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FUKAYA-FLOER HOMOLOGY OF $\Sigma \times \mathbb{S}^1$ AND APPLICATIONS

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Abstract

We determine the Fukaya-Floer (co)homology groups of the three-manifold $Y = \Sigma \times \mathbb{S}^1$, where Σ is a Riemann surface of genus $g \geq 1$. These are of two kinds. For the 1-cycle $\mathbb{S}^1 \subset Y$, we compute the Fukaya-Floer cohomology $HFF^*(Y, \mathbb{S}^1)$ and its ring structure, which is a sort of deformation of the Floer cohomology $HF^*(Y)$. On the other hand, for 1-cycles $\delta \subset \Sigma \subset Y$, we determine the Fukaya-Floer homology $HFF_*(Y, \delta)$ and its $HF^*(Y)$-module structure.

We give the following applications:

- We show that every four-manifold with $b^+ > 1$ is of finite type.
- Four-manifolds which arise as connected sums along surfaces of four-manifolds with $b_1 = 0$ are of simple type and we give constraints on their basic classes.
- We find the invariants of the product of two Riemann surfaces both of genus greater than or equal to one.

1. Introduction

The structure of Donaldson invariants of 4-manifolds has been found out by Kronheimer and Mrowka [16] and Fintushel and Stern [8] for a large class of 4-manifolds (those of simple type with $b_1 = 0$, $b^+ > 1$) making use of universal relations coming from embedded surfaces. In order to analyse general 4-manifolds, we need to set up first the right framework for getting enough universal relations. It is the purpose of

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this work to do this by using the Fukaya-Floer homology, constructed in [12] [2], for the three-manifold $Y = \Sigma \times \mathbb{S}^1$, the product of a surface with a circle. This is obviously not the only way, but it already gives new results.

Donaldson invariants for a 4-manifold X with $b^+ > 1$ are defined as linear functionals

$$D_X^w : \mathbb{A}(X) = \mathrm{Sym}^*(H_0(X) \oplus H_2(X)) \otimes \Lambda^* H_1(X) \to \mathbb{C},$$

where $w \in H^2(X;\mathbb{Z})$. For the homology $H_*(X)$ we shall understand complex coefficients. $\mathbb{A}(X)$ is graded giving degree $4-i$ to the elements in $H_i(X)$. There is a slight difference in our definition of $\mathbb{A}(X)$ with that of Kronheimer and Mrowka [16], as we do not consider 3-homology classes (this is done in this way since the techniques in this paper are not well suited to deal with these classes).

We say that X is of w-**simple type** when $D_X^w((x^2-4)z) = 0$, for any $z \in \mathbb{A}(X)$. If X has $b_1 = 0$ and it is of w-simple type, then it is of w'-simple type for any other w', and X is said to be of simple type for brevity. Analogously, we say that X is of w-**finite type** when there is some $n \geq 0$ such that $D_X^w((x^2-4)^n z) = 0$, for any $z \in \mathbb{A}(X)$. The order is the minimum of such n, so order 1 means simple type and order 0 means that the Donaldson invariants are identically zero. X is of finite type if it is of w-finite type for any w. Indeed the order of w-finite type of X does not depend on w (see [25]). We also introduce the notion of X being of w-**strong simple type** when $D_X^w((x^2-4)z) = 0$, for any $z \in \mathbb{A}(X)$ and $D_X^w(\gamma z) = 0$, for any $\gamma \in H_1(X)$ and any $z \in \mathbb{A}(X)$. This condition is the right one for extending the concept of simple type in the case $b_1 = 0$ to the case $b_1 > 0$.

Let $\Sigma = \Sigma_g$ be a Riemann surface of genus $g \geq 1$ and consider the three-manifold $Y = \Sigma \times \mathbb{S}^1$. In [23] we computed the ring structure of the (instanton) Floer (co)homology of Y, together with the $SO(3)$-bundle with $w_2 = \mathrm{P.D.}[\mathbb{S}^1]$. This gadget encodes all the relations $R \in \mathbb{A}(\Sigma)$ satisfied by all 4-manifolds X containing an embedded surface Σ, representing an odd homology class and with $\Sigma^2 = 0$. More accurately, for such X, $D_X^w(Rz) = 0$, for any $z \in \mathbb{A}(\Sigma^\perp)$ (and $w \in H^2(X;\mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod 2$). This is so since we have a decomposition $X = X_1 \cup_Y A$, where A is a tubular neighbourhood of Σ, and we can consider $R \in \mathbb{A}(A)$ and $z \in \mathbb{A}(X_1)$. Then the (relative) Donaldson invariants for A corresponding to R are already vanishing.

In order to drop the condition $z \in \mathbb{A}(\Sigma^\perp)$, the useful space to work in is no longer the Floer homology, but the extension developed by

Fukaya [2] [12] and known as Fukaya-Floer homology. This one deals with 2-cycles in X cutting Y non-trivially. In our case as $X = X_1 \cup_Y A$, the only possibility for the cutting of a 2-cycle of X with Y is a multiple of $\mathbb{S}^1 \subset Y = \Sigma \times \mathbb{S}^1$. In order to describe the structure of the Fukaya-Floer (co)homology $HFF_g^* = HFF^*(Y, \mathbb{S}^1)$ (with the $SO(3)$-bundle with $w_2 = \text{P.D.}[\mathbb{S}^1]$), we introduce a ring structure provided by the cobordism between $Y \sqcup Y$ and Y (this cobordism is just the pair of pants times Σ). Then the main point is to describe the ring HFF_g^* by finding generators and relations. These relations are translated in a straightforward way into relations satisfied by the Donaldson invariants of 4-manifolds.

The ring structure of HFF_g^* is similar in spirit to the quantum cohomology of a symplectic manifold. Let $\Lambda_q = \mathbb{C}((q))$ be field of formal Laurent series in q. We can lift the $\mathbb{Z}/4\mathbb{Z}$-graded object HFF_g^* to a $\mathbb{Z}$-graded space (with an isomorphism shifting degrees by 4) by putting $\widehat{HFF}_g^* = HFF_g^* \otimes \Lambda_q$, with q of degree 4. Then we have

Theorem 1.1. *Let $\mathcal{N}_g$ be the moduli space of odd degree rank-2 stable vector bundles on $\Sigma = \Sigma_g$. Then $\widehat{HFF}_g^*$ is isomorphic to*

$$H^*(\mathcal{N}_g) \otimes \mathbb{C}[[t]] \otimes \mathbb{C}((q)),$$

with t of degree -2 and q of degree 4. The product of $\widehat{HFF}_g^*$ is a deformation with two parameters t and q of the ring structure of $H^*(\mathcal{N}_g)$, i.e., for $f_1 \in H^i(\mathcal{N}_g)$ and $f_2 \in H^j(\mathcal{N}_g)$, we have the product of f_1 and f_2 in $\widehat{HFF}_g^*$ to be of the form

$$f_1 * f_2 = f_1 \cup f_2 + \sum_{r>0, s\geq 0} \Phi_{rs} q^r t^s,$$

for $\Phi_{rs} \in H^{i+j-4r+2s}(\mathcal{N}_g)$.

The analysis of HFF_g^* carried out in this paper sets up the background work necessary to give a structure theorem of the Donaldson invariants for manifolds not of simple type [17]. Such work which will be carried out in future. Such a structure theorem was conjectured in [17] and presumably, it might follow from the arguments given in [16], [8]. The first result in this direction is the finite type condition for all 4-manifolds with $b^+ > 1$, which we prove (Frøyshov [13] and Wieczorek [30] have given alternative proofs only valid for simply connected 4-manifolds).

Theorem 1.2. *Any* 4*-manifold with* $b^+ > 1$ *is of finite type.*

There is also another possibility for the Fukaya-Floer homology of Y, which is given by a loop $\delta \subset \Sigma \subset Y$, δ primitive in homology. For completeness, we also determine the structure of $HFF_*(Y, \delta)$ and, as an application, we show the following result on the basic classes of 4-manifolds which are connected sums along a surface (see [20] [21] for results in the same direction).

Theorem 1.3. *Let* $\bar{X}_1$, $\bar{X}_2$ *be smooth closed* 4*-manifolds of simple type with* $b_1 = 0$. *Suppose that there are embedded surfaces* $\Sigma \hookrightarrow \bar{X}_i$ *of the same genus* $g \geq 1$, *self-intersection zero and representing odd elements in homology,* $i = 1, 2$. *Let* $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ *be a connected sum along* Σ. *Then* X *is of simple type with* $b_1 = 0$ *and* $b^+ > 1$, *and all its basic classes* K_i *satisfy* $K_i \cdot \Sigma \equiv 2g - 2 \pmod 4$.

Finally, we give the Donaldson invariants of the product of two Riemann surfaces with genus at least one. The basic classes coincide with its Seiberg-Witten basic classes, as expected.

