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A GENERALIZATION OF THE ISOPERIMETRIC INEQUALITY

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1. For a simple closed plane curve of length L bounding an area A the classical isoperimetric inequality asserts that

$$L^2 - 4\pi A \geq 0 ,$$

with equality holding only for a circle. We show here that this inequality remains true for non-simple closed curves where in place of A we take the sum of the areas into which the curve divides the plane, each weighted with the square of the winding number, i.e.,

$$L^2 - 4\pi \int_{E^2} w^2 dA \geq 0 ,$$

where, for $p \in E^2$, $w(p)$ is the winding number of p with respect to the curve. Equality holds if and only if the curve is a circle, or a circle traversed several times or several coincident circles each traversed in the same direction any number of times. Note that this implies that

$$L^2 - 4\pi \int_{E^2} |w|^p dA \geq 0$$

for any $0 < p \leq 2$ and that 2 is here the best possible power.

This may all be generalized to arbitrary dimension and codimension. For the case of closed space curves let G denote the space of lines in E^3 (parallel lines are not identified) and let dG denote its invariant measure [1], [7]. Then

$$L^2 - 4 \int_G \lambda^2 dG \geq 0 ,$$

where $\lambda(l)$ denotes the linking number of $l \in G$ with the curve. Equality holds

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here only for one or several coincident circles, as before. For closed oriented surfaces M in E^3 we obtain

$$\int_{M\times M} r^{-1}dA_1dA_2 - 12\pi \int_{E^3} w^2dV \geq 0 ,$$

where, for $(x, y) \in M \times M$, r is the length of the chord joining x and y, dA_1 and dA_2 denote the elements of area of M at x and y, respectively, and $w(p), p \in E^3$, denotes the winding number of the surface with respect to p. Equality holds only for one or several coincident spheres with coincident orientations. Note that the left-hand integral is essentially the gravitational self-potential of M considered as a thin homogeneous shell. The right-hand integral is just the volume in case M is connected and embedded.

The general result is as follows. Let M be a compact oriented manifold of dimension m, and $f: M \to E^n$ be an immersion of class C^2. For $(x, y) \in M \times M$ let $r(x, y)$ denote the chord length from $f(x)$ to $f(y)$, and let dV_1, dV_2 denote the volume elements on M at x and y respectively. Let $H_{n-m-1,n}$ denote the Grassmann manifold of $(n - m - 1)$-planes in E^n (parallel planes are not identified), and let $|dH_{n-m-1,n}|$ denote its invariant measure [1], [2], [6], [7]. For $h \in H_{n-m-1,n}$ let $\pm\lambda$ denote the linking number of h with f.

Theorem 1.

$$\int_{M\times M} r^{-m+1}dV_1dV_2 - (1 + m)\Sigma_m K_{m,n} \int_{H_{n-m-1,n}} \lambda^2 |dH_{n-m-1,n}| \geq 0 ,$$

where Σ_m denotes the surface volume of the unit m-sphere and $K_{m,n}$ is a constant which depends only on m and n. Equality holds only for one or several coincident spheres with coincident orientations, or ($n = 1$) one or several coincident circles all traversed in the same direction each a number of times.

Let us take the second integral in this inequality and write

$$\mathscr{A}(M) = K_{m,n} \int_{H_{n-m-1,n}} \lambda^2 |dH_{n-m-1,n}| .$$

Then $\mathscr{A}(M)$ may be thought of, in the case of a space curve, as the "area" bounded by the curve, or in general as the "volume" bounded by a submanifold of higher codimension of a euclidean space. There are, of course, several candidates for such a "volume", e.g., the surface volume of the submanifold of dimension $m + 1$ of least surface volume spanning $f(M)$, or the volume of the convex hull, or the surface volume of the convex envelope. $\mathscr{A}(M)$, however, has the following simple properties:

1) $\mathscr{A}(M)$ is just the volume bounded by $f(M)$ if f is an embedding into a linear space of dimension $m + 1$;

2) $\mathscr{A}(M)$ is stable under raising of the codimension, i.e., if $M \subset E^n \subset E^N$, then $\mathscr{A}(M)$ in the sense of submanifolds of E^N is the same as $\mathscr{A}(M)$ in the sense of submanifolds of E^n;

3) $\mathscr{A}(M)$ is finite and is given by an integral over $M \times M$ (cf. Theorem 4 and the Remark following);

4) $\mathscr{A}(M)$ has the "reproductive" property in the sense of Chern [3], i.e., for $q > n - m - 1$, $H_q \in H_{q,n}$, we have

$$l_{m,n,q} \int_{H_{q,n}} \mathscr{A}(M \cap H_q) \, |dH_{q,n}| = \mathscr{A}(M) \ ,$$

where $l_{m,n,q}$ is a constant depending only on m, n and q (cf. Theorem 3);

5) $\mathscr{A}(M)$ satisfies an "isoperimetric" inequality (Theorem 1 above).

This paper is organized as follows. In § 2 we prove a theorem on the convergence of certain sequences of integrals which we shall use in the sequel; in § 3 we establish properties 1)–4) above; the proof of Theorem 1 is given in § 4; and in § 5 we prove some additional inequalities using the same methods. Some of these seem to be new even for convex surfaces. §§ 3 and 4 are based on [6]; however, we give explicit references, so that the present paper may be read without first reading [6], the reader looking up the references as needed. Theorem 3 was proved in a special case in [6, Formula 1]. Our proof here is considerably simpler.

Theorem 1 for the case of plane curves is much simpler to prove than the general result and is suitable for presentation in an elementary course. To extract this simpler proof one uses Theorem 3 in the special case $q = m = 1$, $n = 2$, where it is quite easy to prove; Theorem 4 which is easy to prove in this case, since in the plane dI is essentially the invariant measure for lines; Proposition 5, which is given here a separate simple proof for curves; the local analysis proceeding Proposition 5, which is also done separately for curves; and, finally, the proof of Theorem 1 as it stands in § 4. § 2 is not needed, and a simpler account of $S(M)$ is given in [5].

Our proof of Theorem 1 generalizes one given for plane convex curves by Arne Pleijel [4]. We wish to thank H.Guggenheimer for bringing the work of Pleijel to our attention. We also wish to thank Mario Miranda for help with the proof of Proposition 2.

2. In this section we establish convergence of certain limits of integrals in case $m > 1$. Let $M^m = M$ denote a compact differentiable manifold of dimension m, and let $f_k : M \to E^n$, $k = 1, 2, \cdots$, denote a sequence of C^1 immersions which converge uniformly to an immersion f such that the first derivatives of f_k converge uniformly to the first derivatives of f. Let dV_k and dV denote the volume elements of f_k and f respectively. Let $\pi_1, \pi_2 \colon M \times M \to M$ denote the projection mappings into the first and second factors, respectively, and let

$dV_{ki} = \pi_i{}^*dV_k$, $dV_i = \pi_i{}^*dV$, $i = 1, 2$, $r_k(x, y) = |f_k(y) - f_k(x)|$ and $r(x, y) = |f(y) - f(x)|$.

