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ZERMELO NAVIGATION ON RIEMANNIAN MANIFOLDS

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Abstract

In this paper, we study Zermelo navigation on Riemannian manifolds and use that to solve a long standing problem in Finsler geometry, namely the complete classification of strongly convex Randers metrics of constant flag curvature.

0. Introduction

0.1. Purpose. We have four goals in this paper. The first is to describe Zermelo's problem of navigation on Riemannian manifolds. Zermelo aims to find the paths of shortest travel time in a Riemannian manifold (M, h), under the influence of a wind or a current which is represented by a vector field W on M, with $|W| := \sqrt{h(W,W)} < 1$. We point out that the solutions are the geodesics of a strongly convex Finsler metric, which is of Randers type and is *necessarily* non-Riemannian unless W is zero. Conversely, we show constructively that every strongly convex Randers metric arises as the solution to Zermelo's navigational problem on some Riemannian landscape (M, h), under the influence of an appropriate wind W on M with $|W| < 1$. This is the content of Proposition 1.1 in Section 1.3.

Randers metrics are interesting not only as solutions to Zermelo's problem of navigation. They form a ubiquitous class of metrics with a strong presence in both the theory and applications of Finsler geometry. Of particular interest are Randers metrics of constant flag curvature, the latter being the Finslerian analog of the Riemannian sectional

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curvature. It is the second goal of this paper to describe strongly convex Randers metrics of constant flag curvature via Zermelo navigation. Unlike previous characterization results [**5**, **22**], the navigation description has the advantage of clearly illuminating the underlying geometry. More precisely, suppose (h, W), with $|W| < 1$, is the navigation data of a strongly convex Randers metric F on M. Then: F has constant flag curvature K, if and only if there exists a constant σ, such that h has constant sectional curvature $K + \frac{1}{16}\sigma^2$, and W satisfies the equation $\mathcal{L}_W h = -\sigma h$ (namely, W is an infinitesimal homothety of h). This is Theorem 3.1.

The correspondence between strongly convex Randers metrics and their navigation data is a natural one, in the following sense. Let (M_1, F_1), (M_2, F_2) be strongly convex Randers metrics with navigation data (h_1, W_1), (h_2, W_2), respectively. Then, F_1 and F_2 are isometric as Finsler metrics if and only if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that $\phi^* h_2 = h_1$ and $\phi_* W_1 = W_2$; furthermore, the equation $\mathcal{L}_W h = -\sigma h$ behaves functorially under ϕ^*. This, in conjunction with Theorem 3.1, brings us one step closer to a complete list, up to local isometry, of strongly convex Randers metrics with constant flag curvature. In fact, let (M, F) be any such metric, with navigation data (h, W) such that $|W| < 1$. Then, every point $p \in M$ has an open subset U on which h is isometric to some neighborhood $\tilde{U}$ (depending on p and U) in a standard Riemannian space form (round sphere, Euclidean space, or hyperbolic space), and W restricted to U corresponds to an infinitesimal homothety on that space form. Furthermore, the Randers metric on U is *Finslerian* isometric to its concrete counterpart on $\tilde{U}$.

Our third goal is to work out the formulae for all the infinitesimal homotheties W of the three standard Riemannian space forms h. This is done in Theorem 5.1. The resulting list serves as the genesis, up to local isometry, of strongly convex Randers metrics of constant flag curvature. This classification problem was proposed by Ingarden about half a century ago. (Until 2002, it was erroneously thought to have been solved by Yasuda–Shimada in 1977; see [**3**, **5**, **22**] for references therein.) Every vector field W in our list will perturb its companion Riemannian space form h into a strongly convex Randers metric of constant flag curvature. Also, let D denote the *maximal* domain on which W satisfies the essential constraint $|W| := \sqrt{h(W, W)} < 1$. Each such triplet (h, W, D) will be called a standard "model" for constant flag curvature Randers metrics. It remains to identify the inequivalent ones among these standard models.

For each *standard* Riemannian space form h in dimension n, we partition its infinitesimal homotheties W into equivalence classes: $W_1 \simeq W_2$ if and only if the Randers metrics on the maximal domains D_1 and D_2, generated by the navigation data (h, W_1) and (h, W_2), are globally isometric. These equivalence classes comprise what we call the *moduli space* $\mathcal{M}_K$ for strongly convex n-dimensional Randers metrics of constant flag curvature K. Our fourth goal in the paper is to parametrize $\mathcal{M}_K$ and thereby determine its dimension. It is found that for each non-negative value of K, the Randers moduli space is of dimension $n/2$ when n is even, and $(n+1)/2$ when n is odd. For each $K < 0$: the dimension of $\mathcal{M}_K$ is $n/2$ for even n; but for odd n, the moduli space is a stratified set, with one component of dimension $(n+1)/2$, and another component of dimension $(n-1)/2$. The specifics are detailed in Propositions 6.1, 6.2, and 6.3, respectively for K positive, zero, or negative. This picture is in striking contrast with the Riemannian setting, where the moduli space consists of a single equivalence class for each value of K; the class in question is represented either by the round sphere, or Euclidean space, or the hyperbolic metric, depending on the sign of K.

To conclude the paper, we illustrate the usefulness of the classification and the moduli space analysis by applying them to two special cases. First, those standard models (h, W, D) which effect projectively flat strongly convex Randers metrics of constant flag curvature K are singled out. (Beltrami's theorem assures us that a Riemannian space is of constant curvature if and only if it is projectively flat. The analogous statement does not hold among Randers metrics.) We find that up to isometry, the *non*-Riemannian ones (namely, those with $W \neq 0$) consist of a 1-parameter family of Minkowski metrics when $K = 0$, and a single variant of the Funk metric for each $K < 0$. In particular, while the Riemannian standard sphere is projectively flat, its perturbation by any non-zero W is not. This discussion constitutes Section 7.3. Our conclusion in the $K < 0$ case is then used to shed new light on the main result of Shen in [**29**].

Next, the moduli space analysis is specialized to the setting in which the tensor $\theta_i := b^s \operatorname{curl}_{si}$ vanishes. This enables us to describe explicitly all the Randers metrics addressed by systems of non-linear partial differential equations in the *corrected* Yasuda–Shimada theorem [**5**, **22**, **3**].

Such is the thesis of Section 8.2. We find that by perturbing Riemannian standard models h, the resulting strongly convex *non*-Riemannian Randers metrics of constant flag curvature K and $\theta = 0$ comprise, *up to isometry*, three small but distinguished camps.

- $K < 0$: there is just a single variant of the Funk metric for each value of K.
- $K = 0$: there is simply a 1-parameter family of Minkowski metrics.
- $K > 0$: this is the most enigmatic case. There is exactly a 1-parameter family of the $\theta = 0$ metrics on S^{2k+1}, and none on S^{2k}, regardless of whether the metrics being sought are local or global.

The classification of the $K > 0$ metrics within the $\theta = 0$ family has previously been done by Bejancu–Farran [**9**, **10**]. However, our description of the isometry classes offers a totally different perspective.

The described dimension counts are summarized below:

Table 1.

<table>
<tr><td colspan="2"></td><td colspan="4">Moduli space's dimension</td></tr>
<tr><td rowspan="2">CFC metrics</td><td rowspan="2">dim M</td><td rowspan="2">$K > 0$</td><td rowspan="2">$K = 0$</td><td colspan="2">$K < 0$</td></tr>
<tr><td>$\sigma = 0$</td><td>$\sigma \neq 0$</td></tr>
<tr><td>Riemannian
$b = 0/W = 0$</td><td>$n \geqslant 2$</td><td colspan="3">0</td><td>empty</td></tr>
<tr><td>Projectively flat
$db = 0/dW^\flat = 0$</td><td>$n \geqslant 2$</td><td>0^*</td><td>1</td><td>0^*</td><td>$0^\dagger$</td></tr>
<tr><td rowspan="2">Yasuda–Shimada
$\theta = 0$</td><td>even n</td><td>0^*</td><td rowspan="2">1</td><td rowspan="2">0^*</td><td rowspan="2">$0^\dagger$</td></tr>
<tr><td>odd n</td><td>1</td></tr>
<tr><td rowspan="2">Unrestricted
Randers</td><td>even n</td><td colspan="4">$n/2$</td></tr>
<tr><td>odd n</td><td colspan="3">$(n+1)/2$</td><td>$(n-1)/2$</td></tr>
<tr><td colspan="6">The moduli spaces of dimension 0 consist of a single point.
* The single isometry class is Riemannian.
† The single isometry class is non-Riemannian, of Funk type.</td></tr>
</table>

0.2. Summary of contents. Section 1 presents Zermelo's problem of navigation on Riemannian manifolds, and its solution.

We specialize to concrete 3-dimensional Riemannian space forms in Section 2. These examples deal with Zermelo navigation on spheres, Euclidean space, and the Klein model of hyperbolic geometry. The resulting Finsler metrics of Randers type are categorized into three subsections, depending on the sign of their flag curvature. In Section 2.4, we review the definition of Finsler metrics of constant flag curvature.

Section 3 begins by recalling a previously published characterization result. This is followed by the navigation description of strongly convex Randers metrics of constant flag curvature K. It also includes a Matsumoto identity which exhibits the interplay between the constant σ (in the equation $\mathcal{L}_W h = -\sigma h$) and the constant flag curvature K.

Before presenting the classification theorem, we pause in Section 4 to derive a complete list of allowable vector fields for each of the three standard models of Riemannian space forms. With the list in hand, Section 5 gives the classification of strongly convex Randers metrics of constant flag curvature; both local (Section 5.1) and global (Sections 5.2–5.4) aspects are treated.

The moduli space $\mathcal{M}_K$ for strongly convex Randers metrics of constant flag curvature K is the focus of Section 6. We make explicit the requisite Lie theory (mostly for a non-compact subgroup of the Lorentz group) in the Appendix, and (then) give concrete descriptions of $\mathcal{M}_K$ and its dimension.

Section 7 contains a discussion of projectively flat Randers metrics of constant flag curvature, and shows that our formalism is able to give important information about the metrics in [**29**]. Finally, in Section 8, we specialize our classification to the $\theta = 0$ case, and use that to catalog all the solutions of the partial differential equations in [**5**, **22**].

1. Zermelo navigation

1.1. Perturbing Riemannian metrics by vector fields.

1.1.1. Background metric and perturbing vector field. Given any Riemannian metric h on a differentiable manifold M, denote the corresponding norm-squared of tangent vectors $y \in T_x M$ by

$$|y|^2 := h_{ij}\, y^i y^j = h(y, y).$$

Think of $|y|$ as the *time* it takes, using an engine with a fixed power output, to travel from the base(point) of the vector y to its tip. Note the symmetry property $|-y| = |y|$.

The unit tangent sphere in each $T_x M$ consists of all those tangent vectors u such that $|u| = 1$. Now, introduce a vector field W such that $|W| < 1$, thought of as the spatial velocity vector of a mild wind on the

Riemannian landscape (M, h). Before W sets in, a journey from the base to the tip of any u would take 1 unit of time, say, 1 second. The effect of the wind is to cause the journey to veer off course (or merely off target if u is collinear with W). Within the same 1 second, we traverse not u but the resultant $v = u + W$ instead.

As an example, suppose $|W| = \frac{1}{2}$. If u points along W (that is, $u = 2W$), then $v = \frac{3}{2}u$. Alternatively, if u points opposite to W (namely, $u = -2W$), then $v = \frac{1}{2}u$. In these two scenarios, $|v|$ equals $\frac{3}{2}$ and $\frac{1}{2}$ instead of 1. So, with the wind present, our Riemannian metric h no longer gives the travel time along vectors. This prompts the introduction of a function F on the tangent bundle TM, in order to keep track of the travel time needed to traverse tangent vectors y under windy conditions. For all those resultants $v = u + W$ mentioned above, we have $F(v) = 1$. In other words, within each tangent space T_xM, the unit sphere of F is simply the W-translate of the unit sphere of h. Since this W-translate is no longer centrally symmetric, F cannot possibly be Riemannian.

1.1.2. Formula for the new Minkowski norm F. Start with the fact $|u| = 1$; equivalently, $h(u, u) = 1$. Into this, we substitute $u = v - W$ and then $h(v, W) = |v|\,|W| \cos\theta$. After using the abbreviation $\lambda := 1 - |W|^2$ to reduce clutter, we have $|v|^2 - (2\,|W| \cos\theta)\,|v| - \lambda = 0$. Since $|W| < 1$, the resultant v is never zero, hence $|v| > 0$. This leads to $|v| = |W| \cos\theta + \sqrt{|W|^2 \cos^2\theta + \lambda}$, which we abbreviate as $p + q$. Since $F(v) = 1$, we see that

$$F(v) = |v| \frac{1}{q + p} = |v| \frac{q - p}{q^2 - p^2} = \frac{\sqrt{[h(W, v)]^2 + |v|^2 \lambda}}{\lambda} - \frac{h(W, v)}{\lambda}.$$

It remains to deduce $F(y)$ for an arbitrary $y \in TM$. Note that every non-zero y is expressible as a positive multiple c of some v with $F(v) = 1$. For $c > 0$, traversing cv under the windy conditions should take c seconds. Consequently, F is positively homogeneous. Using this homogeneity and the formula derived for $F(v)$, we find that:

$$F(y) = \frac{\sqrt{[h(W, y)]^2 + |y|^2 \lambda}}{\lambda} - \frac{h(W, y)}{\lambda}.$$

It is now manifest that $F(-y) \neq F(y)$. By hypothesis, $|W| < 1$, hence $\lambda > 0$. We see from the formula for $F(y)$ that it is positive whenever $y \neq 0$. Also, $F(0) = 0$ as expected.

1.1.3. New Riemannian metric and 1-form. Our formula for F has two parts.

- The first term is the norm of y with respect to a new Riemannian metric
$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i}{\lambda}\frac{W_j}{\lambda},$$
where $W_i := h_{ij}\,W^j$ and $\lambda = 1 - W^i W_i$.
- The second term is the value on y of a differential 1-form
$$b_i = \frac{-W_i}{\lambda}.$$

Under the influence of W, the most efficient navigational paths are no longer the geodesics of the Riemannian metric h; instead, they are the geodesics of the Finsler metric F. For $\mathbb{R}^2$, this phenomenon is treated by Carathéodory [**15**] as Zermelo's navigation problem [**32**]. Shen [**30**] showed that the same phenomenon holds for arbitrary Riemannian backgrounds in all dimensions. See also the exposition in [**6**].

1.2. Ubiquitous class of Finsler metrics. The Finsler metric F derived from the perturbation has the simple form $F := \alpha + \beta$, where
$$\alpha(x, y) := \sqrt{a_{ij}(x)\, y^i y^j}\,, \quad \beta(x, y) := b_i(x)\, y^i.$$
This is the defining feature of Randers metrics, which were introduced by Randers in 1941 [**25**] in the context of general relativity, and later named by Ingarden [**20**].

The function F is positive on the manifold $TM \setminus 0$, whose points are of the form (x, y), with $0 \neq y \in T_xM$. Over each point (x, y) of $TM \setminus 0$ (treated as a parameter space), we designate the vector space T_xM as a fiber, and name the resulting vector bundle π^*TM. There is a canonical symmetric bilinear form $g_{ij}\, dx^i \otimes dx^j$ on the fibers of π^*TM, with
$$g_{ij} := \tfrac{1}{2}\left(F^2\right)_{y^i y^j}.$$
The subscripts y^i, y^j signify partial differentiation, and the matrix (g_{ij}) is known as the fundamental tensor. A Finsler metric F is said to be *strongly convex* if the said bilinear form is positive definite, in which case it defines an inner product on each fiber of π^*TM.

For a Randers metric to be strongly convex, it is necessary and sufficient to have
$$\|b\| := \sqrt{b_i\, b^i} < 1, \quad \text{where} \ \ b^i := a^{ij}\, b_j.$$

See [4] or [1] for the proof of this fact. In our case, using the formulae $a^{ij} = \lambda(h^{ij} - W^i W^j)$ and $b^i = -\lambda W^i$, we find that

$$\|b\|^2 := a^{ij}\, b_i b_j = h_{ij}\, W^i W^j =: |W|^2,$$

which is less than 1 by hypothesis. Therefore, the described perturbation of Riemannian metrics h by vector fields W, with $|W| < 1$, always generates strongly convex Randers metrics.

1.3. An inverse problem. A question naturally arises: can every strongly convex Randers metric be realized through the perturbation of some Riemannian metric h by some vector field W satisfying $|W| < 1$?

Happily, the answer to this question is yes. Indeed, let us be given an arbitrary Randers metric F with data a and b, respectively a Riemannian metric and a differential 1-form, such that $\|b\|^2 := a^{ij}\, b_i b_j < 1$. Set $b^i := a^{ij}\, b_j$, and $\varepsilon := 1 - \|b\|^2$. Construct h and W as follows:

$$h_{ij} := \varepsilon\,(a_{ij} - b_i b_j), \qquad W^i := -b^i/\varepsilon.$$

Note that F is Riemannian if and only if $W = 0$, in which case $h = a$. Also, we have $W_i := h_{ij}\, W^j = -\varepsilon\, b_i$. Using this, it can be directly checked that perturbing the above h by the stipulated W gives back the Randers metric we started with. Furthermore,

$$|W|^2 := h_{ij}\, W^i W^j = a^{ij}\, b_i b_j =: \|b\|^2 < 1.$$

Let us summarize:

Proposition 1.1. *A strongly convex Finsler metric F is of Randers type if and only if it solves the Zermelo navigation problem on a Riemannian manifold (M, h), under the influence of a wind W which satisfies $h(W, W) < 1$. Also, F is Riemannian if and only if $W = 0$.*

Incidentally, the inverse of h_{ij} is $h^{ij} = \varepsilon^{-1}\, a^{ij} + \varepsilon^{-2}\, b^i b^j$. This h^{ij}, together with W^i, defines a Cartan metric F^* of Randers type on the cotangent bundle T^*M. A comparison with [19] shows that F^* is the Legendre dual of the Finsler–Randers metric F on TM. It is remarkable that the Zermelo navigation data of any strongly convex Randers metric F is so simply related to its Legendre dual. See also [33] and [30].

1.4. Remark about isometries. Two Finsler spaces (M_1, F_1) and (M_2, F_2) are said to be isometric if there exists a diffeomorphism $\phi : M_1 \to M_2$ which, when lifted to a map between TM_1 and TM_2, satisfies $\phi^* F_2 = F_1$.

Now, consider two strongly convex Randers metrics F_1 and F_2, where F_i has Riemannian data (a_i, b_i). By the above proposition, they arise as solutions to Zermelo's navigation problem with (h_1, W_1) and (h_2, W_2), respectively. A moment's thought (via applying $y \mapsto -y$ to tangent vectors y in the equation $\phi^* F_2 = F_1$) gives the lemma below.

Lemma 1.2. *Let $\phi : M_1 \to M_2$ be a diffeomorphism. The following three statements are equivalent:*

- *ϕ lifts to an isometry between F_1 and F_2.*
- *$\phi^* a_2 = a_1$ and $\phi^* b_2 = b_1$.*
- *$\phi^* h_2 = h_1$ and $\phi_* W_1 = W_2$.*

2. Zermelo navigation on Riemannian space forms

This section illustrates a variety of perturbations on 3-dimensional Riemannian space forms. In each example, with the exception of the radial perturbation on the Euclidean metric (Section 2.3.1), W is an infinitesimal isometry of h. It happens that all the resulting strongly convex Randers metrics are of constant flag curvature (denoted K). The concept of flag curvature is a natural extension of Riemannian sectional curvatures to the Finslerian realm (see Section 2.4 for a review).