Theorem 1.4. *Let* $S = \Sigma_g \times \Sigma_h$ *be the product of two Riemann surfaces of genus* $h \geq g \geq 1$. *Then* S *is of strong simple type and the Donaldson series are as follows:*

$$\begin{cases} \mathbb{D}_S = 4^g e^{Q/2} \sinh^{2g-2}[\Sigma_g] & \text{if } h = 1, \\ \mathbb{D}_S = 2^{7(g-1)(h-1)+3} \sinh K & \text{if } g, h > 1, \text{ both even}, \\ \mathbb{D}_S = 2^{7(g-1)(h-1)+3} \cosh K & \text{if } g, h > 1, \text{ at least one odd}, \end{cases}$$

where $K = K_S$ *is the canonical class of* S.

The paper is organised as follows. In sections 2 and 3 we review the construction of the Floer homology and Fukaya-Floer homology of a three-manifold with $b_1 \neq 0$. Then in section 4 we recall, for the convenience of the reader, the structure of the Floer cohomology $HF^*(\Sigma \times \mathbb{S}^1)$. Section 5 is devoted to studying the Fukaya-Floer cohomology corresponding to the 1-cycle $\mathbb{S}^1 \subset Y$, $HFF_g^* = HFF^*(\Sigma \times \mathbb{S}^1, \mathbb{S}^1)$, constructing its ring structure and proving Theorem 1.1. In section 6 we study the subspace of HFF_g^* which gives the relations for Donaldson invariants of 4-manifolds of strong simple type with $b^+ > 1$. This subspace is determined completely in Theorem 6.2. In section 7 we study the bigger subspace of HFF_g^* corresponding to the analysis of 4-manifolds which are only required to have $b^+ > 1$. This subspace is not determined in full, but Theorem 7.2 provides many useful relations (in the form of eigenvalues of the maps given as multiplication by the natural generators of HFF_g^*). The Fukaya-Floer cohomology $HFF^*(\Sigma \times \mathbb{S}^1, \delta)$ corresponding to $\delta \subset \Sigma \subset \Sigma \times \mathbb{S}^1$, is determined as an $HF^*(\Sigma \times \mathbb{S}^1)$-module in section 8. The proofs of the Theorems 1.2, 1.3 and 1.4 are collected in section 9.

2. Review of Floer homology

In this section we are going to review the construction of the Floer homology groups of a 3-manifold Y with $b_1 > 0$, endowed with an $SO(3)$-bundle P with second Stiefel-Whitney class $w_2 = w_2(P) \neq 0 \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$. Recall that w_2 determines P uniquely. To be more precise, we are going to suppose that w_2 has an integral lift (i.e., that the $SO(3)$-bundle lifts to an $U(2)$-bundle). All the facts stated here are well known. For full treatment and proofs see [10], [3], [5] (the case of Floer homology of rational homology spheres is dealt with in [1]). We shall use complex coefficients for the Floer homology (although it is usually developed over the integers).

2.1. Floer homology. As $w_2 \neq 0 \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$, there are no reducible flat connections on P. Possibly after a small perturbation of the flat equations, there will be finitely many flat connections ρ_j, and they will all be non-degenerate. The Floer complex $CF_*(Y)$ is the complex vector space with basis given by the ρ_j, with a $\mathbb{Z}/4\mathbb{Z}$-grading which is given by the index [3], [4]. Actually, this grading is only defined up to addition of a constant. The complex $CF_*(Y)$ depends on w_2, but in general we will not express this in the notation.

We define the boundary ∂ as follows. For every two flat connections ρ_k and ρ_l there is a moduli space $\mathcal{M}(\rho_k, \rho_l)$ of (perturbed) ASD connections on the tube $Y \times \mathbb{R}$ with limits ρ_k and ρ_l. There is an $\mathbb{R}$-action by translations, and $\mathcal{M}_0(\rho_k, \rho_l)$ shall stand for the quotient $\mathcal{M}(\rho_k, \rho_l)/\mathbb{R}$. This space has components $\mathcal{M}_0^D(\rho_k, \rho_l)$ of dimensions $D \equiv \mathrm{ind}(\rho_k) - \mathrm{ind}(\rho_l) - 1 \pmod 4$, and can be oriented in a compatible way [10]. The boundary map of the Floer complex is then

$$\begin{array}{rcl} \partial : CF_i(Y) & \to & CF_{i-1}(Y) \\ \rho_k & \mapsto & \displaystyle\sum_{\substack{\rho_l \\ \mathrm{ind}(\rho_l) = \mathrm{ind}(\rho_k) - 1}} \#\mathcal{M}_0^0(\rho_k, \rho_l)\rho_l, \end{array}$$

where $\#\mathcal{M}_0^0(\rho_k, \rho_l)$ is the algebraic number of points of the compact zero-dimensional moduli space $\mathcal{M}_0^0(\rho_k, \rho_l)$. One may check [3], [5] that

$\partial^2 = 0$. Therefore $(CF_*(Y), \partial)$ is a complex and we define the Floer homology $HF_*(Y)$ as the homology of this complex (see [10]). It can be proved that these groups do not depend on the metric of Y or on the chosen perturbation of the ASD equations. The groups $HF_*(Y)$ are natural under diffeomorphisms of the pair (Y, P). The Floer cohomology $HF^*(Y)$ is defined analogously out of the dual complex $CF^*(Y)$, and it is naturally isomorphic to $HF_{c-*}(\overline{Y})$, where $\overline{Y}$ denotes Y with reversed orientation (c is a constant that we need to introduce due to the indeterminacy of the grading). The natural pairing $HF_*(Y) \otimes HF^*(Y) \to \mathbb{C}$ yields the pairing $\langle , \rangle : HF_*(Y) \otimes HF_{c-*}(\overline{Y}) \to \mathbb{C}$. It is worth noticing that when Y has an orientation reversing diffeomorphism, i.e., $Y \cong \overline{Y}$, we have a pairing

$$\langle , \rangle : HF_*(Y) \otimes HF_{c-*}(Y) \to \mathbb{C}. \tag{2.1}$$

2.2. Action of $H_*(Y)$ on $HF_*(Y)$. Let $\alpha \in H_{3-i}(Y)$. We have cycles V_α, in the moduli spaces $\mathcal{M}(\rho_k, \rho_l)$, of codimension $i+1$, representing $\mu(\alpha \times \mathrm{pt})$, for $\alpha \times \mathrm{pt} \subset Y \times \mathbb{R}$, much in the same way as in the case of a closed manifold [6], [16]. Using them, we construct a map

$$\begin{array}{rcl} \mu(\alpha) : CF_j(Y) & \to & CF_{j-i-1}(Y) \\ \rho_k & \mapsto & \displaystyle\sum_{\substack{\rho_l \\ \mathrm{ind}(\rho_l) = \mathrm{ind}(\rho_k) - i - 1}} (\# \mathcal{M}^{i+1}(\rho_k, \rho_l) \cap V_\alpha)\, \rho_l \end{array}$$

(this time we do not quotient by the translations as the cycles V_α are not translation invariant). This map satisfies $\partial \circ \mu(\alpha) + \mu(\alpha) \circ \partial = 0$, so it descends to a map

$$\mu(\alpha) : HF_*(Y) \to HF_{*-i-1}(Y).$$

2.3. Products in Floer homology. Suppose that we have an (oriented) four-dimensional cobordism X between two closed oriented 3-manifolds Y_1 and Y_2, i.e., X is a 4-manifold with boundary $\partial X = Y_1 \sqcup \overline{Y}_2$. Suppose that we have an $SO(3)$-bundle P_X over X such that $P_1 = P_X|_{Y_1}$ and $P_2 = P_X|_{Y_2}$ satisfy $w_2(P_i) \neq 0$, $i = 1, 2$, so that we have defined the Floer homologies of (Y_1, P_1) and (Y_2, P_2). Furnishing X with two cylindrical ends, the cobordism X gives a map

$$\begin{array}{rcl} \Phi_X : CF_*(Y_1) & \to & CF_*(Y_2) \\ \rho_k & \mapsto & \displaystyle\sum_{\rho'_l} \# \mathcal{M}^0(X, \rho_k, \rho'_l) \rho'_l, \end{array}$$

where $\mathcal{M}(X, \rho_k, \rho'_l)$ is the moduli space of (perturbed) ASD connections on X with flat limits ρ_k on the Y_1 side and ρ'_l on the Y_2 side. Again $\partial \circ \Phi_X + \Phi_X \circ \partial = 0$, so we have a map $\Phi_X : HF_*(Y_1) \to HF_*(Y_2)$. Also if $\alpha_1 \in H_*(Y_1)$ and $\alpha_2 \in H_*(Y_2)$ define the same homology class in X, then $\mu(\alpha_2) \circ \Phi_X = \Phi_X \circ \mu(\alpha_1)$.

