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CONFORMAL ACTIONS OF SIMPLE LIE GROUPS ON COMPACT PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract

As is well-known, the real rank of a simple Lie group that acts conformally on a pseudo-Riemannian manifold is bounded by means of the signature of the manifold. We give a precise description of the action whenever the real rank of the group reaches that bound, assuming the action is minimal.

1. Introduction and statement of results

1.1 Introduction

The isometry group of a compact Riemannian manifold must be compact by the Arzela-Ascoli Theorem. This is no longer the case for the group of conformal transformations. Indeed, it is well known that, letting $\mathbb{H}^n$ denote hyperbolic n-space, the isometry group $\mathrm{Iso}^o(\mathbb{H}^n) \simeq \mathrm{SO}^o(n,1)$ acts conformally on $\partial\mathbb{H}^n \simeq S^{n-1}$. A well known conjecture of A. Lichnerowicz, considered by J. Lelong-Ferrand and M. Obata, asserts that these are the only examples:

Theorem (Lelong-Ferrand [14, 6], Obata [15])**.** *If M is a compact Riemannian manifold and the group of conformal transformations of M, $\mathrm{Conf}(M)$, is not compact, then $M \simeq S^n$, endowed with the standard conformal structure.*

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It is natural to inquire whether a similar result can be established for the conformal transformation group of a compact Lorentz, and more generally, pseudo-Riemannian manifold. This problem is one source of motivation for the present study.

Given a general compact pseudo-Riemannian manifold M of signature (p,q), the only constraint we are aware of on the (finite dimensional Lie) group $\mathrm{Conf}(M)$ is that the real rank of its semisimple component is bounded by $\min\{p,q\}+1$. This fact follows immediately from the following (denoting the real rank of G by $\mathrm{rk}_{\mathbb{R}}(G)$):

Theorem (Zimmer, [21]). *Assume G is a semisimple Lie group with finite center, acting faithfully on a compact manifold M, and preserving an H-structure, where H is a real-algebraic group. Then $\mathrm{rk}_{\mathbb{R}}(G) \leq \mathrm{rk}_{\mathbb{R}}(H)$.*

The algebraic structure present in the case of pseudo-Riemannian manifold is the real algebraic group of conformal linear transformations $H = \mathrm{CO}(p,q) \simeq \mathrm{SO}(p,q) \times \mathbb{R}_+^*$, and as is well known $\mathrm{rk}_{\mathbb{R}}(H) = \min\{p,q\}+1$.

Let now G be a semisimple Lie group of higher real-rank that acts on M preserving an H-structure and a volume form. In this case, a number of rigidity phenomena arise, both in terms of the topology of the underlying manifold, as well as the structure of the action itself. These have been studied extensively (see e.g., [22, 7]). We note, however, that the proof of these rigidity results depend crucially on the existence of an invariant volume form (or at least a finite invariant measure) on the manifold. It is an interesting problem to consider the case of conformal transformations, which preserve a non-unimodular structure, and establish whether analogous rigidity phenomena persist in some form. This is our second source of motivation for the present study.

1.2 Statement of results

We first note that each of the groups $G = \mathrm{SO}^{\mathrm{o}}(p,q)$ admit a conformal action on a compact homogeneous pseudo-Riemannian manifold of signature $(p-1,q-1)$. This manifold is the product of the two spheres $S^{p-1} \times S^{q-1}$, and is a 2-fold cover of the homogeneous projective variety G/Q, where Q is the maximal parabolic subgroup of least codimension in $\mathrm{SO}^{\mathrm{o}}(p,q)$. The spaces G/Q will be denoted by $C^{p-1,q-1}$, and will constitute our standard models for conformal actions on compact pseudo-Riemannian manifolds. The pseudo-Riemannian manifold $C^{p-1,q-1}$ is

Riemannian if and only if $p = 1$ (or $q = 1$), and in that case, we retrieve S^{q-1} (or S^{p-1}), which is the standard model for a conformal action on a compact Riemannian manifold. For a more explicit description of the standard spaces we refer to §2.4.

Our framework will be that of compact manifolds with a bilinear structure, namely endowed with a bilinear form in each tangent space. We emphasize that we do not require anything further from the bilinear forms, and in particular they need not be symmetric or even nondegenerate. This setup is considerably more general than that of G-actions preserving a pseudo-Riemannian structure. We begin by formulating a general lower bound on the dimension of an isotropic subspace in a bilinear manifold.

Theorem 1. *Let G be a connected almost simple real Lie group with finite center. Assume G acts conformally* (*and nontrivially*) *on a compact manifold with a bilinear structure M. Then there exists some point $m \in M$, where the bilinear form on $T_m(M)$ has an isotropic subspace of dimension at least* $\mathrm{rk}_{\mathbb{R}}(G) - 1$.

As noted already, in §2.4 we will show that the group $\mathrm{SO}(p,q)$ acts on compact homogeneous space, preserving a pseudo-Riemannian structure up to a scalar factor. We will also show that $\mathrm{SL}(3,\mathbb{R})$ acts on a compact homogeneous space, preserving a symplectic structure up to a scalar factor. Clearly, finite covers of these groups will act similarly on finite covers of these spaces. Furthermore, any outer automorphism of the group gives rise to a twisted action on the same space. The actions obtained by this prescription will be called *standard models* of conformal actions. We refer to §2.4 for more details and a complete list. In the presence of an upper bound on the dimension of an isotropic subspace in a bilinear manifold, we will show the following

Theorem 2. *Let G and M be as in Theorem* 1. *Assume that the maximum dimension of an isotropic subspace of $T_m(M)$ is at most* $\mathrm{rk}_{\mathbb{R}}(G) - 1$, *for all $m \in M$. Then:*

- *G must be locally isomorphic to either* $\mathrm{SO}(p,q)$, *or* $\mathrm{SL}(3,\mathbb{R})$.
- *There exists a closed G-orbit in M.*
- *Any closed G-orbit in M is equivariantly and conformally diffeomorphic to a standard model.*

Theorem 2 has the following two corollaries. The first considers pseudo-Riemannian manifold admitting a group of conformal transformations with simple subgroup of real rank which is the maximum allowed by Zimmer's Theorem (or by Theorem 1). It asserts that if the action is minimal, then the group is determined uniquely (up to local isomorphism), and the action, which turns out to be transitive, is also uniquely determined (up to finite covers).

Corollary 3. *Assume that G is a connected almost simple Lie group. Assume that G acts conformally and minimally on a pseudo-Riemannian manifold M of signature (p, q), and has real rank* $\min\{p, q\}$ $+1$. *Then:*

- *G is locally isomorphic to* $\mathrm{SO}(p+1, q+1)$.
- *M is a homogeneous G-space, and M is conformally and equivariantly diffeomorphic to a finite cover of the standard model $C^{p,q}$* (*see* §2.4).

Our second corollary considers even dimensional manifold with a bilinear structure where the form at each tangent space is a nondegenerate symplectic form. Such manifolds we call *symplectic manifolds* (we do not require the 2-form to be closed). Here it turns out that the real rank of a simple subgroup of the conformal group must be strictly smaller than the maximum allowed by Zimmer's Theorem (or by Theorem 1). The only exception arises from a symplectic structure on $\mathbb{P}^2$ arising from the cross-product structure on $\mathbb{R}^3$, an example that will be discussed further in §2.4.

Corollary 4. *Let G be a connected almost simple Lie group of real rank $n+1$. Assume that G acts conformally* (*and nontrivially*) *on a compact symplectic manifold of dimension $2n$. Then $G = \mathrm{SL}(3, \mathbb{R})$ or its universal cover, and M is conformally and equivariantly diffeomorphic to $\mathbb{P}^2$ or its universal cover S^2. The action of G on $\mathbb{P}^2$ is either the one arising from the standard action of G on $\mathbb{R}^3$ or its twist by the nontrivial outer automorphism of G.*

Our proof of Theorem 1 depends on establishing the existence of a continuous nonconstant G-equivariant map from a minimal subset M' of M to a projective algebraic variety U. The variety we choose to consider is the product of the projective space of bilinear forms on the Lie algebra of G with the Grasmann variety consisting of Lie algebras of stability groups of points of M'. The stability group Q of any point u in the image of M' stabilizes the form induced on the Lie algebra (via the orbit map). At the same time, Q normalizes the stabilizer of any point in the preimage of u. In addition, for some point u_0, Q is co-compact (and algebraic), hence contains a maximal $\mathbb{R}$-split torus, A. One constructs from the action of G on M' a conformal faithful linear representation of the torus A. The weight spaces associated to a codimension one subtorus which acts isometrically (with respect to the induced form) are used to construct an isotropic subspace of dimension at least $\mathrm{rk}_{\mathbb{R}}(G) - 1$ in the tangent space of every point of the pre-image of u_0.

The proof of Theorem 2 consists of analyzing further the conformal linear representation of the maximal torus just described. This representation is in fact a subrepresentation of the tensor square of the co-adjoint representation of the torus on the Lie algebra. Inspecting the root systems of real simple groups, one can often construct an isotropic subspace of dimension greater than $\mathrm{rk}_{\mathbb{R}}(G) - 1$, contradicting the additional assumption of an upper bound on the dimension of an isotropic subspace. The proof then proceeds by determining explicitly for which Lie algebras the construction yields an isotropic subspace of dimension exactly $\mathrm{rk}_{\mathbb{R}}(G) - 1$.

1.3 Remarks and relevant references

As noted already, our work is motivated by a conjecture of Lichnerowicz in conformal Riemannian geometry on the one hand, and by rigidity phenomena associated with simple isometry groups in pseudo-Riemannian geometry on the other hand.

We recall that R. Zimmer [20] showed that the noncompact semisimple component of the isometry group of a compact Lorentz manifold is locally-isomorphic to $\mathrm{SL}(2,\mathbb{R})$. This result was generalized by M. Gromov to the case of a pseudo-Riemannian manifold of any signature.

Both of these results are dependent on the fact that the volume form is stabilized by isometries, hence the group preserves a finite measure. This allows the use of variants of the Borel density Theorem (see e.g., [22]).

Another approach, which dispensed with the assumption of finite invariant measure, appeared first in Kowalsky's thesis [12] (see also [13]). This method allowed Kowalsky to characterize the isometry groups of Lorentz manifolds under the sole assumption of non-properness of the action, rather than the compactness of the manifold. Furthermore, this method yielded some results on conformal actions on Lorentzian manifolds.

One idea introduced by Kowalsky is that the non-properness of the action of a split torus implies the existence of large dimensional isotropic subspace of the tangent space to the orbit at some point of the manifold. To obtain this conclusion she used implicitly a map from the manifold to the space of bilinear forms on the Lie algebra, and considered the dynamics of the action of a split torus in the latter space. Our approach is motivated by these ideas, which we develop systematically in the compact, conformal pseudo-Riemannian case. We have also used some ideas from [21, 17, 16].

We note that isometry groups of compact and noncompact Lorentz manifolds were considered by S. Adams and G. Stuck e.g., in [1, 2] (see also [3]). Isometry groups of Lorentz and more generally pseudo-Riemannian manifolds were considered by A. Zeghib e.g., in [18, 19]. Finally, we note that in the case of symplectic conformal actions, if the 2-form is closed, and the manifold is of dimension ≥ 4, then every conformal transformation is isometric. This result is due to P. Libermann, see [11, p. 27].

