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THE SPACE OF KÄHLER METRICS II

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Abstract

This paper, the second of a series, deals with the function space $\mathcal{H}$ of all smooth Kähler metrics in any given n-dimensional, closed complex manifold V, these metrics being restricted to a given, fixed, real cohomology class, called a polarization of V. This function space is equipped with a pre-Hilbert metric structure introduced by T. Mabuchi [10], who also showed that, formally, this metric has nonpositive curvature. In the first paper of this series [4], the second author showed that the same space is a path length space. He also proved that $\mathcal{H}$ is geodesically convex in the sense that, for any two points of $\mathcal{H}$, there is a unique geodesic path joining them, which is always length minimizing and of class $C^{1,1}$. This partially verifies two conjectures of Donaldson [8] on the subject. In the present paper, we show first of all, that the space is, as expected, a path length space of nonpositive curvature in the sense of A. D. Aleksandrov. A second result is related to the theory of extremal Kähler metrics, namely that the gradient flow in $\mathcal{H}$ of the "K energy" of V has the property that it strictly decreases the length of all paths in $\mathcal{H}$, except those induced by one parameter families of holomorphic automorphisms of M.

1. Introduction and main results

1.1 Riemannian metrics and nonpositively curved space.

Let (V, ω_0) be an n-dimensional, compact Kähler manifold without boundary. Consider the space of Kähler distortion potentials

$$\mathcal{H} = \{\varphi \in C^\infty(V) : \omega_\varphi = \omega_0 + \partial\overline{\partial}\varphi > 0 \text{ on } V\}.$$

Clearly, each fibre of the tangent space $T\mathcal{H}$ of $\mathcal{H}$ is $C^\infty(V)$, and each Kähler metric ω_φ defined by any potential $\varphi \in \mathcal{H}$ defines a measure

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$d\,\mu_\varphi = \frac{1}{n!}\omega_\varphi^n$. The L_2 norm of functions in V determined by the measure $d\,\mu_\varphi$ and applied to $T\mathcal{H}$ defines a Weil-Peterson type metric on the function space $\mathcal{H}$ (see Section 2.1 for historical remarks on this metric).

In 1997, following a program of Donaldson [8], the second author proved that this space is convex with respect to geodesics of class $C^{1,1}$, and used this fact to prove that such geodesics (and only they) achieve the infimum length of all paths joining their end points. The strongest result on these geodesics established so far is that they are of class $C^{1,1}$. If one could prove that they are, as expected, of class at least C^4, then the formal calculations in [10] would imply that the curvature of $\mathcal{H}$ is nonpositive in the geometric sense of A. D. Aleksandrov. It is an interesting question whether one could prove that the geometric curvature of $\mathcal{H}$ is nonpositive regardless of the regularity of geodesics. The following theorem confirms that this is indeed the case.

Theorem 1.1. *The space of Kähler potentials in any given, polarized manifold V is a nonpositive curved space in the following sense. Let A, B, C be three points in the space of Kähler potentials and denote $d(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbf{R}^+$ the distance (in terms of the metric aforementioned). For any λ, with $0 \leq \lambda \leq 1$, let P_λ be the point on a geodesic path joining B and C with $d(B, P_\lambda) = \lambda d(B, C)$ and $d(P_\lambda, C) = (1-\lambda)d(B, C)$. Then the following inequality holds:*

$$d(A, P_\lambda)^2 \leq (1-\lambda)d(A, B)^2 + \lambda d(A, C)^2 - \lambda \cdot (1-\lambda)d(B, C)^2.$$

In [4], the second author proved that every geodesic segnent achieves the minimum length of all paths connecting its end points. However, the problem remained unsettled whether any length minimizing sequence of paths joining any two points of $\mathcal{H}$ must include a subsequence converging to the unique geodesic. That question is now settled, according to the following statement:

Theorem 1.2. *Let $\varphi_i \in \mathcal{H}(i = 1, 2)$ be two Kähler distortion potentials, generating two corresponding Kähler metrics in V, and let $\{C_i\}$ be any sequence of paths in $\mathcal{H}$ from φ_0 to φ_1 and whose length converge to the least possible limit. Then $\{C_i\}$ converges, in the topology of Hausdorff distance, to the unique geodesic of class $C^{1,1}$ from φ_0 to φ_1.*

1.2 The gradient flow

The notion of extremal Kähler metrics was first introduced in [2] by the first author of the present article. In order to attack the problem

of the existence of extremal Kähler metrics, he proposed a "steepest descent" process, consisting of the initial value problem corresponding to the following parabolic equation of order four:

$$\frac{\partial \varphi}{\partial s} = R(\varphi) - \underline{R}. \tag{1.1}$$

Here $R(\varphi)$ denotes the scalar curvature of the Kähler metric ω_φ and $\underline{R}$ is the average value over V — a constant depending only the polarized class. The flow in $\mathcal{H}$ represented by the solutions of (1.1) has sometimes been refered as the Calabi flow. In general, a solution to the initial value problem for (1.1) is known to exist for a short interval of the "time" paremeters. In the case of Riemann surfaces ($n = 1$), Chrusciel [7] proved, using the fact that metric of constant curvature are known *a prori* to exist, that the solution of (1.1) exists for all s and converges at infinity.[1] It is remarkable that, under the flow in $\mathcal{H}$ represented by solutions of (1.1), each smooth path in $\mathcal{H}$ has decreasing length, as stated more precisely below.

Theorem 1.3. *Let $\varphi_i \in \mathcal{H}(i = 0, 1)$ be two Kähler distortion potentials, generating two corresponding Kähler metrics in V, and let $\varphi(t)(0 \leq t \leq 1)$ be a smooth path in $\mathcal{H}$ from $\varphi(0) = \varphi_0$ to $\varphi(1) = \varphi_1$. Suppose that, for some constant $s_0 > 0$, there is a smooth family of Kähler distortion potentials $\phi(s,t)(0 \leq s \leq s_0, 0 \leq t \leq 1)$ such that, for each $t \in [0,1]$, $\phi(s,t)$ satisfies th initial value problem $\frac{\partial \phi}{\partial s} = R(\phi) - \underline{R}$, $\phi(0,t) = \varphi(t)$. Then the length $L(s)$ of the path $\{\phi(s,t) : 0 \leq t \leq 1\}$ in $\mathcal{H}$ is strictly decreasing function of s, except for the case where the complex vector field $g^{\lambda\overline{\mu}}(\phi(s,t))\frac{\partial^2 \phi(s,t)}{\partial t \partial \overline{z}_\mu}\frac{\partial}{\partial z^\lambda}$ in V is holomorphic for each (s,t) in a neighborhood of a segement $\{(s,t) : 0 \leq t \leq 1\}$.*

The proof of this statement consists of a formal calculation involving the second order differential operator on real or complex functions, defined below and named after André Lichnérowicz.

