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ON THE STRUCTURE OF SPACES WITH RICCI CURVATURE BOUNDED BELOW. III

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0. Introduction

This paper is the third in a series devoted to the study of the structure of complete connected Riemannian manifolds, M^n, whose Ricci curvature has a definite lower bound and of the Gromov-Hausdorff limits, Y, of sequences of such manifolds.

By [6], in the noncollapsed case, off a subset of codimension ≥ 2, such a limit space, Y, is bi-Hölder equivalent to a connected smooth Riemannian manifold (for the proof of connectedness, see Section 3 of [7].) Additionally, even in the collapsed case, there exist natural renormalized limit measures, ν, with respect to which Y is infinitesimally Euclidean almost everywhere.

In the present paper, we show that the renormalized limit measures determine a unique measure class. Moreover, with respect to any fixed such measure, ν, a limit space, Y, is a finite union of countably ν-rectifiable spaces (in the sense of [18], p. 251) and the measure, ν, is absolutely continuous with respect to the relevant Hausdorff measure. Thus, the regular part of Y is a *finite* union of spaces which, although they are not given as subsets of Euclidean space, have the properties of countably rectifiable varifolds (whose dimensions might not all be equal).

By employing rectifiability and a type $(1,2)$ Poincaré inequality, we give a short direct argument showing that associated to the Dirichlet

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energy, $|\mathrm{Lip}\, f|_2^2$, is a *linear self-adjoint* operator, Δ, the Laplacian on functions; compare [3]. For compact spaces, $(1+\Delta)^{-1}$ is a compact operator. We show in addition, that in the presence of a definite lower Ricci curvature bound, the eigenvalues and eigenfunctions behave continuously under measured Gromov-Hausdorff convergence. From this together with the Cheng-Yau gradient estimate, it follows that eigenfunctions of the Laplacian on our limit spaces are Lipschitz functions. In particular, our results on the Laplace operator verify conjectures of Fukaya; see Conjecture 0.5 of [19].

In order to describe our results in more detail, we will recall some background.

After rescaling the metric, we can assume

$$\mathrm{Ric}_{M^n} \geq -(n-1)\,. \tag{0.1}$$

Let d_{GH} denote Gromov-Hausdorff distance. As indicated above, most of our results are phrased in terms of the structure of pointed Gromov-Hausdorff limits of sequences, $(M_i^n, m_i) \xrightarrow{d_{GH}} (Y, y)$, where

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\,. \tag{0.2}$$

The renormalized limit measures arise as limits of subsequences of normalized Riemannian measures, $\underline{\mathrm{Vol}}_j \to \nu$, where

$$\underline{\mathrm{Vol}}_j(\,\cdot\,) = \frac{1}{\mathrm{Vol}_j(B_1(m_j))}\mathrm{Vol}_j(\,\cdot\,)\,; \tag{0.3}$$

see [19] and [6]. In the noncollapsed case, i.e.,

$$\mathrm{Vol}(B_1(m_i)) \geq v > 0\,, \tag{0.4}$$

the measure, ν, is unique and coincides with normalized Hausdorff measure; [13], [6].

A Radon measure, μ, is said to satisfy a doubling condition if for all $0 < r \leq R$ and some $\kappa = \kappa(R, n)$, we have

$$\mu(B_{2r}(\underline{z})) \leq 2^{\kappa}\mu(B_r(\underline{z}))\,. \tag{0.5}$$

If (0.5) holds for some $\kappa < \infty$ and all $0 < r < \infty$, then μ is said to satisfy a global doubling condition.

The relative volume comparison theorem (see [21]) implies that for Riemannian manifolds satisfying (0.1), relation (0.5) holds with $\mu = \mathrm{Vol}$, for all $r \leq R$, with $\kappa = \kappa(R, n)$. Thus, for renormalized limit measures, ν, relation (0.5) also holds for all $r \leq R$, with $\kappa = \kappa(R, n)$.

For Riemannian manifolds, M^n, satisfying (0.1), the constant in the type $(1, p)$ Poincaré inequality $(1 \leq p < \infty)$ can be bounded in terms of n, p and the diameter d; [2]. When formulated in terms of "upper gradients" (see Section 1) this holds for limit spaces (Y, ν) as well; see [3] $(p > 1)$ and Section 2 below.

A *tangent cone*, Y_y, at $y \in Y$ is the pointed Gromov-Hausdorff limit as $r_i \to 0$, of some sequence, $\{(Y, y, r_i^{-1} d)\}$. Here, d denotes the metric on Y.

A point, y is called *k-regular* if every tangent cone, Y_y, at y, is isometric to $\mathbf{R}^k$. Let $\mathcal{R}_k$ denote the set of k-regular points and $\mathcal{R} = \cup_k \mathcal{R}_k$, the *regular set.* The *singular set*, $Y \setminus \mathcal{R}$, is denoted $\mathcal{S}$. By [7], we have $\nu(\mathcal{S}) = 0$.

In Section 1, we consider pairs, (Z, μ), where Z is a metric space and μ is a Radon measure, satisfying (0.5) and a Poincaré inequality of type $(1, p)$. We record some results of standard type which are required in subsequent sections. In particular, for functions, f, with an upper gradient, $g \in L_p$, we give a bound for the Lipschitz constant of the restriction of f to a set of almost full measure; compare [16], [27], [29], [32].

In Section 2, we consider an inequality called the *segment inequality*, which involves a function, $\mathcal{F}_g(z_1, z_2) : Z \times Z \to \mathbf{R}$, defined by integration of a nonnegative function, $g : Z \to [0, \infty]$, over minimal geodesic segments, γ, from z_1 to z_2. For Riemannian manifolds satisfying (0.1), this inequality was proved in Theorem 2.11 of [5]. The segment inequality implies the existence of a Poincaré inequality of type $(1, p)$, for all $p \geq 1$.

We show that the segment inequality passes to limit spaces under measured Gromov-Hausdorff convergence; see Theorem 2.6. (Under the assumption that the measure is doubling, the corresponding result for the Poincaré inequality $(p > 1)$ is proved in [3].)

In Section 3, we turn to the study of the detailed properties of limit spaces satisfying (0.2).

Note that for $\underline{y}$, a k-regular point, there is no specific requirement on the rate of convergence as $r \to 0$, of the family of rescaled spaces, $\{(Y, y, r_i^{-1} d)\}$, to the tangent cone, $\mathbf{R}^k$. Equivalently, prior to rescaling, on sufficiently small balls, $B_r(\underline{y})$, the convergence to $\mathbf{R}^k$ takes place at the rate $o(r)$. For $\alpha > 0$, a point, $\underline{y}$, is called (k, α)-regular, if on sufficiently small balls, $B_r(\underline{y})$, the convergence to $\mathbf{R}^k$ takes place at the rate, $O(r^{1+\alpha})$. The set of $(\underline{k}, \alpha)$-regular points is denoted $\mathcal{R}_{k;\alpha}$.

In Section 3, we show that $\nu(\mathcal{R}_k \setminus \mathcal{R}_{k;\alpha(n)}) = 0$, for some $\alpha(n) > 0$. We also show that $\mathcal{R}_{k;\alpha(n)}$ is a countable union of sets, each of which is bi-Lipschitz to a subset of $\mathbf{R}^k$.

In Section 4, we show that for limit spaces satisfying (0.2), on the set, $\mathcal{R}_{k;\alpha(n)}$, any of the renormalized limit measures, ν, and Hausdorff measure, $\mathcal{H}_k$, are mutually absolutely continuous. It follows that the the collection of all renormalized limit measures determines a unique measure class.

We also recover the fact that for so called *polar* limit spaces, those for which the base point of every iterated tangent cone is a pole, the Hausdorff dimension is an integer. This was already shown in [7], using the result on renormalized volume convergence proved there.

In Section 5 we introduce μ-rectifiable spaces, which are the abstract version of finite unions of rectifiable varifolds. From the results of Section 4 and the existence of bi-Lipschitz maps to $\mathbf{R}^k$, established in Section 3, it follows that our limit spaces are ν-rectifiable. Indeed, it can be arranged that the bi-Lipschitz constant is as close as one likes to 1.

In Section 6, we consider analysis on μ-rectifiable spaces. It is clear that Rademacher's theorem on the almost everywhere differentiability of Lipschitz functions has a simple extension to such spaces. (More generally, one has the existence of a complex of differential forms which locally are sums of forms type, $f_0\, df_1 \wedge \cdots \wedge df_i$, where the f_j are Lipschitz.)

We give a short direct proof that if for some $p \geq 1$, the type $(1,p)$-Poincaré inequality holds in the sense of Section 1, then when viewed as an unbounded operator, the operator, d, on *functions*, is closeable. Equivalently, strong derivatives are unique. Note that in Section 4 of [3], for $p > 1$, existence and uniqueness of strong derivatives was proved for possibly *nonrectifiable* (Z, μ) satisfying (0.5) and a type $(1,p)$ Poincaré inequality.

Under the assumption that the above mentioned bi-Lipschitz constant can be chosen arbitrarily close to 1 (which holds for limit spaces (Y, ν)) it follows that there is a natural pointwise norm on differential forms. Moreover, for 1-forms, we have $|df| = \operatorname{Lip} f$, where $\operatorname{Lip} f$ denotes the pointwise Lipschitz constant of f. As a consequence, the Dirichlet energy, $|\operatorname{Lip} f|_2^2$ is associated to a *quadratic form* and from the uniqueness of strong derivatives ($p = 2$) it follows that the corresponding Laplace operator is *linear self-adjoint*; compare [3]. If in addition, the underlying space is compact, then $(1 + \Delta)^{-1}$ is a compact operator.

In Section 7, we show that on a compact limit space, $(M_i^n, \underline{\mathrm{Vol}}_i) \xrightarrow{d_{GH}} (Y, \nu)$, for which (0.2) holds, the eigenfunctions and eigenvalues of the Laplacian are precisely the limits of the corresponding objects for the manifolds, M_i^n. The existence of such a self-adjoint operator on $L_2(Y, \nu)$ was conjectured by Fukaya; see [19]. As a particular consequence, it follows that the eigenfunctions are Lipschitz.

We point out that Perelman has exhibited a noncollapsing sequence, $\{M_i^n\}$, of compact manifolds with positive Ricci curvature, converging in the Gromov-Hausdorff sense to some compact space, Y, such that the corresponding sequence, $\{b_2(M_i^n)\}$, of second Betti numbers is unbounded. This shows that our results on functions cannot be extended to 2-forms; compare also [31].

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1. Consequences of the Poincaré inequality

For the convenience of the reader, in this section we collect some standard results concerning upper gradients (which concept was defined in [27]).

Let Z be a metric space and let $f : Z \to \mathbf{R}$ be a Borel function on Z. Let $g : Z \to [0, \infty]$ be a Borel function such that for all points, $z_1, z_2 \in Z$ and all rectifiable curves, $c : [0, \ell] \to Z$, parameterized by arclength, s, with $c(0) = z_1$, $c(\ell) = z_2$, we have

$$|f(z_2) - f(z_1))| \leq \int_0^\ell g(c(s))\, ds\,. \tag{1.1}$$

In this case, g is called an *upper gradient* for f.

As a particular example, consider the case in which f is Lipschitz. We define the Borel function, $\mathrm{Lip}\, f$, by

$$\mathrm{Lip}\, f(z) = \limsup_{r \to 0} \frac{|f(z') - f(z)|}{\overline{z', z}}\,, \tag{1.2}$$

where $z' \in B_r(z)$. Clearly, (1.1) holds with $g = \mathrm{Lip}\, f$; see [3] for further discussion.

