *isotropy representation*, and the *slice representation*, respectively. By the *slice theorem*, a tubular neighborhood of the orbit $Gx \subset M$ is equivariantly diffeomorphic to $G \times_{G_x} T_x^\perp$. In particular, each component of the set M_K of orbits of type (K), i.e., with isotropy group conjugate to K, is a (minimal) submanifold, and $T_xM_{G_x} = T_x^\top + (T_x^\perp)^{G_x}$.

When viewed as a point in the space M/G of orbits, we use the notation $\underline{x}$ for the orbit Gx. The distance between orbits in M defines a length metric on M/G, so that the projection $\pi : M \to M/G$ defined by $\pi(x) = \underline{x}$ is a submetry, i.e., it maps every r-ball $B(x, r)$ in M onto the r-ball $B(\underline{x}, r)$ in M/G. In particular, M/G is a so-called Alexandrov space with positive curvature (cf. [**3**] for general background). To avoid cumbersome notation, we will frequently simply write $\underline{A}$ in place of $\pi(A)$ when $A \subset M$. For the *tangent cone* $T_{\underline{x}}\underline{M}$ ($T_{\underline{x}}$ for short) at $\underline{x} \in \underline{M}$, we have $T_{\underline{x}} \simeq T_x^\perp/G_x$. This is also the euclidean cone on the *space of directions* $S_{\underline{x}}\underline{M} \simeq S_x^\perp/G_x$ ($S_{\underline{x}}$ for short) to $\underline{M}$ at $\underline{x}$. Note that we have the splitting $T_{\underline{x}} = T_{\underline{x}}^\top + T_{\underline{x}}^\perp$, where $T_{\underline{x}}^\top \simeq (T_x^\perp)^{G_x}$ is the tangent space to the stratum, i.e., component of $\underline{M_{G_x}}$, through $\underline{x}$ and $T_{\underline{x}}^\perp \simeq ((T_x^\perp)^{G_x})^\perp/G_x$ is called its *normal cone*.

For any orbit type (K), each component of $M_K/G = \underline{M_K}$ is a manifold which by the slice theorem is locally totally geodesic in $\underline{M}$. Moreover, the orbit map $\pi : M_K \to \underline{M_K}$ is a locally trivial bundle with fiber G/K and structure group $N(K)/K$, where $N(K)$ is the *normalizer* of K in G. The corresponding principal bundle is given by $(M_K)^K \to (M_K)^K/(N(K)/K) = \underline{M_K}$. Also, for each component N of $\underline{M_K}$, we have that $\pi^{-1}(N) = \overline{\pi^{-1}(N)} \cap M_K$ and $N = \overline{\pi^{-1}(N)^K/(N(K)/K)}$. It follows that the closure $cl(N) \subset \underline{M}$ can be described as

$cl(N) = cl(\pi^{-1}(N)^K)/(N(K)/K)$. Note that $cl(\pi^{-1}(N)^K)$ consists of components of M^K each of which have orbit space $cl(N)$ under the subgroup of $N(K)/K$ preserving the component. In particular,

Proposition 1.1 (Stratum lemma)**.** *The closure in $\underline{M} = M/G$ of any connected component of orbits of a fixed type (K) is in its intrinsic length metric an Alexandrov space.*

Recall that when $K = H$ is the principal isotropy group, then M_H is connected and $cl(M_H)^H = {}_cM$ is a manifold called the *core* of M (or *principal reduction*) and $N(H)/H = {}_cG$ is called the *core group*. We will make strong use of the fact that $\underline{M} = M/G$ is isometric to ${}_cM/{}_cG$, which we will also denote by $\underline{{}_cM}$ (cf. [**1**, **22**, **12**]).

If (K) is a maximal orbit type among non-principal orbits, then the slice representation (modulo its kernel) acts either transitively, or freely on the normal sphere to a component of M_K. In the first case, the corresponding stratum is part of the boundary $\partial\underline{M}$ of $\underline{M}$, and we call its closure a *face* of the boundary. The boundary of $\underline{M}$ is a finite union of faces, ∂_i, each of which have non-empty boundary themselves as long as they are in a component of $\partial\underline{M}$ with more than one face. In contrast to what is known for general Alexandrov spaces, we point out that also $\partial\underline{M}$ with its induced length metric is an Alexandrov space. We will not make use of this fact here, however.

We will now formulate two versions of the soul theorem adapted to the setting of positively curved orbit spaces (for general Alexandrov spaces see also [**19**]). For this purpose, we will use CX to denote the cone on X.

Theorem 1.2 (Boundary soul lemma)**.** *Suppose $\underline{M} = M/G$ has non-empty boundary. Then*

(i) *there is a unique point $\underline{s}_o \in \underline{M}$, the soul of $\underline{M}$, at maximal distance to $\partial\underline{M}$, and $S_{\underline{s}_o}$ is homeomorphic to $\partial\underline{M}$;*
(ii) *there is a homeomorphism $C(\partial\underline{M}) \to \underline{M}$ which is the identity on $\partial\underline{M}$, and takes the cone point to $\underline{s}_o$;*
(iii) *a stratum of $\underline{M}$ is either contained in $\partial\underline{M}$, or in the interior $\operatorname{int}\underline{M} = \underline{M} - \partial\underline{M}$. In the latter case, it is either the interior minus cone point of a cone on strata contained in the boundary, the interior of a line through $\underline{s}_o$ with end points at boundary strata points, all of $\operatorname{int}\underline{M}$, or $\underline{s}_o$ by itself;*

(iv) *M is equivariantly diffeomorphic to the union of a tubular neighborhood of the soul orbit Gs_o, and a neighborhood of $\pi^{-1}(\partial \underline{M})$.*

In the second version, it is only a face of the boundary, which plays the "Riemannian" role of the boundary of a convex set.

Theorem 1.3 (Face Soul Lemma)**.** *Suppose $M/G = \underline{M}$ has non-empty boundary with more than one face, and let $\partial_i \subset \partial \underline{M}$ be a face. Then*

(i) *there is a unique point $\underline{s}_i \in \partial \underline{M}$, at maximal distance to ∂_i, and $S_{\underline{s}_i}$ is homeomorphic to ∂_i;*
(ii) *there is a homeomorphism $C(\partial_i) \to \underline{M}$ which is the identity on ∂_i and takes the cone point to $\underline{s}_i$;*
(iii) *a stratum of $\underline{M}$ is either contained in ∂_i, or in $\underline{M} - \partial_i$. In the latter case, it is either the interior minus cone point of a cone on strata contained in ∂_i, or $\underline{s}_i$ by itself;*
(iv) *M is equivariantly diffeomorphic to the union of a tubular neighborhood of the face soul orbit Gs_i, and a neighborhood of $\pi^{-1}(\partial_i)$.*

The first version was presented in [**11**] with a slightly different formulation. Its proof is based on critical point theory for distance functions (cf. [**13**, **9**]), and the fact that the distance function to the boundary is strictly concave (cf. [**5**]). The proof of the second theorem is basically the same and uses no new ingredients other than the observation due to Wilking that the distance function to a face is strictly concave as well. For this, it is important to note that the angle between any two faces is at most $\pi/2$, which is a simple fact about cohomogeneity one actions on spheres (cf. Section 2).

Separately and together, these two theorems provide a powerful tool in the study of positively curved manifolds with isometric group actions, since orbit spaces of such manifolds frequently have boundary. We mention that in [**28**], it is proved, for example, that if the action has non-trivial principal isotropy group, then the orbit space has boundary, and that a positively curved k-dimensional orbit space with boundary and $l+1$ faces is the join of a $k-l-1$ dimensional space and an l simplex. We also point out that by the stratum lemma (1.1), we can analyse each face with non-empty boundary the same way we analyse $\underline{M}$ itself.

We conclude this section with a general extension of the result obtained for fixed point homogeneous manifolds in [**11**]. Although it will become apparent that only simple special cases of what was done in [**11**] would suffice for our purposes, the extension unifies and eases the exposition.

Recall, that if M is a positively curved manifold and $\operatorname{cohomfix}(M, G') = 0, G' \subset \operatorname{Iso}(M)$, then M is G'-equivariantly diffeomorphic to $(\overline{M}, G')$, $G' \subset \operatorname{Iso}(\overline{M})$, where $\overline{M}$ is a compact rank one symmetric space with its standard metric, or a quotient of it by a finite group $\Pi \subset \mathrm{S}^3$, which acts freely on $\overline{M}$ and preserves a pair of dual symmetric subspaces in $\overline{M}$ (see [**11**]). We will refer to any such *model* $\overline{M}$ as a Π-CROSS.

Lemma 1.4 (Reduction lemma)**.** *Let $G \times M \to M$ be an isometric action on a positively curved manifold M, with G a (connected) compact Lie group. Suppose $G' \lhd G$ is a connected normal subgroup of G, which acts fixed point homogeneously on M. Then, M is G-equivariantly diffeomorphic to a Π-CROSS, $\overline{M}$ with $G \subset \operatorname{Iso}(\overline{M})$.*

Proof. Let F' be a maximal dimensional component of $M^{G'}$ so that G' acts transitively on the normal spheres S' of F'. Recall from [**11**] (cf. (1.2)) that there is a unique orbit $F^* = G'/K'$ at maximal distance to F', and that $M - (F' \cup F^*)$ consists of principal orbits $G'/H' \simeq S'$. Moreover, M/G' is a cone on F', which in turn is diffeomorphic to the space of directions S^*/K' at the cone point, where S^* is the normal sphere to F^* at a point with isotropy K'.

When $\dim F' > 0$, H' acts trivially on S^* and $K'/H' = \Gamma \subset S^3$ acts freely. Moreover, in this case M (as well as F') is

- a sphere, when $\Gamma = \{1\}$;
- a space form with fundamental group Γ, when Γ is finite (F' can also be a circle in this case);
- a complex projective space, when $\Gamma = \mathrm{S}^1$;
- the $\mathbb{Z}_2$ quotient of an odd dimensional complex projective space, when $\Gamma = \mathrm{Pin}(2)$; or
- a quaternionic projective space, when $\Gamma = S^3$.

If F' is a point, then $K'/H' = S^*$, M/G' is an interval, and M is either a sphere, a real projective space, a complex projective space, a quaternionic projective space, or the Cayley plane. It is only in this case $\mathbb{C}aP^2 = \mathrm{F}_4 / \mathrm{Spin}(9)$ arises and then $G = \mathrm{Spin}(9)$.