Definition 1.4. For any smooth function $f : V \to \mathbf{C}$ and any Kähler metric g in V, the Lichnérowicz operator D is defined in local terms by $D_g f = \sum_{\alpha,\beta=1}^{n} f_{,\alpha\beta} dz^\alpha \otimes dz^\beta$ where the local functions $f_{,\overline{\alpha}\overline{\beta}}$

[1] The second author gave a new geometric proof to Chrusciel's theorem [6]. Following the approach taken in [6], M. Struwe [13] gave a unified treatment of both Ricci flow and Calabi flow in Riemann surfaces. For higher dimensional Kähler manifolds, very little is known about the global existence of the flow.

are components of the second covariant derivatives of f in terms of g:

$$f_{,\overline{\alpha}\overline{\beta}} = \sum_{\lambda,\mu=1}^{n} g_{\lambda\overline{\alpha}} \frac{\partial}{\partial \overline{z}_\beta} \left(g^{\lambda\overline{\mu}} \frac{\partial f}{\partial \overline{z}_\mu} \right).$$

The length $L(s)$ of the path $\{\phi(s,t) : 0 \leq t \leq 1\}$ in $\mathcal{H}$ is, by definition,

$$L(s) = \int_0^1 \left(\int_V \left(\frac{\partial \phi(s,t)}{\partial t} \right)^2 d\,\mu_{\phi(s,t)} \right)^{\frac{1}{2}} d\,t.$$

If one differentiates both sides with respect to s, one readily obtains from the standard variational formulas the desired expression

$$\frac{d\,L}{d\,s} = -\int_0^1 \left(\int_V \left| D \frac{\partial \phi}{\partial t} \right|^2_{\phi(s,t)} d\,\mu_{\phi(s,t)} \right) \cdot \left(\int_V \left| \frac{\partial \phi}{\partial t} \right|^2 d\,\mu_{\phi(s,t)} \right)^{-\frac{1}{2}} d\,t.$$

Theorem 1.5. *The following two statements are immediate consequences of Theorem* 1.3*.*

1. *If the flow in $\mathcal{H}$ generated by* (1.1) *exists for all time for any smooth initial data, then the distance between any two Kähler metrics decreases under that flow.*

2. *If the K-energy*[2] *is a weakly convex function along geodesics,*[3] *and in particular if the first Chern class of V is nonpositive, then the same flow decreases the distance between metrics.*

The flowing question/conjecture is highly interesting:

Question/Conjecture 1.6. *Is the distance function in $\mathcal{H}$ strictly decreasing under the gradient flow* (1.1)*, in particular, if the two metrics are not equivalent up to some holomorphic isometric transformation?*

Remark 1.7. The only conceivable paths in $\mathcal{H}$ that do not shrink under the flow under discussion are those induced by paths in the group

[2]The K-energy is defined by T. Mabuchi in 1987 [10] while the flow was introduced by the first author in 1982 [2]. And it is commonly known as the "Calabi flow" in the literature.

[3]A function $f(t)(0 \leq t \leq 1)$ is weakly convex if for any t, we have $f(t) \leq (1-t)\, f(0) + t\, f(1)$. For this theorem, we additionally assume that f is differentiable at both end points, i.e., $f'(1) \geq f'(0)$.

of holomorphic transformations of V that are of *hyperbolic* type at each point. A path in that group is said to be of hyperbolic type with respect to a given Kähler metric if the holomorphic vector field in V represented by the tangent to a given Kähler metric is a multiple of a vector field generating a group of holomorphic isometries. A rigorous verification of this conjecture would require locally uniform estimates of the regularity of the flow (1.1), independent of second derivatives of the initial metrics.

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2. $\mathcal{H}$ is a nonpositively curved space

In this section, we want to show that $\mathcal{H}$ is a nonpositively curved space in the sense of Aleksandrov.

2.1 A Riemannian metric in the infinite dimensional space.

In 1987, T. Mabuchi introduced ([10]) the infinite dimensional Riemannian (or pre-Hilbert) structure, in the function space of Kähler metrics in V and showed that the space with this metric is, *formally*, a locally symmetric space with nonpositive curvature. The same metric was rediscovered, apparently unaware of Mabuchi's work, by S. Semmes [12] and Donaldson [8], each describing it from somewhat different viewpoints. We shall briefly review the definition and elemntary properties of this metric.

For each Kähler metric g in V, any smooth, real-valued function ψ in V represents a tangent vector to the space of all Kähler metrics.[4] Thus any "basis" $\{\psi_\lambda\}$ for the space of all smooth functions, attached to each Kähler distortion potential $\varphi_0 \in \mathcal{H}$, constitutes a global field of tangential frames. Since any two of these vector fields obviously

[4]Since constants, and only constants, represent trivial infinitesimal deformations of metric, it is convenient to replace the actual space of Kähler metrics with the product of $\mathcal{H}$ with the real line $\mathbf{R}$, so that, for each metric g, constants act by translation on $\mathbf{R}$ alone, while function φ orthogonal to constants are regarded as infinitesimal Kähler distortion potentials, acting trivially on $\mathbf{R}$.

commute, the total frame may be considered, formally, as a sort of a global, flat affine coordinate system for $\mathcal{H}$, in the following sense: the "point" $\varphi_0 \in \mathcal{H}$ being the origin, the numerical range of the coordinates consists of the set of arrays (y^λ) such that the function $\varphi_1 = \varphi_0 + \sum_\lambda y^\lambda \psi_\lambda$ satisfies the condition to be the Kähler distortion potential for a Kähler metric, that is then assigned to the array (y^λ).