Since all our examples are in three dimensions, we let (x, y, z) denote position coordinates, and expand arbitrary tangent vectors as $u\partial_x + v\partial_y + w\partial_z$. We give expressions for the norm $\alpha := \sqrt{a(y, y)}$ instead of a_{ij} because the former are more compact. The Riemannian metric a (defined in Section 1.2) can be recovered via $a_{ij} = (\frac{1}{2}\alpha^2)_{y^i y^j}$.

2.1. Constant positive flag curvature.

2.1.1. Rotational perturbation of S^3. Let S^3 denote the standard unit sphere in $\mathbb{R}^4$. Using its tangent spaces at the east and west poles, we may parametrize the sphere by

$$(x, y, z) \mapsto \frac{1}{\sqrt{1 + x^2 + y^2 + z^2}} (s, x, y, z);$$

here, $s = \pm 1$, respectively, for the eastern and western hemispheres. Note that the equator corresponds to asymptotic infinity on the above tangent spaces. Fix any constant $0 < \tau < 1$ and perturb via the infinitesimal rotation

$$W = \tau \, (y, -x, 0) \quad \text{with } |W| = \tau \sqrt{\frac{x^2 + y^2}{1 + x^2 + y^2 + z^2}} < 1 \, .$$

The bound on τ is needed to maintain $|W| < 1$ globally on S^3. The resulting Randers metric $F = \alpha + \beta$ has constant flag curvature $K = 1$. Explicitly, with ψ abbreviating $xu + yv$, we have

$$\alpha^2 = \frac{\rho^2(u^2+v^2) - (\rho + \tau^2\varphi)\psi^2 + \eta\left\{(\rho - z^2)w^2 - 2zw\psi\right\}}{\rho\,\eta^2},$$

$$\beta = \frac{\tau\,(-yu + xv)}{\eta},$$

where $\varphi := 1+z^2$, $\rho := 1+x^2+y^2+z^2$, and $\eta := 1+(1-\tau^2)(x^2+y^2)+z^2$.

2.1.2. Perturbing by a privileged Killing field of S^3. Again, start with the unit sphere S^3 in $\mathbb{R}^4$, parametrized as above. For each constant $K > 1$, let h be $\frac{1}{K}$ times the standard Riemannian metric induced on S^3. The re-scaled metric has sectional curvature K.

Perturb h by the Killing vector field

$$W = \sqrt{K-1}\Big(-s(1+x^2), z - sxy, -y - sxz \Big) \quad \text{with} \quad |W| = \sqrt{\frac{K-1}{K}}.$$

This W is tangent to the S^1 fibers in the Hopf fibration of S^3. The resulting Randers metric F has constant flag curvature K (see [8]). Explicitly, $F = \alpha + \beta$, where

$$\alpha = \frac{\sqrt{K(su - zv + yw)^2 + (zu + sv - xw)^2 + (-yu + xv + sw)^2}}{1 + x^2 + y^2 + z^2},$$

$$\beta = \frac{\sqrt{K-1}\,(su - zv + yw)}{1 + x^2 + y^2 + z^2}.$$

2.2. Zero flag curvature.

2.2.1. Perturbing $\mathbb{R}^3$ by a translation. The Riemannian metric h to be perturbed is the standard Euclidean metric δ_{ij} on $\mathbb{R}^3$. Choose any three constants p, q, r which satisfy $p^2 + q^2 + r^2 < 1$. We perturb h by the vector field

$$W = (p, q, r) \qquad \text{with } |W| = \sqrt{p^2 + q^2 + r^2}.$$

The resulting Randers metric $F = \alpha + \beta$ has the form

$$\alpha = \frac{\sqrt{(pu + qv + rw)^2 + (u^2 + v^2 + w^2)\{1 - (p^2 + q^2 + r^2)\}}}{1 - (p^2 + q^2 + r^2)},$$

$$\beta = \frac{-(pu + qv + rw)}{1 - (p^2 + q^2 + r^2)}.$$

This F has constant flag curvature $K = 0$, and is a Minkowski metric.

2.2.2. Rotational perturbation of $\mathbb{R}^3$. As above, h is the Euclidean metric on $\mathbb{R}^3$. The perturbing vector field is the infinitesimal rotation $W := y\,\partial_x - x\,\partial_y + 0\,\partial_z$. The resulting Randers metric [**30**] $F = \alpha + \beta$ solves the least time problem for fish that are *surface*-feeding in a cylindrical tank with a rotational current. F is defined on the open cylinder $x^2 + y^2 < 1$ in R^3, and has constant flag curvature $K = 0$. Explicitly,

$$\begin{aligned}
\alpha &= \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2}, \\
\beta &= \frac{-yu + xv}{1 - x^2 - y^2} \qquad \text{with } |W|^2 = x^2 + y^2.
\end{aligned}$$

2.3. Constant negative flag curvature.

2.3.1. Radial perturbation of $\mathbb{R}^3$. Again, we perturb the Euclidean metric, but this time M is the open ball of radius R in $\mathbb{R}^3$, centered at the origin. The perturbing vector field is the radial $W = \tau(x\partial_x + y\partial_y + z\partial_z)$, where τ is a constant. Impose the constraint $|\tau| \leqslant \frac{1}{R}$ to ensure that $|W| < 1$ on M. The resulting Randers metric $F = \alpha + \beta$ is of constant flag curvature $K = -\frac{1}{4}\tau^2$, and is given by

$$\begin{aligned}
\alpha &= \frac{\sqrt{\tau^2(xu + yv + zw)^2 + (u^2 + v^2 + w^2)\{1 - \tau^2(x^2 + y^2 + z^2)\}}}{1 - \tau^2(x^2 + y^2 + z^2)}, \\
\beta &= \frac{-\tau(xu + yv + zw)}{1 - \tau^2(x^2 + y^2 + z^2)} \qquad \text{with } |W| = \sqrt{\tau^2(x^2 + y^2 + z^2)}.
\end{aligned}$$

When $R = 1$ and $\tau = -1$, the perturbation generates the Funk metric [**18**] on the unit ball in $\mathbb{R}^3$, with constant flag curvature $K = -\frac{1}{4}$. See also [**23**, **28**]. The Funk metric is isometric to the so-called Finslerian Poincaré ball. A 2-dimensional version of the latter is analyzed in [**4**].

2.3.2. Rotational perturbation of Hyperbolic space. Consider the Klein metric

$$h_{ij} = \frac{(1 - x^2 - y^2 - z^2)\delta_{ij} + x_i x_j}{(1 - x^2 - y^2 - z^2)^2}$$

on the unit ball $\mathbb{B}^3 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$. Here $x_i := \delta_{is} x^s$. We perturb by the infinitesimal rotation

$$W = (y, -x, 0) \qquad \text{with } |W| = \sqrt{\frac{x^2 + y^2}{1 - x^2 - y^2 - z^2}}.$$

In order that $|W| < 1$, we restrict to the domain $\{2x^2 + 2y^2 + z^2 < 1\}$. Define $\varphi := 1 - 2x^2 - 2y^2 - z^2$. Perturbing h by W produces a Randers metric $F = \alpha + \beta$, with

$$\begin{aligned} \alpha^2 &= \frac{\varphi\left[\rho(u^2+v^2) + (1-\eta)w^2 + 2zw(xu+yv)\right] + \eta(yu - xv)^2}{(1 - x^2 - y^2 - z^2)\varphi^2} \\ \beta &= \frac{-yu + xv}{\varphi}, \end{aligned}$$

and $\rho := 1 - z^2$, $\eta := x^2 + y^2$. This Randers metric F is of constant flag curvature $K = -1$.

2.4. Finsler metrics of constant flag curvature. Given any Finsler metric F, the Chern connection on the pulled-back tangent bundle $\pi^* TM$ gives rise to two curvature tensors, one of which, $R_j{}^i{}_{kl}$, is analogous to the curvature tensor in Riemannian geometry. Indices on R are raised and lowered by the fundamental tensor g_{ij} and its inverse g^{ij}.

At any point x on M, a flag consists of a flagpole $0 \neq y \in T_xM$, a transverse edge $V \in T_xM$, and $y \wedge V$. The corresponding *flag curvature* depends on x, y, $\mathrm{span}\{y, V\}$, and is defined as

$$K(x, y, V) := \frac{V^i\,(y^j\,R_{jikl}\,y^l)\,V^k}{g(y,y)\,g(V,V) - [g(y,V)]^2}.$$

In the generic Finslerian setting, both the Chern hh-curvature R and the inner product g (given by the fundamental tensor g_{ij}) depend on the flagpole y. This dependence is absent whenever we specialize to the Riemannian realm, in which case the flag curvature becomes the familiar sectional curvature. For details and conventions, see [**4**]. A Finsler metric is said to have constant flag curvature K if $K(x, y, V)$ has the constant value K for all locations $x \in M$, flagpoles y, and transverse edges V.

We note an interesting phenomenon shared by all our examples. In each case, the constant flag curvature of the resulting Randers metric F does not exceed the constant sectional curvature of the original Riemannian metric h which underwent the perturbation. This turns out to be a general phenomenon (see Theorem 3.1).

3. Randers metrics of constant flag curvature

3.1. Characterization. Let $F = \alpha + \beta$, with $\alpha^2 := a_{ij}\,y^i y^j$ and $\beta := b_i\,y^i$, be a Randers metric. Using a^{ij} to raise the index on the components b_j of the 1-form b, we obtain a vector field $b^\sharp = b^i \partial_{x^i}$. Let us introduce the abbreviations

$$\mathrm{curl}_{ij} := \partial_{x^j} b_i - \partial_{x^i} b_j \quad \text{and} \quad \theta_j := b^i \, \mathrm{curl}_{ij}.$$

Note that the tensor curl $:= \mathrm{curl}_{ij}\, dx^i \otimes dx^j$ equals the 2-form $-db$, and interior multiplication of curl by the vector field $b^\sharp$ gives the 1-form θ.

Define the geometric quantity

$$\sigma := \frac{2 \operatorname{div} b^\sharp}{n - \|b\|^2},$$

where the divergence is taken with respect to a. A theorem in [**5**] states that the Randers metric F has constant flag curvature K if and only if the following three conditions hold: σ is constant,

$$\mathcal{L}_{b^\sharp} a = \sigma(a - b \otimes b) - (b \otimes \theta + \theta \otimes b)$$

(where $\mathcal{L}_{b^\sharp} a = b^k \, \partial_{x^k} a_{ij} + a_{kj} \, \partial_{x^i} b^k + a_{ik} \, \partial_{x^j} b^k$ is a Lie derivative), and the Riemann tensor of a has the form

$$\begin{aligned} {}^{a}R_{hijk} \quad = \quad & \xi \, (a_{ij} \, a_{hk} - a_{ik} \, a_{hj}) \\ & - \tfrac{1}{4} \, a_{ij} \, \mathrm{curl}^t{}_h \, \mathrm{curl}_{tk} + \tfrac{1}{4} \, a_{ik} \, \mathrm{curl}^t{}_h \, \mathrm{curl}_{tj} \\ & + \tfrac{1}{4} \, a_{hj} \, \mathrm{curl}^t{}_i \, \mathrm{curl}_{tk} - \tfrac{1}{4} \, a_{hk} \, \mathrm{curl}^t{}_i \, \mathrm{curl}_{tj} \\ & - \tfrac{1}{4} \, \mathrm{curl}_{ij} \, \mathrm{curl}_{hk} + \tfrac{1}{4} \, \mathrm{curl}_{ik} \, \mathrm{curl}_{hj} + \tfrac{1}{2} \, \mathrm{curl}_{hi} \, \mathrm{curl}_{jk} \end{aligned}$$

with

$$\xi := (K - \tfrac{3}{16}\sigma^2) + (K + \tfrac{1}{16}\sigma^2) \, \|b\|^2 - \tfrac{1}{4} \, \theta^i \theta_i.$$

In these equations, all tensor indices are raised and lowered by a. For later purposes, let us refer to the above as the Basic equation and the Curvature equation, respectively.

The Basic equation *alone* is equivalent to the statement that the S-curvature (divided by F) has the constant value $\frac{1}{4}\sigma(n+1)$; see [**16**]. While the Basic equation only makes sense for Randers metrics, its characterization in terms of the S-curvature gives a well-defined criterion which can be imposed on Finsler metrics in general.

In the original statement [**5**] of the characterization above, there is a third equation that a and b must satisfy. As such, the said theorem is equivalent in content to one in [**22**]. Recent work shows that this third equation is derivable from the Basic and Curvature equations with σ constant. Hence it is omitted here. See [**6**] for more discussions.

3.2. Navigation description. According to Proposition 1.1, our strongly convex Randers metric F can be realized as the perturbation of a Riemannian metric h by a vector field W which satisfies $h(W,W) < 1$. Using this fact and Section 1.1.3, the tensors a and b that comprise F are expressible as

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i}{\lambda}\frac{W_j}{\lambda}, \qquad b_i = \frac{-W_i}{\lambda},$$

where $W_i := h_{ij}\,W^j$ and $\lambda := 1 - h(W,W) > 0$. For a^{ij} and b^i, see Section 1.2.

3.2.1. Navigation version of the Basic equation. The Basic equation in the stated characterization involves a, b, $\mathcal{L}_{b^\sharp}a$, and θ. Substituting the above formulae for a, b and computing the requisite partial derivatives in the remaining two tensors, we obtain a much simpler $\mathcal{L}_W$ equation:

$$\mathcal{L}_W h = -\sigma\, h.$$

The left-hand side can be rewritten in terms of the covariant derivative operator "$:$" associated to h, in which case the $\mathcal{L}_W$ equation reads

$$W_{i:j} + W_{j:i} = -\sigma\, h_{ij}\,.$$

From this follows the 'navigation description' of σ as $-2\,\mathrm{div}(W)/n$, where the divergence is taken with respect to h.

Conversely, using $h = \varepsilon(a - b\otimes b)$, $W = -b^\sharp/\varepsilon$, with $\varepsilon := 1 - \|b\|^2$ (see Section 1.3), it has been checked that the Basic equation results from the $\mathcal{L}_W$ equation. Hence the two are equivalent.

In the $\mathcal{L}_W$ equation,

"σ must vanish whenever h is not flat."

Indeed, let φ_t denote the time t flow of the vector field W. The $\mathcal{L}_W$ equation tells us that $\varphi_t^* h = e^{-\sigma t}h$. Since φ_t is a diffeomorphism, $e^{-\sigma t}h$ and h must be isometric; therefore, they have the same sectional curvatures. If h is not flat, this condition on sectional curvatures mandates that $e^{-\sigma t} = 1$, hence $\sigma = 0$. The above argument was pointed out to us by Robert Bryant.

3.2.2. Riemannian connections of a and h. To minimize some anticipated clutter, let us introduce the abbreviations

$$\mathcal{C}_{ij} := \partial_{x^j} W_i - \partial_{x^i} W_j = W_{i:j} - W_{j:i}, \qquad \mathcal{T}_j := W^i\,\mathcal{C}_{ij},$$

and agree to let the subscript 0 denote contraction of any index with y^i. For example, $\mathcal{T}_0 := \mathcal{T}_j\, y^j$. Indices on $\mathcal{C}$, $\mathcal{T}$ are to be manipulated by the Riemannian metric h only.

Let ${}^a\gamma^i{}_{jk}$ and ${}^aG^i := \frac{1}{2}{}^a\gamma^i{}_{00}$ be, respectively, the Christoffel symbols and geodesic spray coefficients of the Riemannian metric a. Likewise, let ${}^hG^i := \frac{1}{2}{}^h\gamma^i{}_{00}$ be the geodesic spray coefficients of h. (The factor of $\frac{1}{2}$ here is absent in some references such as [**4**].) A straight-forward computation, or an application of Rapcsák's identity [**26**], together with the $\mathcal{L}_W$ equation, shows [**6**] that

$$
{}^aG^i = {}^hG^i + \frac{y^i}{2\lambda}(\mathcal{T}_0 - \sigma\, W_0) - \mathcal{T}^i \left(\frac{h_{00}}{4\lambda} + \frac{W_0\, W_0}{2\lambda^2} \right) + \frac{\mathcal{C}^i{}_0\, W_0}{2\lambda},
$$

where $\lambda := 1 - h(W, W)$.

3.2.3. Navigation version of the Curvature equation. Abbreviate the above formula as ${}^aG^i = {}^hG^i + \zeta^i$. We now use it to relate the curvature tensor ${}^aR$ of a to the curvature tensor ${}^hR$ of h. To this end, consider the spray curvature [**12**] tensors ${}^aK^i{}_j = {}^aR_0{}^i{}_{j0}$ and ${}^hK^i{}_j = {}^hR_0{}^i{}_{j0}$. The Riemann tensor can be recovered from the spray curvature through ${}^aR_{hijk} = \frac{1}{3}\{({}^aK_{ij})_{y^k y^h} - ({}^aK_{ik})_{y^j y^h}\}$, where the up index on ${}^aK$ has been lowered by a. A similar formula holds for ${}^hR_{hijk}$ and ${}^hK_{ij}$, with the index on ${}^hK$ lowered by h. The advantage of working with the spray curvature is that it has less indices than the full Riemann tensor.

The Curvature equation of Section 3.1 can be recast into the form

$$
\begin{aligned}
{}^aK^i{}_j \;\; = \;\; & \xi\, (\alpha^2\, \delta^i{}_j - y^i\, {}^ay_j) \\
& + \tfrac{1}{4}\, \mathrm{curl}^s{}_0\, (\mathrm{curl}_s{}^i\, {}^ay_j + y^i\, \mathrm{curl}_{sj} - \mathrm{curl}_{s0}\, \delta^i{}_j) \\
& - \tfrac{1}{4}\, \alpha^2\, \mathrm{curl}^{si}\, \mathrm{curl}_{sj} - \tfrac{3}{4}\, \mathrm{curl}^i{}_0\, \mathrm{curl}_{j0},
\end{aligned}
$$

where ξ is as defined in Section 3.1 and ${}^ay_j := a_{jk}y^k$. Into (the left-hand side of) this, we substitute one version of the split covariantized Berwald formula (see [**6**, **28**] for expositions and references therein), which says that

$$
{}^aK^i{}_j = {}^hK^i{}_j + (2\,\zeta^i)_{:j} - (\zeta^i)_{y^s}(\zeta^s)_{y^j} - y^s(\zeta^i{}_{:s})_{y^j} + 2\,\zeta^s(\zeta^i)_{y^s y^j}.
$$

Here, the subscripts "${}_{y^k}$" mean ∂_{y^k}. This is followed by a tedious calculation, in which all quantities are rewritten in terms of the navigation variables h, W, and the $\mathcal{L}_W$ equation is used prodigiously. A formula for ${}^hK^i{}_j$ then results, from which we compute the Riemann tensor ${}^hR_{hijk}$.