Proposition 2. *Let $\{\Phi_k\}$ denote a sequence of real-valued functions on $M \times M$ such that $|\Phi_k| \leq 1$ and*

$$\lim_{k\to\infty} \Phi_k = \Phi \qquad \text{a.e..}$$

Then

$$\lim_{k\to\infty} \int_{M\times M} \Phi_k r_k{}^{-m+1} dV_{k1} dV_{k2} = \int_{M\times M} \Phi r^{-m+1} dV_1 dV_2 ,$$

and all these integrals are absolutely convergent.

Proof. Let P_1 and P_2 be two m-planes in R^n, and define the angle $\theta = \angle(P_1, P_2)$ between them as follows. The metric in R^n induces a metric in $\wedge^m R^n$ which with suitable normalization is given by the formula $e_1 \wedge \cdots \wedge e_m \cdot f_1 \wedge \cdots \wedge f_m = \det [e_i \cdot f_j]$. If $e_1, \cdots, e_m$ is an orthonormal set of vectors spanning P_1, and $f_1, \cdots, f_m$ an orthonormal set of vectors spanning P_2, then we define θ by $\cos \theta = |e_1 \wedge \cdots \wedge e_m \cdot f_1 \wedge \cdots \wedge f_m|$ and $0 \leq \theta \leq \pi/2$. We may interpret this as follows. $e_1 \wedge \cdots \wedge e_m$ and $f_1 \wedge \cdots \wedge f_m$ have unit norm and hence represent points on the unit sphere in $\wedge^m R^n$. If these points are joined with the origin by lines, then θ is the lesser positive angle between these lines and can be measured as a distance on the unit sphere in $\wedge^m R^n$.

We assert that there is a real number $\varepsilon > 0$ such that for any pair of m-planes P_1 and P_2 in E^n there exists an m-plane Q such that $\angle(P_1, Q) \leq \pi/2 - \varepsilon$ and $\angle(P_2, Q) \leq \pi/2 - \varepsilon$. To show this it suffices to consider only m-planes through the origin, which form a compact Grassmann manifold $G_{m,n}$, since θ is unchanged by parallel translation of P_1 and P_2. Suppose the assertion is false. Then there exist sequences of m-planes $\{P_{1n}\}$, $\{P_{2n}\}$ such that for any n-plane Q either $\angle(P_{1n}, Q) > \pi/2 - 1/n$ or $\angle(P_{2n}, Q) > \pi/2 - 1/n$. We may extract subsequences from $\{P_{1n}\}$ and $\{P_{2n}\}$ which converge to $P_{1\infty}$ and $P_{2\infty}$ respectively, and these have the property that for any m-plane Q, either $\angle(P_{1\infty}, Q) = \pi/2$ or $\angle(P_{2\infty}, Q) = \pi/2$. But this is impossible since $G_{m,n}$ is an irreducible algebraic variety, and the assertion is proven.

For $(x, z) \in M \times M$ let $d_k(x, z)$, $d(x, z)$ denote the distance between x and z in the sense of the Riemannian metric induced on M by f_k and f, respectively, and T_{xk}, T_x the tangent m-planes at x to f_k and f respectively. Since f is differentiable of class C^1 and M is compact, we may find a $\delta > 0$ such that if $d(x, z) < \delta$, then $\angle(T_x, T_z) < \varepsilon/2$. By the uniform convergence of the derivatives we may find a positive integer K such that if $k > K$ then $\angle(T_{zk}, T_z) < \varepsilon/4$ for all z, so that if $d(x, z) < \delta$ then

$$\angle(T_{zk}, T_x) \leq \angle(T_{zk}, T_z) + \angle(T_z, T_x) < 3\varepsilon/4 .$$

On $M \times M$ let us take the metric $D((x_1, y_1), (x_2, y_2)) = (d(x_1, x_2)^2 + d(y_1, y_2)^2)^{1/2}$. Let us now take a partition of unity $\sum \varphi_\alpha = 1$ on $M \times M$, whose supports are contained in open sets $U_1, \cdots, U_N$ of diameter $<\delta$ is the metric D, and for each i choose a point $(x_i, y_i) \in U_i$ and an m-plane Q_i such that

$$\angle(T_{x_i}, Q_i) < \pi/2 - \varepsilon \quad \text{and} \quad \angle(T_{y_i}, Q_i) < \pi/2 - \varepsilon .$$

If $(x, y) \in U_i$, then

$$\angle(Q_i, T_{xk}) \leq \angle(Q_i, T_{x_i}) + \angle(T_{x_i}, T_{xk}) \leq \pi/2 - \varepsilon/4 ,$$
$$\angle(Q_i, T_{yk}) \leq \angle(Q_i, T_{y_i}) + \angle(T_{y_i}, T_{yk}) \leq \pi/2 - \varepsilon/4 .$$

Let $E = \sec(\pi/2 - \varepsilon/4)$. It follows that we can represent $f_k(\pi_1(U_i))$ and $f_k(\pi_2(U_i))$ non-parametrically in E^n with Q_i as base plane; i.e., if we take Cartesian coordinates $x_1, \cdots, x_n$ in E^n such that Q_i is defined by $x_{m+1} = \cdots = x_n = 0$, then we may represent $f_k(\pi_1(U_i))$ and $f_k(\pi_2(U_i))$ as

$$\begin{aligned} x_{m+1} &= f^k_{m+1}(x_1, \cdots, x_m) , \qquad & x_{m+1} &= \bar{f}^k_{m+1}(x_1, \cdots, x_m) , \\ &\vdots & &\vdots \\ x_n &= f^k_n(x_1, \cdots, x_m) , & x_n &= \bar{f}^k_n(x_1, \cdots, x_m) , \end{aligned}$$

respectively, and similarly for $f(\pi_1(U_i))$ and $f(\pi_2(U_i))$.

