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ESSENTIAL LAMINATIONS AND KNESER NORMAL FORM

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0. Introduction

One of the fundamental results in the theory of 3-manifolds is the Haken lemma [19]: "If T is an incompressible surface in the closed irreducible triangulated 3-manifold M, then T is isotopic to a normal surface." This result is crucial for establishing the existence of hierarchies in Haken manifolds [19]. The hierarchy in turn is the starting point for many spectacular results in 3-manifold topology e.g [19], [33], [31].

In 1990 Mark Brittenham [3] observed the following analogue of the Haken lemma: "If λ is an essential lamination in the closed orientable 3-manifold M with triangulation τ, then M has an essential lamination $\mathcal{L}$ normal with respect to τ."

An incompressible surface can be normalized via a finite number of elementary operations; however, these same operations applied to an essential lamination λ may never yield a normal lamination. Nevertheless, Brittenham mysteriously obtains a normal essential lamination $\mathcal{L}$ from an infinite sequence of normalizing isotopies applied to λ. The main technical result of this paper precisely explains the passage from λ to $\mathcal{L}$.

Theorem 4.4. *Let λ be a nowhere dense essential lamination in the closed orientable 3-manifold M with triangulation τ. Then at least one of the following occurs.*

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1) After possibly splitting λ open along a finite number of leaves, λ is isotopic to a normal lamination.

2) There exists a normal essential lamination $\mathcal{L}$ in M such that $\mathcal{GN}(\mathcal{L}) > \mathcal{GN}(\lambda)$ and $\mathcal{L}$ is obtained from λ by first splitting along finitely many leaves, then evacuating a taut sutured manifold (N, γ) and finally isotopy.

3) λ has a generalized cylindrical component (see 4.1). In particular λ has a torus leaf and M is toroidal.

The gut number $\mathcal{GN}(\lambda)$ is a very rough measure of how far a lamination is from being a split open foliation. See Definition 0.1.

Theorem 4.4 together with the Kneser principle yields

Theorem 5.2. *To each closed 3-manifold M there exists a minimal nonnegative integer $\mathcal{GN}(M)$, called the gut number of M, such that if λ is an essential lamination in M, then $\mathcal{GN}(\lambda) \leq \mathcal{GN}(M)$.*

In [17] we use Theorem 5.2 to establish the finiteness of the mapping class group of atoroidal 3-manifolds with genuine laminations, thereby generalizing the similar result for atoroidal Haken 3-manifolds established by Johannson [22].

Corollary 5.3. *If M has an essential lamination, then it has an essential lamination of maximal gut number.*

Corollary 5.4. *If λ is a maximal gut number essential lamination in an atoroidal manifold with triangulation τ, then after possibly splitting along finitely many leaves, λ is isotopic to a normal lamination.*

Corollary 5.5. *If M is laminar, then it has an essential lamination λ such that for any triangulation τ on M, λ is isotopic to a normal lamination.*

Theorem 6.5. *Let M be a closed orientable atoroidal 3-manifold. The collection of nowhere dense essential laminations on M is carried, up to isotopy, by finitely many essential branched surfaces.*

See Theorem 6.13 for a similar statement about Reebless foliations.

Two foliations $\mathcal{F}$ and $\mathcal{G}$ in a Riemannian 3-manifold are ϵ-coarse (resp. coarse) isotopic if up to isotopy, of each foliation, their oriented tangent planes differ pointwise by angle less than ϵ (resp. π).

Theorem 6.15. *Given a closed orientable atoroidal Riemannian 3-manifold, there exists an integer $N(M) > 0$ such that for any $\epsilon > 0$ any taut foliation on M is ϵ-coarse isotopic to one of $N(M)$ taut foliations.*

Note that $N(M)$ is independent of both ϵ and the Riemannian metric.

Corollary 6.16. *If $\epsilon > 0$ and $\mathcal{F}_1, \cdots, \mathcal{F}_{N(M)+1}$ are taut foliations on the closed oriented Riemannian atoroidal 3-manifold M, then there exists $i \neq j$ such that $\mathcal{F}_j$ and $\mathcal{F}_i$ are ϵ-coarse isotopic.*

This result had been previously obtained by Cantwell - Conlon [7] for depth-1 foliations.

Corollary 6.18(Kronheimer - Mrowka [24]). *On a closed orientable 3-manifold, there are only finitely many homotopy classes of plane fields of taut foliations.*

In contrast to Corollary 5.5 we have

Corollary 6.21. *Let M be a closed orientable atoroidal 3-manifold. There exists a triangulation τ on M such that any taut foliation or Reebless foliation or nowhere dense essential lamination can be isotoped to be normal to τ.*

Corollary 6.21 can be viewed as an analogue for laminations of the result of Schoen - Yau [29], Schoen [28] that in a Riemannian 3-manifold, any π_1-injective closed surface is isotopic to one with uniformly bounded normal curvature.

Corollary 6.21 is a positive answer to a question asked by Thurston in the late 1970's.

Corollary 6.22. *If M is a closed orientable atoroidal 3-manifold, then M is covered by a finite set of charts such that any taut foliation or essential lamination can be isotoped so that each of these charts is a foliation chart.*

Conjecturally the bound on the number of foliation charts can be obtained from the topological complexity of M.

This paper is organized as follows. In §1 we provide several examples of infinite passages from λ to $\mathcal{L}$. In particular we show that for any triangulation on the 3-torus there exists an essential lamination which cannot be put into normal form with respect to that triangulation. (The reader who masters §1 can easily read this paper.) In §2 we define an infinite isotopy which attempts to make λ normal. It has the feature that modulo certain compression operations this isotopy is supported in a tiny neighborhood of the 2-skeleton. Let λ_t denote the isotoped λ at time t. In §3 we completely understand $\lambda_t|\eta$ where η is an n-simplex, where $1 \leq n \leq 3$. This enables us to obtain a limit branched lamination λ_∞. We apply the arguments of [3] to obtain a normal essential lamination $\mathcal{L}$ from λ_∞. In §4 we prove our main technical result. In particular we observe that the lamination $\mathcal{L}$ is carried by a branched surface H which is naturally created at some finite moment t_3 of the isotopy process. Roughly speaking, the isotopy after time t_3 fixes H pointwise and the horizontal boundary $R(\gamma)$ of the evacuating sutured manifold is the union of sectors of H which lie on the boundary of regions where the isotopy does not stabilize in finite time. See Examples 1.3 - 1.5 for examples of this phenomena. In §5 - 6 we establish the application cited above.

The main results of this paper concern essential laminations in triangulated 3-manifolds. However all these results generalize to laminations in 3-manifolds with pseudotriangulations, handlebody or regular cell structures.

Historical Remarks. Kneser [23] introduced the idea of normal surface in 1929 in order to establish the prime decomposition of compact triangulated 3-manifolds. He showed how to transform an essential 2-sphere into a finite set of normal essential 2-spheres. It took another 32 years for someone (Haken) to recognize the enormous importance of higher genus normal surfaces.

Acknowledgments. I would like to thank Will Kazez for his constructive comments.

Definition 0.1. Read [18] for the basic facts and definitions about essential laminations and branched surfaces. Define the *closed complement* of a lamination λ in the 3-manifold M to be the metric completion of $M - \lambda$ with respect to the path metric on $M - \lambda$. In a similar manner define the closed complement C of a branched surface $B \subset M$. Such a C is a 3-manifold with *corners*, the corners denoted $s(\partial C)$ arising from the branched locus of B. A *closed complementary region* of a lamination or branched surface is a component of the closed complement.

The closed complement of an essential lamination can be uniquely decomposed (up to isotopy) into a union of $\mathcal{I}(\lambda)$ and $\mathcal{G}(\lambda)$. The *interstitial bundle* $\mathcal{I}(\lambda)$ is π_1-injective and is a maximal union of maximal connected noncompact I-bundle's or I-bundles over closed surfaces or maximal I-bundles over connected surfaces of negative Euler characteristic. The *gut* $\mathcal{G}(\lambda)$ is a compact manifold such that $\mathcal{I}(\lambda) \cap \mathcal{G}(\lambda)$ is a union of properly embedded essential annuli. See [16] for more details.

The *gut number* $\mathcal{GN}(\lambda)$ is the number of components of the gut of λ.

Definition change 0.2. Our definition of gut is different from that of [16] for it allows for components of the interstitial bundle which are I-bundles over surfaces of negative Euler characteristic. The same argument as [16] shows that the gut is unique up to isotopy. (Uniqueness is lost if we allow I-bundles over the annulus or Mobius band.)

Definition 0.3. See 4.1 for the definition of sutured manifold. $\mathring{E}$ denotes the interior of E and $\mid E \mid$ denotes the number of components of E. If τ is a cell complex, then τ^n denotes the n-skeleton. Let σ be a 3-simplex and α and β simple closed curves in $\partial\sigma$ disjoint from σ^0 and transverse to and not disjoint from σ^1. If $\alpha \cap \beta = \emptyset$, then we say α and β are *strongly normally isotopic* if each component of $\sigma^1 \cap A$ is an essential arc, where $A \subset \partial\sigma$ is the annulus cobounded by α and β. If $\alpha \cap \beta \neq \emptyset$, and they can be made strongly normally isotopic after arbitrarily small isotopies (e.g., because they are tangent at a point or coincide along arcs), then we also say that α and β are strongly normally isotopic.

1. Examples

Example 1.1 (A nonnormalizable 2-dimensional lamination). Figure 1.1 shows an annulus 2-complex K together with a Reeb lamination ρ embedded in its interior. Here $\rho = \rho_0$ has 2 compact leaves and 1 noncompact leaf. With respect to the given triangulation on K, the leaves of ρ_0 are in normal form except for one subarc. An isotopy of ρ_0 to ρ_1 eliminates that subarc at the expense of creating a new one. After 3 more such isotopies we obtain ρ_4, which is normally isotopic to ρ_0, and thus have apparently accomplished nothing.

Example 1.2 (A nonnormalizable essential lamination). Let ψ denote the lamination $\rho \times S^1$ on $(S^1 \times I) \times S^1$. Call ψ a *cylindrical lamination.* Let τ be a triangulation on the 3-torus T^3 and let $K(\tau)$ denote its Kneser number. (E.g. see [20]) I.e., if $n > K(\tau)$ and $T_1, \cdots, T_n$ are pairwise disjoint incompressible normal tori in T^3, then some pair of these tori are normally parallel. Partition T^3 into $n > K(\tau), S^1 \times S^1 \times I$ regions which meet only along their boundaries. Laminate each $S^1 \times S^1 \times I$ regions by cylindrical laminations. The resulting lamination ϕ on T^3 is essential, however it cannot be isotoped to be normal to τ. Otherwise a pair of adjacent torus leaves T_1, T_2 of ϕ would be normally isotopic, via an isotopy disjoint from the other torus

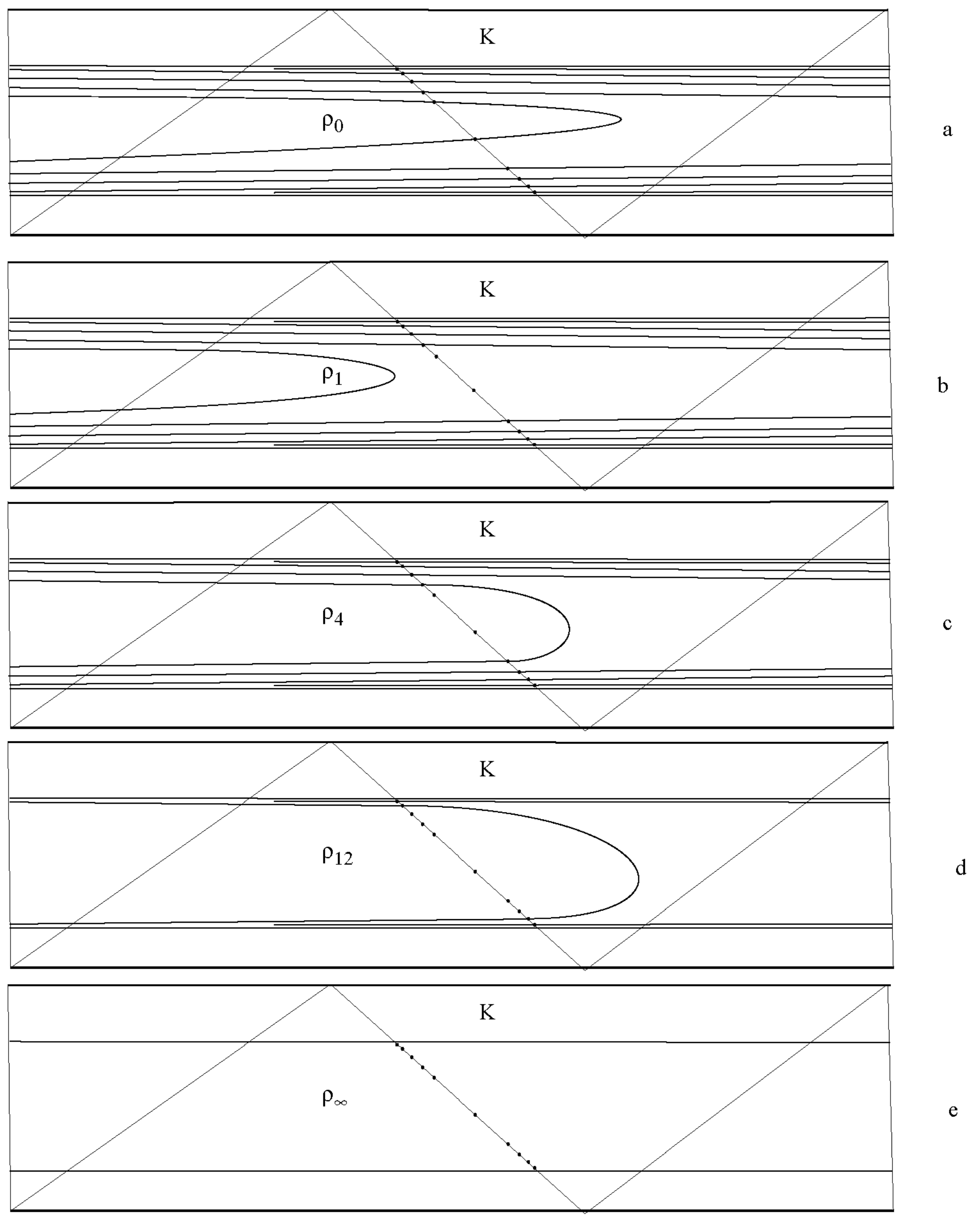


FIGURE 1.1

leaves. If V is the closure of the region bounded by T_1, T_2 and disjoint from the other tori, then one finds nonnormal arcs within $\phi|(\tau^2 \cap V)$ as in Example 1.1.

Example 1.3. To transform the lamination ρ of Example 1.1 to a normal lamination, one invokes the *Brittenham principle* (see Remark 2.5) as follows. Let ρ_t denote the isotoped ρ at time t and $E_t \overset{def}{=} \rho_t \cap K^1$. If $s > t$, then $E_s \subset E_t$. Now define $E_\infty = \cap E_t$ and define ρ_∞ to be the lamination of K which naturally extends E_∞. In this case ρ_∞ consists exactly of the 2-compact leaves of ρ. ρ_∞ is the result of attempting to put ρ into normal form in an infintely fast manner. See Figure 1.1.

Example 1.4 (The key example). Let $\lambda = \lambda_0$ be an essential lamination in M with triangulation τ such that for some $(S^1 \times I) \times [-1,1] \subset M$, $\lambda|(S^1 \times I) \times [-1,1] = \rho \times [-1,1]$ and some subcomplex K of τ^2 meets $(S^1 \times I) \times 0$ as in Example 1.1. If one applies the standard normalizing operations to λ near K, then the lamination $\lambda|(S^1 \times I) \times [-1,1]$ would get isotoped to the laminations shown in Figure 1.2 a),b) at times 1 and 13. In the limit one obtains the lamination $\mathcal{L}'$ of Figure 1.2c. The passage from λ to the limit lamination $\mathcal{L}'$ is obtained by *evacuating* (a term to be defined in §4) the taut sutured manifold (N, γ) shown in Figure 1.3. Indeed $\lambda_8|N$ provides a sufficient hint for describing a taut foliation on (N, γ). A crucial observation is that $\mathcal{G}(\lambda) \cap N = \emptyset$ and so the passage from λ to $\mathcal{L}'$ creates new non I-bundle complementary region. Finally each finite isotopy is supported within N.

In some sense $R(\gamma)$ arises from "blasting open" a (not necessarily connected) leaf of λ.

Example 1.5 (The 2-complex view). The creation of the sutured manifold (N, γ) can already be detected at the 2-skeleton level. If K extends to a subcomplex $J \subset \tau^2$, such that $\lambda_0|J$ appears as in Figure 1.4 a), then $\lambda_8|J$ appears as in Figure 1.4 b) and the limit lamination λ_∞ is the branched lamination appearing in Figure 1.4 c). Finally $(N, \gamma) \cap J$ appears as in Figure 1.4 d), the various arrows indiciting the normal orientation on $R(\gamma)$. Notice that the "arrow in" region i.e., $R_-(\gamma)$ is that part of ∂N where leaves are being "scraped off", while the "arrow out" region i.e., $R_+(\gamma)$ is that part of ∂N where leaves are being "sucked in". In some sense the leaves in N are flowing from $R_-(\gamma)$ to $R_+(\gamma)$.

Exercise 1.6 (A more interesting example). Figure 1.5 shows a lamination restricted to a 2-complex. Show that such a laminated 2-

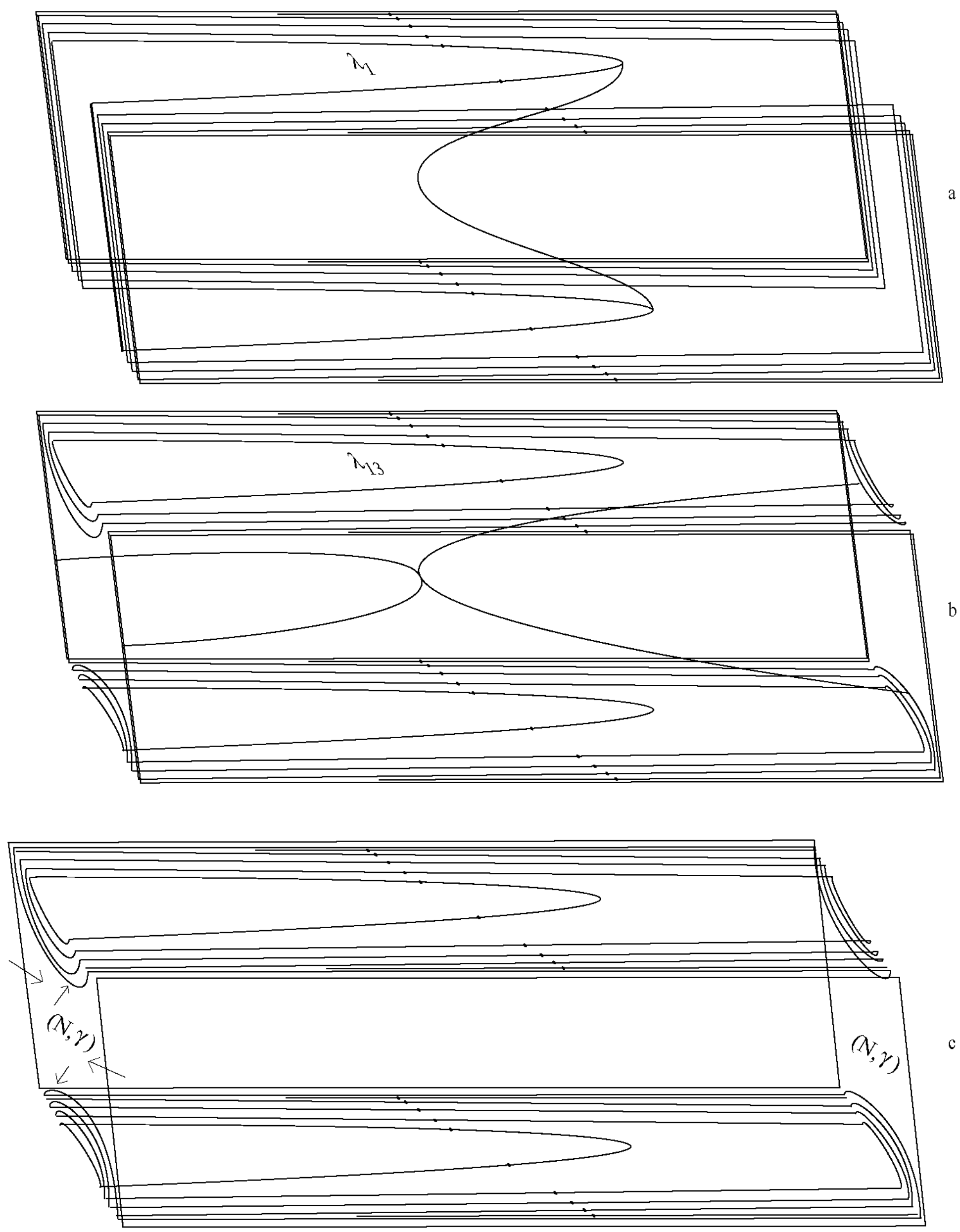


FIGURE 1.2.