It is clear that the group $\hat{G} = G/G'$ acts isometrically on M/G' with quotient M/G. It is also clear that $\hat{G}$ preserves both F' and the cone point. When F' is a point, i.e., $\operatorname{cohom}(M, G') = 1$, it is clear that G

has the same orbits as G', and it is not hard to complete the proof of the lemma in these cases. When $\dim F' > 0$, we see in particular, that $F^* = G'/K' = G/K$ is also a G-orbit at maximal distance to the G-invariant manifold F', and that the slice representation of K induces an isometric action on S^*/Γ, a $\Pi = \Gamma/\Gamma_0$ quotient of a CROSS. It is also clear that there is a G-invariant smooth $\mathrm{dist}(F', \cdot)$ gradient like vector field on M which is radial near F' and F^*.

Let $\overline{M}$ be a Π-CROSS model for M, with $G' \subset \mathrm{Iso}(\overline{M})$ and corresponding dual pairs $\overline{F}'$ and $\overline{F}^*$. From [**11**], there is a G'-equivariant diffeomorphism $f : M \to \overline{M}$ induced via the slice theorem by a G'-equivariant bundle equivalence between the normal bundles of $F^* \subset M$ and $\overline{F}^* \subset \overline{M}$, each of which can presented as $G' \times_{K'} D^*$. By assumption, the normal bundle of $F^* \subset M$ is also described in terms of the extended G-action as $G \times_K D^*$. Note that the slice representation of K induces an isometric action on the space of directions $S^*/K' = (S^*/\Gamma_0)/\Pi = \overline{F}'$. From the description of $\mathrm{Iso}(\overline{M})$ (see e.g. [**21**]), it then follows that the transplanted G-action on the normal bundle to $\overline{F}^* \subset \overline{M}$ extends to an isometric action on $\overline{M}$. Moreover, by construction, the G'-equivariant diffeomorphism f is also G-equivariant in the complements of F' and $\overline{F}'$ and hence globally. q.e.d.

Remark 1.5. If $\hat{G} = G/G'$ acts fixed point homogeneously on F' with $F \subset F'^{\hat{G}}$ a component of maximal dimension, then G acts by fixed point cohomogeneity one on M, and the action of G on the normal space to the fixed point set F is reducible. All but four fixed point cohomogeneity one actions on a positively curved manifold with reducible action on the normal space to the fixed point set F are of this type. Note that when G is connected, so is $\hat{G}$, and we have the following description of the orbit space $\underline{M} = M/G = (M/G')/\hat{G}$:

(1) if $\Gamma = \{1\}$, then $\underline{M}$ has two smooth faces;
(2) if $\Gamma = \mathrm{Pin}(2), F' \simeq \mathbb{C}P^1/\mathbb{Z}_2$, and $\hat{G} = \mathrm{S}^1$, then $\underline{M}$ is a triangle with two right angles and one $\pi/4$ angle;
(3) if $\Gamma = \mathrm{S}^3, F' \simeq \mathbb{H}P^1$, and $\hat{G} = \mathrm{S}^1$, then $\underline{M}$ is 4-dimensional, has one smooth face, and one face with a unique interior singular point, with space of directions the 3-dimensional disc $D^3(1/2)$ with curvature 4;

(4) in all other cases, $\underline{M}$ is either a triangle with three right angles, or it has two faces and a geodesic I of singular points joining the faces perpendicularly at unique singular points of the faces.

In all cases described here, the angle between the faces meeting along F is $\pi/2$.

We will see in Section 3, that except for the possibility of having an angle $\pi/3, \pi/4$, or $\pi/6$ between the two faces meeting along F in the first and last orbit spaces described above, and for the possibility that $\underline{M}$ has one face, this yields a complete description of all possible orbit spaces of positively curved manifolds of fixed point cohomogeneity one.

2. Examples and equivariant sphere diffeomorphisms

Since the linear cohomogeneity one actions on spheres (classified in [**15**] and [**24**]), and cohomogeneity one actions on related manifolds play a central role in our investigations, we will briefly recall the general structure of such manifolds.

We are only interested in those connected Riemannian cohomogeneity one G-manifolds V, where $\underline{V} = V/G$ is an interval. If $\underline{V}$ has length $2a$, to be written $|\underline{V}| = 2a$, we parameterize it as $[-a, a]$. Let c be a normal geodesic perpendicular to all orbits. We denote by H the principal isotropy group $G_{c(0)}$ at $c(0)$, which is equal to the isotropy groups $G_{c(t)}$ for all $t \neq \pm a$, and by $K_\pm$ the isotropy groups at $c(\pm a) = x_\pm$. In terms of this, we have

$$V = G \times_{K_-} D_- \cup_{G/H} G \times_{K_+} D_+ = V_- \cup_{V_0} V_+, \tag{2.1}$$

where $D_\pm$ denotes the normal disc to the orbit $Gx_\pm = G/K_\pm = B_\pm$ at $x_\pm$, and the gluing is done along $V_0 = V_- \cap V_+ = G/H$ with the identity map. It is important to note that $S_\pm = \partial D_\pm = K_\pm/H$, and that the diagram of groups

$$\begin{tikzcd}
& G & \\
K_- \arrow[ur, "j_-"] & & K_+ \arrow[ul, "j_+"'] \\
& H \arrow[ul, "i_-"] \arrow[ur, "i_+"'] &
\end{tikzcd} \tag{2.2}$$

which we also record as $H \subset \{K_-, K_+\} \subset G$, determine V.

Suppose there are groups $L \subset \{L_-, L_+\} \subset G$, where $H \triangleleft L$, $K_\pm \triangleleft L_\pm$ are normal subgroups such that $\Gamma = L/H \to L_\pm/K_\pm$ are isomorphisms. Then, the free actions by Γ on the G-orbits induce a free action on V commuting with G, and $W = V/\Gamma$ is a cohomogeneity one G-manifold with diagram

$$\begin{tikzcd} & G & \\ L_- \arrow[ur, "j_-"] & & L_+ \arrow[ul, "j_+"'] \\ & L \arrow[ul, "i_-"] \arrow[ur, "i_+"'] & \end{tikzcd} \tag{2.3}$$

Now, suppose $H \triangleleft \hat{H} \subset G$ with $\hat{H}/H = \mathbb{Z}_2$, and that $K_+ = wK_-w^{-1}$ with $w \in \hat{H} - H$. Then, the action of $\hat{H}/H$ on G/H induces a fixed point free involution on V, which preserves the middle V_0 and interchanges the two sides $V_\pm$ of V. Moreover, the involution commutes with the G-action on V and the induced cohomogeneity one G-action on the quotient $V/\mathbb{Z}_2$ has the diagram

$$\begin{tikzcd} & G & \\ K_- \arrow[ur, "j_-"] & & \hat{H} \arrow[ul, "j"'] \\ & H \arrow[ul, "i_-"] \arrow[ur, "i"'] & \end{tikzcd} \tag{2.4}$$

The principal case of interest here is when $V = S$ is the unit sphere in a euclidean space, and the action $G \times S \to S$ described by (2.2) is linear. Recall that in this case $|\underline{S}| = \theta$ is one of the numbers $\pi, \pi/2, \pi/3, \pi/4$, or $\pi/6$ (cf. also [**17**] for the general case of isoparametric hypersurfaces in spheres). Here, $|\underline{S}| = \pi$ if and only if G fixes a pair of antipodal points and acts transitively on their equatorial sphere, and $|\underline{S}| = \pi/2$ if and only if G acts reducibly and without fixed points. The simplest class of reducible representations are those called *splitting*, i.e., $G = G_1 \times G_2$ and the representation is an *outer* direct sum $\Phi = \Phi_1 \oplus \Phi_2$. Since one of the factors is allowed to be trivial, this includes the first and not so important case for us. In the remaining reducible cases, the representation is an *inner* direct sum, i.e., $G = \Delta G \subset G \times G$ and $\Phi = (\Phi_1 \oplus \Phi_2)_{|\Delta}$. We point out that from the classification (see [**24**]), one finds that all but four of these cases, and of course, all the splitting cases have normal subgroups which act fixed point homogeneously on S (cf. (1.5) and Section 1). Including the irreducible representations, one also easily finds the following exhaustive list of possibilities for core groups ${}_cG$ (see also [**24**, **25**]):

- If ${}_cG$ is finite, then ${}_cS = S^1$, and ${}_cG = \mathrm{D_q}$ a dihedral group with $q = 2, 3, 4$, or 6, and correspondingly $\theta = \pi/q$.
- If ${}_cG$ is connected, then ${}_cS = S^3, S^5$, or S^7, and ${}_cG = \mathrm{T}^2$, $\mathrm{S}^1 \times \mathrm{S}^3$ or $\mathrm{U}(2)$, or $\mathrm{S}^3 \times \mathrm{S}^3$, respectively. In all cases, at least one singular isotropy group say ${}_cK_+$ (S^1 in the first case and $\mathrm{SU}(2) = \mathrm{S}^3$ in the remaining cases) is a normal subgroup of ${}_cG$ which acts fixed point homogeneously on ${}_cS$. In particular, the action is reducible and $\theta = \pi/2$.
- If ${}_cG$ is neither finite nor connected, then ${}_cG/{}_cG_0 = \mathbb{Z}_2$, and the identity component ${}_cG_0$ of ${}_cG$ is either T^2 from before, or else S^1 or S^3 with suspension actions on S^2 or S^4, respectively. In the latter, two special cases $\theta = \pi/2$, whereas $\theta = \pi/4$ in the first scenario.

Moreover, ${}_cG/{}_cG_0 = \mathbb{Z}_2$ and ${}_cG_0 = S^3$ only happens for connected G, when the action of G is splitting.

When $V = S$, the two situations described above in (2.3), and (2.4), are rather restrictive:
In the first case (2.3), the identity component Γ_0 of Γ must be either $\{1\}, \mathrm{S}^1$, or S^3 and the corresponding action is the Hopf action. When Γ_0 is trivial, the group of components $\Pi = \Gamma/\Gamma_0$ is Γ itself, and $W = S/\Pi$ is a cohomogeneity one space form with fundamental group Π. Since also the normal spheres of the singular orbits are described by $L_\pm/L = K_\pm/H$, we get from the classification of transitive actions on spheres, that $\Pi = L_\pm/K_\pm$ is a finite subgroup of S^3. In case $\Gamma_0 = \mathrm{S}^1$, the action of G is complex, and $W = S/\Gamma$ is either a complex projective space, or the $\mathbb{Z}_2$ quotient of a complex odd dimensional projective space. Finally, when $\Gamma_0 = \mathrm{S}^3$, the G-action is quaternionic, $\Gamma = \Gamma_0$ and $W = S/\Gamma$ is a quaternionic projective space.