The outcome of that calculation is remarkable. It says that given the $\mathcal{L}_W$ equation, the said Curvature equation is equivalent to the statement that h is a Riemannian metric of constant sectional curvature $K + \frac{1}{16}\sigma^2$. Namely,

$$^{h}R_{hijk} = (K + \tfrac{1}{16}\sigma^2)(h_{ij}\, h_{hk} - h_{ik}\, h_{hj}).$$

Conversely, it has been verified that the use of $h = \varepsilon(a - b \otimes b)$, $W = -b^\sharp/\varepsilon$, $\varepsilon := 1 - \|b\|^2$ (see Section 1.3), in conjunction with the $\mathcal{L}_W$ equation, converts the above formula of $^{h}R_{hijk}$ into the Curvature equation of Section 3.1. Thus, the navigation description we have derived is indeed equivalent to the characterization presented in Section 3.1.

3.3. Main geometric content.

Theorem 3.1. *A strongly convex Randers metric F has constant flag curvature K if and only if:*

- *F solves Zermelo's navigation problem on a Riemannian space (M, h) of constant sectional curvature $K + \frac{1}{16}\sigma^2$ for some constant σ, under the influence of a vector field ("wind") W.*
- *The wind W satisfies $h(W, W) < 1$, and is coupled to h and σ in such a way that $\mathcal{L}_W h = -\sigma\, h$, where $\mathcal{L}$ denotes Lie differentiation.*

For non-flat h, σ must vanish, in which case W must be a Killing vector field of h.

The last statement has already been observed in Section 3.2.1. Since the sectional curvature of h is $K + \frac{1}{16}\sigma^2$, that statement is equivalent to the following interplay between the constants σ and K:

$$\sigma(K + \tfrac{1}{16}\sigma^2) = 0.$$

This is sometimes known as a Matsumoto identity (see [**5**] and [**6**]).

Note that K, the flag curvature of F, is bounded above by the sectional curvature $K + \frac{1}{16}\sigma^2$ of h. This proves the phenomenon we noted at the end of Section 2.4. Since $\sigma\,(K + \frac{1}{16}\sigma^2) = 0$, we have the following trichotomy.

(+) For $K > 0$: The quantity $K + \frac{1}{16}\sigma^2$ is positive, hence $\sigma = 0$. Consequently, the sectional curvature of h must equal K, the flag curvature of F.

(0) For $K = 0$: The sectional curvature of h reduces to $\frac{1}{16}\sigma^2$. Matsumoto's identity then implies that $\sigma = 0$. So, h must be flat.

(−) For $K < 0$: There are two viable scenarios. The first is $\sigma = \pm 4\sqrt{|K|}$, in which case h is flat. For the second scenario, the quantity $K + \frac{1}{16}\sigma^2 \neq 0$; hence $\sigma = 0$ and h must have negative sectional curvature K.

4. Complete list of allowable vector fields

Our goal here is towards a classification of Randers metrics of constant flag curvature. By the navigation description, these metrics arise as perturbations of constant curvature Riemannian metrics h by vector fields W satisfying $W_{i:j} + W_{j:i} = -\sigma\, h_{ij}$. For each of the three standard Riemannian space forms (Euclidean, spherical and hyperbolic), we derive a formula for W.

4.1. Setting some notation with a basic lemma.

Lemma 4.1. *Let $P_i = P_i(x)$ be solutions of the following system:*

$$\frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i} = 0.$$

Then

$$P_i = Q_{ij}\, x^j + C_i,$$

where (C_i) is an arbitrary constant row vector and $Q = (Q_{ij})$ is an arbitrary constant skew-symmetric matrix ($Q_{ji} = -Q_{ij}$).

Proof. Using the defining differential equation three times, we have

$$\frac{\partial^2 P_i}{\partial x^k \partial x^j} = -\frac{\partial^2 P_j}{\partial x^k \partial x^i} = \frac{\partial^2 P_k}{\partial x^i \partial x^j} = -\frac{\partial^2 P_i}{\partial x^j \partial x^k}.$$

This shows that all second-order partial derivatives of P_i must vanish. Hence P_i must be linear; that is, it has the form $P_i = Q_{ij} x^j + C_i$, with constants Q_{ij} and C_i. Inserting this expression into the defining PDE shows that $Q_{ij} + Q_{ji} = 0$. q.e.d.

For the rest of the paper: "·" refers to the standard dot product on $\mathbb{R}^n$; indices on Q and C are raised and lowered by the Kronecker delta δ_{ij}; and $Qx + C$ means $(Q^i{}_j\, x^j + C^i)$. We regard (C_i) as a row vector and (C^i) as a column vector.

4.2. The Euclidean case. The first Riemannian space form we consider is the standard Euclidean metric. The admissible perturbing vector fields W are described in the following proposition.

Proposition 4.2. *Let $F = \alpha+\beta$ be a strongly convex Randers metric which results from perturbing the flat metric $h_{ij} = \delta_{ij}$ on $\mathbb{R}^n$ by a vector field $W = (W^i)$. Then F is of constant flag curvature K if and only if W has the form*

$$W^i(x) = -\tfrac{1}{2}\sigma\, x^i + Q^i{}_j\, x^j + C^i,$$

where $(Q^i{}_j)$ is a constant skew-symmetric matrix, (C^i) is a constant column vector, σ is a constant such that $\sigma^2 = -16K$, and

$$(Qx + C)\cdot(Qx + C) + \sigma x \cdot \left(\tfrac{1}{4}\sigma x - C\right) \ < \ 1.$$

Remark. Since $\sigma^2 = -16K$, we see that K must be $\leqslant 0$.

Proof. Being flat, h satisfies the curvature criterion of the navigation description (Theorem 3.1), with $K + \frac{1}{16}\sigma^2 = 0$. The rest of the proof studies the second criterion, which is the equation $\mathcal{L}_W h = -\sigma\, h$.

($\Leftarrow$) Suppose W, with its index lowered by $h_{ij} = \delta_{ij}$, is of the form

$$W_i = -\tfrac{1}{2}\sigma\, \delta_{ij}\, x^j + Q_{ij}\, x^j + C_i.$$

Keeping in mind that the covariant derivative "$_:$" associated with the Euclidean h is simply partial differentiation, together with the skew-symmetry of Q, we immediately obtain

$$(*) \qquad W_{i:j} + W_{j:i} = -\sigma\, \delta_{ij}.$$

Thus the $\mathcal{L}_W$ equation in the navigation description is satisfied, and F has constant flag curvature K by Theorem 3.1.

($\Rightarrow$) Conversely, suppose F has constant flag curvature K. By the navigation description, W must be a solution of $(*)$. Note that

$$W_i = -\tfrac{1}{2}\sigma\, \delta_{ij}\, x^j$$

is a particular solution. Adding to it the solutions of the homogeneous system

$$\frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^j} = 0$$

gives the general solution. According to Lemma 4.1, the latter have the form $P_i = Q_{ij}\, x^j + C_i$, where each C_i is constant and

(Q_{ij}) is a constant skew-symmetric matrix. Using $h^{ij} = \delta^{ij}$, we raise the index on W_i to effect the W^i as claimed.

The inequality satisfied by Q, C, and σ comes from the strong convexity requirement $|W| < 1$. q.e.d.

4.3. The spherical and hyperbolic cases. We now perturb standard models of Riemannian metrics with constant sectional curvature $\kappa \neq 0$. The list of allowable W is given in the following proposition.

Proposition 4.3. *Let $F = \alpha + \beta$ be a strongly convex Randers metric which results from perturbing the standard, complete, simply connected, n-dimensional Riemannian space (M, h) of constant sectional curvature $\kappa \neq 0$ by a vector field W. Then, F is of constant flag curvature K if and only if $K = \kappa$ and W is Killing, with the following description in terms of a constant vector (C^i) and a constant skew-symmetric matrix $(Q^i{}_j)$.*

(a) $K = \kappa > 0$. *Employ a projective coordinate system on the unit n-sphere, one which comes from parametrizing each hemisphere using the tangent space at the pole. Multiply the standard Riemannian metric by $\frac{1}{K}$ to effect constant sectional curvature K. The h-norm of any tangent vector $y \in T_x\mathbb{R}^n \simeq \mathbb{R}^n$ is given by*

$$|y| := \sqrt{h(y,y)} = \frac{1}{\sqrt{K}} \frac{\sqrt{(y \cdot y)(1 + x \cdot x) - (x \cdot y)^2}}{1 + x \cdot x}.$$

With respect to this coordinate system,

$$W^i(x) = Q^i{}_j\, x^j + C^i + (x \cdot C)\, x^i.$$

(b) $K = \kappa < 0$. *Let h be the Klein model of constant sectional curvature K on the unit ball $\mathbb{B}^n$, with the Cartesian coordinates of $\mathbb{R}^n$. The h-norm of any tangent vector $y \in T_x\mathbb{R}^n \simeq \mathbb{R}^n$ is given by*

$$|y| := \sqrt{h(y,y)} = \frac{1}{\sqrt{|K|}} \frac{\sqrt{(y \cdot y)(1 - x \cdot x) + (x \cdot y)^2}}{1 - x \cdot x}.$$

With respect to this coordinate system,

$$W^i(x) = Q^i{}_j\, x^j + C^i - (x \cdot C)\, x^i.$$

In each case, W is subject to the constraint

$$\frac{1}{1 + \psi(x \cdot x)} \{(Qx + C) \cdot (Qx + C) + \psi(x \cdot C)^2\} \ < \ |K| \text{ with } \psi := \frac{K}{|K|}.$$

Proof. Our Riemannian metric h has constant sectional curvature $\kappa \neq 0$. Therefore, it satisfies the curvature criterion of the navigation description (Theorem 3.1), with $K + \frac{\sigma^2}{16} = \kappa$. In particular, we have $K + \frac{\sigma^2}{16} \neq 0$. The Matsumoto identity (Section 3.3) then implies that σ must vanish. Consequently, $K = \kappa$.

According to our navigation description, perturbing the above h by a vector field W (with $|W| < 1$) generates a Randers metric of constant flag curvature K if and only if the equation $\mathcal{L}_W h = -\sigma\, h$ is satisfied. Since $\sigma = 0$ here, that equation reduces to the statement that W is a Killing vector field of h. The proof of this proposition, therefore, concerns the classification of solutions of the Killing field equation:

$$W_{i:j} + W_{j:i} = 0.$$

- To minimize notational clutter, let us introduce the abbreviations

  $$x_i := \delta_{ij}\, x^j, \qquad \rho := 1 + \psi(x \cdot x),$$

  where

  $$\psi := \frac{K}{|K|}.$$

  Then,

  $$h_{ij} = \frac{1}{|K|}\left(\frac{\delta_{ij}}{\rho} - \psi \frac{x_i x_j}{\rho^2}\right), \qquad h^{ij} = \rho\, |K|\, \{\delta^{ij} + \psi\, x^i x^j\}.$$

  The Christoffel symbols of h are given by

  $${}^h\gamma^k{}_{ij} = -\psi \frac{x_i\, \delta^k{}_j + x_j\, \delta^k{}_i}{\rho}.$$

  Hence

  $$W_{i:j} = \frac{\partial W_i}{\partial x^j} + \psi \frac{x_i W_j + x_j W_i}{\rho}.$$

  The Killing field equation now reads

  $$\frac{\partial W_i}{\partial x^j} + \frac{\partial W_j}{\partial x^i} + \frac{2\psi}{\rho}(x_i W_j + x_j W_i) = 0.$$

- To solve it, let us replace the dependent variables W_i by new ones that are named P_i, as follows:

  $$W_i = \frac{1}{\rho\, |K|} P_i\, .$$

(The division by $|K|$ effects a simplification later, when we use h^{ij} to raise the index on W_i.) Computations give:

$$\frac{\partial W_i}{\partial x^j} + \frac{\partial W_j}{\partial x^i} = \frac{1}{\rho\,|K|}\left(\frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i}\right) - \frac{2\psi}{\rho^2\,|K|}(x_i P_j + x_j P_i),$$

$$\frac{2\psi}{\rho}(x_i W_j + x_j W_i) = \frac{2\psi}{\rho^2\,|K|}(x_i P_j + x_j P_i).$$

This change of dependent variables transforms the above equation into

$$\frac{\partial P_i}{\partial x^j} + \frac{\partial P_j}{\partial x^i} = 0.$$

By Lemma 4.1, the solutions P_i have the form

$$P_i = Q_{ij}\,x^j + C_i,$$

where (Q_{ij}) is a constant skew-symmetric matrix, and the C_i are constants.

Thus the covariant form (that is, with index down) of the Killing field W is

$$W_i = \frac{Q_{ij}\,x^j + C_i}{\rho\,|K|}.$$

To obtain the contravariant form (namely, with index up) of W, we raise its index using $h^{ij} = \rho\,|K|\,\{\delta^{ij} + \psi\,x^i x^j\}$. The result reads:

$$W^i := h^{ij} W_j = Q^i{}_j\,x^j + C^i + \psi(x\cdot C)\,x^i,$$

where $Q^i{}_j := \delta^{is} Q_{sj}$ and $C^i := \delta^{is} C_s$.

Finally, the constraint on Q and C comes from $|W| < 1$, namely, the strong convexity of F. q.e.d.

4.4. Identifying the vector field W in examples. Note that in the case of flat h, both W_i and W^i are polynomials of degree 1 in the position variables x. For non-flat h, W_i is a rational function in x of degree -1, while W^i is a polynomial of degree 2 in x whenever $C \neq 0$.

We tabulate below the constant skew-symmetric matrix Q, the constant vector C, the value of the constant σ, and the constant flag curvature K, for all the examples of Section 2. To reduce clutter, let $0_{3\times 3}$ denote the 3-by-3 zero matrix, and

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Table 2.

Example	(Q_{ij})	(C_i)	σ	K
2.1.1	$\tau J \oplus 0$	(0,0,0)	0	1
2.1.2	$0 \oplus \sqrt{K-1}\,J$	$(-s\sqrt{K-1}, 0, 0)$	0	> 1
2.2.1	$0_{3\times 3}$	(p, q, r)	0	0
2.2.2	$J \oplus 0$	(0,0,0)	0	0
2.3.1	$0_{3\times 3}$	(0,0,0)	-2τ	$-\frac{1}{4}\tau^2$
2.3.2	$J \oplus 0$	(0,0,0)	0	-1

5. Classification of strongly convex Randers metrics with constant flag curvature

5.1. The classification theorem. We now combine the navigation description (see Section 3.3) and the work of Section 4 to classify Randers metrics of constant flag curvature. Before stating the theorem, we recall that:

- the skew-symmetric matrix $Q = (Q^i{}_j)$ and the vector $C = (C^i)$ are constant;
- Qx denotes $(Q^i{}_j\, x^j)$, and $x := (x^i)$;
- all indices on Q, C, x are manipulated by the Kronecker deltas δ_{ij} and δ^{ij};
- "$\cdot$" is the standard Euclidean dot product.

Theorem 5.1 (Classification)**.** *Let $F(x,y) = \sqrt{a_{ij}(x)\, y^i\, y^j} + b_i(x)\, y^i$ be a strongly convex Randers metric on a smooth manifold M of dimension $n \geqslant 2$. Then, F is of constant flag curvature K if and only if the following conditions are satisfied.*

(1) The Riemannian metric a and 1-form b have the representation

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i}{\lambda}\frac{W_j}{\lambda}, \qquad b_i = \frac{-W_i}{\lambda},$$

where h is a Riemannian metric of constant sectional curvature and $W = W^i \partial_{x^i}$ is an infinitesimal homothety ($\mathcal{L}_W h = -\sigma h$) of h, both globally defined on M. Here, $W_i := h_{ij} W^j$ and $\lambda := 1 - h(W,W) > 0$.

(2) Up to local isometry, the constant curvature Riemannian metric h and the vector field W must belong to one of the following four families.

(+) *When* $K > 0$**:** *h is $\frac{1}{K}$ times the standard metric on the unit n-sphere S^n in projective coordinates, and $W = Qx + C + (x \cdot C)\, x$ is Killing, with*

$$\frac{1}{1 + (x \cdot x)} \{(Qx + C) \cdot (Qx + C) + (x \cdot C)^2\} < K.$$

In those coordinates, the quadratic form of h, evaluated on $y \in T_x S^n$, reads

$$h(y, y) = \frac{1}{K} \left\{ \frac{(y \cdot y)(1 + x \cdot x) - (x \cdot y)^2}{(1 + x \cdot x)^2} \right\}.$$

(0) *When* $K = 0$**:** *h is the Euclidean metric δ_{ij} on $\mathbb{R}^n$ and $W = Qx+C$ is Killing, with*

$$(Qx + C) \cdot (Qx + C) < 1.$$

(−) *When* $K < 0$**:**

$(-)_e$ *either h is the Euclidean metric δ_{ij} on $\mathbb{R}^n$, and the infinitesimal homothety $W = -\frac{1}{2}\sigma x + Qx + C$ satisfies*

$$(Qx + C) \cdot (Qx + C) + \sigma x \cdot (\tfrac{1}{4}\sigma x - C) < 1$$

with $\sigma = \pm 4\sqrt{|K|}$;

$(-)_k$ *or h is the Klein model of sectional curvature K on the unit ball $\mathbb{B}^n$ (of $\mathbb{R}^n$) in projective coordinates, and the Killing field $W = Qx + C - (x \cdot C)x$ satisfies*

$$\frac{1}{1 - (x \cdot x)} \{(Qx + C) \cdot (Qx + C) - (x \cdot C)^2\} < |K|.$$

In those coordinates, the quadratic form of h, evaluated on $y \in T_x\mathbb{B}^n$, reads

$$h(y, y) = \frac{1}{|K|} \left\{ \frac{(y \cdot y)(1 - x \cdot x) + (x \cdot y)^2}{(1 - x \cdot x)^2} \right\}.$$

Proof.

- By Proposition 1.1, every strongly convex Randers metric has the representation, stipulated in (1), in terms of the Zermelo navigation variables (h, W).
- Theorem 3.1 tells us that h must be a Riemannian metric of constant sectional curvature. The discussion after the statement of Theorem 3.1 reduces the landscape to only four families, in keeping with (2). They are as follows.

(+) For $K > 0$: h must have sectional curvature K and W is Killing.
(0) For $K = 0$: h must be flat and W is Killing.
(−) For $K < 0$: there are two scenarios,
$(-)_e$ either h is flat, $\sigma = \pm 4\sqrt{|K|}$, and $\mathcal{L}_W h = -\sigma\, h$ (in which case W turns out to be $-\frac{1}{2}\sigma$ times the radial vector $x = (x^i)$, plus an arbitrary Killing field);
$(-)_k$ or h has sectional curvature K and W is Killing.

- Up to (Riemannian) isometry, there are only three standard models for Riemannian metrics h of constant sectional curvature K. They are: $\frac{1}{K}$ times the standard metric on the unit n-sphere, Euclidean $\mathbb{R}^n$, and the Klein metric with sectional curvature K on the unit ball in $\mathbb{R}^n$. By Lemma 1.2 (see also Sections 0.1 and 6.1), when classifying F up to Finslerian isometry, it suffices to list the allowable vector fields W for each of the three specific models. For the families (+) and $(-)_k$, this has been done by Proposition 4.3. Families (0) and $(-)_e$ are handled by Proposition 4.2, with $\sigma = 0$ and $\sigma = \pm 4\sqrt{|K|}$, respectively.
- In each of the four families, the constraint that must be satisfied by Q, C and x is equivalent to $|W| < 1$, which characterizes the strong convexity of the Randers metric in question. The table in Section 4.4 shows that this constraint admits non-trivial solutions for all four families. In Sections 6.2–6.4 we enumerate, with the help of normal forms, all the Q, C for which there exists a neighborhood D of x on which $|W| < 1$.

q.e.d.