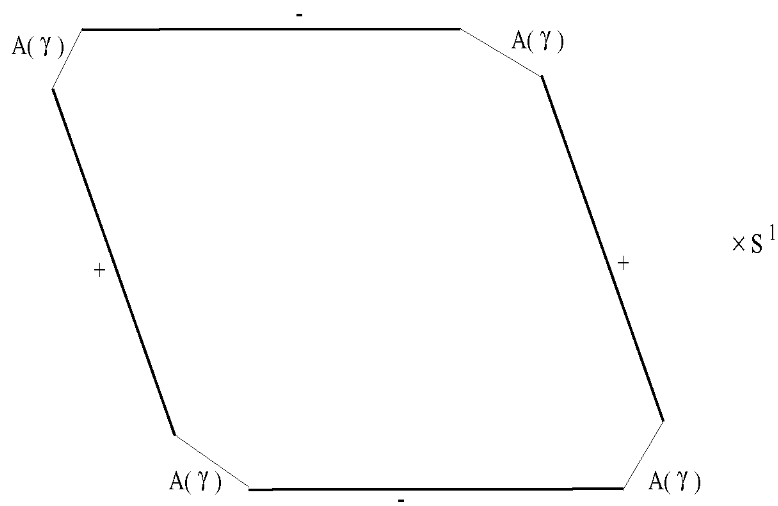


FIGURE 1.3.

complex might embed in a triangulated manifold with essential lamination. Analyze the resulting sutured manifold evacuation and construct the resulting limit lamination.

Example 1.7 (Creating Reeb laminations). It is possible that ρ and K might embed in a manifold with essential lamination as in Figure 1.6. In that case the resulting limit lamination $\mathcal{L}'$ (constructed as in Example 1.4) will contain a Reeb lamination. The lamination $\mathcal{L}$ obtained by deleting the Reeb lamination is just λ with a leaf split open.

Remark 1.8. There are more interesting ways of obtaining Reeb laminations in the limit lamination.

2. The infinite isotopy

Theorem 2.1(Brittenham [3]). *Let λ be an essential lamination in the closed orientable 3-manifold M with triangulation τ. Then M contains a normal essential lamination $\mathcal{L}$.*

Defintion 2.2. A *local leaf* in the lamination λ is a leaf of $\lambda|\sigma$, where σ is a 3-simplex.

The following standard result follows from the Reeb stability theorem and the fact that no leaf of $\lambda|\sigma$ has holonomy, since λ is essential

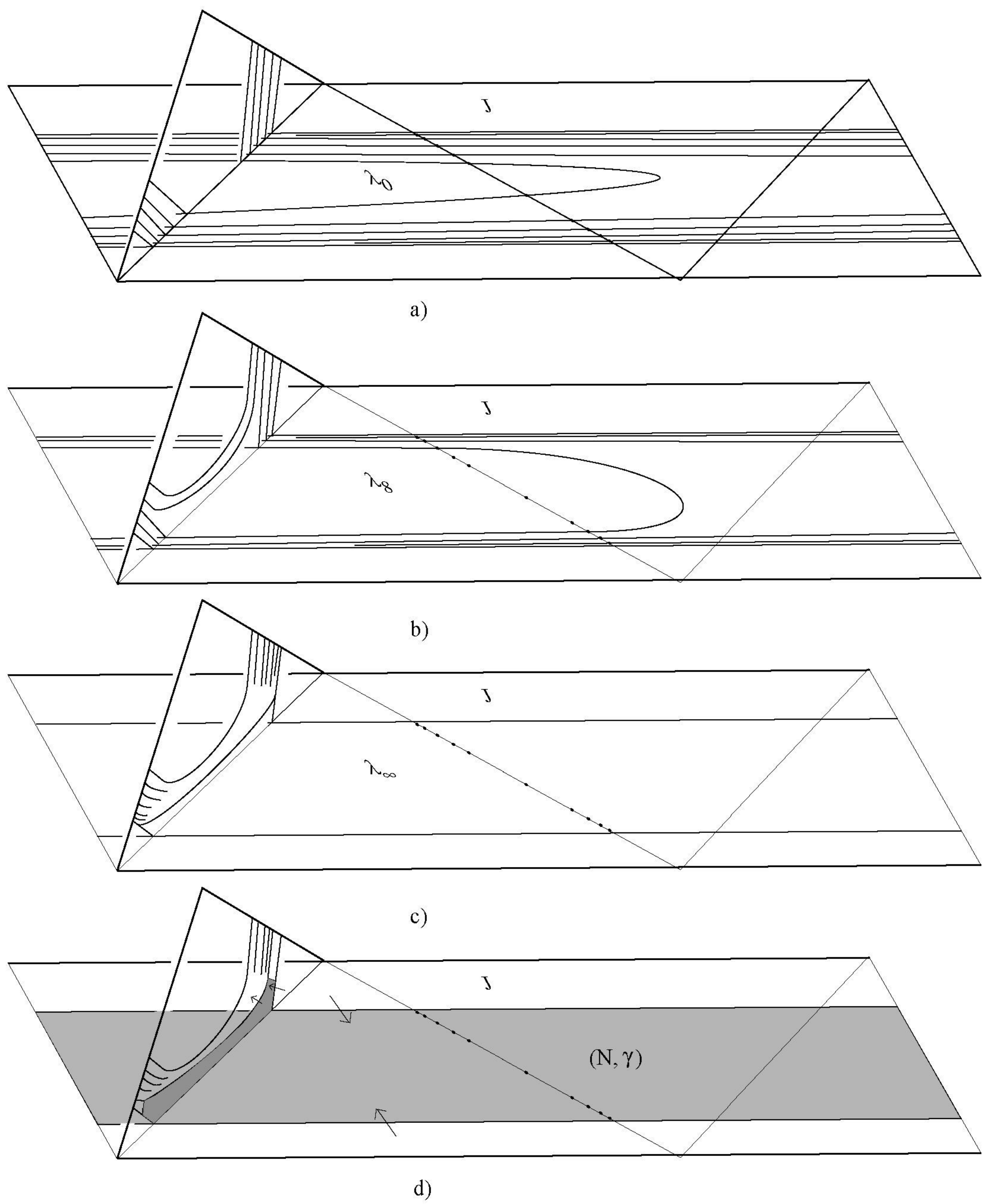


Figure 1.4.

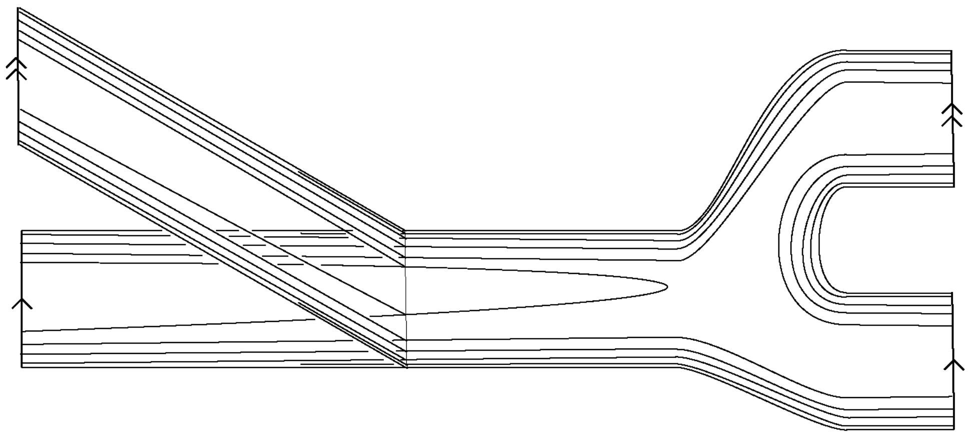


FIGURE 1.5.

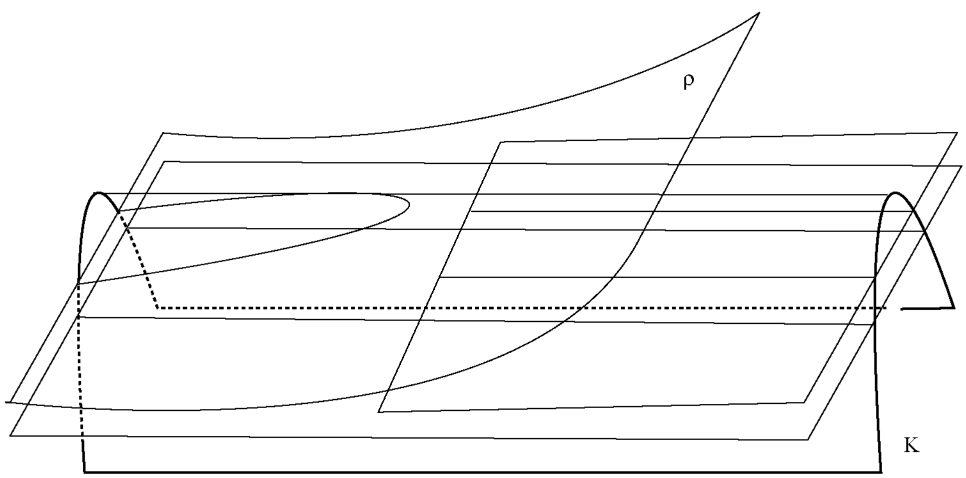


FIGURE 1.6.

and a 3-simplex is simply connected.

Packet Lemma 2.3. *Let τ be a triangulation of the closed orientable 3-manifold M. If λ_t is an essential lamination transverse to $\tau^n, n \leq 2$, then for each 3-simplex (resp. 2-simplex) η, $\lambda_t|\eta$ canonically decomposes into a finite set of sublaminations of the form $T_i \times K_i \subset T_i \times [0,1] \subset \eta$, such that for each $s \in [0,1]$, $T_i \times s$ is a properly embedded compact surface (resp. interval) transverse to τ^2 (resp. τ^1), K_i is a closed subset of $[0,1]$, and if $i \neq j$, then $(T_i \times [0,1]) \cap (T_j \times [0,1]) = \emptyset$.*
q.e.d.

Condition (2.1). The essential lamination λ is nowhere dense and has no isolated leaves.

Remark 2.4. All laminations in this chapter will satisfy the above Condition (2.1). This is not a serious constraint, for any lamination can be transformed into one satisfying (2.1) by replacing each isolated leaf by an I-bundles worth of leaves and then sufficiently splitting the resulting lamination.

We review the procedure of [3] for transforming λ into $\mathcal{L}$. Given a 3-simplex σ one isotopes λ to a lamination, also called λ such that $\lambda|\sigma$ is a lamination by normal discs. Now normalize λ with respect to one 3-simplex after the next, ignoring the fact that λ may now be nonnormal on previously cleaned up 3-simplices. Do this for each 3-simplex of τ, and then repeatedly cycle through the 3-simplicies cleaning them up one at a time. These elementary normalizing operations have the property that if λ_t denotes the lamination λ at time t and $E_t = \lambda_t \cap \tau^1$, then $E_t \subset E_s$ for $t > s$. Thus if $E_\infty = \cap E_t$, then E_∞ is a nonempty compact set. Brittenham shows that E_∞ extends to a branched lamination λ_∞ which is normal with respect to τ. The branched leaves naturally split open to create a normal lamination $\mathcal{L}'$, and after passing to the sublamination obtained by deleting all the Reeb laminations, one obtains the desired normal essential lamination $\mathcal{L}$.

Remark 2.5. Let λ_0 be an essential lamination and Δ be a 2-complex in the 3-manifold M. The following 3-step process will be called the *Brittenham principle.* See [2]–[6] for various applications.

i) Deform λ_0 to $\lambda_t, t \geq 0$ so that $\lambda_t \cap \Delta^1 \subset \lambda_s \cap \Delta^1$ for $t > s$.

ii) Extend $\cap(\lambda_t \cap \Delta^1)$ to a, possibly branched, lamination λ_∞.

iii) Derive an essential lamination from this λ_∞.

The remainder of §2 is devoted to refining the isotopy process of [3].

Definition 2.6. The branched surface C *compatibly* carries the essential lamination λ, if C carries λ in a manner compatible with $\mathcal{I}(\lambda)$. I.e., if $\mathcal{V}$ is the I-fibring of $N(C)$, then up to isotopy of $\mathcal{I}(\lambda)$, for each closed complementary region V of λ, $\mathcal{V} \mid V$ is a sub I-bundle of $\mathcal{I}(V)$.

For example if C has a disc of contact and compatibly carries λ, then the I-fibres of the corresponding complementary $D^2 \times I$ region of $N(C) - \lambda$ is a sub-I-bundle of $\mathcal{I}(\lambda)$.

It is routine to show that if λ satisfies (2.1), then λ is isotopic to a lamination λ_0 which satisfies the following Condition (2.2) with $t = 0$.

Condition (2.2). λ_t is fully and compatibly carried by a branched surface B_t with fibred neighborhood $N(B_t)$ such that $\partial_h N(B_t) \subset \lambda_t$. Also assume that $\tau^0 \cap N(B_t) = \emptyset$, and both τ^1 and τ^2 intersect $N(B_t)$ in a union of I-fibres and τ^2 is transverse to λ_t.

Remark 2.7. If B_t is a branched surface which satisfies (2.2) and B_{t+1} and λ_{t+1} are obtained by any of the following operations, then B_{t+1} satisfies the first sentence of Condition (2.2).

i) B_{t+1} is obtained by λ_t-splitting B_t, see [18].

ii) B_{t+1} is obtained by squeezing B_t along product discs, i.e., a squeezing corresponding to a properly embedded $I \times I \subset M - \mathring{N}(B_t)$ with $I \times \partial I$ vertical arcs in $\partial_v N(B_t)$ and $\partial I \times I \subset \partial_h N(B_t)$.

The following condition for λ_0 follows from the end-incompressibility of λ_t and the compatibility of $\mathcal{V}$ with $\mathcal{I}(\lambda_0)$.

Condition (2.3). If $\mathcal{V}_t$ is the vertical fibering of $N(B_t)$, then no subinterval of a fibre of $\mathcal{V}$ with endpoints in λ_t can be homotoped rel endpoints to an arc lying in a leaf of λ_t.

We will also assume:

Condition (2.4). The number of components of $\tau^1 \cap N(B_t)$ is minimal. I.e., if μ is isotopic to λ_t and B is a branched surface carrying μ satisfying Condition (2.2), then $|\tau^1 \cap N(B_t)| \leq |\tau^1 \cap N(B)|$.

Definition 2.8. The passage from λ_0 to λ_∞ will consist of an infinite sequence of normal isotopies and three other types of isotopies called *compressions*, *boundary-compressions*, and *general-boundary-compressions*, which are the laminations versions of the standard normalizing moves of Kneser and Haken. A compression is the isotopy shown in Figure 2.1 a). A *full compression* is a finite sequence of compressions such that each local leaf of the resulting lamination is a disc.

Remark 2.9. i) Any essential lamination λ transverse to τ^2 admits a full compression. To see this observe that if σ is a 3-simplex of τ, μ is obtained from compressing λ and each leaf of $\lambda|\sigma$ is a disc, then so is each leaf of $\mu|\sigma$. Thus by cleaning up one 3-simplex at a time, any essential lamination transverse to τ^2 can be transformed via compressions to a lamination with only disc local leaves.

ii) Full compressions are canonical. I.e., if λ is essential, and μ_1, μ_2 are obtained by fully compressing λ, then μ_1 is normally isotopic to μ_2. We will not be using this fact.

Definition 2.10. A *boundary compression* is supported in a small neighborhood of a 2-simplex κ and corresponds to pushing an I-fibred set of nonnormal arcs of $\lambda_t|\kappa$ across a 1-simplex $e \subset \partial\kappa$. The effect on the 2-simplices which meet e is shown in Figure 2.1. Suppose that D is an embedded disc in a 3-simplex σ such that ∂D consists of 2 arcs α and β where α lies in a 1-simplex e and β lies in a leaf of λ_t. Also $D \cap \partial\sigma = \alpha \subset e$ and $\lambda_t|D$ is a union of parallel arcs. Then the isotopy that pushes $\lambda_t|D$ across e and is supported in a very small neighborhood of D is called a *general-boundary-compression.* A boundary-compression differs from a general-boundary-compression in that the former is associated to a disc D which lies in a 2-simplex. General-boundary-compressions are needed to normalize local leaves whose boundaries are normally embedded. For example an almost normal octagon is not normal, yet it's boundary is a normal curve. Call a *∂-compression* an operation which is either a boundary-compression or a general-boundary-compression. Remark 2.17 i) explains our interest in distinguishing the two types of ∂-compressions.

Lemma 2.11. *If λ_t is an essential lamination satisfying Conditions (2.1)-(2.4), then the lamination μ obtained by fully compressing λ_t satisfies Conditions (2.1)-(2.4) and $\mu|\tau^2 \subset \lambda_t|\tau^2$.* q.e.d.