Let Z carry a Borel measure for which $0 < r < \infty$ implies $0 < \mu(B_r(\underline{z})) < \infty$. For $f \in L_1$, set

$$\fint_U f\,d\mu = \frac{1}{\mu(U)} \int_U f\,d\mu\,, \tag{1.3}$$

and

$$f_{\underline{z},r} = \fint_{B_r(\underline{z})} f\,d\mu\,. \tag{1.4}$$

Let $f : B_{2r}(\underline{z}) \to \mathbf{R}$, $g : B_{2r}(\underline{z}) \to [0,\infty]$. Assume that (1.1) holds for all $c \subset B_{2r}(\underline{z})$. We say that (Z,μ) satisfies a *weak Poincaré inequality of type* $(1,p)$, if there exists $\tau = \tau(r',p)$, such that for all $0 < r \leq r' < \infty$ and f, g as above, we have

$$(|f - f_{\underline{z},r}|)_{\underline{z},r} \leq \tau r((g^p)_{\underline{z},2r})^{1/p}\,. \tag{1.5}$$

If

$$(|f - f_{\underline{z},r}|)_{\underline{z},r} \leq \tau r((g^p)_{\underline{z},r})^{1/p}\,. \tag{1.6}$$

we say that a *Poincaré inequality of type* $(1,p)$ holds.

Remark 1.7. Since in the present paper, we consider only balls of finite radius, the above terminology is convenient (if not quite standard). If $\tau(\infty,p) < \infty$ we say that a *global* Poincaré inequality (respectively weak Poincaré inequality) holds.

According to [28], [24] if μ satisfies a doubling condition (respectively, global doubling condition), then the existence of a weak Poincaré inequality implies the corresponding Poincaré inequality (respectively, global Poincaré inequality). For length spaces, under these circumstances, a Poincaré-Sobolev inequality actually holds; see [24].

For applications in subsequent sections, we will need the following two results, Theorem 1.8 (see Proposition 2.5 of [14]) and Theorem 1.16.

Theorem 1.8. *Let (Z,μ) satisfy (0.5), (1.6), for some $1 \leq p < \infty$, with Z compact, of diameter d.*

Let $\mathcal{V} \subset L_p$ be a linear subspace such that for some $0 \leq K \leq \infty$, and all $f \in \mathcal{V}$, there exists an upper gradient, g, satisfying,

$$|g|_p \leq K|f|_p\,. \tag{1.9}$$

Then

$$\dim \mathcal{V} \leq N(\kappa,\tau,d,K)\,. \tag{1.10}$$

Proof. We follow closely the proof given in Proposition 2.5 of [14]. From the doubling condition on μ and the bound on the diameter, it follows that for all $r > 0$, there exists a covering of Z by $N_1 = N_1(\kappa, d)$ balls $B_r(z_i)$, whose multiplicity is at most $N_2 = N_2(\kappa)$. Consider the map, $\phi : \mathcal{V} \to R^{N_1}$, given by

$$\phi(f) = \left(\fint_{B_r(z_1)} f \, d\mu, \cdots, \fint_{B_r(z_{N_1})} f \, d\mu \right) . \tag{1.11}$$

Clearly,

$$\int_Z |f|^p \, d\mu \le \sum_i \int_{B_r(z_i)} |f|^p \, d\mu \, . \tag{1.12}$$

By the Poincarè-Sobolev inequality (see the discussion after Remark 1.7 and [24]),

$$\begin{aligned} \sum_i \int_{B_r(z_i)} |f - \fint_{B_r(z_i)} f|^p \, d\mu &\le \tau^p r^p \sum_i \int_{B_r(z_i)} |g|^p \, d\mu \\ &\le \tau^p r^p N_2 \int_Z |g|^p \, d\mu \\ &\le \tau^p r^p N_2 K \int_Z |f|^p \, d\mu \, . \end{aligned} \tag{1.13}$$

It follows from (1.12), (1.13) that for $r < \tau^{-1}(N_2 K)^{-\frac{1}{p}}$, the map, ϕ is an injection. This suffices to complete the proof. q.e.d.

Let (Z, μ) satisfy (0.5). Let $\tilde{f} : Z \to \mathbf{R}$ be a locally integrable Borel function. Recall that (0.5) implies that the Vitali covering theorem, and hence, the Lebesgue differentiation theorem holds; compare e.g. [9]. From the Lebesgue differentiation theorem, it follows that we can replace $\tilde{f}$ by a function, f, which coincides μ a.e. with $\tilde{f}$, such that for μ-a.e. $z \in Z$

$$f(z) = \lim_{r \to 0} f_{z,r} \, . \tag{1.14}$$

Recall that if (1.14) holds for $z \in Z$, then z is called a *Lebesgue point* of f. After replacing $\tilde{f}$ by f, we may assume that (1.14) holds whenever the limit exists; otherwise we set $f(z) = 0$. This assumption will be in force without further mention below.

We now recall a basic estimate on the Lipschitz constant of the restriction to a set of almost full measure, of a function which possesses an upper gradient in L_p, where $1 \leq p < \infty$. For the proof, see [29] and compare [16], [32].

Given $g : B_d(\overline{z}) \to [0, \infty]$, we put

$$u_s(z) = \begin{cases} 0, & \text{if } g^p(z) < s/2\,, \\ g^p(z), & \text{if } g^p(z) \geq s/2\,. \end{cases} \tag{1.15}$$

Theorem 1.16. *Let (Z, μ), satisfy (0.5), (1.5). Let $f : B_d(\overline{z}) \to \mathbf{R}$ and let $g \in L_p(B_d(\overline{z}))$ be an upper gradient for f. Then there exists a collection of balls, $\{B_{5r_i}(q_i)\}$, satisfying*

$$\max\{r_i\} \leq \left(\frac{(4d)^\kappa}{K}\right)^{1/\kappa}\,. \tag{1.17}$$

and

$$\sum_i \frac{\mu(B_{5r_i}(q_i))}{\mu(B_d(\overline{z}))} \leq K^{-1}(u_{K/2})_{\overline{z},d}\,, \tag{1.18}$$

such that:

i) If $z \in B_d(\overline{z}) \setminus \cup_i B_{5r_i}(q_i)$, and $t \leq \overline{z, \partial B_d(\overline{z})}$, then for all $j > 0$,

$$[(g^p)_{z,2^{-j}t}]^{1/p} \leq K^{1/p}\,, \tag{1.19}$$

ii) Moreover, z (as in i)) is a Lebesgue point of f and

$$|f(z)| \leq \left(\frac{4d}{t}\right)^\kappa (|f|)_{\overline{z},d} + tK^{1/p}2^{\kappa+1}\,\tau\,. \tag{1.20}$$

iii) If $z_1, z_2 \in B_d(\overline{z}) \setminus \cup_i B_{5r_i}(q_i)$, and $\overline{z_1, \partial B_d(\overline{z})} \geq 8\overline{z_1, z_2}$, then

$$|f(z_1) - f(z_2)| \leq K^{1/p}\, 2^{4\kappa+7}\tau\overline{z_1, z_2}\,. \tag{1.21}$$

Remark 1.22. Note that for all p, the left hand side of (1.18) contains the factor, K^{-1}, while the right hand side of (1.20) contains the factor, $K^{1/p}$. Thus, as the value of p is increased, the conclusion of Theorem 1.16 becomes stronger (while for fixed f, the hypothesis of Theorem 1.16 becomes more difficult to satisfy). For the application to the case, $p = 1$, of Theorem 6.7, it is crucial that the right-hand side of (1.18) contains the factor, $(u_{K/2})_{\overline{z},d}$, which vanishes in the limit as $K \to \infty$.

2. The segment inequality and its stability under limits

In this section we consider an inequality which we call the *segment inequality*. According to Theorem 2.11 of [5], for smooth Riemannian manifolds satisfying (0.1), the segment inequality holds with constant depending only on the dimension and the size of the sets in question. If the segment inequality holds, then so does the type $(1,p)$ Poincaré inequality, for any $p \geq 1$. We will show that the segment inequality is stable under measured Gromov-Hausdorff limits. In this way, for all $1 \leq p < \infty$, we extend to the case of limit spaces satisfying (0.2), the type $(1,p)$ Poincaré inequality, proved in [2] for manifolds satisfying (0.1); compare [3].

In order to state the segment inequality, we need a few preliminaries.

Let Z be a length space. Let $Q \subset V \times V$ be a compact subset which is the closure of its interior, such that Q is invariant under the involution which interchanges the factors. Let $\pi(Q)$ denote the projection of Q on one of the factors. For $(z_1, z_2) \in Q$, let $\gamma : [0, \ell] \to V$, denote a minimal geodesic segment parameterized by arclength, s, with $(\gamma(0), \gamma(\ell)) = (z_1, z_2)$. Assume that for all $(z_1, z_2) \in Q$ we have $\gamma \subset V$. Let $g : Z \to [0, \infty)$. For all $(z_1, z_2) \in Q$, put

$$\mathcal{F}_g(z_1, z_2) = \inf_{\gamma} \int_0^{\ell} g(\gamma(s))\, ds\,, \tag{2.1}$$

where the infimum is over all minimal geodesic segments γ as above.

Let $\mu \times \mu$ denote the product measure on $Z \times Z$. Put

$$d = \sup_{(z_1, z_2) \in Q} \overline{z_1, z_2}\,. \tag{2.2}$$

We say that the *segment inequality* holds, if there exists $\tau = \tau(d)$, such that for all g, Q, V,

$$\int_Q \mathcal{F}_g(z_1, z_2)\, d(\mu \times \mu) \leq \frac{1}{2} \tau d \cdot \mu(\pi(Q)) \int_V g\, d\mu\,. \tag{2.3}$$

(If τ can be chosen independent of d, we will say that a *global* segment inequality holds.)

According to Theorem 2.11 of [5], the estimate, (2.3), holds for manifolds satisfying (0.1), with $\tau = \tau(n, d)$.

If we replace g by g^p and apply Hölder's inequality, we get for all $p \geq 1$,

$$\left(\int_Q (\mathcal{F}_g(z_1, z_2))^p\, d(\mu \times \mu) \right)^{1/p} \leq d \left(\frac{1}{2} \tau \mu(\pi(Q)) \int_V g^p\, d\mu \right)^{1/p}. \tag{2.4}$$

Note that if (1.1) holds and in (2.4), we take $Q = U \times U$, then since $|f(z_1) - f(z_2)| \leq \mathcal{F}_g(z_1, z_2)$, it follows by Fubini's theorem, that there exists z_2, such that

$$\left(\fint_U |f(z) - f(z_2)|^p \, d\mu \right)^{1/p} \leq d \left(\frac{\tau}{2\mu(U)} \int_V g^p \, d\mu \right)^{1/p} . \tag{2.5}$$

From (2.5), we conclude that (1.5), the weak Poincaré inequality, holds. Note however, that (2.5) might actually hold with τ replaced by some smaller constant.

We will now show that the segment inequality is stable under measured Gromov-Hausdorff convergence. The argument is completely analogous to that of [3], Theorem 9.5; compare also the proof of Ziemer's theorem on the equality of the capacity and the modulus for condensers (see [34, p. 54]).

Theorem 2.6. *Let Z be a metric space for which closed balls are compact and let $\{(Z_i, z_i, \mu_i)\}$ converge to (Z, z, μ) in the measured pointed Gromov-Hausdorff sense. If μ and μ_i (for all i) are Radon measures such that for some τ, the inequality, (2.3), holds for all (Z_i, μ_i) and all functions, g_i, F_{g_i}, then (2.3) holds for (Z, μ), and all functions, g, F_g, with the same constant τ.*

Proof. The main part of the argument below is concerned with a reduction to the case in which g is continuous. In this case, the proof is quite straightforward.

The Vitali-Carathéodory theorem, asserts that given a Borel regular measure and a function, $g \in L_p(Z, \mu)$, for all $\epsilon > 0$, there exists a lower semicontinuous function, $g_\epsilon \in L_p(Z, \mu)$, with $g_\epsilon \geq g$ and $|g_\epsilon|_{L^p} < |g|_{L_p} + \epsilon$. Thus, as observed in [27], we can and will assume that g is lower semicontinuous. As a standard consequence, it follows g is the pointwise limit of a *nondecreasing* sequence of Lipschitz functions $\{h_j\}$; see p. 467 of [3].

We claim that for all $(z_1, z_2) \in Q$, we have

$$\lim_{j \to \infty} \mathcal{F}_{h_j}(z_1, z_2) = \mathcal{F}_g(z_1, z_2) . \tag{2.7}$$

To verify that (2.7) holds, note first that it suffices to assume that there exists $L < \infty$, such that $\lim_{j \to \infty} \mathcal{F}_{h_j}(z_1, z_2) = L$. Otherwise, there is nothing to prove.