5.2. Globally defined solutions on the standard S^n. We see in the previous section that all strongly convex Randers metrics of constant flag curvature $K > 0$ arise locally as solutions to Zermelo's problem of navigation on the unit sphere S^n, under the influence of a Killing field (an infinitesimal isometry) of $\frac{1}{K}$ times the standard metric on S^n. Let us show that each strongly convex solution on any *closed* hemisphere has a unique smooth extension to a globally defined strongly convex solution on S^n. There is no restriction on the dimension n.

5.2.1. An extension. Without loss of generality, let us assume that the hemisphere in question is the *closed* eastern hemisphere. Parametrize the eastern ($s = +1$) and western ($s = -1$) open hemispheres, as submanifolds of the ambient $\mathbb{R}^{n+1}$, by the maps

$$x \mapsto \psi^{\pm}(x) := \frac{1}{\sqrt{1 + x \cdot x}}(s, x) \text{ with } x \in \mathbb{R}^n.$$

Geometrically, the tangent space at the east pole (resp. west pole) is identified with $\mathbb{R}^n$. Each point q on an open hemisphere lies on a unique ray which emanates from the center of the sphere. This ray intersects the copy of $\mathbb{R}^n$ tangent to the pole, at a point x. The above parametrization expresses q in terms of x.

According to Theorem 5.1, on the *open* eastern hemisphere, the given Randers metric has navigation data (h, W), where h is $\frac{1}{K}$ times the standard Riemannian metric of S^n, and $W(x) = Qx + C + (x \cdot C)x$. We find that it is easier to visualize $W(x)$ by considering its image under ψ^+_*. Motivated by a Lie-theoretic reason that will be pointed out in Section 6.2, we convert the image point $p := \psi^+(x)$ into a position *row* vector p^t of $\mathbb{R}^{n+1}$. A computation gives

$$[\psi^+_* W(x)]^t = p^t\, \Omega\,,$$

where

$$\Omega = \begin{pmatrix} 0 & C^t \\ -C & -Q \end{pmatrix}$$

is an $(n+1) \times (n+1)$ skew-symmetric *constant* matrix, C is a *constant* column vector in $\mathbb{R}^n$, and ^{t} means transpose. The continuity of W on the closed hemisphere implies that its value at any point p on the equator is also the matrix product $p^t\, \Omega$.

We extend W to the open western hemisphere by insisting that the equation

$$[\psi^-_* W(x)]^t = [\psi^-(x)]^t\, \Omega \qquad (\text{with the } \textit{above}\ \Omega)$$

holds. The result is $W(x) = Qx + sC + (x \cdot sC)x$ with $s = -1$.

It is an artifact of local coordinates that W is constructed from the data (Q, C) on the eastern hemisphere, but from $(Q, -C)$ on the western hemisphere. The actual Killing field on the embedded unit sphere in $\mathbb{R}^{n+1}$ has the value $p^t\, \Omega$ at any point p, including the equator. Since the matrix Ω is constant, there is no question that the constructed W is globally defined and smooth.

5.2.2. Uniqueness of the extension. Let W be any global extension of the given Killing field. The isometries of (S^n, h) consist of rigid rotations, implemented by constant $(n+1)\times(n+1)$ orthogonal matrices right multiplying the row vectors of $\mathbb{R}^{n+1}$. Since W is an infinitesimal isometry, it is the initial tangent to a curve of isometries. Thus, it also corresponds to a *constant* matrix which right multiplies *all* row vectors. For points p of the eastern hemisphere, we have determined the matrix in question to be the above Ω. Constancy dictates that the same Ω must be used for the western hemisphere as well. This proves that every global extension agrees with the one we presented. In particular, any global W with data (Q, C) on some hemisphere must have data $(Q, -C)$ on the complement.

5.2.3. Strong convexity. The strong convexity criterion reads $|W| < 1$. On the two *open* hemispheres, Proposition 4.3 helps us deduce that

$$|W(x)|^2 = \frac{1}{K\{\, 1 + (x \cdot x)\}} \{(Qx + sC) \cdot (Qx + sC) + (x \cdot sC)^2\}.$$

Using this formula, it is straight-forward to check that $|W(x)|^2$ is equal to $(p^t\,\Omega) \cdot (p^t\,\Omega)$, where $p = \psi^{\pm}(x)$. Before the extension, our Randers metric is strongly convex on the closed eastern hemisphere. In particular, $(p^t\,\Omega) \cdot (p^t\,\Omega) < 1$ for all points p of the open eastern hemisphere. Replacing p by $-p$ generates all the points of the open western hemisphere, but does not alter $(p^t\,\Omega) \cdot (p^t\,\Omega)$. Therefore, the extended metric is also strongly convex on the open western hemisphere, and hence on all of S^n.

5.2.4. Discussion. The examples of Sections 2.1.1 and 2.1.2 determine globally defined Randers metrics of constant positive flag curvature on S^3. The first example illustrates the necessity of assuming strong convexity on a closed hemisphere. Had we permitted $\tau = 1$, the norm of W would have been less than 1 on the open (eastern and western) hemispheres; but strong convexity would fail at the points $(0, p_1, p_2, p_3)$ on the equator.

5.3. Globally defined solutions on Euclidean $\mathbb{R}^n$. Because Euclidean $\mathbb{R}^n$ is covered by a single coordinate chart, globality is relatively easy to address. According to scenarios (0) and $(-)_e$ of Theorem 5.1, navigation on $\mathbb{R}^n$ under an infinitesimal homothety W produces a strongly convex Randers metric of constant flag curvature $K \leqslant 0$ wherever $|W| < 1$.

In particular, the Randers metric is defined globally on $\mathbb{R}^n$ if and only if

$$|W(x)|^2 = (Qx + C) \cdot (Qx + C) + \sigma x \cdot (\tfrac{1}{4}\sigma x - C) < 1 \quad \text{for all } x \in \mathbb{R}^n.$$

Here, σ is zero if $K = 0$, and has the values $\pm 4\sqrt{|K|}$ if $K < 0$. Since $|W(x)|^2$ is polynomial in x, the displayed criterion is possible if and only if both σ and Q vanish, in which case $W = C$, with $C \cdot C < 1$. The resulting Randers metric is Minkowski.

This conclusion is consistent with Section 2.2, where the only globally defined example is that of Section 2.2.1.

5.4. Globally defined solutions on the Klein model. It remains to discuss global solutions to Zermelo's problem of navigation on the Klein model with constant sectional curvature $K < 0$, under the influence of a Killing vector field W. Theorem 5.1 says that the resulting Randers metric has constant negative flag curvature K. Strong convexity of the Randers metric is equivalent to $|W| < 1$. In this subsection, we will show that requiring strong convexity on the entire open unit ball forces $W = 0$, whence the negatively curved Randers metric is simply the Klein model itself.

Suppose $|W| < 1$ holds on the entire open unit ball. It is implicit in Proposition 4.3 that

$$|W(x)|^2 = \frac{(Qx + C) \cdot (Qx + C) - (x \cdot C)^2}{|K|\,(1 - x \cdot x)}\,.$$

Note that $|K|(1 - x \cdot x) > 0$ because K is negative and x is confined to the unit ball. Multiplying the inequality $0 \leqslant |W|^2 < 1$ by this positive denominator yields

$$0 \leqslant (Qx + C) \cdot (Qx + C) - (x \cdot C)^2 < |K|\,(1 - x \cdot x).$$

Letting $x \cdot x \to 1$ leads to $(Qx + C) \cdot (Qx + C) - (x \cdot C)^2 = 0$ for all unit x. In particular, $(Qx + C) \cdot (Qx + C) = (-Qx + C) \cdot (-Qx + C)$, which is equivalent to $Qx \cdot C = 0$. The equality above then simplifies to $Qx \cdot Qx + C \cdot C - (x \cdot C)^2 = 0$, again for all unit x.

Since we are in dimension at least two, there exists a unit x_0 such that $x_0 \cdot C = 0$. The ensuing equation $Qx_0 \cdot Qx_0 + C \cdot C = 0$ tells us that C must have been zero to begin with. This reduces our original equality to $Qx = 0$ for all unit x, implying that $Q = 0$. Thus, W is identically zero, and our assertion follows.

6. The moduli space $\mathcal{M}_K$

6.1. The strategy. Theorem 3.1 characterizes the navigation data (h, W) of strongly convex Randers metrics with constant flag curvature K. It says that h must be a Riemannian metric with constant sectional curvature $K + \frac{1}{16}\sigma^2$, and W must be an infinitesimal homothety of h. Also, we observed that σ can be non-zero only when h is flat.

Consider any Randers metric (M, F) of constant flag curvature K, with navigation data (h, W). There exists a Riemannian local isometry φ between (M, h) and one of the three standard Riemannian space forms:

- the sphere (S^n, h_+) of constant curvature K when $K > 0$;
- Euclidean space $(\mathbb{R}^n, h_0)$ when $K = 0$, or when $K < 0$ and $\sigma = \pm 4\sqrt{|K|}$;
- the Klein model $(\mathbb{B}^n, h_-)$ of constant curvature K when $K < 0$ and $\sigma = 0$.

Lemma 1.2 assures us that φ lifts to a local Finslerian isometry between (M, F) and the Randers metric on $S^n/\mathbb{R}^n/\mathbb{B}^n$ generated by the navigation data $(h_+/h_0/h_-\,,\ \varphi_* W)$. Thus, there is no loss of generality by working with the latter picture, which is more concrete.

Given each Riemannian space form, which for notational simplicity we again denote by h, Theorem 5.1 lists its infinitesimal homotheties W, with implicit *maximal* domains D on which $|W| := \sqrt{h(W, W)} < 1$. That list contains a good amount of redundancy, because it includes isometric (in the Finslerian sense) Randers metrics. The redundancy comes from the symmetry/isometry group G of h, consisting of diffeomorphisms ϕ that leave h invariant. Since $\phi^* h = h$, the action of the Lie group G on the navigation data is $(h, W, D) \mapsto (h, \phi_* W, \phi D)$. According to Lemma 1.2, the standard "models" (h, W, D) and $(h, \phi_* W, \phi D)$ generate isometric Randers metrics on D and ϕD. That is, all navigation data which lie on the same G-orbit correspond to mutually isometric Randers metrics. The redundancy we described can therefore be eliminated by collapsing each G-orbit to a point. These "points" constitute the elements of our moduli space $\mathcal{M}_K$ for strongly convex Randers metrics with constant flag curvature K. It is the goal of this section to parametrize $\mathcal{M}_K$ and thereby count its dimension.

To this end, we begin with a standard Riemannian space form h ($= h_+$, or h_0, or h_-). Identify the isometry group G of h with a matrix subgroup of $GL_{n+1}\mathbb{R}$. The infinitesimal homotheties W of h comprise a representation of a matrix Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_{n+1}\mathbb{R}$. The push-forward action $W \mapsto \phi_* W := \phi_* \circ W \circ \phi^{-1}$ on the manifold then corresponds to the "adjoint action"

$$\Omega \;\mapsto\; Ad_g \Omega := g\,\Omega\,g^{-1}$$

of G on $\mathfrak{h}$. Here:

(1) $g \in GL_{n+1}\mathbb{R}$ is the matrix which corresponds to the isometry map ϕ, and $\Omega \in \mathfrak{h}$ is the matrix analog of the infinitesimal homothety W (which is a vector field on the manifold).
(2) $Ad : \mathfrak{h} \to \mathfrak{h}$ is well defined because the equation $\mathcal{L}_W h = -\sigma h$, being tensorial, becomes $\mathcal{L}_{\phi_* W} h = -\sigma h$ under the action of the isometry map ϕ. Thus, $\phi_* W$ is an infinitesimal homothety of h whenever W is, and the value of σ is invariant under isometries.
(3) According to Theorem 3.1, when h is not flat, its infinitesimal homotheties are simply its Killing vector fields. In that case, $\mathfrak{h}$ equals the Lie algebra $\mathfrak{g}$ of G, and Ad is the standard adjoint action of a Lie group on its Lie algebra.

The adjoint action Ad described above partitions $\mathfrak{h}$ into orbits. Each orbit corresponds to a distinct isometry class of Randers metrics with constant flag curvature K. For each orbit, matrix theory singles out a privileged representative $\tilde{\Omega}$, to be referred to as a normal form. These normal forms provide a concrete parametrization of the points in the moduli space $\mathcal{M}_K$, and the number of parameters constitutes its dimension. The linear algebra behind the construction of $\mathcal{M}_K$ depends on the sign of K. Here is an overview.

- For $K > 0$, $h = h_+$ is $\frac{1}{K}$ times the standard metric on the unit n-sphere. The orbits are those which result from the adjoint action of the orthogonal group $O(n+1)$ on its Lie algebra $\mathfrak{o}(n+1)$.
- For $K = 0$, we have $h = h_0$, the standard flat metric on $\mathbb{R}^n$. The orbits come from the adjoint action of the Euclidean group $E(n)$ on its Lie algebra $\mathfrak{e}(n)$. Here, $E(n)$ is comprised of $O(n)$ and the additive group $\mathbb{R}^n$ of translations.
- For $K < 0$, the orbits consist of two camps. (i) $h = h_-$ is the Klein model, and the Ad orbits arise from the orthochronous subgroup of the Lorentz group $O(1, n)$, acting on the Lie algebra $\mathfrak{o}(1, n)$. (ii) $h = h_0$ is the standard Euclidean metric, and the Ad orbits are those of $E(n)$ acting on a matrix description of the infinitesimal homotheties, with $\sigma = \pm 4\sqrt{|K|}$.

The Lie theory necessary for determining the normal form $\tilde{\Omega}$ is relegated to the Appendix. The material there will be called upon frequently in the following three subsections as we enumerate the elements of $\mathcal{M}_K$.

6.2. The n-sphere. The isometry group G of (S^n, h_+) is $O(n+1)$, whose elements are orthogonal matrices which implement rigid rotations by right multiplying the row vectors of $\mathbb{R}^{n+1}$. As explained in Section 5.2, each Killing vector field W of (S^n, h_+) also corresponds to a constant matrix which right multiplies those row vectors, and we have identified that skew-symmetric $(n+1)\times(n+1)$ matrix to be

$$\Omega := \begin{pmatrix} 0 & C^t \\ -C & -Q \end{pmatrix},$$

an element of the Lie algebra $\mathfrak{o}(n+1)$. This correspondence between the Killing fields of (S^n, h_+) and $\mathfrak{o}(n+1)$ is a Lie algebra isomorphism. (Incidentally, if we had let the group $O(n+1)$ act on column vectors instead, then the matrix $-\Omega$ would correspond to W, while the *negative* of the commutator $[-\Omega_1, -\Omega_2]$ would represent the Lie bracket $[W_1, W_2]$, rendering the correspondence a Lie algebra anti-isomorphism.)

Applying Section 9.2 (with $\ell := n+1$) to Ω, we see that there exists a $g \in O(n+1)$ so that $\tilde{\Omega} = g\Omega g^{-1}$ is in normal form. Explicitly:

when n is even, $\tilde{\Omega} = a_1 J \oplus \cdots \oplus a_m J \oplus 0$ with $m = n/2$,
when n is odd, $\tilde{\Omega} = a_1 J \oplus \cdots \oplus a_m J$ with $m = (n+1)/2$.

Here, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 0$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrix $\tilde{\Omega}$ represents the Killing field $\tilde{W} = \phi_* W$, where ϕ is the map which corresponds to the orthogonal matrix g (Section 6.1). According to Theorem 5.1, $\tilde{W}$ has the form $\tilde{Q}x + \tilde{C} + (x \cdot \tilde{C})x$ with respect to the projective coordinates x which parametrize the eastern hemisphere. Comparing the matrix analog

$$\begin{pmatrix} 0 & \tilde{C}^t \\ -\tilde{C} & -\tilde{Q} \end{pmatrix}$$

of $\tilde{W}$ with $\tilde{\Omega}$, we conclude that $\tilde{C}^t = (a_1, 0, \ldots, 0)$ and

for n even, $-\tilde{Q} = 0 \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0$, with $m = n/2$,
for n odd, $-\tilde{Q} = 0 \oplus a_2 J \oplus \cdots \oplus a_m J$ with $m = (n+1)/2$.

The Randers metric which solves Zermelo's problem of navigation on (S^n, h_+) under the influence of $\tilde{W}$ must satisfy the strong convexity criterion $|\tilde{W}| < 1$. In terms of the data $(\tilde{Q}, \tilde{C})$ for $\tilde{W}$, inequality (2,+) of Theorem 5.1 expresses this criterion as follows.
For n even:

$$a_1^2(1 + x_1^2) + a_2^2(x_2^2 + x_3^2) + \cdots + a_m^2(x_{n-2}^2 + x_{n-1}^2) < K(1 + x \cdot x).$$

For n odd:

$$a_1^2(1 + x_1^2) + a_2^2(x_2^2 + x_3^2) + \cdots + a_m^2(x_{n-1}^2 + x_n^2) \quad < K(1 + x \cdot x).$$

We wish to demarcate those a_i which allow the above inequalities to hold on some open subset of S^n.

6.2.1. Locally defined metrics when n is even. Consider the point $x = (0, \ldots, 0, x_n)$. Here, the condition $|\tilde{W}(x)| < 1$ simplifies to $a_1^2 < K(1 + x_n^2)$, which can be made to hold for arbitrary, but fixed, a_1 by choosing $|x_n|$ large enough. Once we have $|\tilde{W}(x)| < 1$, the continuity of $\tilde{W}$ effects $|\tilde{W}| < 1$ on a neighborhood about this x. Thus, for even n, the moduli space is parametrized by

$$a_1 \geqslant \ldots \geqslant a_m \geqslant 0$$

with no upper bound on any a_i.

6.2.2. Locally defined metrics when n is odd. Suppose $|W| < 1$ holds at some point x. Then $0 \leqslant a_m \leqslant a_i$ implies that

$$\begin{aligned} a_m^2(1 + x \cdot x) &\leqslant a_1^2(1 + x_1^2) + a_2^2(x_2^2 + x_3^2) + \cdots + a_m^2(x_{n-1}^2 + x_n^2) \\ &< K(1 + x \cdot x). \end{aligned}$$

In particular, we obtain the necessary condition $a_m < \sqrt{K}$. Conversely, given $a_m < \sqrt{K}$, let us consider a point x of the form $(0, \ldots, 0, x_n)$. At this x, the desired condition $|\tilde{W}(x)| < 1$ simplifies and can be rearranged to read $a_1^2 < K + (K - a_m^2)x_n^2$. Since $a_m^2 < K$, the inequality can be made to hold by choosing $|x_n|$ large enough. Continuity then extends $|\tilde{W}| < 1$ from this x to a neighborhood containing it. Therefore, the isometry classes of locally defined Randers metrics on the odd dimensional spheres are parametrized by

$$a_1 \geqslant \ldots \geqslant a_m \geqslant 0 \quad \text{with} \quad a_m < \sqrt{K}.$$

6.2.3. Globally defined metrics. Here, the criterion $|\tilde{W}(x)| < 1$ must hold on the entire sphere. In particular, it must hold for all $x \in \mathbb{R}^n$ parametrizing the open eastern hemisphere. Setting $x = 0$ in the inequalities immediately before Section 6.2.1 gives $a_1 < \sqrt{K}$.