The second case (2.4) is only exhibited in those cohomogeneity one actions on S with $\theta = \pi/3$. In those cases, the antipodal map will interchange the two singular orbits, and the induced action on the corresponding real projective space is of the above type.

Example 2.5 (Type L)**.** Let S be a euclidean sphere and $G \times S \to S$ a linear cohomogeneity one action with diagram (2.2). For any euclidean sphere $F = S^*$ (including S^0), the sum of the trivial action of G on S^* with the G-action on S defines an action on the sphere $M = S^* * S$ with fixed point cohomogeneity one.

Example 2.6 (Type SL)**.** Let $G \times S \to S$ be as before and $G \times S/\Gamma \to S/\Gamma$ the cohomogeneity one action with diagram (2.3). For any euclidean sphere S^* on which Γ acts freely, the sum of the trivial action of G on S^* with the G-action on S induces a fixed point cohomogeneity one G-action on $M = (S^* * S)/\Gamma$ with fixed point set $F = S^*/\Gamma$.

Example 2.7 (Type PL)**.** Let $G \times S \to S$ be a linear cohomogeneity one action with $|\underline{S}| = \pi/3$ and $G \times S/\mathbb{Z}_2 \to S/\mathbb{Z}_2$ the cohomogeneity one action on the real projective space with corresponding diagram (2.4). For any euclidean sphere S^*, the sum of the trivial action of G on S^* with the G-action on S induces a fixed point cohomogeneity one G-action on the real projective space $M = (S^* * S)/\mathbb{Z}_2$ with fixed point set $F = S^*/\mathbb{Z}_2$.

In addition to these examples, there are fixed point cohomogeneity one actions that do not arise from any of these simple sum actions on spheres with fixed point cohomogeneity one.

Example 2.8 (Type T)**.** Let $G' \times S' \to S'$ be the standard action of $\mathrm{Sp}(n)$ on S^{4n-1} inducing a transitive isometric action on $\mathbb{C}P^{2n-1}/\mathbb{Z}_2$, and on $\mathbb{H}P^{n-1}$. Consider the corresponding fixed point homogeneous action of G' on $\mathbb{C}P^{2n+1}/\mathbb{Z}_2$, and on $\mathbb{H}P^{n+1}$ with fixed point set $\mathbb{C}P^1/\mathbb{Z}_2$, and $\mathbb{H}P^1$, respectively. Then, $G = \mathrm{S}^1 \times G' \subset \mathrm{Sp}(n+2)$, with S^1 embedded diagonally in $\mathrm{S}^1 \times \mathrm{Sp}(1) \subset \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ acts with fixed point cohomogeneity one on $\mathbb{C}P^{2n+1}/\mathbb{Z}_2$, and on $\mathbb{H}P^{n+1}$ with fixed point set a point $\{p\} \in \mathbb{C}P^1/\mathbb{Z}_2$ and $S^2 \subset \mathbb{H}P^1$, respectively. The normal sphere S to the fixed point set is $S^1 * S'$.

Finally, note the cohomogeneity two actions on $\mathbb{C}P^n$ and on $\mathbb{H}P^n$ in (2.6) above only intrinsically can have an analog on $\mathbb{C}aP^2$. Indeed, we have

Example 2.9 (Type ET)**.** On the Cayley plane $\mathbb{C}aP^2 = \mathrm{F}_4 / \mathrm{Spin}(9)$, the sub-action by $G = \mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ is of cohomogeneity two and has three fixed points. Also, the sub-actions by $\mathrm{SU}(4) = \mathrm{Spin}(6) \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(8)$ are of fixed point cohomogeneity one with fixed point sets S^2 and S^1, respectively. The two latter do not have analogs on $\mathbb{C}P^n$ or $\mathbb{H}P^n$.

We can now formulate our main result.

Theorem 2.10. *Any fixed point cohomogeneity one manifold with positive curvature is equivariantly diffeomorphic to a manifold of one of the above four types.*

This yields a complete classification based on the classification of linear cohomogeneity one actions on spheres in [**24**].

The following more detailed version of Theorem B will play an important role for getting the classification result above up to equivariant diffeomorphism.

Theorem 2.11. *Let G be a connected compact Lie group which acts linearly by cohomogeneity one on a round sphere S. Then, the group $\mathrm{Diff}^G(S)$ of G-equivariant diffeomorphisms of S has the subgroup $\mathrm{O}^G(S)$ of G-equivariant linear maps as a weak deformation retract.*

Proof. Let $f : S \to S$ be a G-equivariant diffeomorphism. Clearly, f induces a diffeomorphism $[f] : \underline{S} \to \underline{S}$.

Assume first that $[f]$ preserves the end points of the interval $\underline{S} = [-\theta/2, \theta/2]$ individually. There is then a canonical G-equivariant isotopy from f to $\hat{f}$, where $[\hat{f}]$ is the identity map, i.e., we can assume from the outset that f preserves all G-orbits individually. As in [**25**], we denote the group of such diffeomorphisms by $\mathrm{Diff}^G_I(S)$. These are described by paths $n : \underline{S} \to N(H)$ with $n(\pm\theta/2) \in N(H) \cap N(K_\pm)$, so that $f(c(t)) = n(t)c(t)$, where $c : \underline{S} \to S$ is a minimal geodesic perpendicular to all orbits, $G_{c(t)} = H, t \neq \pm\theta/2$ and $G_{c(\pm\theta/2)} = K_\pm$. Since any path in H induces the identity map, it is really the induced map, also called $n : \underline{S} \to {}_cG = N(H)/H$, which determines f. Then n starts at the subgroup ${}_cB_- = (N(K_-) \cap N(H))/H \supset (K_- \cap N(H))/H = {}_cK_-$ and ends at the subgroup ${}_cB_+ = (N(K_+) \cap N(H))/H \supset (K_+ \cap N(H))/H = {}_cK_+$. Also, ${}_cB_\pm \subset N({}_cK_\pm) \subset {}_cG$. Clearly, f is linear if and only if n is constant. To avoid smoothing problems near the non principle orbits, we first make an isotopy so that n is constant near the end points of the interval. This can clearly be done uniformly on any compact family of such diffeomorphisms.

From the considerations above, we get immediately that if the core group ${}_cG$ is finite, then f is linear, and no further deformations are needed. In the remaining cases, we also use our knowledge of ${}_cG$, and of ${}_cB_\pm$ from [**25**]. Of particular importance to us is the fact, that when the core group has positive dimension, then either $\dim {}_cB_- = \dim {}_cG$ or $\dim {}_cB_+ = \dim {}_cG$ in all but one case (see [**25**] page 37). In the exceptional case, $G = \mathrm{SU}(4)$ with the representation $\rho_6 + [\mu_4]_{\mathbb{R}}$, the core group is ${}_cG = \mathrm{U}(2), {}_cB_- = \mathrm{SU}(2)$, and ${}_cB_+ = \mathrm{U}(1)^2$.

We now equip ${}_cG$ with its standard bi-invariant metric. It is then apparent in all cases, that there are no non-trivial geodesics in ${}_cG$ connecting the totally geodesic subsets ${}_cB_\pm$ perpendicularly, and the trivial geodesics correspond to the points of ${}_cB_- \cap {}_cB_+ \subset {}_cG$, i.e., to the linear G-maps of S. By classical Morse theory for the energy function E on the path space of paths n considered here (cf. [**16**]), this means that there are no critical points for E other than the trivial null-curves in ${}_cB_- \cap {}_cB_+ \subset {}_cG$. Thus, the integral curves for the negative of the gradient of E provides a canonical retraction of this path space to the trivial paths in ${}_cB_- \cap {}_cB_+ \subset {}_cG$. For any map $S^k \to \mathrm{Diff}^G_I(S)$, it is now clear how this allows us to deform it to a map into $\mathrm{O}^G(S)$ without changing it on the closed set that maps into $\mathrm{O}^G(S)$.

Suppose now that $[f]$ interchanges the end points of $\underline{S}$. Since f preserves isotropy groups, and $S^{K_\pm} \subset G/K_\pm$ except for the cases where ${}_cG = D_3$, it follows that indeed ${}_cG = D_3$. But in those cases, the antipodal map A commutes with the G action and takes one singular orbit to the other. We then apply the above considerations to $f \circ A$ and we are done. q.e.d.

It is the following application of this result that we will use later.

Corollary 2.12. *Let $E \to B$ and $\overline{E} \to \overline{B}$ be two Riemannian G-vector bundles locally G-equivalent to the trivial G-bundle $\mathbb{R}^n \times \mathbb{R}^{\dim S+1} \to \mathbb{R}^n$, where G acts trivially on the base factor, and linearly by cohomogeneity one on the unit sphere $S \subset \mathbb{R}^{\dim S+1}$. Then, any G-equivariant fiber preserving diffeomorphism $\phi : SE \to S\overline{E}$ between the sphere bundles is G-isotopic to the restriction of a G-bundle isomorphism. In particular, ϕ extends to a fiber preserving G-diffeomorphism between the total spaces.*

Proof. Since by assumption ϕ takes fibers to fibers, it induces a diffeomorphism between the base manifolds. Without changing this diffeomorphism, we can use the above theorem to construct isotopies with support inside any small ball of B, so that the resulting diffeomorphism of sphere bundles is linear on the concentric ball with, say, half the radius. It is important that this isotopy does not alter ϕ where it is already linear. Since the fibers where an equivariant diffeomorphism is linear is a closed set, our claim follows by a simple maximal extension argument. q.e.d.

3. Orbit space structures

In this section, we will determine the possible structures for $\underline{M} = M/G$. Although the case where the fixed point set M^G is finite, or equivalently $\dim \underline{M} = 2$, is somewhat special, we will treat this case together with the general case $\dim M^G > 0$ to the extent possible.

Let F be a component of the fixed point set of maximal dimension. Then, by assumption $F \subset \underline{M} = M/G$ has codimension 2, and hence G acts by cohomogeneity one on the normal sphere $S = S_x$ at any point of $x \in F$. If (2.2) is the cohomogeneity one diagram for $G \times S \to S$, we see in particular that $\partial \underline{M} \neq \emptyset$. In fact, a neighborhood of F in $\underline{M}$ is a bundle over F with fiber $T^{\perp}_{\underline{x}} = C(S/G) = C\underline{S}$, and hence any $\underline{x} \in F$ as well as any $\underline{x}_{\pm} \in \underline{M}$ corresponding to orbits near F with orbit type $(K_{\pm})$ belong to $\partial \underline{M}$. We denote the face(s) of $\partial \underline{M}$ containing the latter orbits by $\partial_{\pm}$. Near F, each of these faces is a manifold with totally geodesic boundary F, and the boundary of a small neighborhood of F in $\partial \underline{M}$ is a two-fold cover F'' of F. F'' can also be identified with the boundary of space of normal directions to F in $\underline{M}$.