Since h_j is continuous, it follows from (2.1) and the assumption that closed balls in Z are compact, that there exists $\gamma_j : [0, \ell_j] \to V$ be such that

$$\mathcal{F}_{h_j}(z_1, z_2) = \int_0^{\ell_j} g(\gamma_j(s))\, ds\,. \tag{2.8}$$

Since, we have $\ell_j \leq d < \infty$, and since closed balls are compact, it follows that there exists $\gamma_\infty : [0, \ell_\infty] \to V$ such that after passing to a subsequence, we can assume that $\ell_j \to \ell_\infty$ and $\gamma_j \to \gamma_\infty$. By the monotone convergence theorem, for all $\eta > 0$, there exists N_1, such that

$$\int_0^{\ell_\infty} g(\gamma_\infty(s))\, ds \leq \int_0^{\ell_\infty} h_{N_1}(\gamma_\infty(s))\, ds + \frac{1}{2}\eta\,. \tag{2.9}$$

From the continuity of h_{N_1}, it follows that there exists N_2 such that for $j \geq N_2$, we have

$$\int_0^{\ell_\infty} g(\gamma_\infty(s))\, ds \leq \int_0^{\ell_j} h_{N_1}(\gamma_j(s))\, ds + \eta\,. \tag{2.10}$$

Since the sequence, $\{h_j\}$, is *nondecreasing*, for all $j \geq \max(N_1, N_2)$, we get

$$\int_0^{\ell_\infty} g(\gamma_\infty(s))\, ds \leq \int_0^{\ell_j} h_j(\gamma_j(s))\, ds + \eta \tag{2.11}$$

$$= \mathcal{F}_{h_j}(z_1, z_2) + \eta\,, \tag{2.12}$$

which suffices to establish our claim.

It follows from (2.7) and the monotone convergence theorem, that it suffices to assume that g is continuous.

Now let (Z_i, μ_i) be as in the hypothesis. Recall that for $g_i : Z_i \to [0, \infty)$, a sequence of continuous functions, we write $g_i \xrightarrow{d_{GH}} g$, if for suitable Gromov-Hausdorff approximations, $\psi_i : Z \to Z_i$, the sequence, $\{g \circ \psi_i\}$, converges (uniformly) to $g : Z \to [0, \infty)$.

From the continuity of g, it follows by a standard approximation argument (using partitions of unity and standard covering theorems) that there exist sequences of compact sets which are the closures of their interiors, $Q_i \xrightarrow{d_{GH}} Q$, $V_i \xrightarrow{d_{GH}} V$ and a sequence of continuous functions, $\{g_i\}$, where $g_i : Z_i \to [0, \infty)$, such that $g_i \xrightarrow{d_{GH}} g$ and

$$\lim_{i \to \infty} \int_{V_i} g_i\, d\mu_i = \int_V g\, d\mu\,, \tag{2.13}$$

$$\lim_{i\to\infty}\int_{Q_i}\mathcal{F}_{g_i}\,d(\mu_i\times\mu_i)=\int_Q\mathcal{F}_g\,d(\mu\times\mu)\,. \tag{2.14}$$

This suffices to complete the proof. q.e.d.

From Theorem 2.6 and Theorem 2.11 of [6], we immediately obtain:

Theorem 2.15. *If (Y,y,ν) is the pointed Gromov-Hausdorff limit of a sequence of pointed Riemannian manifolds satisfying (0.2), then the segment inequality, (2.3), holds for (Y,ν). Hence, for all $1\le p<\infty$, the Poincaré inequality (1.6) holds as well.*

3. (k,α)-regularity and bi-Lipschitz maps to $\mathbf{R}^k$

In this section, we introduce the (k,α)-regular set $\mathcal{R}_{k;\alpha}$. For limit spaces satisfying (0.2), we show that $\nu(\mathcal{R}\setminus\mathcal{R}_{k;\alpha(n)})=0$, for some $\alpha(n)>0$. We also show that up to a set of measure zero, $\mathcal{R}_k$ can be expressed as a countable union of sets, each of which is bi-Lipschitz to a subset of $\mathbf{R}^k$.

The results of the present section on the (k,α)-regular set should be compared to the main theorem of [1]. There, C^α estimates on the metric tensor in harmonic coordinates on balls of a definite size are obtained, for Riemannian manifolds satisfying (0.1), under the assumption that there is a definite lower bound on the injectivity radius. Of course, the assumptions here are much weaker than those of [1].

Initially, we will discuss strongly Euclidean points, of which strongly regular points are a special case. Let $0<\alpha<1$ and let k be a positive integer. Let $(\mathcal{E}_{k;\alpha})_r$ denote the set of points, y, such that for all $0<s\le r$, there exists a metric space, X_s and $(0,x_s)\in\mathbf{R}^k\times X_s$, such that

$$d_{GH}(B_s(y),B_s((0,x_s)))\le s^{1+\alpha}\,. \tag{3.1}$$

We put $\mathcal{E}_{k;\alpha}=\cup_r(\mathcal{E}_{k;\alpha})_r$, the (k,α)-*Euclidean set.*

Our results on the (k,α)-Euclidean set, depend on a corresponding quantitative statement for smooth Riemannian manifolds satisfying (0.1). The proofs of these statements involve the interplay between the existence of harmonic functions, $\mathbf{b}$, such that $||\nabla\mathbf{b}|-1|_{L_2}$, $|\mathrm{Hess}_{\mathbf{b}}|_{L_2}$ are small, and the existence of approximate isometric splittings. (Here and below, L_2-norms should be understood as being suitably normalized by volume.) This result in part relies on, and in part complements, the almost splitting theorem of [5]).

First, we will introduce some notation which will be in force throughout the remainder of the paper. We denote by $\Psi(\epsilon_1, \dots \epsilon_k \,|\, c_1, \cdots, c_\ell)$, a nonnegative function, such that when the parameters, $c_1, \cdots, c_\ell$, are held fixed, we have

$$\lim_{\epsilon_1, \dots \epsilon_k \to 0} \Psi(\epsilon_1, \dots \epsilon_k \,|\, c_1, \cdots, c_\ell) = 0\,. \tag{3.2}$$

Theorem 3.3. *There exists $\delta(n) > 0$ such that the following holds: Let M^n be a Riemannian manifold satisfying (0.1), and let $m \in M$. If for some metric space X, $x \in X$, and for some $0 < r < 1, 0 < \delta \leq \delta(n)$, we have*

$$d_{GH}(B_{2r}(m), B_{2r}((0,x))) < \delta r\,, \tag{3.4}$$

then there exists $\alpha(n) \leq 1$, such that for all $u > 0$, the set, $B_r(m) \setminus (\mathcal{E}_{k;\alpha(n)})_{ur}$, can be covered by a collection of balls $\{B_{5s_i r}(q_i)\}$ satisfying

$$\sum_i \frac{Vol(B_{5s_i r}(q_i))}{Vol(B_r(m))} \leq \Psi(\delta|u,n) + \Psi(u|n)\,. \tag{3.5}$$

Proof. By scaling, it suffices to consider $B_2(m) \subset M^n$, such that for some X, $(0,x) \in X$,

$$d_{GH}(B_2(m),\, B_2((0,x))) < \delta\,. \tag{3.6}$$

Fix a constant, $\epsilon(n) > 0$, to be specified later. By the proof of Theorem 6.62 of [7] (see also [13]) the following holds: Given $p \in B_1(m)$, $0 < u < c(n)$, where $0 < c(n) < 1/2$ is sufficiently small, there exist harmonic functions, $\mathbf{b}_1, \dots, \mathbf{b}_k$ on $B_{2u}(p)$, such that

$$\fint_{B_{2u}(p)} \Big(\sum_i \big|\, |\nabla \mathbf{b}_i|^2 - 1 \,\big| + \sum_{i \neq j} |\langle \nabla \mathbf{b}_i,\, \nabla \mathbf{b}_j \rangle\,| + \sum_i u^{-2}\, |\mathrm{Hess}_{\mathbf{b}_i}|^2\Big)$$
$$\leq \epsilon(n)\Phi\,, \tag{3.7}$$

where from now on, we write

$$\Phi = \Psi(\delta|u,n) + \Psi(u|n)\,. \tag{3.8}$$

Furthermore, there exists $C = C(n)$ such that for $i = 1, \cdots, k$

$$|\nabla \mathbf{b}_i| \leq C\,. \tag{3.9}$$

Given $0 < u < 1/2$, we can find a finite collection of balls, $\{B_{2u}(p_j)\}$, such that the collection, $\{B_u(p_j)\}$, covers $B_1(m)$, with multiplicity, $N(n)$ (independent of u). Thus, in order to obtain (3.5), it will suffice to prove (3.5), with $B_u(p_j)$ in place of $B_r(m)$.

After rescaling the Riemannian metric by a factor, u^{-2}, and making the replacement, $\mathbf{b}_i \to u^{-1}\mathbf{b}_i$, we are reduced to considering a ball such that

$$\begin{aligned}&\fint_{B_2(p_j)} \Big(\sum\nolimits_i \big|\,|\nabla \mathbf{b}_i|^2 - 1\,\big| + \sum\nolimits_{i\neq j} \,|\langle \nabla \mathbf{b}_i,\, \nabla \mathbf{b}_j\rangle\,| + \sum\nolimits_i \,|\mathrm{Hess}_{\mathbf{b}_i}|^2\Big) \\ &\qquad \le \epsilon(n)\Phi\,. \qquad (3.10)\end{aligned}$$

We now consider the restrictions of the functions, $\mathbf{b}_i$, to sub-balls, $B_{2s}(z) \subset B_2(p_j)$, where $z \in B_1(p_j)$. Our aim is to control the measure of the union of those sub-balls for which the restricted functions do *not* satisfy an estimate like (3.10). If on the other hand, such an estimate holds on a ball, $B_{2s}(z)$, the desired approximate splitting of $B_s(z)$ will turn out to hold as well.

Set

$$f = \sum\nolimits_i \big|\,|\nabla \mathbf{b}_i|^2 - 1\,\big| + \sum\nolimits_{i\neq j} \,|\langle \nabla \mathbf{b}_i,\, \nabla \mathbf{b}_j\rangle\,|\,, \qquad (3.11)$$

$$h_{i,j} = \langle \nabla \mathbf{b}_i,\, \nabla \mathbf{b}_j\rangle\,, \qquad (3.12)$$

and

$$g^2 = \sum_i |\mathrm{Hess}_{\mathbf{b}_i}|^2\,. \qquad (3.13)$$

In view of (3.9), an easy calculation shows that for all i, j, there exists $C = C(n)$, such that

$$|\nabla f|,\, |\nabla h_{i,j}| \le Cg\,. \qquad (3.14)$$

Equivalently, Cg is an upper gradient of f and h.

Apply Theorem 1.16, with f, g as above, $p = 2$ and $K = \epsilon(n)$. Thus, we obtain balls, $B_{5s_i}(q_i)$, such that

$$\sum_i \frac{\mu(B_{5s_i}(q_i))}{\mu(B_1(z))} \le \Phi\,, \qquad (3.15)$$

and for $z \in B_1(p_j) \setminus \cup B_{5s_i}(q_i)$, and all $0 < s < 1$

$$(g^2)_{z,2s} \leq \epsilon(n)\,, \tag{3.16}$$

$$f_{z,2s} \leq \epsilon(n)\,, \tag{3.17}$$

$$|h_{i,j} - (h_{i,j})_{z,2s}|_{z,2s} \leq c(n)(\epsilon(n))^{\frac{1}{2}} s\,. \tag{3.18}$$

Here, we have used (3.10), to estimate the right-hand sides. Also, κ, the doubling constant, and τ, the constant in the Poincaré inequality satisfy $\kappa = \kappa(n)$, $\tau = \tau(n)$.