Let $q_1, q_2 \colon Q_i \times Q_i \to Q_i$ denote the projections into the first and second factor, respectively, and let $y_i = x_i \circ q_1$, $z_i = x_i \circ q_2$. Let $p \colon E^n \to Q_i$ denote orthogonal projection and let

$$C_k = (p \circ f_k \circ \pi_1) \times (p \circ f_k \circ \pi_2) \colon U_i \to Q_i \times Q_i .$$

Let χ_k denote the characteristic function of $C_k(U_i)$, so that

$$\chi_k(p) = \begin{cases} 1, & p \in C_k(U_i) , \\ 0, & p \in Q_i \times Q_i - C_k(U_i) . \end{cases}$$

Let $U'_i = \cup_k C_k(U_i)$, and let χ denote its characteristic function. We can choose K' so large that if $k > K'$ then the Euclidean distance between $f(x)$ and $f_k(x)$ is less than $\delta/2$ for all x. We henceforth assume $k > K'$. Then the diameter of U'_i is less than 4δ, and

$$\int_{U_i} \varphi_i \, |\Phi_k| \, r_k^{-m+1} dV_{k1} dV_{k2}$$
$$= \int_{Q_i \times Q_i} \chi_k [\varphi_i \, |\Phi_k| \, r_k^{-m+1}] \circ C_k^{-1} \sec \angle(T_k, Q_i) \sec \angle(\bar{T}_k, Q_i) dy_1 \cdots dy_m dz_1 \cdots dz_m ,$$

where $T_k(p) = T_{qk}$, $q = \pi_1 C_k^{-1}(p)$, and $\bar{T}_k(p) = T_{sk}$, $s = \pi_2 C_k^{-1}(p)$. Now for p a variable point in $Q_i \times Q_i$

$$\chi_k[\varphi_i |\Phi_k| r_k^{-m+1}] \circ C_k^{-1} \sec \angle(T_k, Q_i) \sec \angle(\bar{T}_k, Q_i)$$
$$\leq \chi\left(\sum_{j=1}^{m} (y_j - z_j)^2\right)^{(-m+1)/2} E^2 ,$$
$$\int_{Q_i \times Q_i} \chi\left(\sum_{j=1}^{m} (y_j - z_j)^2\right)^{(-m+1)/2} E^2 dy_1 \cdots dy_m dz_1 \cdots dz_m \leq \Sigma_{m-1}(4\delta)^{m+1} E^2 ,$$

where Σ_{m-1} is the surface volume of the unit $(m-1)$-sphere, as is shown by a standard computation using polar coordinates. Thus by the Lebesgue bounded convergence theorem

$$\lim_{k\to\infty} \int_{U_i} \varphi_i \Phi_k r_k{}^{-m+1} dV_{k1} dV_{k2} = \int_{U_i} \varphi_i \Phi r^{-m+1} dV_1 dV_2 ,$$

and all these integrals are absolutely convergent. Hence

$$\lim_{k\to\infty} \int_{M\times M} \Phi_k r_k{}^{-m+1} dV_{k1} dV_{k2} = \lim_{k\to\infty} \sum_i \int_{U_i} \varphi_i \Phi_k r_k{}^{-m+1} dV_{k1} dV_{k2}$$
$$= \sum_i \int_{U_i} \varphi_i \Phi r^{-m+1} dV_1 dV_2 = \int_{M\times M} \Phi r^{-m+1} dV_1 dV_2 ,$$

which establishes Proposition 2.

3. Let $f\colon M^m \to E^n$ be an immersion of a compact oriented differentiable manifold M^m. Let $H_{n-m-1,n}$ denote the Grassmann manifold of all (unoriented) $(n-m-1)$-planes in E^n (it being understood that parallel planes are not identified), and $|dH_{n-m-1,n}|$ the Euclidean invariant measure on $H_{n-m-1,n}$ (this is worked out explicitly in [6]). Consider the quantity

$$\mathscr{A}(M^m) = K_{m,n} \int_{H_{n-m-1,n}} \lambda^2 |dH_{n-m-1,n}| ,$$

where $\pm\lambda(h) = \pm\lambda(h, M^m)$ denotes the linking number of $h \in H_{n-m-1,n}$ with M^m, $K_{m,n} = k_{m,m+2} k_{m,m+3} \cdots k_{m,n}$, and

$$k_{m,j} = \pi^{-(m+1)/2} \frac{\Gamma((j+1)/2)}{\Gamma((j-m)/2)} .$$

In particular $K_{m,m+1} = 1$, so that $\mathscr{A}(M^m)$ generalizes the volume bounded by a simple closed hypersurface. Furthermore, as is proved in [6, pp. 1341–1342] $\mathscr{A}(M^m)$ is *stable* under raising of the codimension, that is, if E^n is regarded as an n-plane in E^N so that $f(M^m) \subset E^n \subset E^N$, then

$$K_{m,N} \int_{H_{N-m-1,N}} \lambda^2 |dH_{N-m-1,N}| = K_{m,n} \int_{H_{n-m-1,n}} \lambda^2 |dH_{n-m-1,n}| \,.$$

The proof given in [6] has a slight error, however, which affects the constant, so we give a corrected version here.

Let H denote a family of $(n-m)$-planes in E^{n+1} and let $h = H \cap E^n \subset E^{n+1}$. To each $(n-m)$-plane assign an orthonormal frame $Xb_1 \cdots b_{n+1}$ such that $X \in h$, $b_1, \cdots, b_{n-m-1}$ span h, b_{n-m} in H perpendicular to h, and $b_{n-m+2}, \cdots, b_{n+1} \perp h$ in E^n. Let $Xa_1 \cdots a_{n+1}$ be another family of frames such that $a_i = b_i$, $1 \le i \le n-m-1$, $a_\alpha = b_\alpha$, $n-m+2 \le \alpha \le n+1$, (we use these ranges of i, α until further notice), a_{n-m} is along the orthogonal projection of b_{n-m} into E^n, a_{n-m+1} is the unit normal to E^n in E^{n+1}, so that a_{n-m+1} is constant. We may write

$$\begin{aligned} b_{n-m} &= \cos\varphi\, a_{n-m} + \sin\varphi\, a_{n-m+1} \,, \\ b_{n-m+1} &= -\sin\varphi\, a_{n-m} + \cos\varphi\, a_{n-m+1} \,. \end{aligned}$$