Given any tangent vector on $\mathcal{H}$, represented by a smooth function $\psi : \mathcal{H} \times V \to \mathbf{R}$, the square length of ψ at the point $\varphi \in \mathcal{H}$ is defined to be

$$\|\psi\|_\varphi^2 = \int_V \psi^2 \, d\mu_\varphi,$$

where $d\mu_\varphi = \frac{1}{n!}\omega_\varphi^n$. The corresponding symmetric bilinear form for any two tangent "vectors" f_1, f_2 at the point $\varphi \in \mathcal{H}$, is written as

$$\langle f_1, f_2 \rangle_\varphi = \int_V f_1 \cdot f_2 \, d\mu_\varphi.$$

When no confusion is arisen, we just write

$$\langle f_1, f_2 \rangle = \int_V f_1 \cdot f_2 \, d\mu_\varphi.$$

The "tensor notation" for the metric, in terms of the global tangent frame $\{\psi_\lambda\}$ is then $\mathcal{G}_{\psi_\lambda \, \psi_\mu} = \langle \psi_\lambda, \psi_\mu \rangle$, $d\mathcal{S}^2 = \mathcal{G}_{\psi_\lambda \, \psi_\mu} \, dy^\lambda d\, y^\mu$.

One can calculate directly the Christoffel symbols of the first kind by differentiating formally $\}_{\psi_\lambda \, \psi_\mu}$ in the direction represented by any ψ_ν:

$$\partial_{\psi_\nu} \mathcal{G}_{\psi_\lambda \, \psi_\mu} = \int_V \psi_\lambda \psi_\mu \triangle_\varphi \psi_\nu \, d\mu_\varphi,$$

where

$$\begin{aligned}
\{\psi_\lambda, \psi_\mu; \psi_n u\} &= \frac{1}{2} \left(\partial_{\psi_\lambda} \mathcal{G}_{\psi_\nu \, \psi_\mu} + \partial_{\psi_\nu} \mathcal{G}_{\psi_\lambda \, \psi_\mu} - \partial_{\psi_\nu} \mathcal{G}_{\psi_\lambda \, \psi_\mu} \right) \\
&= \frac{1}{2} \int_V \left(\psi_\mu \psi_\nu \triangle_\varphi \psi_\lambda \; + \psi_\lambda \psi_\nu \triangle_\varphi \psi_\mu \; - \psi_\lambda \psi_\mu \triangle_\varphi \psi_\nu \; \right) \, d\mu_\varphi \\
&= -\frac{1}{2} \int_V \psi_\lambda g_\varphi^{\alpha \bar{\beta}} \left(\frac{\partial \psi_\mu}{\partial z_\alpha} \frac{\partial \psi_\nu}{\partial \bar{z}_\beta} + \frac{\partial \psi_\nu}{\partial z_\alpha} \frac{\partial \psi_\mu}{\partial \bar{z}_\beta} \right) \, d\mu_\varphi.
\end{aligned}$$

The last expression then yields the covariant derivative formula for tangent vector fields ψ_λ:

$$\nabla_{\psi_\mu}\psi_\lambda(\varphi) = \nabla_{\psi_\lambda}\psi_\mu(\varphi) = \frac{1}{2}g_\varphi^{\alpha\bar{\beta}}\left(\frac{\partial\psi_\mu}{\partial z_\alpha}\frac{\partial\psi_\lambda}{\partial\bar{z}_\beta} + \frac{\partial\psi_\lambda}{\partial z_\alpha}\frac{\partial\psi_\mu}{\partial\bar{z}_\beta}\right).$$

Similarly, the equation for a geodesic path $\varphi(t)$ in $\mathcal{H}$ in terms of a parameter $t(0 \le t \le 1)$ proportional to the arc length is

$$\frac{\partial^2\varphi}{\partial t^2} = g_\varphi^{\alpha\bar{\beta}}\frac{\partial^2\varphi}{\partial t\partial z_\alpha}(t)\frac{\partial^2\varphi}{\partial t\partial\bar{z}_\beta}(t) = \frac{1}{2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|^2_{\varphi(t)}, \tag{2.1}$$

where the norm of the gradient in the right-hand side is taken in V with respect to the metric associated with the form $\omega_{\varphi(t)}$. One may derive the same equation, alternatively, as the Euler-Lagrange equation for the variational problem of minimizing the energy integral

$$\int_0^1 \left\langle\frac{\partial\varphi}{\partial t}(t), \frac{\partial\varphi}{\partial t}(t)\right\rangle^2_{\varphi(t)} dt = \int_0^1\int_V \left(\frac{\partial\varphi}{\partial t}\right)^2 d\mu_{\varphi(t)}\, dt,$$

over the set of smooth paths in $\mathcal{H}$ connecting the fixed end points $\varphi(0)$ and $\varphi(1)$. A calculation with the derivatives of the Christoffel symbols, analogous to the familiar formalism of the Ricci calculus, enables us to verify, *formally*, that each 2-dimensional plane in the tangent bundle of $\mathcal{H}$ has nonpositive curvature. The details may be found in [10], [12] and [8].

In [8], S. K. Donaldson also formulated serval important conjectures about geometric properties and their possible relation with properties of the complex manifold V. In 1997, the second author proved some of his conjectures, whose contents are summarized by the following theorem:

Theorem B ([4])**.** *The following statements are true:*

1. *The space of Kähler potentials $\mathcal{H}$ is convex by $C^{1,1}$ geodesics. More specifically, if $\varphi_0, \varphi_1 \in \mathcal{H}$ and $\varphi(t)$ $(0 \le t \le 1)$ is a geodesic connecting these two points in $\mathcal{H}$, then the second order mixed covariant derivatives of $\varphi(t)$ are uniformly bounded from above.*

2. *$\mathcal{H}$ is a metric space.*[5] *In other words, the infimum of the lengths of all possible curves between any two different points in $\mathcal{H}$ is strictly positive.*

3. *If $C_1(V) < 0$, then the extremal Kähler metric is unique in each Kähler class.*

[5]This is a conjecture of Donaldson[8].