Fix $\epsilon(n)$, sufficiently small. By orthonormalizing the collection of functions, $\{\mathbf{b}_i\}$, with respect to the global inner product defined by the normalized measure on $B_s(z)$, it follows from (3.17), (3.18) that for all s, there exist harmonic functions, $\mathbf{b}_{1,s}, \ldots, \mathbf{b}_{k,s}$, each of which is a linear combination (with coefficients bounded independent of s) of the functions, $\mathbf{b}_1, \ldots, \mathbf{b}_k$, such that

$$\begin{aligned} ⨍_{B_s(z)} f = ⨍_{B_s(z)} \sum\nolimits_i \big|\, |\nabla \mathbf{b}_{i,s}|^2 - 1 \,\big| \\ + \sum\nolimits_{i \neq j} \,|\langle \nabla \mathbf{b}_{i,s},\, \nabla \mathbf{b}_{j,s}\rangle\,| \leq c(n)s\,. \end{aligned} \tag{3.19}$$

By Theorem 16.32 of [3], there exists $\alpha_1(n) > 0$, such that for all i, there exist distance functions, $\rho_{i,s} : B_{2s}(z) \to \mathbf{R}$, with the properties of the distance functions used in the proof of the almost splitting theorem of [5], such that

$$|\mathbf{b}_{i,s} - \rho_{i,s}| \leq c(n) s^{1+\alpha_1(n)}\,. \tag{3.20}$$

In more detail, in [3], the hypotheses of Theorem 16.32 are (16.3), (16.4), (16.12), (16.33). Relation (16.3) is implied by (3.14) above. Relations, (16.4), (16.12) are implied by (3.19) above. Finally, (16.33) follows immediately from the fact that the functions, $\mathbf{b}_{i,s}$, are harmonic.

Relation (3.20) follows directly from (16.13), (16.14), (8.19), (8.20) of [3], which are the conclusions of Theorem 16.32. (Note that in [3], (16.13), (16.14) occur initially in Lemma 16.11.)

From (3.14), (3.16), (3.19), (3.20), together with the (proof of the) almost splitting theorem, we obtain the desired approximate splitting of $B_s(z)$. After rescaling the Riemannian metric on $B_{2s}(z)$ by the factor s^{-2}, and making the replacement, $\mathbf{b}_{i,s} \to s^{-1}\mathbf{b}_{i,s}$, these relations are quantitative versions of (6.28), (6.61), (6.26), (6.16) of [5] respectively. Given these quantitative estimates, inspection of the proof of the almost splitting theorem easily reveals that the convergence to the precise splitting takes place at the rate $s^{1+\alpha(n)}$. q.e.d.

Fix $\alpha > 0$. Let $(\mathcal{R}_{k;\alpha})_r$, denote the set of points, y, such that for all $0 < s \leq r$, we have

$$d_{GH}(B_s(y), B_s(0)) \leq s^{\alpha+1}\,, \tag{3.21}$$

where $B_s(0) \subset \mathbf{R}^k$.

Definition 3.22. The (k,α)-regular set, $\mathcal{R}_{k;\alpha}$, is the set of all y such that $y \in (\mathcal{R}_{k;\alpha})_r$, for some $r > 0$.

By a straightforward limiting argument, the conclusion of Theorem 3.3 also holds for limit spaces satisfying (0.2). From this we get the following:

Theorem 3.23. *There exists $\alpha(n) > 0$, such that if (Y,ν) satisfies (0.2), then for all $\alpha' < \alpha(n)$,*

$$\nu(\mathcal{R}_k \setminus \mathcal{R}_{k;\alpha'}) = 0\,. \tag{3.24}$$

Proof. Since by Section 2 of [6], we have $\nu(\mathcal{S}) = 0$, it follows from Theorem 3.3 and the Lebesgue differentiation theorem, that for $\alpha(n)$ as in Theorem 3.3, we have $\nu(\mathcal{R}_k \setminus \mathcal{E}_{k;\alpha(n)}) = 0$.

On the other hand, $y \in \mathcal{R}_k \cap \mathcal{E}_{k;\alpha(n)}$, implies $y \in \mathcal{R}_{k;\alpha'}$, for all $\alpha' < \alpha(n)$. To see this, assume to the contrary that there exists a sequence, $r_i \to 0$, such that $\mathrm{diam}(X_{r_i}) \geq c r_i^{1+\alpha'}$, where X_r is as in (3.1) By considering the family of rescalings by $r_i^{-(1+\beta)}$, where $\alpha' < \beta < \alpha(n)$, we find that there exists a tangent cone Y_y, not isometric to $\mathbf{R}^k$. This contradiction suffices to complete the proof. q.e.d.

Let $(\mathcal{R}_k)_{\delta,r}$, denote the set of points, $y \in Y$, such that for all $0 < s \leq r$,

$$d_{GH}(B_s(y), B_s(0)) < \delta\, s\,, \tag{3.25}$$

where $B_r(0) \subset \mathbf{R}^k$. Clearly, $\cup_r (\mathcal{R})_{\delta,r} \supset \mathcal{R}_k$.

From the proof of Theorem 3.3, we immediately obtain:

Theorem 3.26. *Let $\nu((\mathcal{R}_k)_{\delta,r}) > 0$, and let y be a Lebesgue point of $(\mathcal{R}_k)_{\delta,r}$. Then for all $\eta > 0$, there exists $r_\eta(y) > 0$, such that for all*

$0 < t \le r_\eta(y)$, there exists a subset, $U_t \subset B_t(y) \cap (\mathcal{R}_k)_{\delta,r}$. with

$$\frac{\nu(B_t(y) \setminus U_t)}{\nu(B_t(y))} \le \eta\,, \tag{3.27}$$

and functions, $\mathbf{b}_1, \dots, \mathbf{b}_k$, on $B_t(y)$, such that the map given by, $\mathbf{b}(w) = (\mathbf{b}_1(w), \dots, \mathbf{b}_k(w))$, determines a $e^{\pm\Psi(\delta|n)}$–bi-Lipschitz equivalence from from U_t to $\mathbf{b}(U_t)$.

Proof. Since y is a Lebesgue point of $(\mathcal{R}_k)_{\delta,r}$, there exists $r_\eta(y) > 0$, such that for all $0 < t \le r_\eta(y)$, there exists $U_t \subset (\mathcal{R}_k)_{\delta,r}$, satisfying (3.27).

From the proof of Theorem 3.3, we can assume in addition that for $w \in U_t$, there exist functions $\mathbf{b}_{1,t}, \dots, \mathbf{b}_{k,t}$, on $B_t(y)$, such that for $w \in U_t$, the restrictions of these functions to a ball, $B_s(w)$, determine $\Psi(\delta|n)s$-Gromov-Hausdorff equivalence from $B_s(w)$ to $B_s(0)$. If we now consider $w_1, w_2 \in U_t$ and take $s = \overline{w_1, w_2}$, then it follows that $\overline{\mathbf{b}(w_1), \mathbf{b}(w_2)}$ is as asserted. q.e.d.

4. Renormalized limit measures and Hausdorff measure

In this section, we show that if (Y, ν) satisfies (0.2) and $y \in (\mathcal{R}_{k;\alpha})_r$, then for some $c = c(n, \alpha, r)$

$$0 < c^{-1}\nu(B_1(y)) \le \lim_{s \to 0} \frac{\nu(B_s(y))}{s^k} \le c\,\nu(B_1(y))\,. \tag{4.1}$$

In particular, the limit in (4.1) exists. This result can be viewed as a partial generalization to the collapsed case, of the volume convergence conjecture of Anderson-Cheeger, which was proved by the second author in [13]; see also [6] Theorem 5.4. For an additional result concerning renormalized volume convergence, see Section 2 of [7].

By bringing in the results of Section 3, we find that any two renormalized limit measures, ν_1, ν_2, are mutually absolutely continuous. Recall in this connection that the renormalized limit measure on a limit space, Y, might depend on the particular sequence, $(M_i^n, m_i) \xrightarrow{d_{GH}} (Y, y)$; see Example 1.24 of [6]. We show in addition that, when restricted to $\cup_\alpha \mathcal{R}_{k;\alpha}$, any such measure ν and k-dimensional Hausdorff measure, $\mathcal{H}_k$, are mutually absolutely continuous. Finally, $\mathcal{H}_k$ is σ-finite on $\cup_\alpha \mathcal{R}_{k;\alpha}$, Recall that for all $y \in Y$ and $0 < s \leq 1$,

$$\frac{\nu(B_s(y))}{\nu(B_1(y))} \geq c(n)s^n \,. \tag{4.2}$$

Also, if say $\partial B_2(y) \neq \emptyset$, then for $0 < s \leq 1$,

$$\frac{\nu(B_s(y))}{\nu(B_1(y))} \leq c(n)s \,. \tag{4.3}$$

As indicated above, we will show (in Theorem 4.6 below) that if $y \in (\mathcal{R}_{k;\alpha})_r$, for some $0 < \alpha, r$, then the exponents in (4.2), (4.3), can each be replaced by k, provided the constants are allowed to depend on n, α, r.

The idea of the proof is very simple. It can be illustrated as follows: Instead of balls we consider cubes in $\mathbf{R}^k$ and denote by $C_r^k \subset \mathbf{R}^k$, the cube of side, r, with center at the origin. Note that given $\mathrm{Vol}(C_1^k) = 1$, we can prove by induction that $\mathrm{Vol}(C_{3^{-i}}^k) = (3^{-i})^k$, for all i, by observing that $\mathrm{Vol}(\cdot)$ is translation invariant and that up to a set of measure zero, $C_{3^{-j}}^k$ is the disjoint union of 3^k translates of $C_{3^{-(j+1)}}^k$.

In proving Theorem 4.6, a similar argument can be applied, although there are error terms whose cumulative effect must be controlled. Since at the j-th stage, we consider a ratio (of the measures of two sets), the required error control corresponds to the convergence to a finite nonzero limit, of the infinite product of these ratios. The hypothesis, $y \in (\mathcal{R}_{k;\alpha})_r$, is easily seen to guarantee this convergence. To show that at the j-th stage, the measure is almost translation invariant in the appropriate sense, we use directionally restricted relative volume comparison, just as in the proof of Proposition 1.35 of [7].

The error terms which arise in the proof of Theorem 4.6 can all be bounded by suitable applications of Lemma 4.4 below (in each instance, after suitable rescaling of the metric.)

Let Z be a metric space. For $\epsilon \leq \frac{1}{20}$, let $\phi : B_2(0) \to B_2(\phi(0))$ be an ϵ-Gromov-Hausdorff equivalence, where $0 \in \mathbf{R}^k$, $\phi(0) \in Z$. Let $w \in \mathbf{R}^k$, with $\overline{0, w} = 1$ and let z_1 satisfy $\overline{\phi(w), z_1} \leq \delta$ where $\delta \leq \frac{1}{20}$. Let $\tau : [0, \ell] \to Z$ be a minimal geodesic parameterized by arclength with $\tau(0) = \phi(0)$, $\tau(\ell) = z_1$.

Lemma 4.4. *For all $0 \leq t \leq 1/2$, there exists a point, q_t, lying on the line segment from 0 to w, such that*

$$\overline{\tau(\ell - t), \phi(q_t)} \leq 2(t + 3\epsilon + \delta)^{1/2}(5\epsilon + 2\delta)^{1/2} \,. \tag{4.5}$$

Proof. By definition, there exists $p \in B_2(0)$, such that $\overline{\phi(p), \tau(\ell - t)} \leq \epsilon$. It follows from the triangle inequality that $\overline{\phi(0), \phi(w)} + 2(\epsilon + \delta) \geq \overline{\phi(0), \phi(p)} + \overline{\phi(p), \phi(w)}$. Thus, letting e denote the excess of the triangle with vertices, $0, w, p$, we get $e \leq 5\epsilon + 2\delta$. Let q be the point on the line segment from 0 to w, such that $\overline{q, w} = \overline{p, w}$. By applying the law of cosines to the triangles, q_t, w, p, and $0, w, p$, we get $\overline{q, p}^2 = e(2 + e - 2\overline{p, w})\overline{p, w}$. The lemma easily follows. q.e.d.