If we let $dX \cdot a_i = \pi_i$, $dX \cdot b_i = \rho_i$, $da_i \cdot a_j = \pi_{ij}$, $db_i \cdot b_j = \rho_{ij}$, then we have

$$\begin{aligned} \rho_{n-m+1} &= -\sin\varphi\, \pi_{n-m} \,, \\ \rho_{n-m,\alpha} &= \cos\varphi\, \pi_{n-m,\alpha} \,, \\ \rho_{i,n-m+1} &= -\sin\varphi\, \pi_{i,n-m} \,, \\ \rho_{n-m,n-m+1} &= d\varphi \,, \end{aligned}$$

so that

$$\begin{aligned} &|dH_{n-m,n+1}| \\ &\quad = |\rho_{n-m+1} \wedge \cdots \wedge \rho_{n+1} \wedge \prod \rho_{i\alpha} \wedge \prod \rho_{n-m,\alpha} \wedge \prod \rho_{i,n-m+1} \wedge \rho_{n-m,n-m+1}| \\ &\quad = |\sin^{n-m}\varphi \cos^m\varphi \, d\varphi \wedge d\Sigma_m \wedge dH_{n-m-1,n}| \,, \end{aligned}$$

where $d\Sigma_m = \prod \pi_{n-m,\alpha}$ is the volume element of the unit m-sphere Σ_m perpendicular to h in E^n. Hence by Fubini's theorem

$$\begin{aligned} &\int_{H_{n-m,n+1}} \lambda^2 |dH_{n-m,n+1}| \\ &\qquad = \int_0^{\pi/2} |\sin^{n-m}\varphi \cos^m\varphi| \, d\varphi \int_{\Sigma_m} d\Sigma_m \int_{H_{n-m-1,n}} \lambda^2 |dH_{n-m-1,n}| \,. \end{aligned}$$

Now

$$\int_0^{\pi/2} |\sin^{n-m}\varphi \cos^m \varphi|\, d\varphi \int_{\Sigma_m} d\Sigma_m$$
$$= \frac{1}{2}\,\frac{\Gamma(\frac{1}{2}(m+1))\Gamma(\frac{1}{2}(n-m+1))}{\Gamma(\frac{1}{2}(n+2))}\,\frac{2\pi^{(m+1)/2}}{\Gamma(\frac{1}{2}(m+1))} = k_{m,n+1}{}^{-1}\ .$$

This establishes the assertion for $N = n + 1$, and the general result follows by induction on N.

In considering $\mathscr{A}(M^m)$ the case $m = 0$ is special and requires an additional assumption. A compact oriented 0-manifold is nothing more than a finite set of points, to each of which is assigned a multiplicity ± 1. We shall require that the sum of these multiplicities be zero. Thus the simplest case under consideration is that of a pair of points with opposite multiplicities so that here $\mathscr{A}(M^0)$ is essentially the measure of the hyperplanes that meet the line segment joining the two points, which by the generalized Crofton-Cauchy formula is essentially the distance between the two points.

More generally, suppose M^0 consists of points $x_1, \cdots, x_q$ with multiplicities $i_1, \cdots, i_q$. Orient E^n and choose an oriented hyperplane h_0; then orient each hyperplane so that it makes a positive acute angle with h_0. Of course, this orientation is indeterminate for hyperplanes perpendicular to h_0, but these form a set of measure zero and so we can neglect them. Each oriented hyperplane divides E^n into two half-spaces, one of which is canonically designated the left-hand half-space h^+, and the other the right-hand half-space h^-. It is now readily seen that

$$\lambda(h, M^0) = \sum_{f(x_i)\in h^+} i_i = -\sum_{f(x_i)\in h^-} i_i\ , \qquad 1 \le i \le q\ ,$$

so that

$$\int_{H_{n-1,n}} \lambda^2 |dH_{n-1,n}| = -\int_{H_{n-1,n}} \sum_{f(x_i)\in h^+} i_i \sum_{f(x_j)\in h^-} i_j\, |dH_{n-1,n}|$$
$$= -\sum_{i,j} i_i i_j \int_{H_{n-1,n}} F_{ij}\, |dH_{n-1,n}|\ ,$$

where $F_{ij}(h) = 1$ if $f(x_i) \in h^+$ and $f(x_j) \in h^-$, and $F_{ij}(h) = 0$ otherwise. But by the generalized Crofton-Cauchy formula,

$$\int_{H_{n-1,n}} (F_{ji} + F_{ij})\, |dH_{n-1,n}| = d_{n-1} r(x_i, x_j)\ ,$$

where $d_{n-1} = K_{0,n}^{-1}$ equals the volume of the disc bounded by a unit sphere of dimension $n - 2$, and $r(x_i, x_j)$ is the distance between $f(x_i)$ and $f(x_j)$. Consequently,

$$\mathscr{A}(M^0) = K_{0,n} \int_{H_{n-1,n}} \lambda^2 |dH_{n-1,n}| = -\tfrac{1}{2} \sum_{i,j} r(x_i, x_j) i_i i_j \ .$$

Theorem 3. *$\mathscr{A}$ is reproductive in the sense of Chern* [3], *i.e., if $q > n - m - 1$, then*

$$l_{m,n,q} \int_{H_{q,n}} \mathscr{A}(M^m \cap H_q) |dH_{q,n}| = \mathscr{A}(M^m) \ ,$$

where

$$l_{m,n,q} = \frac{K_{m,n}}{K_{q-n+m,q}} \frac{\Sigma_{q-n+m} \cdots \Sigma_0 \Sigma_{n-q-1} \cdots \Sigma_0}{\Sigma_n \cdots \Sigma_0} \ ,$$

and $\Sigma_j = \dfrac{2\pi^{(j+1)/2}}{\Gamma((j+1)/2)}$ is the surface volume of the unit j-sphere. In particular,

$$\mathscr{A}(M^m) = -\tfrac{1}{2} l_{m,n,n-m} \int_{H_{n-m,n}} \sum_{x_i, x_j} r(x_i, x_j) i_i i_j |dH_{n-m,n}| \ ,$$

where $x_1, x_2, \cdots$ are the points of intersection of M^m with a moving $(n-m)$-plane H_{n-m}, $r(x_i, x_j)$ is the Euclidean distance from x_i to x_j, and i_i is the intersection number of H_{n-m} (with some orientation) and M^m at x_i.

Proof. Let us observe first that if h_{n-m-1} is a linear space of dimension $n - m - 1$ and h_q is a linear space of dimension q containing h_{n-m-1}, then

$$\lambda(M^m, h_{n-m-1}) = \lambda(M^m \cap h_q, h_{n-m-1}) \ , \tag{3.1}$$

where the first linking number is in the sense of submanifolds of E^n and the second in the sense of submanifolds of h_q.

Let us call this configuration (h_q, h_{n-m-1}), $h_{n-m-1} \subset h_q$, a *banner*. The totality of banners forms a differentiable manifold B, and has a measure which is invariant under the action of the group of rigid motions of E^n. This measure is constructed, according to the method of Chern [2], as follows. Let us assign to each banner (locally) a frame $Xe_1 \cdots e_n$ so that $X \in h_{n-m-1}$, $e_1, \cdots, e_{n-m-1}$ are parallel to h_{n-m-1}, and $e_{n-m}, \cdots, e_q$ are parallel to h_q. Let $dX \cdot e_i = \omega_i$, $de_i \cdot e_j = \omega_{ij}$. The measure is then given by

$$|dB| = |\omega_{n-m} \wedge \cdots \wedge \omega_n \wedge \prod_{\substack{1 \le i \le n-m-1 \\ n-m \le j \le n}} \omega_{ij} \wedge \prod_{\substack{n-m \le k \le q \\ q+1 \le l \le n}} \omega_{kl}| \ .$$

To prove the theorem we integrate $\lambda^2(M^m, h_{n-m-1}) |dB|$ in two ways and equate the results.