Lemma 2.12 *Suppose that σ and σ' are 3-simplices such that σ and σ' meet along the edge e. If L (resp. L') is a disc leaf of $\lambda_t|\sigma$ (resp. $\lambda_t|\sigma'$), such that $L \cap L' \cap e \neq \emptyset$, then $|(L \cup L') \cap e| \leq 2$. In particular $|L \cap e| \leq 2$.*

Proof. We will show that the failure of Lemma 2.12 violates (2.4). By (2.3) no pair of distinct points of $L \cap e$ can lie in the same component of $N(B_t) \cap e$. Suppose $|L \cap e| > 2$. By Reeb stability and the nowhere density of λ there exist leaves L_1, L_2 of $\lambda_t|\sigma$ which are normally parallel to L and together bound a closed complementary region of $\lambda_t|\sigma$. Let x be a point of $L \cap e$ which separates, within e, other points of $L \cap e$. Since

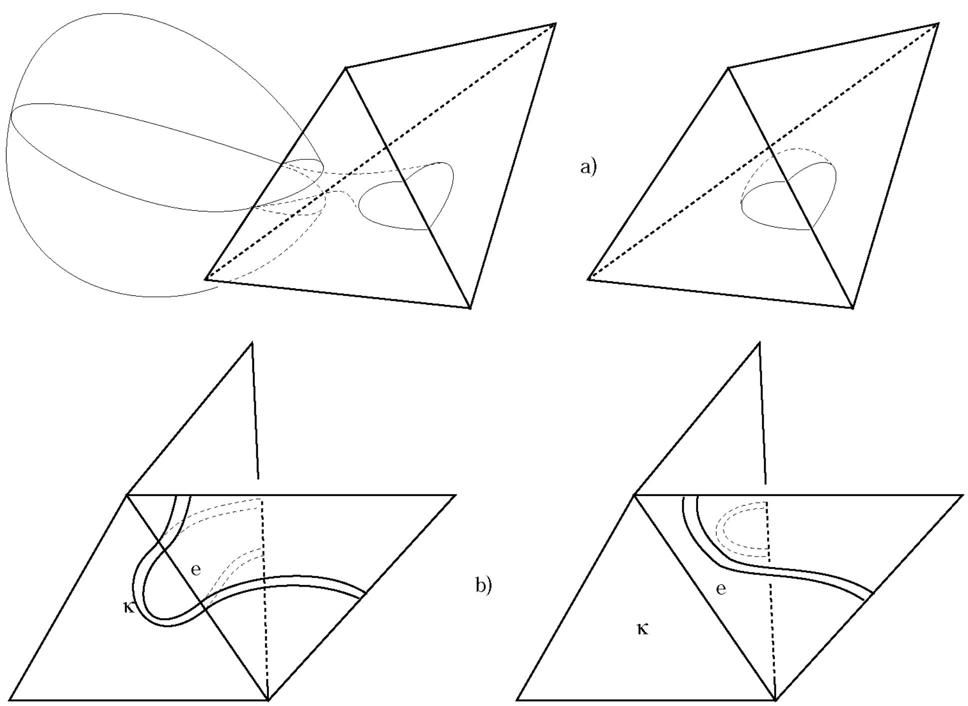


FIGURE 2.1.

$L \cap e > 2$, there are two different ∂-compressions which can eliminate x from $L \cap e$. To see these ∂-compressions think of L as lying very close and parallel to a disc in $\partial\sigma$ that ∂L bounds. Each of the two choices for the disc suggests the various choices of ∂-compression. By first doing one such ∂-compression to L_1 and then doing the other ∂-compression to L_2, then splitting B_t and isotoping the resulting $N(B_t)$ to satisfy (2.2), one obtains a contradiction to (2.4).

A similar argument works if $|(L \cup L') \cap e| = 3$ and $|L \cap e| = |L' \cap e| = 2$. If L(resp. L') hits e in points x and y (resp. y and z), then again by (2.3) and the essentiality of λ_t x, y and z lie in different components of $N(B_t) \cap e$. If say z separates x and y, then ∂-compressions in σ (in order to ∂-compress L) gives rise to a violation of (2.4). If y separates, then ∂-compressions in σ and σ' give rise to a violation of (2.4). Use the fact that one can find local leaves $L_1, L_2 \subset \sigma$, (resp. $L'_1, L'_1 \subset \sigma'$) normally parallel to L (resp. L') such that $L_1 \cup L_2$ (resp. $L'_1 \cup L'_2$) bound a closed complementary region of $\lambda_t|\sigma$ (resp. $\lambda_t|\sigma'$) and $(L_1 \cap L'_1) \cap e \neq \emptyset$ and $(L_2 \cap L'_2) \cap e \neq \emptyset$. The various ∂-compressions correspond to doing ∂-compressions to L_1 and L'_2 (or L_2 and L'_1) within σ and σ'. q.e.d.

Condition (2.5). If σ is a 3-simplex and L is a local leaf of the 3-simplex σ with respect to the essential lamination λ_t, then for each component β of ∂L there exists an edge e of σ such that $|\beta \cap e| = 0$.

Condition (2.5) is useful because of

Lemma 2.13. *A properly embedded disc D in the 3-simplex σ is normal if and only if ∂D is a normal curve disjoint from some edge e of ∂D.* q.e.d.

Lemma 2.14. *λ_0 can be isotoped to λ_1 which satisfies (2.1)-(2.5).*

Proof. Suppose that μ_0 is a lamination which satisfies (2.1)-(2.4) and such that each local leaf is a disc, for example, the lamination obtained by fully compressing λ_0. Packet Lemma 2.3 asserts that the collection of local leaves can be partitioned into a finite set of normally isotopic families of discs. We prove Lemma 2.14 by induction on the number $C(\mu_0)$ of such families whose boundaries fail to satisfy (2.5). In fact let $F \subset \sigma$ be one such family. Being connected, each leaf of ∂F must cross some edge e at least two times, hence by Lemma 2.12 it crosses e exactly two times. A single ∂-compression eliminates the intersections of F with e. Any local leaf in M involved in this ∂-compression is made disjoint from e. Thus, the number of families whose boundaries fail to satisfy (2.5) has been reduced. Since a full compression does not increase this number, it follows that there is a lamination μ_1 isotopic to μ_0 such that $C(\mu_1) < C(\mu_0)$. q.e.d.

Lemma 2.15. *Suppose that λ_t satisfies (2.1)-(2.5) and that every local leaf of λ_t is a disc. If μ is obtained by either compressing or ∂-compressing λ_t, then μ satisfies (2.1)-(2.5).*

Proof. As in the proof of Lemma 2.14, if μ was obtained from λ_t by a ∂-compression across the edge e, then any local leaf in M involved in that ∂-compression will give rise to local leaves of μ disjoint from e. Thus (2.5) holds for μ. It is routine to show the other conclusions of Lemma 2.15. q.e.d.

Construction of the infinite isotopy 2.16. Cyclically order the edges of the 2-simplices of τ by $(e_1, \kappa_1), \cdots, (e_n, \kappa_n)$ where κ_i is a 2-simplex and e_i is an edge of κ_i. Thus if edge e lies on n 2-simplices, then it will appear as the first term of the sequence exactly n times. Let $\lambda_{1.1}$ be obtained from λ_1 by doing a boundary-compression to eliminate a maximal I-fibred collection of nonnormal arcs of κ_1 with endpoints in e_1. Let $\lambda_{1.2}$ be obtained by fully compressing $\lambda_{1.1}$. After finitely many pairs of isotopies we obtain a lamination λ_2 such that each local leaf is a disc and each leaf of $\lambda_2|\kappa_1$ with endpoints in e_i is normal. In this way we obtain an infinite sequence $\lambda_1, \lambda_2, \cdots$, where λ_{k+1} is obtained by normalizing λ_k on the e_k edge of κ_k, where indices of (e_k, κ_k) are taken mod n. q.e.d.

Remark 2.17. i) By Lemmas 2.14-2.15, once λ_1 has been constructed, all future isotopies consist only of normal isotopies, compressions and boundary-compressions.

ii) Consequently, the infinite isotopy can be more or less understood by staring at the 2-skeleton. We shall see that the limiting behavior is basically no more complicated than that exhibited in Example 1.5.

Lemma 2.18. *If B_t is a branched surface satisfying (2.2) - (2.4) which carries λ_t, then a branched surface B_{t+1} satisfying (2.2) - (2.4) carrying λ_{t+1} is obtained by a finite λ-splitting of B_t followed by an isotopy.* q.e.d.

3. Proof of Brittenham's Theorem

In this chapter we understand how to take the limit of λ_t as $t \to \infty$. We will observe that if σ is a 3-simplex, then for t sufficiently large and integral, $\lambda_t|\sigma$ decomposes into a finite set of sublaminations called *walls*. In time the collection of walls stabilizes except possibly for at most two walls, which are ignored. This enables us to construct $\lambda_\infty|\sigma$ which is the limit of $\lambda_t|\sigma$ as $t \to \infty$. To carry out the above plan we will first analyze $\lambda_t|e$ for t sufficiently large, where e is an edge of τ, and then $\lambda_t|\kappa$ for t sufficiently large, where κ is a 2-simplex of τ.

The limit λ_∞ is a branched lamination, which naturally splits to a normal lamination $\mathcal{L}'$. By [3] the desired normal essential lamination $\mathcal{L}$ is obtained by deleting the Reeb laminations of $\mathcal{L}'$.

Analysis of $\lambda_t|e$ where e is a 1-simplex of τ, 3.1. By (2.4) and Remark 2.17 i), if $[a_t, d_t]$ parametrizes a component of $N(B_t) \cap \tau^1$ and $C_t = \lambda_t \cap [a_t, d_t]$, then for $1 \le t < \infty, C_t = \lambda_1 \cap [a_t, d_t]$ and for $s \le t, a_s \le a_t < d_t \le d_s$. Call such a C_t a *clump*. Thus $E_t = \lambda_t \cap \tau^1$ is a disjoint union of a finite number c of clumps. By Condition (2.4) the number of clumps is constant, independent of t. As t increases, a clump may shrink from its ends, but never vanishes or becomes a point, since λ has no isolated leaves. Let $E_\infty = \cap E_t$ and $C_\infty = \cap C_t$. Again E_∞ is naturally partitioned into a disjoint union of limits of clumps. It may happen that a limit clump C_∞ may equal one point.

The following Packet Lemma 3.2 is just Packet Lemma 2.3 where the clump structure is taken into account.

Packet Lemma 3.2. *Let τ be a triangulation of the closed orientable 3-manifold M. If λ_t is an essential lamination transverse to $\tau^n, n \le 2$, then for each 3-simplex (resp. 2-simplex) η, $\lambda_t|\eta$ canonically decomposes into a finite set of maximal sublaminations of the form $T_i \times K_i \subset T_i \times [0,1]$, such that for each $s \in [0,1]$, $T_i \times s$ is a properly embedded compact surface (resp. interval) transverse to τ^2 (resp. τ^1), K_i is a closed subset of $[0,1]$, and if $i \ne j$, then $(T_i \times [0,1]) \cap (T_j \times [0,1]) = \emptyset$. Finally if e is a 1-simplex, then $(T_i \times K_i) \cap e$ lies in a clump of e.* q.e.d.

Analysis of $\lambda_t|\kappa$ where κ is a 2-simplex of τ, 3.3. We shall see that the normal arcs of the restriction of λ_t to a 2-simplex naturally decomposes into a finite set of sublaminations called *planks*. In time the collection of planks stabilizes except possibly for at most one plank, which is ignored. This enables us to take a limit of $\lambda_t \cap \tau^2$ as $t \to \infty$. Here are the details.

Let κ be a 2-simplex of τ, let K_t denote $\lambda_t \cap \kappa$ and let α_1, α_2, and α_3 denote the edges of κ. As in Packet Lemma 3.2, the non-circle leaves of

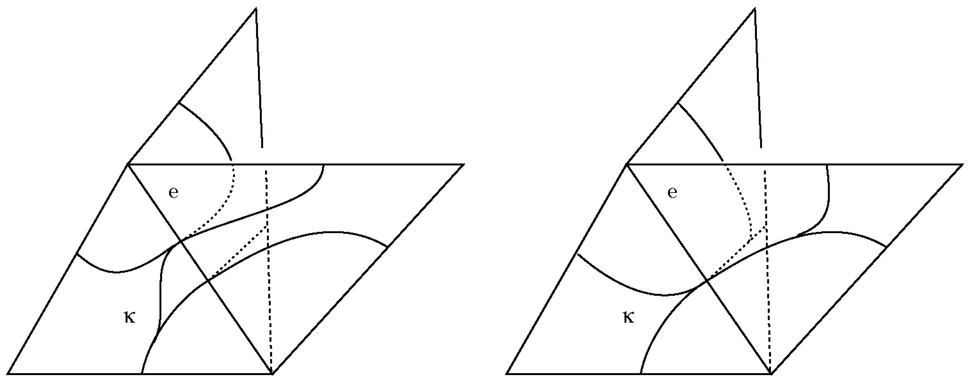


FIGURE 3.1

K_t form a finite union of laminations of the form $I \times C_t \subset I \times [a_t, b_t]$, where C_t is closed, $\{a_t, b_t\} \subset C_t$ and for $i \in \{0,1\}, i \times C_t$ lies in a clump. Assume that these laminations are pairwise disjoint and maximal. Such a lamination $C_t \times I$ is called a *plank*, if it connects clumps lying on distinct edges of κ. If it connects to clumps lying on the same edge it is called a *nonnormal plank*. A leaf of a plank is called a *grain*. The grains of the plank P of the form $a_t \times I$ or $b_t \times I$ are called the *sides* of P. Note that leaves of K_t not lying in planks get eventually isotoped away, for such a leaf has both endpoints on some edge e of κ, and unless it gets eliminated by earlier compressions or boundary-compressions, it will get eliminated exactly when it is time to normalize arcs in κ with endpoints in e.

Lemma 3.4. *If $P_1, P_2 \subset \kappa$ are distinct planks emanating from the same clump C, then $P_1 \cup P_2$ intersects every edge of κ. At most 2 planks can emanate from a clump.*

Proof. If P_1 and P_2 connect to clumps C_1 and C_2 on the edge α, then by squeezing the branched surface B_t which carries λ_t and satisfies (2.2) - (2.4), we obtain a new one B'_t satisfying (2.2) - (2.4) such that C_1 and C_2 are coalesced into the same clump and the other clumps are unchanged. Since B'_t has 1 fewer clump than B_t we obtain a contradiction to (2.4). See Figure 3.1. q.e.d.

Let $\phi : M \times [0, \infty) \to M$ denote the infinite isotopy such that $\phi_0 = \mathrm{id}_M$ and for each $t, \phi_t(\lambda) = \lambda_t$. Call a point $x \in M$ *t-stable*, if for all $h > 0, \phi_{t+h}(\phi_t^{-1}(x)) = x$. A point $x \in M$ is *stable* if it is t-stable for some t. Call a clump C a *κ-spread clump* if two planks emanate from C and lie in κ. By Lemma 3.4, at any time t, an edge α_i can contain

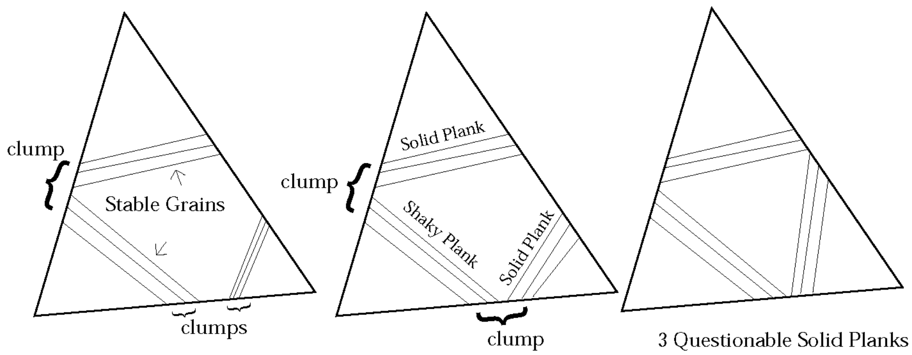


FIGURE 3.2

at most one κ-spread clump and κ can contain at most three κ-spread clumps. We will usually suppress the κ in the expression "κ-spread clump", since κ will always be understood from context. The following lemma also follows from (2.4).

Lemma 3.5. *If κ contains a unique spread clump, then the points lying on the "inside" 2 grains are stable. See Figure 3.2.* q.e.d.

Definition 3.6. We classify the planks of κ into two categories, *shaky* and *solid* and some solid planks will also be called *questionable.* These terms are meant to reflect what might happen to planks during future isotopies.

If κ contains exactly two spread clumps, then call the plank that connects them *shaky.* Call the side of a shaky plank which faces these other two planks the *+ side.* Call all the other planks in κ *solid.* If κ does not contain exactly two spread clumps, then call all the planks solid. If κ has 3 spread clumps, then call the 3 planks emanating from these clumps *questionable.* See Figure 3.2.

Lemma 3.7. *The effect of compressing λ_t on a plank P is to either delete grains from its ends or to eliminate it. I.e., if $P_t = I\times([a_t, b_t]\cap\lambda_1)$ and λ'_t is obtained by compressing λ_t, then the associated plank P' is of the form $I \times ([a'_t, b'_t] \cap \lambda_1)$ with $[a'_t, b'_t] \subset [a_t, b_t]$. A solid plank cannot get eliminated, unless it is questionable. At most one questionable plank can get eliminated from a given 2-simplex.* q.e.d.

Lemma 3.8. *The effect of boundary-compressing λ_t on a solid plank P is to delete grains from its ends or to eliminate it. Only questionable solid planks can get eliminated, and at most one such plank can be eliminated per 2-simplex. The effect of boundary-compressing a shaky* *plank is to either eliminate it or delete grains from the non +-side or to add grains to the +-side. Boundary-compressing may create new shaky planks, but never creates solid planks.* q.e.d.

Lemma 3.9. *Once a questionable plank has been eliminated, the two other former questionable planks remain solid and are no longer questionable. Shaky planks never become solid, and conversely solid planks never become shaky.* q.e.d.

Analysis of the limiting behavior of planks within the 2-simplex κ, 3.10. Since λ_1 has only finitely many questionable planks, and questionable planks are never created during the isotopy process, it follows that after some time t_0, no questionable planks can get eliminated. Thus, if $s > t > t_0$, and P_t is the solid plank $I \times (\lambda_1 \cap [p_t, q_t])$, then at time s there exists a solid plank P_s of the form $I \times (\lambda_1 \cap [p_s, q_s])$ where $[p_s, q_s] \subset [p_t, q_t]$. Thus P_∞, the limit of $P_t, t > t_0$ is a nonempty set of the form $I \times (\lambda_1 \cap [p_\infty, q_\infty])$, which may consist only of a single grain.

Now consider the case that the clump C_t which hits P_t also hits a shaky plank Q_t. In this case parametrize C_t by $[a_t, d_t] \cap \lambda_1$ so that $C_t \cap [a_t, b_t] \subset P_t$ and $C_t \cap [c_t, d_t] \subset Q_t$ where $a_t < b_t < c_t < d_t$ and $b_t \subset P_t$ and $c_t \subset Q_t$. Again by Lemmas 3.7-3.8, $[a_t, b_t]$ is a nested sequence of nonempty intervals and if Q_s exists at some $s > t$, then $c_s \le c_t$ and $d_s \le d_t$. For $e \in \{a, b, c, d\}$, define $e_\infty = \lim e_t$, if such limit exists. Call a plank *enduring* if it is either a solid plank which never gets eliminated or a shaky plank Q_t with $c_t \le d_\infty$ for some $t < \infty$. Since a given 2-simplex can have at most 1 enduring shaky plank, it follows that at some time $t_1 > t_0$, the set of enduring planks are determined, in particular no new enduring planks are created after time t_1 and none are eliminated.

If Q_t is a enduring shaky plank, connecting the clumps C_t and C'_t and $Q_t \cap C_t = [c_t, d_t] \cap \lambda_1, Q_t \cap C'_t = [c'_t, d'_t] \cap \lambda_1$, then define the limit plank Q_∞ to be a plank connecting $[c_\infty, d_\infty] \cap \lambda$ with $[c'_\infty, d'_\infty] \cap \lambda$ where the grains connect in the natural way.

The union of the various limit planks is a branched lamination K_∞ of κ such that $K_\infty \cap \partial\kappa = E_\infty \cap \kappa$. Indeed branching happens exactly when (in the above coordinates) $c_\infty = b_\infty$. See Figure 3.3 and compare with Example 1.5.

Analysis of $\lambda_t|\sigma$ where σ is a 3-simplex of τ, 3.11. Define S'_t to be the sublamination of $S_t \stackrel{def}{=} \lambda_t|\sigma$ consisting of normal cells.