On the other hand, suppose that Y_1 and Y_2 are oriented 3-manifolds, and P_1 and P_2 are $SO(3)$-bundles with $w_2(P_i) \neq 0$, $i = 1, 2$. Consider the $SO(3)$-bundle $P = P_1 \sqcup P_2$ over $Y = Y_1 \sqcup Y_2$. Every flat connection on P is of the form (ρ^1_l, ρ^2_m) and $\operatorname{ind}(\rho^1_l, \rho^2_m) = \operatorname{ind}(\rho^1_l) + \operatorname{ind}(\rho^2_m)$. So we have naturally $CF_*(Y) = CF_*(Y_1) \otimes CF_*(Y_2)$. It is easy to check that $\partial_{CF_*(Y)} = \partial_{CF_*(Y_1)} + \partial_{CF_*(Y_2)}$, so that

$$HF_*(Y) = HF_*(Y_1) \otimes HF_*(Y_2).$$

Putting the above together, a product for $HF_*(Y)$ might arise as follows. Suppose that there is a cobordism between $Y \sqcup Y$ and Y, i.e., an oriented 4-manifold X with boundary $\partial X = Y \sqcup Y \sqcup \overline{Y}$. Then there is a map

$$HF_*(Y) \otimes HF_*(Y) \to HF_*(Y).$$

In some particular cases, this gives an associative and graded commutative ring structure on $HF_*(Y)$. We shall prove it for the particular 3-manifold $Y = \Sigma \times \mathbb{S}^1$ using an argument along different lines (see section 4).

2.4. Relative invariants of 4-manifolds. Let us recall the definition of Donaldson invariants of an (oriented) 4-manifold X with boundary $\partial X = Y$, for any $w \in H^2(X; \mathbb{Z})$ such that $w|_Y = w_2 \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$. These invariants will not be numerical (in contrast with the case of a closed 4-manifold), instead they live in the Floer homology $HF_*(Y)$.

We give X a cylindrical end and consider the moduli spaces $\mathcal{M}(X, \rho_l)$ of (perturbed) ASD connections with finite action and asymptotic to ρ_l. $\mathcal{M}(X, \rho_l)$ has components $\mathcal{M}^D(X, \rho_l)$ of dimensions $D \equiv \operatorname{ind}(\rho_l) + C \pmod 4$, for some fixed constant C only dependent on X. The spaces $\mathcal{M}(X, \rho_l)$ can be oriented coherently and, for $z = \alpha_1 \alpha_2 \cdots \alpha_r \in \mathbb{A}(X)$ of degree d, we can choose (generic) cycles $V_{\alpha_i} \subset \mathcal{M}(X, \rho_l)$ representing $\mu(\alpha_i)$, so that we have defined an element

$$\phi^w(X, z) = \sum_{\substack{\rho_l \\ \operatorname{ind}(\rho_l) + C = d}} (\#\mathcal{M}^d(X, \rho_l) \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_r})\, \rho_l \in CF_*(Y).$$

This element has boundary zero and hence defines a homology class, which is called the **relative invariants** of X, denoted again by $\phi^w(X,z)$, in $HF_*(Y)$ (see [3] [5]).

We have a gluing theorem for these relative invariants. Suppose that a closed 4-manifold X is obtained as the union of two 4-manifolds with boundary, $X = X_1 \cup_Y X_2$, where $\partial X_1 = Y$ and $\partial X_2 = \overline{Y}$. Let $w \in H^2(X;\mathbb{Z})$ with $w|_Y = w_2 \in H^2(Y;\mathbb{Z}/2\mathbb{Z})$ as above (this implies in particular $b^+(X) > 0$, so the Donaldson invariants of X are defined; in the case $b^+ = 1$ relative to chambers [15], [22]). We need another bit of terminology from [20].

Definition 2.1. (w,Σ) is an **allowable** pair if $w,\Sigma \in H^2(X;\mathbb{Z})$, $w \cdot \Sigma \equiv 1 \pmod 2$ and $\Sigma^2 = 0$. Then we define

$$D_X^{(w,\Sigma)} = D_X^w + D_X^{w+\Sigma}.$$

Usually, for $X = X_1 \cup_Y X_2$, we have $w \in H^2(X;\mathbb{Z})$ with $w|_Y = w_2$ as above, and $\Sigma \in H^2(X;\mathbb{Z})$ whose Poincaré dual lies in the image of $H_2(Y;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, and satisfies $w \cdot \Sigma \equiv 1 \pmod 2$. Then (w,Σ) is an allowable pair. The series $D_X^{(w,\Sigma)}$ behaves much in the same way as the Kronheimer-Mrowka [16] series $\mathbb{D}_X^w(\alpha) = D_X^w((1+\frac{x}{2})e^\alpha)$ (they are equivalent for manifolds of simple type with $b_1 = 0$ and $b^+ > 1$, see [21] for an explicit formula), but it is a more efficient way of collecting the information in general.

When $b^+ = 1$, the Donaldson invariants depend on the choice of metric for X. In general, we shall consider a family of metrics g_R, $R > 1$, giving a neck of length R, i.e., $X = X_1 \cup (Y \times [0,R]) \cup X_2$, where the metrics on X_1 and X_2 are fixed, and the metric on $Y \times [0,R]$ is of the form $g_Y + dt \otimes dt$, for a fixed metric g_Y on Y. Then for large enough R (depending on the degree of $z \in \mathbb{A}(X)$), the metrics g_R stay within a fixed chamber and $D_X^w(z)$ is well defined. We shall refer to these metrics as *metrics on X giving a long neck.* Note that in this case $\phi^w(X_i,z_i)$ also depends on the metric on X_i.

Theorem 2.2. *Let $X = X_1 \cup_Y X_2$ be as above and $w \in H^2(X;\mathbb{Z})$ with $w|_Y = w_2$. Take $\Sigma \in H^2(X;\mathbb{Z})$ whose Poincaré dual lies in the image of $H_2(Y;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, and satisfies $w \cdot \Sigma \equiv 1 \pmod 2$. Put $w_i = w|_{X_i} \in H^2(X_i;\mathbb{Z})$. For $z_i \in \mathbb{A}(X_i)$, $i = 1,2$, we have*

$$D_X^{(w,\Sigma)}(z_1\, z_2) = \langle \phi^{w_1}(X_1,z_1), \phi^{w_2}(X_2,z_2)\rangle.$$

When $b^+ = 1$ the invariants are calculated for metrics on X giving a long neck.

This is a standard and well known fact [4]. The only not-so-standard fact is the appearance of (w, Σ). This is so since we are working with $SO(3)$-Floer theory instead of $U(2)$-Floer theory which would give Floer groups graded modulo 8. When we glue the $SO(3)$-bundles over X_1 and X_2 with second Stiefel-Whitney classes w_1 and w_2 we can do it in different ways, as there is a choice of gluing automorphism of the bundles along Y, and both w and $w + \Sigma$ are two different possibilities for the resulting $SO(3)$-bundle whose difference in the indices of both is 4 (see [3], [4]).

In general we shall write

$$\phi^w(X, e^{t\alpha}) = \sum_d \frac{\phi^w(X, \alpha^d)}{d!} t^d,$$

as an element living in $HF_*(Y) \otimes \mathbb{C}[[t]]$. Theorem 2.2 can be rewritten as

Theorem 2.3. *Let $X = X_1 \cup_Y X_2$ be as above and $w \in H^2(X; \mathbb{Z})$ with $w|_Y = w_2$. Take $\Sigma \in H^2(X; \mathbb{Z})$ whose Poincaré dual lies in the image of $H_2(Y; \mathbb{Z}) \to H_2(X; \mathbb{Z})$, and satisfies $w \cdot \Sigma \equiv 1 \pmod 2$. Put $w_i = w|_{X_i} \in H^2(X_i; \mathbb{Z})$. Then for $\alpha_i \in H_2(X_i)$, $i = 1, 2$, we have*

$$D_X^{(w,\Sigma)}(e^{t(\alpha_1+\alpha_2)}) = \langle \phi^{w_1}(X_1, e^{t\alpha_1}), \phi^{w_2}(X_2, e^{t\alpha_2}) \rangle.$$

When $b^+ = 1$ the invariants are calculated for metrics on X giving a long neck.

3. Review of Fukaya-Floer homology

Now we pass on to the definition of the Fukaya-Floer homology groups, which are a refinement of the Floer homology groups of a 3-manifold Y with $b_1 > 0$. The construction is initially given by Fukaya in [12] and explained by Braam and Donaldson [2] in a paper worth reading. The origin of the Fukaya-Floer homology is the need of defining relative invariants (and establishing the appropriate gluing theorem) for 2-homology classes crossing the neck in a splitting $X = X_1 \cup_Y X_2$. They are in some sense more natural than the Floer homology from the point of view of the Donaldson invariants of 4-manifolds.