Let us choose a fixed frame $0i_1\cdots i_n$ in E^n, and let $G_{q-n+m+1,m+1}$ denote the Grassmann manifold of $(q-n+m+1)$-spaces through 0 lying in the span of $i_{n-m},\cdots,i_n$. To each $(n-m-1)$-plane h_{n-m-1} in E^n assign locally a frame $Xa_1\cdots a_n$ such that $X\in h_{n-m-1}$, and $a_1,\cdots,a_{n-m-1}$ are parallel to h_{n-m-1}. For each h_{n-m-1} let $G(h_{n-m-1})$ denote the rigid motion which takes $0i_1\cdots i_n$ to $Xa_1\cdots a_n$. Define a map $H\colon H_{n-m-1,n}\times G_{q-n+m+1,m+1}\to B$ by $H(h_{n-m-1},g_{q-n+m+1})=(\text{span}\,(h_{n-m-1},G(g_{q-n+m+1})),h_{n-m-1})$. Taking frames $Xe_1\cdots e_n$ as before so that $e_i=a_i$, $i\le n-m-1$, we find that

$$\omega_{n-m}\wedge\cdots\wedge\omega_n\wedge\prod_{\substack{1\le i\le n-m-1\\ n-m\le j\le n}}\omega_{ij}=dH_{n-m-1,n}$$

and $\prod_{\substack{n-m\le k\le q\\ q+1\le l\le n}}\omega_{kl}=dG_{q-n+m+1,m+1}+$ terms involving the differentials on $H_{n-m-1,n}$, so that

$$H^*\,|dB|=|dH_{n-m-1,n}\wedge dG_{q-n+m+1,m+1}|\ ,$$

and, by Fubini's theorem,

$$\text{(3.2)}\qquad \int_B\lambda^2(M^m,h_{n-m-1})\,|dB|=g_{m,n,q}\int_{H_{n-m-1,n}}\lambda^2(M^m,h_{n-m-1})\,|dH_{n-m-1,n}|\ ,$$

where $g_{m,n,q}$ is the total volume of $G_{q-n+m+1,m+1}$. To evaluate this we observe that $G_{a,b}=O_b/O_a\times O_{b-a}$, where O_j denotes the orthogonal group in j variables, and $O_j/O_{j-1}=S^{j-1}$, the unit sphere of dimension $j-1$, so that the volume $\mathcal{O}_j$ of O_j is given by

$$\mathcal{O}_j=\Sigma_{j-1}\mathcal{O}_{j-1}=\Sigma_{j-1}\Sigma_{j-2}\cdots\Sigma_0\ ,$$

$$\text{(3.3)}\qquad g_{m,n,q}=\frac{\Sigma_m\cdots\Sigma_0}{\Sigma_{q-n+m}\cdots\Sigma_0\Sigma_{n-q-1}\cdots\Sigma_0}\ .$$

Now let us parametrize the banners in another way. To each $h_q\subset E^n$ let us associate a frame $Yb_1\cdots b_n$ so that $Y\in h_q$, and $b_1,\cdots,b_q$ are parallel to h_q. Let $H_{n-m-1,q}$ denote the Grassmann manifold of linear spaces of dimension $n-m-1$ contained in the linear space through O spanned by $i_1\cdots i_q$, and let $J(h_q)$ denote the rigid motion of E^n which takes $Oi_1\cdots i_n$ to $Yb_1\cdots b_n$. Define a map $K\colon H_{q,n}\times H_{n-m-1,q}\to B$ by $K(h_q,h_{n-m-1})=(h_q,J(h_q)(h_{n-m-1}))$, and take frames $Xe_1\cdots e_n$ as before. Then

$$dH_{q,n}=\omega_{q+1}\wedge\cdots\wedge\omega_n\wedge\prod_{\substack{1\le i\le q\\ q+1\le j\le n}}\omega_{ij}\ ,$$

$$\omega_{n-m} \wedge \cdots \wedge \omega_q \wedge \prod_{\substack{1\le i\le n-m-1 \\ n-m\le j\le q}} \omega_{ij}$$

$$= dH_{n-m-1,q} + \text{terms involving the differentials on } H_{q,n}.$$

Hence

$$K^* |dB| = |dH_{q,n} \wedge dH_{n-m-1,q}| ,$$

and by Fubini's theorem and (3.1) we find that

$$\int_B \lambda^2(M^m, h_{n-m-1}) |dB| = \int \lambda^2(M^m \cap h_q, h_{n-m-1}) |dB|$$

$$= \int_{H_{q,n}} K^{-1}_{q-n+m,q} \mathscr{A}(M^m \cap h_q) |dH_{q,n}| ,$$

which, together with (3.2) and (3.3), proves the theorem.

Remark. By a similar argument, together with the generalized Crofton-Cauchy formula, one may show that the m-dimensional surface volume $V(M^m)$ of M^m is reproductive, i.e.,

$$p_{m,n,q} \int_{H_{q,n}} V(M^m \cap h^q) |dH_{q,n}| = V(M) , \tag{3.4}$$

where $p_{m,n,q} = \dfrac{m\Sigma_{r-1}k_{m-1,m+2}\cdots k_{m-1,n}}{r\Sigma_{m-1}k_{r-1,r+2}\cdots k_{r-1,q}g_{m-1,n,q}}$ and $r = m - n + q$. This generalizes the generalized Crofton-Cauchy formula.

Our proof of the isoperimetric inequality depends on another formula for $\mathscr{A}(M^m)$. Let $G \subset M^m \times M^m$ denote the set of all (x, y) such that $f(x) \neq f(y)$, and note that $M^m \times M^m - G$ is a set of measure zero. Let $l(x, y)$ denote the line joining $f(x)$ and $f(y)$ oriented from x to y, so that $l\colon G \to H_{1,n}$, the Grassmann manifold of oriented lines in E^n, and let $e_1(x, y)$ denote the unit vector oriented along $l(x, y)$. Let $X(x, y) = f(x)$, $\omega_1 = dX \cdot e_1$ and $dl = d\omega_1$. It was shown in [6] that dl comes from an invariant two-form on $H_{1,n}$, and that

$$(dl)^m = m!(-1)^{m(m-1)/2}r^{-m} \cos\tau \sin\sigma_1 \sin\sigma_2 dV_2 \wedge dV_1 , \tag{3.5}$$

where the dV_i and r are as in § 2 above, and σ_i, τ are certain angles which we shall discuss below.