In the following theorem, without loss of generality, we assume $0 < r \leq 1$ and $\partial B_2(y) \neq \emptyset$.

Theorem 4.6. *For all $0 < \alpha < 1$, there exists $c = c(n) > 0$, such that if (Y, ν) satisfies (0.2), $y \in (\mathcal{R}_{k;\alpha})_r$, then*

$$c^{-1} r^{\frac{2(1+\alpha)}{2+\alpha}(n-k)} s^k \leq \frac{\nu(B_s(y))}{\nu(B_1(y))} \leq c\, r^{\frac{2(1+\alpha)}{2+\alpha}} s^k \qquad (0 < s \leq r^{1+\frac{\alpha}{2+\alpha}(1-k)} \leq 1)\,. \tag{4.7}$$

Moreover, $\lim_{s \to 0} \nu(B_s(y))/s^k = v(y)$ *exists and for some $c = c(n)$, $\beta = \beta(n, \alpha) > 0$,*

$$(1 - c\, s^\beta) \leq \frac{\nu(B_s(y))}{s^k v(y))} \leq (1 + c\, s^\beta) \qquad (0 < s \leq r^{1+\frac{\alpha}{2+\alpha}} \leq 1)\,. \tag{4.8}$$

Proof. By (4.2), (4.3), it suffices to consider $0 < s \leq r^{1+\frac{\alpha)}{2+\alpha}}$.

Set $s_j = 3^{-j} r^{1+\frac{\alpha}{2+\alpha}}$. For $j = 1, 2, \ldots$, we will construct cube-like sets, $\mathcal{C}_j$, such that $B_{\frac{1}{2}s_j}(y) \subset \mathcal{C}_j \subset B_{2ks_j}(y)$, and for some $c = c(n)$, $\beta = \beta(n, \alpha) > 0$, we have

$$c^{-1} \leq \frac{\nu(\mathcal{C}_j)}{\nu(B_{s_j}(y))} \leq c\,, \tag{4.9}$$

and

$$(1 - cs_j^\beta) \leq \frac{\nu(\mathcal{C}_{j+1})}{3^{-k}\nu(\mathcal{C}_j)} \leq (1 + cs_j^\beta)\,. \tag{4.10}$$

Note that by relative volume comparison, for $c = c(n)$, we have

$$c^{-1}\nu(\mathcal{C}_j) \leq \nu(B_s(y)) \leq c\nu(\mathcal{C}_j) \qquad (s_{j+1} \leq s \leq s_j)\,. \tag{4.11}$$

Since $\beta > 0$, if we take $j = 1, \ldots,$ in (4.10) and multiply the resulting inequalities together, then the infinite products corresponding to the left-hand and right-hand sides converge. Thus, if we take $j = 1, \ldots,$ multiply these inequalities together, and use (4.9), (4.11), we get (4.7). The existence of $\lim_{s\to 0} \nu(B_s(y))/s^k$, which is discussed at the end of the proof, follows from a slight refinement of the argument.

The sets, $\mathcal{C}_i$, will be fattened images of the honest cubes, $C^k_{s_j} \subset \mathbf{R}^k$, under suitable Gromov-Hausdorff equivalences which carry $0 \in \mathbf{R}^k$ to $y \in Y$.

Let $\underline{\psi}_j$, denote a $s_j^{1+\alpha}$-Gromov-Hausdorff equivalence, $\underline{\psi}_j : B_{s_j}(0) \to B_{s_j}(y)$. $\underline{\psi}_j$, After successively replacing the maps, $\underline{\psi}_j$, by maps, $\underline{\psi}_j \circ \theta_j$, for suitable $\theta_j \in O(k)$, we can (for convenience) glue together the maps, $\underline{\psi}_j \circ \theta_j$, to obtain a single map, $\psi : B_{s_0}(0) \to B_{s_0}(y)$, whose restriction to every ball, $B_{s_j}(0)$, determines a $cs_j^{1+\alpha}$-Gromov-Hausdorff equivalence, where $c = c(k)$.

Let $C^k_s + v$ denote the translate of the cube, C^k_s, by the vector, $v \in \mathbf{R}^k$. For all $s < r$ and $|v| < 10ks$, we let $\underline{\mathcal{C}}_s(v)$ denote the set of points at distance $2s^{1+\alpha}$ from $\psi(C^k_s + v)$. These sets will be called *pseudo-cubes.* We define the set, $\mathcal{C}_j$, in (4.9), (4.10) by $\mathcal{C}_j = \underline{\mathcal{C}}_{s_j}(0)$.

Now let $K \subset Y$ be a compact subset and let s denote the smallest positive number such that $K \subset B_s(y)$. Let F be a $(k-1)$-dimensional face of the cube, C^k_1, and let N_F denote the outward normal to F. For $b < c(k)s^{1-\frac{\alpha}{2+\alpha}}$, we now define a notion which we call *pseudo-translation*, of the set K, by b units in the direction N_F.

For all $\underline{y} \in \underline{\mathcal{C}}_{s_j}(v)$, choose a minimal geodesic parameterized by arclength, σ, from $\underline{y}$ to the point $\psi(v + s^{(1-\frac{\alpha}{2(1+\alpha)})}N_F)$. Put

$$\mathcal{T}_{bN_F}(K) = \cup_\sigma \sigma(b)\,. \tag{4.12}$$

Let T_{bN_F} denote translation by bN_F in $\mathbf{R}^k$. By using Lemma 4.4, it is easy to check that

$$\overline{\mathcal{T}_{bN_F}, \psi \circ T_{bN_F}} < cs^{1+\frac{\alpha}{4}}\,. \tag{4.13}$$

Note in this connection, that the exponent, $\frac{\alpha}{2(1+\alpha)}$, in the definition of $\mathcal{T}_{bN_F}$, was chosen such that $s^{(1+\alpha)(1-\frac{\alpha}{2(1+\alpha)})} = s^{1+\frac{\alpha}{2}}$, where $1 + \frac{\alpha}{2} > 1$.

Let $c = c(n) > 0$ and $\beta = \beta(n, \alpha) > 0$ denote generic constants whose values depend only on n and n, α respectively. As in the model case described above, we would in principle, like to cover $\mathcal{C}_j$ by 3^k pseudo-translates of $\mathcal{C}_{j+1}$, whose mutual intersections have measure 0.

While it is not precisely true that pseudo-translation is measure preserving and carries pseudo-cubes to pseudo-cubes, this does hold up to errors of size $cs^{1+\beta}$.

If for suitable c, β, we replace $\mathcal{C}_{j+1}$, by a slightly larger, pseudo-cube, $\underline{\mathcal{C}}_{s_{j+1}(1+cs^{\beta}_{j+1})}$, centered at y, then it follows from (4.13) that the iterated pseudo-translates of this pseudo-cube cover $\mathcal{C}_j$. Similarly, if instead, we use a suitable pseudo-cube $\underline{\mathcal{C}}_{s_{j+1}(1-cs^{\beta}_{j+1})}$, the corresponding pseudo-translates will be disjoint. By employing Lemma 4.4, we can choose these three pseudo-cubes can be chosen to be linearly ordered by inclusion and by using directionally restricted relative volume comparison, we can guarantee that the ratio of the measures of any two of them lies between $1 - cs^{\beta}_{j+1}$ and $1 + cs^{\beta}_{j+1}$.

Thus, it suffices to show that the pseudo-translations we are employing, distort the measures of our pseudo-cubes by a factors which lie between $1 - cs^{\beta}_{j+1}$ and $1 + cs^{\beta}_{j+1}$.

By directionally restricted relative volume comparison, it follows that for $K \subset B_{ks}(y)$, we have,

$$\nu(\mathcal{T}_{bN_F}(K)) \geq (1 - c(n))s^{\frac{\alpha}{2(\alpha+1)}}\nu(K)\,. \tag{4.14}$$

If pseudo-translation actually mapped pseudo-cubes to pseudo-cubes preserving the side length, then for a K a pseudo-cube, (4.14) would lead to a two sided inequality, and hence, to the conclusion that pseudo-translations were measure preserving, in case K were a pseudo-cube. Namely, we could follow $\mathcal{T}_{bN_F}$ by $\mathcal{T}_{-bN_F}$ and apply (4.14) a second time. Although pseudo-translations do not map pseudo-cubes (precisely) to pseudo-cubes, this difficulty can be circumvented by considering slightly smaller and slightly larger concentric pseudo-cubes as above; compare the proof of Proposition 1.35 of [7]. This suffices to complete the proof of (4.7).

To obtain the existence of $\lim_{s\to 0} \nu(B_s(y))/s^k$, as well as (4.8), we proceed in a fashion similar to the above. We almost cover the ball, $B_s(y)$, by suitable pseudo-translates of the pseudo-cube, $\mathcal{C}_j$, where as $s \to 0$, we also let $s_j/s \to 0$ at a suitable rate. This proceedure can be made arbitrarily accurate for s sufficiently small, and the desired conclusion follows. The details, which are straightforward, are left to the reader. q.e.d.

From Theorem 4.6 together with Theorem 3.23, and the Lebesgue differentiation theorem, we easily get:

Theorem 4.15. *For all $n, \alpha, r > 0$, there exists $c = c(n, \alpha, r) > 0$, such that*

$$c^{-1}\nu \leq \mathcal{H}_k \leq c\,\nu \qquad (on\ (\mathcal{R}_{k;\alpha})_r). \tag{4.16}$$

In particular:

i) When restricted to $\cup_\alpha \mathcal{R}_{k;\alpha}$, the measures, ν and $\mathcal{H}_k$, are mutually absolutely continuous.

ii) When restricted to $\mathcal{R}_k$, the measure, ν, is absolutely continuous with respect to $\mathcal{H}_k$.

iii) When restricted to $\mathcal{R}_k$, the measure, $\mathcal{H}_k$, is σ-finite.

By Theorem 4.15 and the fact (proved in Theorem 2.1 of [6]) that $\nu(\mathcal{S}) = 0$, for any ν, we obtain:

Theorem 4.17. *Any two renormalized limit measures, ν_1, ν_2, are mutually absolutely continuous.*

Remark 4.18. Note that in Theorem 4.17, the renormalized limit measures, ν_1, ν_2, might arise from entirely different sequences of Riemannian manifolds converging to the given limit space Y; compare Example 1.24 of [6].

Remark 4.19. *em From the argument of [7], but using Theorem 4.6 in place of Theorem 1.2 of that paper, we recover the result proved there, asserting that the Hausdorff dimension of a so called polar limit space is an integer; see Section 1 of [7] for further details.*

5. ν-rectifiablity of limit spaces

Let X be a metric space and μ a Radon measure on X. In this section we introduce the concept of μ-rectifiability of X. It is then immediate from the results of Sections 3 and 4 that limit spaces, Y, satisfying (0.2) are ν-rectifiable for any renormalized limit measure, ν.

The concept of μ-rectifiability involves two conditions. The first of these requires that after removing a set of measure zero, the remaining part of our space can be written as a countable union of sets, $C_{k,i}$, each of which is bi-Lipschitz to a subset of $\mathbf{R}^k$, where $k \leq m < \infty$. The second condition requires that when restricted to $C_{k,i}$, the measure, μ, is absolutely continous with respect to k-dimensional Hausdorff measure $\mathcal{H}^k$.

In order to ensure that the Laplacian on a rectifiable space is a *linear* operator, an additional condition iii) must be added; see Section 6 and compare [3]. Namely, we assume that for all $\lambda > 0$, we can choose the above mentioned bi-Lipschitz equivalences, such that they and their inverses have Lipschitz constants between e^{λ} and $e^{-\lambda}$.

Again, it is clear from the results of Sections 3 and 4 that this condition holds for our limit spaces.

Definition 5.1. *em The measure, μ, is Ahlfors k-regular at $x \in X$, if there exists $K = K(x)$, such that for $r \leq 1$,*

$$K^{-1} r^k \leq \mu(B_r(x)) \leq K r^k \,. \tag{5.2}$$

Definition 5.3. The space, X, is *μ-rectifiable*, if there exists an integer, m, a countable collection of Borel subsets, $C_{k,i} \subset X$, where $k \leq m$ and bi-Lipschitz maps, $\phi_{k,i} : C_{k,i} \to \phi_{k,i}(C_{k,i}) \subset \mathbf{R}^k$, such that

i) $\mu(X \setminus \cup_{k,i} C_{k,i}) = 0$,

ii) μ is Ahlfors k-regular at x, for all $x \in C_{k,i}$.