3.1. Fukaya-Floer homology. The input is a triple (Y, P, δ), where P is an $SO(3)$-bundle with $w_2 \neq 0$ over an oriented 3-manifold

Y, and δ is a loop in Y, i.e., an (oriented) embedded $\delta \cong \mathbb{S}^1 \hookrightarrow Y$. The complex $CFF_*(Y, \delta)$ will be the total complex of the double complex

$$CF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty),$$

where $\hat{H}_*(\mathbb{CP}^\infty)$ is the completion of $H_*(\mathbb{CP}^\infty)$, i.e., the ring of formal power series. Recall that $H_i(\mathbb{CP}^\infty) = 0$ for i odd and $\mathbb{C}$ for i even (we are using complex coefficients). Therefore

$$\begin{aligned} CFF_i(Y,\delta) = CF_i(Y) \times CF_{i-2}(Y)t \times CF_{i-4}(Y)\frac{t^2}{2!} \\ \times CF_{i-6}(Y)\frac{t^3}{3!} \times \cdots . \end{aligned} \tag{3.1}$$

The labels $\frac{t^k}{k!}$ must be understood as the generators of $H_{2k}(\mathbb{CP}^\infty)$ and have an assigned (homological) degree $2k$. So

$$CFF_*(Y,\delta) = CF_*(Y) \otimes \mathbb{C}[[t]],$$

i.e., Fukaya-Floer chains are infinite sequences of (possibly non-zero) Floer chains. This complex is also graded over $\mathbb{Z}/4\mathbb{Z}$. To construct the boundary ∂ we work as follows. For every pair of flat connections ρ_k and ρ_l we have the moduli space $\mathcal{M}_0(\rho_k, \rho_l)$ of section 2 and we consider $\mu(\delta \times \mathbb{R})$ as the first Chern class of the determinant line bundle $\mathcal{L}_{\delta\times\mathbb{R}}$ (for Dirac operators on the surface $\delta \times \mathbb{R} \subset Y \times \mathbb{R}$ coupled to connections in the moduli space, with asymptotic decay conditions at the ends). We choose representatives $V_{\delta\times\mathbb{R}}$ for $\mu(\delta \times \mathbb{R})$ conveniently, which are compatible with the compactification of $\mathcal{M}_0(\rho_k, \rho_l)$. As explained in [2], we also have to trivialize $\mathcal{L}_{\delta\times\mathbb{R}}$ over each $\mathcal{M}_0(\rho_k, \rho_l)$, in a compatible way. The boundary of $CFF_*(Y)$ is defined as (see [2])

$$\begin{array}{rcl} \partial : CFF_i(Y) & \to & CFF_{i-1}(Y) \\ \rho_k \dfrac{t^a}{a!} & \mapsto & \displaystyle\sum_{\substack{\rho_l \\ b \geq a}} \binom{b}{a} \left(\# \mathcal{M}_0^{2(b-a)}(\rho_k, \rho_l) \cap V_{\delta\times\mathbb{R}}^{b-a} \right) \rho_l \frac{t^b}{b!} \end{array}$$

for $\rho_k \in CF_{i-2a}$, $\rho_l \in CF_{i-1-2b}$. Here $V_{\delta\times\mathbb{R}}^{b-a}$ means the intersection of $b-a$ different generic representatives (we only have added the labels to the formula in [2]). The proof of $\partial^2 = 0$ is given in [2] and runs as follows. Consider two flat connections ρ_k and ρ_l, such that $\mathrm{ind}(\rho_l) = \mathrm{ind}(\rho_k) - 2 - 2e$. Then the moduli space $\mathcal{M}_0^{2e+1}(\rho_k, \rho_l) \cap V_{\delta\times\mathbb{R}}^e$ is a one-dimensional manifold. We compactify it and count the boundary points in the same way as in the case of Floer homology to get

$$\sum_{\substack{\rho_m \\ \mathrm{ind}(\rho_m)=\mathrm{ind}(\rho_k)-1-2f}} \binom{e}{f} \# \mathcal{M}_0^{2f}(\rho_k,\rho_m) \cap V^f_{\delta\times\mathbb{R}} \cdot \# \mathcal{M}_0^{2(e-f)}(\rho_m,\rho_l) \cap V^{e-f}_{\delta\times\mathbb{R}} = 0,$$

equivalently $\partial^2 \rho_k = 0$.

We have thus defined the Fukaya-Floer homology $HFF_*(Y,\delta)$ as the homology of the complex $(CFF_*(Y,\delta),\partial)$. These groups are independent of metrics and of perturbations of equations [12]. For the effective computation of $HFF_*(Y,\delta)$, we construct a spectral sequence next. There is a filtration $(K^{(i)})_* = CF_*(Y)\otimes(\prod_{*\geq i}\hat{H}_*(\mathbb{CP}^\infty))$ of $CFF_*(Y,\delta)$ inducing a spectral sequence whose E_3 term is $HF_*(Y)\otimes \hat{H}_*(\mathbb{CP}^\infty)$ and converging to the Fukaya-Floer groups (there is no problem of convergence because of the periodicity of the spectral sequence). The boundary d_3 turns out to be

$$\mu(\delta) : HF_i(Y)\otimes H_{2j}(\mathbb{CP}^\infty) \to HF_{i-3}(Y)\otimes H_{2j+2}(\mathbb{CP}^\infty).$$

The obvious $\mathbb{C}[[t]]$-module structure of $CFF_*(Y,\delta) = CF_*(Y)\otimes\mathbb{C}[[t]]$ descends to give a $\mathbb{C}[[t]]$-module structure for $HFF_*(Y,\delta)$ (the boundary ∂ is $\mathbb{C}[[t]]$-linear thanks to the choice of denominators in (3.1)).

The Fukaya-Floer cohomology will be defined as the homology of the dual complex $CFF^*(Y,\delta) = \mathrm{Hom}_{\mathbb{C}[[t]]}(CFF_*(Y,\delta),\mathbb{C}[[t]])$. We remark that this is a different definition from that of [2]. There is a pairing $\langle,\rangle : HFF_*(Y,\delta)\otimes HFF^*(Y,\delta) \to \mathbb{C}[[t]]$ and an isomorphism $HFF_*(\overline{Y},-\delta) \cong HFF^*(Y,\delta)$, where $-\delta$ is δ with reversed orientation, hence a pairing for the Fukaya-Floer homology groups

$$\langle,\rangle : HFF_*(Y,\delta)\otimes HFF_*(\overline{Y},-\delta) \to \mathbb{C}[[t]].$$

This can be defined through the spectral sequence from the natural pairing in $HF_*(Y)$. Also it is a nice way of collecting all the pairings σ_m in [2].

The Fukaya-Floer homology may also be defined for (Y,P,δ) where $\delta \cong \mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1 \hookrightarrow Y$ is a collection of finitely many disjoint loops (possibly none). In particular, for $\delta=\emptyset$, $HFF_*(Y,\emptyset) = HF_*(Y)\otimes\mathbb{C}[[t]]$ naturally.

3.2. Action of $H_*(Y)$ on $HFF_*(Y,\delta)$. This is explained in [19, section 5.3]. Let $\alpha \in H_{3-i}(Y)$. We define $\mu(\alpha)$ at the level of chains as

$$\begin{aligned}\mu(\alpha): CFF_j(Y) &\to CFF_{j-i-1}(Y)\\ \rho_k\frac{t^a}{a!} &\mapsto \sum_{\substack{\rho_l\\ b\geq a}}\binom{b}{a}\Big(\#\mathcal{M}^{2(b-a)+i+1}(\rho_k,\rho_l)\\ &\qquad\qquad \cap V^{b-a}_{\delta\times\mathbb{R}}\cap V_{\alpha\times \mathrm{pt}}\Big)\rho_l\frac{t^b}{b!},\end{aligned}$$

for $\rho_k\in CF_{j-2a}$, $\rho_l\in CF_{j-i-1-2b}$. Again $\partial\circ\mu(\alpha)+\mu(\alpha)\circ\partial=0$ and $\mu(\alpha)$ descends to a map

$$\mu(\alpha): HFF_*(Y,\delta)\to HFF_{*-i-1}(Y,\delta).$$

For instance, for $HFF_*(Y,\emptyset)=HF_*(Y)\otimes\mathbb{C}[[t]]$, the map $\mu(\alpha)$ is the one induced from $HF_*(Y)$. In general, the induced map in the term $E_3=HF_*(Y)\otimes\mathbb{C}[[t]]$ of the spectral sequence computing $HFF_*(Y,\delta)$ is $\mu(\alpha)$ in Floer homology. The structure of the map $\mu(\alpha)$ is the cornerstone of the analysis in [19, chapter 5] and the seed of this work.

3.3. Products in Fukaya-Floer homology. We can extend the arguments of section 2. Suppose that we have an (oriented) four-dimensional cobordism (X,D,P) between two triples (Y_1,δ_1,P_1) and (Y_2,δ_2,P_2) as above. Then Φ_X is defined at the level of chains by

$$\begin{aligned}\Phi_X: CFF_*(Y_1) &\to CFF_*(Y_2)\\ \rho_k\frac{t^a}{a!} &\mapsto \sum_{\rho_l'}\binom{b}{a}(\#\mathcal{M}^{2(b-a)}(X,\rho_k,\rho_l')\cap V_D^{b-a})\rho_l'\frac{t^b}{b!}.\end{aligned}$$

As $\partial\circ\Phi_X+\Phi_X\circ\partial=0$, Φ_X defines a map

$$\Phi_X: HFF_*(Y_1,\delta_1)\to HFF_*(Y_2,\delta_2).$$

In particular, this proves that $HFF_*(Y,\delta)$ only depends on the homology class given by δ, up to isomorphism. Also if $\alpha_1\in H_*(Y_1)$ and $\alpha_2\in H_*(Y_2)$ define the same homology class in X, then $\mu(\alpha_2)\circ\Phi_X=\Phi_X\circ\mu(\alpha_1)$.