Theorem 4. *$\mathscr{A}(M^m)$ is finite and*

$$\mathscr{A}(M^m) = \frac{(-1)^{1+m(m+1)/2}}{m!\Sigma_m} \int_{M^m \times M^m} r(dl)^m . \tag{3.6}$$

Remark. The finiteness does not seem to have an altogether trivial proof. For it is easy to construct a C^∞ closed immersed space curve which has arbitrarily high linking numbers with lines. However, it may be shown that any C^1-immersed hypersurface has bounded winding number with points and any C^2-immersed space curve with nowhere vanishing curvature has bounded linking number with lines.

Proof. In case f is an embedding this follows from [6, (2.16)] but for the determination of the constant. It remains for us here to show that the proof of (2.16) of [6] is valid in the present more general case and to determine the constant.

In [6, pp. 1329–1330] a certain subset $G_+ \subset G_{n-m-1,n-1}$ and a differential form $F^*dH_{n-m,n}$ on $(M^m \times M^m - D) \times G_+$ are defined, where D is the diagonal. (What we call M^m here is called P there and is there of dimension $m+1$.) The definition of $F^*dH_{n-m,n}$ is valid in the present case as long as we restrict ourselves to $G \times G_+$. Now $F^*dH_{n-m,n}$ is the sum of terms of the form

$$\omega_2 \wedge \cdots \wedge \omega_{m+1} \wedge (\omega_{1k_1} \wedge \cdots \wedge \omega_{1k_m}) \cos \varphi_1 \cos \varphi_2 dG_{n-m-1,n-1}, \qquad 2 \leq k_1 < \cdots < k_m \leq n \,,$$

where φ_1 and φ_2 are certain angles. (Read "$\sum$" for "$\prod$" on lines 14 and 16 of p. 1330 of [6].) Now following the argument used to establish (2.3) and (2.4) of [6] we find that

$$|rF^*dH_{n-m,n}| \leq \binom{n-1}{m} r^{-m+1} dV_1 dV_2 \, |dG_{n-m-1,n-1}| \,.$$

But this is integrable on $G \times G_+$ by Fubini's theorem and Proposition 2 above. Hence the integrations over the fibre used to establish (2.16) of [6] are valid and lead to finite quantities, and the rest is valid without change.

To determine the sign and constant in (3.6) we observe that both $\mathscr{A}(M^m)$ and [6, p. 1341] $\int r(dl)^m$ are stable under raising of the codimension. Hence the sign and constant are stable and can be found by checking the formula for a sphere $S^m \subset E^{m+1}$. The sign is checked using (3.5) and the facts that $\cos \tau = -1$, and $\sin \sigma_i \geq 0$ for the sphere. To evaluate the constant now, we use the fact that $\pm (m!)^{-1}(dl)^m$ is essentially the invariant measure for oriented lines in E^{m+1}; and for each oriented line L meeting S^m there is a unique ordered pair of points $(x, y) \in S^m \times S^m$ such that $l(x, y) = L$, so that the integration can be made on $H_{1,m+1}$. To each oriented line in E^{m+1} we associate a frame $Xe_1 \cdots e_{m+1}$ such that e_1 is directed along the line and X is a point on the line. Let $dX \cdot e_i = \omega_i$, $de_i \cdot e_j = \omega_{ij}$. Then $(m!)^{-1}(dl)^m = \pm \omega_2 \wedge \cdots \wedge \omega_{m+1} \wedge \omega_{12} \wedge \cdots \wedge \omega_{1m+1}$. If the direction of the line is held fixed, $\omega_2 \wedge \cdots \wedge \omega_{m+1}$ becomes the volume element on a perpendicular m-plane, so that

$$\int_{L \cap S^m \neq 0} r\omega_2 \wedge \cdots \wedge \omega_{m+1} \wedge \omega_{12} \wedge \cdots \wedge \omega_{1\,m+1}$$

$$= \int \mathscr{A}(S^m)\omega_{12} \wedge \cdots \wedge \omega_{1\,m+1} = \Sigma_m \mathscr{A}(S^m) ,$$

which completes the proof of the theorem.

4. In this section we prove Theorem 1. Basic to the proof is an integral formula obtained by integration over a certain "secant space". We begin by defining this space.

Let $M^m = M$ be a differentiable manifold of class C^2 and dimension m. Let $S(M)$ denote the two-fold cartesian product of M with the diagonal replaced by its bundle of oriented normal directions. This space, which is fully explained in [6], is a differentiable manifold, with boundary, of class C^1 and has the universal property that if $f: M^m \to E^n$ is an embedding, then there is an induced smooth map L_f: $S(M) \to H_{1,n}$ which assigns to each $(x, y) \in M \times M$, $x \neq y$, the line directed from $f(x)$ to $f(y)$, and to the boundary points of $S(M)$ the corresponding tangent lines of M. Let π: $S(M) \to M \times M$ be the canonical projection map, and π_i: $M \times M \to M$ be the projection into the i-th factor, $i = 1, 2$.

Let us now assume that M is compact and oriented with $1 \leq m < n$. For $z \in S(M)$ let $e_1(z)$ denote the unit vector in E^n directed along $L_f(z)$. Let $X(z) = f\pi_1\pi(z)$ and $\omega_1 = dX \cdot e_1$, which is a differential 1-form on $S(M)$. Denote $d\omega_1$ by dI, and the (absolute) euclidean distance from $f\pi_1\pi(z)$ to $f\pi_2\pi(z)$ by $r(z)$. Consider the differential form $r\omega_1 \wedge (dI)^{m-1}$. The orientation of M induces an orientation on $M \times M$ and hence on $S(M)$. By applying Stokes' theorem we find that

$$\int_{T(M)} r\omega_1 \wedge (dI)^{m-1} = \int_{S(M)} dr \wedge \omega_1 \wedge (dI)^{m-1} + \int_{S(M)} r(dI)^m .$$

Now the left-hand term is zero, since $r = 0$ on $T(M)$. Also, $S(M)$ and $M \times M$ differ by sets of measure zero. Hence we may write

$$(4.1) \qquad - \int_{M \times M} dr \wedge \omega_1 \wedge (dI)^{m-1} = \int_{M \times M} r(dI)^m .$$

Let us now give a local analysis of these differential forms. $r(dI)^m$ has been analyzed in [6, pp. 1324–1326], and our analysis of the other follows the same procedure and uses the same frames. It is more convenient to give a separate analysis of the case $m = 1$. Let $(x, y) \in M \times M$ be such that $f(x) \neq f(y)$ and that the tangent spaces to f at x and y are in general position with respect to $L_f(x, y)$ and are not perpendicular to $L_f(x, y)$. (All other points of $S(M)$ form a set of measure zero which we ignore.) We now drop the requirement that f be an embedding.