Conversely, if $a_1 < \sqrt{K}$, then those inequalities are satisfied for all x because $a_1 \geqslant a_i \geqslant 0$. Hence the constraint $a_1 < \sqrt{K}$ is both necessary and sufficient for strong convexity on the open eastern hemisphere. By virtue of Section 5.2.3, the same bound on a_1 effects $|\tilde{W}| < 1$ on the open western hemisphere. Thus, strong convexity holds on the open hemispheres if and only if the condition $a_1 < \sqrt{K}$ is met.

It turns out that $a_1 < \sqrt{K}$ ensures strong convexity on the equator as well. To see this, let u be any unit vector in the copy of $\mathbb{R}^n$ tangent to the poles. Our parametrization (see Section 5.2.1) of the open hemispheres says that $\lim_{t\to\infty} tu$ corresponds asymptotically to some point p on the equator. In fact, $p = \lim_{t\to\infty}(1+tu\cdot tu)^{-1/2}(s,tu) = (0,u)$. Calculating with the norm $|y|^2 := h(y,y)$ given in part (a) of Proposition 4.3, we find that

$$|\tilde{W}(p)|^2 \;=\; \lim_{t\to\infty} |\tilde{W}(tu)|^2 = \frac{1}{K}\left\{ (\, u\cdot s\tilde{C}\,)^2 + |\,\tilde{Q}u\,|^2 \right\},$$

which is independent of $s = \pm 1$. A direct computation, using the fact that a_1 dominates all other a_i, and $u \cdot u = 1$, yields the strong convexity criterion $|\tilde{W}(p)|^2 \leqslant (a_1)^2/K < 1$.

Thus, the moduli space for the isometry classes of globally defined constant flag curvature $K > 0$ Randers metrics on S^n is given by the polytope

$$\sqrt{K} > a_1 \geqslant \cdots \geqslant a_m \geqslant 0.$$

6.2.4. Global versus local. For the locally defined metrics, the upper bound $a_1 < \sqrt{K}$ is not necessary because the strong convexity criterion $|\tilde{W}| < 1$ only has to hold on some open subset of S^n. However, when n is odd, all local solutions have to satisfy $a_m < \sqrt{K}$.

The metric of Section 2.1.1 illustrates these nuances well. The table in Section 4.4 tells us that $C^t = (0,0,0)$ and $Q = \tau J \oplus 0$. Using the data (Q,C), construct Ω as in Section 6.2. Almost by inspection, the normal form is $\tilde{\Omega} = \tau J \oplus 0J$, thus $a_1 = \tau$ and $a_m \equiv a_2 = 0$. Since K here is 1, the theory assures us that a locally defined strongly convex solution exists for any τ, while strongly convex global solutions are characterized by $\tau < 1$.

Indeed, Section 5.2.3 tells us that $\tilde{W}(p) = p^t\,\tilde{\Omega}$, and $|\tilde{W}(p)|^2$ is equal to $(p^t\,\tilde{\Omega})\cdot(p^t\,\tilde{\Omega}) = \tau^2(p_0^2 + p_1^2)$, where $p^t = (p_0,p_1,p_2,p_3)$ gives the coordinates of an *arbitrary* point on the embedded S^3 in $\mathbb{R}^4$. So $|\tilde{W}| < 1$ globally, as long as $\tau < 1$. On the other hand, if $\tau \geqslant 1$, then $|\tilde{W}(p)| < 1$ holds only at those points p on S^3 where $p_0^2 + p_1^2 < 1/\tau^2$.

6.2.5. The moduli space for $K > 0$.

Proposition 6.1. *The moduli space $\mathcal{M}_K$ for n-dimensional strongly convex Randers metrics of constant flag curvature $K > 0$ is parametrized by $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ as follows.*

- *When n is even, $m = n/2$ and the parameter space is given by*
$$a_1 \geqslant \cdots \geqslant a_m \geqslant 0.$$
- *When n is odd, $m = (n+1)/2$ and the parameter space is given by*
$$a_1 \geqslant \cdots \geqslant a_m \geqslant 0, \quad \textit{with } \sqrt{K} > a_m.$$
- *The globally defined metrics on S^n are parametrized by the polytope*
$$\sqrt{K} > a_1 \geqslant \cdots \geqslant a_m \geqslant 0.$$

6.3. Euclidean space. The isometry group of $(\mathbb{R}^n, h_0)$ consists of rotations, reflections, and translations; it is the Euclidean group $E(n)$. Though the action of $E(n)$ on $\mathbb{R}^n$ is affine, it can be implemented by matrix multiplication. To this end, we first represent elements ϕ of $E(n)$ by matrices $g \in GL_{n+1}\mathbb{R}$ of the form
$$g = \begin{pmatrix} A & 0 \\ b^t & 1 \end{pmatrix},$$
where
$$A \in O(n) \text{ and } b \in \mathbb{R}^n.$$
Next, we embed Euclidean n-space into $\mathbb{R}^{n+1}$ by assigning to each point x the column position vector $\psi(x) = \binom{x}{1} =: p$. The matrix action we have in mind is then
$$p^t \mapsto p^t g = (x^t A + b^t, 1).$$
Here, b^t, the input p^t, and the output $p^t g$ are all row vectors.

The image of an infinitesimal homothety $W = -\frac{1}{2}\sigma x + Qx + C$ under the described representation is $[\psi_* W(x)]^t = p^t\,\Omega$, where
$$\Omega := \begin{pmatrix} -\frac{1}{2}\sigma I_n - Q & 0 \\ C^t & 0 \end{pmatrix} \quad \text{and} \quad C^t \text{ is a row vector.}$$
Such matrices, with $\sigma \in \mathbb{R}$, $C \in \mathbb{R}^n$ and $Q \in \mathfrak{o}(n)$, form a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_{n+1}$. The correspondence between the infinitesimal homotheties W of $(\mathbb{R}^n, h_0)$ and the subalgebra $\mathfrak{h}$ is a Lie algebra isomorphism. When $\sigma = 0$, $\mathfrak{h}$ is the Lie algebra $\mathfrak{e}(n)$ of $E(n)$.

The vector field $\tilde{W} = -\frac{1}{2}\tilde{\sigma}x + \tilde{Q}x + \tilde{C}$ is the push forward of W under an isometry $\phi \in G$ if and only if its matrix representative $\tilde{\Omega}$ is given by $g\Omega g^{-1}$. Since

$$g^{-1} = \begin{pmatrix} A^t & 0 \\ -b^t A^t & 1 \end{pmatrix},$$

we have

$$\begin{pmatrix} -\frac{1}{2}\tilde{\sigma} I_n - \tilde{Q} & 0 \\ \tilde{C}^t & 0 \end{pmatrix} = \tilde{\Omega} = g\Omega g^{-1} = \begin{pmatrix} -\frac{1}{2}\sigma I_n - AQA^t & 0 \\ [AW(b)]^t & 0 \end{pmatrix},$$

where $W(b) = -\frac{1}{2}\sigma b + Qb + C$. Thus, $\tilde{\sigma} = \sigma$, $\tilde{Q} = AQA^t$, and $\tilde{C} = AW(b)$; in particular, the value of σ remains unchanged under any isometry, a general fact we pointed out in Section 6.1. Our objective is to find A and b, equivalently $g \in E(n)$, so that $\tilde{\Omega}$ takes on a simplest form.

6.3.1. The case of $\sigma = 0$ and the moduli space for $K = 0$. The Randers metrics of constant flag curvature zero arise as perturbation of the Euclidean metric under an infinitesimal isometry. This corresponds to the $\sigma = 0$ case in the above discussion.

To conserve space, we abbreviate group elements $g \in E(n)$ as $\{A, b\}$, Lie algebra elements $\Omega \in \mathfrak{e}(n)$ as $\{-Q, C\}$, and write column vectors horizontally.

(1) By Section 9.2, we can find an $R \in O(n)$ which puts $-Q$ into the normal form $-\tilde{Q} = \rho_1 J \oplus \cdots \oplus \rho_h J \oplus 0_{n-2h}$, with $\rho_1 \geqslant \cdots \geqslant \rho_h > 0$. Thus, $g_1 := \{R, 0\}$ conjugates Ω into $\tilde{\Omega}_1 := \{-\tilde{Q}, RC\}$.

(2) Choose $r \in O(n-2h)$ to transform the last $n-2h$ components of RC into $(0, \ldots, 0, \xi \geqslant 0)$, without affecting its first $2h$ components $D := (D_1, \ldots, D_h)$, listed pairwise for convenience as $D_i = [C_{2i-1}, C_{2i}]$. The corresponding group element $g_2 := \{I_{2h} \oplus r, 0\}$ conjugates $\tilde{\Omega}_1$ into $\tilde{\Omega}_2 := \{-\tilde{Q}, (D, 0, \ldots, 0, \xi)\}$.

(3) Pick $b := (\frac{-JD_1}{\rho_1}, \ldots, \frac{-JD_h}{\rho_h}, 0, \ldots, 0)$ and observe that we have $\tilde{Q}b = (-D, 0, \ldots, 0)$. Then $g_3 := \{I_n, b\}$ conjugates $\tilde{\Omega}_2$ into the element $\tilde{\Omega}_3 := \{-\tilde{Q}, (0, \ldots, 0, \xi)\}$.

In short, using $g := g_3 g_2 g_1 \in E(n)$, letting $0_{p,q}$ denote the p-by-q zero matrix, and abbreviating $0_{p,p}$ as 0_p, we get

$$\tilde{\Omega} := g\Omega g^{-1} = \begin{pmatrix} \rho_1 J \oplus \cdots \oplus \rho_h J & 0_{2h,n-2h} & 0 \\ 0_{n-2h,2h} & 0_{n-2h} & 0 \\ 0 \ldots & \ldots 0 \;\; \xi & 0 \end{pmatrix}.$$

Thus $\tilde{C}^t = (0, \ldots, 0, \xi)$. A moment's thought tells us that $\xi = 0$ whenever $C \in \operatorname{Range} Q$, and $\xi > 0$ otherwise. The strong convexity condition $|\tilde{W}| < 1$ restricts our domain to those x which satisfy

$$|\tilde{W}(x)|^2 = (\tilde{Q}x + \tilde{C}) \cdot (\tilde{Q}x + \tilde{C}) = \xi^2 + \sum_{i=1}^{h} \rho_i^2 (x_{2i-1}^2 + x_{2i}^2) < 1 \,.$$

In particular, we must have $\xi < 1$. Conversely, as long as $\xi < 1$, strong convexity will always hold on some neighborhood of the origin in $\mathbb{R}^n$, and globally on $\mathbb{R}^n$ only if all ρ_i are zero.

The 0_{n-2h} in $\tilde{\Omega}$ contains the direct sum of copies of 0 times J. This realization, followed by some appropriate relabeling, simplifies $\tilde{\Omega}$ to

$$\begin{pmatrix} a_1 J \oplus \cdots \oplus a_m J & 0 \\ 0 \ldots\ldots\ldots 0 \;\; a_0 & 0 \end{pmatrix}_{\substack{\text{for} \\ \text{even } n}} , \qquad \begin{pmatrix} a_2 J \oplus \cdots \oplus a_m J & 0 & 0 \\ 0 \ldots\ldots\ldots 0 & a_1 & 0 \end{pmatrix}_{\substack{\text{for} \\ \text{odd } n}} .$$

Here, *a priori* we have

$$\begin{array}{lll} 1 > a_0 \geqslant 0, \quad a_1 \geqslant \cdots \geqslant a_m \geqslant 0 & \text{and } m = n/2 & \text{for even } n, \\ 1 > a_1 \geqslant 0, \quad a_2 \geqslant \cdots \geqslant a_m \geqslant 0 & \text{and } m = (n+1)/2 & \text{for odd } n. \end{array}$$

However:

- When n is even, a_0 and a_m cannot both be non-zero for any fixed Ω. Indeed, if $a_0 > 0$, then C is not in $\operatorname{Range} Q$ and we must at least have $a_m = 0$. On the other hand, if $a_m \neq 0$, then Q is surjective; chasing through steps (1), (2), (3) with $2h = n$ shows that the last row of $\tilde{\Omega}$ is zero, that is, a_0 must vanish.
- When n is odd, the displayed normal form precludes any sort of rigid coupling between a_1 and a_m.

For the even n case, whenever $a_0 > 0$ (so that $a_m = 0$), let us agree to relabel the remaining parameters $a_0, a_1, \ldots, a_{m-1}$ as $a_1, a_2, \ldots, a_m$.

Proposition 6.2. *The moduli space $\mathcal{M}_K$ for n-dimensional strongly convex Randers metrics of constant flag curvature $K = 0$ is parametrized by $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ as follows.*

- *When n is even, $m = n/2$ and the parameter space is the disjoint union of*

$$a_1 \geqslant \cdots \geqslant a_m \geqslant 0 \quad \textit{and} \quad 1 > a_1 > 0 \;,\; a_2 \geqslant \cdots a_m \geqslant 0.$$

- *When n is odd, $m = (n+1)/2$ and the parameter space is given by*

$$1 > a_1 \geqslant 0, \;\; a_2 \geqslant \cdots \geqslant a_m \geqslant 0.$$

- *The globally defined metrics on* $\mathbb{R}^n$ *are parametrized by*
$$1 > a_1 \geqslant 0\ ,\ a_2 = \cdots = a_m = 0\,.$$

6.3.2. When σ is non-zero. Refer to the general discussion at the beginning of Section 6.3, and the abbreviation introduced in Section 6.3.1. Conjugating $\Omega = \{-\frac{1}{2}\sigma I_n - Q, C\}$ by any $g := \{A, b\} \in E(n)$ converts it to $\{-\frac{1}{2}\sigma I_n - AQA^t,\, AW(b)\}$. Select $A \in O(n)$ to cast $-Q$ into the following normal form.

When n is even:
$$-\tilde{Q} = -AQA^t = a_1 J \oplus \cdots \oplus a_m J \quad \text{with } m = n/2.$$
When n is odd:
$$-\tilde{Q} = -AQA^t = a_1 J \oplus \cdots \oplus a_m J \oplus 0 \quad \text{with } m = (n-1)/2.$$
Here, $a_1 \geqslant \cdots \geqslant a_m \geqslant 0$. Note that $W(b) = (Q - \frac{1}{2}\sigma I_n)b + C$. The linear operator $Q - \frac{1}{2}\sigma I_n$ is invertible because the spectrum of Q is pure imaginary (Section 9.2) whereas σ is real and non-zero. Therefore, we may select b so that $W(b) = 0$. With this choice of A and b, $g := \{A, b\}$ conjugates Ω into the normal form
$$\tilde{\Omega} = g\Omega g^{-1} = \begin{pmatrix} -\frac{1}{2}\sigma I_n - \tilde{Q} & 0 \\ 0 & 0 \end{pmatrix}.$$

The corresponding infinitesimal homothety has $\tilde{C} = 0$ and its formula is $\tilde{W}(x) = -\frac{1}{2}\sigma x + \tilde{Q}x$. Navigating on Euclidean $\mathbb{R}^n$ subject to the wind $\tilde{W}$ generates a Randers metric of negative flag curvature $K = -\frac{1}{16}\sigma^2$. This metric is strongly convex wherever
$$|\tilde{W}(x)|^2 = \tilde{Q}x \cdot \tilde{Q}x + \tfrac{1}{4}\sigma^2 x \cdot x = \tfrac{1}{4}\sigma^2 x \cdot x \ + \ \sum_{i=1}^{m} a_i^2 (x_{2i-1}^2 + x_{2i}^2) \ < \ 1\,.$$

- For any choice of $\sigma \neq 0$ and a_i, this condition will be satisfied on some neighborhood of the origin in $\mathbb{R}^n$. That is, strong convexity does not create any further constraint on the a_i. Therefore, the space of normal forms is parametrized by the original chamber obtained through Q:
$$a_1 \geqslant \cdots \geqslant a_m \geqslant 0\,,$$
with $m = n/2$ when n is even, and $m = (n-1)/2$ when n is odd.
- Since $|\tilde{W}(x)|^2$ is a non-trivial ($\sigma \neq 0$) quadratic form in x, we see that strong convexity will never hold globally on $\mathbb{R}^n$.

The above arguments handle the moduli space analysis for the $(-)_e$ family described in Theorem 5.1. In order to complete our parametrization of the moduli space for Randers spaces with constant negative flag curvature, it remains to analyze the $(-)_k$ family in Theorem 5.1, namely, perturbations of the Klein model.

6.4. Hyperbolic space. In analogy with the spherical (Sections 5.2 and 6.2) and Euclidean (Section 6.3) cases, we embed the Klein model of hyperbolic geometry into an ambient $(n+1)$ dimensional space. To that end, consider $\mathbb{R}^{n+1}$ equipped with the scalar product $\langle v, w\rangle := v^t E w$, where $E = -1 \oplus I_n$. The isometry group of this space is the Lorentz group $O(1, n)$.

For $K < 0$, define the subspace $H_K := \{p \in \mathbb{R}^{n+1} \mid \langle p, p\rangle = \frac{1}{K}\}$. We make three observations [**24**]:

- H_K consists of two components, each diffeomorphic to $\mathbb{R}^n$.
- $\langle\, ,\, \rangle$ restricts to a Riemannian metric of constant sectional curvature K on H_K.
- $O(1, n)$ preserves H_K, but is only a proper subgroup of the isometry group of H_K.

Let H_K^+ denote the component which passes through $(1/\sqrt{|K|}, 0, \ldots, 0)$. Then, H_K^+ is a complete, simply connected model of hyperbolic space. The isometry group G of H_K^+ consists of those matrices $g \in O(1, n)$ such that $g(H_K^+) = H_K^+$. This identifies G as the orthochronous subgroup $O_+(1, n)$ of $O(1, n)$ (see [**24**]). Its Lie algebra is $\mathfrak{o}(1, n)$ (see Section 9.3).

Let us determine the relationship between Killing vector fields on the Klein model and the Lie algebra $\mathfrak{o}(1, n)$. Introduce the diffeomorphism

$$\psi(x) = \frac{1}{\sqrt{|K|}\sqrt{1 - x \cdot x}}(1, x),$$

which maps the open unit ball $\mathbb{B}^n$ (in $\mathbb{R}^n$) onto H_K^+. The map ψ is an isometry between the Klein model and H_K^+. Let $p := \psi(x)$ abbreviate the position column vector of the image point. Then, Killing vector fields $W(x) = Qx + C - (x \cdot C)x$ of the Klein model are associated with elements

$$\Omega := \begin{pmatrix} 0 & C^t \\ C & -Q \end{pmatrix} \quad \in \ \mathfrak{o}(1, n)$$

via $[\psi_* W(x)]^t = p^t\, \Omega$, where the column $C \in \mathbb{R}^n$ and Q is real $n \times n$ skew-symmetric. This correspondence is a Lie algebra isomorphism.

In Section 9.3 of the Appendix, we show that there exists a $g \in O_+(1,n)$ so that $\tilde{\Omega} = g\Omega g^{-1}$ assumes one of three possible block diagonal forms, as follows.