From now on, we will refer to $F \subset \partial \underline{M}$ as the *edge* E of $\underline{M}$ (when $\dim \underline{M} = 2$, the edge may contain one additional point). As we shall see, only a short list of types of orbit spaces $\underline{M}$ can occur. For convenience, we will list and baptize them here

- $\underline{M}$ is called a *lens* if its boundary has two smooth faces.
  In particular, the faces are discs, the edge their boundary sphere, and $\underline{M}$ has no interior singular points. (The two-dimensional version of a lens is a biangle)
  The orbit spaces of examples of type L in (2.5) are lenses (Figure 1).

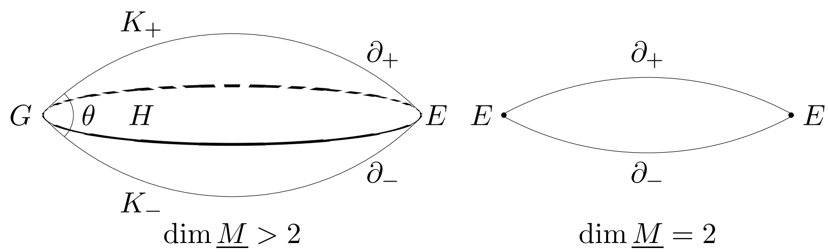

Figure 1. Type L

- $\underline{M}$ is called a *singular lens* if its boundary has two faces, and there is a geodesic I of singular points, called the *spine* of $\underline{M}$, joining the two face soul points $\underline{s}_\pm$ and making an angle $\pi/2$ with the faces.
  In particular, the spaces of directions at the face soul points, are spherical cones on the spaces of face directions. (The correct two-dimensional analog of a singular lens is a triangle with two right angles at the "spine", i.e. the face. ∂_0 opposite the "edge" F.)
  The orbit spaces of examples of type SL in (2.6) are singular lenses (Figure 2).

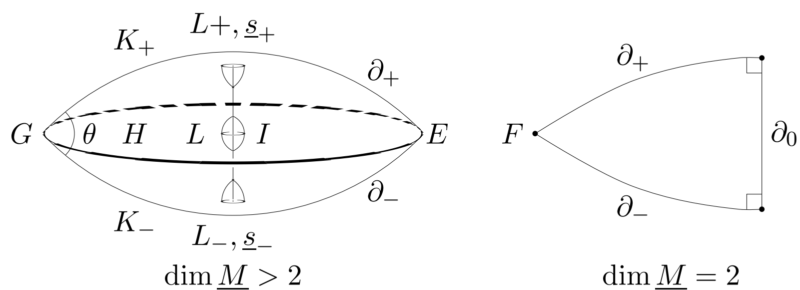

Figure 2. Type SL

- $\underline{M}$ is called a *top* if its boundary has two faces, one of which is smooth and the other singular.
  In particular, the faces are discs (one non-smooth), the edge their boundary sphere, and $\underline{M}$ has no interior singular points. The space of directions at the singular face soul point, say $\underline{s}_-$ called

the *vertex*, is a hemisphere of constant curvature 4. The two-dimensional version of a top is a triangle with two right angles at the "edge" consisting of two points, and vertex angle π/q.
The orbit spaces of examples of type T and ET in (2.8) and (2.9) are tops (Figure 3).

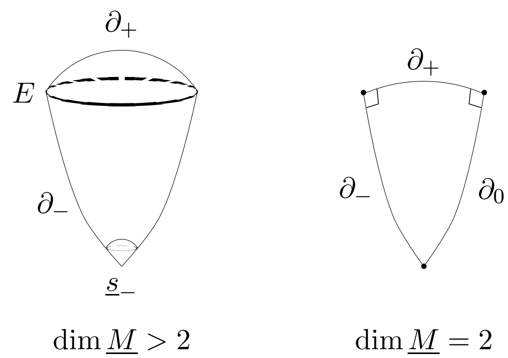

Figure 3. Type T and ET

Note that an isometric face preserving $\mathbb{Z}_2$ quotient of a lens, whose restriction to the edge is free (equivalent to the antipodal map), is a singular lens, with edge a real projective space. Also, an isometric face interchanging $\mathbb{Z}_2$ quotient of a lens, or a singular lens, fixing the edge, is a lens, or a singular lens.

- $\underline{M}$ is called a *projective lens* if it is an isometric face interchanging $\mathbb{Z}_2$ quotient of a lens, whose restriction to the edge is free (equivalent to the antipodal map).
  In particular, a projective lens has only one face, exactly one interior singular point $\underline{s}_o$, and edge a real projective space. The space of directions at the soul point is a real projective space of curvature 1. The two dimensional version of a projective lens is a loop with one interior singular point with vertex angle π.
  The orbit spaces of examples of type PL in (2.7) are projective lenses.

In view of this, it is may be surprising that a *projective singular lens* and a *singular top* will not arise as the orbit space of an action (by a connected group) of fixed point cohomogeneity one.

To see that the above structures are the only possible ones for orbits spaces of positively curved manifolds with fixed point cohomogeneity one, we proceed by analyzing the *core*, ${}_cM \subset M$ with its induced action by the core group ${}_cG = N(H)/H$. This suffices since $\underline{M}$ is isometric to the orbit space ${}_cM/{}_cG$, which we will denote by $\underline{{}_cM}$. Note that ${}_cG$ is also the core group of the linear cohomogeneity one action (2.2) of G on S. In particular, since the core ${}_cS$ of S is connected, so is ${}_cM$.

We will now determine ${}_cM$ and $\underline{{}_cM}$ corresponding to the different possibilities for ${}_cG$ exhibited in Section 2.

Lemma 3.1 (Polar action)**.** *Suppose ${}_cG$ is finite, i.e., ${}_cG = D_q$, with $q = 2, 3, 4$, or 6. Then, $|\pi_1({}_cM)| \leq 2$ and the universal cover ${}_c\widetilde{M}$ is diffeomorphic to S^{n+2}.*

If ${}_cM$ is simply connected, then $\underline{{}_cM} = \underline{M}$ is a lens *with edge angle π/q.*

If $\pi_1({}_cM) = \mathbb{Z}_2$, then $\underline{{}_cM} = \underline{M}$ is either a singular lens *with edge $\mathbb{R}P^n$ and edge angle $\pi/q, q$ even, or a* projective lens *with edge $\mathbb{R}P^n$ and edge angle $\pi/3$.*

Proof. When ${}_cG$ is finite, the action of G on M is polar (cf. [**18**]) with section ${}_cM$, and the component F of ${}_cM^{{}_cG}$ of maximal dimension has codimension two in ${}_cM$. From the description of $\underline{M}_c = M/{}_cG$ near F, it follows that ${}_cG = D_q$ is a dihedral group generated by two involutions $I_\pm$, where $< I_\pm >$ are the isotropy groups corresponding to the faces $\partial_\pm$. Let $F_\pm$ be the codimension one fixed point sets of $I_\pm$ containing F. Then, either both hypersurfaces $F_\pm \subset {}_cM$ are *two-sided*, or they are both *one-sided*.

In the first case, ${}_cM$ is the union of two discs with common boundary F_+ (or F_-) by the soul theorem. Moreover, since the *reflection* I_+ (resp. I_-) provides a canonical identification between the two smooth discs, the argument from [**6**], which in general is incomplete, shows in this case that ${}_cM$ must be diffeomorphic to the standard sphere. The intersection of two "hemispheres" determined by $F_\pm$ is easily seen to be isometric to ${}_cM/{}_cG$. In fact, the action of D_q on ${}_cM$ is equivalent to the standard linear action of D_q on the sphere $S^n * S^1 \simeq F * S = {}_cM$ fixing S^n, and hence $\underline{{}_cM} = \underline{M}$ is a lens. Note that $E = M^G \supset F$ consists of two points when $\dim \underline{M} = 2$.

In the second case, it follows from the soul theorem that ${}_cM$ is double covered by $S^{n+2} \simeq {}_c\widetilde{M}$, and that the inverse images $F' \subset F'_\pm \subset {}_c\widetilde{M}$ of $F \subset F_\pm \subset {}_cM$ are standard spheres $S^n \subset S^{n+1}_\pm \subset S^{n+2}$. Note that each of $I_\pm$ fix a unique point $s_\mp \in {}_cM$ at maximal distance to $F_\pm$. If $A : {}_c\widetilde{M} \to {}_c\widetilde{M}$ is the non-trivial element in the deck group $\pi_1({}_cM)$, we let $(s'_\mp, A(s'_\mp))$ be the corresponding unique pairs of points in ${}_c\widetilde{M}$ at maximal distance to $F'_\pm$. If $\widetilde{I}_\pm : {}_c\widetilde{M} \to {}_c\widetilde{M}$ are lifts of $I_\pm$, we note that $\widetilde{I}_\pm$ is either a *reflection* $R_\pm$ in $F'_\pm$, or a "*rotation*" $S_\pm$ fixing $(s'_\mp, A(s'_\mp))$ and acting by A on $F'_\pm$. In either case, A commutes with both of $\widetilde{I}_\pm$, and ${}_cM/{}_cG = {}_c\widetilde{M}/{}_cG'$, where ${}_cG'$ is the group generated by $\widetilde{I_-}, \widetilde{I_+}$, and A. Since $AS_\pm = R_\pm$, we see that ${}_cG' = < R_-, R_+, A >$, and hence $\underline{{}_cM} = ({}_c\widetilde{M}/ < R_-, R_+ >)/\mathbb{Z}_2$, where ${}_c\widetilde{M}/ < R_-, R_+ >$ is a lens from the first case.

If the involution preserves the two faces of the lens ${}_c\widetilde{M}/ < R_-, R_+ >$, it will clearly fix a line between the face souls and act as the antipodal map on the edge S^n. We conclude that $\underline{{}_cM} = \underline{M}$ is a singular lens with edge $\mathbb{R}P^n$. Note that in this case, the isotropy groups of ${}_cG$ corresponding to the face souls, i.e., the fixed points $s_\mp$ of $I_\pm$, are D_2, and hence in particular q is even. Conversely, it is not hard to see that when q is even the involution will preserve the faces of the lens.