2.2 A lemma on approximate geodesics

In a local coordinate system of V, let $\omega_0 = \sum_{\alpha,\beta=1}^{n} {g_0}_{\alpha\bar{\beta}}\, dz_\alpha\, \overline{dz_\beta}$ and

$$\omega_\varphi = \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}\, dz_\alpha\, \overline{dz_\beta}, \qquad \text{where} \qquad g_{\alpha\bar{\beta}} = {g_0}_{\alpha\bar{\beta}} + \frac{\partial^2\varphi}{\partial z_\alpha \partial\overline{z_\beta}}.$$

In this subsection, $z_1, z_2, \ldots, z_n$ are local coordinates in V. For any path $\varphi(\cdot, t) : [0,1] \to \mathcal{H}$, we can view it as a function defined in the product manifold $V \times [0,1]$. Following an idea of S. Semmes, we introduce a dummy variable θ such that $V \times ([0,1] \times S^1)$ is a $(n+1)$-dimensional Kähler manifold and $t = \mathrm{re}(z_{n+1})$. Here S^1 is the unit circle, and we always use the following notations:

$$i, j, k = 1, 2, \ldots, n, n+1 \qquad \text{and} \qquad \alpha, \beta, \gamma = 1, 2, \ldots, n.$$

Consider the projection

$$\begin{aligned} \pi : V \times ([0,1] \times S^1) &\to V \\ (z, t, \theta) &\to z. \end{aligned}$$

Consider the pullback metric $\pi^* g_0$. Note that $\pi^*\omega_0$ is a degenerate Kähler form of co-rank 1 in $V \times ([0,1] \times S^1)$.

Definition 2.1. A path $\varphi(t)(0 < t < 1)$ in $\mathcal{H}$ is a convex path if

$$\det\left(\pi^* {g_0}_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i \partial\overline{z_j}}\right)_{(n+1)(n+1)} > 0, \qquad \text{in } V \times (I \times S^1).$$

Definition 2.2. A convex path $\varphi(t)$ in the space of Kähler metrics is called an ϵ-approximate geodesic if the following holds:

(2.2)
$$\begin{aligned} \det\left(\pi^* {g_0}_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i \partial\overline{z_j}}\right)_{(n+1)(n+1)} &= \left(\frac{\partial^2\varphi}{\partial t^2} - \frac{1}{2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|^2_{g(t)}\right) \det\, g(t) \\ &= \epsilon \cdot \det\, g_0 \end{aligned}$$

where $g(t)_{\alpha\bar{\beta}} = {g_0}_{\alpha\bar{\beta}} + \frac{\partial^2\varphi}{\partial z_\alpha \partial\overline{z}_\beta}$ $(1 \le \alpha, \beta \le n)$.

Also from [4], we have the following, which plays a crucial role in the proof in the next subsection.

Lemma 2.3 (Geodesic approximation lemma, [4]). *Suppose given two smooth curves $\phi_1(\cdot,s),\phi_2(\cdot,s):[0,1]\to\mathcal{H}$. For ϵ_0 small enough, there exist two-parameter smooth families of curves*

$$\varphi(\cdot,t,s,\epsilon):[0,1]\times[0,1]\times(0,\epsilon_0](0\le t,s\le 1,\ \text{and } 0<\epsilon\le\epsilon_0)\to\mathcal{H}$$

(from $\phi_1(\cdot,s)$ to $\phi_2(\cdot,s)$) such that the following properties hold:

1. *For any fixed s and ϵ, $\varphi(\cdot,t,s,\epsilon)$ is an ϵ-approximate geodesic connecting $\varphi_1(\cdot,s)$ and $\varphi_2(\cdot,s)$. More precisely, $\varphi(\cdot,t,s,\epsilon)$ solves the corresponding Monge-Ampere equation*

$$\det\left(\pi^*g_{0\,i\bar{j}}+\frac{\partial^2\varphi}{\partial z_i\partial\overline{z}_j}\right)=\epsilon\cdot\det(g_0),\ \text{in } V\times\mathbf{R}; \tag{2.3}$$

   and

$$\varphi(\cdot,0,s,\epsilon)=\phi_1(\cdot,s),\quad \varphi(\cdot,1,s,\epsilon)=\phi_2(\cdot,s).$$

   Here φ is independent of $\mathrm{Im}(z_{n+1})$.

2. *There exists a uniform constant C which depends only on $\phi_1(\cdot,s),\phi_2(\cdot,s)$ such that*

$$|\varphi|+\left|\frac{\partial\varphi}{\partial s}\right|+\left|\frac{\partial\varphi}{\partial t}\right|<C;\qquad 0<\frac{\partial^2\varphi}{\partial t^2}<C,\qquad \frac{\partial^2\varphi}{\partial s^2}<C.$$

3. *For fixed s, as $\epsilon\to 0$, the ϵ-approximating geodesic $\varphi(\cdot,t,s,\epsilon)$ converges to the unique geodesic between $\phi_1(\cdot,s)$ and $\phi_2(\cdot,s)$ in weak $C^{1,1}$ topology.*

4. *Define the energy element along $\varphi(\cdot,t,s,\epsilon)$ by*

$$E(t,s,\epsilon)=\int_V\left|\frac{\partial\varphi}{\partial t}\right|^2 dg(t,s,\epsilon),$$

   where $g(t,s,\epsilon)$ is the corresponding Kähler metric defined by the Kähler potentials $\varphi(t,s,\epsilon)$. Then there exists a uniform constant C such that

$$\max_{t,s}\left|\frac{\partial E}{\partial t}\right|\le\epsilon\cdot C.$$

   In other words, both the energy and length element converge to a constant along each convex curve if $\epsilon\to 0$.

2.3 Length of Jacobi vector field grows supre-linearly

In this subsection, we use the same notation as in Lemma 2.3. We want to prove that the Jacobi vector field along any geodesic grows supre-linearly.