- $i\Omega$ has a timelike eigenvector.
  n even: $\tilde{\Omega} = 0 \oplus a_1 J \oplus \cdots \oplus a_m J, \quad m = n/2,$
  n odd: $\tilde{\Omega} = 0 \oplus a_1 J \oplus \cdots \oplus a_m J \oplus 0, \quad m = (n-1)/2.$
  Here, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 0$. See Section 2.3.2 for an example.
- $i\Omega$ has a null eigenvector with non-zero eigenvalue. (This assumption automatically rules out timelike eigenvectors; see Section 9.3.5.)
  n even: $\tilde{\Omega} = a_1 S \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0, \quad m = n/2,$
  n odd: $\tilde{\Omega} = a_1 S \oplus a_2 J \oplus \cdots \oplus a_m J, \quad m = (n+1)/2.$
  Here, $a_1 > 0$ and $a_2 \geqslant \cdots \geqslant a_m \geqslant 0$. See [7] for an example.
- $i\Omega$ has a null eigenvector with zero eigenvalue but no timelike eigenvector.
  n even: $\tilde{\Omega} = a_1 T \oplus a_2 J \oplus \cdots \oplus a_m J, \quad m = n/2,$
  n odd: $\tilde{\Omega} = a_1 T \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0, \quad m = (n-1)/2.$
  Here, $a_1 > 0$ and $a_2 \geqslant \cdots \geqslant a_m \geqslant 0$. See [7] for an example.

In the above description, J, S and T denote the matrices

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We declare this $\tilde{\Omega}$ to be the normal form of Ω. It remains to determine how the criterion $|\tilde{W}|^2 < 1$, with $\tilde{W}(x) = \tilde{Q}x + \tilde{C} - (x \cdot \tilde{C})x$, constrains the parameters that describe these normal forms. By $(-)_k$ of Theorem 5.1, that inequality reads: $(\tilde{Q}x + \tilde{C}) \cdot (\tilde{Q}x + \tilde{C}) - (x \cdot \tilde{C})^2 < |K|(1 - x \cdot x)$.

6.4.1. When $i\Omega$ has a timelike eigenvector. The type (J) normal form $\tilde{\Omega}$ is derived in Section 9.3.4. The corresponding Killing field is given by $\tilde{C} = 0$ and

when n is even, $-\tilde{Q} = a_1 J \oplus \cdots \oplus a_m J$ with $m = n/2$,
when n is odd, $-\tilde{Q} = a_1 J \oplus \cdots \oplus a_m J \oplus 0$ with $m = (n-1)/2$.

Here, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 0$. Since the strong convexity criterion $|\tilde{W}|^2 < 1$ now reads $(\tilde{Q}x) \cdot (\tilde{Q}x) < |K|(1 - x \cdot x)$, it will always be satisfied in some neighborhood of the origin in $\mathbb{B}^n$. Therefore, the moduli space is parametrized by

$$a_1 \geqslant \cdots \geqslant a_m \geqslant 0\,.$$

6.4.2. When $i\Omega$ has a null eigenvector with non-zero eigenvalue. The type (S) normal form $\tilde{\Omega}$ is given in Section 9.3.5. The associated Killing field has data $\tilde{C}^t = (a_1, 0, \ldots, 0)$ and

$$\begin{array}{lll} \text{for } n \text{ even}, & -\tilde{Q} = 0 \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0 & \text{with } m = n/2, \\ \text{for } n \text{ odd}, & -\tilde{Q} = 0 \oplus a_2 J \oplus \cdots \oplus a_m J & \text{with } m = (n+1)/2. \end{array}$$

Here, $a_1 > 0$ and $a_2 \geqslant \cdots \geqslant a_m \geqslant 0$. The condition $|\tilde{W}| < 1$ is equivalent to

$$a_1^2(1 - x_1^2) \ + \ \sum_{j=2}^{m} a_j^2 (x_{2j-2}^2 + x_{2j-1}^2) \ < \ |K|(1 - x \cdot x)\,.$$

In particular, we must have $a_1^2(1 - x_1^2) < |K|(1 - x \cdot x)$, which implies $a_1^2(1 - x_1^2) < |K|(1 - x_1^2)$. This forces $a_1 < \sqrt{|K|}$ because $x \in \mathbb{B}^n$. Conversely, as long as a_1 satisfies this bound, we shall have $|\tilde{W}| < 1$ on a neighborhood of the origin. Hence, the moduli space is parametrized by

$$\sqrt{|K|} > a_1 > 0\ , \quad a_2 \geqslant \cdots \geqslant a_m \geqslant 0\,.$$

6.4.3. When $i\Omega$ has a null eigenvector with zero eigenvalue but no timelike eigenvector. For this case, the normal form $\tilde{\Omega}$ is of type (T) and is determined in Section 9.3.6. The corresponding Killing field $\tilde{W}$ has $\tilde{C}^t = (a_1, 0, \ldots, 0)$ and

$$\begin{array}{lll} \text{when } n \text{ is even}, & -\tilde{Q} = a_1 J \oplus \cdots \oplus a_m J & \text{with } m = n/2, \\ \text{when } n \text{ is odd}, & -\tilde{Q} = a_1 J \oplus \cdots \oplus a_m J \oplus 0 & \text{with } m = (n-1)/2. \end{array}$$

Here, $a_1 > 0$ and $a_2 \geqslant \cdots \geqslant a_m \geqslant 0$. Given this data, it can be checked that $|\tilde{W}| < 1$ precisely when

$$a_1^2(1 - x_2)^2 \ + \ \sum_{j=2}^{m} a_j^2 (x_{2j-1}^2 + x_{2j}^2) \ < \ |K|(1 - x \cdot x)\,.$$

(All $a_1^2 x_1^2$ terms cancel out.) At any $x \in \mathbb{B}^n$ of the type $(0, x_2, 0, \ldots, 0)$, our inequality simplifies to $a_1^2(1 - x_2) < |K|(1 + x_2)$, which always holds provided that x_2 is sufficiently close to 1. Continuity then extends the inequality to a neighborhood of that x. Thus, demanding strong convexity locally does not impose any additional constraint on the a_i. We conclude that the moduli space is parametrized by

$$a_1 > 0\ , \qquad a_2 \geqslant \cdots \geqslant a_m \geqslant 0\,.$$

6.4.4. The moduli space for $K < 0$. Unlike those of positive and zero flag curvature, Randers spaces of negative constant flag curvature may arise in two different fashions, corresponding to the cases $\sigma \neq 0$ and $\sigma = 0$. Since σ is invariant under isometries (Section 6.1), it makes sense to talk about the isometry classes, and hence the moduli spaces, for these two families.

- Zermelo navigation on Euclidean space under an infinitesimal homothety with $\sigma \neq 0$ produces a metric with flag curvature $K = -\frac{1}{16}\sigma^2$. The moduli space for these metrics has been parametrized in Section 6.3.2.
- For $\sigma = 0$, the perturbation of the Klein model of negative sectional curvature K by infinitesimal isometries generates metrics with flag curvature K. The moduli space is parametrized, up to isometry, in Sections 6.4.1–6.4.3. A quick glance at the explicit form of $|\tilde{W}(x)|^2 < 1$, with $x \in \mathbb{B}^n$, shows that having strong convexity globally on $\mathbb{B}^n$ is only possible in the scenario Section 6.4.1, with $a_1 = \cdots = a_m = 0$; in that case, our Randers metric is simply the Klein model itself.

Together, the Euclidean and hyperbolic parametrizations provide a complete description of the isometry classes.

Proposition 6.3. *The moduli space $\mathcal{M}_K$ for n-dimensional strongly convex Randers metrics of constant flag curvature $K < 0$ is parametrized by $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ as follows:*

(e) *For those obtained by perturbing the standard Euclidean metric on $\mathbb{R}^n$, using infinitesimal homotheties with $\sigma = \pm 4\sqrt{|K|}$, the parameter space is*

$$a_1 \geqslant \cdots \geqslant a_m \geqslant 0,$$

where $m = n/2$ when n is even, and $m = (n-1)/2$ when n is odd. These metrics cannot be extended to all of $\mathbb{R}^n$.

(k) *For those obtained by perturbing the Klein model on the open unit ball $\mathbb{B}^n$, the parameter space is the disjoint union of three sets.*

  ∘ *When n is even, $m = n/2$ and the three sets are*

$$\begin{gathered} a_1 \geqslant \cdots \geqslant a_m \geqslant 0, \\ \sqrt{|K|} > a_1 > 0,\ a_2 \geqslant \cdots a_m \geqslant 0 \\ \textit{and} \qquad a_1 > 0,\ a_2 \geqslant \cdots \geqslant a_m \geqslant 0. \end{gathered}$$

- *When n is odd, $m = (n+1)/2$ and the three sets are*

$$a_1 \geqslant \cdots \geqslant a_{m-1} \geqslant 0 =: a_m,$$
$$\sqrt{|K|} > a_1 > 0,\ a_2 \geqslant \cdots a_m \geqslant 0$$
$$\textit{and} \quad a_1 > 0,\ a_2 \geqslant \cdots \geqslant a_{m-1} \geqslant 0 =: a_m.$$

Among such Randers metrics, the only globally defined one on $\mathbb{B}^n$ is the Klein model itself, corresponding to $a_1 = \cdots = a_m = 0$.

7. Restricting to projectively flat metrics

Let M be an n-dimensional differentiable manifold. A metric on M is said to be projectively flat if M can be covered by coordinate charts in which the geodesics of the metric are straight lines. For Riemannian metrics, Beltrami's theorem says that the only projectively flat ones are those with constant sectional curvature. There are Finsler metrics of constant flag curvature which are not projectively flat; see for example [**13**, **14**] and [**8**]. Thus, Beltrami's theorem does not extend to the Finslerian setting.

7.1. Douglas' theorem. A theorem due to Douglas [**17**] states that a Finsler metric F is projectively flat if and only if two special curvature tensors are zero. The first is the Douglas tensor. The second is the projective Weyl tensor for $n \geqslant 3$, and the Berwald–Weyl tensor [**11**] for $n = 2$. (The projective Weyl tensor automatically vanishes when $n = 2$, thereby predicating the need for a different invariant in that dimension.) A complete statement of Douglas' theorem can be found on p. 144 of [**27**].

The projective Weyl tensor vanishes when and only when the flag curvature of F is merely a function of the position x and the flagpole y (that is, no dependence on $\text{span}\{y, V\}$, with V transversal to y); see Section 2.4 and [**31**, **21**, **1**]. That vanishing is automatic in dimension 2 because, once x and y are specified, the said span is always the tangent plane T_xM, independent of V.

The Berwald–Weyl tensor is defined for all n, though only relevant in Douglas' theorem when $n = 2$. It is explicitly given in formula (8.27) on p. 144 of [**27**]. The criterion of a Finsler metric F having constant flag curvature K may be recast into the form $K^i{}_k = K(\delta^i{}_k - \ell^i \ell_k)F^2$; for an exposition, see [**6**, **4**]. From this, a straightforward computation shows that the Berwald–Weyl tensor vanishes for all such metrics.

7.2. Specializing to Randers metrics. For Randers metrics of constant flag curvature, there is certainly no dependence on the transverse edges, hence the projective Weyl tensor vanishes. Also, as remarked above, the Berwald–Weyl tensor in two dimensions is zero as well.

According to [**2**], a Randers metric F has vanishing Douglas tensor if and only if the 1-form $b := b_i\, dx^i$ is closed. Let $W^\flat$ denote the 1-form $W_i\, dx^i := h_{ij} W^j\, dx^i$, where (h, W) is the Zermelo navigation data of F. Using the equation $\mathcal{L}_W h = -\sigma\, h$ with constant σ, it can be checked that the 2-forms curl $:= -db$ (Section 3.1) and $\mathcal{C} := -dW^\flat$ (Section 3.2.2) are related through $\text{curl}^{ij} = -\lambda\, \mathcal{C}^{ij}$ (indices on curl, $\mathcal{C}$ are raised, resp., by $a^\sharp$, $h^\sharp$), where $\lambda := 1 - |W|^2$ is positive because of strong convexity (Section 1.2). In particular, $db = 0 \Leftrightarrow dW^\flat = 0$, whenever the above $\mathcal{L}_W$ equation holds.

If the Randers metric F has constant flag curvature, then Theorem 3.1 (Section 3.3) avails us of this $\mathcal{L}_W$ equation; in that case, the vanishing of the Douglas tensor is equivalent to the condition $dW^\flat = 0$.

7.3. Projectively flat strongly convex Randers metrics of constant flag curvature. By virtue of Douglas' theorem, we see that a Randers metric F of constant flag curvature and navigation data (h, W) is projectively flat if and only if the 1-form $W^\flat$ is closed, namely, $\partial_{x^j} W_i - \partial_{x^i} W_j = 0$. Let us apply this criterion to the models (h, W, D) discussed in Section 6.1.

- Suppose F is obtained by perturbing the Euclidean metric. Using the formula for W_i given in the proof of Proposition 4.2, we see that $W^\flat$ is closed if and only if (Q_{ij}) is the zero matrix. Hence W simplifies to $-\frac{1}{2}\sigma x + C$, with $\sigma = \pm 4\sqrt{|K|}$.
- Suppose F is obtained by perturbing the standard sphere or the Klein model. Since W is Killing (Theorem 3.1) and $W^\flat$ is closed, it must be parallel; that is, $W_{i:j} = 0$. In this case, the standard Ricci identity $W^j{}_{:j:i} - W^j{}_{:i:j} = -{}^h\text{Ric}_i{}^s\, W_s$, in conjunction with ${}^h\text{Ric}_i{}^s = (n-1)K\, \delta_i{}^s$ (because h is a space form), reduces to $K\, W_i = 0$. Since $K \neq 0$, W must vanish identically on the maximal domain D.

The above information, together with the classification given in Theorem 5.1, tells us the following. Each projectively flat strongly convex *non*-Riemannian Randers metrics of constant flag curvature K is locally isometric to one of the two types listed below:

(1) $K = 0$**:** Zermelo navigation on Euclidean space with a constant vector field $W = C$ satisfying $0 < |C| < 1$. These are Minkowski spaces (see Section 2.2.1). A rotation transforms the vector field W into $(0, \ldots, 0, |C|)$ without causing the Minkowski metric in question to leave its isometry class. Thus $|C|$ parametrizes the moduli space $(0, 1) \subset \mathbb{R}$. Alternatively, Proposition 6.2 with $Q = 0$ handles the projectively flat metrics; among those, the non-Riemannian ones are parametrized by $1 > a_1 > 0$, which is consistent with the above conclusion.

(2) $K < 0$**:** Zermelo navigation on Euclidean $\mathbb{R}^n$ with $W = -\frac{1}{2}\sigma x + C$, $\sigma = \pm 4\sqrt{|K|}$, and $C \cdot C + \sigma x \cdot (\frac{1}{4}\sigma x - C) < 1$. This camp includes the Funk metric of Section 2.3.1. A translation transforms W into $\tilde{W} = -\frac{1}{2}\sigma\tilde{x}$. By Sections 6.3 (with $Q = 0$) and 6.1, the corresponding metrics F and $\tilde{F}$ are isometric. Closer examination of $\tilde{F}$ reveals that it is a $\tilde{x}$-scaled variant of the Funk metric, one which lives on the open ball of radius $1/(2\sqrt{|K|})$ centered at the origin of $\mathbb{R}^n$. In particular, the moduli space consists of only one point, as predicted by case (e) of Proposition 6.3 (with all a_i set to zero because $Q = 0$ here).

As a corollary of this itemization,

Every projectively flat, strongly convex Randers metric of constant positive flag curvature must be locally isometric to a Riemannian standard sphere.

We see from the table in Section 4.4 that among the examples in Section 2, only 2.2.1 and 2.3.1 are projectively flat.

7.4. Comments, and a fine point. The above conclusions about projectively flat Randers metrics F of constant flag curvature are consistent with the main result of [**29**]. However, other than the fact that the two papers use totally different methods, some further distinctions are worth noting.

- Here, the $K < 0$ camp has simple navigation data (h, W), where h is the Kronecker delta; but the resulting F, when generated with Section 1.1.3, shows a certain amount of complexity. In [**29**], a simple expression is derived for F in the $K < 0$ camp; but, upon the use of Section 1.3 to recover the navigation data (h, W), we find that h, though isometric to the Euclidean metric, takes on a complicated form.

- We have just seen that the moduli space for projectively flat strongly convex Randers metrics of constant flag curvature $K < 0$ consists of a single point. This is not manifest in [**29**] because there, each metric in question was parametrized by a vector $\vec{a}$ of $\mathbb{R}^n$, and no attempt was made to ascertain whether metrics corresponding to different $\vec{a}$ were in fact isometric.

We hasten to belabor a nuance. Take any projectively flat strongly convex Randers space (M, F) with constant flag curvature $K < 0$. Let $\tilde{F}$ be the $\tilde{x}$-scaled variant of the Funk metric which lives on the open ball B of radius $1/(2\sqrt{|K|})$ centered at the origin of $\mathbb{R}^n$. The above discussion says that given any point $p \in M$, there exists a point $\tilde{p} \in B$, and open sets $U \subset M$, $\tilde{U} \subset B$ containing p and $\tilde{p}$, respectively, such that (U, F) is isometric to $(\tilde{U}, \tilde{F})$. If we move to a different vantage point $q \in M$, there would likewise be an isometry between some (V, F) and $(\tilde{V}, \tilde{F})$, where V contains q. It can be shown (say, by computing a geometric invariant such as the Cartan tensor) that for the Funk metric, unlike its Riemannian counterpart the Klein metric, $(\tilde{U}, \tilde{F})$ is typically *not* isometric to $(\tilde{V}, \tilde{F})$. Consequently, (U, F) is in general *not* isometric to (V, F).

8. Restricting to the $\theta = 0$ family

Recall the tensor $\theta_i := b^s \operatorname{curl}_{si}$ encountered in Section 3.1. Strongly convex Randers metrics of constant flag curvature and satisfying the additional condition $\theta = 0$ have previously been characterized by the *corrected* Yasuda–Shimada theorem in terms of non-linear partial differential equations. See [**5**, **22**] for details and references therein, and [**3**] for a historical account. Here, we compute the moduli space for all the solutions of these PDEs.

8.1. Necessary and sufficient conditions for $\theta = 0$. It can be shown (using the machinery in [**6**]) that the tensor θ for Randers metrics of constant flag curvature has the navigation description $(1 - |W|^2)\theta_j = (|W|^2)_{:j} + \sigma W_j$. Since our Randers metrics are always presumed to be strongly convex ($|W| < 1$), we see that

$$\theta = 0 \quad \Leftrightarrow \quad (|W|^2)_{:j} + \sigma W_j = 0.$$

8.1.1. The Euclidean case. When h is the standard Euclidean metric, Proposition 4.2 says that $W = -\frac{1}{2}\sigma x + Qx + C$. The equation

$(|W|^2)_{:j} + \sigma W_j = 0$ is polynomial in the local coordinates (x^i). By considering the coefficients of this polynomial, one can establish that $\theta = 0$ if and only if

- $Q = 0$ when $\sigma \neq 0$,
- $Q^2 = 0$ and $QC = 0$ when $\sigma = 0$.