If the involution interchanges the faces of the lens ${}_c\widetilde{M}/ < R_-, R_+ >$, it will clearly fix its soul point, and act as the antipodal map on the edge S^n. We conclude that $\underline{{}_cM} = \underline{M}$ is a projective lens with edge $\mathbb{R}P^n$. Note that this case corresponds exactly to q odd. q.e.d.

Remark 3.2. The above Lemma actually gives a partial answer to the question of what positively curved manifolds with fixed point cohomogeneity one are, when the definition is extended to finite groups G. Since here $M^G \subset M$ has codimension two, it follows that G is either a cyclic group, or a dihedral group. In the latter case, we just determined both M and the action. However, when G is cyclic, $\underline{M}$ does not have boundary and the situation is rather open, except for what follows from Wilking's connectedness lemma [**27**].

When ${}_cG$ is not finite, i.e., $\dim {}_cG > 0$, we note that its identity component ${}_cG_0$ acts on the core ${}_cM$ with fixed point cohomogeneity one or zero, and that $\underline{{}_cM} = ({}_cM/{}_cG_0)/\mathbb{Z}_2$.

Lemma 3.3 (Non-polar action I)**.** *Suppose* $\operatorname{cohomfix}({}_cM, {}_cG_0) = 1$. *Then* ${}_cM$ *is either a sphere, a space form with fundamental group* $\Gamma \subset S^3$*, a complex or quaternionic projective space, or the* $\mathbb{Z}_2$ *quotient of an odd dimensional complex projective space.*

Moreover, $\underline{M} = \underline{{}_cM}$ *is either a* lens *with edge* S^n*, a* singular lens (including a triangle) *with edge either* $S^n/\Gamma, \mathbb{C}P^{n/2}, \mathbb{H}P^{n/4}$*, or* $\mathbb{C}P^{n/2}/\mathbb{Z}_2$ *when* $n/2$ *is odd, or a* top *with edge* S^2 *or* S^0*(and vertex angle* $\pi/4$*). The angle between the faces is* $\pi/2$*, or* $\pi/4$ *in case* ${}_cG$ *is disconnected. In the exceptional top cases,* ${}_cM$ *is either* $\mathbb{H}P^2$*, or* $\mathbb{C}P^3/\mathbb{Z}_2$*, respectively, and* ${}_cG$ *is one of* $\mathrm{S}^1 \times \mathrm{S}^3$*, or* $\mathrm{U}(2)$*.*

Proof. By assumption ${}_cG_0$ is one, the groups $\mathrm{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1,\ \mathrm{S}^1 \times \mathrm{S}^3$ or $\mathrm{U}(2)$, or $\mathrm{S}^3 \times \mathrm{S}^3$ with a reducible cohomogeneity one action on ${}_cS$. Since one of the singular isotropy groups, say ${}_cK'_+ \subset {}_cG_0$ of the ${}_cG_0$-action on ${}_cS$, is a normal subgroup (S^1 in the first case and $\mathrm{S}^3 = \mathrm{SU}(2)$ in the remaining cases), it follows that it acts fixed point homogeneously on ${}_cM$, and we therefore know what both ${}_cM$ and ${}_cM/{}_cK'_+$ are from (1.4). Moreover, ${}_cG_0/{}_cK'_+ = \mathrm{S}^1$ or S^3 acts on ${}_cM/{}_cK'_+$ preserving the cone point and the boundary, on which it acts fixed point homogeneously (cf. (1.5)). For $({}_cM/{}_cK'_+)/({}_cG_0/{}_cK'_+) = {}_cM/{}_cG_0 = \underline{{}_cM'}$, we therefore explicitly have the following cases:

When ${}_cG_0 = \mathrm{T}^2$, $\underline{{}_cM'}$ is either a lens with edge S^n, or a singular lens with edge either $S^n/\mathbb{Z}_q$ or $\mathbb{C}P^{n/2}$, corresponding to ${}_cM$ being either S^{n+4}, $S^{n+4}/\mathbb{Z}_q$, or $\mathbb{C}P^{\frac{n}{2}+2}$. When $\underline{{}_cM'}$ is a triangle (in the second case $q = 2$), all angles are $\pi/2$.

Similarly, when ${}_cG_0$ is $\mathrm{S}^1 \times \mathrm{S}^3$, or $\mathrm{U}(2)$, $\underline{{}_cM'}$ is in general either a lens with edge S^n, or a singular lens with edge either $S^n/\mathbb{Z}_q$ or $\mathbb{C}P^{n/2}$. In addition, there are two exceptional cases, where $\underline{{}_cM'}$ is a top with edge S^2, or a triangle (a two-dimensional top with vertex angle $\pi/4$). The general cases correspond to ${}_cM$ being either S^{n+6}, $S^{n+6}/\mathbb{Z}_q$, or $\mathbb{C}P^{n/2+3}$. In the exceptional cases, ${}_cM$ is $\mathbb{H}P^2$, or $\mathbb{C}P^3/\mathbb{Z}_2$, respectively.

Finally, when ${}_cG_0 = \mathrm{S}^3 \times \mathrm{S}^3$, $\underline{{}_cM'}$ is either a lens with edge S^n, or a singular lens with one of the following edges: (1) S^n/Γ, where $\Gamma \subset S^3$ is a finite subgroup acting freely and linearly on S^n, (2) $\mathbb{C}P^{n/2}$, (3) $\mathbb{C}P^{n/2}/\mathbb{Z}_2$ where $n/2$ is odd, or (4) $\mathbb{H}P^{n/4}$. These orbit spaces correspond to ${}_cM$ being either S^{n+8}, S^{n+8}/Γ, $\mathbb{C}P^{\frac{n}{2}+4}$, $\mathbb{C}P^{\frac{n}{2}+4}/\mathbb{Z}_2$ ($n/2$ odd), or $\mathbb{H}P^{\frac{n}{4}+2}$.

If ${}_cG$ is connected, this completes the proof.

Now suppose ${}_cG$ is not connected. Then, ${}_cG_0 = \mathrm{T}^2$, and the induced involution on $\underline{{}_cM'}$ fixes the edge E and interchanges the faces. From the above, we know then that $\underline{{}_cM} = \underline{{}_cM'}/\mathbb{Z}_2$, where $\underline{{}_cM'}$ is a lens, or a singular lens (including the triangle case), and the involution must be a reflection in a hyperplane through the interior of $\underline{{}_cM'}$ reflecting the spine I'. Clearly, $\underline{{}_cM}$ is again either a lens or a singular lens with angle $\pi/4$ between the faces. Note that the face of $\underline{{}_cM}$ corresponding to the fixed point set of the reflection, is the closure of the set of orbits with isotropy group $\mathbb{Z}_2$. q.e.d.

Lemma 3.4 (Non-polar action II)**.** *Suppose* cohomfix$({}_cM, {}_cG_0) = 0$. *Then, ${}_cM$ is either a sphere, a real projective space, or it is one of the exceptional spaces $S^3/\mathbb{Z}_q$, or $\mathbb{C}P^2$.*

In the general cases, $\underline{M} = \underline{{}_cM}$ is either a lens *with edge S^n, or a* singular lens *with edge $\mathbb{R}P^n$ (including a triangle).*

In the exceptional cases, $\underline{M} = \underline{{}_cM}$ is either a two-, or three-dimensional top*, and ${}_cG_0 = \mathrm{S}^1$.*

In all cases, the angle between the faces at the edge is $\pi/2$.

Proof. By assumption ${}_cG_0$ is either S^1 or S^3 with the suspension actions on ${}_cS = S^2$ or S^4, and ${}_cG/{}_cG_0 = \mathbb{Z}_2$. Moreover, ${}_cG_0 = \mathrm{S}^3$ only when the G-action is splitting, and hence ${}_cG = \mathrm{S}^3 \times Z_2$. In both cases, ${}_cG_0$ acts on ${}_cM$ with fixed point cohomogeneity zero, and hence $\underline{{}_cM'} = {}_cM/{}_cG_0$ is homeomorphic to the cone $C\hat{F}$, where $\hat{F} = \partial \underline{{}_cM'} \subset \underline{{}_cM'}$ is the component of ${}_cM^{{}_cG_0}$ containing F. Moreover, only the soul point $\underline{s}_o$ is possibly singular, and $S_{\underline{s}_o}\underline{{}_cM'} = S^{\perp}_{s_o}/({}_cG_0)_{s_o}$ is diffeomorphic to $\hat{F}$. Since $F \subset \hat{F}$ has codimension one, it follows as before from the soul theorem, that $\hat{F}$ or a two-fold cover of it is the standard sphere S^{n+1}.

First, assume that $\underline{{}_cM'}$ has no interior singular points, i.e., it is a smooth disc. The induced involution is then a reflection in a codimension one disc with boundary $E = F$ (or E is F union a point when $n = 0$). In particular, $\underline{{}_cM} = \underline{M}$ is a lens with angle $\pi/2$ between the faces.

Next, let us consider the exceptional cases, where $\hat{F}$ is a sphere, but the soul point is singular. Metrically, the only possibilities for the space of directions at the soul point are $S^1/\mathbb{Z}_q \simeq S^1, \mathbb{C}P^1 \simeq S^2$, or $\mathbb{H}P^1 \simeq S^4$, corresponding to ${}_cM = S^3/\mathbb{Z}_q$, or $S^5/\mathbb{Z}_q$, ${}_cM = \mathbb{C}P^2$, or ${}_cM = \mathbb{H}P^2$, respectively (see [11]). We note that the three- and four-dimensional cores have ${}_cG_0 = \mathrm{S}^1$, and the five- and eight-dimensional cores have ${}_cG_0 = \mathrm{S}^3$. Again, the induced involution is a reflection in a codimension one cone on the boundary F (or F union a point, when F is a point). In particular, this easily rules out the five-, and eight-dimensional cores, and in the remaining two cases $\underline{{}_cM'}/\mathbb{Z}_2 = \underline{{}_cM}$ is a top of dimension two, three. In the two-dimensional case, the angles at the two fixed points of ${}_cG$ are $\pi/2$, and the vertex angle is π/q.

It remains to consider the situations, where $\hat{F}$ is not a sphere, i.e., $\pi_1(\hat{F}) = \mathbb{Z}_2$ and its universal cover is diffeomorphic to S^{n+1} with $n \geq 1$. Because $F \subset \hat{F}$ is one-sided, the induced involution on $\underline{{}_cM'}$ restricted to $\hat{F}$ will fix F and one additional point $\underline{s}$ at maximal distance from F. Since the involution also fixes the soul point $\underline{s}_o \in \underline{{}_cM'}$ at maximal distance to $\hat{F}$, it will fix a geodesic I from $\underline{s}$ to $\underline{s}_o$. In other words, $\underline{{}_cM}$ will always have interior singular points and two faces.