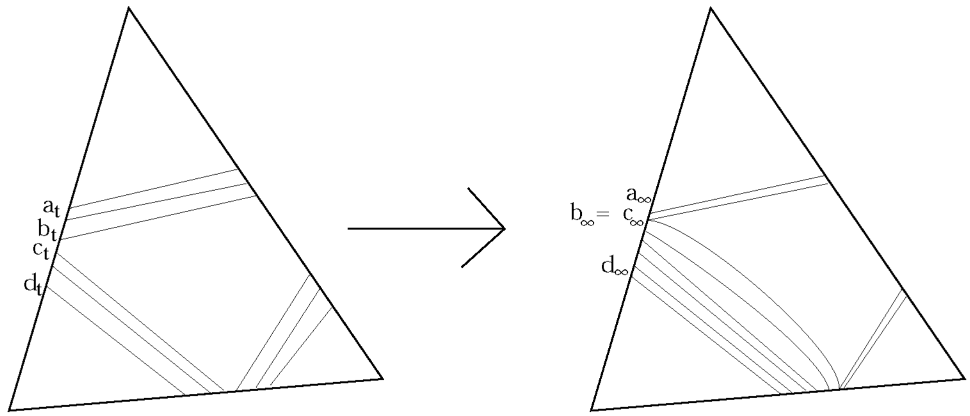


FIGURE 3.3

By Packet Lemma 3.2 the lamination S_t decomposes canonically into finitely many sublaminations. Call such a sublamination a *wall* (resp. *nonnormal wall*) if all its leaves are normal (resp. nonnormal). Each wall (resp. nonnormal wall) is a maximal $(D^2 \times I, \partial D^2 \times I) \subset (\sigma, \partial\sigma)$ laminated by $D^2 \times K$, where K is a Cantor set in I containing 0 and 1 and for $x \in \partial D^2 - \tau^1, x \times K$ lies in exactly one plank (resp. plank or nonnormal plank). Define an *edge* of a wall or nonnormal wall to be the intersection of w with a plank or nonnormal plank. The *sides* of a wall w are the discs $D^2 \times t, t \in \{0,1\}$. The collection of walls is uniquely determined and is called a *wall decomposition* of S'_t.

Define an equivalence relation on the set of walls of S'_t, generated by the rule that two walls are equivalent if they intersect the same plank. There are 25 possible combinatorial types of classes. There are 5 classes which contain a wall of quadralaterals such that all the other walls in its class lie on one side of the quadralateral wall. Figure 3.4 shows how these five classes intersect $\partial\sigma$. There are 15 classes which contain quadralateral walls. There are 6 classes (resp. 2,1,1) which exactly involve 4 walls (resp. 3,2,1) walls of triangles. Observe that a clump (resp. plank) can meet up to 3 (resp. 2) walls. Typically an equivalence class contains exactly one wall, however S'_t may contain as many as two classes which contain more than one wall.

Definition 3.12. i) Each of the 25 combinatorial classes of walls in σ corresponds to a branched surface in σ which we call a *3-simplex local branched surface.*

ii) Call a normal wall w *shaky* if some edge of w lies on a shaky plank

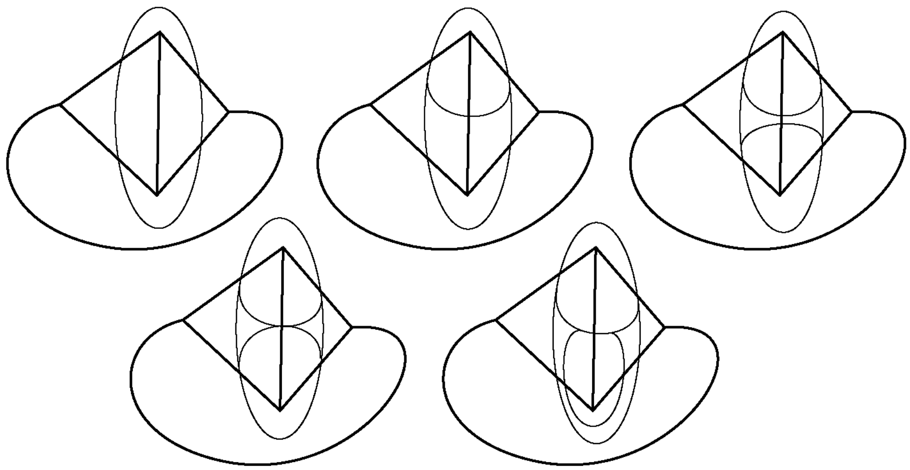


FIGURE 3.4

and all other edges lie on either a shaky plank or on a plank shared by another wall of S'_t. We say that the side of a wall is $+$ if it contains the $+$-side of a shaky plank. Note that a shaky wall has at least one $+$-side and can have two $+$-sides. Call a normal nonshaky wall *solid* and call a solid wall w *questionable* if the other members of its equivalence class lie in one component of $\sigma - w$ and deleting w does not reduce the number of clumps on σ^1.

Shaky, solid and questionable walls earn their names because they satisfy the conclusions of the following elementary lemmas. The next two results are the analogues of Lemmas 3.7-3.9.

Lemma 3.13. *The effect on a solid wall W by compressing or boundary compressing λ_t is to delete sheets from its ends. If W is questionable, then it might get eliminated. The effect on a shaky wall is to either eliminate it or delete sheets from a non $+$-side or to add sheets to the $+$-side. Compressing or boundary compressing may create new shaky walls, but never creates solid walls.* q.e.d.

Lemma 3.14. *Shaky walls never become solid and conversely solid walls never become shaky.* q.e.d.

Lemma 3.15. *If $x \in E_\infty, x \in \partial\sigma$, then for t sufficiently large, there is a stable (normal) local leaf of $\lambda_t|\sigma$ which contains x. (See 3.1.)*

Proof. Let κ_1 be a 2-simplex face of σ which contains x. It follows from Lemmas 3.7-3.8 that there exists a stable grain $g_1 \subset \kappa_1$ which contains x. Set $x_2 = \partial g_1 - x$, and $\kappa_2 \neq \kappa_1$ the 2-simplex of σ which contains x_2. Let g_2 be a stable grain which lies in κ_2 and contains x_2. Continuing in this manner, for each n we find a path of stable grains $\gamma_n = g_1 * g_2 * \cdots * g_n$ which begins at x and lies in $\partial\sigma$. By Lemma 2.12, for some $n \leq 12$, γ_n is an embedded loop of stable grains through x which lies in $\partial\sigma$. Also γ_n misses some edge of σ^1. For t a sufficiently large integer this loop necessarily bounds a stable leaf of $\lambda_t|\sigma$ which is normal by Lemma 2.13. q.e.d.

Definition 3.16. i) Define the notion of a *enduring wall* in a manner analogous to that of an enduring plank. Since at any moment a given 3-simplex can have only a finite number of walls, it follows that after some time t_2, no new enduring walls are created and no questionable walls get eliminated. Assume that t_2 has the property that for each enduring shaky plank, $c_{t_2} \leq d_\infty$, with notation as in 3.10. Also a similar property holds for enduring shaky walls. Thus after time t_2 every enduring wall has a stable leaf.

ii) In a natural way solid walls limit to walls and a limit wall may consist of a single sheet. Define the limit of enduring shaky walls in a manner analogous to that of shaky planks. The union of the limits of enduring walls of σ is a branched lamination S'_∞. As with planks, branching will only occur on the edges of a +-side.

iii) Define λ_∞ to be the branched lamination of M obtained by taking the union of the limit walls.

iv) *(A thick handle structure on $N(\lambda_t), t \geq 1$)* Let $C_t = [a_t, d_t] \cap \lambda_1$ be a clump of λ_t which lies on the edge e. Construct a small $D^2 \times [a_t, d_t]$ which is transverse to e, intersects e in $0 \times [a_t, d_t]$, and intersects λ_t in $D^2 \times ([a_t, d_t] \cap \lambda_1)$. The set $\hat{C}_t \overset{def}{=} D^2 \times ([a_t, d_t] \cap \lambda_1)$ is called a *thick clump* and the set $C_t^F \overset{def}{=} D^2 \times [a_t, d_t]$ is called the *fibred neighborhood* of the thick clump $\hat{C}_t$. Each $D^2 \times s$ in a thick clump should be viewed as a 2-dimensional 0-handle.

Let $P_t = I \times ([p_t, q_t] \cap \lambda_1)$ be a plank, possibly nonnormal, of λ_t which lies on the 2-simplex κ and connects the clumps C_0 and C_1. Let $I' = [1/4, 3/4]$. Construct a small $[-1,1] \times I' \times [p_t, q_t]$ which is transverse to κ, intersects κ in $0 \times I' \times [p_t, q_t]$, intersects λ_t in $[-1,1] \times I' \times ([p_t, q_t] \cap \lambda_1)$ and intersects $\hat{C}_0 \cup \hat{C}_1$ in $[-1,1] \times \partial I' \times ([p_t, q_t] \cap \lambda_1)$. See Figure 3.5. The set $\hat{P}_t \overset{def}{=} [-1,1] \times I' \times ([p_t, q_t] \cap \lambda_1)$ is called a *thick plank* and the set $P_t^F \overset{def}{=} [-1,1] \times I' \times [p_t, q_t]$ is called the *fibred neighborhood* of the thick plank $\hat{P}_t$. Each $[-1,1] \times I' \times s$ should be viewed as a 2-dimensional 1-handle.

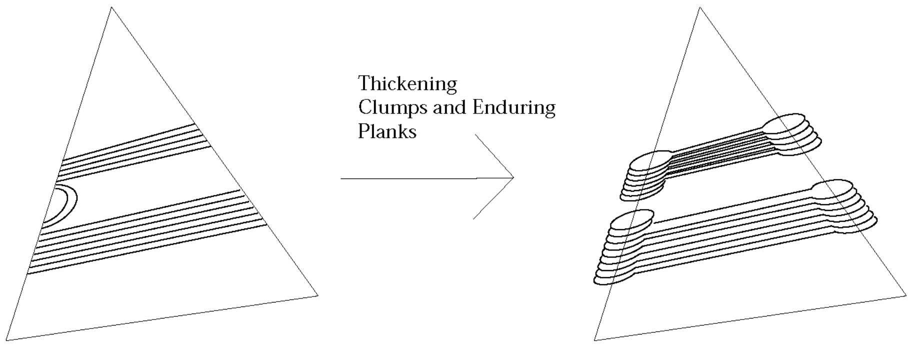


FIGURE 3.5

In a similar manner to each wall, possibly nonnormal, $w_t \subset \sigma$, construct a *thick wall* $\hat{w}_t = D^2 \times K$ where $K \subset [r_t, s_t]$ is a Cantor set containing $\{r_t, s_t\}$ and $\partial D^2 \times K$ attaches to the various thick clumps and thick planks, possibly nonnormal as determined by w_t. Call $w_t^F \stackrel{def}{=} D^2 \times [r_t, s_t]$ a *fibred neighborhood* of $\hat{w}_t$.

The handle structure on λ_{t_2} induced by the collection of handles will be called a *thick handle structure* on λ_t. A thick clump, plank, or wall, possibly nonnormal, will be called a *thick handle*. The union of the fibred neighborhoods of the thick handles will be denoted $N(\lambda_t)$ and is a fibred neighborhood of λ_t. Note that $N(\lambda_t) = N(B_t)$ where $N(B_t)$ is a fibred neighborhood of a branched surface B_t carrying λ_t. The map which contracts to a point each I-fibre of a fibred neighborhood of a thick i-handle induces the projection of $N(B_t)$ onto B_t.

Lemma 3.17. *If B is any branched surface which carries λ_t and satisfies (2.2), (2.3) and (2.4), then after λ-splitting or squeezing along bigons and discs, B is isotopic to B_t. Furthermore B_t satisfies (2.2), (2.3) and (2.4).*

Proof. By (2.2) and (2.4) each of $B \cap \tau^1$ and $B_t \cap \tau^1$ are cannonically in 1-1 correspondence with the clumps of λ_t. If $P \subset \kappa$ is a plank, possibly nonnormal, connecting two clumps, then associated to P and B are finitely many arcs in κ whose ends coincide in a neighborhood of τ^1 and these arcs can be transformed into a single arc via a finite sequence of squeezing along bigons. By (2.2) and (2.4) if P and P' are distinct planks (possibly nonnormal), then the associated arcs of B are either disjoint or coincide along a single subarc eminating from τ^1. It follows that we can assume that after λ-splitting, squeezing along bigons and normal isotopy of B that $B|N = B_t|N$ where N is a neighborhood of τ^2. An analogous argument for the normal and nonnormal walls shows that after splitting and squeezing along bigons and discs B is isotopic to B_t. By Remark 2.7 and construction, B_t satisfies (2.2) and (2.4). q.e.d.

A thick handle will often be denoted $D^2 \times [x, y] \cap \lambda_1$ even if it is a thick 1-handle. The discs $D^2 \times \{x, y\}$ of the fibred neighborhood $D^2 \times [x, y]$ of a thick handle will be called the *sides* of the thick handle.

If $t \geq t_2$, call the union of all the thick handles associated to clumps, enduring planks and enduring walls a *thick partial handle structure* on λ_t. The union of all the fibred neighborhoods of such thick handles is denoted by $N^p(\lambda_t)$ and called the *fibrelike neighborhood* of λ_t even though $\lambda_t \not\subset N^p(\lambda_t)$.

Remark 3.18. It is routine to modify ϕ so that for any integral $t \geq t_2$, the isotopy transforms the thick handle structure of λ_{t_2} into a thick handle structure on λ_t. This means that if $s > t$, then the isotopy transforms B_t into B_s. Also if $C_{t_2} = [a_{t_2}, d_{t_2}] \cap \lambda_1 \subset e$ is a clump of λ_{t_2} and $C_t = [a_t, d_t] \cap \lambda_1$ denotes the corresponding clump at time $t \geq t_2$, with $[a_t, d_t] \subset [a_{t_2}, d_{t_2}]$, then the associated thick clump $\hat{C}_t$ is of the form $D^2 \times ([a_t, d_t] \cap \lambda_1) \subset D^2 \times ([a_{t_2}, d_{t_2}] \cap \lambda_1) = \hat{C}_{t_2}$ and the associated fibred neighborhood is of the form $D^2 \times [a_t, d_t]$. Similarly if the thick solid plank P_{t_2} is of the form $[-1, 1] \times I' \times ([a_{t_2}, b_{t_2}] \cap \lambda_1)$, then the thick solid plank P_t is of the form $[-1, 1] \times I' \times ([a_t, b_t] \cap \lambda_1)$ with $[a_t, b_t] \subset [a_{t_2}, b_{t_2}]$. If the thick shaky plank Q_{t_2} is of the form $[-1, 1] \times I' \times ([c_{t_2}, d_{t_2}] \cap \lambda_1)$, then the thick shaky plank Q_t is of the form $[-1, 1] \times I' \times ([c_t, d_t] \cap \lambda_1)$ where $a_{t_2} \leq b_\infty \leq b_t < c_t \leq c_{t_2} \leq d_\infty \leq d_t \leq d_{t_2}$ and $b_t \leq b_{t_2}$ (same notation as in 3.10). A similar statement holds for how thick walls evolve over time.

A thick handle structure on the fibrelike neighborhood $N(\lambda_\infty)$, 3.19. We now define the thick handle structure on λ_∞ which is the limit of the above thick partial handle structures. Define $\hat{C}_\infty = D^2 \times ([a_\infty, d_\infty] \cap \lambda_1)$ and $C^F_\infty = D^2 \times [a_\infty, d_\infty]$, to be respectively the *thick clump* and is associated *fibered neighborhood* of the clump C_∞ of λ_∞. In a similar manner, define the limit of thick planks and thick walls as well as the limit of fibred neighborhoods of the thick planks and thick walls. A thick handle in the limit may be a single D^2. In that case the limit fibred neighborhood consists of a single 2-disc. In that case the *sides* of the limit fibred neighborhood are two distinct 2-discs that trivially 2-fold cover the given 2-disc. Define $N(\lambda_\infty)$, the *fibrelike neighborhood* of λ_∞ to be the union of the limit fibred neighborhoods of all the clumps, enduring planks and enduring walls. Let B_∞ denote the limit branched surface. Define the horizontal boundary $\partial_h N(\lambda_\infty)$ of the fiberlike neighborhood to be the union of the various sides of the thick handles of λ_∞. These sides glue together in the natural way to make $\partial_h N(\lambda_\infty)$ a compact surface, possibly with boundary. The immersion $\partial_h N(\lambda_\infty) \to N(\lambda_\infty)$ is an embedding away from the degenerate thick handles, and maps 2-1 on the degenerate thick handles. Define $\partial_v N(\lambda_\infty)$ to be the closure of those points on the boundary of $N(\lambda_\infty)$ which do not lie on $\partial_h N(\lambda_\infty)$. Define $\partial N(\lambda_\infty) = \partial_h N(\lambda_\infty) \cup \partial_v N(\lambda_\infty)$ where the boundaries of $\partial_h N(\lambda_\infty)$ and $\partial_v N(\lambda_\infty)$ are identified in the natural way. Except along circles corresponding to the branch locus of $\lambda_\infty, \partial_h N(\lambda_\infty)$ is a smooth manifold with boundary that immerses into $\partial N(\lambda_\infty)$. Let $s(\partial N(\lambda_\infty))$ denote this branch locus. Note that $\partial_v N(\lambda_\infty)$ is a disjoint union of annuli, and $\partial_v N(\lambda_\infty)$ meets $\partial_h N(\lambda_\infty)$ transversely along a finite set of circles. See Figure 3.6.

Definition 3.20. Let $\mathring{V}$ be a component of $M - N(\lambda_\infty)$ and V denote its closure with respect to the path metric. We will call such a V a *closed complementary region* of $N(\lambda_\infty)$. ∂V inherits from $N(\lambda_\infty)$ the sets $\partial_h V, s(\partial V)$, and $\partial_v V$. Call V *active* if it is not stable. Call $x \in \partial V$ active, if no neighborhood of x in V is stable.

Lemma 3.21. *i) The closed complementary region V of $N(\lambda_\infty)$ is active if and only if each $x \in \partial V$ is active.*
ii) If V is active, then $\partial_v V = \emptyset$. If V is not active, then $s(\partial V) = \emptyset$.

Proof. i) The triangulation τ induces a cell structure τ_V on a closed complementary region V. Since ϕ is an infinite composition of compressions, boundary compressions and normal isotopies, one readily checks that if R_0 is a 3-cell of τ_V that contains $x \in V$ and some neighborhood in V of x is stable, then R_0 is stable. Similarly if R_1 is a 3-cell of τ_V that hits R_0, then R_1 is stable. Conclusion i) now follows by induction.

ii) If $\partial_v V \neq \emptyset$, then $\partial_v V$ meets the two inside grains of a spread clump of λ_t for t sufficiently large, and thus some neighborhood in V of a vertical fibre of $\partial_v V$ is stable. By i) V is not active. By construction if $s(\partial V) \neq \emptyset$, then no neighborhood of $x \in s(\partial V)$ in V is stable. By i) V is active. q.e.d.