1.4 The next chapters

The paper is organized as follows. In Chapter 2 we present the necessary definitions relating to G-actions on manifold with a bilinear structure, and describe the standard models mentioned above. In Chapter 3 we present some preliminary facts on almost simple algebraic groups that will be needed later. In Chapter 4, we establish the existence of the G-equivariant map to the projective variety, and proceed to prove Theorem 1. Chapter 5 is devoted to a streamlined proof of Theorem 2. The technical details relating to analysis of the root systems are relegated to the Appendix. Chapter 6 is devoted to some further remarks on related matters, including the case of isometric actions.

2. Manifolds with bilinear structure

2.1 Lie group actions

Definition 2.1. An action of a Lie group G on a C^∞-manifold is a C^∞-*action* if the action map $G \times M \to M$ is a C^∞-map. That is, the induced map $G \to \mathrm{Diff}(M)$ is a group homomorphism.

For every $g \in G$ we have the action map, denoted also by g:

$$g : M \to M, \quad g(m) = gm$$

and for every $m \in M$ we have the orbit map, denoted also by m:

$$m : G \to M, \quad m(g) = gm.$$

The G-action gives a bundle automorphism of the tangent bundle TM, defined by $g(m, V) = (gm, dg_m(V))$. G also acts by automorphism on the trivial bundle $M \times \mathfrak{g}$ by $g(m, X) = (gm, \mathrm{Ad}(g)X)$. The orbit map gives a bundle morphism $\phi : M \times \mathfrak{g} \to TM$, defined by $\phi(m, X) = (m, dm_e(X))$.

Denoting the inner automorphism of G, $h \mapsto ghg^{-1}$ by $\mathrm{inn}(g)$, the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{inn}(g)} & G \\ \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle gm} \\ M & \xrightarrow{g} & M. \end{array}$$

By differentiating at $e \in G$ we deduce the commutativity of:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{g} \\ \Big\downarrow{\scriptstyle dm_e} & & \Big\downarrow{\scriptstyle d(gm)_e} \\ T_mM & \xrightarrow{dg_m} & T_{gm}M. \end{array}$$

which gives the formula:

$$\forall X \in \mathfrak{g} : \quad dg_m \cdot dm_e(X) = d(gm)_e(\mathrm{Ad}(g)X). \tag{1}$$

We conclude that ϕ is a G-equivariant bundle morphism.

Define $\psi : M \to \mathrm{Gr}(\mathfrak{g})$ by $\psi(m) = \ker(dm_e)$, where $\mathrm{Gr}(\mathfrak{g}) = \bigcup_{k=0}^{\dim(\mathfrak{g})} \mathrm{Gr}_k(\mathfrak{g})$ denotes the space of all linear subspaces of $\mathfrak{g}$. Note the action is locally free if and only if the map ψ takes the constant value

$\{0\} \in \mathrm{Gr}_0(\mathfrak{g})$. It follows similarly that ψ is a G-map, where $\mathrm{Gr}(\mathfrak{g})$ is a G-space via the adjoint action. Note further that the stability group of $\psi(m)$ is the normalizer of the stability group of m.

Denote $G_m = \mathrm{Stab}_G(m)$. There is a canonical representation, induced by the action, $\pi_m : G_m \to \mathrm{GL}(T_m M)$ defined by $\pi_m(g) = dg_m$.

2.2 Manifolds with bilinear structure

Definition 2.2. A *manifold with a bilinear structure* is a couple (M, s), such that M is a C^∞-manifold and $s : M \to TM^* \otimes TM^*$ is a C^∞section. We call s *the structure of* (M, s).

When s is clear from the context we will denote (M, s) by M, and $s_m(X, Y)$ by $\langle X, Y\rangle_m$.

The main examples are:

1. (M, s) is called a *Riemannian manifold* if for all $m \in M$, s_m is a positive definite symmetric form on $T_m M$.

2. (M, s) is called a *pseudo-Riemannian manifold* if M is connected, and $\forall m \in M$, s_m is a nondegenerate symmetric form on $T_m M$. The signature $\sigma(s_m)$ is a constant, denoted $\sigma(M)$.

3. (M, s) is called a *symplectic manifold* if for all $m \in M$, s_m is a nondegenerate anti-symmetric form on $T_m M$ (we do not require the form s to be closed).

Recall that given a vector space with a bilinear form (V, B), a subspace $W < V$ is called isotropic if every two vectors in W are perpendicular, that is $B|_{W \times W} = 0$.

2.3 Lie groups action on manifolds with bilinear structure

Let (M, s) be a manifold with a bilinear structure. Given a diffeomorphism $t : M \to M$ we define a new structure on M, ts, by:

$$(ts)_m(X, Y) = s_{t(m)}(dt_m(X), dt_m(Y)).$$

Definition 2.3. For a manifold with a bilinear structure (M, s) and a diffeomorphism t, we call t an *isometry* if $ts = s$. We call t a *conformal map* if $ts = \lambda s$ for some scalar valued function $\lambda : M \to \mathbb{R}^*_+$.

Definition 2.4. Let G be a Lie group acting on M. If for all $g \in G$, $g : M \to M$ is an isometry, we say that G *acts isometrically on* M. If for all $g \in G$, $g : M \to M$ is a conformal map, we say that G *acts conformally on* M.

Notice that G acts isometrically on M if and only if the section s is a G-map, with respect to the natural action of G on the bundle $T^*M \otimes T^*M$. In this case we construct a G-equivariant map $r : M \to \mathfrak{g}^* \otimes \mathfrak{g}^*$. The G-bundle-map $\phi : M \times \mathfrak{g} \to TM$ gives rise to a G-bundle-map $\phi^* : T^*M \otimes T^*M \to M \times (\mathfrak{g}^* \otimes \mathfrak{g}^*)$. Denoting $r = \mathrm{pr}_2 \circ \phi^* \circ s$, where pr_2 denotes the projection on the second coordinate, we obtain the desire map.

Similarly, if G acts conformally on M then the map $\bar{s} : M \to \mathbb{P}(T^*M \times T^*M)$ is a G-equivariant map (if $\bar{s}$ is known to exist then the converse holds also). If for all $m \in M$, $dm_e(\mathfrak{g}) = T_m(Gm)$ is not isotropic (under s_m) then one can define $\bar{\phi}^* : \mathbb{P}(T^*M \otimes T^*M) \to M \times \mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g}^*)$, induced by ϕ. This is also a G-equivariant map. Composing with $\bar{s}$ and projecting on the second coordinate, we obtain the G-equivariant map $\bar{r} : M \to \mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g}^*)$.

We keep this conclusion for future reference as:

Lemma 2.5. *If G acts conformally on the manifold with a bilinear structure M, such that the map r defined above is non-vanishing (that is, $r(m) \neq 0$ for all $m \in M$) then $\bar{r} : M \to \mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g}^*)$ is a well defined C^∞ G-map.*

The following Lemma establishes the fact that in the homogeneous case the bilinear structure is determined uniquely by its value at any point.

Lemma 2.6. *Let G be a Lie group. Let M be a G-homogeneous space. Let m_0 be a point in M, and denote $H = \mathrm{Stab}_G(m_0)$. Assume s_0 is a bilinear form on $T_{m_0}M$. We then have:*

- *If the representation $\pi_{m_0} : H \to \mathrm{GL}(T_{m_0}M)$ is* orthogonal *then there is a unique bilinear structure s on M such that $s(m_0) = s_0$, and that G acts isometrically on (M, s).*

- *If the representation $\pi_{m_0} : H \to \mathrm{GL}(T_{m_0}M)$ is* conformal *then (up to a conformal equivalence) there is a unique bilinear structure s on M such that $s(m_0) = s_0$, and that G acts conformally on (M, s).*

2.4 The standard models for conformal actions

Let $\mathbb{R}^{p,q}$ be the vector space $\mathbb{R}^{p+q}$ endowed with the bilinear form $B^{p,q}$ defined by $B^{p,q}(x,y) = \langle x,y\rangle = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$. The signature of this form is denoted by (p,q). The group of linear orthogonal transformation of $\mathbb{R}^{p,q}$ is denoted by $\mathrm{O}(p,q)$, and the group of linear conformal transformation is denoted by $\mathrm{CO}(p,q)$.

The additive group $\mathbb{R}^{p+q}$ acts on itself, so we have an action of $\mathrm{O}(p,q) \ltimes \mathbb{R}^{p+q}$ on the homogeneous space $\mathrm{O}(p,q) \ltimes \mathbb{R}^{p,q}/\mathrm{O}(p,q) \simeq \mathbb{R}^{p+q}$ (we are identifying the coset $\mathrm{O}(p,q)$ with $0 \in \mathbb{R}^{p+q}$). The form $s_0 = B^{p,q}$ on $T_0\mathbb{R}^{p+q}$ is $\mathrm{O}(p,q)$-invariant. Using Lemma 2.6 we get a bilinear structure on $\mathbb{R}^{p,q}$. We call $\mathbb{R}^{p+q}$ with this bilinear structure $\mathbb{E}^{p,q}$. $\mathbb{E}^{p,q}$ is a pseudo-Riemannian manifold of signature (p,q). In fact $\mathbb{E}^{p,q}$ is the manifold with a bilinear structure obtained by identifying every tangent space of $\mathbb{R}^{p+q}$ with $\mathbb{R}^{p,q}$.

What are the $\mathrm{O}(p,q)$ orbits in $\mathbb{E}^{p,q}$? Identifying $\mathbb{E}^{p,q}$ with $\mathbb{R}^{p,q}$ one can define the norm function $f(x) = \langle x,x\rangle$ on $\mathbb{E}^{p,q}$. Define $F_t = \{x \in \mathbb{E}^{p,q} - \{0\} \mid f(x) = t\}$. The tangent spaces to F_t are subspaces of the tangent spaces to $\mathbb{E}^{p,q}$, so the submanifold F_t inherits a bilinear structure from $\mathbb{E}^{p,q}$. What is the restricted form on F_t? Observing that

$$\forall x \in \mathbb{E}^{p,q},\ \forall y \in T_x\mathbb{E}^{p,q} \quad \mathrm{grad}(f)_x(y) = \langle 2x, y\rangle$$

we see that $T_xF_{f(x)}$ is parallel to $\{x\}^{\perp}$, hence both subspaces have the same bilinear form. If $t > 0$, then F_t is a sub-pseudo-Riemannian manifold of signature $(p-1,q)$. If $t < 0$, then F_t is a sub-pseudo-Riemannian manifold of signature $(p,q-1)$. F_0 is not a pseudo-Riemannian manifold but has a bilinear structure of signature $(p-1,q-1,1)$ (by this notation we mean that the restricted form has a one dimensional radical in every tangent space). Observe that for all $x \in F_0$, $\mathbb{R}x - \{0\} \subset F_0$, and $\mathbb{R}x = rad(T_xF_0)$. Passing to projective quotient we get the manifold $\overline{F}_0$. $\overline{F}_0$ has a natural bilinear structure defined up to a scalar multiple (at any point). Fixing such a structure we obtain our canonical examples of conformal actions, which we denote by $C^{p-1,q-1}$. Since $\mathrm{O}(p,q)$ acts isometrically on F_0, its action on $C^{p-1,q-1}$ is conformal.