In the general case, where $S_{\underline{s}_o}\underline{cM'} = \mathbb{R}P^{n+1}$, we get that $\underline{{}_cM} = \underline{{}_cM'}/\mathbb{Z}_2$ is a singular lens with edge $\mathbb{R}P^n$ and angle $\pi/2$ between faces. Moreover, the spaces of directions at the face souls are metrically spherical cones on $\mathbb{R}P^n$.

As in the "spherical" case above, exceptional cases may be possible here as well, if $\underline{s}_o$ metrically is more singular, but, of course, $\pi_1(S_{\underline{s}_o}\underline{{}_cM'}) = \mathbb{Z}_2$ and $S_{\underline{s}_o}\underline{{}_cM'}$ is two-fold covered by S^{n+1}. It is easy to see that the only possibility is that $S_{\underline{s}_o}\underline{{}_cM'} = \mathbb{C}P^1/\mathbb{Z}_2 \simeq \mathbb{R}P^2$, corresponding to $({}_cG_0)_{s_o} = \mathrm{Pin}(2) \subset {}_cG_0 = \mathrm{S}^3$ and ${}_cM = \mathbb{C}P^3/\mathbb{Z}_2$ according to [**11**]. Since, however, ${}_cG = \mathrm{S}^3 \times \mathbb{Z}_2$, this is easily seen to be impossible. q.e.d.

We point out that it is not *a priori* clear if the above exhaustive list of potential cores, are actually all realized as the core of a fixed point cohomogeneity one manifold with positive curvature and connected G.

4. Lenses and projective lenses

In this section, we will classify positively curved manifolds of fixed point cohomogeneity one, whose orbit space is either a lens or a projective lens. We note that only in these cases do the exceptional edge angles $\pi/3$ and $\pi/6$ occur.

We begin with a sphere recognition result:

Theorem 4.1 (Lens case)**.** *Suppose $\underline{M}$ is a lens with edge $E \simeq S^n$. Then, M is equivariantly diffeomorphic to $S^n * S^m$, where $S = S^m$ and the action is of type* L.

Proof. By assumption, each face $\partial_\pm$ is diffeomorphic to the disc D^{n+1} with common boundary sphere $E \simeq S' = S^n$. Also, $\underline{M} = C\partial_+ = CCE = E * I$. We can achieve this description in a particular way.

First, choose a smooth gradient like vector field for $\mathrm{dist}(\partial_+, \cdot)$ which is radial near its maximum $\underline{s}_- \in \partial_-$, is the unit normal field to the hypersurfaces parallel to ∂_+ near ∂_+ and away from E, and on ∂_- near E is the unit normal field to the hypersurfaces parallel to E near E. Now, let I be an integral curve from say $\underline{s}_+ \in \partial_+$ to $\underline{s}_- \in \partial_-$. Note that the normal bundle to I in $\underline{M}$ is trivial. I together with each integral curve on ∂_- spans a two dimensional surface, which is normal to I and to the integral curve, and near whose endpoints agrees with radial normal slices. To achieve this description globally, first choose piecewise smooth trivializations of tubular neighborhoods of ∂_- and of I slightly away from E and from the endpoints of I with the desired properties, and then apply, e.g., standard center of mass mollifier smoothing techniques. We finally modify the gradient like vector field for $\mathrm{dist}(\partial_+, \cdot)$ so that near I, E and ∂_-, it is tangential to this family of two-dimensional surfaces. With such a choice, each two-dimensional sector of the tangent space at $\underline{s}_-$ spanned by I and a tangent vector to ∂_- sweeps out a triangular region in $\underline{M}$ with one vertex at a point of E and the other two vertices the endpoints of I. We now change our viewpoint and view each of these triangles as cones on I. In M, the corresponding set is a disc equivariantly diffeomorphic to the normal disc of E at the point.

We now claim that M is equivariantly diffeomorphic to the standard sphere $S' * S$, where G acts trivially on S' and by cohomogeneity one on S. First, note that $\pi^{-1}(I) \subset M$ is an invariant submanifold equivariantly diffeomorphic to S. Moreover, a tubular neighborhood of this submanifold is equivariantly diffeomorphic to the product $S \times D'$, where G acts trivially on the D' factor. The same is obviously true for the submanifold $S \subset S' * S$, and we thus have a G-diffeomorphism Φ between these two tubular neighborhoods. However, the boundary of each of these tubular neighborhoods are also the boundaries of tubular neighborhoods of $E \subset M$ and of $S' \subset S' * S$. The boundaries of each of these latter tubular neighborhoods are G-locally equivalent to the trivial G-bundle $\mathbb{R}^n \times S$ (cf. e.g. [8]), and the restriction ϕ of Φ to the total spaces $S(E)$ and $S(S')$ of these bundles is a G-diffeomorphism. From our choice of gradient like vector field, it follows that ϕ preserves the fibers, and we can use (2.12) to see that ϕ extends to a G-diffeomorphism between the tubular neighborhoods $D(E)$ and $D(S')$, and hence together with Φ gives the desired G-diffeomorphism between M and $S' * S$. q.e.d.

Note that without a careful choice of gradient like vector field in the above proof, the equivariant diffeomorphism ϕ would not preserve fibers, but will only induce a pseudo-isotopy $E \times [-\theta/2, \theta/2] \simeq S(E)/G \to S(S')/G \simeq S' \times [-\theta/2, \theta/2]$. For $\dim E \geq 5$, one could then appeal to the work of Cerf [**4**] that this map is isotopic to an isotopy to complete the proof.

We now proceed to consider the only one face case, where as we have seen in the proof of the polar lemma (3.1), that the core group ${}_cG$ is the dihedral group D_3, and hence $G \times S \to S$ is one of the four exceptional actions where $\underline{S}$ has length $\pi/3$.

Theorem 4.2 (Projective lens case)**.** *Suppose $\underline{M}$ is a projective lens with edge $\mathbb{R}P^n$. Then, M is equivariantly diffeomorphic to the real projective space $S^n * S^m / < -id >$, where $S = S^m$ and the action is of type* PL.

Proof. First, let us consider the case $\dim \underline{M} > 2$, where, by assumption, we know that $\pi_1(\partial \underline{M}) = \mathbb{Z}_2$. Let $\alpha : S^1 \to F = E$, and suppose it represents the trivial element in $\pi_1(M)$. We claim that α is then also trivial in $\pi_1(\partial \underline{M})$. Let $A : D^2 \to M$ be an extension of α. Since by assumption the soul orbit Gs_o has codimension at least 3, it follows by transversality that we can assume that $A(D)$ does not meet Gs_o and hence by (1.2) it can be arranged that $\pi(A(D)) \subset \partial \underline{M}$. By a similar argument, we see that $\pi_1(F) \to \pi_1(M)$ is onto and hence $\pi_1(M) = \pi_1(F)/ker = \mathbb{Z}_2$.

Let $\widetilde{M}$ be the two-fold universal cover of M. Allowing G to act almost effectively on M, we can assume that the action lifts to $\widetilde{M}$ and that clearly $\underline{M}$ is the $\mathbb{Z}_2$ quotient of $\underline{\widetilde{M}}$. This means that $\underline{\widetilde{M}}$ is a lens and hence $\widetilde{M}$ is a sphere with the action described above. In addition, the "spine" I in $\tilde{M}$ constructed above can be assumed to be invariant under the involution with quotient $I' \subset \underline{M}$. Consequently, the corresponding cohomogeneity one manifold P in M is diffeomorphic to the deck group quotient of S. From its group diagram, we see that the deck group restricted to the "spine manifold" S is the antipodal map. Combining this with the initial fact that the deck group restricted to the two-fold cover $\hat{F}$ of F is also the antipodal map, we get that M is the real projective space with an action of type PL as claimed.

It remains to consider the two dimensional case. Let I' be a minimal geodesic from the singular soul point to $\partial \underline{M}$, and pick another geodesic from the soul point to the point F. This way, $\underline{M}$ is the union of two geodesic triangles with two sides in common. This description yields that M is the union of the disc with boundary S, and a tubular neighborhood of the cohomogeneity one manifold P corresponding to I. Again, it follows from the group diagram that P is the real projective space and the projection map from the boundary of its tube is the quotient map by the antipodal map. Thus, M is the real projective space $S^0 * S/ < -id >$ with the induced action of type PL. q.e.d.

5. Singular lenses

In this section, we will classify the large subclass of positively curved fixed point cohomogeneity one manifolds corresponding to the ones having orbit space a singular lens.

We begin by describing all common features and setting up appropriate notation.

Denote the isotropy groups corresponding to the face soul points $\underline{s}_\pm$ and to the interior points of the spine I by $L_\pm$ and L, respectively (when $\underline{M}$ is a triangle, we use the same notation, where now $I = \partial_0$ is the face opposite the point edge $E = F$, and $\underline{s}_\pm$ its endpoints) (see Figure 2). Since I is perpendicular to the faces $\partial_\pm$, we see that the slice representation of $L_\pm$ is reducible and has three orbit types. In fact, it acts on the unit sphere $S'_\pm$ of one subspace corresponding to the face $\partial_\pm$ with only one orbit type $(K_\pm)$, and transitively with isotropy group L on the unit sphere of its orthogonal complement. In particular, $\pi^{-1}(I) = P$ is a G-invariant cohomogeneity one submanifold of M with diagram (2.3). We also note that the slice representation of L restricted to the unit sphere S' of the orthogonal complement to P has only one orbit type (H). A simple space of directions argument yields $S'_+/L_+ = S'/L = S'_-/L_-$ and that this manifold by the soul lemma is diffeomorphic to F. Since $S'_\pm$ and S' are all normal spheres to $P \subset M$, it follows that $L_+/K_+ = L/H = L_-/K_-$.

We conclude from the above, that either $L_\pm$ and L act transitively on $S'_\pm$ and S' respectively, corresponding to the case where $\underline{M}$ is a triangle, or else $K_\pm \subset L_\pm$ and $H \subset L$ are all normal subgroups, and the quotient group Γ acts freely on the sphere $S'_\pm = S'$. In the latter situation, the identity component of Γ is either $\{1\}$, S^1, or S^3 and consequently $E = F$ is a finite quotient of either a sphere, a complex projective space, or a quaternionic projective space. We will let $\Pi = (L/H)/(L/H)_0$ denote the group of components of $\Gamma = L/H$.