It is clear, from the normal form $\tilde{Q}$ (Section 9.2) of Q, that $Q^2 = 0$ if and only if $Q = 0$. Hence the two cases can be unified into a single criterion $Q = 0$, which is in turn equivalent to the 1-form $W^\flat := W_i dx^i$ being closed (Section 7.3). We conclude that, for strongly convex constant flag curvature Randers metrics which are generated by navigating on Euclidean $\mathbb{R}^n$ under the influence of an infinitesimal homothety W,

$$\theta = 0 \quad \text{if and only if} \quad dW^\flat = 0.$$

Such metrics are precisely the projectively flat ones enumerated in Section 7.3. It is worth recollecting (Section 7.2) that in the present context, $dW^\flat = 0$ is equivalent to $db = 0$.

8.1.2. The spherical and Klein models. When h is either the spherical or hyperbolic metric, σ must vanish (Section 3.3), and we see that

$$\theta = 0 \quad \Leftrightarrow \quad (|W|^2)_{:j} = 0 \quad \Leftrightarrow \quad |W|^2 \text{ is constant.}$$

Proposition 4.3 says that $W_i = (Q_{ij}x^j + C_i)/\{|K|(1 + \psi x \cdot x)\}$ and $W^i = Q^i{}_k x^k + C^i + \psi(x \cdot C)x^i$, where $\psi := K/|K|$. Consequently, the constancy of $|W|^2$ can be re-expressed as a polynomial equation in the local coordinates (x^i). That polynomial's coefficients lead to the following necessary and sufficient conditions for $\theta = 0$:

$$QC = 0 \quad \text{and} \quad Q^2 = \psi \left(CC^t - |C|^2 I_n\right).$$

Here, C is a column and C^t is a row.

The above equations are invariant in form under any orthogonal transformation $R \in O(n)$. Indeed, multiplying each term by R on the left, and also by R^t on the right for matrices, those equations become $\tilde{Q}\tilde{C} = 0$ and $\tilde{Q}^2 = \psi\,(\tilde{C}\tilde{C}^t - |\tilde{C}|^2 I_n)$, where $\tilde{Q} = RQR^t$, $\tilde{C} = RC$.

- Therefore, without any loss of generality, we may assume that Q is already in the normal form derived in Section 9.2. Namely,

$$Q = q_1 J \oplus \cdots q_k J \oplus 0_{n-2k} \quad \text{with} \quad q_1 \geqslant \cdots \geqslant q_k > 0.$$

- With this Q, the equation $QC = 0$ can be solved immediately to find that the first $2k$ components of C are zero. Its remaining components can be transformed by any $r \in O(n - 2k)$ without altering Q. Thus, we may assume that the column vector C which solves $QC = 0$ has the simplified form

$$C = (0, \dots, 0, |C|).$$

We now substitute the displayed Q and C into the equation $Q^2 = \psi\,(CC^t - |C|^2 I_n)$. The outcome reads

$$(*) \qquad -q_1^2 I_2 \oplus \cdots \oplus -q_k^2 I_2 \oplus 0_{n-2k} = -\psi\,|C|^2\,I_{n-1} \oplus 0,$$

where I_j denotes the $j \times j$ identity matrix.

- By inspection, all the q_i are zero if and only if $|C| = 0$. In other words, $Q = 0 \Leftrightarrow C = 0$. The Killing field corresponding to $Q = 0$, $C = 0$ is $W = 0$. In that case, the associated Randers metric is simply the original Riemannian space form h.
- It remains to examine the scenario in which C is non-zero. Equation $(*)$ then implies that all the q_i are non-zero as well, and forces three restrictions.
  (1) $\psi := K/|K| = 1$, hence $K > 0$ and h is the spherical metric.
  (2) $q_1 = \cdots = q_k = |C|$.
  (3) $2k = n - 1$; equivalently, $n = 2k + 1$ is odd.
  Up to isometry, the strongly convex Randers metric in question must have arisen from navigation on an *odd* dimensional sphere, under the influence of a one parameter family (indexed by $|C|$) of winds W.

We hasten to reiterate that these restrictions are obtained from local considerations only, on spheres and open balls; globality is not needed in their derivation.

8.2. The corrected Yasuda–Shimada family. Taken together, Sections 8.1.1, 8.1.2 and 7.3 allow us to enumerate the moduli space for strongly convex Randers metrics with constant flag curvature K and $\theta = 0$. They are obtained by Zermelo navigation on Riemannian space forms h, subject to the influence of appropriate winds W which satisfy $|W| < 1$. The *non*-Riemannian ones are as follows:

- When $K < 0$: h is the standard metric on Euclidean $\mathbb{R}^n$, and $W = -\frac{1}{2}\sigma x + C$, with $\sigma = \pm 4\sqrt{|K|}$. As explained in Section 7.3, the resulting Randers metric is isometric to a position-scaled variant of the Funk metric, one which is generated by $\tilde{W} = -\frac{1}{2}\sigma\tilde{x}$ and lives on the open ball of radius $1/(2\sqrt{|K|})$.

- When $K = 0$: h is the standard metric on Euclidean $\mathbb{R}^n$, and $W = C$ with $0 < |C| < 1$. We saw in Section 7.3 that up to isometry, this family, which consists of Minkowski metrics, is parametrized by a single parameter $|C|$.
- When $K > 0$: h is $1/K$ times the standard metric on the unit sphere S^n, with $n = 2k + 1$ odd. The wind W is given in projective coordinates (Sections 4.3 and 5.2.1) as $Qx + C + (x \cdot C)x$, where Q and C are specially related on account of $\theta = 0$. In fact (Section 8.1.2), there is an $R \in O(n)$ such that $\tilde{C} := RC = (0, \ldots, 0, |C|)$ and $\tilde{Q} := RQR^t = |C|(J \oplus \cdots \oplus J) \oplus 0$, respectively. This is equivalent to conjugating the matrix representative (Section 6.2) of W by the element $1 \oplus R$ in the isometry group of h. Thus (Section 6.1) the Randers metric generated by $\tilde{W} := \tilde{Q}x + \tilde{C} + (x \cdot \tilde{C})x$ lies in the same isometry class as that from W. Applying the analysis in Section 6.2.2 to $\tilde{W}$, we see that strong convexity mandates $|C| < \sqrt{K}$, which as a bonus (Section 6.2.3) ensures that the metric is global on S^n. Thus, up to isometry, there is only a one parameter family (indexed by $|C|$) of *non*-Riemannian strongly convex Randers metrics with constant flag curvature K and $\theta = 0$ on the odd dimensional spheres. By contrast, no such metric exists on the even dimensional spheres, regardless of whether it is locally or globally defined.

Strongly convex non-Riemannian Randers metrics with constant flag curvature K and $\theta = 0$ are characterized by the corrected Yasuda–Shimada theorem [**5**, **22**, **3**]. The conclusion for the $K = 0$ case is as described above. For non-zero K, the characterization is in terms of coupled systems of non-linear partial differential equations. Our discussion above may be viewed as a complete list of solutions to those partial differential equations.

Bejancu–Farran [**9**, **10**], assisted by the corrected Yasuda–Shimada theorem, have recently established a bijection between Sasakian space forms of constant ϕ-sectional curvature $c \in (-3, 1)$, and Randers metrics of constant flag curvature $K = 1$ with $\theta = 0$. In the course of their study, they showed that the underlying manifold M must be of odd dimension, and is necessarily diffeomorphic to a sphere when it is simply connected and complete with respect to a. These results can be made equivalent to what we have described for the $K > 0$ case. Our θ is denoted by β in the Bejancu–Farran papers, and their c is $1 - 4\|b\|^2$ in our notation.

It is worth mentioning here that all spheres, of both odd and *even* dimensions, admit a wealth of non-Riemannian globally defined Randers metrics of constant positive flag curvature, provided that the restriction $\theta = 0$ is lifted. Here is a straightforward example on S^4. Following the treatment of Section 5.2, we let $p = (p^0, p^1, p^2, p^3, p^4)$ denote the canonical coordinates on $\mathbb{R}^5$. The infinitesimal rotation

$$W(p) = \tau(-p^2\partial_{p^1} + p^1\partial_{p^2})\,, \quad \tau \text{ constant},$$

restricts to a globally defined Killing field on the standard unit sphere S^4. As long as $|\tau| < 1$, we have $|W| < 1$ on the entire sphere. Hence W induces a globally defined, strongly convex Randers metric with constant flag curvature $+1$ on S^4. Notice, however, that $\theta \neq 0$. This is immediate from the statement displayed at the beginning of Section 8.1.2, which says that θ vanishes if and only if $|W|$ is constant. The norm of our W is certainly not constant. Hence θ is non-zero.

9. Appendix: Some Lie theory

Recall from Section 6.1 that the symmetry/isometry groups G (of the Riemannian space forms) act on the Lie algebras of infinitesimal homotheties, via the adjoint action *Ad*. Our analysis of the moduli space (Section 6) for constant flag curvature Randers metrics requires detailed knowledge of each *Ad* orbit, in order to pinpoint a distinguished representative.

Though the Lie theory for the orthogonal group is well known, it is invoked in so many different contexts that we feel obligated to at least set the notation (Section 9.2). In the non-compact case $G = O_+(1,n)$, the orthochronous Lorentz group, the information we need is not available in a form that we could use without substantial modification or synthesis. Since this material plays such a pivotal role in our geometric conclusions, we are compelled to sketch a cohesive account (Section 9.3). Finally, our exposition is cast in matrix language for the sake of concreteness.

9.1. Scalar products and the "perp argument". By a *scalar product* on any complex vector space $\mathcal{V}$, we mean a pairing $\langle\,,\,\rangle$ which is $\mathbb{C}$-linear in the first factor, satisfies $\langle u,v\rangle = \overline{\langle v,u\rangle}$, and is non-degenerate (namely, if $\langle u,v\rangle = 0$ for all $v \in \mathcal{V}$, then u must vanish). Inner products are simply positive definite scalar products. For example, if E is the diagonal matrix $-1 \oplus I_n$, then $\langle u,v\rangle := u^t E\overline{v}$ is a scalar product on $\mathbb{C}^{1+n}$, whereas replacing that -1 by $+1$ gives the canonical inner product $u^t\overline{v}$ on $\mathbb{C}^{1+n}$.

In any scalar product space, a non-zero vector v is said to be spacelike, null, or timelike, respectively, if $\langle v, v\rangle$ is positive, zero, or negative. The zero vector is by definition spacelike.

Using the fact that $\langle u, v\rangle = \mathrm{Re}\langle u, v\rangle + i\mathrm{Re}\langle u, iv\rangle$, together with the polarization identity $\mathrm{Re}\langle p, q\rangle = \frac{1}{4}\{\langle p+q, p+q\rangle - \langle p-q, p-q\rangle\}$, one can check by contradiction (of non-degeneracy) that:

If $\dim\mathcal{V} \geqslant 1$, then every scalar product on $\mathcal{V}$ admits either a timelike vector, or a non-zero spacelike vector.

Let $\mathcal{W}$ be any subspace of a scalar product space $\mathcal{V}$. Its perp $\mathcal{W}^\perp$ is $\{v \in \mathcal{V} : \langle v, w\rangle = 0$ for all $w \in \mathcal{W}\}$. Adapting the arguments in [**24**] to complex vector spaces, one can check that

$$\dim\mathcal{W} + \dim\mathcal{W}^\perp = \dim\mathcal{V} \qquad \text{and} \qquad (\mathcal{W}^\perp)^\perp = \mathcal{W}\,.$$

The restriction of $\langle\,,\,\rangle$ to $\mathcal{W}^\perp$ may be degenerate when $\mathcal{W}$ contains a null vector. For instance, in $\mathbb{C}^{1+2}$ with $E = \mathrm{diag}(-1, 1, 1)$, if $\mathcal{W} = \mathrm{span}\{(1, 1, 0)\}$, then $\langle\,,\,\rangle$ is degenerate on $\mathcal{W}^\perp = \mathrm{span}\{(1, 1, 0), (0, 0, 1)\}$. On the other hand, if $\mathcal{W} = \mathrm{span}\{(1, 1, 0), (1, -1, 0)\}$, then non-degeneracy holds on $\mathcal{W}^\perp = \mathrm{span}\{(0, 0, 1)\}$. These examples illustrate the following lemma that we shall invoke repeatedly without mention.

Lemma 9.1. *Let $\mathcal{W}$ be any subspace in a complex scalar product space $(\mathcal{V}, \langle\,,\,\rangle)$. Then, the following three statements are equivalent:*

(1) *$\mathcal{W}$ admits a $\langle\,,\,\rangle$ orthonormal basis (note, $|w| := \sqrt{|\langle w, w\rangle|}$).*
(2) $\mathcal{W} \cap \mathcal{W}^\perp = \{0\}$.
(3) *$\langle\,,\,\rangle_{|\mathcal{W}^\perp}$ is non-degenerate, hence defines a scalar product on $\mathcal{W}^\perp$.*

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are simple. Once we have (1) $\Rightarrow$ (3), it can be used, in conjunction with the automatic existence of non-null vectors, to establish inductively the following useful fact.

If $\mathcal{U}$ is any subspace with dimension $\geqslant 1$ on which $\langle\,,\,\rangle$ is non-degenerate, then there is a $\langle\,,\,\rangle$ orthonormal basis for $\mathcal{U}$.

This then effects (3) $\Rightarrow$ (1). Indeed, given (3), the above fact provides $\mathcal{W}^\perp$ with an orthonormal basis. Applying (1) $\Rightarrow$ (3) to $\mathcal{W}^\perp$ (instead of $\mathcal{W}$), we see that $\langle\,,\,\rangle$ is non-degenerate on $(\mathcal{W}^\perp)^\perp = \mathcal{W}$. By the above fact again, $\mathcal{W}$ has an orthonormal basis, which is (1). Our reasoning is synthesized from that in [**24**], and adapted to the complex case.

Let A be a self-adjoint linear operator on the scalar product space $\mathcal{V}$. Suppose the subspace $\mathcal{W}$ is invariant under A. Then, so is $\mathcal{W}^\perp$, because $\langle Av, w\rangle = \langle v, Aw\rangle$. Hence the restriction of A to $\mathcal{W}^\perp$ makes sense. If, in addition, $\langle\, ,\, \rangle$ is non-degenerate on $\mathcal{W}^\perp$, then the restricted A is again operating on a scalar product space, albeit a smaller one. We shall repeatedly invoke this "perp argument".

9.2. A compact case: skew-symmetric real matrices. Let Ω be any real $\ell \times \ell$ skew-symmetric matrix. Then $A := i\Omega$ is a self-adjoint linear operator on the inner product space $\mathbb{C}^\ell$, with $\langle u, v\rangle := u^t\overline{v}$. Thus each eigenvalue of A is real, and eigenspaces corresponding to distinct eigenvalues are $\langle\, ,\, \rangle$ orthogonal.

Since $A = i\Omega$ where Ω is real, the non-zero eigenvalues of A occur in pairs $\pm a$ $(a > 0)$, with $\langle\, ,\, \rangle$ orthogonal eigenvectors z and $\overline{z}$. The real vectors $v := (z+\overline{z})/2$ and $u := (z-\overline{z})/(2i)$ satisfy $Au = -iav$, $Av = iau$, and $\langle z, \overline{z}\rangle = 0$ implies that $\langle u, u\rangle = \langle v, v\rangle$ and $\langle u, v\rangle = 0$. Hence the normalized versions $\hat{u}$, $\hat{v}$ still satisfy $A\hat{u} = -ia\hat{v}$ and $A\hat{v} = ia\hat{u}$.

The "perp argument" (Section 9.1) implies that each eigenvalue of A with multiplicity s has an eigenspace of the same dimension. Enumerating the non-zero eigenvalues of A as $\pm a_1, \dots, \pm a_k$, where $a_1 \geqslant \cdots \geqslant a_k > 0$, we get a *real* orthonormal set $\{\hat{u}_1, \hat{v}_1, \dots, \hat{u}_k, \hat{v}_k\}$ such that $A\hat{u}_k = -ia_k\hat{v}_k$ and $A\hat{v}_k = ia_k\hat{u}_k$. If $2k < \ell$, then 0 is an eigenvalue of A with eigenspace spanned by a *real* orthonormal set $\{\xi_{2k+1}, \dots, \xi_\ell\}$ because Ω is real. These two sets comprise a real orthonormal basis in which the matrix representation of A is $ia_1J \oplus \cdots \oplus ia_kJ \oplus 0_{\ell-2k}$, where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Correspondingly, that of Ω is $\tilde{\Omega} := a_1J \oplus \cdots \oplus a_kJ \oplus 0_{\ell-2k}$, with $2k$ being its rank. Suppressing the rank of Ω, we see that

$$\begin{array}{lll} \text{when } \ell \text{ is even,} & \tilde{\Omega} = a_1J \oplus \cdots \oplus a_mJ & \text{with } m = \ell/2, \\ \text{when } \ell \text{ is odd,} & \tilde{\Omega} = a_1J \oplus \cdots \oplus a_mJ \oplus 0 & \text{with } m = (\ell-1)/2, \end{array}$$

where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 0$. This is the desired normal form of Ω. Note that $\tilde{\Omega} = B^{-1}\Omega B$, where B is the orthogonal matrix whose columns are given by the vectors in our real orthonormal basis.

In terms of Lie theory, a skew-symmetric matrix Ω is an element in the Lie algebra $\mathfrak{o}(\ell)$ of the orthogonal group $O(\ell)$. The fact that $\exp(a_iJ)$ is the 2×2 rotation matrix with angle a_i tells us that $\exp(\tilde{\Omega})$ lands in a maximal torus of $O(\ell)$, and $\tilde{\Omega}$ itself belongs to a Cartan subalgebra $\mathcal{H}$

of $\mathfrak{o}(\ell)$. The condition $a_1 \geqslant \cdots \geqslant a_m \geqslant 0$ singles out the fundamental closed Weyl chamber of $\mathcal{H}$. Our arguments show that every real skew-symmetric Ω can be conjugated by the $O(\ell)$ element $g = B^{-1}$ into this closed Weyl chamber.

9.3. A non-compact case. Let E denote the diagonal matrix $-1 \oplus I_n$. The elements of $\mathfrak{o}(1,n)$ are real $(n+1)\times(n+1)$ matrices Ω which satisfy the condition $\Omega^t = -E\Omega E$; equivalently, Ω has the defining form

$$\Omega = \begin{pmatrix} 0 & C^t \\ C & -Q \end{pmatrix},$$

where Q, C are real, and Q is $n\times n$ skew-symmetric. The Lorentz group $O(1,n)$ is a non-compact Lie group with Lie algebra $\mathfrak{o}(1,n)$. Elements of $O(1,n)$ are real $(n+1)\times(n+1)$ matrices g such that $g^{-1} = Eg^tE$. With respect to the scalar product $\langle v, w\rangle := v^t E w$ of $\mathbb{R}^{n+1}$, the columns of g comprise a $\langle\,,\,\rangle$ orthonormal basis, with the first column being timelike, and the rest spacelike. In particular, the top left entry of g satisfies $(g^0{}_0)^2 \geqslant 1$. We described in Section 6.4 a model H_K^+ for n-dimensional hyperbolic space. The isometry group of H_K^+ is the orthochronous subgroup $G := O_+(1,n)$, whose matrices g have top left entry $g^0{}_0 \geqslant 1$.