Lemma 2.4. *Let $\varphi(\cdot, t, s, \epsilon)$ be the two-parameter families of approximating geodesics defined as in Lemma 2.3. Let $Y(\cdot, t, s, \epsilon) = \frac{\partial \varphi}{\partial s}$ be the deformation vector fields and $X(\cdot, t, s, \epsilon) = \frac{\partial \varphi}{\partial t}$ the tangential vector fields along the approximating geodesics. Then the second derivatives of Y along the approximating geodesics are positive:*

$$\nabla_X \nabla_X Y \geq 0.$$

Note that Y converges to a Jacobi vector field as $\epsilon \to 0$. Moreover, we have

$$\langle Y, \nabla_X Y \rangle \geq \langle Y, Y \rangle.$$

Proof. The equation for a family of ϵ-approximate geodesics is:

$$\left(\frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2_{g(t)} \right) \det\, g(t) = \epsilon \cdot \det\, g_0, \quad 0 \leq\ s,\ t\ \leq 1.$$

Denote

$$X = \frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial s}, \qquad Y' = \nabla_X Y,$$

and

$$H = \frac{\det\, g_0}{\det\, g}, \qquad f = \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{2} \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2_{g(t)} = \nabla_X X.$$

Then the approximating geodesic equation becomes

$$f = \nabla_X X = \epsilon \cdot H.$$

Note that any two-parameter family of smooth functions $F(s, t)$ can be viewed as a two parameter family of tangent vectors at $T_{\varphi(\cdot,t,s,\epsilon)}\mathcal{H}$. Then the Riemannian metric in $\mathcal{H}$ gives the following covariant derivatives:

$$\nabla_X F = \nabla_{\frac{\partial}{\partial t}} F(s, t) = \frac{\partial F}{\partial t} - \frac{1}{2} \nabla_g \frac{\partial \varphi}{\partial t} \cdot \nabla_g \frac{\partial F(s, t)}{\partial t}$$

and

$$\nabla_Y F = \nabla_{\frac{\partial}{\partial s}} F(s,t) = \frac{\partial F}{\partial s} - \frac{1}{2}\nabla_g \frac{\partial \varphi}{\partial s} \cdot \nabla_g \frac{\partial F(s,t)}{\partial s}.$$

Clearly, as $\epsilon \to 0$, Y is the Jacobi vector field along the geodesic. By definition, the length of Y at t is:

$$|Y|^2(t,s) = \int_V \left|\frac{\partial \varphi}{\partial s}\right|^2 \det\, g.$$

Then

$$\frac{1}{2}\frac{\partial}{\partial t}|Y|^2 = \langle \nabla_X Y, Y\rangle = \langle \nabla_Y X, Y\rangle.$$

Let $K(X,Y)$ denote the sectional curvature of the space of Kähler metrics at point $\varphi(\cdot,t,s,\epsilon)$. By a formal calulation (cf. [10], [12] and [8]), we have[6]

$$K(X,Y) = -|\{X,Y\}_\varphi|_g^2 \le 0.$$

Therefore, we have

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial t^2}|Y|^2 &= \langle \nabla_Y X, \nabla_X Y\rangle + \langle \nabla_X\, \nabla_Y X, Y\rangle \\
&= |Y'|^2 - K(X,Y) + \langle \nabla_Y\, \nabla_X X, Y\rangle \\
&\ge |Y'|^2 + \int_V \epsilon \frac{\partial \varphi}{\partial s} \nabla_{\frac{\partial}{\partial s}} H \ \det\, g \\
&= |Y'|^2 + \int_V \epsilon \frac{\partial \varphi}{\partial s} \cdot \left(\frac{\partial H}{\partial s} - \frac{1}{2}\nabla \frac{\partial \varphi}{\partial s} \cdot \nabla H\right) \det\, g \\
&= |Y'|^2 + \frac{\epsilon}{2}\int_V \left|\nabla \frac{\partial \varphi}{\partial s}\right|^2 H \cdot \det\, g \ \ge |Y'|^2.
\end{aligned}$$

The last equality holds since

$$\frac{\partial H}{\partial s} = \frac{\partial}{\partial s}\left(\frac{\det\, g_0}{\det\, g}\right) = -\triangle_g \frac{\partial \varphi}{\partial s} \cdot H$$

and

$$\begin{aligned}
&-\int_V \frac{\partial \varphi}{\partial s} \triangle_g \frac{\partial \varphi}{\partial s} \cdot H \ \det\, g \\
&\qquad = \frac{1}{2}\int_V \left|\nabla \frac{\partial \varphi}{\partial s}\right|^2 H \cdot \det\, g + \frac{1}{2}\int_V \frac{\partial \varphi}{\partial s} \nabla \frac{\partial \varphi}{\partial s} \nabla H \ \det\, g.
\end{aligned}$$

[6] For any two functions f_1, f_2 and a Kähler form ω_φ, the term $\{f_1, f_2\}_\varphi$ is defined to be the Possion bracket of f_1 and f_2 with respect to the sympletic form ω_φ.

It follows that

$$\frac{\partial^2}{\partial t^2}|Y| \geq 0.$$

In other words, $|Y(t)|(0 \leq t \leq 1)$ is a convex function of t. Since $Y(0) = 0$, we have

$$\frac{\partial}{\partial t}|Y(t)|_{t=1} \geq \frac{|Y(1)|}{1}.$$

or at time $t = 1$

$$\langle Y, Y' \rangle \geq \langle Y, Y \rangle. \tag{2.4}$$

q.e.d.

2.4 Proofs of Theorems 1.1 and 1.2

In this subsection, we want to show that $\mathcal{H}$ is a nonpositively curved space. We follow again the notations in Lemma 2.3 and the preceding subsection.

Proof of Theorem 1.1. Consider a special case of Lemma 2.3 when $\phi_1(\cdot, s) = \phi_1$ is one point on $\mathcal{H}$ (instead of a curve). We denote this point as P. Let $Q = \phi_2(\cdot, 0) \in \mathcal{H}$ and $R = \phi_2(\cdot, 1) \in \mathcal{H}$. Furthermore, we assume that $\phi_2(\cdot, s)$ (denoted as by QR) is an ϵ-approximate geodesic connecting Q and R. In other words, it satisfies the following equation:

$$\nabla_Y Y \cdot \det\, g = \left(\frac{\partial^2 \varphi}{\partial t^2}_{ss} - \frac{1}{2} \left| \nabla \frac{\partial \varphi}{\partial t}_s \right|_g^2 \right) \det\, g = \epsilon \cdot \det\, g_0.$$

Let $Q(s)$ denote the point $\phi_2(\cdot, s)$ and denote by $E(s)$ the energy of the ϵ-approximate geodesic from P to $Q(s)$. As $\epsilon \to 0, E(s) \to$ a constant which, by our normalization, is the square of the geodesic distance from P to $Q(s)$. Thus it is enough to work with $E(s)$. Next

$$E(s) = \int_0^1 \langle X, X \rangle dt = \int_0^1 \int_V \left(\frac{\partial \varphi}{\partial t} \right)^2 \det\, g\, dt$$

and

$$E(QR) = \int_0^1 \langle Y, Y \rangle ds = \int_0^1 \int_V \left(\frac{\partial \varphi}{\partial s} \right)^2 \det\, g\, ds.$$