Standard models for conformal actions on compact pseudo-Riemannian manifolds.

We call the action of $\mathrm{SO}(p,q)$ (as well as locally-isomorphic groups with finite center) on $C^{p,q}$ (as well as its finite covers) the standard models of conformal action on a compact pseudo-Riemannian manifold of signature (p,q). In the case where $G = \mathrm{SO}(4,4)$ (and in this case only) there exist three non-equivalent actions of G on $C^{3,3}$ obtained by twisting the standard action by an outer automorphism of G of order three. We consider these additional actions as standard models too.

We note that all the groups $\mathrm{SO}(q,q)$ admit a nontrivial outer automorphism, but if $q \neq 4$, the twisted action obtained is equivalent to the standard one. This fact will be observed in the course of the proof of Theorem 2.

We now consider the canonical example of a conformal action on a symplectic manifold.

Standard models for conformal actions on compact symplectic manifolds.

Identify the tangent planes to S^2 with their parallel copies through the origin. For $m \in S^2$ and $x, y \in T_m S^2$ denote $s_m(x,y) = \langle x \times y, m\rangle$. (M,s) is a two-dimensional symplectic manifold. The action of $\mathrm{SL}(3,\mathbb{R})$ (or its 2-fold cover group) on S^2 (or $\mathbb{P}^2$) which arises from the natural action on rays in $\mathbb{R}^3$ is conformal with respect to this symplectic form. In addition there is another action obtained on the same spaces by twisting the above action by the nontrivial outer automorphism of $\mathrm{SL}(3,\mathbb{R})$. We call these actions the standard models of conformal action on compact symplectic manifolds.

3. Preliminary results

We list here some simple facts which are well known. We provide short proofs for the sake of completeness.

By a *real algebraic group* we mean a group of finite index in the group of real points of a complex algebraic group defined over $\mathbb{R}$. For such a group G, we denote by $G_{\mathbb{C}}$ the associated connected complex algebraic group.

Recall that a (complex) solvable algebraic group is a semi-direct product of a subtorus and of its unipotent radical. We use the term *$\mathbb{R}$-split solvable algebraic group* to denote a subgroup of finite index in the set of real points of a solvable algebraic group defined over $\mathbb{R}$, if it has a maximal subtorus which is $\mathbb{R}$-split. Such a connected group, G, admits a composition series $G = G_0 \supset G_1 \supset \cdots \supset G_s = \{e\}$ consisting of connected closed real algebraic subgroups such that $G_i/G_{i+1} \simeq (\mathbb{R},+)$ or $(\mathbb{R}^*_+,\cdot)$ for $0 \leq i < s$. The following Lemma is a simple variation on Borel's fixed point theorem [4, §15.2].

Lemma 3.1. *Let G be a connected $\mathbb{R}$-split solvable algebraic group. Let V be an algebraic variety, defined over $\mathbb{R}$, and assume $G_{\mathbb{C}}$ acts algebraically on V, and that the action is defined over $\mathbb{R}$. Let U be a G-invariant, (Hausdorff) compact subset of the real points of V. Then there exist a G-fixed point in U.*

Proof. We argue by induction on the dimension of G. Assume $G = G_0 \supset G_1 \supset \cdots \supset G_s = \{e\}$ is a composition series as above, and assume G_1 has a fixed point in U. We then have an algebraic action of $G/G_1 \simeq (\mathbb{R},+)$ or $(\mathbb{R}^*_+,\cdot)$ on the compact subset U' consisting of G_1-fixed points in U. A minimum dimensional orbit in U' must be closed in U', since otherwise its boundary, which consists of smaller dimensional orbits, lies outside U'. But a compact algebraic factor of $(\mathbb{R},+)$ or $(\mathbb{R}^*_+,\cdot)$ is a single point. We end up with a G/G_1-fixed point in U', hence a G-fixed point in U. q.e.d.

Recall that an almost-simple group G is of finite index in the set of real-points of a semisimple algebraic group defined over $\mathbb{R}$, i.e., it is a real algebraic group.

Lemma 3.2. *Let G be an almost-simple Lie group, and $Q < G$ a cocompact proper algebraic subgroup. Then:*

1. *Q contains a maximal $\mathbb{R}$-split torus.*
2. *Q is contained in a proper parabolic subgroup of G.*
3. *If G is a split form, then Q is of finite index in a parabolic subgroup.*

Proof. Let A be a maximal $\mathbb{R}$-split torus. Let Φ be the root system of G with respect to A, and Π the set of positive roots. Let $N < G$ be the connected unipotent subgroup with Lie algebra $\mathfrak{n} = \bigoplus_{\alpha\in\Pi} \mathfrak{g}_\alpha$. Let $N' < G$ be the connected unipotent subgroup with Lie algebra $\mathfrak{n}' = \bigoplus_{\alpha\in-\Pi} \mathfrak{g}_\alpha$. AN is a connected $\mathbb{R}$-split solvable algebraic group which acts algebraically on G/Q. By Lemma 3.1, AN has a fixed point in G/Q, hence AN is contained in a conjugate of Q. We can assume (by picking a conjugate of A in the first place, and if necessary changing the notion of positivity in Φ) that $AN < Q$. Now:

1. Follows immediately from the discussion above.

2. Q contains N. If it was reductive it would contain N' too. Indeed, A is a maximal torus of Q, and the root system of A in Q is invariant under multiplication by -1. Furthermore, the multiplicities of α and $-\alpha$ must be the same. The group generated by N and N' is normal, hence it is G itself. It follows that Q contains a nontrivial unipotent-radical. Q is contained in the normalizer of this radical, and this normalizer itself is contained in a parabolic subgroup (Borel and Tits, [5, §3.1]).

3. If G is a split form then AN is of finite index in a minimal parabolic subgroup. Hence a finite extension of Q is a parabolic subgroup, since it contains a minimal parabolic subgroup.

q.e.d.

Lemma 3.3. *Let G be an almost-simple Lie group, $H < G$ a proper subgroup, $A \leq N_G(H)$ an $\mathbb{R}$-split torus. Then the representation induced by the adjoint representation of A on $\mathfrak{g}/\mathfrak{h}$ is faithful.*

Proof. Assuming the representation is not faithful, $\mathfrak{h}$ contains all eigenspaces of some $t \in A$ with nonzero eigenvalues. The algebra generates by all such spaces is a nontrivial ideal, hence $\mathfrak{h} = \mathfrak{g}$, contradicting the properness assumption. q.e.d.

It will be significant in our considerations below to have a lower bound for the codimensions of subgroups of a simple group.

Lemma 3.4.

1. *Let H be an almost simple complex algebraic group, H' a proper Zariski closed subgroup. Then* $\mathrm{codim}_{\mathbb{C}}(H') \geq \mathrm{rk}(H)$.

2. *Let G be an almost-simple Lie group, G' a proper closed subgroup. Then* $\mathrm{codim}_{\mathbb{R}}(G') \geq \mathrm{rk}(G_{\mathbb{C}}) \geq \mathrm{rk}_{\mathbb{R}}(G)$.

Proof.

1. Assume $H' < H$ is a proper algebraic subgroup of minimal codimension. As is well known [4, §5.1], there exists a representation $H \to GL(V)$ for some complex vector space V, and there exists a line $L < V$ such that H' is the stabilizer of L in H. By passing to a quotient space if necessary one can assume there are no H-invariant lines in V. Stating the above in other words, there exist an algebraic action of H on a projective space $P = \mathbb{P}(V)$

with no H-fixed points, and there is a point $p = [L] \in P$ such that $H' = \mathrm{Stab}_H(p)$. In the Zariski closure of the orbit $H \cdot p$ there are no H-fixed points, and no orbits of dimension smaller than $\mathrm{codim}_{\mathbb{C}}(H')$ (by minimality of codimension). It follows that the orbit $H \cdot p \simeq H/H'$ is closed, hence it is a projective variety. Therefore H' is a parabolic subgroup, hence contains a maximal torus A. Restricting the adjoint representation of H to A, and passing to the representation on the quotient space $\mathfrak{h}/\mathfrak{h}'$, (which must be faithful by Lemma 3.3), we conclude that $\mathrm{rk}(H) = \dim_{\mathbb{C}}(A) \leq \mathrm{rk}(\mathrm{GL}(\mathfrak{h}/\mathfrak{h}')) = \dim_{\mathbb{C}}(\mathfrak{h}/\mathfrak{h}') = \mathrm{codim}_{\mathbb{C}}(H')$.

2. Assume now that G is a real algebraic group which is almost simple. Let $G' < G$ be a proper subgroup of minimal codimension. Assume also, without loss of generality, that G' is connected. By the simplicity assumption $N_G(G')^{\mathrm{o}} = G'$, and therefore $\mathrm{Ad}(G')$ is the connected component of $\mathrm{Ad}(G) \cap \mathrm{Stab}_{\mathrm{GL}(\mathfrak{g})}(\mathfrak{g}')$. The latter group is the real points of an algebraic group defined over $\mathbb{R}$, so G' is real-algebraic. By the foregoing result $\mathrm{codim}_{\mathbb{R}}(G', G) = \mathrm{codim}_{\mathbb{C}}(G'_{\mathbb{C}}, G_{\mathbb{C}}) \geq \mathrm{rk}(G_{\mathbb{C}})$, and of course $\mathrm{rk}(G_{\mathbb{C}}) \geq \mathrm{rk}_{\mathbb{R}}(G)$.

q.e.d.

Finally, we recall the following result (and its proof) from [21].

Lemma 3.5. *Let G be a simple Lie group acting on the connected manifold M. Let $m \in M$ be a G-fixed point, and assume $\pi_m = 1$ (see* §2.1*). Then G acts trivially on M.*

Proof. Let K be a maximal compact subgroup in G. Using the compactness of K one can assume M is a Riemannian manifold and K acts on it isometrically. It follows, using the exponential map, that K acts trivially in a neighborhood of $n \in M$ if and only if n is fixed and $\pi_n = 1$. The set of points satisfying both conditions is obviously open and closed. It is not empty by assumption, so the connectivity of M implies the triviality of the action of K. The action of G on M has a nontrivial kernel so it is trivial by simplicity. q.e.d.

4. Proof of Theorem 1

We begin with the following linear version of Theorem 1. Note that the bilinear form we consider below need not be symmetric, nor nondegenerate.

Lemma 4.1. *Let* (V, B) *be a real vector space with a bilinear form. Let* $\mathrm{O}(B)$, $\mathrm{CO}(B)$ *be the orthogonal and the conformal groups of* B, *respectively. Then* V *has an isotropic subspace* U *of dimension* $\dim(U) = \mathrm{rk}_{\mathbb{R}}(\mathrm{O}(B)) \geq \mathrm{rk}_{\mathbb{R}}(\mathrm{CO}(B)) - 1$.