9.3.1. An available simplification. Our goal here is to select a simplest representative along the G adjoint orbit of Ω. To that end, we first invoke Section 9.2 to find an element $R \in O(n)$ such that $RQR^{-1} = q_1 J \oplus \cdots \oplus q_h J \oplus 0_{n-2h}$, where $q_1 \geqslant \cdots \geqslant q_h > 0$. This has the effect of changing C to RC. Next, we use an element $r \in O(n-2h)$ to transform the last $n-2h$ components of RC into $(0,\dots,0,\xi)$ without affecting its first $2h$ components. In terms of matrix conjugation, set $g_1 := 1 \oplus R$ and $g_2 := 1 \oplus I_{2h} \oplus r$, then $(g_2 g_1)\Omega(g_2 g_1)^{-1}$ has the simplified form

$$\begin{pmatrix} 0 & D^t & 0 & \xi \\ D & -(q_1 J \oplus \cdots \oplus q_h J) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 \end{pmatrix}.$$

Here, D is a column of $2h$ entries listed pairwise; in other words, it has the form $D = (D_1, \dots, D_h)$, with $D_j := [(RC)_{2j-1}, (RC)_{2j}]$. Since $g_2 g_1 \in G$, the above matrix lies on the same Ad orbit as Ω. When necessary, we can use this simplified form for Ω with no loss of generality.

9.3.2. Preliminaries about eigenvalues and eigenvectors. Given any element $\Omega \in \mathfrak{o}(1,n)$, the matrix $A := i\Omega$ is a self-adjoint linear operator on the scalar product space $\mathbb{C}^{1+n}$, with $\langle U, V\rangle := U^t E \overline{V}$. Let $V = (v_0, v)$ be an arbitrary (possibly complex) eigenvector of A with eigenvalue λ. Then

(1) $A\overline{V} = -\overline{\lambda}\,\overline{V}$ holds, besides $AV = \lambda V$,
(2) we have $\lambda v_0 = iC^t v$ and $\lambda v = iv_0 C - iQv$,
(3) the skew-symmetry of Q, together with item (2), implies that $\lambda(v_0^2 - v^t v) = 0$.

The following three conclusions are about eigenvectors V with $\lambda \neq 0$. In the derivations, keep in mind that by (3), we have $v_0^2 = v^t v$.

(4) *V must either be spacelike or null. (Consequently, all timelike eigenvectors must have zero eigenvalue; though the converse might not be true.)* This comes about because $\langle V, V\rangle = -|v_0|^2 + |v|^2$ and $|v_0|^2 = |v^t v| = |(v, \overline{v})| \leqslant |v|\,|\overline{v}| = |v|^2$, where the Cauchy–Schwarz inequality is being applied to the canonical inner product $(v, w) := v^t \overline{w}$ on $\mathbb{C}^n$.
(5) *The spacelike eigenvectors have real eigenvalues, which must occur in pairs $\pm a$ $(a > 0)$, with corresponding $\langle\,,\,\rangle$ orthogonal eigenvectors V, $\overline{V}$.* The self-adjointness of A implies that $\lambda\langle V, V\rangle = \overline{\lambda}\,\langle V, V\rangle$, hence λ is real whenever V is not null. The rest follows from item (1), $\lambda = a > 0$, and $\langle AV, \overline{V}\rangle = \langle V, A\overline{V}\rangle$.
(6) *The null eigenvectors have pure imaginary eigenvalues, and can always be standardized into the form $V = (1, v)$ with v real.* Indeed, $V = (v_0, \tilde{v})$ being non-zero and null means that $|v_0|^2 = |\tilde{v}|^2$ with $v_0 \neq 0$; dividing by v_0 gives $(1, v)$, where $v^t\overline{v} = |v|^2 = 1$. Yet, (3) says that $v^t v = 1$. Substituting $v = \mathrm{Re}\, v + i\mathrm{Im}\, v$ into these two equations gives $\mathrm{Im}\, v = 0$. Then, (1) tells us that $\lambda = -\overline{\lambda}$.

9.3.3. Categorizing the normal forms of $A = i\Omega$. Let us first establish that *if A has no timelike eigenvector, then it must admit a null eigenvector.*

Given the absence of timelike eigenvectors, suppose there were no null eigenvectors either. Then, all eigenvectors of A would have to be spacelike. Applying the perp argument (Section 9.1) n times would produce a $\langle\,,\,\rangle$ orthonormal basis B which is entirely spacelike (and which diagonalizes A). With respect to B, the matrix of $\langle\,,\,\rangle$ would be I_{n+1} instead of $E = -1 \oplus I_n$, contradicting the invariance of the index of $\langle\,,\,\rangle$.

Thus, it is reasonable to split our derivation of the normal forms of A into three camps.

- When A has a timelike eigenvector, the normal form is of type (J).
- In the absence of timelike eigenvectors:
  * If A has a null eigenvector with non-zero eigenvalue, then its normal form is of type (S).
  * If A has a null eigenvector with eigenvalue zero, then its normal form is of type (T).

These types will be defined and discussed separately in Sections 9.3.4–9.3.6. After those discussions, the following will be apparent:

(a) The three types of normal forms are mutually exclusive.
(b) Having a null eigenvector with *non-zero* eigenvalue automatically rules out timelike eigenvectors; hence the assumption about timelike eigenvectors being absent is not needed in the type (S) case.
(c) On the other hand, the absence of timelike eigenvectors is essential for the type (T) normal form to surface.

9.3.4. In the presence of a timelike eigenvector for A. Call this eigenvector U; by item (4) of Section 9.3.2, its eigenvalue must be 0. This puts U in the null space of A and hence that of Ω. Since the latter is real, U can be chosen real. Being timelike, the first component u_0 of U cannot vanish. Replace U by $-U$ if necessary to effect $u_0 > 0$, and scale U to unit length.

Set $\mathcal{U} := \text{span}\{U\}$. Since U is timelike, $\langle\, ,\, \rangle_{|\mathcal{U}^\perp}$ is non-degenerate by Lemma 9.1. According to Section 9.1, $\mathcal{U}^\perp$ then admits a $\langle\, ,\, \rangle$ orthonormal basis B. All vectors in B must be spacelike, or else $\{U\} \cup B$ contradicts the invariance of $\langle\, ,\, \rangle$'s index. This shows that $\langle\, ,\, \rangle_{|\mathcal{U}^\perp}$ is positive-definite. Hence the analysis of $A_{|\mathcal{U}^\perp}$ reduces to the compact case considered in Section 9.2. So, there is a real orthonormal basis B for $\mathcal{U}^\perp$, with respect to which $A_{|\mathcal{U}^\perp}$ has the normal form $ia_1 J \oplus \cdots \oplus ia_k J \oplus 0_{n-2k}$.

The collection $\mathcal{B} := \{U\} \cup B$ is a real $\langle\, ,\, \rangle$ orthonormal basis which puts Ω into the normal form $\tilde{\Omega} := 0 \oplus a_1 J \oplus \cdots \oplus a_k J \oplus 0_{n-2k}$, with $a_1 \geqslant \cdots \geqslant a_k > 0$. Denote also by $\mathcal{B}$ the matrix whose columns are the vectors in our real $\langle\, ,\, \rangle$ orthonormal basis. Then, $\tilde{\Omega} = g\Omega g^{-1}$, where $g := \mathcal{B}^{-1}$. Since $u_0 > 0$, (the matrix $\mathcal{B}$ and hence) g belongs to $O_+(1, n)$. Suppressing the rank of Ω gives the following "type (J)" normal form

$$\text{for } n \text{ even,} \quad \tilde{\Omega} = 0 \oplus a_1 J \oplus \cdots \oplus a_m J \qquad \text{with } m = n/2,$$
$$\text{for } n \text{ odd,} \quad \tilde{\Omega} = 0 \oplus a_1 J \oplus \cdots \oplus a_m J \oplus 0 \quad \text{with } m = (n-1)/2.$$

Here, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 0$.

9.3.5. When A has a null eigenvector with non-zero eigenvalue. Take any such null eigenvector and call it X. According to item (6) of Section 9.3.2, the eigenvalue in question has the form ia with $0 \neq a \in \mathbb{R}$, and X can be chosen as $(1, x)$, where x is real and $|x|^2 = 1$. Incidentally, item (2) of Section 9.3.2 characterizes x by the equations $a = C^t x$ and $ax = C - Qx$.

There is, in fact, a companion real null eigenvector Y with the standardized form $(1, y)$, and which has eigenvalue $-ia$. To see this, it suffices to solve $-a = C^t y$ and $-ay = C - Qy$ for a real y. These equations and $a \neq 0$ then imply $|y|^2 = y^t y = 1$.

Since $Q^t = -Q$, we can rewrite the second equation as $y^t(Q + aI) = -C^t$. Also, $Q + aI$ is invertible because the spectrum of Q is pure imaginary (Section 9.2). Thus $y^t = -C^t(Q+aI)^{-1}$, which is real because Q and C are. Finally, with the help of the hypothesized x, we have $C^t y = y^t C = y^t(Q + aI)x = -C^t x = -a$. This proves that the asserted Y exists. (Since y is not a multiple of x, we have $\langle X, Y\rangle = -1 + x \cdot y < -1 + |x|\,|y| = 0$; thus X, Y are *not* $\langle\, ,\, \rangle$ orthogonal.)

By interchanging X with Y if necessary, we may assume that $a > 0$. For later purposes, relabel it as a_1. Define $U := X + Y = (2, x + y)^t$ and $V := X - Y = (0, x - y)^t$. Observe that

* $\langle U, U\rangle = 2(-1 + x \cdot y) < 0$ and $\langle V, V\rangle = 2(1 - x \cdot y) > 0$,
* U and V are $\langle\, ,\, \rangle$ orthogonal,
* $AU = ia_1 V$ and $AV = ia_1 U$. Since $|\langle U, U\rangle| = \langle V, V\rangle$, that pair of equations remains valid for the normalized vectors $\hat{U}$ and $\hat{V}$.

Set $\mathcal{W} := \operatorname{span}\{\hat{U}, \hat{V}\}$. Since $\hat{U}$ is timelike, a (by now) familiar argument shows that $\langle\, ,\, \rangle$ becomes positive definite on the $(n-1)$-dimensional $\mathcal{W}^\perp$, which is invariant under the self-adjoint A. In view of Section 9.2, there is a real orthonormal basis B for $\mathcal{W}^\perp$, with respect to which the restricted A has the normal form $ia_2 J \oplus \cdots \oplus ia_k J \oplus 0_{n-1-2(k-1)}$.

The collection $\mathcal{B} := \{\hat{U}, \hat{V}\} \cup B$ is a real $\langle\, ,\, \rangle$ orthonormal basis which puts Ω into the normal form $\tilde{\Omega} := a_1 S \oplus a_2 J \oplus \cdots \oplus a_k J \oplus 0_{n+1-2k}$, where

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $a_1 > 0$, $a_2 \geqslant \cdots \geqslant a_k > 0$. Denote also by $\mathcal{B}$ the matrix whose columns are the vectors in our real $\langle\, ,\, \rangle$ orthonormal basis. Then $\tilde{\Omega} = g\Omega g^{-1}$, where $g := \mathcal{B}^{-1}$. Since the first component of $\hat{U}$ is positive, (the matrix $\mathcal{B}$ and hence) g belongs to $O_+(1,n)$. Suppressing the rank of Ω gives the following "type (S)" normal form

$$\begin{array}{lll} \text{for } n \text{ even}, & \tilde{\Omega} = a_1 S \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0 & \text{with } m = n/2, \\ \text{for } n \text{ odd}, & \tilde{\Omega} = a_1 S \oplus a_2 J \oplus \cdots \oplus a_m J & \text{with } m = (n+1)/2. \end{array}$$

Here, $a_1 > 0$ and $a_2 \geqslant \cdots \geqslant a_m \geqslant 0$.

This normal form explains why there was no need to hypothesize the absence of timelike eigenvectors here. Indeed, any such eigenvector would have to have zero eigenvalue (by item (4) of Section 9.3.2), putting it in the null space of $\tilde{\Omega}$. But then, its first two components would have to vanish (on account of $a_1 S$), which is incompatible with being timelike.

9.3.6. When A has a null eigenvector with zero eigenvalue but no timelike eigenvector. Let V be such an eigenvector of $A = i\Omega$. Since $\Omega V = 0$ and Ω is real, V can be chosen real. Being null, V must have non-zero first component; hence it can be standardized into the form $(1, v)$, where v is real and $v \cdot v = 1$. By (2) of Section 9.3.2, we also have $Qv = C$ and $C \cdot v = 0$. Section 9.3.1 says there is no loss of generality in assuming that Q and C have already been simplified to $q_1 J \oplus \cdots \oplus q_h J \oplus 0_{n-2h}$ and $(D_1, \ldots, D_h, 0, \ldots, 0, \xi)$, respectively. Here, $q_1 \geqslant \cdots \geqslant q_h > 0$ and $D_j = [C_{2j-1}, C_{2j}]$. The hypothesized existence of V implies that $Qv = C$ admits a solution. Hence C is in the range of Q and ξ must vanish. The use of $J^2 = -I$ solves the equation $Qv = C$ to give

$$v = \left(\frac{-JD_1}{q_1}, \ldots, \frac{-JD_h}{q_h}, v_{2h+1}, \ldots, v_n \right).$$

This v automatically satisfies $C \cdot v = 0$ because of the skew-symmetry of J, and its last $n - 2h$ components are constrained by the requirement $v \cdot v = 1$.

For further discussions, set

$$z := \left(\frac{-JD_1}{q_1}, \ldots, \frac{-JD_h}{q_h}, 0, \ldots, 0 \right).$$

The null space $\mathcal{N}_1$ of $A = i\Omega$ consists of eigenvectors $U = (u_0, u)$ with eigenvalue 0, which are characterized by $Qu = u_0 C$ and $C^t u = 0$. Since Ω is real, U may be chosen to be real. A calculation like the one above tells us that $\mathcal{N}_1$ admits a basis $\{(1, z), (0, e_j), j = 2h+1, \ldots, n\}$, where e_j has a 1 in the jth entry, and 0 elsewhere. In particular, $(1, z)$ is an eigenvector of A with eigenvalue 0.

If $(1, z)$ were not null, then the components $v_{2h+1}, \ldots, v_n$ of the hypothesized null eigenvector $(1, v)$ could not all be zero, whence $|z|^2 < |v|^2 = 1$. This would force the eigenvector $(1, z)$ to be timelike, a scenario forbidden by our hypothesis. Thus, $(1, z)$ has to be null; that is, $z \cdot z = 1$. Since $|JD_i| = |D_i|$, the condition $z \cdot z = 1$ is equivalent to

$$(*) \qquad \frac{|D_1|^2}{q_1^2} + \cdots + \frac{|D_h|^2}{q_h^2} = 1.$$

In particular, some $|D_j|^2$ must be positive.

Introduce the column vectors (written here as rows)

$$z_1 := \left(\frac{D_1}{q_1^2}, \ldots, \frac{D_h}{q_h^2}, 0, \ldots, 0 \right), \qquad z_2 := \left(\frac{JD_1}{q_1^3}, \ldots, \frac{JD_h}{q_h^3}, 0, \ldots, 0 \right).$$

Let $\mathcal{N}_i$ be the null space of A^i, equivalently that of Ω^i. Abbreviate the vectors $(1, z), (0, e_j), j = 2h+1, \ldots, n$ collectively as B_0. Using the simplified form of Ω (Section 9.3.1) with $\xi = 0$ (as explained above), we get:

$$\begin{aligned} &\mathcal{N}_1 = \operatorname{span}\{B_0\}, \\ &\mathcal{N}_2 = \operatorname{span}\{(0, z_1), B_0\}, \\ &\mathcal{N}_3 = \operatorname{span}\{(0, z_2), (0, z_1), B_0\}; \\ &\mathcal{N}_p = \mathcal{N}_3 \text{ for any } p \geqslant 3. \end{aligned}$$

The first three follow from $Qz = C$, $Qz_1 = -z$, $Qz_2 = -z_1$, and $C \cdot z = 0$, $C \cdot z_1 = 1$, $C \cdot z_2 = 0$. The fourth is essentially due to the fact that, while certainly there is a z_3 such that $Qz_3 = -z_2$, it is unable to satisfy $C \cdot z_3 = 0$ because $(*)$ above implies that $|D_j|^2 > 0$ for some j. The union of all the $\mathcal{N}_i$ is the generalized null space $\mathcal{N}$ of A. It is invariant under A.

Normalize $(0, z_1)$, $(0, z_2)$ to yield two real $\langle\, , \rangle$ orthonormal spacelike vectors X_1, X_2. A routine calculation produces the unit timelike real vector

$$X_0 := \frac{|z_2|}{|z \cdot z_2|}(1, z) + X_2 = \frac{\sqrt{\sum_{i=1}^h |D_i|^2/q_i^6}}{\sqrt{\sum_{i=1}^h |D_i|^2/q_i^4}}\,(1, z) + X_2,$$

which is $\langle\, , \rangle$ orthogonal to X_1, X_2. Also, with $a_1 := |z_1|/|z_2|$, we have $AX_0 = ia_1 X_1$, $AX_1 = ia_1(X_0 - X_2)$, and $AX_2 = ia_1 X_1$. Let B_1 be the real $\langle\, , \rangle$ orthonormal basis $\{X_0, X_1, X_2, (0, e_j), j = 2h+1, \ldots, n\}$ for the generalized null space $\mathcal{N}$. With respect to B_1, the matrix of $A_{|\mathcal{N}}$

has the form $ia_1 T \oplus 0_{n-2h}$, where

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Correspondingly, the matrix of $\Omega_{|\mathcal{N}}$ is $a_1 T \oplus 0_{n-2h}$, with $a_1 > 0$.

Since X_0 is timelike, a (by now) familiar argument shows that $\langle\, , \rangle$ becomes positive definite on $\mathcal{N}^\perp$, which is invariant under the self-adjoint A. By Section 9.2, there is a real orthonormal basis B_2 for $\mathcal{N}^\perp$ which puts $A_{|\mathcal{N}^\perp}$, and hence $\Omega_{|\mathcal{N}^\perp}$, into normal form. Incidentally, this normal form must look like $a_2 J \oplus \cdots \oplus a_{m'} J$, where $a_2 \geqslant \cdots \geqslant a_{m'} > 0$, because the kernel of Ω has already been accounted for in $\mathcal{N}$.

Let $\mathcal{B} := \{X_0, X_1, X_2\} \cup B_2 \cup \{(0, e_j), j = 2h+1, \ldots, n\}$. Denote also by $\mathcal{B}$ the matrix whose columns are the vectors in this real $\langle\, , \rangle$ orthonormal basis. Then, the normal form of Ω is $\tilde{\Omega} = g \Omega g^{-1}$, where $g := \mathcal{B}^{-1}$. Since the first component of X_0 is positive, (the matrix $\mathcal{B}$ and hence) g belongs to $O_+(1,n)$. Suppressing the rank of Ω gives the following "type (T)" normal form

$$\begin{array}{lll} \text{for } n \text{ even}, & \tilde{\Omega} = a_1 T \oplus a_2 J \oplus \cdots \oplus a_m J & \text{with } m = n/2, \\ \text{for } n \text{ odd}, & \tilde{\Omega} = a_1 T \oplus a_2 J \oplus \cdots \oplus a_m J \oplus 0 & \text{with } m = (n-1)/2. \end{array}$$

Here, $a_1 > 0$ and $a_2 \geqslant \cdots \geqslant a_m \geqslant 0$.

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