Thus

$$\begin{aligned}\frac{1}{2}\frac{d\,E(s)}{d\,s} &= \int_0^1 \langle \nabla_Y X, X\rangle dt = \int_0^1 (X\langle X, Y\rangle - \langle \nabla_X X, Y\rangle) dt \\ &= \langle X, Y\rangle_{t=1} - \int_0^1 \int_V \frac{\partial \varphi}{\partial s} \cdot \epsilon\, H \ \det\, g\, dt \\ &= \langle X, Y\rangle_{t=1} - \epsilon \cdot \int_0^1 \int_V \frac{\partial \varphi}{\partial s} \ \det\, g_0\, dt.\end{aligned}$$

Now the second derivatives:

$$\begin{aligned}\frac{1}{2}\frac{d^2\,E(s)}{d\,s^2} &= \frac{d}{d\,s}\langle X, Y\rangle_{t=1} - \epsilon \cdot \int_0^1 \int_V \frac{\partial^2 \varphi}{\partial s^2} \ \det\, g_0\, dt \\ &\geq \langle Y', Y\rangle_{t=1} + \langle X, \nabla_Y Y\rangle_{t=1} - C\,\epsilon \int_V \det\, g_0 \\ &\geq \langle Y, Y\rangle_{t=1} + \int_V \frac{\partial \varphi}{\partial t}\, \epsilon \cdot H \cdot \det g - C\,\epsilon \int_V \det\, g_0 \\ &\geq \langle Y, Y\rangle_{t=1} + \int_V \frac{\partial \varphi}{\partial t}\, \epsilon \cdot \det\, g_0 - C\,\epsilon \int_V \det\, g_0 \\ &\geq E(QR) - C \cdot \epsilon \cdot \mathrm{vol}(V).\end{aligned}$$

Here we have used the inequality (2.4) in the second inequality from the top. And $E(QR)$ denotes the energy of the path $\phi_2(\cdot, s)$. For the energy elements of curves, the following inequality holds:

$$E(s) \leq (1-s)E(0) + sE(1) - s(1-s)(E(QR) - C \cdot \epsilon \cdot \mathrm{vol}(V)).$$

Now fix s;[7] as $\epsilon \to 0$, each energy element of a path approaches the square of the length of that path. Thus the above inequality reduces to

$$|PQ(s)|^2 \leq (1-s)|PQ|^2 + s|PR|^2 - s(1-s)|QR|^2. \tag{2.5}$$

Thus the space of Kähler metrics satisfies the defining inequality for a nonpositively curved space and hence it is a nonpositive space. Here $|PQ(s)|$ represents the distance from P to $Q(s)$; $|PQ|$ represents the distance from P to Q; $|PR|$ represents the distance from P to R; and $|QR|$ represents the distance from Q to R. q.e.d.

Next we prove Theorem 1.2.

[7] Actually, using successive subdivision one sees that knowing the inequality (2.5) holds for $s = \frac{1}{2}$ is enough to prove it for all $0 < \lambda < 1$, cf. [9].

Proof of Theorem 1.2. Let φ_0 and φ_1 be two points in $\mathcal{H}$ with distance $l > 0$. Suppose that $\varphi(\cdot, t)(0 \leq t \leq 1)$ is a $C^{1,1}$ geodesic which connects these two points in $\mathcal{H}$. Let $\varphi_i(t)(0 \leq t \leq 1)$ be an arbitrary family $(i = 1, 2, \dots, n, \dots)$ of curves between φ_0 and φ_1 with length $l_i \geq l > 0$. Next we assume that this is a distance minimizing sequence of curves. In other words,

$$\lim_{i\to\infty} l_i = l.$$

Then, we need to show that $\varphi_i(\cdot, t)(0 \leq t \leq 1)$ converges to $\varphi(\cdot, t)(0 \leq t \leq 1)$ in some reasonable topology. For convenience, we assume that every curve involved has been parameterized proportionally to the arc-length. Then we only need to show that for each fixed $s > 0$, we have

$$\lim_{i\to\infty} d(\varphi_i(\cdot, s), \varphi(\cdot, s)) \to 0.$$

Since $\mathcal{H}$ is a nonpositively curved space, we have (comparing with the Euclidean space):

$$d(\varphi_i(\cdot, s), \varphi(\cdot, s)) \leq \sqrt{\frac{l_i^2 - l^2}{4}} \to 0.$$

Theorem 1.2 is then proved. q.e.d.

3. The gradient flow of the K-energy

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. We follow the notations in Theorem 1.3. Let $\phi(\cdot, t) : [0, 1] \to \mathcal{H}$ be any smooth curve in $\mathcal{H}$ and let $\phi(\cdot, t, s)$ be the image of this curve under the gradient flow after time s. Recall

$$\frac{\partial \phi}{\partial s} = R(\phi) - \underline{R}.$$

Denote by $g(s, t)$ the Kähler metric associated with the Kähler potentials $\phi(s, t)$. Use $\triangle$ to denote the complex Laplacian operator of metric $g(s, t)$. Following a calculation in [2], we have

$$\frac{\partial R}{\partial t} = -D^* D \frac{\partial \phi}{\partial t} + \sum_{\alpha=1}^{n} \left(\frac{\partial \phi}{\partial t}\right) \phi^\alpha R_{,\alpha}$$

and

$$\frac{\partial}{\partial t}\det\, g(s,t) = \triangle\, \frac{\partial \phi}{\partial t}\ \det\, g.$$

Recall that the energy of the path $\phi(\cdot, t, s)$ (at time s fixed) is:

$$E(s) = \int_0^1 \int_V \left(\frac{\partial \phi}{\partial t}\right)^2 \det g\ dt.$$