Proof. First observe that $\mathrm{rk}_{\mathbb{R}}(\mathrm{O}(B)) = \mathrm{rk}_{\mathbb{R}}(\mathrm{CO}(B)) - 1$, unless B is the zero form and then obviously, $\mathrm{rk}_{\mathbb{R}}(\mathrm{O}(B)) = \mathrm{rk}_{\mathbb{R}}(\mathrm{CO}(B)) = \mathrm{GL}(V) = \dim(V)$. Let $l = \mathrm{rk}_{\mathbb{R}}(\mathrm{O}(B))$. Let A be a maximal $\mathbb{R}$-split torus of $\mathrm{O}(B)$, and denote $\mathfrak{a} = \mathrm{Lie}(A)$. Then $\mathfrak{a}$ is $\mathbb{R}$-diagonalizable in V. Let α, β be weights of $\mathfrak{a}$, and V_α, V_β be the corresponding weight spaces. Observe that if $\alpha + \beta \neq 0$ then $V_\alpha \perp V_\beta$, and $V_\beta \perp V_\alpha$, because for $x \in V_\alpha,\ y \in V_\beta$ and $t \in \mathfrak{a}$ such that $(\alpha + \beta)(t) \neq 0$,

$$B(x, y) = B(\exp(t)x, \exp(t)y) = e^{(\alpha+\beta)(t)} B(x, y).$$

The representation of $\mathfrak{a}$ on V is faithful, so the weights of this representation span $\mathfrak{a}^*$. Let the weights $\alpha_1, \ldots, \alpha_l$ be a basis of $\mathfrak{a}^*$. $U = \bigoplus V_{\alpha_j}$ is isotropic subspace of V of dimension at least l. q.e.d.

Our next Lemma is essential. It ensures the existence of an equivariant map to a projective G-space. This map will be our main tool in the proofs of Theorems 1 and Theorem 2. Recall the definition of the map $r : M \to \mathfrak{g}^* \otimes \mathfrak{g}^*$, given at §2.3.

Lemma 4.2. *Let* G *and* M *be as in Theorem* 1. *Assume that the maximum dimension of an isotropic subspace of* $T_m(M)$ *is at most* $\mathrm{rk}_{\mathbb{R}}(G) - 1$, *for all* $m \in M$. *Then the map* $r : M \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ *is non-vanishing.*

Proof. It follows from Lemma 3.4 that $r(m) \neq 0$ whenever m is not a fixed point. Indeed the dimension of the orbit Gm is at least $\mathrm{rk}_{\mathbb{R}}(G)$, and therefore the tangent space to the orbit cannot be isotropic (by assumption), so $r(m)$ is nonzero.

We will show there are no fixed points. Assume m is a fixed point. G acts conformally, but since every character of G is trivial, its representation π_m on $T_m M$ is actually orthogonal. It follows that the representation π_m is trivial, by our assumption on the maximal dimension of an isotropic subspaces and by Lemma 4.1. Using Lemma 3.5 we get that the action of G on M is trivial, a contradiction. q.e.d.

Proof of Theorem 1. Assume there is no isotropic subspace of $T_m(M)$ of dimension $\mathrm{rk}_{\mathbb{R}}(G) - 1$, for all $m \in M$. Using Lemma 4.2 r is non-vanishing so the continuous G-map $\bar{r} : M \to \mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g}^*)$ is well defined (see Lemma 2.5).

We now define another continuous G-map to a different projective space. Let $M' \subset M$ be the set of points with stability group of maximal dimension. M' is a compact G-invariant subset (see e.g., [17]). Recall the map $\psi : M \to \mathrm{Gr}(\mathfrak{g})$, which assigns to each point in M the Lie algebra of its stability group. ψ is a continuous map on M' (but not on M) by [17, Lemma 2.1]. From now on we consider the restriction of ψ to M'.

The map $\bar{r} \times \psi : M' \to \mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g}^*) \times \mathrm{Gr}(\mathfrak{g})$ is a continuous G-map. Denote the image of M' by U. G acts algebraically on the space $\mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g}^*) \times \mathrm{Gr}(\mathfrak{g})$. In particular any minimum dimensional orbit in U is closed (in the relative topology), hence compact. The stabilizer Q of a point in such an orbit is real algebraic and cocompact, hence contains a maximal $\mathbb{R}$-split torus (by Lemma 3.2, part I). We then have a point $u \in U$ and a maximal $\mathbb{R}$-split torus $A < Q$ that fixes u.

Let m be a point in the pre-image of u. Denote $H = \mathrm{Stab}_G(m)$, $\mathfrak{h} = \mathrm{Lie}(H)$, and note that $H < G$ since there are no G-fixed points (see the proof Lemma 4.2). By construction, the adjoint action of A on $\mathfrak{g}$ is conformal with respect to the form $r(m)$ since $A < Q$. $\mathfrak{h}$ is clearly contained in the radical of $r(m)$, and is normalized by Q and hence also by A. Therefore A acts on $\mathfrak{g}/\mathfrak{h}$ as well, the form $r(m)$ descends to a form on $\mathfrak{g}/\mathfrak{h}$, and A acts conformally on the latter. Furthermore the representation of A on $\mathfrak{g}/\mathfrak{h}$ is faithful by Lemma 3.3. By Lemma 4.1, $\mathfrak{g}/\mathfrak{h}$ contains an isotropic subspace of dimension $\mathrm{rk}_{\mathbb{R}}(G) - 1$.

Finally, $\mathfrak{g}/\mathfrak{h}$ is naturally identified with the tangent space to the orbit $T_m(Gm)$, and we obtain a contradiction to the assumption that there is no $(\mathrm{rk}_{\mathbb{R}}(G) - 1)$-dimensional isotropic subspace of $T_m M$. q.e.d.

5. Proof of Theorem 2

5.1 Beginning the proof of Theorem 2

Through out this section we will denote $l = \mathrm{rk}_{\mathbb{R}}(G)$. The proof begins exactly as the proof of Theorem 1, except now our leading assumption is that we do not have any l-dimensional (instead of $l - 1$) isotropic subspaces of any of the tangent-spaces. We pick-up at the last sentence of the proof, just before using Lemma 4.1, and keep on from there. To summarize: we have a point $m \in M$ and a cocompact algebraic subgroup $Q < G$ which both normalizes the stabilizer H of m, and acts conformally on $\mathfrak{g}$ with respect to the bilinear form $r(m)$ via the adjoint representation.

We note that the upper bound on the dimension of an isotropic subspace implies that Q is a proper subgroup of G. Indeed, $Q = N_G(\mathfrak{h}) \cap \mathrm{Stab}_G(\bar{r}(m))$, so in particular if $Q = G$ then $\mathfrak{h}$ is an ideal. Since there are no fixed points and G is simple, $\mathfrak{h} = 0$, and we can identify $T_m(Gm)$ with $\mathfrak{g}$. Furthermore, G acts conformally with respect to the form $r(m)$ on $\mathfrak{g}$. Since G is simple G must preserve the form, hence by Lemma 4.1 there exists an isotropic subspace of dimension l, a contradiction.

Recall that by Lemma 3.2, Q is contained in a proper parabolic subgroup. Fix such a parabolic subgroup and denote it by P. Q also contains a maximal $\mathbb{R}$-split torus, which we denote A. We fix a root system Φ of A in $\mathfrak{g} = \mathrm{Lie}(G)$, a set of positive roots Π, and a set of simple roots Δ. We assume from now on that $\mathfrak{p} = \mathrm{Lie}(P)$ and all other parabolic subalgebra that occurs are standard (for the construction of the standard parabolic subalgebra see the Appendix).

Let $\mathfrak{a} = \mathrm{Lie}(A), \mathfrak{h} = \mathrm{Lie}(H), \mathfrak{p} = \mathrm{Lie}(P)$ and $\mathfrak{q} = \mathrm{Lie}(Q)$. $\mathfrak{h} < \mathfrak{q} < \mathfrak{p} < \mathfrak{g}$ are all semisimple $\mathfrak{a}$-modules, hence admit direct sum complements: $\breve{\mathfrak{h}} > \breve{\mathfrak{q}} > \breve{\mathfrak{p}} > \breve{\mathfrak{g}} = 0$. $\breve{\mathfrak{h}} \simeq \mathfrak{g}/\mathfrak{h}$, $\mathfrak{g} = \mathfrak{h} \oplus \breve{\mathfrak{h}}$, and similarly for $\breve{\mathfrak{q}}$ and $\breve{\mathfrak{p}}$. $r(m)$ restricts to a form on $\breve{\mathfrak{h}}$. $\breve{\mathfrak{h}}$ might be identified with a subspace of T_mM as well, and under this identification $s(m)|_{\breve{\mathfrak{h}}} = r(m)|_{\breve{\mathfrak{h}}}$.

The weights of the natural representation of G on the space of all bilinear forms, $\mathfrak{g}^* \otimes \mathfrak{g}^*$, are simply $\hat{\Phi} + \hat{\Phi}$ (where $\hat{\Phi} = \Phi \cup \{0\}$). As an $\mathfrak{a}$-module we have the decomposition

$$\mathfrak{g}^* \otimes \mathfrak{g}^* = \bigoplus_{\lambda \in \hat{\Phi} + \hat{\Phi}} V_\lambda$$

where V_λ is the set of bilinear forms, B, on $\mathfrak{g}$ satisfying for all $t \in \mathfrak{a}$, and $u, v \in \mathfrak{g}$

$$B(\exp(t)u, \exp(t)v) = e^{\lambda(t)} B(u, v).$$

(Note that V_λ is not the weight space of λ, but of $-\lambda$, since the action on $\mathfrak{g}^*$ involves taking an inverse. Nevertheless, we use the above convention, for convenience.) Keeping in mind that for $u \in \mathfrak{g}_\alpha$ and $v \in \mathfrak{g}_\beta$ we have

$$B(\exp(t)u, \exp(t)v) = e^{(\alpha+\beta)(t)} B(u, v)$$

it follows that the forms in V_λ are characterized by:

$$\alpha + \beta \neq \lambda \Rightarrow \mathfrak{g}_\alpha \perp \mathfrak{g}_\beta.$$

We apply this fact to the form $r(m) \in \mathfrak{g}^* \otimes \mathfrak{g}^*$. By our choice of m the form $r(m)$ is A-invariant up to a scalar multiple, hence there is a weight $\lambda \in \mathfrak{a}^*$ such that $r(m) \in V_\lambda$.

The proof will now proceed according to the following three steps. In step one we show that $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(p,q)$ or $\mathfrak{sl}(3,\mathbb{R})$. This fact is established by showing that only in these Lie algebras the weights, Λ, appearing in the $\mathfrak{a}$-module $\breve{\mathfrak{p}}$ give rise to an isotropic subspace, W, whose dimension is bounded by $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) - 1$. We also show that the parabolic subalgebra $\mathfrak{p}$ is uniquely determined (for a given $\mathfrak{g}$). Step one will take most of the effort and will be divided into two parts, one dealing with all root systems except A_l, and the other dealing with the case where the root system is of type A_l. In step two we show that $\mathfrak{p} = \mathfrak{h}$, so that $H = \mathrm{Stab}_G(m)$ is cocompact. In step three we show that the compact orbit $G/H \subset M$ is conformally equivalent to one of the standard models.

First we note the following facts and we refer to [8, Ch. X, pp. 530-535] for their verification.