On the other hand, suppose that we have (Y_1,δ_1,P_1) and (Y_2,δ_2,P_2) and consider (Y,δ,P) with $Y=Y_1\sqcup Y_2$, $P=P_1\sqcup P_2$ and $\delta=\delta_1\sqcup\delta_2$. One can prove easily that

$$HFF_*(Y,\delta)=HFF_*(Y_1,\delta_1)\otimes_{\mathbb{C}[[t]]}HFF_*(Y_2,\delta_2).$$

Finally, in case that there is a cobordism between $(Y,\delta) \sqcup (Y,\delta)$ and (Y,δ), we have a map

$$HFF_*(Y,\delta) \otimes_{\mathbb{C}[[t]]} HFF_*(Y,\delta) \to HFF_*(Y,\delta), \tag{3.2}$$

which in some cases it may give an associative and graded commutative ring structure on $HFF_*(Y,\delta)$. Also note that if there is a cobordism between $(Y,\delta) \sqcup (Y,\emptyset)$ and (Y,δ), then there will be a map

$$HF_*(Y) \otimes HFF_*(Y,\delta) \to HFF_*(Y,\delta), \tag{3.3}$$

which may lead to a module structure of $HFF_*(Y,\delta)$ over $HF_*(Y)$.

3.4. Relative invariants of 4-manifolds. To define relative invariants, let X be a 4-manifold with $\partial X = Y$ and $w \in H^2(X;\mathbb{Z})$ such that $w|_Y = w_2 \in H^2(Y;\mathbb{Z}/2\mathbb{Z})$. We give X a cylindrical end. Let $D \subset X$ be a 2-cycle such that $\partial D = D \cap Y = \delta$ (more accurately, $D \cap (Y \times [0,\infty)) = \delta \times [0,\infty)$). One has the moduli spaces $\mathcal{M}(X,\rho_k)$ and can choose generic cycles $V_D^{(i)}$ representing $\mu(D)$ and intersecting transversely in the top stratum of the compactification of $\mathcal{M}(X,\rho_k)$ (see [2]). Then we have an element

$$\phi^w(X,D^d) = \sum_{\rho_k} (\#\mathcal{M}^{2d}(X,\rho_k) \cap V_D^{(1)} \cap \cdots \cap V_D^{(d)})\rho_k$$

in $CF_*(Y) \otimes H_{2d}(\mathbb{CP}^\infty) \subset CFF_*(Y,\delta)$. We remark that this is **not** a cycle. Then we set $\phi^w(X,D) = \prod_d \phi^w(X,D^d)$, which is a cycle. We also denote by $\phi^w(X,D) \in HFF_*(Y,\delta)$ the Fukaya-Floer homology class it represents. Alternatively, we denote this same element as

$$\phi^w(X,e^{tD}) = \phi^w(X,D) = \sum_d \phi^w(X,D^d)\frac{t^d}{d!}.$$

Formally this element lives in $HF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty)$, the E_3 term of the spectral sequence alluded above, but represents the same Fukaya-Floer homology class. The definition of $\phi^w(X,D^d)$ depends on some choices [2], but the homology class $\phi^w(X,D)$ only depends on (X,D). Moreover if we have a homology of D which is the identity on the cylindrical end of X, $\phi^w(X,D)$ remains fixed. Analogously, for any $z \in \mathbb{A}(X)$, we define $\phi^w(X,z\,D^d) \in CF_*(Y) \otimes H_{2d}(\mathbb{CP}^\infty) \subset CFF_*(Y,\delta)$ and $\phi^w(X,z\,e^{tD})$. The relevant gluing theorem is [2] [19]:

Theorem 3.1. *Let $X = X_1 \cup_Y X_2$ and $w \in H^2(X;\mathbb{Z})$ with $w|_Y = w_2$. Take $\Sigma \in H^2(X;\mathbb{Z})$ whose Poincaré dual lies in the image of $H_2(Y;\mathbb{Z}) \to H_2(X;\mathbb{Z})$, and satisfies $w \cdot \Sigma \equiv 1 \pmod 2$. Put $w_i = w|_{X_i} \in H^2(X_i;\mathbb{Z})$. Let $D \in H_2(X)$ be decomposed as $D = D_1 + D_2$ with $D_i \subset X_i$, $i = 1,2$, 2-cycles with $\partial D_1 = \delta$, $\partial D_2 = -\delta$. Choose $z_i \in \mathbb{A}(X_i)$, $i = 1,2$. Then*

$$D_X^{(w,\Sigma)}(z_1 z_2 e^{tD}) = \langle \phi^{w_1}(X_1, z_1 e^{tD_1}), \phi^{w_2}(X_2, z_2 e^{tD_2}) \rangle.$$

When $b^+ = 1$, the invariants are calculated for metrics on X giving a long neck.

4. Floer homology of $\Sigma \times \mathbb{S}^1$

We want to specialise to the case relevant to us. Let $\Sigma = \Sigma_g$ be a Riemann surface of genus $g \geq 1$ and let $Y = \Sigma \times \mathbb{S}^1$ be the trivial circle bundle over Σ. Over this 3-manifold, we fix the $SO(3)$-bundle with $w_2 = \mathrm{P.D.}[\mathbb{S}^1] \in H^2(Y;\mathbb{Z}/2\mathbb{Z})$, which satisfies the hypothesis of section 2. Therefore the instanton Floer homology $HF_*(Y)$ is well-defined. As $Y = \Sigma \times \mathbb{S}^1$ admits an orientation reversing self-diffeomorphism, given by conjugation on the $\mathbb{S}^1$ factor, there is a Poincaré duality isomorphism of $HF^*(Y)$ with $HF_*(Y)$ (this identification will be done systematically and without further notice) and a pairing $\langle , \rangle : HF^*(Y) \otimes HF^*(Y) \to \mathbb{C}$. We introduce a multiplication on $HF^*(Y)$ using the cobordism between $Y \sqcup Y$ and Y given by the 4-manifold which is a pair of pants times Σ. This yields a map $HF^*(Y) \otimes HF^*(Y) \to HF^*(Y)$. We shall prove later explicitly that this is an associative and graded commutative ring structure on $HF^*(Y)$. As a shorthand notation, we shall write henceforth $HF_g^* = HF^*(Y)$, making explicit the dependence on the genus g of the Riemann surface Σ.

The Floer cohomology of $Y = \Sigma \times \mathbb{S}^1$ has been completely computed thanks to the works of Dostoglou and Salamon [7] and its ring structure has been found by the author in [23] and turns out to be isomorphic to the quantum cohomology of the moduli space $\mathcal{N}_g$ of stable bundles of odd degree and rank two over Σ (with fixed determinant), i.e., $QH^*(\mathcal{N}_g) \cong HF^*(\Sigma_g \times \mathbb{S}^1)$, as the author has proved in [24].

Here we shall recall the result stated in [23]. We fix some notation. Let $\{\gamma_1, \ldots, \gamma_{2g}\}$ be a symplectic basis of $H_1(\Sigma;\mathbb{Z})$ with $\gamma_i \gamma_{i+g} = \mathrm{pt}$, for $1 \leq i \leq g$. Also x will stand for the generator of $H_0(\Sigma;\mathbb{Z})$. First we recall the usual cohomology ring of $\mathcal{N}_g$, because of its similarity with the Floer cohomology HFF_g^* and for later use in section 5.