Under the gradient flow (1.1), we have

$$\begin{aligned}
\frac{dE}{d\,s} &= \int_0^1 \int_V 2\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t \partial s}\ \det\, g\ dt + \int_0^1 \int_V \left(\frac{\partial \phi}{\partial t}\right)^2\ \triangle \frac{\partial \phi}{\partial s} \det\, g\ dt \\
&= \int_0^1 \int_V 2\frac{\partial \phi}{\partial t}\frac{\partial R}{\partial t}\ \det\, g\ dt - \int_0^1 \int_V 2\frac{\partial \phi}{\partial t}\left(\frac{\partial \phi}{\partial t}\right)^{\alpha}\left(\frac{\partial \phi}{\partial s}\right)_{\alpha} \det\, g\ dt \\
&= \int_0^1 \int_V 2\frac{\partial \phi}{\partial t}\left(-D^*D\ \frac{\partial \phi}{\partial t} + \phi^{,\alpha} R_{,\alpha}\right)\ \det\, g\ dt \\
&\qquad - \int_0^1 \int_V 2\frac{\partial \phi}{\partial t}\left(\frac{\partial \phi}{\partial t}\right)^{\alpha} R_{\alpha} \det\, g\ dt \\
&= -\int_0^1 \int_V \left|D\frac{\partial \phi}{\partial t}\right|_g^2\ \det\, g\ dt.
\end{aligned}$$

It follows that

$$\frac{dL}{d\,s} = -\int_0^1 \left(\int_V \left|D\frac{\partial \phi}{\partial t}\right|^2_{\phi(s,t)}\ dg(s,t) \cdot \sqrt{\int_V \left|\frac{\partial \phi}{\partial t}\right|^2\ dg(s,t)}^{-\frac{1}{2}}\right)\ dt,$$

where $L(s)$ is the length of the evolved curve at time $s > 0$. From this formula, if the length of a smooth curve is not decreasing, then

$$\int_0^1 \left(\int_V \left|D\frac{\partial \phi}{\partial t}\right|^2_{\phi(s,t)}\ \det\, g(s,t) \sqrt{\int_V \left|\frac{\partial \phi}{\partial t}\right|^2\ dg(s,t)}^{-\frac{1}{2}}\right)\ dt = 0.$$

It follows that

$$\int_0^1 \int_V \left|D\frac{\partial \phi}{\partial t}\right|^2_{\phi(s,t)} = 0$$

or

$$\left(\frac{\partial \phi}{\partial t}\right)_{,\alpha\beta} \equiv 0, \qquad \forall\ \alpha, \beta = 1, 2, \ldots n;\ \forall\ t \in [0, 1].$$

In other words, the curve $\phi(t)(0 \leq t \leq 1)$ is either trivial (depending only on t) or it represents a family of holomorphic transformations. Theorem 1.3 is then proved. q.e.d.

Next we give a proof of the first part of Theorem 1.5.

Proof of Theorem 1.5(1). For any $\varphi_0, \varphi_1 \in \mathcal{H}$, consider the space of all smooth curves which connect φ_0 with φ_1. We denote it by $\mathcal{L}(\varphi_0, \varphi_1)$. For any curve $c \in \mathcal{L}(\varphi_0, \varphi_1)$, we denote its length by $L(c)$. Then the distance between the two points φ_0 and φ_1 can be defined as

$$d(\varphi_0, \varphi_1) = \inf_{c \in \mathcal{L}(\varphi_0, \varphi_1)} L(c).$$

We also define a map in $\mathcal{H}$ via the gradient flow (1.1): for a fixed time s, and for any $\varphi \in \mathcal{H}$, we define that the image of φ under the map π_s is the image of φ along the gradient flow after time $s > 0$, provided the gradient flow initiated at φ does exist for time $s > 0$. It is clear that for any φ, the map is defined for small $s > 0$. However, for a fixed $s > 0$, π_s is not necessarily defined for all $\varphi \in H$ since we don't know the global existence of the gradient flow.

On the other hand, if the gradient flow exists for all the time for any smooth initial metric, then this induces a well-defined map from $\mathcal{L}(\varphi_0, \varphi_1)$ to $\mathcal{L}(\pi_s(\varphi_0), \pi_s(\varphi_1))$ for any $s > 0$. Since the length of any smooth curve in $\mathcal{H}$ is decreased under the gradient flow, we have

$$\inf_{c \in \mathcal{L}(\pi_s(\varphi_0), \pi_s(\varphi_1))} L(c) \leq \inf_{c \in \mathcal{L}(\varphi_0, \varphi_1)} L(c), \qquad \forall\ s > 0.$$

Thus,

$$d(\pi_s(\varphi_0), \pi_s(\varphi_1)) \leq d(\varphi_0, \varphi_1), \qquad \forall\ s > 0.$$

q.e.d.

Before proving the second part of Theorem 1.5, we need to use a theorem in [4] where an explicit formula for the first derivatives of the distance function in $\mathcal{H}$ is given. For the convenience of the readers, we will include this theorem here. The second part of Theorem 1.5 is essentially a corollary of this theorem.

Theorem 3.1 ([4]). *For any two Kähler potentials φ_0, φ_1, the distance function $d(\varphi_0, \varphi_1)$ is at least C^1. More specifically, if φ_0, φ_1 move along two curves $\varphi_0(s)$, and $\varphi_1(s)$ respectively, and if we denote the distance between $\varphi_1(s)$ and $\varphi_2(s)$ by $L(s)$, then*

$$\begin{aligned}\frac{d\,L(s)}{d\,s}\bigg|_{s=0} &= \langle X, Y_1\rangle |X|^{-\frac{1}{2}}\Big|_{t=1} - \langle X, Y_0\rangle |X|^{-\frac{1}{2}}\Big|_{t=0} \\ &= \int_V \frac{\partial\varphi_1}{\partial s}\,\frac{\partial\,\varphi}{\partial\,t}\,dg(s)\cdot\left\{\int_V\left|\frac{\partial\varphi}{\partial t}\right|^2 dg(s)\right\}^{-\frac{1}{2}}\Bigg|_{t=1} \\ &\quad - \int_V \frac{\partial\varphi_0}{\partial s}\,\frac{\partial\,\varphi}{\partial\,t}\,dg(s)\cdot\left\{\int_V\left|\frac{\partial\varphi}{\partial t}\right|^2 dg(s)\right\}^{-\frac{1}{2}}\Bigg|_{t=0}.\end{aligned}$$

Here $\varphi(t)(0 \le t \le 1)$ denotes the $C^{1,1}$ geodesic connecting the two metrics φ_0 and φ_1; and $X = \frac{\partial\varphi}{\partial t} \in T_{\varphi(t)}\mathcal{H}$ and $Y_i = \frac{\partial\varphi_i}{\partial s} \in T_{\varphi_i}\mathcal{H}$ $(i = 0, 1)$.