Fact 5.1.

- A real simple Lie algebra is determined uniquely by the type of its real root system, together with the multiplicities of the long and the short roots [8, p. 535, Ex. 9].
- If the type of the real root system is B_l and the long roots have multiplicity one, then the Lie algebra is isomorphic to $\mathfrak{so}(l, l+n)$, where n is the multiplicity of the short roots [8, Table VI, see also Ex. 8].
- If the type of the real root system is D_l there is only one root lengths. If the multiplicity is one, then the Lie algebra is isomorphic to $\mathfrak{so}(l,l)$ [8, Table VI, see also Ex. 8].
- If the type of the real root system is A_l there is only one root length. If the multiplicity is one, then the Lie algebra is isomorphic to $\mathfrak{sl}(l+1,\mathbb{R})$ [8, Table VI].

Our strategy is to use the information that the dimension of an isotropic subspace is bounded by $l-1$ in order to determine the type of the Lie algebra, together with the multiplicities.

We will now focus our attention on the $\mathfrak{a}$-module $\breve{\mathfrak{p}}$. Denote the restriction of the form from $\breve{\mathfrak{h}}$ to $\breve{\mathfrak{p}}$ by $\langle\cdot,\cdot\rangle$. From now on we will consider only this form and only the space $\breve{\mathfrak{p}}$ unless stating otherwise. We suggest that it may be helpful to read the first paragraph of the Appendix for some definitions and notations.

Denote by Λ the set of weights (or roots) of the $\mathfrak{a}$-module $\breve{\mathfrak{p}}$. The existence of λ gives a natural splitting of the set Λ into three parts, $\Lambda = \Lambda_1 \cup \Lambda_3 \cup \Lambda_2$, where:

- $\Lambda_1 = \{\alpha \in \Lambda \mid \lambda - \alpha \in \Lambda, \lambda - \alpha \neq \alpha\}$. Λ_1 itself splits into *pairs of distinct roots*, $\{\alpha, \beta\}$ such that $\alpha + \beta = \lambda$. For such a pair, $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$ is perpendicular to its complement (in $\breve{\mathfrak{p}}$), and each of $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ is isotropic.

- $\Lambda_2 = \{\frac{1}{2}\lambda\} \cap \Lambda$. In this case we can not tell a-priori whether the space $\mathfrak{g}_{\frac{1}{2}\lambda}$ is isotropic or not. (Note that $\frac{1}{2}\lambda$ may fail to be a root.)

- $\Lambda_3 = \{\alpha \in \Lambda \mid \lambda - \alpha \notin \Lambda\}$. For such roots, $\mathfrak{g}_\alpha$ is radical with respect to the form $\langle \cdot\,, \cdot \rangle$ on $\breve{\mathfrak{p}}$.

Notice that any of the above sets might be empty.

Denote by Λ_1' a set consisting of one weight of each of the pairs in Λ_1. The following subspace of $\breve{\mathfrak{p}}$,

$$W = \bigoplus_{\alpha \in \Lambda_1'} \mathfrak{g}_\alpha \oplus \bigoplus_{\beta \in \Lambda_3} \mathfrak{g}_\beta$$

is an isotropic subspace of dimension at least $\lceil \frac{1}{2}(|\Lambda| - 1) \rceil$. Indeed, the dimension of the isotropic subspace W is at least

$$|\Lambda_1'| + |\Lambda_3| \geq |\Lambda_1'| + \frac{1}{2}|\Lambda_3| = \frac{1}{2}(|\Lambda| - |\Lambda_2|) \geq \frac{1}{2}(|\Lambda| - 1).$$

On the other hand, by assumption $\dim(W) \leq l - 1$, hence it follows that $|\Lambda| \leq 2l - 1$.

It turns out that the three conditions above, namely the upper bound on the number of weights appearing in the complement of a parabolic subalgebra, the additional structure imposed on Λ by the weight λ, and the upper bound on the dimension of an isotropic subspace, are very restrictive. We now turn to a case by case analysis to determine the possible couples $(\mathfrak{g}, \mathfrak{p})$ that might occur.

5.2 Beginning Step 1: Root systems other than A_l

We will make use of the following lemma, which is an immediate consequence of Lemma A.1 in the Appendix.

Lemma 5.2. *Let Φ be a root system of rank l, whose type is not A_l. Assume that for some maximal parabolic subalgebra $\mathfrak{p}$ the number of weights in the module $\breve{\mathfrak{p}}$ is less than $2l$. Then Φ is of type B_l, C_l or D_l, and $\mathfrak{p}$ is the standard parabolic subalgebra associated with the simple root α_1 (under the standard ordering of the simple roots, see* [8], *the table in page* 470 *and the discussion preceding).*

Utilizing this lemma we now analyze the set Λ and determine the multiplicities of the long and the short roots.

B_l $(l \geq 2)$**:** By Lemma 5.2, $\Lambda = \{e_1, e_1 \pm e_i \mid 1 < i \leq l\}$. We claim that the only possible splitting of Λ obtains for $\lambda = 2e_1$, and:

$$\Lambda_1 = \{e_1 \pm e_i \mid 1 < i \leq l\}$$

and

$$\Lambda_2 = \{e_1\},\ \Lambda_3 = \emptyset.$$

This claim follows because for any other choice of λ, $\Lambda_1' \cup \Lambda_2$ will contain at most one weight (indeed, the equation $\alpha + \beta = \lambda$ has at most one solution in Λ). Therefore

$$\dim(W) \geq |\Lambda_1'| + |\Lambda_3| = |\Lambda| - |\Lambda_1' \cup \Lambda_2| = 2l - 2$$

and $2l - 2 \geq l$ for $l \geq 2$.

Assuming $\lambda = 2e_1$, in order to have $\dim(W) \leq l-1$, the dimension of $\mathfrak{g}_{e_1-e_i}, \mathfrak{g}_{e_1+e_i}$ must be exactly one. The latter dimension is the multiplicity of the long roots. Denoting the multiplicity of the short roots by n, Fact 5.1 implies that $\mathfrak{g} \simeq \mathfrak{so}(l+n, l)$. This possibility is realized in the action of $\mathrm{SO}(l+n, l)$ on the standard model $C^{l+n-1,l-1}$ (see §2.4).

C_l $(l \geq 3)$**:** By Lemma 5.2, $\Lambda = \{2e_1, e_1 \pm e_i \mid 1 < i \leq l\}$. One can see, as in the previous case, that λ must be equal to $2e_1$. On the other hand, λ can not be equal to $2e_1$, because then $2e_1 \in \Lambda_3$, and choosing $e_1 + e_i \in \Lambda_1'$ for $1 < i \leq l$, we get $\dim(W) \geq l$, contradicting our assumptions.

Hence the case that Φ is of type C_l, $l \geq 3$ can not arise (observe that the case C_2 is covered by B_2).

D_l $(l \geq 4)$**:** By Lemma 5.2, $\Lambda = \{e_1 \pm e_i \mid 1 < i \leq l\}$. As in the previous cases it is easy to see that in order to have $\dim(W) \leq l - 1$

we must have $\lambda = 2e_1$. In this case, again, the *pairs* must be $\{e_1 + e_i, e_1 - e_i\}$, but in contrast to the B_l case, Λ_2 is empty. Clearly, the dimension of $\mathfrak{g}_{e_1-e_i}, \mathfrak{g}_{e_1+e_i}$ must be exactly one. This latter dimension is the multiplicity of the long (and all) roots. Once again, by Fact 5.1, we get $\mathfrak{g} \simeq \mathfrak{so}(l,l)$. This possibility is realized in the action of $\mathrm{SO}(l,l)$ on the standard model $C^{l-1,l-1}$ (see §2.4).

For $l = 4$ the parabolic subalgebras associated with α_1, α_3 and α_4 are mapped to one another by an outer automorphism of order three. Hence there exist three distinct homogeneous actions of $\mathrm{SO}(4,4)$ on $C^{3,3}$ (see §2.4). Finally, we note that the case of D_3 is covered by the A_3 case discussed below.

5.3 Concluding Step 1: The root system A_l

We now turn to the case where Φ is of type A_l. We have already seen examples of conformal actions of groups of type A_l, for $l = 1, 2, 3$. Recall that $\mathrm{SO}(n,1)$ is of type A_1 and it acts conformally on S^{n-1}, $\mathrm{SL}(3,\mathbb{R})$ is of type A_2 and it acts conformally on the symplectic manifold S^2, and $\mathrm{SO}(3,3)$ is of type A_3 and it acts conformally on $C^{2,2}$ (see §2.4). We will show that groups of type A_l, $l > 3$ can not act conformally without having an l-dimensional isotropic subspace. Also in the cases $l = 1, 2, 3$ we determine what the parabolic subgroup $\mathfrak{p}$ must be. We continue our analysis of the set Λ of weights appearing in $\check{\mathfrak{p}}$ and the multiplicities of the roots, as in the §5.2.

We first consider the case $l = 1$.

$l = 1$: Denoting the roots multiplicity by n, it follows that $\mathfrak{g} \simeq \mathfrak{so}(n+1,1)$. $\mathfrak{p}$ must be the only proper standard parabolic subalgebra.

This possibility is realized in the action of $\mathrm{SO}(n+1,1)$ on the standard model $C^{n,0} \simeq S^n$ (see §2.4).

In order to handle the case where $l > 1$ we will use the following lemma, which follows immediately from Lemma A.2 in the Appendix.

Lemma 5.3. *Let Φ be a root system of type A_l. Let $\mathfrak{p}$ be a maximal parabolic subalgebra which is different than those associates to the simple roots α_1 or α_l. We then have the following:*

- *The number of weights in the module $\check{\mathfrak{p}}$ satisfies $|\Lambda| \geq l+1$.*

- *Assume $l \geq 3$. If $|\Lambda| = l+1$ then $l = 3$ and $\mathfrak{p}$ is the maximal parabolic subalgebra associated with α_2.*

The roots of A_l are $e_i - e_j$ $(1 \leq i, j \leq l+1)$. One can easily see that Λ_1 contains at most two pairs. The case of two pairs arises for example when $\lambda = e_1 + e_2 - e_3 - e_4$, and $\Lambda_1 = \{e_1 - e_3, e_2 - e_4, e_1 - e_4, e_2 - e_3\}$. Then

$$\lambda = (e_1 - e_3) + (e_2 - e_4) = (e_1 - e_4) + (e_2 - e_3).$$

One can see also that Λ_1 and Λ_2 can not both be nonempty, because all roots are of the same length (the equation $\alpha + \beta = 2\gamma$ does not have a nontrivial solution where α, β and γ are on the circle).

In case Λ_1 is not empty we have (since Λ_1 has at most two pairs)

$$l - 1 \geq \dim(W) \geq |\Lambda_1'| + |\Lambda_3||\Lambda| - \frac{1}{2}|\Lambda_1| \geq |\Lambda| - 2.$$

In case Λ_2 is not empty we have

$$l - 1 \geq \dim(W) \geq |\Lambda_3| = |\Lambda| - |\Lambda_2| \geq |\Lambda| - 1.$$

Either way we get $|\Lambda| \leq l+1$. By Lemma 5.3, this inequality can occur in only three cases:

$l = 3$ and $\mathfrak{p}$ is the maximal parabolic subalgebra associated to α_2.