Note that by ii), we $C_{k,i} \cap C_{k',i'} = \emptyset$, if $k \neq k'$. Moreover, on $C_{k,i}$ the measure, μ, and Hausdorff measure, $\mathcal{H}^k$, are mutually absolutely continuous.

Remark 5.4. Our concept of "μ-rectifiability", differs a bit from that of [18], p. 251 (where the term "countably (μ, m)-rectifiable" is employed.) There, it is stipulated that $k = m$, for all k, while condition ii) is omitted. Note that if one makes this stipulation while keeping ii), then the resulting conditions on the union of the sets, $C_{k,i}$, would correspond to those which, for subsets of $\mathbf{R}^k$, define a countably rectifiable varifold.

In the present abstract setting, conditions i), ii), do not imply the existence of regular points. However, if μ satisfies (0.5), then Vitali's covering theorem holds and we can assume without loss of generality that each $C_{k,i}$ consists entirely of Lebesgue points. It follows easily that for all $x \in C_{k,i}$, every tangent cone at x is bi-Lipschitz to $\mathbf{R}^k$. Thus, the doubling condition together with condition i) already implies that $C_{k,i} \cap C_{k',i'} = \emptyset$, if $k \neq k'$.

Additionally, by the Lebesgue differentiation theorem, for almost all points at which μ is k-regular, $\lim_{r \to 0} \mu(B_r(x))/\mathcal{H}^k(B_r(x)) < \infty$, exists, is finite and nonzero. Thus, we can assume that this holds at all points of $C_{k,i}$.

As an immediate consequence of Theorems 3.3, 3.26 and 4.6 we get:

Theorem 5.5. *If (Y, ν) is the measured Gromov-Hausdorff limit of a a sequence satisfying, (0.2), then Y is ν-rectifiable.*

Remark 5.6. Recall that for limit spaces satisfying (0.2), the limit, $\lim_{r \to 0} \mu(B_r(y))/r^k$ exists and is finite and nonzero for all $y \in \mathcal{R}_{k;\alpha(n)}$, where $\nu(Y \setminus \cap_k \mathcal{R}_{k;\alpha(n)}) = 0$; see Theorem 4.6.

We now introduce an additional requirement on the bi-Lipschitz maps, $\phi_{k,i}$, of Definition 5.3.

iii) For all $x \in \cup_{k,i} C_{k,i}$ and all $\lambda > 0$, there exists $C_{k,i}$ such that $x \in C_{k,i}$ and the map, $\phi_{k,i} : C_{k,i} \to \phi_{k,i}(C_{k,i}) \subset \mathbf{R}^k$, is $\mathrm{e}^{\pm\lambda}$-bi-Lipschitz.

As an immediate consequence of Theorems 3.3, 3.26 and 4.6 we get:

Theorem 5.7. *If (Y, ν) is the measured Gromov-Hausdorff limit of a a sequence satisfying, (0.2), then condition iii) holds.*

6. The Laplacian on rectifiable spaces

In this section, we study analysis on μ-rectifiable spaces as defined in Section 5. By Theorems 5.5 and 5.7, the results of this section apply to limit spaces for which (0.2) holds.

We begin by considering spaces which satisfy the two conditions of Definition 5.3. Under these hypotheses, we observe the existence of a complex of Lipschitz differential forms.

Next, we assume in addition that the measure is doubling and that for some $p \geq 1$, a Poincaré inequality of type $(1, p)$ holds for balls of a definite size, say $r \leq 1$. For all $p \geq 1$, we show the uniqueness of the differential of a function in the strong L_p sense.

From the uniqueness of strong derivatives for the case, $p = 2$, and condition iii), it follows that the natural Laplace operator Δ, on functions is *linear* and self-adjoint. (By definition, Δ is a nonnegative operator.) At this point, we are effectively in an abstract situation in which our measure, μ, has all the properties of an *admissable weight*, in the sense of Chapter 1 of [25]. From Theorem 1.8, we conclude that if the underlying space is compact, then $(1+\Delta)^{-1}$ is a compact operator. The eigenfunctions are Hölder continuous, and the degree of Hölder regularity can be bounded below in terms of the doubling constant of μ, the constant in the Poincaré inequality and the corresponding eigenvalue; see Theorems 5.27 and 6.6 of [25]. It will be shown in Section 7, that for the case of limit spaces satisfying (0.2), the eigenfunctions are actually *Lipschitz.*

Note that for limit spaces satisfying (0.2), the existence of a self-adjoint Laplace operator was conjectured in [19]. There, it was also conjectured that the eigenfunctions and eigenvalues should behave continuously under measured Gromov-Hausdorff convergence. This will be proved in Section 7.

Let Z be a metric space and $f : A \to \mathbf{R}$, is Lipschitz, for some $A \subset Z$, We will denote by $\tilde{f} : Z \to \mathbf{R}$, some Lipschitz function such that $\tilde{f}\,|\,A = f\,|\,A$ and the Lipschitz constant of $\tilde{f}$ is equal to that of f. By a well known elementary lemma ("MacShane's lemma") such a function always exists; see 2.10.44 of [18].

In particular, if $A \subset \mathbf{R}^k$, then by Rademacher's theorem, $\tilde{f}$ is differentiable almost everywhere. Moreover, if $\underline{f} : \mathbf{R}^k \to \mathbf{R}$ is another Lipschitz extension of f, then it is clear that for almost all $p \in A$ we have $d\tilde{f}(p) = d\underline{f}(p)$. In particular, this holds for all Lebesgue points, $p \in A$, at which $d\tilde{f}(p)$ and $d\underline{f}(p)$ exist. Thus, for f as above, we will simply write df, which is defined at almost all points of A.

Let X be μ-rectifiable. Assume that for all Borel subsets, $C \subset X$, and all bi-Lipschitz maps, $\phi : C \to \mathbf{R}^k$, we are given a Borel function, $f_\phi : \phi(C) \to \mathbf{R}$, such that for all $C_1, C_2 \subset X$ (for which the corresponding values of k are equal) we have the compatibility condition,

$$f_{\phi_1} \circ \phi_1 = f_{\phi_2} \circ \phi_2\,|(C_1 \cap C_2) \qquad \text{(a.e.)}\,. \tag{6.1}$$

Then for some μ-a.e. uniquely defined Borel function, $f : X \to \mathbf{R}$, we have $f \circ \phi^{-1} = f_\phi$.

By MacShane's lemma, there exists a Lipschitz extension to $\mathbf{R}^k$, of the Lipschitz map, $\phi_2^{-1} \circ \phi_1$. The extended map is differentiable almost everywhere and at almost all points of $\phi(C_1 \cap C_2)$ the differential is independent of the extension.

Consider a collection of i-forms, $\{\omega_\phi\}$, such that ω_ϕ can be written as a finite sum of forms,

$$\omega_\phi = \sum_\alpha f_{0,\alpha}\, df_{1,\alpha} \wedge \cdots \wedge df_{i,\alpha} \qquad \text{(a.e.)}\,, \tag{6.2}$$

with $f_{j,\alpha}$ Lipschitz, for $1 \le j \le i$ and all α. Suppose in addition that the functions, $f_{0,\alpha}$, are either measurable for all α, or that that $f_{0,\alpha}$ is Lipschitz for all α.

A collection of i-forms as above is said to determine an i-form, ω, on X, if for all $C_1, C_2 \subset X$, we have

$$\omega_{\phi_1} \circ d\phi_1 \circ d\phi_2^{-1} = \omega_{\phi_2} \mid \phi_2(C_1 \cap C_2) \qquad \text{(a.e.)}\,. \tag{6.3}$$

Two such collections determine the same i-form if and only if the corresponding forms, say $\omega_\phi, \omega'_\phi$, agree a.e. on $\phi(C)$, for all (C, ϕ). Thus, an i-form is actually an equivalence class, the pointwise values of a representative of which are only well defined μ-a.e.; see ii) above. Such an i-form is called measurable (respectively Lipschitz) if the functions, $f_{0,\alpha}$ are all measurable (respectively Lipschitz).

If $f : X \to \mathbf{R}$ is Lipschitz, then it is clear that there exists a μ-a.e. well defined L_∞ 1-form, df, such that $df_\omega = d(f \circ \phi^{-1})$. Moreover, if the i-form, ω, is Lipschitz, there is a well defined L_∞ $(i+1)$-form $d\omega$, which for all (C, ϕ), has the representation,

$$d\omega_\phi = \sum_\alpha df_{0,\alpha} \wedge df_{1,\alpha} \wedge \cdots \wedge df_{i,\alpha}\,. \qquad \text{(a.e.)}\,. \tag{6.4}$$

To verify that this holds, it is enough to check that the form in (6.4) does not depend on the particular representation as a sum, of the form, ω_ϕ, on the right hand side of (6.2). Note however, that if we chose Lipschitz extentions, $\tilde{f}_{j,\alpha}$, and denote by $\tilde{\omega}$, the corresponding extension of ω, then by a standard regularization argument, it follows that at almost all points of $\phi(C)$, the distributional exterior derivative of $\tilde{\omega}$ is independent of the particular extension, and is given by the expression in (6.4).

Without loss of generality, we can assume that the sets, $C_{k,i}$, have been chosen such that $C_{k,i} \cap C_{k',i} = \emptyset$, unless $(k, i) = (k', i')$. Then for μ-a.e. $x \in X$, we can define the pointwise norm of a form, ω by $|\omega(x)| = |\omega_\phi(\phi(x))|$. Note that a different choice of sets, $C_{k,i}$, would lead to the same pointwise norm, for μ-a.e. $x \in X$.

In the usual fashion, we can define, $|\omega|_{L_p}$, the L_p norm of ω. Note the specific choice of norm we have made is not essential, but rather, the corresponding notion of L_p convergence, $\omega_j \xrightarrow{L_p} \omega$ (i.e., we could as well have used an equivalent norm). From i), ii), it is clear that if $\omega_j \xrightarrow{L_p} \omega$, then for all $\phi : C \to \mathbf{R}^k$ as above, we have $(\omega_j)_\phi \xrightarrow{L_p} \omega_\phi$. By completing the L_∞ forms with respect to this norm, we obtain a vector space, which we call the space of L_p forms. Clearly, the space of Lipschitz forms is dense in this space.

If ω is an L_p form for which there exists a sequence of Lipschitz forms, $\omega_j \xrightarrow{L_p} \omega$, such that $d\omega_j \xrightarrow{L_p} \theta$, for some L_p form θ, then we say that θ is a *strong L_p exterior derivative of* ω. If the form, θ (when it exists), is always uniquely determined (μ-a.e.) by the form, ω, then we say that *strong L_p exterior derivatives are unique.*

Fix $p \geq 1$. We now show that if the measure, μ, is doubling and if in addition, a Poincaré inequality of type $(1,p)$ holds, then for functions, strong L_p exterior derivatives are unique. Note in this connection that the usual argument, proving the uniqueness of strong (or more generally, weak) derivatives, for (smooth or Lipschitz) functions defined on some open subset of $\mathbf{R}^k$, depends on integration by parts against a smooth test function. Since in our situation, we must consider functions which are defined only on measurable subsets, $\phi_{k,i}(C_{k,i}) \subset \mathbf{R}^k$, this argument can not be applied; compare Examples 6.20, 6.21.

Remark 6.5. The argument that we will provide works as well for forms of arbitrary degree, given the appropriate a priori estimate (for forms) which corresponds to the Poincaré inequality (for functions). However, since we do not know of a general sufficient condition which guarantees the existence of such an estimate, we will just state Theorem 6.7 below for functions.

Remark 6.6. Theorem 6.7 below, is closely related to a result of Semmes on the uniqueness of strong derivatives with respect to measures in $\mathbf{R}^n$ which are absolutely continuous with respect to Lebesgue measure. Semmes' result appears as Theorem 5.1 of [26]. Heinonen has pointed out to us that properly understood, Semmes' argument can also be applied in more general contexts.