Now we complete the proof of Theorem 1.5.

Proof of Theorem 1.5(2). If the gradient flow (1.1) exists for all the time, then it is straightforward to show that flow (1.1) decreases the distance between any two points in $\mathcal{H}$ unless they are connected by a holomorphic transformation. Thus, we only deal with the case when the K-energy is weakly convex. By definition, for any curve $\varphi(t) \in \mathcal{H}$, the K-energy is defined as

$$\frac{dM(\varphi(t))}{dt} = -\int_V \frac{\partial\varphi}{\partial t}\,(R - \underline{R})\,\det g.$$

Along a $C^{1,1}$ geodesic, the second derivative of the K-energy is convex in the weak sense that

$$\frac{d^2\,M(\varphi(t))}{d\,t^2} \ge 0.$$

In particular, we have

$$\frac{dM(\varphi(t))}{dt}\bigg|_{t=1} \ge \frac{dM(\varphi(t))}{dt}\bigg|_{t=0}. \tag{3.1}$$

Suppose that $\varphi(t)(0 \le t \le 1)$ is the unique $C^{1,1}$ geodesic which connects φ_1 and φ_2, and suppose it is parameterized proportionally to arc length. If we flow φ_1 and φ_2 by the gradient flow (1.1), we have

$$\frac{\partial \varphi_1}{\partial s} = R(\varphi_1(s)) - \underline{R} \qquad \text{and } \frac{\partial \varphi_2}{\partial s} = R(\varphi_2(s)) - \underline{R}.$$

Plugging this into the corresponding formula in Theorem 3.1, we have

$$\begin{aligned}
\frac{d\,L(s)}{ds} &= \left\{ \int_V \left| \frac{\partial \varphi}{\partial t} \right|^2 d\,g(s) \right\}^{-\frac{1}{2}} \\
&\qquad \cdot \left(\int_V (R(\varphi_2) - \underline{R}) \ \frac{\partial \varphi}{\partial t}\, dg(s) \right|_{t=1} \\
&\qquad\qquad - \int_V (R(\varphi_1) - \underline{R}) \ \frac{\partial \varphi}{\partial t}\, dg(s) \Big|_{t=0} \Big) \\
&= - \left\{ \int_V \left| \frac{\partial \varphi}{\partial t} \right|^2 d\,g(s) \right\}^{-\frac{1}{2}} \cdot \left(\frac{dM}{dt} \Big|_{t=1} - \frac{dM}{dt} \Big|_{t=0} \right) \\
&= - \left\{ \int_V \left| \frac{\partial \varphi}{\partial t} \right|^2 dg(s) \right\}^{-\frac{1}{2}} \cdot \int_{t=0}^{1} \frac{d^2 M}{dt^2}\, dt \le 0.
\end{aligned}$$

q.e.d.

4. Some further corollaries, remarks and the relationship with stability

Corollary 4.1 ([8]). *If all geodesics are smooth, then any extremal Kähler metric is unique up to a holomorphic automorphism.*

Proof. Suppose that there exist two extremal Kähler metrics in a fixed Kähler class. It was proved in [3] that any extremal Kähler metric must be symmetric with respect to a maximal compact subgroup. Without loss of generality, one may assume that both metrics are symmetric under the same maximal compact subgroup. Then every metric in the geodesic which connects this two extremal metrics must also have the same symmetry group (via the Maximum Principle). If the scalar curvature is constant, then an argument of Donaldson [8] on the convexity of the K energy implies that the extremal metric must be unique. If the scalar curvature is not a constant, then the gradient vector field of the scalar curvature is a holomorphic vector field and it is unique in each Kähler class once the maximal compact subgroup is fixed. In particular,

the gradient flow (1.1) restricted to the two extremal metrics induces the same holomorphic transformation. Thus the distance of these two extremal Kähler metrics under the gradient flow is unchanged. Suppose φ_1, φ_2 are the two extremal metrics and $\varphi(\cdot, s)$ is the unique geodesic connecting them. Since the distance of φ_1 and φ_2 is not decreased under the gradient flow, by Theorem 1.3, the path $\varphi(\cdot, s)$ must either be totally trivial or represent a holomorphic transformation. q.e.d.

Remark 4.2. For the uniqueness of the extremal Kähler metrics, the known results are as follows:

1) In the 1950s, the first author showed the uniqueness of Kähler-Einstein metric if $C_1 \leq 0$.

2) In 1987, T. Mabuchi and S. Bando [1] showed the uniqueness of Kähler-Einstein metric up to a holomorphic transformation if the first Chern class is positive.

3) In [4], the second author proved that any metric with constant scalar curvature is unique in each Kähler class if $C_1(V) < 0$.

The problem for the general case is still open. However, the second author [5] had examples of nonuniqueness of some degenerated extremal Kähler metrics in S^2.

S.-T. Yau predicted in [14] that the existence of a Kähler-Einstein metric is related to the stability in the sense of Hilbert schemes and geometric invariant theory. His conjecture should be extended to include the case of extremal Kähler metrics. From Theorem 1.3, we observe some kind of link, perhaps still a bit mysterious, between the the existence of extremal metrics and "stability" of the infinite dimensional space $\mathcal{H}$ in some sense. At least formally, it fits nicely in the general picture that Yau's conjecture describes. The following paragraph is essentially speculative in trying to explain this point. If we are willing to put aside the regularity issue, then Theorem 1.3 implies that the gradient flow of the K energy is a distance contracting flow in $\mathcal{H}$. In this infinite dimensional path length space $\mathcal{H}$, we choose a large enough ball, which hopefully contains any possible candidates for extremal Kähler metrics. Now flow the entire ball by this gradient flow. If a global solution of the gradient flow always exists for all smooth initial metrics, then the contracting nature of the flow will shrink the size of the ball. In the limit, the ball will shrink to a point, which must be an extremal

Kähler metric we are looking for. However, this formal picture is not quite complete. A dichotomy can possibly take place: As the size of the ball shrinks, the ball may also drift away to infinity. In the first possibility when the ball stays in a finite domain, the infinite dimensional manifold is considered "stable" in some sense and we arrive at the unique extremal Kähler metric in the limit of the flow. In the second case when the ball drifts to infinity, then the infinite dimensional space is considered "unstable" in some sense, and the gradient flow converges to an extremal Kähler metric in a different Kähler manifold.

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