4.1. Cohomology ring of $\mathcal{N}_g$. (See [14], [27], [24].) The ring $H^*(\mathcal{N}_g)$ is generated by the elements

$$\begin{cases} a = 2\,\mu(\Sigma) \in H^2(\mathcal{N}_g), & \\ c_i = \mu(\gamma_i) \in H^3(\mathcal{N}_g), & 1 \leq i \leq 2g, \\ b = -4\,\mu(x) \in H^4(\mathcal{N}_g), & \end{cases}$$

where the map $\mu : H_*(\Sigma) \to H^{4-*}(\mathcal{N}_g)$ is, as usual, given by $-\frac{1}{4}$ times slanting with the first Pontrjagin class of the universal $SO(3)$-bundle over $\Sigma \times \mathcal{N}_g$. Thus there is a basis $\{f_s\}_{s\in\mathcal{S}}$ for $H^*(\mathcal{N}_g)$ with elements of the form

$$f_s = a^n b^m c_{i_1} \cdots c_{i_r}, \tag{4.1}$$

for a finite set $\mathcal{S}$ of multi-indices of the form $s = (n, m; i_1, \ldots, i_r)$, $n, m \geq 0$, $r \geq 0$, $1 \leq i_1 < \cdots < i_r \leq 2g$. There is an epimorphism of rings $\mathbb{A}(\Sigma) \twoheadrightarrow H^*(\mathcal{N}_g)$. The mapping class group $\mathrm{Diff}(\Sigma)$ acts on $H^*(\mathcal{N}_g)$, with the action factoring through the action of the symplectic group $\mathrm{Sp}\,(2g, \mathbb{Z})$ on $\{c_i\}$. The invariant part, $H_I^*(\mathcal{N}_g)$, is generated by a, b and $c = -2\sum_{i=0}^{g} c_i c_{i+g}$. Then

$$\mathbb{C}[a, b, c] \twoheadrightarrow H_I^*(\mathcal{N}_g), \tag{4.2}$$

which allows us to write

$$H_I^*(\mathcal{N}_g) = \mathbb{C}[a, b, c]/I_g,$$

where I_g is the ideal of relations satisfied by a, b and c. The space $H^3 = H^3(\mathcal{N}_g)$ has a basis $c_1, \ldots, c_{2g}$, so $\mu : H_1(\Sigma) \xrightarrow{\cong} H^3$. For $0 \leq k \leq g$, the primitive component of $\Lambda^k H^3$ is

$$\Lambda_0^k H^3 = \ker(c^{g-k+1} : \Lambda^k H^3 \to \Lambda^{2g-k+2} H^3).$$

The spaces $\Lambda_0^k H^3$ are irreducible $\mathrm{Sp}\,(2g, \mathbb{Z})$-representations. This follows from the fact that they are irreducible $\mathrm{Sp}\,(2g, \mathbb{R})$-representations [11, Theorem 17.5] and $\mathrm{Sp}\,(2g, \mathbb{Z})$ is Zariski dense in $\mathrm{Sp}\,(2g, \mathbb{R})$. The description of the cohomology ring $H^*(\mathcal{N}_g)$ is given in the following

Proposition 4.1 ([27] [14]). *The cohomology ring of the moduli space $\mathcal{N}_g$ of stable bundles of odd degree and rank two over Σ with fixed determinant has a presentation*

$$H^*(\mathcal{N}_g) = \bigoplus_{k=0}^{g} \Lambda_0^k H^3 \otimes \mathbb{C}[a,b,c]/I_{g-k},$$

where $I_r = (q_r^1, q_r^2, q_r^3)$, and q_r^i are defined recursively by setting $q_0^1 = 1$, $q_0^2 = 0$, $q_0^3 = 0$ and then for all $r \geq 0$

$$\begin{cases} q_{r+1}^1 = a q_r^1 + r^2 q_r^2, \\ q_{r+1}^2 = b q_r^1 + \frac{2r}{r+1} q_r^3, \\ q_{r+1}^3 = c q_r^1. \end{cases}$$

The basis $\{f_s\}_{s\in\mathcal{S}}$ of $H^*(\mathcal{N}_g)$ can be chosen to be as follows. Choose, for every $0 \leq k \leq g-1$, a basis $\{x_i^{(k)}\}_{i\in B_k}$ for $\Lambda_0^k H^3$. Then

$$\{x_i^{(k)} a^n b^m c^r / k = 0, 1, \ldots, g-1,\ n+m+r < g-k,\ i \in B_k\} \tag{4.3}$$

is a basis for $H^*(\mathcal{N}_g)$, as proved in [27]. Also Proposition 4.1 gives us the relations for $H^*(\mathcal{N}_g)$. If we set $x_0^{(k)} = c_1 c_2 \cdots c_k \in \Lambda_0^k H^3$, then the relations are given by

$$x_0^{(k)} q_{g-k}^i, \qquad 1 \leq i \leq 3, \quad 0 \leq k \leq g,$$

and their transforms under the $\mathrm{Sp}\,(2g, \mathbb{Z})$-action.

4.2. Floer cohomology HF_g^*. The description of the Floer cohomology $HF_g^* = HF^*(Y)$ of $Y = \Sigma \times \mathbb{S}^1$, where $\Sigma = \Sigma_g$ is a Riemann surface of genus g, is given in [23]. Consider the manifold $A = \Sigma \times D^2$, Σ times a disc, with boundary $Y = \Sigma \times \mathbb{S}^1$, and let $\Delta = \mathrm{pt} \times D^2 \subset A$ be the horizontal slice. Let $w \in H^2(A; \mathbb{Z})$ be any odd multiple of P.D.$[\Delta]$, so that $w|_Y = w_2$. Clearly

$$\mathbb{A}(A) = \mathbb{A}(\Sigma) = \mathrm{Sym}^*(H_0(\Sigma) \oplus H_2(\Sigma)) \otimes \textstyle\bigwedge^* H_1(\Sigma).$$

Define the following elements of $HF^*(Y)$ as in [23]

$$\begin{cases} \alpha = 2\,\phi^w(A, \Sigma) \in HF_g^2, \\ \psi_i = \phi^w(A, \gamma_i) \in HF_g^3, \qquad 0 \leq i \leq 2g, \\ \beta = -4\,\phi^w(A, x) \in HF_g^4. \end{cases} \tag{4.4}$$

The relative invariants of section 2 give a map

$$\begin{array}{rcl} \mathbb{A}(\Sigma) & \to & HF_g^*, \\ z & \mapsto & \phi^w(A,z). \end{array} \tag{4.5}$$

For every $s \in \mathcal{S}$ and f_s as in (4.1), we define

$$\begin{array}{rcl} z_s & = & \Sigma^n x^m \gamma_{i_1} \cdots \gamma_{i_r} \in \mathbb{A}(\Sigma), \\ e_s & = & \phi^w(A, z_s) \in HF_g^*. \end{array} \tag{4.6}$$

As a consequence of [21, Lemma 21], $\{e_s\}_{s\in\mathcal{S}}$ is a basis for HF_g^*. Hence (4.5) is surjective. Now it is easy to check that

$$\phi^w(A,z)\phi^w(A,z') = \phi^w(A,zz'),$$

as for any $s \in \mathcal{S}$, the gluing Theorem 2.2 implies

$$\begin{aligned} \langle \phi^w(A,z)\phi^w(A,z'), \phi^w(A,z_s)\rangle =& D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(zz'z_s) \\ =& \langle \phi^w(A,zz'), \phi^w(A,z_s)\rangle. \end{aligned}$$

In particular this implies that the product of HF_g^* is graded commutative and associative, and that (4.5) is an epimorphism of rings. The neutral element of the product is $\mathbf{1} = \phi^w(A,1)$. The mapping class group Diff(Σ) acts on HF_g^*, with the action factoring through the action of $\mathrm{Sp}\,(2g,\mathbb{Z})$ on $\{\psi_i\}$. It also acts on $\mathbb{A}(\Sigma)$, and (4.5) is $\mathrm{Sp}\,(2g,\mathbb{Z})$-equivariant. The invariant part, $(HF_g^*)_I = HF_I^*(Y)$, is generated by α, β and $\gamma = -2\sum_{i=0}^g \psi_i\psi_{i+g}$, so that there is an epimorphism

$$\mathbb{C}[\alpha,\beta,\gamma] \twoheadrightarrow (HF_g^*)_I,$$

which allows us to write

$$(HF_g^*)_I = \mathbb{C}[\alpha,\beta,\gamma]/J_g,$$

where J_g is the ideal of relations satisfied by α, β and γ. As a matter of notation, let H^3 denote the $2g$-dimensional vector space generated by $\psi_1,\ldots,\psi_{2g}$ in HF^3. Then $H^3 \cong H^3(\mathcal{N}_g)$ and $\phi^w(A,\cdot) : H_1(\Sigma) \stackrel{\cong}{\to} H^3$. No confusion should arise from this multiple use of H^3. Then from [23], a basis for HF_g^* is given by

$$\{x_i^{(k)}\alpha^a\beta^b\gamma^c / k = 0,1,\ldots,g-1,\ a+b+c < g-k,\ i \in B_k\},$$

where $x_i^{(k)} \in \Lambda_0^k H^3$ are interpreted now as Floer products. The explicit description of HF_g^* is given in [23, Theorem 16]

Proposition 4.2. *The Floer cohomology of $Y = \Sigma\times\mathbb{S}^1$, for $\Sigma = \Sigma_g$ a Riemann surface of genus g, and $w_2 = P.D.[\mathbb{S}^1] \in H^2(Y;\mathbb{Z}/2\mathbb{Z})$, has a presentation*

$$HF^*(\Sigma\times\mathbb{S}^1) = \bigoplus_{k=0}^{g} \Lambda_0^k H^3 \otimes \mathbb{C}[\alpha,\beta,\gamma]/J_{g-k}.$$

where $J_r = (R_r^1, R_r^2, R_r^3)$ and R_r^i are defined recursively by setting $R_0^1 = 1$, $R_0^2 = 0$, $R_0^3 = 0$ and putting for all $r \geq 0$

$$\begin{cases} R_{r+1}^1 = \alpha R_r^1 + r^2 R_r^2, \\ R_{r+1}^2 = (\beta + (-1)^{r+1}8)R_r^1 + \frac{2r}{r+1}R_r^3, \\ R_{r+1}^3 = \gamma R_r^1. \end{cases}$$

The meaning of this proposition is the following. The Floer (co)homology HF_g^* is generated as a ring by α, β and ψ_i, $1\leq i\leq 2g$, and the relations are

$$x_0^{(k)} R_{g-k}^i, \qquad 1\leq i\leq 3, \quad 0\leq k\leq g,$$

where $x_0^{(k)} = \psi_1\psi_2\cdots\psi_k \in \Lambda_0^k H^3$, and the $\mathrm{Sp}\,(2g,\mathbb{Z})$-transforms of these. Also if we write

$$F_r = \mathbb{C}[\alpha,\beta,\gamma]/J_r = (HF_r^*)_I,$$

then $HF_g^* = \oplus \Lambda_0^k H^3 \otimes F_{g-k}$. We finish the section with two technical results about the quotient $\bar{F}_r = F_r/\gamma F_r$, which will be necessary in section 7.