This is case two of Lemma 5.3. Here

$$\Lambda = \Lambda_1 = \{e_1 - e_3, e_2 - e_4, e_1 - e_4, e_2 - e_3\}$$

and $\dim(W) = |\Lambda_1'| = 2 = l-1$ only if the roots multiplicity is one. In this case (according to Fact 5.1) $\mathfrak{g} \simeq \mathfrak{sl}(4, \mathbb{R}) \simeq \mathfrak{so}(3, 3)$. This possibility is realized in the action of $\mathrm{SO}(3,3)$ on the standard model $C^{2,2}$ (see §2.4).

l is arbitrary, and $\mathfrak{p}$ is the maximal parabolic associated with α_1.

In this case

$$\Lambda = \{e_1 - e_j \mid 2 \leq j \leq l+1\}$$

and $|\Lambda| = l$. Since the sums of distinct weights in Λ are distinct, Λ_1 can contain at most one pair. It follows that $|\Lambda_1'| + |\Lambda_3| \geq l-1$ (whether Λ_1 is empty or Λ_2 is empty). The dimension of W is then greater or equal to $l-1$ times the roots multiplicity. At the same time, by assumption, it is smaller or equal to $l-1$ so the roots multiplicity is one, hence (by Fact 5.1) $\mathfrak{g} \simeq \mathfrak{sl}(l+1, \mathbb{R})$.

From now on we will consider the form $r(m)$ on $\mathfrak{g}$. First notice that in the case $\mathfrak{g} \simeq \mathfrak{sl}(l+1, \mathbb{R})$, we can and will assume that the parabolic subalgebra $\mathfrak{p}$ actually coincides with $\mathfrak{q}$, by Lemma 3.2. In particular $\mathfrak{p}$ acts conformally on $\mathfrak{g}$ (by the definition of $\mathfrak{q}$). Equivalently, we have that $\langle [X_\alpha, X_\beta], X_\gamma\rangle + \langle X_\beta, [X_\alpha, X_\gamma]\rangle$ is a constant multiple of $\langle X_\beta, X_\gamma\rangle$, for $\beta, \gamma \in \Lambda$ and $\alpha \notin \Lambda$ (namely when $X_\alpha \in \mathfrak{p}$). We always assume $X_\alpha \in \mathfrak{g}_\alpha$ and so on.

We now proceed to show that if $l > 2$ then the form has an l-dimensional isotropic subspace, namely $\breve{\mathfrak{p}}$ itself, which contradicts our assumption. We focus our attention on the radical of the form. First note that if $l > 2$ then Λ_3 is nonempty, hence the radical is nonzero. Indeed, Λ has l weights, and $|\Lambda_1 \cup \Lambda_2|$ is at most two. On the other hand, we claim that if the radical is nonzero, it must coincide with $\mathfrak{p}$.

Recall that, since any root space is one dimensional, we have that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta, \alpha+\beta \in \Phi$.

Assume $\beta \in \Lambda_3$ (and therefore, $\mathfrak{g}_\beta$ is radical) and $\gamma \in \Lambda$ is arbitrary. We will show that $\mathfrak{g}_\gamma$ is radical. Let $\delta \in \Lambda$ be arbitrary root. If $\delta = \beta$ then $\langle X_\gamma, X_\delta\rangle = \langle X_\delta, X_\gamma\rangle = 0$. Assume $\delta \neq \beta$, and note that $\delta, \beta \in \Lambda$, hence they are of the form $e_1 - e_i, e_1 - e_j$. So $\delta - \beta = e_j - e_i$ is in $\Phi - \Lambda$. Denoting $\alpha = \delta - \beta \in \Phi - \Lambda$, given $X_\beta \in \mathfrak{g}_\beta$ and $X_\delta \in \mathfrak{g}_\delta$ there exist X_α such that $[X_\alpha, X_\beta] = X_\delta$. Since $\alpha \in \Phi - \Lambda$, X_α is in $\mathfrak{p}$, hence acts conformally. Since X_β is radical we have

$$\langle [X_\alpha, X_\gamma], X_\beta\rangle + \langle X_\gamma, [X_\alpha, X_\beta]\rangle = C \cdot \langle X_\gamma, X_\beta\rangle = 0$$

hence

$$\langle X_\gamma, X_\delta\rangle = \langle X_\gamma, [X_\alpha, X_\beta]\rangle = -\langle [X_\alpha, X_\gamma], X_\beta\rangle = 0$$

and similarly

$$\langle X_\delta, X_\gamma\rangle = \langle [X_\alpha, X_\beta], X_\gamma\rangle = -\langle X_\beta, [X_\alpha, X_\gamma]\rangle = 0.$$

This proves that $\mathfrak{p}$ is contained in the radical if $l > 2$.

We conclude that $l = 2$. Since the parabolic subalgebra we consider here is the one associated to α_1, we have $\Lambda = \Lambda_1 = \{e_1 - e_2, e_1 - e_3\}$. This possibility is realized in the action of $\mathrm{SL}(3,\mathbb{R})$ on the standard model S^2, (see §2.4).

l is arbitrary, and $\mathfrak{p}$ is the maximal parabolic associated with α_l.

The parabolic subalgebra associated with α_l is the image of the parabolic subalgebra associated with α_1 under an outer automorphism of the Lie algebra. This implies that for $l = 2$ (and only in this case) there is another conformal action of $\mathrm{SL}(3,\mathbb{R})$. This action is on the Grasmann variety of two planes in $\mathbb{R}^3$ which is one of the standard models in §2.4.

This concludes the discussion of all possible cases, and the determination of the possible pairs $(\mathfrak{g},\mathfrak{p})$ promised in Step 1.

5.4 Steps 2 and 3: Existence of a compact orbit, finitely covering a standard model

We first summarize the notations and conclusion of the discussion so far. We are given a compact manifold with a bilinear structure M, and a simple group G of real rank l, acting conformally on M, and we assume that the dimension of a maximal isotropic subspace is bounded by $l-1$. We have focused our attention on a certain point $m \in M$ and denoted its stability group by H. We have seen that H is contained in a proper parabolic subgroup which we have denoted by P. The pair $(\mathfrak{g},\mathfrak{p})$ must be one of the pairs described in Step 1 (§5.2, 5.3). Furthermore, each of those pairs is realized by an action on a standard model.

We now show that the Lie algebras $\mathfrak{h}$ and $\mathfrak{p}$ coincide. Indeed, the dimension of the isotropic subspace $W < \breve{\mathfrak{p}}$ is exactly $l-1$. This follows from examination of all possible cases in step one. Another fact that follows from this examination is the following observation: in all possible cases $\lambda - \Lambda = \Lambda$, or in other words, $\alpha \in \Lambda$ if and only if $\lambda - \alpha \in \Lambda$. As a result, if $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{p}$ then there exists a root α such that $\mathfrak{g}_\alpha < \breve{\mathfrak{h}}$ and $\mathfrak{g}_{\lambda-\alpha} \cap \breve{\mathfrak{p}} = 0$. Since $\alpha + \beta \neq \lambda \Rightarrow \mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$, It follows that $W \oplus \mathfrak{g}_\alpha$ is an isotropic subspace that contain W properly, hence its dimension is at least l, which is a contradiction.

We conclude that $\mathfrak{h} = \mathfrak{p}$ and H is of finite index in the parabolic subgroup P of G. The orbit Gm is isomorphic to a finite cover G/P, and in particular is compact. The same conclusion holds for any closed orbit. To see that, assume $M_1 \subset M$ is a closed orbit. Note first that we have already seen that M contains no fixed points (see the proof of Lemma 4.2). Upon reflection one sees that the arguments used so far in the proof of Theorem 2 apply equally well to the subbundle of the tangent bundle which is tangent to the orbits. Hence they apply equally well when we consider the action of G on M_1. It follows that the stability group of a point in M_1 is of finite index in P. This conclude the proof of step two, namely the fact that every closed G-orbit is isomorphic as a G-space to a finite cover of G/P.

It remains to prove step three, namely that a closed orbit is *conformally* equivalent to a finite cover of one of the standard models. First notice that one can substitute a standard model instead of M in the arguments of step one and two. This is permissible since the bound $l-1$ on the dimension of an isotropic subspace is satisfied for the standard models, as we invite the reader to verify. Since the action on the standard model is transitive, we deduce that the model action is conformally and equivariantly isomorphic to G/P. Clearly, it remains to show that this conformal structure on G/P (the one which makes it conformally equivalent to the standard model) is unique.

This follows from the fact that $\mathfrak{p} < \mathfrak{so}(p+1, q+1)$ contains a copy, $\mathfrak{s}$, of $\mathfrak{so}(p,q)$ (or $\mathfrak{sp}(1,\mathbb{R})$ if $\mathfrak{g} \simeq \mathfrak{sl}(3,\mathbb{R})$) whose action on $T_m(G \cdot m) \simeq \mathfrak{g}/\mathfrak{p}$ must be faithful. Indeed, otherwise, by Lemma 3.5, its action on the orbit is trivial, hence the action of G itself is trivial because it has a nontrivial kernel. The conformal action of $\mathfrak{s}$ on $\mathfrak{g}/\mathfrak{h} = \mathfrak{g}/\mathfrak{p} \simeq \mathbb{R}^{p+q}$ (or the action of $\mathfrak{sp}(1,\mathbb{R})$ on $\mathfrak{g}/\mathfrak{h} = \mathfrak{g}/\mathfrak{p} \simeq \mathbb{R}^2$) is actually orthogonal, by simplicity. It is easy to see that any $\mathfrak{s}$-invariant form on $\mathbb{R}^{p+q}$ (or $\mathfrak{sp}(1,\mathbb{R})$-invariant form on $\mathbb{R}^2$) is a constant times the usual one. To see this fact, note that $\mathfrak{s}$ is a real form of $\mathfrak{s}_{\mathbb{C}} = \mathfrak{so}(p+q,\mathbb{C})$ (or $\mathfrak{sp}(1,\mathbb{C})$), which acts irreducibly on $\mathbb{C}^{p+q}$ (or $\mathbb{C}^2$), and use the following:

Claim. If S is a semisimple subgroup of $\mathrm{GL}(n,\mathbb{R})$ and $S_{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ then the space of S-invariant forms on $\mathbb{R}^n$ is at most one dimensional.

The claim follows from the following three facts: Every two linearly independent complex forms have a degenerate (complex) linear combination, the radical of an invariant complex form is an invariant complex subspace, and finally, S is Zariski dense in $S_{\mathbb{C}}$.

We see that the conformal structure on G/P coincides with that of the standard model in the tangent space to a certain point, hence, by Lemma 2.6, it coincides everywhere.

This concludes the proof of Theorem 2.

6. Some further results and remarks

We list here for completeness some results that follow quite easily from the discussion above, and will be very brief in their explanation.

6.1 A generalization of Theorem 1

Notice that in the proofs of Lemma 4.1 and hence of Theorem 1, no harm is done if we assume that the form is not necessarily bilinear, but multilinear. We will define:

Definition 6.1. A *manifold with a k-linear structure* is a couple (M, s), such that M is a C^∞-manifold and $s : M \to T^*M^{\otimes k}$ is a C^∞-section. We call s *the structure of* (M, s).