In case $m = 1$, let a_2, b_2 denote the unit tangent vectors to f at x and y respectively such that e_1 makes positive acute angles σ_1, σ_2 with either, and let $X = f(x)$, $Y = f(y)$. Then $dX = \pi_2 a_2$, $dY = \rho_2 b_2$, and $\omega_1 = dX \cdot e_1 = \cos\sigma_1 \pi_2$. Now $(Y - X) = re_1$, and $dr = d(re_1)\cdot e_1$, so that

$$dr = \cos\sigma_2\,\rho_2 - \cos\sigma_1\,\pi_2\ , \qquad dr \wedge \omega_1 = (\cos\sigma_1 \cos\sigma_2)\rho_2 \wedge \pi_2\ .$$

We set $\cos\nu = 1$ if a_2 and b_2 are both positively directed or both negatively directed tangent vectors to our oriented curve, and $\cos\nu = -1$ if one of a_2, b_2 is positively directed and the other negatively directed. From this we get

$$dr \wedge \omega_1 = -(\cos\nu \cos\sigma_1 \cos\sigma_2) ds_1 \wedge ds_2\ , \tag{4.2}$$

where ds_1, ds_2 denote the positively directed elements of arc at x and y respectively. By [6] we can also write

$$rdI = -(\cos\tau \sin\sigma_1 \sin\sigma_2) ds_1 \wedge ds_2\ , \tag{4.3}$$

where $\cos\tau$ is the angle between the 2-planes spanned by $e_1 a_2$ and $e_1 b_2$ with orientations $e_1 t_1$ and $e_1 t_2$, where t_1 and t_2 are the positively directed unit tangent vectors to f at x and y, respectively.

If $m > 1$, let T_1 denote the (oriented) tangent space to f at x, and S_1 the linear space of dimension $m + 1$ spanned by T_1 and $L_f(x, y)$ with orientation $e_1 T_1$. Let Q_1 be the 2-plane spanned by e_1 and its orthogonal projection on T_1. Choose frames $Xa_1 \cdots a_n$ so that $a_1 \perp T_1$ in S_1, a_2 in Q_1, and $a_3, \cdots, a_{m+1}$ in T_1, and by reversing a_2 or a_3 if necessary we arrange that $a_1 \cdots a_{m+1}$ agrees with the orientation $e_1 T_1$, and $a_2 \cdots a_{m+1}$ agrees with the orientation of T_1, and that e_1 and a_2 make a positive acute angle σ_1. Let $dX \cdot a_i = \pi_i$. Then

$$\omega_1 = dX \cdot e_1 = \cos\sigma_1\,\pi_2\ . \tag{4.4}$$

Let T_2 denote the oriented tangent space to f at y, S_2 the $(m + 1)$-plane spanned by T_2 and $L_f(x, y)$ with orientation $e_1 T_2$, and Q_2 the 2-plane containing $L_f(x, y)$ and its orthogonal projection on T_2. Let us take another family of frames $Yb_1 \cdots b_n$ such that $Y = f(y)$, $b_1 \perp T_2$ in S_2, b_2 in Q_2, and $b_3, \cdots, b_{m+1}$ in T_2, and by reversing b_2 or b_3 if necessary we arrange that $b_1 \cdots b_{m+1}$ agrees with the orientation $e_1 T_2$, and $b_2 \cdots b_{m+1}$ agrees with the orientation of T_2, and that e_1 and b_2 make a positive acute angle σ_2. Let $dY \cdot b_i = \rho_i$. Then

$$dr = d(re_1)\cdot e_1 = \cos\sigma_2\,\rho_2 - \cos\sigma_1\,\pi_2\ ,$$

whence

$$dr \wedge \omega_1 = -(\cos\sigma_1 \cos\sigma_2)\pi_2 \wedge \rho_2\ . \tag{4.5}$$

Now from the computation in [6, pp. 1324–1326] it follows that

$$(4.6)\qquad \begin{aligned}(dI)^{m-1} \equiv (m-1)!(-1)^{(m-1)(m-2)/2}r^{-m+1}\cos\nu\,\rho_3\wedge\cdots\\ \wedge\,\rho_{m+1}\wedge\pi_3\wedge\cdots\wedge\pi_{m+1}\,, \qquad \bmod \rho_2, \pi_2\,,\end{aligned}$$

where $\cos\nu = a_3\wedge\cdots\wedge a_{m+1}\cdot b_3\wedge\cdots\wedge b_{m+1}$. From (4.5) and (4.6) we obtain

$$(4.7)\qquad \begin{aligned}&dr\wedge\omega_1\wedge(dI)^{m-1}\\ &\quad=(m-1)!(-1)^{m(m-1)/2}r^{-m+1}\cos\nu\cos\sigma_1\cos\sigma_2 dV_2\wedge dV_1\,,\end{aligned}$$

where dV_1, dV_2 denote the elements of volume on M at x, y, respectively. Note that (4.7) agrees with (4.2), so that we need not further distinguish the case $m=1$, except when we interpret ν. Let us recall [6] that

$$(4.8)\qquad r(dI)^m = m!(-1)^{m(m-1)/2}r^{-m+1}\cos\tau\sin\sigma_1\sin\sigma_2 dV_2\wedge dV_1\,,$$

where τ is the angle between the oriented planes S_1 and S_2.

Proposition 5. *Let $f\colon M^m\to E^n$ be an immersion of class C^2. Then*

$$\begin{aligned}-\Sigma_m\mathscr{A}(M^m) &= \int_{M\times M} r^{-m+1}\cos\tau\sin\sigma_1\sin\sigma_2\,dV_1dV_2\\ &= -\frac{1}{m}\int_{M\times M} r^{-m+1}\cos\nu\cos\sigma_1\cos\sigma_2 dV_1dV_2\,.\end{aligned}$$

Proof. For $m=1$ the proof is simple. On $S(M)$ we consider the form $dX\cdot(Y-X)$. Where e_1 is defined, i.e., for $(x,y)\in M\times M$ such that $f(x)\neq f(y)$, $dX\cdot(Y-X)=r\omega_2$. Hence by (4.1) we obtain

$$-\int_{S(M)} dr\wedge\omega_1 = \int_{S(M)} rdI\,.$$

The second equality follows from this, together with (4.7) and (4.8), while the first follows from Theorem 4 and (4.8).