To recover the structure of M, we proceed as in the proof of (4.1) by providing a more detailed description of the structure of $\underline{M}$. For this purpose, we first note, that $I \subset \underline{M}$ has trivial "tubular neighborhood" $I \times C(S'/L) \simeq I \times CF$. Moreover, we can choose a gradient like vector field for say $\mathrm{dist}(\partial_+, \cdot)$ as in the proof of (4.1), and as a result, write $\underline{M} = F * I$ as a union of triangular surfaces with common base I and vertices in F, and respecting the normal structure near F, and the trivialization near I.

For simplicity, we now separate the discussion into two scenarios corresponding to the edge being simply connected or not.

Theorem 5.1 (Singular lens case I)**.** *Suppose $\underline{M}$ is a singular lens with simply connected edge E. Then, M is equivariantly diffeomorphic to a projective space with an action of type* SL*, or to the Cayley plane with the* Spin(8)*-action of type* ET.

Proof. From our discussion above, we see that our assumption implies that E is either a point, a complex, or a quaternionic projective space.

If E is a point, and thus $\underline{M}$ a triangle with right angles at the base $I = \partial_0$, we see that S is the boundary of a tubular neighborhood of P. It follows that the equivariant projection $S \to P$ is one of the standard Hopf maps of the sphere to a projective space, including the quotient map by the antipodal map, and the exceptional Hopf map $S^{15} \to S^8$. In all but the last case, M is a projective space with an action of type *SL*.

In the last exceptional case, it follows from the classification of linear cohomogeneity one actions on spheres, that the Hopf map $S = S^{15} \to S^8 = P$ is equivariant only under the cohomogeneity one action of $G = \mathrm{Spin}(8)$, the sub-action of the transitive $\mathrm{Spin}(9)$ action on S^{15}. The induced action of $\mathrm{Spin}(8)$ on $S^8 = P$ is the suspension action with kernel $\mathbb{Z}_2$. Consequently, M is equivariantly diffeomorphic to $\mathbb{C}aP^2 = \mathrm{F}_4 / \mathrm{Spin}(9)$ with the sub-action of $\mathrm{Spin}(8) \subset \mathrm{Spin}(9) \subset \mathrm{F}_4$. We note that all angles in the triangle $\underline{M}$ are $\pi/2$, and that each vertex correspond to a fixed point of $G = \mathrm{Spin}(8)$. In particular, this case could/should be viewed at an exceptional top case (see Section 6).

Now, assume that E is a complex or a quaternionic projective space of positive dimension. As we saw above, this is equivalent to $\Gamma = L/H$ being either S^1 or S^3 respectively, and as a consequence, also, P is a projective space of the same type. Let us say that the complex, respectively quaternion dimensions of E and P are k' and k. The remaining part of the proof proceeds along the exact same lines in both cases. By construction, we can pick $k'+1$ of the triangular surfaces as above, such that the angle between any two along I is $\pi/2$. This corresponds to $k'+1$ complex (resp. quaternionic) G-invariant line sub-bundles of the normal bundle to $P \subset M$, each isomorphic to the canonical line bundle and any two being perpendicular to one another. In particular, a tubular neighborhood of P in M is G-diffeomorphic to a tubular neighborhood of P^k in $P^{k+k'+1}$ with the action induced by the linear action on $CS' \times CS$, which is trivial on the first factor and the cone on (2.2) on the second factor. As in the proof of (4.1) the above construction yields a fiber preserving G-equivariant diffeomorphism between the boundaries of tubular neighborhoods of F in M and of $P^{k'}$ in $P^{k+k'+1}$, and the proof is completed as there. q.e.d.

Theorem 5.2 (Singular lens case II)**.** *Suppose $\underline{M}$ is a singular lens with non-simply connected edge E. Then, M is equivariantly diffeomorphic to a space form with fundamental group $\Pi \subset S^3$, or to the $\mathbb{Z}_2$ quotient of an odd dimensional complex projective space with action induced from a type* L*, or* SL *action on its universal cover.*

Proof. From what we have seen, our assumption means that E is either S^1, a space form with fundamental group $\Pi \subset S^3$, or the $\mathbb{Z}_2$ quotient of an odd dimensional complex projective space. Since E is also the quotient of the normal sphere S' to $P \subset M$ by the group Γ, $S' = S^1$ and $\Gamma = \mathbb{Z}_q$ in the first case. Therefore, P is a lens space, and by transversality $\pi_1(P) = \pi_1(M)$. When E is not a circle, i.e., it has dimension at least two, we use transversality to see that $\pi_1(E) = \pi_1(M)$. Hence, in all cases we have $\pi_1(M) = \Pi$.

Allowing G to act almost effectively, we can assume that G acts by fixed point cohomogeneity one on the universal cover $\widetilde{M}$ of M, with fixed point set $\hat{E}$ a Π-fold cover of E. Clearly, $\underline{\widetilde{M}}$ is a lens when E and P are lens spaces, and $\underline{\widetilde{M}}$ is a singular lens with edge a complex projective space, when E and P are $\mathbb{Z}_2$ quotients of such spaces. Consequently, $\widetilde{M}$ is either a sphere, or a complex projective space, with G-action of type L or SL, respectively, and inducing the G-action on their sub-covers $M = \widetilde{M}/\Pi$ of type SL as claimed. q.e.d.

6. Tops

In this final section, we will classify the more exceptional positively curved fixed point cohomogeneity one manifolds, i.e., those whose orbit spaces are tops.

First, observe that a common feature for all these cases is that the G-action on the normal space to F is reducible. From the classification of such actions, we know that all but four of them satisfy the assumption of the reduction lemma (1.4). We will refer to them as *general type* and *exceptional type*, respectively. We also note from the proof of (1.4) (see (1.5)) that for actions of general type, only the two- and four-dimensional tops exhibited in (3.3) can occur. In these situations, a special case of (1.4) yields:

Theorem 6.1 (General tops)**.** *Suppose $\underline{M}$ is a* top *with edge $E \supset F$, and the representation of G on $F^{\perp}$ is of general type. Then, M is equivariantly diffeomorphic to $\mathbb{H}P^{n+1}$, or to $\mathbb{C}P^{2n+1}/\mathbb{Z}_2$ with an action of type* T*.*

It remains to consider the cases, where $\underline{M}$ is a *top* with edge $E \supset F$, and the representation of G on $F^{\perp}$ is of exceptional type. According to the classification in [**24**], there are only four such representations $G \to \mathrm{SO}(\dim S + 1)$ which we list here:

- $G = \mathrm{Spin}(8)$ with representation $\rho_8 + \Delta_{\pm}$ on $\mathbb{R}^8 + \mathbb{R}^8 = \mathbb{R}^{16}$,
- $G = \mathrm{Spin}(7)$ with representation $\rho_7 + \Delta_7$ on $\mathbb{R}^7 + \mathbb{R}^8 = \mathbb{R}^{15}$,
- $G = \mathrm{Spin}(6) = \mathrm{SU}(4)$ with representation $\rho_6 + \Delta_6 = \rho_6 + [\mu_4]_{\mathbb{R}}$ on $\mathbb{R}^6 + \mathbb{R}^8 = \mathbb{R}^{14}$,

or, the maximal action orbit equivalent to the latter $\mathrm{SU}(4) = \mathrm{Spin}(6)$ action

- $G = \mathrm{U}(1) \times \mathrm{SU}(4)$ with representation $\rho_6 + [\mu_1 \otimes_{\mathbb{C}} \mu_4]_{\mathbb{R}}$ on $\mathbb{R}^6 + \mathbb{R}^8 = \mathbb{R}^{14}$.

We point out that the representations of $\mathrm{Spin}(6) \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(8)$ all come as subrepresentations of the first one, fixing a two-, or a one-dimensional subspace of $\mathbb{R}^8$. In particular, if $\mathrm{Spin}(8)$ acts by fixed point cohomogeneity one on M, so does $\mathrm{Spin}(7)$, and if $\mathrm{Spin}(7)$ does, so does $\mathrm{Spin}(6)$. To understand M, it therefore suffices to consider $\mathrm{Spin}(6) = \mathrm{SU}(4)$. Before analyzing this situation, we point out that the principal isotropy groups H for the above actions are G_2 for $\mathrm{Spin}(8)$, $\mathrm{SU}(3)$ for $\mathrm{Spin}(7)$, $\mathrm{SU}(2)$ for $\mathrm{Spin}(6)$, and $\mathrm{U}(1) \times \mathrm{SU}(2)$ for $\mathrm{U}(1) \times \mathrm{SU}(4)$. The corresponding core groups ${}_cG$ are $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, Pin(2), U(2), and T^2 respectively (cf. e.g. [**24**]). In particular, we see from (3.3) that the orbit space by $\mathrm{U}(1) \times \mathrm{SU}(4)$ is not a top. By (3.1) the orbit space by Spin(8) is a top only if it is a triangle with three right angles. By (3.4) a top orbit space by Spin(7) may be either three-dimensional, or two-dimensional with vertex angle π/q. Here, the two-dimensional top is ruled out, since by (3.3) the sub-action by Spin(6) must have a three-dimensional lens or singular lens as orbit space because its core group is connected. Either of those possibilities are incompatible with a two-dimensional top. Finally, if an orbit space by Spin(6) is a top, it must be either four-dimensional, or two-dimensional with vertex angle $\pi/4$ according to (3.3). Here again, it turns out that the two-dimensional top can be ruled out and we will prove the following:

Theorem 6.2 (Exceptional top)**.** *Suppose $\underline{M}$ is a* top *with edge $E \supset F$, and the representation of G on $F^{\perp}$ is of exceptional type. Then, M is equivariantly diffeomorphic to $\mathbb{C}aP^2$ with the action of $G = \mathrm{Spin}(k), k = 6, 7$, or 8 of type* ET.