Definition 3.22. Define $X = \{x \in \lambda_\infty | x \text{ stable } \}$ and $Y = \lambda_\infty - X$. The points Y of λ_∞ are called *new points.* By construction Y is a compact surface, and each "leaf" of X is a complete surface

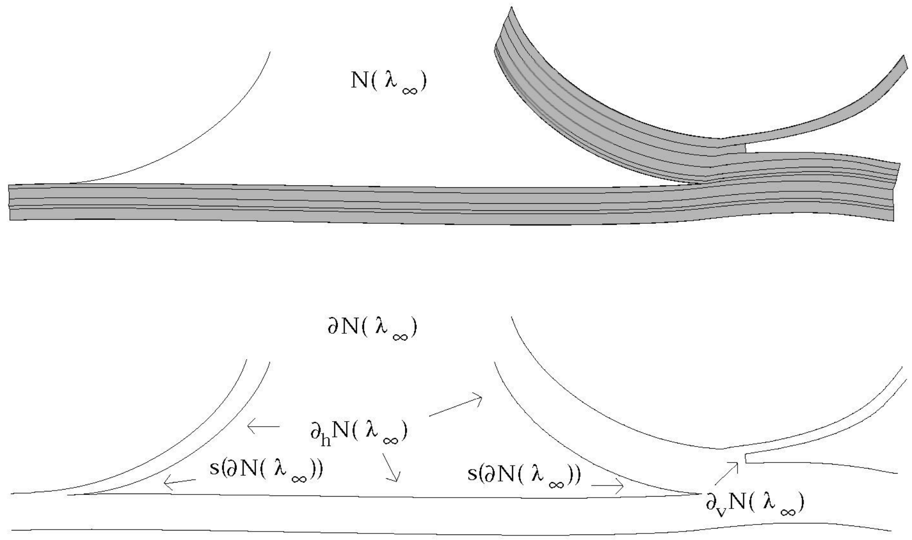


FIGURE 3.6

injectively immersed in M. Define J to be the union of leaves of X which nontrivially intersect $\bar{Y}$. Note that $J \cup Y$ are the branched leaves of λ_∞.

Lemma 3.23. *With respect to the path metric, each leaf of X is complete, injectively immerses in M and the induced map on π_1 is injective.*

Proof. The first two conclusions follow by construction, the last conclusion follows exactly as in the Lemma of p. 224 [3]. q.e.d.

Definition 3.24. *(Creating the lamination $\mathcal{L}'$ from λ_∞)* Let μ be the branched lamination obtained by first splitting λ_∞ along J and then adding a leaf called J_{middle}. This is the usual operation of replacing the leaves J by $\partial N(J)$ and then adding the "zero section" of the I-bundle on $N(J)$. Let $\mathcal{L}'$ be the (unbranched) lamination obtained from μ by deleting $int(V \cap \partial N(J))$, where V is the union of closed complementary regions of μ corresponding to the union of the active closed complementary regions of λ_∞. See Figure 3.7.

Remark 3.25. Replacing J by the triple cover prevents disjoint complementary regions of λ_∞ from connecting during the passage of λ_∞ to $\mathcal{L}'$. This would happen if a thick wall of λ_∞ was degenerate and both sides met active complementary regions. For example such a situation could occur if the wall structure within a simplex appears as in Figure

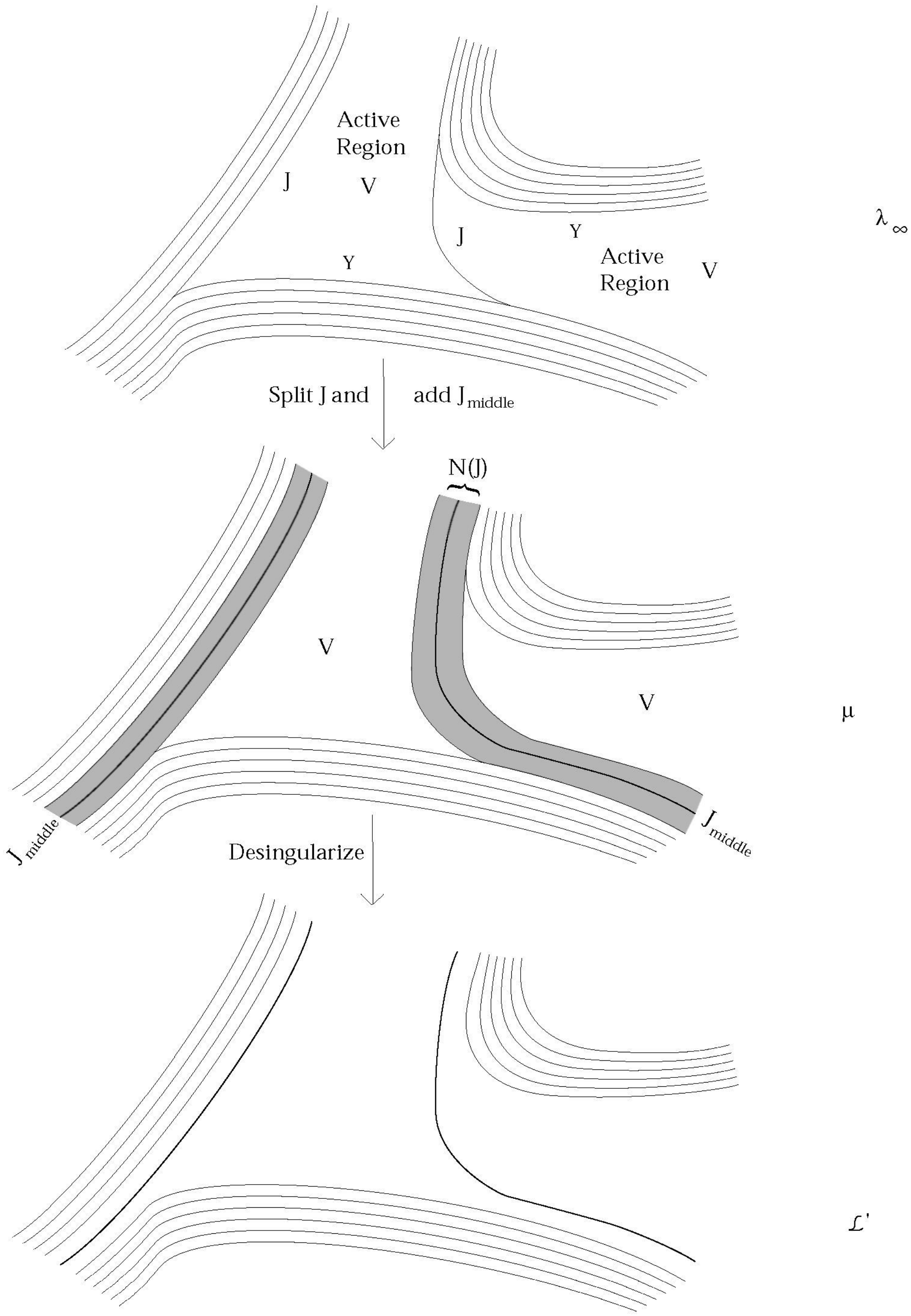

Active
Region
J
V
J
Y
Y
Active
Region
V
λ ∞
Split J and
add J middle
N(J)
V
V
μ
J middle
J middle
Desingularize
ℒ'

Figure 3.7

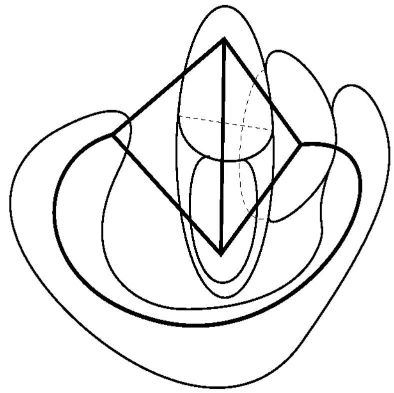


FIGURE 3.8

3.8. This local picture can be part of an example where the branch locus of λ_∞ is not embedded.

Lemma (Brittenham) 3.26. *If T' is a compressible torus in $\mathcal{L}'$, then T' bounds a solid torus W', $T \cap Y \neq \emptyset$, and T' is isolated exactly on the non-W' side. Here $T \subset \lambda_\infty$ is the immersed torus corresponding to T' and we denote by W the corresponding immersed solid torus. If V is a closed complementary region of λ_∞ and $\overset{\circ}{V} \cap W = \emptyset$, then $V \cap W \subset \bar{Y}$. There are only finitely many such tori T', and they bound pairwise disjoint solid tori which are disjoint from the various J_{middle}'s. Finally the lamination $\mathcal{L}$ obtained from $\mathcal{L}'$ by deleting these solid tori is a normal essential lamination.*

Proof. Apply the argument of p. 229-233 [3], noting that what he calls the "L of $N(L)$" is what we call J_{middle}. Actually that argument only asserts that compressible tori bound either solid tori or cubes with knotted hole. If T' bounded a cube with knotted hole C, then by Lemma 3.23 and the proof of the Theorem of p.616-617 [13], it follows that $\mathcal{L}|\overset{\circ}{C}$ extends to a foliation by planes and hence by [21] $\pi_1(C)$ is abelian (see also [13]) and so C is a solid torus. There is a much more elementary yet verbose proof of this fact.

Note also that the interior of each W' is nontrivially laminated by planes. The nontriviality follows from the fact $T \cap Y \neq \emptyset$ and Lemma 3.23. The π_1-incompressibility of X together with the end-incompressibility of λ implies that each leaf of $\mathcal{L}'|\overset{\circ}{W}{}'$ is a plane. q.e.d.

This completes our rendering of Brittenham's Theorem.

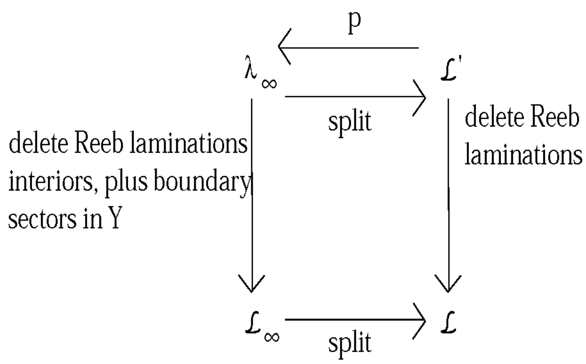


FIGURE 3.9

Definition 3.27. By Lemma 3.26, the collection of solid tori $W_1', \cdots, W_m'$ in M bounded by leaves of $\mathcal{L}'$ is finite and pairwise disjoint. These tori correspond to immersed solid tori $W_1, \cdots, W_m$ whose boundaries lie in λ_∞ and whose interiors are pairwise disjoint. Define $\mathcal{W} = W_1 \cup \cdots \cup W_m$. Letting $p : M \to M$ denote the map which collapses $N(J)$ to J and hence projects $\mathcal{L}'$ onto λ_∞, define $\mathcal{L}_\infty = p(\mathcal{L}' - \cup W_i')$. In words $\mathcal{L}_\infty$ is the branched lamination obtained from λ_∞ by splitting off and deleting the Reeb solid tori. These operations are summarized in the commutative diagram of Figure 3.9. A *Reeb lamination* on a solid torus W', is a lamination such that $\partial W'$ is a leaf and $\overset{\circ}{W'}$ is nontrivially laminated by planes. Lemma 3.26 implies that the solid tori bounded by compressible torus leaves of $\mathcal{L}'$ have Reeb laminations. Let $\mathcal{K}$ denote the leaves of $\mathcal{L}$ which intersect $N(J) \cup Y$, and let $\mathcal{K}_0 \subset \mathcal{K}$ be the leaves which intersect Y.

Lemma 3.28. *$\mathcal{K}_0 \subset \mathcal{L}$ is non-isolated on exactly one side.*

Proof. The Lemma of p.229 [3] exactly proves the analogous result for $\mathcal{L}'$. Since $\mathcal{L}$ is obtained from $\mathcal{L}'$ by deleting laminated solid tori which are isolated to the outside (p. 229-230 [3]), Lemma 3.28 follows. q.e.d.

Definition 3.29. By thick handle structures on $\mathcal{L}_\infty$ and $\mathcal{L}$ we mean the thick handle structures induced by λ_∞. Also let $N(\mathcal{L}_\infty)$ and $N(\mathcal{L})$ denote the induced fibrelike neighborhoods of $\mathcal{L}_\infty$ and $\mathcal{L}$. Note that $N(\mathcal{L}_\infty) = N(\lambda_\infty) - (\mathcal{W} - \mathcal{L}_\infty)$. Call a closed complementary region of $N(\mathcal{L}_\infty)$ *active* if it contains an active closed complementary region of $N(\lambda_\infty)$.

Lemma 3.30. *i) The closed complementary region V of $N(\mathcal{L}_\infty)$ is active if and only if for each $x \in V$ either $x \in \mathcal{W}$ or some neighborhood of x in V is not stable.*

ii) If V is active, then $\partial_v(V) = \emptyset$. Furthermore V is a union of W_i's and active closed complementary regions of λ_∞ and contains at least one of the latter.

iii) If V is inactive, then $V \cap int(\mathcal{W}) = \emptyset$.

Proof. If $\mathcal{W} = \emptyset$, then this is Lemma 3.21. Otherwise combine Definition 3.24 with the conclusions of Lemma 3.21 and Lemma 3.26.

q.e.d.

Remark 3.31. i) The reader should check that the no-isolated leaves requirement of Condition (2.1), was used purely to simplify the notation in §2-3. For example, it allowed us to construct natural fibred neighborhoods and thus equate the closed complementary regions of $N(B_t)$ with the union of $\mathcal{G}(\lambda_t)$ and a compact part of the $\mathcal{I}(\lambda_t)$. It allowed us to avoid the annoying situation of clumps of λ_t being reduced to points for $t < \infty$, which in turn implied that the various closed complementary regions of $\lambda_t, t < \infty$ are injectively immersed in M.

ii) The nowhere densitiy of λ was used only to construct a branched surface carrying λ. Many locally dense laminations are carried by branched surfaces. Our argument could have been readily carried out for such laminations.

4. Evacuating the sutured manifold

The reader is advised to go directly to Theorem 4.4, referring back only as needed.

Definition 4.1. A *sutured manifold* [10] (N, γ) is a compact oriented 3-manifold N together with a collection of pairwise disjoint tori $T(\gamma) \subset \partial N$ and annuli $A(\gamma) \subset \partial N$, where the core of each component of $A(\gamma)$ is oriented. Also $\partial N - \mathring{A}(\gamma) \stackrel{def}{=} R(\gamma)$ is the disjoint union of oriented surfaces $R_-(\gamma)$ and $R_+(\gamma)$ where the orientations on $\partial R_-(\gamma)$ and $\partial R_+(\gamma)$ are induced from the orientations on the cores of $A(\gamma)$. Think of N as a manifold with corners $\partial A(\gamma)$, possessing a vector field defined near ∂N, pointing in along $R_-(\gamma)$, out along $R_+(\gamma)$ and tangent to $A(\gamma) \cup T(\gamma)$. A *product sutured manifold* is one of the form $R \times I$, where $A(\gamma) = \partial R \times I$. A *product disc* (resp. *product annulus*) is a properly embedded $I \times I \subset N$ (resp. $S^1 \times I \subset N$) such that $\partial I \times I$ are

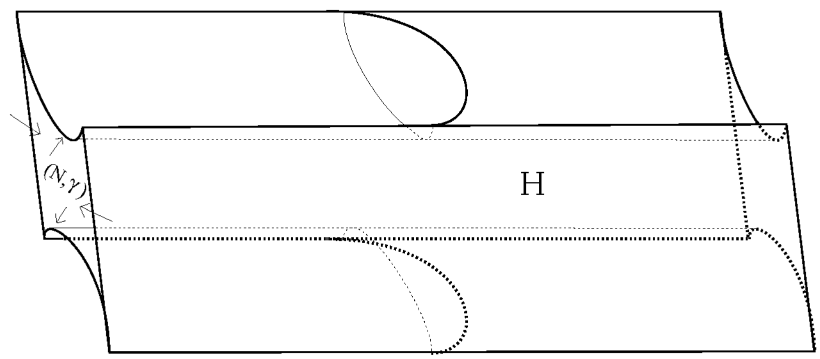


FIGURE 4.1

essential arcs in $A(\gamma)$ and $I \times \partial I \subset R(\gamma)$ (resp. $S^1 \times 0 \subset R_-(\gamma)$ and $S^1 \times 1 \subset R_+(\gamma)$.)

A *generalized cylindrical component* is an essential lamination ψ on a manifold F which is either the torus$\times I$ or the nonorientable I-bundle over the Klein Bottle, such that ∂F is a union of leaves of ψ, ψ has a transverse orientation which points in along ∂F and $\psi|\mathring{F} \neq \emptyset$.

Definition 4.2. We say that the essential lamination μ is obtained from the essential lamination λ by *evacuating the taut sutured manifold* (N, γ) if λ (resp. μ) is fully carried by a branched surface B (resp. H) such that

i) H is obtained by splitting a proper subbranched surface of B along (possibly zero) discs of contact.

ii) N is the union of the closed complementary regions of $M - H$ which contain deleted sectors of B. The branching of H induces the sutured structure on N and (N, γ) is taut. Finally N is π_1-injectively embedded in M.

iii) $\mathcal{G}(\lambda)$ lies in the union of components of $M - N(H)$ which are disjoint from N. $\mathcal{G}(\mu)$ is the disjoint union of $\mathcal{G}(\lambda)$ and the gut of the closed complementary regions of μ which contain N.

Remark 4.3. The motivating example is shown in Figure 1.2b (Example 1.4). Here H is the branched surface of Figure 4.1 which carries the lamination μ and B is obtained by attaching a single saddle shaped disc sector to H. The λ of Definition 4.2 is λ_{13} of Figure 1.2.

Theorem 4.4. *Let λ be an essential lamination in the closed orientable 3-manifold M with triangulation τ. Then at least one of the following occurs.*

1) After possibly splitting λ open along a finite number of leaves, λ is isotopic to a normal lamination.

2) There exists a normal essential lamination $\mathcal{L}$ in M such that $\mathcal{GN}(\mathcal{L}) > \mathcal{GN}(\lambda)$ and $\mathcal{L}$ is obtained from λ by first splitting along finitely many leaves, then evacuating a taut sutured manifold (N, γ) and finally isotopy.

3) λ has a generalized cylindrical component. In particular λ has a torus leaf and M is toroidal.

Idea of Proof. Given λ, let $\phi : M \times [-1, \infty) \to M$ be the infinite isotopy which attempts to normalize it. If this isotopy becomes constant after finite time, then 1) holds without any splitting of leaves. Otherwise as in §3 we obtain the limit branched lamination λ_∞, the branched lamination $\mathcal{L}_\infty$ obtained by deleting the Reeb laminations of λ_∞, and $\mathcal{L}$ the normal essential lamination obtained by splitting $\mathcal{L}_\infty$. We saw that the limit clumps, enduring planks and enduring walls gave rise to a *thick handle structure* on $N(\lambda_\infty)$, where $N(\lambda_\infty)$ (in the non-degenerate cases) is a fibred neighborhood of λ_∞. This structure in turn gave rise to a thick handle structures on $N(\mathcal{L}_\infty)$ and $N(\mathcal{L})$. In Step 2 we see that the isotopy ϕ is supported (for $t \geq t_3$) in a very small closed neighborhood N of the active closed complementary regions of $N(\mathcal{L}_\infty)$ and that N possesses a natural sutured manifold structure (N, γ). The crucial observation is that after time t_3 the isotopy pushes the leaves of λ_t only in one direction, thus $\lambda_t|N$ obtains a natural transverse orientation, even though λ itself may not be transversly orientable. It also suggests the fact that (N, γ) has a taut foliation. The leaves of $\lambda_t|N$ get spun around and around and get washed out in the limit, creating a new complementary region whose gut is equal to the gut of (N, γ). $N(\mathcal{L}_\infty)$ is a useful technical device, serving to separate the active complementary regions of $\mathcal{L}_\infty$ from the gut of λ_{t_3}.

For the remainder of §4 we will assume without loss of generality that $\mathcal{L}_\infty$ has no degenerate thick handles. This allows us to avoid the annoying but easily understood situation of a degenerate thick handle of $\mathcal{L}_\infty$ meeting an active closed complementary region of $\mathcal{L}_\infty$ on both sides. We now begin the proof of Theorem 4.4.

Step 1. Construction of the foliations $\mathcal{F}(N(\lambda_\infty))$ and $\mathcal{F}(N(\mathcal{L}_\infty))$.