For general m the first equality follows also from Theorem 4 and (4.8); the second from (4.1), (4.7) and (4.8) provided that f is an embedding. If f is not an embedding, we consider E^n as contained in E^N, where $N\geq 2n+2$. By the Thom transversality theorem we can find a sequence of embeddings $f_k\colon M\to E^N$ which converge to f uniformly, and whose first derivatives also converge uniformly to those of f. Since the second equality holds for f_k, it holds for f by Proposition 2.

We come now to the proof of Theorem 1. From the elementary identity

$$2\sin^2\tfrac{1}{2}(\sigma_1-\sigma_2) = 1-\cos\sigma_1\cos\sigma_2-\sin\sigma_1\sin\sigma_2$$

and Proposition 5, the convergence of the integrals being guaranteed by Proposition 2, we obtain

$$
\begin{aligned}
&\int_{M\times M} r^{-m+1} 2\sin^2 \tfrac{1}{2}(\sigma_1-\sigma_2)dV_1dV_2 \\
&\quad + \int_{M\times M} r^{-m+1}(1+\cos\tau)\sin\sigma_1\sin\sigma_2 dV_1dV_2 \\
&\quad + \int_{M\times M} r^{-m+1}(1-\cos\nu)\cos\sigma_1\cos\sigma_2 dV_1dV_2 \\
&= \int_{M\times M} r^{-m+1}dV_1dV_2 + \int_{M\times M} r^{-m+1}\cos\tau\sin\sigma_1\sin\sigma_2 dV_1dV_2 \\
&\quad - \int_{M\times M} r^{-m+1}\cos\nu\cos\sigma_1\cos\sigma_2 dV_1dV_2 \\
&= \int_{M\times M} r^{-m+1}dV_1dV_2 - (1+m)\Sigma_m\mathscr{A}(M)\ .
\end{aligned}
\tag{4.9}
$$

Now the expression (4.9) is nonnegative, since σ_1 and σ_2 are acute and positive almost everywhere. Hence we have

$$\int_{M\times M} r^{-m+1}dV_1dV_2 - (1+m)\Sigma_m\mathscr{A}(M) \geq 0\ .$$

Equality holds if and only if $\sigma_1 \equiv \sigma_2$, $\tau \equiv \pi$, and $\nu \equiv 0$. This holds for a sphere, and in fact for several coincident spheres with coincident orientations or, in case $m = 1$, for one or several coincident circles each gone around in the same direction any number of times. Suppose it holds for $f\colon M \to E^n$. Then let $p \in M$ and let T_p denote the tangent m-plane to f at p. Consider all m-spheres in E^n tangent to T_p at $f(p)$. Through any point of $E^n - T_p$ there passes a unique such sphere and hence their tangent spaces form a field of m-planes on $E^n - T_p$. The conditions $\sigma_1 \equiv \sigma_2$, $\tau \equiv \pi$, $\nu \equiv 0$ imply that $f(M) \cap E^n - T_p$ is an integral submanifold of this field of m-planes. Now no component of M can be mapped into T_p by f, for then this component would not be immersed. Hence each component of M has a point which is mapped into $E^n - T_p$. By the uniqueness of integral submanifolds, then, each component of M is mapped onto a sphere through $f(p)$ tangent to T_p at p. But $p \in M$ is arbitrary. Hence $f(M)$ is a single sphere. The condition $\tau \equiv \pi$ implies that the orientations of coincident branches coincide, and the theorem is proven.

5. In this section we use the methods of the preceeding section to prove some additional similar inequalities.

Replacing $r\omega_1 \wedge (dI)^{m-1}$ by $r^q\omega_1 \wedge (dI)^{m-1}$, $q \geq 1$, in the proof of Proposition 5, we obtain

$$-q(m-1)!\int_{M\times M} r^{-m+q}\cos\nu\cos\sigma_1\cos\sigma_2 dV_1dV_2$$
$$= m!\int_{M\times M} r^{-m+q}\cos\tau\sin\sigma_1\sin\sigma_2 dV_1dV_2$$
$$= (-1)^{m(m+1)/2}\int_{M\times M} r^q(dI)^m .$$

Hence

$$\int_{M\times M} 2r^{-m+q}\sin^2\tfrac{1}{2}(\sigma_1-\sigma_2)dV_1dV_2$$
$$+\int_{M\times M} r^{-m+q}(1+\cos\tau)\sin\sigma_1\sin\sigma_2 dV_1dV_2$$
$$+\int_{M\times M} r^{-m+q}(1-\cos\nu)\cos\sigma_1\cos\sigma_2 dV_1dV_2$$
$$=\int_{M\times M} r^{-m+q}dV_1dV_2+\int_{M\times M} r^{-m+q}\cos\tau\sin\sigma_1\sin\sigma_2 dV_1dV_2$$
$$-\int_{M\times M} r^{-m+q}\cos\nu\cos\sigma_1\cos\sigma_2 dV_1dV_2 ,$$

so that

$$\int_{M\times M} r^{-m+q}dV_1dV_2+(-1)^{m(m+1)/2}\frac{m+q}{m!q}\int_{M\times M} r^q(dI)^m \geq 0 ,$$

with equality holding for one or several coincident spheres with coincident orientations, or ($m=1$) for one or several coincident circles each traversed a number of times in the same direction. For convex hypersurfaces, i.e., for $n=m+1$, the last integral is a familiar object. In fact for $n=m+1$

$$-\frac{1}{m!}\left|\int_{M\times M} r^q(dI)^m\right| = \int_{H_{1,m+1}} \sum r^q(x_i,x_j)i_ii_j dH_{1,m+1} ,$$

where for each line $l \in H_{1,m+1}$, x_i are the points of intersection of l and $f(M)$, and i_i is the intersection number of l with $f(M)$ at x_i. For convex hypersurfaces we have

$$\frac{1}{m!}\left|\int_{M\times M} r^q(dI)^m\right| = 2\int_{H_{1,m+1}} r^q dH_{1,m+1} ,$$

and therefore

$$\int_{M\times M} r^{-m+q}dV_1dV_2 - \frac{2(m+q)}{q}\int_{H_{1,m+1}} r^q dH_{1,m+1} \geq 0 .$$

For $m = 2$, $q = 2$ we get

$$A^2 - 4\int_{H_{1,3}} r^2 dH_{1,3} \geq 0 ,$$

where A is the area of the surface. By a formula of Herglotz [1, vol. 2, p. 77] we have

$$V^2 = \frac{1}{6}\int_{H_{1,3}} r^4 dH_{1,3} ,$$

and therefore, for $m = 2$, $q = 4$,

$$\int_{M\times M} r^2 dA_1 dA_2 - 18V^2 \geq 0 ,$$

where V denotes the volume bounded by the surface. These are just samples of the various inequalities which can be obtained by these methods, and of course the last three formulas may be generalized to higher dimensions.

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