Proposition 4.3. *Let $\bar{F}_r = F_r/\gamma F_r$, $r\geq 0$. Then $\bar{F}_r$ has basis $\alpha^a\beta^b$, $a+b<r$. We have $\bar{F}_r = \mathbb{C}[\alpha,\beta]/\bar{J}_r$, where $\bar{J}_r = (\bar{R}_r^1, \bar{R}_r^2)$, and $\bar{R}_r^i$ are determined by $\bar{R}_0^1 = 1$, $\bar{R}_0^2 = 0$ and then recursively for all $r\geq 0$,*

$$\begin{cases} \bar{R}_{r+1}^1 = \alpha\bar{R}_r^1 + r^2\bar{R}_r^2, \\ \bar{R}_{r+1}^2 = (\beta + (-1)^{r+1}8)\bar{R}_r^1. \end{cases}$$

Proof. The $\binom{r+1}{2}$ elements $\alpha^a\beta^b$, $a+b<r$, generate $\bar{F}_r$. Also Poincaré duality identifies $\bar{F}_r = F_r/\gamma F_r$ with $\ker(\gamma : F_r \to F_r)$ which equals J_{r-1}/J_r, by [23, Corollary 18]. So

$$\begin{aligned} \dim \bar{F}_r &= \dim(\mathbb{C}[\alpha,\beta,\gamma]/J_r) - \dim(\mathbb{C}[\alpha,\beta,\gamma]/J_{r-1}) \\ &= \dim F_r - \dim F_{r-1} = \binom{r+1}{2}, \end{aligned}$$

and therefore $\alpha^a\beta^b$, $a+b<r$, form a basis for $\bar{F}_r$. q.e.d.

Lemma 4.4. *We have*

$$\bar{J}_r/\bar{J}_{r+1} = \ker(\bar{F}_{r+1} \twoheadrightarrow \bar{F}_r) = \bigoplus_{\substack{-r \le i \le r \\ i \equiv r \pmod 2}} R_{r+1,i},$$

where $R_{r+1,i}$ is a 1-dimensional vector space such that

$$R_{r+1,i} = \mathbb{C}[\alpha,\beta]/(\alpha - 4i\sqrt{-1}, \beta - 8)$$

for r even,

$$R_{r+1,i} = \mathbb{C}[\alpha,\beta]/(\alpha - 4i, \beta + 8)$$

for r odd.

Proof. The first equality follows from the exact sequence

$$\frac{\bar{J}_r}{\bar{J}_{r+1}} \hookrightarrow \bar{F}_{r+1} = \frac{\mathbb{C}[\alpha,\beta]}{\bar{J}_{r+1}} \twoheadrightarrow \bar{F}_r = \frac{\mathbb{C}[\alpha,\beta]}{\bar{J}_r}.$$

Next we claim that

$$(\beta + (-1)^{r+1}8)\bar{J}_r \subset \bar{J}_{r+1} \subset \bar{J}_r,$$

$r \geq 0$. The second inclusion is obvious as $\bar{R}^i_{r+1}$ are written in terms of $\bar{R}^i_r$ by Proposition 4.3. The first inclusion follows from $(\beta + (-1)^{r+1}8)\bar{R}^1_r = \bar{R}^2_{r+1} \in \bar{J}_{r+1}$ and then multiplying the first equation in Proposition 4.3 by $(\beta + (-1)^{r+1}8)$ to get $(\beta + (-1)^{r+1}8)\bar{R}^2_r \in \bar{J}_{r+1}$.

Now

$$\bar{J}_r/\bar{J}_{r+1} = \ker(\beta + (-1)^{r+1}8 : \bar{F}_{r+1} \to \bar{F}_{r+1}).$$

This is seen by factoring the map $\beta + (-1)^{r+1}8$ as

$$\mathbb{C}[\alpha,\beta]/\bar{J}_{r+1} \twoheadrightarrow \mathbb{C}[\alpha,\beta]/\bar{J}_r \overset{\beta+(-1)^{r+1}8}{\hookrightarrow} \mathbb{C}[\alpha,\beta]/\bar{J}_{r+1}.$$

The second map is well defined by the claim above and it is a monomorphism since $\alpha^a\beta^b$, $a+b<r$, form a basis for $\mathbb{C}[\alpha,\beta]/\bar{J}_r$, and their image under $\beta + (-1)^{r+1}8$ are linearly independent in $\mathbb{C}[\alpha,\beta]/\bar{J}_{r+1}$. As $\bar{F}_{r+1}$ is a Poincaré duality algebra (being a complete intersection algebra),

$$\ker(\beta + (-1)^{r+1}8 : \bar{F}_{r+1} \to \bar{F}_{r+1})$$

is dual to

$$\bar{F}_{r+1}/(\beta + (-1)^{r+1}8) = F_{r+1}/(\beta + (-1)^{r+1}8, \gamma).$$

Using the computations in the proof of [23, Proposition 20], we get finally

$$\begin{aligned}\bar{J}_r/\bar{J}_{r+1} =& F_{r+1}/(\beta+(-1)^{r+1}8,\gamma)\\ =& \begin{cases} \mathbb{C}[\alpha]/\left((\alpha^2+r^2 16)\cdots(\alpha^2+2^2 16)\alpha\right), & r \text{ even}\\ \mathbb{C}[\alpha]/\left((\alpha^2-r^2 16)\cdots(\alpha^2-1^2 16)\right), & r \text{ odd}\end{cases}\end{aligned}$$

as required. q.e.d.

5. Fukaya-Floer homology $HFF_*(\Sigma\times\mathbb{S}^1,\mathbb{S}^1)$

In this section we are going to describe the Fukaya-Floer (co)homology of the 3-manifold $Y=\Sigma\times\mathbb{S}^1$ with the $SO(3)$-bundle with $w_2=$ P.D.$[\mathbb{S}^1]\in H^2(Y;\mathbb{Z}/2\mathbb{Z})$ and loop $\delta=\mathrm{pt}\times\mathbb{S}^1\subset Y=\Sigma\times\mathbb{S}^1$, together with its ring structure. As Y admits an orientation reversing self-diffeomorphism, we can identify its Fukaya-Floer homology and Fukaya-Floer cohomology through Poincaré duality, as we shall do. From now on we fix the genus $g\geq 1$ of Σ and denote $HFF_g^*=HFF^*(\Sigma\times\mathbb{S}^1,\mathbb{S}^1)$.

5.1. The vector space HFF_g^*. The following argument is taken from [21]. The spectral sequence computing HFF_g^* has E_3 term $HF_g^*\otimes\mathbb{C}[[t]]$. All the differentials in this E_3 term are of the form $HF_g^{\text{odd}}\to HF_g^{\text{even}}$ and $HF_g^{\text{even}}\to HF_g^{\text{odd}}$. As $\mathbb{S}^1$ is invariant under the action of the mapping class group $\text{Diff}(\Sigma)$ on $Y=\Sigma\times\mathbb{S}^1$, the differentials commute with the action of $\text{Diff}(\Sigma)$. Since there are elements $f\in\text{Diff}(\Sigma)$ acting as -1 on $H_1(\Sigma)$, we have that f acts as -1 on HF_g^{odd} and as 1 on HF_g^{even}. Therefore the differentials are zero. Analogously for the higher differentials. So the spectral sequence degenerates in the third term and

$$HFF_g^*=HF_g^*\otimes\mathbb{C}[[t]]=HF_g^*[[t]].$$

The pairing in HFF_g^* is induced from that of HF_g^* by coefficient extension to $\mathbb{C}[[t]]$.

For a 4-manifold X with boundary $\partial X=Y$, $w\in H^2(X;\mathbb{Z})$ with $w|_Y=w_2$ and $D\in H_2(X)$ with $\partial D=\mathbb{S}^1$, the relative invariants will be $\phi^w(X,e^{tD})\in HF_g^*[[t]]$, i.e., formal power series with coefficients in the Floer cohomology HF_g^*.