The I-fibred structure on the nondegenerate thick handles of $N(\lambda_\infty)$ induces an I-fibering on the closed complementary regions of λ_∞ restricted to $N(\lambda_\infty)$. By filling in these I-bundles in the natural way (e.g.

by product foliations on the trivial I-bundles) we obtain a branched foliation $\mathcal{F}(N(\lambda_\infty))$ of $N(\lambda_\infty)$ which is tangent to $\partial_h N(\lambda_\infty)$ and transverse to $\partial_v N(\lambda_\infty)$. Define $\mathcal{F}(N(\mathcal{L}_\infty))$ to be the restriction of $\mathcal{F}(N(\lambda_\infty))$ to $N(\mathcal{L}_\infty)$. Of course these are foliations in the usual sense away from $\partial\bar{Y}$.

Step 2. Construction of the sutured manifold (N, γ).

Let V be the union of the active closed complementary regions of $\mathcal{L}_\infty$. V is a compact set by Lemma 3.30 ii). Let V^* denote the union of closed complementary regions of $\mathcal{L}$ which contain V. Each component of $s(\partial V)$ corresponds to a properly embedded annulus α_i in V^*, and V corresponds to a compact submanifold $V_1 \subset V^*$ which is bounded by these annuli. If some component of $\partial\alpha_i$ bounds a disc in a leaf of $\mathcal{L}$, then by essentiality of $\mathcal{L}$ so does the other component (on the same side of α_i) and the two discs together bound a $D^2 \times I$ with $\alpha_i = \partial D^2 \times I$ and $D^2 \times \mathring{I} \cap \mathcal{L} = \emptyset$. Let N_1 be the union of V_1 together with all such $D^2 \times I$ components. Again without loss of generality we will assume that N_1 is embedded in M for it is routine to extend to the degenerate case of ∂N_1 being immersed and nonembedded in M. Define a sutured structure (N_1, γ_1) on N_1 as follows. Let $\mathcal{A}_1$ denote the union of those annuli α_i lying in ∂N_1. Let $R_-(\gamma_1) = (\partial N_1) \cap J_{middle}$ and $R_+(\gamma_1) = \partial N_1 - int(R_-(\gamma_1) \cup \mathcal{A}_1)$. Here is another description of the sutured structure. To start with assume that $\mathcal{W} = \emptyset$ and no α_i bounds a $D^2 \times I$. Let Δ denote the cell structure on ∂V induced by the sides of the thick handles. Label a 2-cell $+$ if it corresponds to the $+$-side of a shaky wall or shaky plank, otherwise label the 2-cell $-$. Thus $s(\partial V)$ separates ∂V into two (not necessarily connected) surfaces of $-$ or $+$ type. $R_-(\gamma_1)$ (resp. $R_+(\gamma_1)$) consists of those components of $\partial N_1 - int(\mathcal{A}_1)$ which contain a surface of $-$ (resp. $+$) type. If $\mathcal{W} \neq \emptyset$, then all of $\partial\mathcal{W} \cap \partial N_1$ is labelled $-$. If some α_i's bound $D^2 \times I$'s, then in the natural way extend the sutured structure on V_1 to N_1.

We now isotope N_1 slightly to a manifold called N which among other things has the property that for t sufficiently large $M - \mathring{N}$ is t-stable. See Figure 4.2. By construction $R_-(\gamma_1)$ is stable, however no point of $R_-(\gamma_1)$ has a stable neighborhood. Our desired N (constructed in the next paragraph) is obtained by first thinking of N_1 as a closed complementary region of $\mathcal{L}_\infty$ and then pushing both $A(\gamma_1)$ and $R_+(\gamma_1)$ out a little bit. In what follows it is helpful to remember the following basic fact of foliation theory. If T is a compact oriented surface and $\mathcal{F}$ is a foliation defined in an open neighborhood X of $T \times 0 \subset T \times I$, such

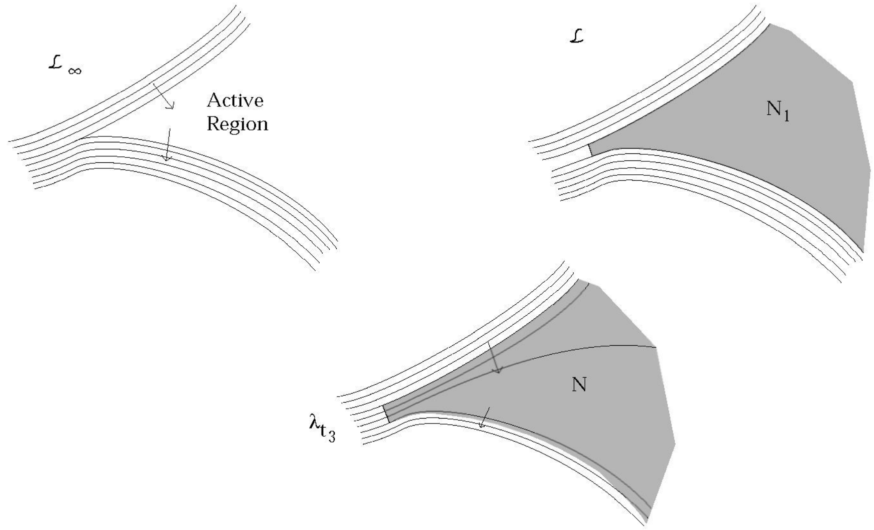


FIGURE 4.2

that $T \times 0$ is a leaf and $\mathcal{F}$ is transverse to $\partial T \times [0, \epsilon)$, then there exists a properly embedded surface $T' \subset X$ transverse to the I-fibres of $T \times I$ such that either T' is a leaf or T' is transverse to $\mathcal{F}$ except at isolated saddle singularites in $\mathring{T}'$. Furthermore each component of $\partial T'$ is either transverse to $\mathcal{F}$ or lies in a leaf of $\mathcal{F}$.

A small isotopy takes N_1 to a manifold N satisfying the following properties. $R_-(\gamma_1)$ is isotoped to a surface called $R_-(\gamma) = p(R_-(\gamma_1))$, where p is defined in 3.27. By Lemma 3.30, $N(\mathcal{L}_\infty)$ contains a neighborhood of $p(R_+(\gamma_1))$, thus we can isotope $\mathcal{A}_1$ to a union of thin annuli $\mathcal{A} = A(\gamma) \subset N(\mathcal{L}_\infty)$ which have the property that $\mathcal{F}(N(\mathcal{L}_\infty))$ is transverse to the I-fibres of $\mathcal{A}$ and if β is a component of $\partial\mathcal{A} - R_-(\gamma)$, then β is either transverse to or lies in a leaf of $\mathcal{F}(N(\mathcal{L}_\infty))$. Finally $R_+(\gamma_1)$ is isotoped to a surface $R_+(\gamma) \subset N(\mathcal{L}_\infty)$ such that either $R_+(\gamma)$ is a leaf of $\mathcal{F}(N(\mathcal{L}_\infty))$ or $R_+(\gamma)$ is transverse to $\mathcal{F}(N(\mathcal{L}_\infty))$ except at isolated saddle tangencies in $\mathring{R}_+(\gamma)$ with points of $\mathcal{F}(N(\mathcal{L}_\infty)) - \mathcal{L}_\infty$. Also there is a normal vector field to $R_+(\gamma)$ which is transverse to $\mathcal{F}(N(\mathcal{L}_\infty))$.

By construction there exists an integer t_3 such that $M - \mathring{N}$ is t_3-stable. q.e.d.

For the remainder of §4 we will assume that N is both connected and embedded in M. We will also assume that $\mathcal{W} = \emptyset$ and no α_i bounds a $D^2 \times I$. After reading the whole proof in this special case, it should be routine for the reader to promote our argument to the general case.

Step 3. If (N, γ) is a product sutured manifold (i.e., $N = R \times I$, with $A(\gamma) = \partial R \times I, R_-(\gamma) = R \times 0$ and $R_+(\gamma) = R \times 1$), then after splitting λ open along a finite number of leaves, a (finite) isotopy takes λ to a normal lamination.

Proof. First isotope λ to λ_t, for some $t \geq t_3$. By construction λ_t is transverse to the I-fibres of N near ∂N. We now show that we can isotope λ_t to μ, so that $\mu|N$ is transverse to the I-fibres of N. If $\partial R \neq \emptyset$, then there is a finite set (possibly empty) of pairwise disjoint product discs which decompose (N, γ) to a $D^2 \times I$. If C is a product disc, use the π_1-injectivity of leaves of λ_t, the second sentence and Lemma 2.3 to isotope λ_t to μ_1 so that $\mu_1|C$ is transverse to the I-fibres. Again use Packet Lemma 2.3 to isotope μ_1 within the $D^2 \times I$ to finally make it transverse to the I-fibres of N. If $\partial R = \emptyset$, then an inspection of $\lambda_t \cap \tau^2$ shows that $\partial N = R_-(\gamma)$ and hence (N, γ) is not a product sutured manifold.

Let μ_2 be the lamination obtained by splitting μ open along the leaf which contains $R_-(\gamma) = R \times 0$. By pushing up along the I-fibres isotope μ_2 to μ_3 via an isotopy supported in the union of N and a small neighborhood of $\partial R \times I$ so that $\mu_3|N$ consists of $R \times 0$ together with leaves that lie very close to $R \times 1$ and have tangent planes almost parallel to those of $R \times 1$. μ_3 is the desired normal lamination. q.e.d.

From now on we will assume that (N, γ) is not a product sutured manifold.

Remark 4.5. If the annulus K of Figure 1.6 was triangulated as in Figure 1.2 and appeared as part of a 2-subcomplex of the 2-skeleton of a triangulation, then the sutured manifold (N, γ) arising from the limit lamination λ_∞ would be a product. (Assuming no other nonnormality phenomena.) Also $\mathcal{L}$ would be obtained from $\mathcal{L}'$ by deleting a Reeb lamination.

Step 4. If N is an active region, then for $t \geq t_3$ the leaves of $\lambda_t|N$ are π_1-injective in M. The surfaces $R_+(\gamma), R_-(\gamma)$ are π_1-injective in M.

Proof. By the essentiality of $\mathcal{L}$ and construction, the surfaces $R_+(\gamma), R_-(\gamma)$ are π_1-injective in M. (Technical point: We included in N the $D^2 \times I$ components bounded by α_i's to obtain this π_1-injectivity condition.) Since the isotopy is supported in N for $t \geq t_3$, it suffices to establish Step 4 for $t = t_3$. Since each leaf of λ_{t_3} is π_1-injective, it suffices to show that if α is a embedded circle lying in a leaf of $\mathcal{F}(N(\mathcal{L}_\infty))|\partial N$, then either α is homotopically nontrivial in M or α bounds a disc in a leaf of $\mathcal{F}(N(\mathcal{L}_\infty))|N$. If α is homotopically trivial in ∂N, then being simple it bounds a disc D in ∂N. This would imply that either $\mathcal{F}(N(\mathcal{L}_\infty))|N$ is the product foliation $D^2 \times I$ or that there exists a center tangency of $\mathcal{F}(N(\mathcal{L}_\infty))$ with ∂N. Either case is a contradiction. If α is homotopically non trivial in ∂N, then it is homotopic to a nontrivial element of $\pi_1(R_+(\gamma))$ and hence is homotopically non trivial in M.

q.e.d.

Step 5. If N is an active region, then for $t \geq t_3, \lambda_t|N$ is transversely orientable. The isotopy can be chosen so that after time t_3, points only move in the direction of the transverse orientation.

Proof. If L is a non normal local disc leaf of λ_{t_3} in the 3-simplex σ, and L could be transformed to the (possibly disconnected) leaves K_1, K_2 via distinct boundary-compression (or even ∂-compression) operations, then K_1 and K_2 must both lie on the same side of L. This follows by enumerating the various possibilities for L using Lemmas 2.12 and 2.14 which assert that L is disjoint from an edge e of σ and can intersect any other edge at most 2 times. (One readily enumerates such L's by first labeling all the vertices of σ with x or y. Second labeling an edge 1 if it connects vertices labeled x and y otherwise labeling it 0 or 2, but label at least one edge 0. Third draw a circle ($= \partial L$) in $\partial\sigma$ which intersects the various edges the indicated number of times.) This assertion would be false if either of these conditions was false, e.g. consider either the almost normal octagon or any disc which hits an edge 3 times. Thus in a well defined manner we can transversely orient all nonnormal local leaves of λ_{t_3} so that the orienting vector points in the direction of normalizing operations.

Transversely orient each leaf of a non-enduring shaky wall w so that the orienting vector points into the $+$-side of w and out the other side which is necessarily an unlabeled side since w is non-enduring. Any other leaf of $\lambda_{t_3}|N$ can be normally isotoped into $R(\gamma)$. Transversely orient such a leaf consistantly with that of $R(\gamma)$, i.e., at points near of $R_-(\gamma)$ (resp. $R_+(\gamma)$) the orienting vectors should point into (resp. out of) N.

Given λ_{t_3}, the isotopy starts off by normalizing some nonnormal local leaves, i.e., boundary compressing say leaves in a 3-simplex σ. With the above conventions, one readily checks that the boundary compression can be executed so that points move infinitesimally only in the direction of the transverse orientation. Also (using (2.4)) if the isotopy gives rise to nonsimply connected local leaves, the compression needed to normalize these leaves can be forced to respect the transverse orientation. Finally if a leaf L is transformed to a leaf K under the isotopy and K is a disc, then the transverse orientation induced on K from L is consistant with the transverse orientation mandated in the previous two paragraphs. q.e.d.

Remark 4.6. The fact "for $t \geq t_3$ the isotopy pushes leaves in one direction only, (i.e., there is never backtracking)" is the most important technical observation of this paper.

Step 6. $\mathcal{G}(\lambda_{t_3}) \cap N = \emptyset$.

Proof. By construction for $t \geq t_3$, $\partial N \subset N(B_t)$ and $N(B_t) \cap \mathcal{G}(\lambda_t) = \emptyset$. For $t \geq t_3$ define the sutured manifold (A_t, α_t) where A_t denotes the closure of $N - N(B_t)$ and the transverse orientation on $\lambda_t | N$ induces a sutured structure (A_t, α_t) on A_t, i.e., $R_-(\alpha_t)$ (resp. $R_+(\alpha_t)$) consists of those $x \in \partial A_t$ where the transverse orienting vector points into (resp. out of) A_t. To complete the proof it suffices to show that (A_{t_3}, α_{t_3}) is a product sutured manifold, since $R(\alpha_{t_3}) \subset \lambda_{t_3}$ and $A(\alpha_{t_3})$ is a union of vertical fibres of $\mathcal{I}(\lambda_{t_3})$ implies that $A_{t_3} \subset \mathcal{I}(\lambda_{t_3})$ and hence $\mathcal{G}(\lambda_{t_3}) \cap N = \emptyset$.

No component R of $R_-(\alpha_{t_3})$ is closed else R would have a stable neighborhood, thereby contradicting Lemma 3.30.

If κ is a 2-simplex, then each arc of $\kappa \cap R_+(\alpha_{t_3})$ is a properly embedded arc in $R_+(\alpha_{t_3})$. Furthermore the collection of such arcs $a_1, \cdots, a_n$ coming from all the 2-simplices cuts $R_+(\alpha_{t_3})$ into a union of discs. We need to show that for all i, there exists maps $f_i : I \times I \to A_{t_3}$ such that $f_i | I \times 0$ is an embedding onto a_i, $f_i | \partial I \times I$ are embeddings onto I-fibres of $\mathcal{I}(\lambda_{t_3})$ and $f_i | I \times 1 \subset R_-(\alpha_{t_3})$. The desired homotopy of a_1 is suggested in Figure 4.3. By sliding the ends of a_1 up and off the interstitial fibre through ∂a_1 we obtain a homotopy of a_1 to an arc $b_1 \subset A_{t_3}$ with endpoints in $R_-(\alpha_{t_3})$. We now show how to homotope b_1 into $R_-(\alpha_{t_3})$ rel ∂b_1. Since b_1 lies in an active region and our infinite isotopy is a composition of normal isotopies, compressions and boundary compressions, there must be a time $s_1 \geq t_3$ when b_1 is part of a compression or boundary compression. At that moment one sees how to homotope b_1 into $R_-(\alpha_{s_1})$. By playing this homotopy backwards we see how to homotope b_1 into $R_-(\alpha_{t_3})$.

By the loop theorem and the usual innermost disc arguments there exists pairwise disjoint product discs $D_1, \cdots D_m \subset (A_{t_3}, \alpha_{t_3})$ which in-

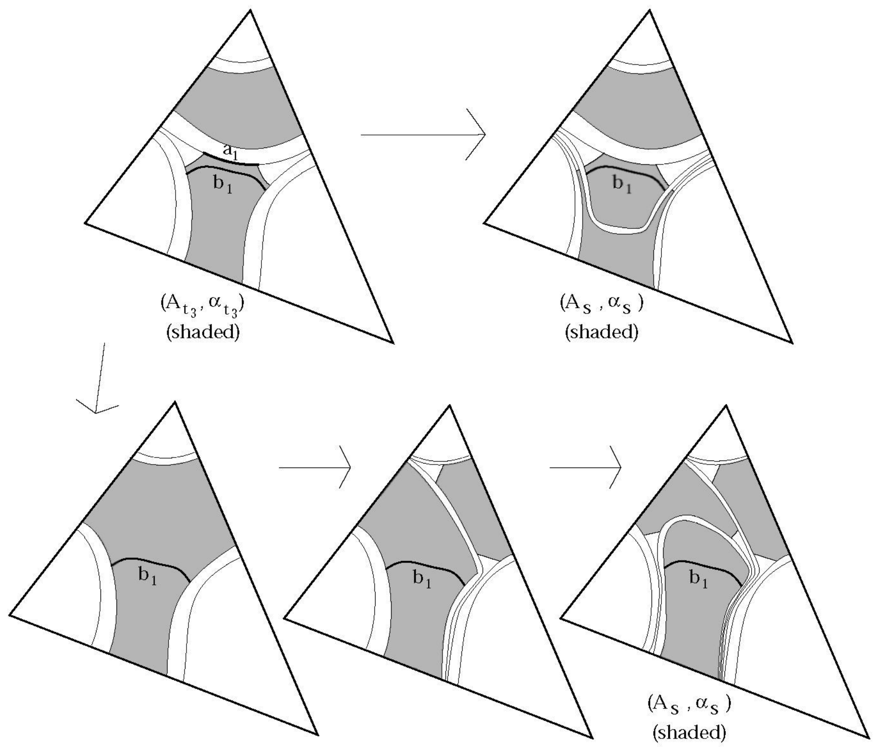


FIGURE 4.3

tersect $R_+(\alpha_{t_3})$ in $c_1, \cdots, c_m$ where $R_+(\alpha_{t_3}) - \cup c_i$ is a union of discs. Since A_{t_3} is irreducible, $R_-(\alpha_{t_3})$ is π_1-injective in A_{t_3} and no component of $R(\alpha_{t_3})$ is closed, these discs cut (A_{t_3}, α_{t_3}) into a union of product sutured manifolds $D^2 \times I$ and hence (A_{t_3}, α_{t_3}) itself is a product, [11].

q.e.d.

Step 7. Either $\lambda|N$ is a generalized cylindrical component or (N, γ) is not an I-bundle.