A subspace of a vector space with a multilinear structure is called *isotropic* if the restriction of the form to this subspace is the zero form. The proof of Theorem 1 applies without change to show the following:

Theorem 5. *Let G be a connected almost simple real Lie group with finite center. Assume G acts conformally (and nontrivially) on a compact manifold with a multilinear structure M. Then there exists some point $m \in M$, where the multilinear form on $T_m(M)$ has an isotropic subspace of dimension at least* $\mathrm{rk}_{\mathbb{R}}(G) - 1$.

6.2 The isometric case

We now comment briefly on the case where the action of the almost simple group G on the manifold M is isometric. The result below should be compared with [7, 5.3] and [18] where it is obtained for pseudo-Riemannian manifolds under the assumption that the volume induced by the pseudo-Riemannian structure is finite, and by use of the Borel density theorem. Below, we do not need an invariant volume form, and we allow the bilinear (or multilinear) forms to be singular, for example.

Theorem 6. *Let G be a connected almost simple noncompact real Lie group with finite center. Assume G acts* isometrically *(and nontrivially) on a compact manifold with a multilinear structure M. Then for* *every point $m \in M$, the form induces on $\mathfrak{g}$, $r(m)$, is a G-invariant form (with respect to the adjoint representation of G on $\mathfrak{g}$).*

Proof. Consider the equivariant map $r : M \to \mathfrak{g}^{*\otimes k}$ (see §2.3). Clearly $U = \mathrm{span}(r(M))$ is a G invariant subspace, and every G-orbit in it is bounded. This property is established as follows. If $u_1 \dots u_k \in r(M)$ span U, then writing $u = b_1 u_1 + \dots b_k u_k$, we have that $G \cdot u$ is bounded since each $G \cdot u_i$ is bounded, being contained in the compact set $r(M)$. Therefore given an $\mathbb{R}$-split torus, its connected component must act trivially (i.e., with eigenvalues equal to 1). It follows that the action of G itself is trivial. q.e.d.

Note that a similar proof can be given for any real algebraic group which does not have a compact factor.

Consider again the case of bilinear forms. Here it is easy to classify real G-invariant forms on $\mathfrak{g}$. For an almost simple *complex* group, there is evidently a unique (up to a scalar multiple) invariant complex form, that is the Killing form. If G is real there are two possibilities.

$\mathfrak{g}$ has no complex structure: $G_{\mathbb{C}}$ is then almost simple as well (see e.g., [10, 6.94]). Because G is Zariski dense in $G_{\mathbb{C}}$ there is a unique (up to scalars) G-invariant complex bilinear form on $\mathfrak{g}_{\mathbb{C}}$. It follows that the space of real invariant forms on $\mathfrak{g}$ is also one dimensional, and so must be spanned by the Killing form on $\mathfrak{g}$.

$\mathfrak{g}$ has a complex structure: Denote $\mathfrak{g} = \mathfrak{s}_{\mathbb{R}}$ for some complex simple Lie algebra $\mathfrak{s}$. $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{s} \oplus \mathfrak{s}$ (see e.g., [10, 6.94]). So the space of real invariant forms on $\mathfrak{g}$ is 2-dimensional and is obviously spanned by the real and imaginary parts of the Killing form of $\mathfrak{s}$. Observe that the real part of the Killing form of $\mathfrak{s}$ is equal the Killing form of $\mathfrak{g}$.

We call G *absolutely simple* if $\mathfrak{g}$ has no complex structure. We can now state (compare [7, 5.3] and [18]).

Corollary 7. *Let G be a connected almost simple noncompact real Lie group with finite center. Assume G acts* isometrically *(and non-trivially) on a compact manifold with a bilinear structure M. If G is absolutely almost simple , then for every point $m \in M$, the form induced on $\mathfrak{g}$, $r(m)$, is a scalar multiple of the Killing form. In general, the induced form is invariant, symmetric, nondegenerate and of the same signature as the Killing form, unless it is the zero form.*

Finally, we observe that if the bilinear structure on M gives rise to an invariant volume form of finite total mass, then the foregoing result can easily be deduced from Borel's density Theorem. Here one need not assume that M is compact. For the proof, one notes that the image of the measure on M under the map r is supported on G-fixed forms on $\mathfrak{g}$. Hence for almost every point in M the induced form on $\mathfrak{g}$ is G-invariant. The set of such points is obviously dense and closed. Hence the same conclusion applies to every point in M.

6.3 A lower bound for minimum codimension of closed subgroups

Lemma 3.4 gives the lower bound $\mathrm{rk}_{\mathbb{R}}(G)$ for the codimension of closed subgroups of an almost simple group. The proof shows also that the subgroup of minimal codimension is always a maximal parabolic subgroup. An elaborate list of the maximal parabolic subgroup of minimum codimension in every simple group appears in [16]. We need only an estimate which we establish directly by an elementary argument.

In the Appendix below we actually calculate a lower bound to the codimension of all maximum dimensional parabolic subgroups of every simple group. In particular, it turns out that the bound in Lemma 3.4 is realized only for groups locally isomorphic to $\mathrm{SL}(n,\mathbb{R})$. We conclude:

Corollary 8. *Let G be an almost-simple Lie group which is not locally isomorphic to $\mathrm{SL}(n,\mathbb{R})$, G' a proper closed subgroup of G. Then*

$$\mathrm{codim}(G') \geq 2 \cdot \mathrm{rk}_{\mathbb{R}}(G) - 2.$$

This bound is realized only in groups locally isomorphic $SO(n,n)$.

We remark that the foregoing corollary implies that the map $\bar{r} : M \to \mathbb{P}(\mathfrak{g}^* \otimes \mathfrak{g}^*)$ exists under weaker conditions than those stated in Lemma 4.2 (or the assumptions of Theorem 2). This map gives a nontrivial smooth projective factor defined on M.

Appendix

A. Standard parabolic subalgebras

Let us recall some facts about the structure of standard parabolic subalgebras first (compare e.g., [10, 9]). Let $\mathfrak{g}$ be a simple real Lie algebra. Let $\mathfrak{a}$ be a maximal $\mathbb{R}$-split abelian subalgebra of $\mathfrak{g}$, and Let $\Phi \subset \mathfrak{a}^*$ be the root system related to $\mathfrak{a}$. Let Π be the set of positive roots associated to a fixed chosen positivity, and Δ be the related set of simple roots. For every subset $I \subset \Delta$ one can define $\Phi_I = \Phi \cap \operatorname{span}(I)$ and $\Pi_I = \Pi \cap \operatorname{span}(I)$. Define also

$$\mathfrak{p}_I = \mathfrak{g}_0 \oplus \bigoplus_{-\alpha\in\Pi} \mathfrak{g}_\alpha \oplus \bigoplus_{\beta\in\Pi_I} \mathfrak{g}_\beta$$

$\breve{\mathfrak{p}}_I$, the direct $\mathfrak{a}$-module complement to $\mathfrak{p}_I$, is then given by

$$\breve{\mathfrak{p}}_I = \bigoplus_{\alpha\in\Pi-\Pi_I} \mathfrak{g}_\alpha$$

Write $\breve{\Pi}_I = \Pi - \Pi_I$. It is clear that $\dim(\breve{\mathfrak{p}}_I) \geq |\breve{\Pi}_I|$. The latter is very easy to compute given the type of Φ and I, using the fact that if Γ is the Dynkin diagram of Φ then Γ_I, the diagram of the root system Φ_I, is obtained from Γ by erasing all vertices not in I, and all edges connected to them. $|\Pi|$ is given by the following table (see e.g., [9, p. 66]):

Type	A_l	B_l, C_l	$(BC)_l$	D_l	E_6	E_7	E_8	F_4	G_2
Number of Positive Roots	$\binom{l+1}{2}$	l^2	l^2+l	l^2-l	36	63	120	24	6

Notice that Φ_I is not irreducible in general, and we have $\Phi_I = \prod_i \Phi_{J_i}$, where Γ_{J_i} are the connected components of Γ_I. We deduce $|\Pi_I| = \Sigma_i |\Pi_{J_i}|$.

In what follows, the simple roots of a given root system are ordered according to the standard order as in [8, Ch. X, §3].

Lemma A.1. *Let Φ be a root system of rank l, whose type is not A_l. Let Δ be the set of simple roots, and I a subset of Δ such that $|\breve{\Pi}_I| < 2l$. It follows that Φ is of type B_l, C_l or D_l, and $I = \Delta - \{\alpha_1\}$, unless $\Phi = D_4$. In the latter case $I = \Delta - \{\alpha_3\}$, and $I = \Delta - \{\alpha_4\}$ occur also.*

Proof. Denote $I_i = \Delta - \{\alpha_i\}$. It is enough to proof the Lemma assuming $I = I_i$ for some $i \in \Delta$, for then if $I \neq I_1$ then $I \subset I_i$ for some $i > 1$ and $|\breve{\Pi}_I| \geq |\breve{\Pi}_{I_i}| \geq 2l$.

We do a case by case analysis.

B_l, C_l ($l \geq 2$): Assume first that $l \geq 3$ and $I = I_i = \Delta - \{\alpha_i\}$ for some $2 \leq i \leq l$. Define $J_1 = \{1, \ldots, i-1\}, J_2 = \{i+1, \ldots, l\}$. $I_i =$

$J_1 \cup J_2$ and $\Gamma_{J_1}, \Gamma_{J_2}$ are the connected components of Γ_I. $\Gamma_{J_1}, \Gamma_{J_2}$ are Dynkin diagrams of root systems of type A_{i-1} and B_{l-i} (or C_{l-i}) respectively, so Φ_I is of type $A_{i-1} \times B_{l-i}$ (or $A_{i-1} \times C_{l-i}$). From the table above we deduce $|\breve{\Pi}_I| = |\Pi| - (|\Pi_{J_1}| + |\Pi_{J_2}|) = l^2 - (\binom{(i-1)+1}{2} + (l-i)^2) = \frac{1}{2}i(4l-3i+1) \geq 2l$.

For $l = 2$, $|\breve{\Pi}_\emptyset| = 4 = 2l$ and $|\breve{\Pi}_{I_1}| = |\breve{\Pi}_{I_2}| = 3 = 2l - 1$.

Note that B_2 and C_2 are the same root system, but with a different order of the two simple roots. Therefore the pair B_2, I_2 is equivalent to C_2, I_1, and the pair C_2, I_2 is equivalent to B_2, I_1. Hence these extra solutions are redundant.

We summarize: $|\breve{\Pi}_I| \leq 2l - 1$ implies $I = I_1$ (and $|\breve{\Pi}_{I_1}| = 2l - 1$).

$(BC)_l$ $(l \geq 1)$**:** Assume $l = 1$. Then $\Pi = \{\alpha, 2\alpha\}$, $\Delta = \{\alpha\}$, and I, which is a proper subset of Δ, must be the empty set. Then $|\breve{\Pi}_I| = 2 = 2l$.