Theorem 6.7. *Let X be μ-rectifiable. If (0.5), (1.5) hold, for some $1 \leq p < \infty$, then strong L_p exterior derivatives of functions are unique.*

For the proof of Theorem 6.7, we will need Lemma 6.8 below. Note that in (6.11), (6.13) below, we regard df as the vector valued function, $(\partial_1 f, \ldots, \partial_k f)$. All integrals in Lemma 6.8 are with respect to Lebesgue measure on $\mathbf{R}^k$.

Lemma 6.8. *Let $A \subset B_1(0) \subset \mathbf{R}^k$ and let f be Lipschitz, $f : A \to \mathbf{R}$. If*

$$(1-\delta) \leq \frac{Vol(A)}{Vol(B_1(0))}\,, \tag{6.9}$$

$$Lip f \leq L\,, \tag{6.10}$$

$$\tag{6.11} \fint_A \left| df - \fint_A df \right| \leq \epsilon \,,$$

$$\tag{6.12} \fint_A |f| \leq \eta \,,$$

then

$$\tag{6.13} \left| \fint_A df \right| \leq \frac{C(k)}{(1-\delta)} \left(\delta L + \eta + \epsilon \right) \,.$$

Proof. To simplify the notation, we will just write B for $B_1(0)$.

As previously noted, we can extend f to a Lipschitz function, $\tilde{f}$, on $\mathbf{R}^k$, such that $\mathrm{Lip}\tilde{f} \leq L$.

Let $(x_1, \ldots, x_k)$ the position vector and define the homogeneous linear function, $\tilde{\ell}$, by

$$\tag{6.14} \tilde{\ell}(x_1, \ldots, x_k) = \langle \fint_B d\tilde{f}, (x_1, \ldots, x_k) \rangle \,.$$

If we put $\hat{f} = \tilde{f} - \tilde{\ell}$, then

$$\tag{6.15} \fint_B \hat{f} = \fint_B \tilde{f} \,,$$

$$\tag{6.16} d\hat{f} = d\tilde{f} - \fint_B d\tilde{f} \,,$$

and on B,

$$\tag{6.17} \tilde{f} \leq \eta + 2L \,.$$

(Although (6.17) can sharpened for δ small, it will suffice for our purposes.)

Let $\lambda(k)$ denote the constant in the type $(1,1)$ Poincaré inequality for $B = B_1(0)$. If we apply this inequality to the function, $\hat{f}$, then by using (6.16), we get

$$
\begin{aligned}
\frac{1}{\lambda(k)} \fint_B \left|\hat f - \fint_B \hat f\right| &\le \fint_B \left|d\hat f\right| \\
&= \fint_B \left|d\tilde f - \fint_B d\tilde f\right| \\
&\le \fint_A \left|df - \fint_B d\tilde f\right| + \delta \fint_{B\setminus A} \left|d\tilde f - \fint_B d\tilde f\right| \\
&\le \fint_A \left|df - \fint_A df\right| + \delta \fint_A |df| \\
&\quad + \delta \fint_{B\setminus A} \left|d\tilde f\right| + \delta \fint_{B\setminus A} \left|d\tilde f - \fint_B d\tilde f\right| \\
&\le 4\delta L + \epsilon\,.
\end{aligned}
\tag{6.18}
$$

Also, by using (6.15), (6.17), we have

$$
\begin{aligned}
\fint_B \left|\hat f - \fint_B \hat f\right| &= \fint_B \left|\tilde f - \tilde \ell - \fint_B \tilde f\right| \\
&\ge \fint_B \left|\tilde \ell\right| - 2\fint_B \left|\tilde f\right| \\
&\ge \frac{(1-\delta)}{c(k)} \left|\fint_A df\right| - \frac{1}{c(k)}\delta L - 2\eta - \delta(\eta + 2L)\,.
\end{aligned}
\tag{6.19}
$$

From (6.18), (6.19), we get (6.13), which concludes the proof. q.e.d.

Proof of Theorem 6.7. Assume that $\{\underline{f}_j\}$ is a sequence of functions such that $\underline{f}_j : X \to \mathbf{R}$ and $\underline{f}_j \xrightarrow{L_p} 0$, $d\underline{f}_j \xrightarrow{L_p} \underline{\theta} \neq 0$. We will deduce a contradiction.

By (1.18), (1.21), for all $L, j < \infty$, there exists, $Z_{L,j}$, such that the Lipschitz constant of $f_j \,|\, Z_{L,j}$ is bounded by L, and for fixed j, we have $\lim_{L\to\infty} \mu(X \setminus Z_{L,j}) = 0$. Thus, there exists x, $\{L_j\}$, $(C_{k,i}, \phi_{k,i})$ such that $x \in C_{k,i} \cap Z_{L_j,j}$, for all j, and $\underline{\theta}_{\phi_{k,i}}(\phi_{k,i}(x)) \neq 0$.

Without loss of generality, we can assume that $\phi_{k,i}(\underline{x})$ is a Lebesgue point of the set, $\phi_{k,i}(C_{k,i} \cap Z_{L_j,j})$, for all j, and of the form (i.e. vector valued function) $(\underline{\theta})_{\phi_{k,i}}$.

For $s > 0$, put $A = B_s(\phi_{k,i}(\underline{x})) \cap \phi_{k,i}(C_{k,i} \cap Z_{L,j})$. Rescale the ball, $B_s(\phi_{k,i}(\underline{x}))$ to unit size and replace f_j by $s^{-1} f_j$.

From the previous paragraph, it follows that if we let $s \to 0$, then $j \to \infty$ and apply Lemma 6.8, we contradict the assumption, $\underline{\theta}_{k,i}(\phi_{k,i}(x)) \neq 0$. q.e.d.

In Theorem 6.7, the global assumption that the Poincaré inequality holds, is actually neccessary. In order to illustrate this point, we now give some examples which show that the conditions, i), ii) are not sufficient to guarentee that strong derivatives are unique. The examples which follow are similar to the one presented in the context of weights on pp. 91–92 of [17].

Example 6.20. We construct a Cantor-like set, $J_\infty \subset \mathbf{R}$, of positive measure, such that if we take μ to be the measure, $\mathcal{H}^1$, then the densely defined operator, d, on Lipschitz functions is not closable.

Let $\epsilon_i > 0$ and $\epsilon_1 + \epsilon_2 \cdots < 1$. Start with the interval, $[0,1]$, and remove from the center, an open interval of length, ϵ_1, leaving two intervals of equal lengths. Remove from the center of each of these intervals, an interval of length $2^{-1}\epsilon_2$, leaving in each case, two intervals of equal lengths. Proceed in this way by induction, removing at the i-th stage, a total of 2^i intervals, each of length $2^{-i}\epsilon_i$. Set $J_0 = [0,1]$ and let J_i denote the closed set produced at the i-th stage of the above construction. Put $J_\infty = \cap_i J_i$.

It follows easily from Lusin's Theorem, that for all $f \in L_2(J_\infty)$, there exists a sequence, $\{f_i\}$, of functions on J_∞, such that $\{f_i\}$ converges to f in $L_2(J_\infty)$ and such that each f_i is the restriction to J_∞, of a function which is constant on each component of J_i. For each Lipschitz function, f_i, we have $df_i = 0$ and it follows that strong L_2 derivatives are not unique.

Example 6.21. By a simple modification of the construction of Example 6.20, we obtain a measure, μ, on $[0,1]$, which is *absolutely continuous* with respect to Lebesgue measure, but for which strong L_2 derivatives are not unique. Let I_i denote the open set consisting of the 2^i disjoint intervals which are removed at the i-th stage of the construction in Example 6.20; $I_i = [0,1] \setminus J_i$. Let $\delta_i > 0$, for all i, and $\delta_i \to 0$. Let μ denote the measure with density, ϕ, with respect to $\mathcal{H}^1$, where $\phi \,|\, I_i = \delta_i$ and $\phi \,|\, J_\infty = 1$. Let f_i denote the Lipschitz function determined by the following conditions:

(a) The restriction of f_i to each closed interval of the set, J_i, is the linear function which vanishes at the left hand end point and has derivative $\equiv 1$.

(b) The restriction of f_i to each interval of $[0,1] \setminus J_\infty$ is linear.

Note that $f_i \to 0$, uniformly, as $i \to \infty$. Moreover, $f_i' \,|\, J_i \equiv 1$, and $f_i' \,|\, I_j \to 0$, for fixed j. Finally, if $\delta_i \to 0$, sufficiently fast, then $\{f_i'\}$ converges in $L_2([0,1])$ to the characteristic function of J_∞. Thus, it follows that strong L_2 derivatives are not unique.

Remark 6.22. Examples 6.20 and 6.21 should also be compared to Theorem 2.1.4 of [20].

From now on, we assume that condition iii) holds. Let ω_1, ω_2 be L_∞ forms. Note that condition iii) enables us to define almost everywhere (in an obvious fashion) the pointwise inner product, $\langle \omega_1, \omega_2 \rangle$. Moreover, it is clear that μ-a.e., we have $\operatorname{Lip} f(x) = |df(x)|$.

We now introduce a self-adjoint Laplace operator on functions. Consider the Hilbert space of 1-forms associated to the global norm, $\int_X |\omega|^2 \, d\mu$. We can view the operator, d, on Lipschitz functions, as a densely defined unbounded operator. By Theorem 6.7 this operator is closable.

In our situation, the Dirichlet energy can be expressed in terms of the norm of the differential,

$$\int_X (\operatorname{Lip} f)^2 \, d\mu = \int_X |df|^2 \, d\mu \,. \tag{6.23}$$

As a consequence of ii), iii), the right hand side of (6.23) defines a quadratic form which is obtained from the bilinear form,

$$\int_X \langle df_1, df_2 \rangle \, d\mu \,. \tag{6.24}$$

In view of Theorem 6.7, we get:

Theorem 6.25. *Let (X, μ) be as in Theorem 6.7. The bilinear form in (6.24) is closable. Hence, there is a unique self-adjoint operator, Δ (associated to the minimal closure) such that*

$$\int_X |df|^2 \, d\mu = \langle \Delta^{\frac{1}{2}} f, \Delta^{\frac{1}{2}} f \rangle \,. \tag{6.26}$$

Proof. This is an immediate consequence of standard results on bilinear forms; see [20], Chapter 1. q.e.d.

The self-adjoint operator whose existence is guarenteed by Theorem 6.25 should be thought of as the Laplacian with respect to generalized absolute boundary conditions (equivalently Neumann boundary conditions).

From Theorem 1.8 together with the arguments concerning Hölder continuity given in [25] (see in particular, Theorem 6.6 there) we immediately obtain:

Theorem 6.27. *Let (X, μ) be as in Theorem 6.7. If the space, X, is compact, then $(I + \Delta)^{-1}$ is a compact operator. Moreover, every eigenfunction is Hölder continuous, for some exponent, $\alpha > 0$.*

7. Continuity of the Laplacian under limits

In this section, we prove the upper semicontinuity of the spectrum of the Laplacian under measured Gromov-Hausdorff convergence for rectifiable spaces, and the continuity of the eigenfunctions and eigenvalues for limit spaces satisfying (0.2).

As previously mentioned, for limit spaces satisfying (0.2), the existence of a canonical self-adjoint Laplace operator was conjectured by Fukaya, who also conjectured that the associated the eigenvalues and eigenfunctions should behave continuously under measured Gromov-Hausdorff convergence; see Conjecture 0.5 of [19]. Thus, the results of Section 6, together with those of the present section, yield a proof of Fukaya's conjectures; compare problem 88 of [35] for a related question in a different context.