Proof. A connected non-product sutured manifold can only be an I-bundle if $A(\gamma) = \emptyset$. By construction $R_+(\gamma) \neq \emptyset$ implies that $A(\gamma) \neq \emptyset$ and hence if (N, γ) is a non-product I-bundle, then $R_-(\gamma) = \partial N$. By Steps 5-6, $\mathcal{F}(N(\mathcal{L}_\infty))$ is defined on all of N and is transversely orientable, such that the orientation points in along $R_-(\gamma)$. Therefore $0 = \chi(N) = \chi(R_-(\gamma))$ and hence $R_-(\gamma)$ is a union of tori. Finally $\lambda_{t_3}|\overset{\circ}{N} \neq \emptyset$, since N contains an active region. Therefore if N is a nontrivial I-bundle, then $\lambda|N$ is a generalized cylindrical component.

q.e.d.

From now on we also assume that λ contains no generalized cylindrical component.

Step 8. $\mathcal{GN}(\mathcal{L}) \geq \mathcal{GN}(\lambda)$ with equality holding if and only if (N, γ) is an I-bundle.

Proof. The closed complementary region $\mathcal{X}$ of $\mathcal{L}$ which contains N_1 lies in the union of N_1 and $N(J)$. Therefore, the non-I-bundle closed complementary regions of $\mathcal{L}$ are of two mutually disjoint types, the $\mathcal{X}$ which contains N_1 and those which contain closed complementary regions of $N(\mathcal{L}_\infty) - N$. Let $\mathcal{C}$ denote the collection of closed complementary regions of the second type. The fiberlike structure on $N(\mathcal{L}_\infty)$ together with $\mathcal{I}(\lambda_{t_3})$ induce an I-bundle structure $\mathcal{I}$ on all but a compact set C of $\mathcal{C}$. Combining this with Step 6 we conclude that $C = \mathcal{G}(\lambda_{t_3})$. To conclude the proof that $C = \mathcal{G}(\mathcal{C})$ we must show that no component $\mathcal{B}$ of $\mathcal{I}$ is an I-bundle over the annulus, Mobius band or disc. Suppose that such a $\mathcal{B}$ exists. Since $\mathcal{B} \cap C$ is t-stable, $t \geq t_3$, the proof of Lemma 3.21 shows that $\mathcal{B}$ is t-stable for t sufficiently large, say $t \geq s$. Since every product dics of C is ∂-parallel and $\partial_v(\mathcal{B}) \subset \mathcal{I}(\lambda_s)$, it follows that $\mathcal{B}$ is a component of $\mathcal{I}(\lambda_s)$, which is a contradiction.

Thus distinct components of $\mathcal{G}(\lambda)$ correspond to distinct components of $\mathcal{G}(\mathcal{L})$ and these components are distinct from the closed complementary region $\mathcal{X}$. Since by construction each component of $A(\gamma_1)$ is essential in $\mathcal{X}$ and $cl(\mathcal{X} - N_1)$ is an I-bundle extending an I-bundle structure on $A(\gamma_1)$ it follows that $\mathcal{X}$ is an I-bundle if and only if (N_1, γ_1) is an I-bundle if and only if (N, γ) is an I-bundle. q.e.d.

Remark 4.7. The proof of Step 8 shows that $\mathcal{G}(\mathcal{L})$ is diffeomorphic to (as a manifold with corners) to the disjoint union of $\mathcal{G}(\lambda)$ and the gut of the closed complementary region of $\mathcal{L}$ which contains N.

Step 9. (N, γ) is taut.

Proof of Step 9. It suffices to show that N is irreducible and $R(\gamma)$ is π_1-injective and Thurston norm minimizing as an element of $H_2(N, A(\gamma))$. Irreducibility follows from the π_1-injectivity of leaves of $\lambda_{t_3}|N$ in M and the essentiality of λ_{t_3}. The π_1-injectivity of $R(\gamma)$ follows from Step 4. In particular this implies that if some component of $R(\gamma)$ is a disc D, then (N, γ) is the product sutured manifold $(D^2 \times I, \partial D^2 \times I)$. Now assume that no component of $R(\gamma)$ is a disc. To complete the proof we need to show that if T is an embedded incompressible surface in N, such that $\partial T \subset A(\gamma)$ and $[T] = [R(\gamma)] \in H_2(N, A(\gamma))$, then $\chi(R(\gamma)) \geq \chi(T)$.

In the usual way construct a partial foliation $\mathcal{F}'$ on $M - \mathcal{G}(\lambda_{t_3})$ by filling in $\mathcal{I}(\lambda_{t_3})$. It follows from Steps 5-6 that $\mathcal{F} = \mathcal{F}'|N$, is defined on all of N, is transversely oriented and is tangent to $R_-(\gamma)$, transverse to $A(\gamma)$ and almost tangent to $R_+(\gamma)$. Additionally, the normal vectors to $\mathcal{F}$ point in along $R_-(\gamma)$, out along $R_+(\gamma)$ and are tangent along $A(\gamma)$.

Isotope T within N so that each component of ∂T is either a leaf of $\mathcal{F}|A(\gamma)$ or is transverse to $\mathcal{F}|A(\gamma)$. Since the leaves of $\mathcal{F}$ are π_1-injective we can apply the Roussarie - Thurston [27], [30] isotopy to transform each component of T to either a leaf of $\mathcal{F}$ or a surface transverse to $\mathcal{F}$ except at isolated saddle and circle tangencies. It is crucial to observe that the isotopy never pushes T outside of N. Indeed, as discussed in [12], a partially isotoped T called T_s can be viewed as a compact submanifold of T together with a finite number of subsurfaces, called *plateaus*, which lie in leaves of $\mathcal{F}$. If the isotopy pushed T out of N, there would be a moment where $T_s \subset N$, and a plateau of T_s would be tangent to an interior point of $R(\gamma)$ which contradicts the fact that $R_+(\gamma)$ is transverse to $\mathcal{F}$ except at isolated saddle tangencies. Finally by considering the Euler class of $\mathcal{F}$, Thurston's argument (Corollary 2, p. 119 [32]), shows that $\chi(R(\gamma)) \geq \chi(T)$ and hence $R(\gamma)$ is Thurston norm minimizing. q.e.d.

Step 10. The essential lamination $\mathcal{L}$ is obtained (up to isotopy) from the essential lamination λ by evacuating the taut sutured manifold

(N, γ).

Proof. We check that i)-iii) of Definition 4.2 hold.

i) By construction $\mathcal{L}$ is carried by the branched surface H obtained by splitting open the branched surface corresponding to the clumps, enduring planks and enduring walls of λ_{t_3}.

ii) All but π_1-injectivity of N follows by construction. Again by construction, (N, γ) is isotopic to (N_1, γ_1) which is a closed complementary region of a fibred neighborhood of H. N_1 has the feature that if $\mathcal{X}$ is the closed complementary region of $\mathcal{L}$ which contains N_1, then $cl(\mathcal{X} - N_1)$ has no $D^2 \times I$ components. It follows from [18] that complementary regions of essential laminations in M are π_1-injective in M. Thus N_1 and hence N is π_1-injective in M.

iii) This follows by Step 8.

Suppose that λ has no cylindrical components. If (N, γ) has multiple components then the above argument shows that each component corresponds to either splitting of leaves or sutured manifold evacuation, depending on whether or not the component is a product sutured manifold. Thus up to isotopy, λ can be transformed into a normal lamination $\mathcal{L}$ by first splitting along finitely many leaves and then perfoming $N < \infty$ sutured manifold evacuations. Thus $\mathcal{GN}(\mathcal{L}) \geq \mathcal{GN}(\lambda) + N$ and the proof of Theorem 4.4 is complete. q.e.d.

Corollary 4.8. *Let λ be a nowhere dense essential lamination in the closed orientable 3-manifold M with triangulation τ. Then λ can be transformed into a normal essential lamination μ by doing or skipping in turn the following operations 1) - 4).*

1) Deleting the interior of finitely many generalized cylindrical components.

2) Splitting open along a finite number of leaves.

3) Evacuating a taut sutured manifold (N, γ).

4) Isotopy.

Proof. Being π_1-injective and embedded, the torus leaves of λ can be partitioned into finitely many parallel families [19]. Thus one can obtain an essential sublamination μ_0 of λ without generalized cylindrial components, by deleting the leaves in the interior of finitely many generalized cylindrical components of λ. Corollary 4.8 now follows by applying the proof of Theorem 4.4 to μ_0. q.e.d.

Remark 4.9. The proof of Theorem 4.4 shows that one can permute the above operations 1) - 3).

Lemma 4.10. *If μ is obtained from λ by deleting a generalized cylindrical component, then $\mathcal{GN}(\mu) = \mathcal{GN}(\lambda)$.* q.e.d.

Problem 4.11. Classify the evacuable sutured manifolds (N, γ) which arise from the normalization procedure of Theorem 4.4. Are they all depth-1 sutured manifolds?

5. The gut number

Definintion 5.1. Define

$$\mathcal{GN}(M) = max\{\mathcal{GN}(\lambda) | \lambda \text{ is essential in } M\}$$

to be the *gut number* of the closed orientable 3-manifold M. The essential lamination λ in M is said to have *maximal gut number* if $\mathcal{GN}(\lambda) = \mathcal{GN}(M)$.

Theorem 5.2. $\mathcal{GN}(M) < \infty$.

Proof. Fix a triangulation τ on M. By Theorem 4.8 and Lemma 4.9, if λ is an essential lamination in M, then there exists a normal essential lamination $\mathcal{L}$ on M such that $\mathcal{GN}(\mathcal{L}) \geq \mathcal{GN}(\lambda)$. Now Kneser's argument [23], [20] shows that $\mathcal{GN}(\mathcal{L}) \leq 6(|\text{3-simplices in } \tau|)$. q.e.d.

Corollary 5.3. *If M has an essential lamination, then it has an essential lamination of maximal gut number.* q.e.d.

Corollary 5.4. *If λ is a maximal gut number essential lamination in an atoroidal 3-manifold with triangulation τ, then after possibly splitting along finitely many leaves, λ is isotopic to a normal lamination.*

q.e.d.

Corollary 5.5. *If M is laminar, then it has an essential lamination λ such that for any triangulation τ on M, λ is isotopic to a lamination normal with respect to τ.*

Proof. Let μ be a maximal gut number essential lamination in M without generalized cylindrical components. Let $\tau_1, \tau_2, \cdots$ be a series of triangulations which contains a representative of each isotopy class of triangulation on M. By Theorem 4.4, after possibly splitting along finitely many leaves, μ can be isotoped to a τ_i-normal lamination. Let $\{F_j\}$ be the countable union of these finite sets of leaves. Let λ be the lamination obtained by splitting μ along $\{F_j\}$. q.e.d.

In [17] we use Theorem 5.2 to obtain the following result which generalizes the similar result for Haken manifolds due to Johannson [22].

Theorem 5.6 [17]. *If the atoroidal 3-manifold M contains a genuine essential lamination, then the mapping class group (the group of homeomorphisms modulo isotopy) of M is finite.* q.e.d.

Remark 5.7. See Remark 3.6 i) [15] for another possible application of Theorem 5.2. This paper was motivated by that application.

6. Local regularity of essential laminations and taut foliations

Definition 6.1. Let τ be a triangulation on the 3-manifold M. If S is a normal immersion of a compact surface S whose boundary is a union of normally immersed curves, then define length(∂f) to be the number of 1-cells in the induced triangulation on ∂S, and area(f) to be the number of 2-cells in the induced cellulation on S.

The following definition is meant to locally describe the types of branched surfaces B_∞ that can carry our limit laminations λ_∞.

Definition 6.2. The branched surface B in the 3-manifold M with triangulation τ is said to be a *standard normal branched surface* if it satisfies the following conditions.

i) B is transverse to the 0, 1 and 2-skeleta and $\partial B \subset \partial M$.

ii) If σ is a 3-simplex, then each component of $\sigma \cap B$ is a 3-simplex local branched surface. (See 3.12.)

The branched surfaces B_{t_2} are the motivating examples of standard branched surfaces which are defined as follows. See Remark 6.4.

Definition 6.3. The branched surface B' in the 3-manifold M with triangulation τ is said to be a *standard branched surface* if it satisfies the following conditions.

i) B' is transverse to the 0, 1 and 2-skeleta and $\partial B' \subset \partial M$.

ii) B' is the union of a standard normal branched surface B and finitely many discs $D_1, \cdots, D_m$ such that each D_i is either normal or a properly embedded disc in a 3-simplex σ_i such that ∂D_i is transverse to σ_i^1, ∂D_i crosses each edge of σ_i at most twice, ∂D_i crosses some edge of σ_i exactly twice and ∂D_i misses some edge of σ_i. If D_i and D_j are embedded in the same 3-simplex σ and D_i is not normal, then ∂D_i

is not strongly normally isotopic to ∂D_j. (See 0.3.) Furthermore D_i lies to one side of D_j in σ, although D_i and D_j may coincide along a compact set. The various discs are identfied with each other and B in the standard manner. (See Remark 6.4 ii).)

iii) There exists a neighborhood U of τ^1 such that $U \cap B = U \cap B'$.

Remark 6.4. i) As in 3.18, $B_t|\sigma$ is obtained by identifying finitely many discs, one disc for each equivalence class of walls of $\lambda_t|\sigma$. If $t \geq t_2$ and B_t^* denotes the branched surface obtained by just using the discs arising from the enduring walls in the various 3-simplices, then $B_t^* = B_\infty$ is a standard normal branched surface. The $D_1, \cdots, D_n$ are the various discs corresponding to the equivalence classes of non enduring normal walls (at most 2 per 3-simplex) and the equivalence classes of the nonnormal walls.

ii) The local models for the identifications of 6.3 ii) are given by the possibilities in the passage of $B_{t_2}^*|\sigma$ to B_{t_2}.

Theorem 6.5. *Let M be a closed orientable atoroidal 3-manifold. The collection of nowhere dense essential laminations on M is carried, up to isotopy, by finitely many essential branched surfaces.*

Proof of Theorem 6.5. By replacing isolated leaves by Cantor sets of leaves it suffices to consider laminations without isolated leaves. Fix a triangulation τ on M. To avoid notation such as $\lambda_{t_2(\lambda)}$, we abuse notation by letting t_2 denote the time that an isotoped lamination satisfies the properties described in 3.16 i), irrespective of the lamination in question. We will assume that $\mathcal{L}' = \mathcal{L}$ for again the extension to the general case is routine. Thus B_∞ carries the essential lamination $\mathcal{L}$.

Step 1. There are only finitely many possibilities for B_∞.

Proof of Step 1. The triangulation τ induces a cellulation Δ on the complementary space $C(N(B)) = M - \mathring{N}(B)$ of a standard normal branched surface B. If a 3-cell d of Δ has the property that $d \cap \partial_h N(B)$ equals two normally isotopic discs, then d has a natural I-bundle structure. The union of all such cells induces an I-bundle structure on a subset $J(B)$ of $C(N(B))$. Now let B be a branched surface which arises from an infinite normalizing isotopy of the essential lamination λ. Such a branched surface will be called a *B_∞-branched surface.*

If X is a component of $J(B)$ let $Z \subset C(N(B))$ be the maximal connected space which contains X, has an I-bundle structure extending that of X, $\partial Z \subset \partial X \cup \partial_h N(B)$ and $\partial_v Z \subset \partial_v X$. Here $\partial_v X \overset{def}{=} \partial X -$

$\text{int}(\partial_h(N(B)) \cap X)$, and $\partial_v Z$ is defined similarly. By thickening near finitely many I-fibres in $X \cap \tau^1$ we will assume that Z is an I-bundle over a surface Z_0 (rather than a possibly pinched surface). We now show that Z_0 is either a disc, annulus or Mobius band. Since B carries an essential lamination and M is atoroidal, Z is not an I-bundle over a closed surface of non-negative Euler characteristic. If $\chi(Z_0) < 0$, then using the essentiality of λ and (2.3) , one can isotope λ_{t_2} to a lamination μ which is carried by a standard branched surface C disjoint from $\mathring{Z}$. Furthermore, C is obtained from B by the standard splitting, isotopy, and squeezing along bigon operations. Also both C and B have the same underlying standard normal branched surface. It follows that Z can be incorporated into the interstitial bundle $\mathcal{I}(\mu)$ and that the I-bundle structure on X can be made compatible with $\mathcal{I}(\mu)$. By squeezing C along X one obtains a branched surface D carrying μ which satisfies (2.2) and (2.3) and has fewer clumps than C. This contradicts (2.4).

To complete the proof of Step 1 it suffices to show that $C(N(B))$ has bounded combinatorial complexity. If t is the number of tetrahedra of τ, then there are less than 6^t non-I-bundle 3-cells of $C(N(B))$ and each has small combinatorial complexity. Thus we need to show that $\text{area}(J_0)$ is uniformly bounded where J_0 is the 0-section of $J(B)$. If not there is a sequence of essential laminations $\lambda^1_{t_2}, \lambda^2_{t_2}, \cdots$, and branched surfaces $F^1_{t_2}, F^2_{t_2}, \cdots, F^1_\infty, F^2_\infty, \cdots$ which are the corresponding branched surfaces arising at times t_2 and ∞ such that $\text{area}(J^i_0) \to \infty$. Here J^i_0 is the 0-section of $J(F^i_\infty)$. By the previous paragraph, each component of J^i_0 is either a disc with holes or Mobius band with holes. In either case the number of boundary components is bounded by $4 \cdot 6^t$, furthermore $\text{length}(\partial J^i_0) \leq 4 \cdot 6^t$. By the Plante [26] argument, after passing to a subsequence the J^i_0's converge to an embedded measured Euler characteristic 0 normal lamination ϕ. An analysis of ϕ and J^i_0 for i large shows that one can obtain a branched surface E^i carrying an isotoped $\lambda^i_{t_2}$ satisfying (2.2) and (2.3) but E^i has fewer clumps than $F^i_{t_2}$. Again we obtain a contradiction to (2.4). Here E^i is more or less obtained from $F^i_{t_2}$ by either unrolling a Reeblike subdisc of J^i_0 or unrolling a (monogon with long tail)$\times S^1 \subset J^i_0$. q.e.d.

Let N denote the maximal number of 3-cells that can arise in the cellulation Δ of a $C(N(B))$, where B is a B_∞-branched surface.

Step 2. There are only finitely many possibilities for B_{t_2}.

Proof of Step 2. Remark 6.4 explains how B_{t_2} is obtained from

B_∞ by adding at most $2t + 1000N$ sectors, where the t sectors arise from the non-enduring walls of λ_{t_2} and the other sectors arising from the nonnormal walls. Since each such sector is of uniformly bounded complexity, Step 2 follows. q.e.d.

Step 3. If C is a B_{t_2} branched surface in M, then there exists essential branched surfaces $C_1, \cdots, C_m$, such that every λ_{t_2} carried by C is carried by some C_i.