We now assume $l \geq 2$. As before, we assume $I = I_i$ for some i. The root systems of types B_l and C_l are subsystems of Φ so $|\breve{\Pi}_I^{(BC)_l}|$ is at least $|\breve{\Pi}_I^{B_l}|$ or $|\breve{\Pi}_I^{C_l}|$. Recall that $\breve{\Pi}_I^{(BC)_l}$ is exactly the set of positive roots whose expansion as a linear combination of simple roots contains a nonzero multiple of α_i. α_i belongs to one of the subsystems B_l or C_l, so the claim follows. In light of the previous discussion of B_l and C_l, we may assume $I = I_1$. We conclude by exhibiting a root in $\breve{\Pi}_I^{(BC)_l}$, which is not in B_l. Indeed, it is easy to check that $2\alpha_1$ is a positive integral linear combination of simple roots, with a nonzero α_1 component.

D_l $(l \geq 4)$**:** Assume first that $I = I_i = \Delta - \{\alpha_i\}$ for some $2 \leq i \leq l-2$. Define $J_1 = \{1, \ldots, i-1\}, J_2 = \{i+1, \ldots, l\}$. $I_i = J_1 \cup J_2$ and $\Gamma_{J_1}, \Gamma_{J_2}$ are the connected components of Γ_I. $\Gamma_{J_1}, \Gamma_{J_2}$ are Dynkin diagrams of root systems of type A_{i-1} and D_{l-i} correspondingly, so Φ_I is of type $A_{i-1} \times D_{l-i}$. From the table above we deduce $|\breve{\Pi}_I| = |\Pi| - (|\Pi_{J_1}| + |\Pi_{J_2}|) = (l^2 - l) - (\binom{(i-1)+1}{2} + (l-i)^2 - (l-i)) = \frac{1}{2}i(4l - 3i - 1) \geq 2l$.

Assume $l \geq 5$. If $i = l-1$ or l then Φ_{I_i} is of type A_{l-1} and $|\breve{\Pi}_I| = (l^2 - l) - \binom{l}{2} = \frac{1}{2}l(l-1) \geq 2l$.

We summarize: For $l > 4$, $|\breve{\Pi}_I| \leq 2l - 1$ implies $I = I_1$ (and $|\breve{\Pi}_{I_1}| = 2l - 2$). For $l = 4$ the possibilities $I = I_3$ and $I = I_4$ also occur.

E_6**:** Deleting α_1 or α_6 leaves D_5, and $|\breve{\Pi}_I| = 14$.
Deleting α_2 leaves A_5. $|\breve{\Pi}_I| = 21$.
Deleting α_3 or α_5 leaves $A_1 \times A_4$. $|\breve{\Pi}_I| = 25$.
Deleting α_4 leaves $A_2 \times A_1 \times A_2$, and $|\breve{\Pi}_I| = 29$.
In any case $|\breve{\Pi}_I| > 12 = 2l$.

E_7**:** $|\breve{\Pi}_{I_1}| = 33$ $|\breve{\Pi}_{I_2}| = 42$ $|\breve{\Pi}_{I_3}| = 47$
$|\breve{\Pi}_{I_4}| = 53$ $|\breve{\Pi}_{I_5}| = 50$ $|\breve{\Pi}_{I_6}| = 42$
$|\breve{\Pi}_{I_7}| = 27$
In any case $|\breve{\Pi}_I| > 14 = 2l$.

E_8**:** $|\breve{\Pi}_{I_1}| = 78$ $|\breve{\Pi}_{I_2}| = 92$ $|\breve{\Pi}_{I_3}| = 98$
$|\breve{\Pi}_{I_4}| = 106$ $|\breve{\Pi}_{I_5}| = 104$ $|\breve{\Pi}_{I_6}| = 97$
$|\breve{\Pi}_{I_7}| = 83$ $|\breve{\Pi}_{I_8}| = 57$
In any case $|\breve{\Pi}_I| > 16 = 2l$.

F_4**:** $|\breve{\Pi}_{I_1}| = 15$ $|\breve{\Pi}_{I_2}| = 20$
$|\breve{\Pi}_{I_3}| = 20$ $|\breve{\Pi}_{I_4}| = 15$
In any case $|\breve{\Pi}_I| > 8 = 2l$

G_2**:** $|\breve{\Pi}_{I_1}| = |\breve{\Pi}_{I_2}| = 5 > 4 = 2l$

q.e.d.

Note that in all cases above $|\breve{\Pi}_I| \geq 2l - 2$. We deduce that if for some I, $|\breve{\Pi}_I| < 2l - 2$ than Φ is of type A_l.

Lemma A.2. *Let Φ be a root system of type A_l. Let Δ be the set of simple roots, and I a subset of Δ which is not equal to $\Delta - \{\alpha_1\}$ or $\Delta - \{\alpha_l\}$. We then have the following:*

- $|\breve{\Pi}_I| \geq l + 1$.
- *Assume $l \geq 3$. If $|\breve{\Pi}_I| = l + 1$ then $l = 3$ and $I = \Delta - \{\alpha_2\}$.*

Proof. We prove the first statement first. Denote $I_i = \Delta - \{\alpha_i\}$. $|\breve{\Pi}_{I_1}| = |\breve{\Pi}_{I_l}| = l$, hence it is enough to proof the Lemma assuming $I = I_i$ for some $2 \leq i \leq l - 1$, because for any other I, $I \subsetneq I_i$ for some i, and then $|\breve{\Pi}_I| > |\breve{\Pi}_{I_i}|$. Assuming $I = I_i$ we can assume $l \geq 3$.

Define $J_1 = \{1, \ldots, i-1\}, J_2 = \{i+1, \ldots, l\}$. $I_i = J_1 \cup J_2$ and $\Gamma_{J_1}, \Gamma_{J_2}$ are the connected components of Γ_I. $\Gamma_{J_1}, \Gamma_{J_2}$ are Dynkin diagrams of root systems of type A_{i-1} and A_{l-i} correspondingly, so Φ_I is of type $A_{i-1} \times A_{l-i}$. From the table above we deduce

$$\begin{aligned}|\breve{\Pi}_I| &= |\Pi| - (|\Pi_{J_1}| + |\Pi_{J_2}|) \\ &= \binom{l+1}{2} - \binom{(i-1)+1}{2} - \binom{(l-i)+1}{2} \\ &= i(l+1-i).\end{aligned}$$

Assuming $l \geq 3$, $2 \leq i \leq l-1$ we obtain

$$|\breve{\Pi}_{I_i}| = i(l+1-i) \geq 2(l-1) \geq l+1. \tag{2}$$

For proving the second statement observe first that for $I = \Delta - \{\alpha_1, \alpha_l\}$, $|\breve{\Pi}_I| = 2l-1 > l+1$. Therefore, it is enough to assume that $I = I_i$ for some $2 \leq i \leq l-1$, because for any other I, $I \subsetneq I_i$ for some $2 \leq i \leq l-1$, and then $|\breve{\Pi}_I| > |\breve{\Pi}_{I_i}|$.

In order to have equality in equation (2), one must have $2(l-1) = l+1$, which implies $l = 3$, and $i(4-i) = i(l+1-i) = 2(l-1) = 4$, which implies $i = 2$. q.e.d.

References

[1] S. Adams & G. Stuck, *The isometry group of a compact Lorentz manifold*, I, Invent. Math. **129** (1997) 239–262, MR 98i:53092, Zbl 0897.53046.

[2] S. Adams & G. Stuck, *The isometry group of a compact Lorentz manifold*, II, Invent. Math. **129** (1997) 263–287, MR 98i:53092, Zbl 0897.53047.

[3] S. Adams & G. Stuck, *Conformal action of* $\mathfrak{sl}(n,\mathbb{R})$ *and* $\mathrm{SL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$ *on Lorentz manifolds*, Trans. Amer. Math. Soc. **352**(**9**) (2000), 3913–3936. MR 2000m:53100, Zbl 0976.53076.

[4] A. Borel, *Linear Algebraic Groups*, Springer-Verlag, 1991, MR 92d:20001, Zbl 0726.20030.

[5] A. Borel & J. Tits, *Elements unipotents et sous-groupes paraboliques de groupes reductifs*, I, Invent. Math. **12** (1971) 95–104, MR 45 #3419, Zbl 0238.20055.

[6] J. Ferrand, *The action of conformal transformations on a Riemannian manifold*, Math. Annalen **304** (1996) 277–291, MR 97c:53044, Zbl 0866.53027.

[7] M. Gromov, *Rigid transformation groups*, Geometrie Differntielle (Paris, 1986), Travaux en Cours, **33**, Hermann, Paris, 1988, 65–139, MR 90d:58173, Zbl 0652.53023.

[8] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, 1978, MR 80k:53081, Zbl 0451.53038.

[9] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972, MR 48 #2197, Zbl 0254.17004.

[10] A.W. Knapp, *Lie Groups Beyond an Introduction*, Birkhauser, 1996, MR 98b:22002, Zbl 0862.22006.

[11] S. Kobayashi, *Transformation Groups in Differential Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete., **70**, Springer-Verlag, Berlin, 1972, MR 50 #8360, Zbl 0246.53031.

[12] N. Kowalsky, *Actions of non-compact simple groups on Lorentz manifolds*, C. R. Acad. Sci. Paris Sér. I Math. **321**(**5**) (1995), 595–599, MR 96g:53093, Zbl 0836.53040.

[13] N. Kowalsky, *Noncompact simple automorphism groups of Lorentz manifolds and other geometric manifolds*, Ann. of Math. **144**(**3**) (1996), 611–640, MR 98g:57059, Zbl 0871.53048.

[14] J. Lelong-Ferrand, *Transformations conformes et quesi-conformes des varietes Riemanniennes compactes*, Mem. Acad. Royale de Belgique **39**(**5**) (1971) 1–44, MR 48 #1100, Zbl 0215.50902.

[15] M. Obata, *The conjectures about conformal transformation*, J. Differential Geometry **6** (1971) 247–258, MR 46 #2601, Zbl 0236.53042.

[16] G. Stuck, *Low dimensional actions of semisimple groups*, Israel Journal of Mathematics **76** (1991) 27–71, MR 93i:57049, Zbl 0769.57022. Correction: Israel Journal of Mathematics **80** (1992), 255, CMP 1 248 938.

[17] G. Stuck, *Minimal actions of semisimple groups*, Ergodic Theory and Dynamical Systems **16** (1996) 821–831, MR 98a:57046, Zbl 0912.57022.

[18] A. Zeghib, *On affine actions of Lie groups*, Mathematisch Zeitschrift **227** (1998) 245–262, MR 99c:53077, Zbl 0914.57026.

[19] A. Zeghib, *The identity component of the isometry group of a compact Lorentz manifold*, Duke Math. J. **92** (1998) 321–333, MR 98m:53091, Zbl 0976.53077.

[20] R.J. Zimmer, *On the automorphism group of a compact Lorentz manifold and other geometric manifolds*, Invent. Math. **83** (1986) 411–424, MR 87j:58019, Zbl 0591.53026.

[21] R.J. Zimmer, *Split rank and semisimple automorphism groups of G-structures*, J. Differential Geometry **26** (1987) 169–173, MR 88i:22018, Zbl 0644.22003.

[22] R.J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhauser, 1984, MR 86j:22014, Zbl 0571.58015.

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