Proof. The action of $G = \mathrm{Spin}(6) = \mathrm{SU}(4)$ on $S = S^{13}$ has the following cohomogeneity one diagram

(6.3)
$$\begin{tikzcd}
 & \mathrm{SU}(4) = \mathrm{Spin}(6) & \\
\mathrm{SU}(3) \arrow[ur, "j_-"] & & \mathrm{Spin}(5) \arrow[ul, "j_+"'] \\
 & \mathrm{SU}(2) = \mathrm{Sp}(1) \arrow[ul, "i_-"] \arrow[ur, "i_+"'] &
\end{tikzcd}$$

Suppose first that $\underline{M}$ is a two-dimensional top with vertex angle $\pi/4$, and hence the core of M^{14} is $\mathbb{C}P^3/\mathbb{Z}_2$ by (3.3). It follows from the orbit structure of M and of ${}_cM$, that the invariant manifold corresponding to the face ∂_+, i.e., to the closure of orbits of type (Spin(5)) is $\mathbb{R}P^6$. A simple transversality argument then shows that M is not simply connected, and in fact $\pi_1(M) = \mathbb{Z}_2$ by Synge's theorem. (One could also use Wilking's connectedness lemma [**27**] on the inclusion ${}_cM \to M$ to get that M is not simply connected.) As a consequence, we also get a fixed point cohomogeneity one action of Spin(6) on the universal cover $\widetilde{M}$ with finite fixed point set. It is then clear that $\underline{\widetilde{M}}$ is a triangle with three right angles. In particular, the action of $\mathrm{Spin}(6)$ on $\tilde{M}$ is of type SL, which is impossible according to the group diagram above and (5.1).

We now know that $\underline{M}$ is a four-dimensional top with $E = F \simeq S^2$, and hence $\dim M = 16$. Clearly, the smooth face, ∂_+ of $\underline{M}$ is the closure of orbits of type $(\mathrm{Spin}(5))$, and the singular face, ∂_- is the closure of orbits of type $(\mathrm{SU}(3))$. The vertex point of ∂_- corresponds to the isolated face soul orbit which then must have isotropy group $\mathrm{U}(3)$ in order for the space of directions to be $D^3(1/2)$. Note that the collection of orbits corresponding to ∂_+ is a manifold N (actually $S^2 * S^5$), and M is equivariantly diffeomorphic to the union of a tubular neighborhood of N with a tubular neighborhood of the face soul orbit $Gs_- = \mathrm{SU}(4)/\mathrm{U}(3)$.

One has the exact same orbits space structure and isotropy groups for the sub-action of $\mathrm{SU}(4) = \mathrm{Spin}(6) \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ on the model space $\mathbb{C}aP^2 = \mathrm{F}_4 / \mathrm{Spin}(9)$. In particular, the tubular neighborhoods of the face soul orbits in M and in $\overline{M} = \mathbb{C}aP^2$ are equivariantly diffeomorphic by the slice theorem. As in the proof of the reduction lemma (1.4), the restriction of this diffeomorphism Φ to the boundaries $S(N)$ and $S(\overline{N})$ of tubular neighborhoods of $N \subset M$ and of $\overline{N} = S^2 * S^5 \subset \mathbb{C}aP^2 = \overline{M}$ yields an equivariant diffeomorphism ϕ between the total spaces of the sphere bundles $S(N) \to N$ and $S(\overline{N}) \to \overline{N}$ with fiber S^7. In contrast to the proof of the reduction lemma, the equivariance does not imply that ϕ preserves fibers, except for the fibers over the edge. Of course, ϕ takes principal orbits to principal orbits, but the restrictions of $S(N) \to N$ and $S(\overline{N}) \to \overline{N}$ to corresponding principal orbits $G/H = \mathrm{Spin}(6)/\mathrm{Sp}(1)$ is given by $G/H \to G/K_+$ and $G/H \to G/nK_+n^{-1}$, where $K_+ = \mathrm{Spin}(5) = \mathrm{Sp}(2)$ and $n \in N(H)$. For such a particular principal orbit, fibers are mapped to fibers if and only if $n \in N(H) \cap K_+$. Thus, the failure of taking fibers to fibers is given by a map $\eta : D^3(1/2) \to N(H)/(N(H) \cap K_+) = \mathrm{S}^1$, where $D^3(1/2)$ is the space of directions at the face soul orbits, and η maps the boundary $S^2(1/2)$ to $\{1\}$. Since $\pi_3(\mathrm{S}^1) = 0$, we can thus construct a G-isotopy of ϕ which at the end is a G-diffeomorphism ϕ', which takes fibers to fibers. Because the isotropy groups K_+ along N and $\overline{N}$ act transitively on the normal sphere fibers, we conclude as in [**11**] (cf. (1.4)) that ϕ' is linear on fibers and hence extends to a G-diffeomorphism between the tubular neighborhoods of N and $\overline{N}$. This then yields the desired extension of Φ to a G-equivariant diffeomorphism between M and $\overline{M} = \mathbb{C}aP^2$.

The same statement for the bigger groups $\mathrm{Spin}(7) \subset \mathrm{Spin}(8)$ follows in exactly the same way. Note that the two-, three-, and four-dimensional tops $M/\mathrm{Spin}(8) \subset M/\mathrm{Spin}(7) \subset M/\mathrm{Spin}(6)$ sit as cross sections inside one another. q.e.d.

We conclude by pointing out that the exceptional representation of $\mathrm{U}(1) \times \mathrm{SU}(4)$ can only be realized as the representation on the normal space to the fixed point set F of a fixed point cohomogeneity one action on a positively curved manifold M, when M is either a sphere or a real projective space. This easily follows from its cohomogeneity one diagram together with the fact that $\underline{M}$ is either a lens or a singular lens.

References

[1] A. Back & W.-Y. Hsiang, *Equivariant geometry and Kervaire spheres*, Trans. Am. Math. Soc. **304** (1987), 207–227, MR 0906813, Zbl 0632.53048.

[2] G. E. Bredon, *On the structure of orbit spaces of generalized manifolds*, Trans. Am. Math. Soc. **100** (1961), 162–196, MR 0126503, Zbl 0134.19403.

[3] Y. Burago, M. Gromov, & G. Perel'man, *A. D. Aleksandrov spaces with curvature bounded below*, Uspekhi Mat. Nauk **47** (1992), 3–51; translation in Russian Math. Surveys **47** (1992), 1–58, MR 1185284, Zbl 0802.53018.

[4] J. Cerf, *La stratification naturelle des espaces de fonctions différentiables rélles et le théorème de la pseudo-isotopie*, Inst. Hautes Études Sci. Publ. Math. **39** (1970), 5–173, MR 0292089, Zbl 0213.25202.

[5] J. Cheeger & D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. Math. **96** (1972), 413–443, MR 0309010, Zbl 0169.24101.

[6] H.-I. Choi, S.-M. Kim & S.-H. Park, *Remarks on sphere-type theorems*, Proc. Am. Math. Soc. **125** (1997), 569–572, MR 1343684, Zbl 0948.53016.

[7] F. Fang & X. Rong, *Homeomorphism classification of positively curved manifolds with almost maximal symmetry rank*, preprint, arXiv.org, math.DG/0311183.

[8] K. Grove, *Center of mass and G-local triviality of G-bundles*, Proc. Am. Math. Soc. **54** (1976), 352–354, MR 0394712, Zbl 0293.57018.

[9] K. Grove, *Critical point theory for distance functions*, Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), Am. Math. Soc. Proc. Sympos. Pure Math. **54** (1993), 357–385, Zbl 0806.53043.

[10] K. Grove, *Geometry of, and via, symmetries*, *Conformal, Riemannian and Lagrangian geometry (Knoxville, TN, 2000)*, Am. Math. Soc. Univ. Lect. Ser. **27** (2002), 31–53, MR 1922721, Zbl 1019.53001.

[11] K. Grove & C. Seale, *Differential topological restrictions by curvature, and symmetry*, J. Differential Geom. **47** (1997), 530–559, MR 1617636, Zbl 0929.53017.

[12] K. Grove & C. Seale, *Global G-manifold reductions and resolutions*, Ann. Global Anal. Geom. **18** (2000), 437–446, MR 1795106, Zbl 0982.53048.

[13] K. Grove & K. Shiohama, *A generalized sphere theorem*, Ann. Math. **106** (1977), 201–211, MR 0500705, Zbl 0341.53029.

[14] A. Hatcher, *A proof of a Smale conjecture* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. Math. **117** (1983), 553–607, MR 0701256, Zbl 0531.57028.

[15] W.-Y. Hsiang & H. B. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Diff. Geom. **5** (1971), 1–38, MR 0298593, Zbl 0219.53045.

[16] J. Milnor, *Morse Theory*, Ann. of Math. Studies **51** (1963), Princeton University Press, vi+153pp, MR 0163331, Zbl 0108.10401.

[17] H. F. Münzner, Isoparametrische hyperflächen in sphären, Math. Ann. **251** (1980), 57–71, MR 0583825, Zbl 0438.53050; *Über die zerlegung der Sphäre in Ballbündel. II*, Math. Ann. **256** (1981), 215–232, MR 0620709, Zbl 0438.53050.

[18] R. S. Palais & C.-L. Terng, *A general theory of canonical forms*, Trans. Am. Math. Soc. **300** (1987), 771–789, MR 0876478, Zbl 0652.57023.

[19] G. Perel'man, *Alexandrov spaces with curvature bounded below, II*, preprint (1991).

[20] X. Rong, *Positively curved manifolds with almost maximal symmetry rank*, Geom. Dedicata, **95** (2002), 157–182, MR 1950889, Zbl pre01879691.

[21] K. Shankar, *Isometry groups of homogeneous spaces with positive sectional curvature*, Diff. Geom. Appl. **14** (2001), 57–78, MR 1812525, Zbl 0991.53023.

[22] T. Skjelbred & E. Straume, *On the reduction principle for compact transformation groups*, preprint (1993).

[23] S. Smale, *Diffeomorphisms of the* 2*-sphere*, Proc. Am. Math. Soc. **10** (1959), 621–626, MR 0112149, Zbl 0118.39103.

[24] E. Straume, *Compact connected Lie transformation groups on spheres with low cohomogeneity, I*, Mem. Am. Math. Soc. **119** (1996), no. 569, vi+93pp, MR 1297539, Zbl 0854.57033.

[25] E. Straume, *Compact connected Lie transformation groups on spheres with low cohomogeneity, II*, Mem. Am. Math. Soc. **125** (1997), no. 595, viii+76pp, MR 1346930, Zbl 0876.57053.

[26] B. Wilking, *The normal homogeneous space* $(\mathrm{SU}(3) \times \mathrm{SO}(3))/\mathrm{U}^\bullet(2)$ *has positive sectional curvature*, Proc. Am. Math. Soc. **127** (1999), 1191–1194, MR 1469441, Zbl 0948.53025.

[27] B. Wilking, *Torus actions on manifolds of positive sectional curvature*, Acta Math., **191** (2003), 259–297, MR 2051400.

[28] B. Wilking, *Positively curved manifolds with symmetry*, Ann. of Math., to appear.

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