Proof of Step 3. By hypothesis C carries no S^2 and fully and compatibly carries an essential lamination μ. By construction and definition any λ_{t_2} essential lamination carried by C is compatibly carried by C. (Recall Definition 2.6.) The branched locus of C is a compact 1-complex b. Let $n(b(C)) = |b(C) - b(C)^0|$, where $b(C)^0$ is the set of nonmanifold points. The branched surface C might fail to be essential because it contains discs of contact, or monogons or it might carry a torus bounding a solid torus. If an essential lamination μ' is compatibly carried by C and C_1 is obtained by splitting C along a disc of contact, then μ' is compatibly carried by C_1 and $n(b(C_1)) < n(b(C))$. It follows by induction on $n(b(C))$ that there exists a branched surface D such that D has no discs of contact, and every essential lamination compatibly carried by C is compatibly carried by D. Each lamination carried by D is fully carried by one of finitely many subbranched surfaces. Passing to a subbranched surface neither increases $n(b(D))$ and nor destroys compatibility. By repeatedly splitting along contact discs and passing to subbranched surfaces we conclude by induction on $n(b(C))$ that the λ_{t_2} laminations are fully and compatibly carried by one of finitely many branched surfaces without discs of contact. The operations of splitting along a disc of contact or passing to a subbranched surface does not increase the number of complementary regions of the branched surface. By (2.3) a branched surface which compatibly and fully carries an essential lamination has no monogons. Now suppose that D carries a torus bounding a solid torus V. Since D has neither monogons nor discs of contact, each component of $C(N(D)) \cap V$ is a $D^2 \times I$. Thus by squeezing all but one of the $D^2 \times I$ components of $D|V$ and "rolling up" $D|V$ we obtain a new branched surface D_1 such that $D_1|(M - \mathring{V}) = D|(M - \mathring{V})$ and $D_1|V$ is the standard Reeb branched surface. Also any lamination compatibly and fully carried by D is compatibly and fully carried by D_1. Now consider a lamination μ fully and compatibly carried by D_1. The effect on D_1 by suitably unrolling" μ inside V is to create a new branched surface D_2 compatibly and fully carrying μ which has

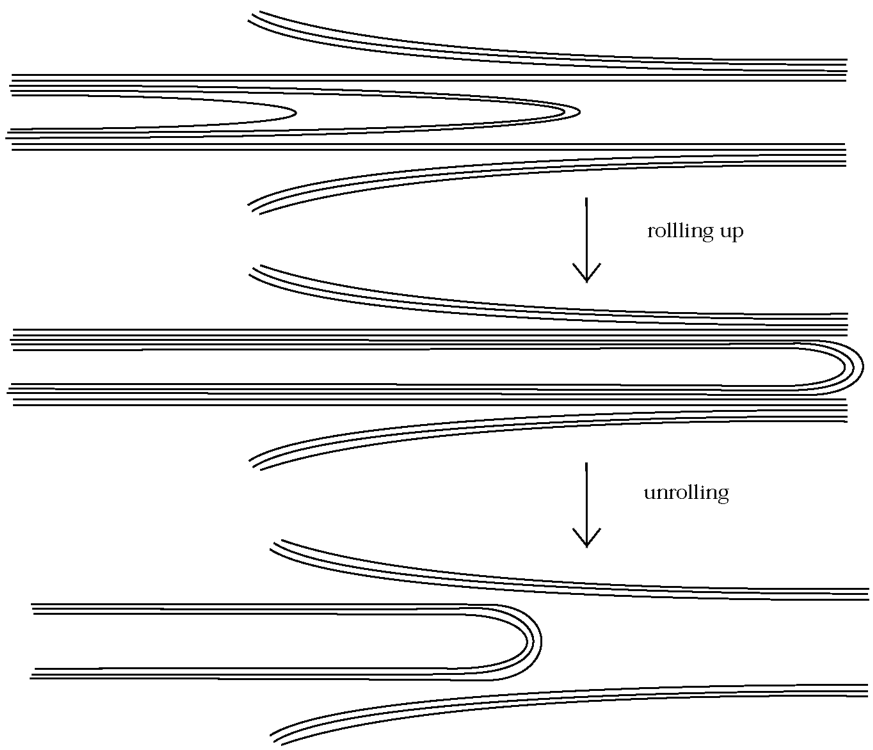


FIGURE 6.1

the Reeb branched surface broken open and

$$|C(N(D_2))| < |C(N(D_1))| \leq |C(N(D))| \leq |C(N(C))|.$$

See Figure 6.1. As shown in that figure, one may need to rechoose the complementary region of μ within the solid torus to exactly meet a complementary region of μ outside of the torus. Since D is of bounded combinatorial complexity, the number of such branched surfaces D_2 that can arise in this manner is finite. Thus Step 3 follows by induction on pairs of nonnegative integers $(|C(N(C))|, n(b(C)))$ lexicographically ordered. q.e.d.

Definition 6.6. The foliation $\mathcal{F}$ is said to be *carried* by the branched surface B, if there exists a lamination λ obtained by splitting $\mathcal{F}$ along a countable number of leaves and λ is carried by B. Define a *foliation branched surface* to be a branched surface such that each closed complementary region of a fibred neighborhood is a product sutured manifold.

Proposition 6.7. *If B is a foliation branched surface in the closed orientable 3-manifold M, then there exists an integrable plane field $\mathcal{D}_B \subset M$ whose integral surfaces are smooth and consist of B and the leaves of a branched product foliation $\mathcal{F}(X)$ on the closed complement X of B. I.e., if $y \in B$, then $\mathcal{D}_B(y) = T_y(B)$. If X is the closed complement of B, then $X = S \times [0,1]/\sim$ where S is a compact surface and $(x,t) \sim (y,s)$ if and only if $x = y$ and either $t = s$ or $x \in \partial S$. The foliation $\mathcal{F}(x)$ is induced from the product foliation on $S \times [0,1]$.* q.e.d.

Example 6.8. Here is an example in dimension 2, which provides the idea for the proof of Proposition 6.7. The train track T of Figure 6.2 is a foliation branched surface on the torus. The closed complement of T is a bigon. Filling in the bigon with a branched product foliation gives rise to the branched foliation of Figure 6.2 whose tangent plane field is a nonLipshitz, tangentially smooth, integrable line field on the torus. (A standard result in differential geometry asserts that there exists a unique integral curve through any point of a Lipshitz line field.)

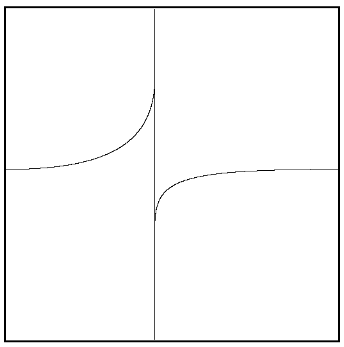
A model for the nonuniquely integral points is given by the following vector field on $\mathbb{R}$. Take a vector field consisting of unit tangent vectors to the following 3 families of curves; $g_v(x) = v, v \in (-\infty, 0]; g_u(x) = f(x) + u$, where $u \in [0,\infty)$; and $f_s(x) = sf(x)$, where $s \in [0,1]$ and where

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

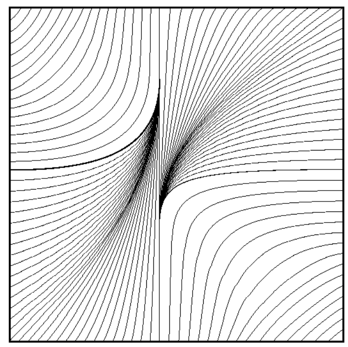
Lemma 6.9. *A foliation branched surface B has a sequence of nested fibred neighborhoods $N_t(B)$ such that for each $t, \partial_h N_t(B)$ is an integral surface of $\mathcal{D}_B$ and each vertical fibre of $N_t(B)$ is transverse to $\mathcal{D}_B$. Finally $\cap_t N_t(B) = B$. (See Figure 6.3.)* q.e.d.

Definition 6.10. Let $\mathcal{F}(\mathcal{D}_B)$ denote the branched foliation consisting of integral surfaces of $\mathcal{D}_B$. The foliation $\mathcal{F}$ is *strongly carried* by the foliation branched surface B if for some fibred neighborhood $N(B), \mathcal{F}|(M - \mathring{N}(B)) = \mathcal{F}(\mathcal{D}_B)|(M - \mathring{N}(B))$ and the vertical fibres of $N(B)$ are transverse to $\mathcal{F}$.

Proposition 6.11. *If the foliation $\mathcal{F}$ in the compact 3-manifold M is strongly carried by a foliation branched surface B with associated plane field $\mathcal{D}_B$, then for every $\epsilon > 0$ there exists a smooth ambiant isotopy of M taking $\mathcal{F}$ to a foliation also called $\mathcal{F}$ such that if $x \in M$*



Train Track on Torus



A non Lipshitz tangentially smooth line field

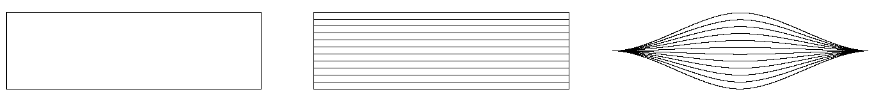


Figure 6.2

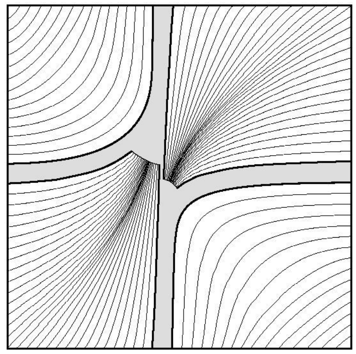


FIGURE 6.3

the angle between the unoriented tangent plane to $\mathcal{F}$ at x is ϵ-close to the plane $\mathcal{D}_B(x)$. q.e.d.

The following result roughly says that any two linear foliations on the torus can be isotoped so that their tangent line fields are ϵ-close.

Corollary 6.12. *If $\mathcal{F}_1, \mathcal{F}_2$ are linear foliations on the torus T and $\epsilon > 0$, then there exist foliations $\mathcal{G}_1, \mathcal{G}_2$ respectively isotopic to $\mathcal{F}_1, \mathcal{F}_2$ such that for each $x \in T$ the tangent line field of $\mathcal{G}_1$ at x is ϵ-close to the tangent line field of $\mathcal{G}_2$ at x.*

Proof. If the foliations have slopes s_1 and s_2, then after applying an element A of $\mathrm{SL}(2, \mathbb{Z})$ we can assume that each of s_1 and s_2 is fully carried by the branched surface of Figure 6.2. Let r_1 and r_2 be the new slopes with corresponding foliations $\mathcal{F}_1, \mathcal{F}_2$. For a given δ one can individually isotope $\mathcal{F}_1, \mathcal{F}_2$ to $\mathcal{G}_1, \mathcal{G}_2$ so that the angles between the tangent line fields of $\mathcal{G}_1, \mathcal{G}_2$ are δ close to that of the branched foliation of Figure 6.2. Since the linear map A boundedly distorts angles, the line fields of the foliations $A^{-1}(\mathcal{G}_1), A^{-1}(\mathcal{G}_2)$ are ϵ-close provided that δ is sufficiently small. q.e.d.

Theorem 6.13. *Up to isotopy any Reebless foliation $\mathcal{F}$ on a closed atoroidal orientable 3-manifold M is strongly carried by one of finitely many foliation branched surfaces.*

Proof. Let λ be an essential lamination carried by a foliation branched surface B satisfying (2.3) such that λ is obtained by splitting $\mathcal{F}$ along finitely many leaves. Indeed, if M is covered by n foliation charts, then any finite set of leaves of $\mathcal{F}$ whose union meets these charts will suffice. Since Reebless foliations on atoroidal 3-manifolds have no torus leaves we can apply the machinery of this paper to show that λ is carried by a B_{t_2} branched surface. Recall that Step 2 of the proof of Theorem 6.5 shows that the number of such surfaces is finite. To check that such a branched surface B^* is actually a foliation branched surface, note that B is a foliation branched surface and that, up to isotopy, B^* is obtained from B by finitely many λ-splittings and squeezing along product discs.

Let L denote the union of leaves on which $\mathcal{F}$ was split. We can assume that L has trivial holonomy since by [8] such leaves are dense in M.

We will now show that after isotopy $\mathcal{F}$ is strongly carried by B^*. First observe that $B = B_0$ has the property that there is a small product neighborhood $T_0 \times I$ of a compact subsurface $T_0 \subset L$ such that T_0 is identified with $T_0 \times 1/2, \mathcal{F}|T_0 \times I$ is the product foliation and B_0 has a fibred neighborhood $N(B_0)$ such that $\partial_h N(B_0) = T_0 \times \partial I, \partial_v N(B_0) = \partial T_0 \times I$ and $F|N(B_0)$ is transverse to the I-fibres of $N(B_0)$. Now if B_1 is obtained from B_0 by λ splitting, then $N(B_1)$ is obtained from $N(B_0)$ by deleting a compact I-bundle. Since L has no holonomy, we can enlarge T_0 to a compact surface $T_1 \subset L$, and shrink I to $I_1 \subset I$ such that $\mathcal{F}|T_1 \times I_1$ has the product foliation, $\partial_h N(B_1) = F \times \partial I_1, \partial_v N(B_1) = \partial F \times I_1$ and $F|N(B_1)$ is transverse to the I-fibres of $N(B_0)$. A similar statement holds if B_1 was obtained by squeezing B_0 (except that $T_1 \subset T_0$). Thus up to isotopy of $\mathcal{F}$ there exists a compact suface $T^* \subset L$, a product neighborhood $T^* \times I^*$ of T^* with T^* identified with $T^* \times 1/2$ such that $\mathcal{F}|T^* \times I^*$ is the product foliation etc. By [1] one can isotope $\mathcal{F}$ to have the above properties such that if X is the closed complement of $N(B)$, then $\mathcal{F}|X = \mathcal{F}(\mathcal{D}_B)|X$. q.e.d.

Definition 6.14. The transversely orientable foliations $\mathcal{G}$ and $\mathcal{F}$ in the Riemannian 3-manifold M are said to be *ϵ-coarse isotopic* if $\mathcal{F}$ and $\mathcal{G}$ can be respectively isotoped to foliations $\mathcal{F}^*$ and $\mathcal{G}^*$ such that for each $x \in M$ the angle between the transverse orienting orthogonal vectors (to $\mathcal{F}^*$ and $\mathcal{G}^*$) is less than ϵ. We say that $\mathcal{G}$ and $\mathcal{F}$ are *coarse isotopic* if $\epsilon < \pi$. I.e., for each $x \in M$ either $\mathcal{G}$ is transverse to $\mathcal{F}$ at x or $\mathcal{G}$ is tangent to $\mathcal{F}$ at x and at x the normal orientations agree.

The next two results follow directly from Proposition 6.11 and Theorem 6.13.

Theorem 6.15. *Given a closed, orientable, atoroidal 3-manifold, there exists an integer $N(M) > 0$ such that for any $\epsilon > 0$ any taut foliation on M is ϵ-coarse isotopic to one of $N(M)$ taut foliations.* q.e.d.

Corollary 6.16. *If $\epsilon > 0$ and $\mathcal{F}_1, \cdots, \mathcal{F}_{N(M)+1}$ are taut foliations* *on the closed oriented atoroidal 3-manifold M, then there exists $i \neq j$ such that up to isotopy the tangent plane fields of $\mathcal{F}_i$ and $\mathcal{F}_j$ are ϵ-close. I.e., for $k \in \{i, j\}, \mathcal{F}_k$ is isotopic to $\mathcal{G}_k$ such that for each $x \in M$ the oriented orthogonal to the tangent plane of $\mathcal{G}_i$ at x is ϵ-close to that of $\mathcal{G}_j$ at x.* q.e.d.

Question 6.17. Do higher order jets allow one to obtain a finer measure of distance between isotopy classes of foliations.

Since ϵ-close tangent plane fields are homotopic, via the straight line homotopy we obtain the following result.

Corollary 6.18 (Kronheimer–Mrowka [24]). *On a closed orientable 3-manifold, there are only finitely many homotopy classes of plane fields of taut foliations.* q.e.d.

Remark 6.19. The proof we gave required that M be atoroidal, however it is not difficult to obtain a proof of the toroidal case using our technology.

Definition 6.20. We say that the foliation or lamination $\mathcal{F}$ is *normal* to the triangulation τ on the 3-manifold M if for each 3-simplex σ there exists a topological foliation chart $\mathbb{R}^2 \times \mathbb{R}$ such that in local coordinates σ is a linear 3-simplex with vertices at distinct z-coordinates. If $\mathcal{F}$ is nowhere dense, then we require that its support be disjoint from the 0-skeleton.

Theorem 6.21. *Let M be a closed, orientable, atoroidal 3-manifold. Then there exists a triangulation τ on M such that any essential lamination or taut or Reebless foliation can be isotoped to be normal to τ.*

Proof. Given a branched surface B_0, it is easy to construct by hand a triangulation τ such that any lamination carried by B_0 or any foliation strongly carried by B_0 is normal to τ. Also if λ is normal with respect to τ, and τ_1 is any *linear* subdivision of τ, then one can isotope λ to be normal with respect to τ_1. Theorem 6.21 now follows from Theorem 6.5, Theorem 6.13 and the fact [25] that any two triangulations have isomorphic linear subdivisions. q.e.d.

Corollary 6.22. *Given a closed, orientable, atoroidal 3-manifold M, there exists a finite set of manifold charts $(U_1, \cdots, U_n)$ which cover M such that if $\mathcal{F}$ is a taut foliation, then $\mathcal{F}$ is isotopic to $\mathcal{G}$ such that each U_i is a foliation chart for $\mathcal{G}$. (And so for each i, each leaf of $\mathcal{G}|U_i$ is a disc.)* q.e.d.

Theorem 6.21 positively answers a question asked by Thurston in the late 1970's. Later Larry Conlon independently asked this question and I thank him for a discussion during 1994.

Remark (Thurston 1970's) 6.23. Every 3-manifold has a triangulation τ such that no taut foliation is normal to τ. Indeed let τ be a triangulation such that in some coordinate chart there exists a knotted simple closed curve which is the union of three 1-simplices. If that coordinate chart was also a foliation chart, then the knot would have at least four critical points with respect to the third coordinate. However, at most three of these extrema can occur at vertices, thus there has to be a tangency of a 1-simplex with a leaf. If the coordinate chart is not a foliation chart, then a more elaborate argument using the Roussarie - Thurston [27], [30] isotopy theorem, applied to a 2-sphere which bounds a ball and contains the knot, reduces to the foliation chart case.

Remark 6.24. Theorem 6.5 and Theorem 6.13 can be viewed as discrete analogues of Schoen's Theorem [28] to essential laminations in 3-manifolds. Schoen asserts that any least area surface in a fixed closed Riemannian 3-manifold has bounded normal curvature, i.e., it cannot locally bend too much. In the same manner, our results assert that up to isotopy, essential laminations and in particular taut foliations have uniformly bounded normal curvature.

7. Problems and conjectures

In what follows triangulations also mean pseudo-triangulations. (I.e., two simplices are allowed to meet along more than one face.)

Problem 7.1. Let τ be a triangulation on a closed oriented atoroidal 3-manifold M.

a) Is every nowhere dense essential lamination isotopic to a normal lamination?

b) Is there an explicit example requiring splitting of leaves?

c) Is there an explicit example requiring sutured manifold evacuation?

d) Is there a number n such that for every closed oriented 3-manifold N with triangulation τ, every nowhere dense essential lamination on N can be isotoped to be normal with respect to the n'th barycentric subdivision of τ.

Problem 7.2. Compute an explicit function $f : \mathbb{N} \to \mathbb{N}$ such that if M is an atoroidal 3-manifold with a triangulation of n simplices, then every essential lamination is carried by one of $f(n)$ explicitly described essential branched surfaces.

Conjecture 7.3. *Let λ be a nowhere dense essential lamination in the closed orientable atoroidal Riemannian 3-manifold M. Then after possibly splitting along leaves and/or collapsing along I-bundles, λ can be isotoped to a lamination by stable minimal surfaces. (Is splitting ever necessary?)*

A positive proof of this conjecture together with Schoen's theorem [28] could be used to give another proof that an essential lamination can be isotoped to a lamination with bounded normal curvature, and hence another proof that every essential lamination is carried by one of finitely many branched surfaces. Indeed one could then derive an explicit bound on the number of such surfaces.

Conjecture 7.4. *Let $\mathcal{F}$ be a taut foliation on the closed orientable 3-manifold M. Suppose that no leaf or pair of leaves bounds an I-bundle in M. If M has a generic Riemannian metric, then $\mathcal{F}$ naturally fractures into a nowhere dense essential lamination by stable minimal surfaces.*

Question 7.5. How does the splitting of leaves of $\mathcal{L}$ depend on the Riemannian metric on M?

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