Let $(W_1, \sigma_1), (W_2, \sigma_2)$ denote metric measure spaces satisfying conditions ii), iii), of Section 5, and let $f_1, \ldots, f_j$ denote the first j eigenfunctions of the Laplacian on W_1. By the minimax principle, roughly speaking, to prove upper semicontinuity of the spectrum, we must show that if $(W_1, \sigma_1), (W_2, \sigma_2)$ are sufficiently close in the measured Gromov-Hausdorff sense, then there exist functions, $\hat{f}_1, \ldots, \hat{f}_j$, on W_2, such that $\overline{f_k, \hat{f}_k}$ is small and $|d\hat{f}_k|^2_{L_2}$ does not appreciably exceed $|df_k|^2_{L_2}$, $1 \leq k \leq j$. Intuitively, given bounds on the geometry, the required closeness would depend on bounds on the oscillation of f_k, $|df_k|$.

For the case of limit spaces satisfying (0.2), the continuity under measured Gromov-Hausdorff convergence of the eigenfunctions and eigenvalues will follow from a quantitative version of the above transplantation argument, with errors controlled in terms of a priori bounds on the oscillation of eigenfunctions and their differentials. The consequences of (0.1) on which the required a priori bounds depend are the doubling condition, the Poincaré inequality, the segment inequality, the gradient estimate and Bochner's formula. In addition, we use results from [3] on the approximation and transplantation of functions, f, with bounds on $|\mathrm{Lip}\, f|_{L_\infty}$, $|\mathrm{Lip}(\mathrm{Lip}\, f)|_{L_q}$, for some q.

We begin by considering the upper semicontinuity of the spectrum.

Let $(Z_i, \mu_i) \overset{d_{GH}}{\longrightarrow} (Z, \mu)$, where Z_i, Z are length spaces, $diam(Z) < \infty$ and μ_i, μ are Radon measures. Let (Z_i, μ_i) (for all i) and (Z, μ) satisfy the following conditions: the doubling condition and type (1,2) Poincaré inequality, with constants, τ, κ independent of i, and conditions ii), iii), of Section 5.

Let $\{\lambda_1, \ldots, \}$, $\{\lambda_{1,i} \ldots, \}$ denote the eigenvalues for Δ, Δ_i.

Theorem 7.1. *Let (Z_i, μ_i) (for all i) and (Z, μ) satisfy the above conditions, with constants, τ, κ in the doubling condition and type (1, 2) Poincaré inequality, independent of i. Then for all j, we have*

$$\limsup_i \lambda_{j,i} \leq \lambda_j \,. \tag{7.2}$$

Proof. This is a direct consequence of Lemma 10.7 of [3], together with the minimax principle.

In more detail, note first of all, that the inequality, (10.11), of Lemma 10.7 of [3], which involves th quantity $\mathrm{Lip}\, f$, can be rewritten in terms of $|df|$ in our case; see the discussion prior to (6.23) of Section 5 above.

Let $f_1, f_2, \ldots$ be an orthonormal basis of eigenvectors for Δ. Since the space of all Lipschitz functions is dense in the domain of Δ, it follows that for all $\epsilon > 0, \ell$, there exists a Lipschitz function, $f_{\ell,\epsilon}$, such that $|f_\ell - f_{\ell,\epsilon}|_{L_2} + |df_\ell - df_{\ell,\epsilon}|_{L_2} \leq \epsilon$.

Fix $1 \leq j < \infty$. Let $V_{j,\epsilon}$ denote the subspace spanned by $f_{1,\epsilon}, \ldots, f_{j,\epsilon}$, and put $S_{j,\epsilon} = \{f \in V_{j,\epsilon} \,|\, |f|_{L_2} = 1\}$. Let $\{f_{j,k,\epsilon}\}$, $1 \leq k \leq N(j,\epsilon)$, denote an ϵ-dense subset of $S_{j,\epsilon}$, with respect to the norm $|f|_{L_\infty} + |df|_{L_\infty}$. We can (and will) assume $f_{j,\ell,\epsilon} = f_{\ell,\epsilon}$, for all $1 \leq \ell \leq j$.

Let $\widehat{f_{j,k,\epsilon}}$ be the Lipschitz function on Z_i associated to $f_{j,k,\epsilon}$, constructed in Lemma 10.7 of [3]. From now on, we put $\widehat{f_{j,k,\epsilon}} = \hat{f}_{j,k,\epsilon,i}$. By (10.10), (10.11) of [3], we have $||\hat{f}_{j,k,\epsilon,i}|_{L_2} - 1| \leq \epsilon$, and

$$|d\hat{f}_{j,k,\epsilon,i}|_{L_2} \leq |df_{j,k,\epsilon}|_{L_2} + \epsilon \leq \lambda_j + 2\epsilon,$$

provided i is sufficiently large. Moreover, by (10.9) of [3], if

$$f_{j,k,\epsilon} = a_{1,j,k,\epsilon} f_{1,\epsilon} + \cdots + a_{j,j,k,\epsilon} f_{j,\epsilon},$$

then

$$\lim_{1 \to \infty} |\hat{f}_{j,k,\epsilon,i} - (a_{1,j,k,\epsilon}\hat{f}_{j,1,\epsilon,i} + \cdots + a_{j,j,k,\epsilon}\hat{f}_{j,j,\epsilon,i})|_{L_\infty} = 0.$$

Let $W_{j,i}$ denote the direct sum of the first j eigenspaces of the Laplacian, Δ_i, on Z_i. Let h be a unit vector in the (j-dimensional) subspace spanned by $\hat{f}_{j,1,\epsilon,i}, \cdots, \hat{f}_{j,j,\epsilon,i}$, which is orthogonal to the (($j-1$)-dimensional) subspace $W_{j-1,i}$. Denote by $\hat{f}_{j,\underline{k},\epsilon,i}$, the function, $\hat{f}_{j,k,\epsilon,i}$, which best approximates h in the L_2 norm. It follows from the previous paragraph that for i sufficiently large, $| < \hat{f}_{j,\underline{k},\epsilon,i}, w > | \leq \epsilon$, for every unit vector, $w \in W_{j-1,i}$, and in addition, $||\hat{f}_{j,\underline{k},\epsilon,i}| - 1|_{L_2} \leq \epsilon$, $|d\hat{f}_{j,\underline{k},\epsilon,i}|_{L_2} \leq \lambda_j + \epsilon$. Since ϵ is arbitrary, by considering the projection of $\hat{f}_{j,\underline{k},\epsilon,i}$, onto the orthogonal complement of $W_{j-1,i}$ and applying the minimax principle, the proof is completed. q.e.d.

As indicated above, in order to obtain the continuity of the eigenfunctions and eigenvalues under measured Gromov-Hausdorff convergence, the estimates which imply Theorem 7.1 (and in particular, those which imply the conclusions of Lemma 10.7 of [3]) must hold uniformly in i.

Denote by $\{\underline{\lambda}_{j,i}\}$, the collection $\{\lambda_{j,i}\}$, but possibly arranged in a different order. The slightly awkward statements below involving the underlined eigenvalues arise from the possibility that there might exist multiple eigenvalues, or more generally, that there might exist very small spectral gaps.

Theorem 7.3. *Let $(Z_i, \mu_i) \xrightarrow{d_{GH}} (Z, \mu)$, and put $d_{GH}((Z_i, \mu_i), (Z, \mu)) = \rho_i$. In addition to the assumptions of Theorem 7.1, assume that the segment inequality, (2.3), holds for all (Z_i, μ_i), with constant, τ, independent of i. Also assume that there exist constants, L_j, A_j, such that if $f_{j,i}$ denotes the j-th eigenfunction on Z_i, then for all $f_{j,i}$, we have*

$$|Lip\, f_{\ell,i}|_{L_\infty} \leq L_j \qquad (\ell \leq j)\,, \tag{7.4}$$

and for some q,

$$|Lip(Lip\, f_{\ell,i})|_{L_q} \leq A_j \qquad (\ell \leq j)\,. \tag{7.5}$$

Then

$$|\lambda_j - \underline{\lambda}_{j,i}| < \Psi(\kappa, \tau, L_j, A_j)\,. \tag{7.6}$$

Moreover, given an orthonormal basis of eigenfunctions, $\{f_j\}$ there exist orthonormal bases of eigenfunctions, $\{f_{j,i}\}$, such that with respect to the (uniform) Gromov-Hausdorff distance,

$$\overline{f_j, f_{j,i}} < \Psi(\rho_i \,|\, \kappa, \tau, L_j, A_j)\,, \tag{7.7}$$

and for which the corresponding eigenvalues, $\{\underline{\lambda}_{i,j}\}$, satisfy

$$|\lambda_j - \underline{\lambda}_{j,i}| < \Psi(\rho_i \,|\, \kappa, \tau, L_j, A_j)\,. \tag{7.8}$$

Proof. If we grant for the moment that the quantities occuring in the hypothesis of Lemma 10.7 of [3] are uniformly bounded, then for $j = 1$, our assertion is an easy consequence of that lemma. Moreover, as in the proof of Theorem 7.1 above, the general case follows by a straightforward argument based on the minimax principle.

The conclusions of Lemma 10.7 are stated in terms of a function, $\Psi(\epsilon|R, \kappa, L, r_f)$, where in our situation, R is the diameter, L is bounded by (7.4) (and the notation, $\Psi(\,\cdot\,|\,\cdot\,)$, is as in the present paper; see (3.2)). Thus, it is enough to bound (from below) the function r_f. (See Lemma 6.24 of [3] for the definition of r_f and Lemma 6.30 of [3], for the role played by r_f in Lemma 10.7.)

According to Lemma 16.39, of [3], the function, r_f, can be replaced by a certain function, R_f, provided (as we have assumed) the segement inequality holds. (The proof of Lemma 16.39 depends on Lemma 15.6 of [3], the hypothesis of which includes the segment inequality, or more generally what is refered to there as an (ϵ, δ)-inequality.) Thus, to complete the proof, it suffices to estimate from below, the function, R_f. The required estimate is provided by Proposition 16.43 of [3]. Relative to Lemma 16.39, this proposition requires an additional assumption, (16.44), which corresponds to (7.5) above. This suffices to complete the proof. q.e.d.

In our next result, we equip the Riemannian manifold, M_i, with the renormalized volume element, $\underline{\mathrm{Vol}}_i$, as in (0.3).

Theorem 7.9. *Let diam*$(Y) < \infty$ *and let* $(M_i, m_i, \underline{\mathrm{Vol}}_i) \xrightarrow{d_{GH}} (Y, y, \nu)$ *satisfy (0.2). Then (7.6)–(7.8) hold.*

Proof. . It suffices to verify the hypotheses of Theorem 7.3. By Theorem 2.11 of [5], the segment inequality holds with constant τ, independent of i. According to the Cheng-Yau gradient estimate, [8], the bound, (7.4), is valid for Riemannian manifolds, satisfying (0.2). If f is an eigenfunction of the Laplacian on such a manifold, (7.5) (for $q = 2$) is an immediate consequence of Bochner's formula. This suffices to complete the proof. q.e.d.

Remark 7.10. It is easy to check that the above mentioned bounds on eigenfunctions for Riemannian manifolds satisfying (0.1), pass to the eigenfunctions on the limit spaces, (Y, ν), for which (0.2) holds; compare Theorem 6.27. In particular, the eigenfunctions on the limit space are Lipschitz.

We also have the following obvious variant of Theorem 7.9, neither the statement nor the proof of which depends on the existence of an intrinsically defined self-adjoint Laplacian on limit spaces.

Theorem 7.11. *Let M_1^n, M_2^n be Riemannian manifolds satisfying (0.1), $diam(M_1^n) \leq d < \infty$. Then for all $N < \infty$, $\epsilon > 0$, there exists $\delta(n, d, \epsilon, N) > 0$, such that if the measured Gromov-Hausdorff distance satisfies $d_{GH}(M_1^n, M_2^n) < \delta$, then for $j \leq N$, we have $|\lambda_{j,1} - \lambda_{j,2}| < \epsilon$. Moreover, given an orthonormal basis, $\{f_{j,1}\}$ for the eigenspaces corresponding to the $\{\lambda_{j,1}\}$, there exists an orthornormal basis of eigenfunctions, $\{f_{j,2}\}$, $j = 1, \ldots, N$, for which the corresponding eigenvalues, $\underline{\lambda}_{2,j}$ satisfy $|\lambda_{j,1} - \underline{\lambda}_{j,2}| < \epsilon$.*

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