Recall the manifold $A=\Sigma\times D^2$, with boundary $Y=\Sigma\times\mathbb{S}^1$, and let $\Delta=\mathrm{pt}\times D^2\subset A$ be the horizontal slice with $\partial\Delta=\mathbb{S}^1$. Let $w\in H^2(A;\mathbb{Z})$ be any odd multiple of P.D.$[\Delta]$, so that $w|_Y=w_2\in H^2(Y;\mathbb{Z}/2\mathbb{Z})$. The elements

$$\hat{e}_s=\phi^w(A,z_s\,e^{t\Delta})\in HFF_g^* \tag{5.1}$$

analogous to the elements e_s of (4.6), for $s \in \mathcal{S}$, are a basis of HFF_g^* as $\mathbb{C}[[t]]$-module (see [21, Lemma 21]). There is a well defined map $HFF_g^* = HF_g^* \otimes \mathbb{C}[[t]] \twoheadrightarrow HF_g^*$ formally obtained by equating $t=0$. It takes $\phi^w(A, z\, e^{t\Delta}) \mapsto \phi^w(A, z)$, for any $z \in \mathbb{A}(\Sigma)$. This map intertwines the μ actions on HFF_g^* and HF_g^*, and respects the pairings.

5.2. The ring HFF_g^*. The ring structure of HFF_g^* comes from the cobordism between $(Y, \mathbb{S}^1) \sqcup (Y, \mathbb{S}^1)$ and $(Y, \mathbb{S}^1)$, given by the pair of pants times (Σ, pt). This yields

$$HFF_g^* \otimes HFF_g^* \to HFF_g^*,$$

which is an associative and graded commutative ring structure on HFF_g^*. We prove this as for the case of Floer homology by showing first that $\phi^w(A, z\, e^{t\Delta})\phi^w(A, z' e^{t\Delta}) = \phi^w(A, zz'\, e^{t\Delta})$, so that

$$\begin{array}{ccc} \mathbb{A}(\Sigma) \otimes \mathbb{C}[[t]] & \to & HFF_g^*, \\ z & \mapsto & \phi^w(A, z\, e^{t\Delta}) \end{array} \tag{5.2}$$

is a $\mathbb{C}[[t]]$-linear epimorphism of rings. The map $HFF_g^* \twoheadrightarrow HF_g^*$ mentioned above is a ring epimorphism.

Lemma 5.1. *The product on HFF_g^* extends the action of $H_*(\Sigma)$ in HFF_g^*. More specifically, $\mu(\Sigma)$ is Fukaya-Floer multiplication by $\phi^w(A, \Sigma\, e^{t\Delta})$, and analogously for $\mu(\gamma_i)$, $1 \le i \le 2g$, and $\mu(x)$.*

Proof. We only need to check the statement for elements of the form $\phi^w(A, z\, e^{t\Delta})$, as they generate the whole of HFF_g^* as a vector space. For instance,

$$\mu(\Sigma)\phi^w(A, z\, e^{t\Delta}) = \phi^w(A, \Sigma z\, e^{t\Delta}) = \phi^w(A, \Sigma\, e^{t\Delta})\phi^w(A, z\, e^{t\Delta}),$$

and analogously in the other cases. q.e.d.

We define the following elements of HFF_g^* which are generators as $\mathbb{C}[[t]]$-algebra,

$$\left\{ \begin{array}{ll} \hat{\alpha} = 2\, \phi^w(A, \Sigma\, e^{t\Delta}) \in HFF_g^2, & \\ \hat{\psi}_i = \phi^w(A, \gamma_i\, e^{t\Delta}) \in HFF_g^3, & 0 \le i \le 2g \\ \hat{\beta} = -4\, \phi^w(A, x\, e^{t\Delta}) \in HFF_g^4. & \end{array} \right. \tag{5.3}$$

The mapping class group $\mathrm{Diff}(\Sigma)$ acts on both sides of (5.2) with the action factoring through an action of $\mathrm{Sp}\,(2g, \mathbb{Z})$. The invariants parts surject

$$\mathbb{C}[\hat{\alpha}, \hat{\beta}, \hat{\gamma}] \otimes \mathbb{C}[[t]] = \mathbb{C}[[t]][\hat{\alpha}, \hat{\beta}, \hat{\gamma}] \twoheadrightarrow (HFF_g^*)_I, \tag{5.4}$$

where $\hat{\gamma} = -2\sum_{i=0}^{g} \hat{\psi}_i \hat{\psi}_{i+g}$. Thus we can write

$$(HFF_g^*)_I = \mathbb{C}[[t]][\hat{\alpha}, \hat{\beta}, \hat{\gamma}]/\mathcal{J}_g, \tag{5.5}$$

where $\mathcal{J}_g$ is the ideal of relations of the generators $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$. Recall that t has homological degree 2 and hence cohomological degree -2. The other cohomological degrees are $\deg \hat{\alpha} = 2$, $\deg \hat{\psi}_i = 3$, $\deg \hat{\beta} = 4$ and $\deg \hat{\gamma} = 6$.

The ring structure of HFF_g^*, which is in some sense equivalent to the determination of the kernel of (5.4), runs closely parallel to the arguments in [23] to find out the ring structure of $HF_g^* = HF^*(Y)$. We recommend the reader to have [23] at hand.

Consider the ring $H^*(\mathcal{N}_g)[[t]]$, where t is given degree -2. The elements in $H^i(\mathcal{N}_g)[[t]]$ are thus sums $\sum_{n \geq 0} s_{i+2n} t^n$, where $\deg(s_{i+2n}) = i + 2n$. Note that all such elements are finite sums, although $H^i(\mathcal{N}_g)[[t]] \neq 0$ for arbitrarily negative i. The following result is an analogue of [23, Theorem 5] and provides a proof of Theorem 1.2.

Proposition 5.2. *Denote by $*$ the product induced in $H^*(\mathcal{N}_g)[[t]]$ by the product in HFF_g^* under the $\mathbb{C}[[t]]$-linear isomorphism $H^*(\mathcal{N}_g)[[t]] \overset{\cong}{\to} HFF_g^*$ given by $f_s \mapsto \hat{e}_s$, $s \in \mathcal{S}$. Then $*$ is a deformation of the cup-product graded modulo 4, i.e., for $f_1 \in H^i(\mathcal{N}_g)[[t]]$, $f_2 \in H^j(\mathcal{N}_g)[[t]]$, it is $f_1 * f_2 = \sum_{r \geq 0} \Phi_r(f_1, f_2)$, where $\Phi_r \in H^{i+j-4r}(\mathcal{N}_g)[[t]]$ and $\Phi_0 = f_1 \cup f_2$.*

Proof. To start with, let us fix some notation. The choice of basis (5.1) gives a splitting $\imath : H^*(\mathcal{N}_g) \to \mathbb{A}(\Sigma)$, $f_s \mapsto z_s$, satisfying the property that $f \mapsto \phi^w(A, \imath(f)\, e^{t\Delta})$ is the isomorphism of the statement.

Now we claim that for any $s, s' \in \mathcal{S}$ we have

$$\begin{aligned} \langle \hat{e}_s, \hat{e}_{s'} \rangle =& D^{(w,\Sigma)}_{\Sigma \times \mathbb{CP}^1}(z_s z_{s'} e^{t\mathbb{CP}^1}) \\ =& -\langle f_s, f_{s'} \rangle + O(t^{(6g-6-(\deg(f_s)+\deg(f_{s'})))/2+1}), \end{aligned} \tag{5.6}$$

where $O(t^r)$ means any element in $t^r\mathbb{C}[[t]]$ (note that (5.6) vanishes when $\deg(f_s) + \deg(f_{s'}) \not\equiv 0 \pmod 2$). If $\deg(f_s) + \deg(f_{s'}) > 6g - 6$ then the statement is vacuous. For $\deg(f_s) + \deg(f_{s'}) \leq 6g - 6$ it follows from the fact that the dimensions of the moduli spaces of anti-self-dual connections on $\Sigma \times \mathbb{CP}^1$ are $6g - 6 + 4r$, $r \geq 0$, and the $(6g - 6)$-dimensional moduli space is $\mathcal{N}_g$, as remarked in [23], so that for

$$\deg(f_s) + \deg(f_{s'}) + 2d = 6g - 6,$$

it is

$$D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'}(\mathbb{CP}^1)^d) = 0,$$

unless $d = 0$, and in that case it gives $-\langle f_s, f_{s'}\rangle$ (the minus sign is due to the different convention orientation for Donaldson invariants).

We shall check the statement of the proposition on basic elements f_s and $f_{s'}$ of degrees i and j respectively. Put $f_s * f_{s'} = \sum_{m\leq M} g_m$, where $g_m \in H^m(\mathcal{N}_g)[[t]]$ and $g_M \neq 0$ is the leading term. By definition, $\hat{e}_s\hat{e}_{s'} = \sum_{m\leq M} \hat{g}_m$ (with $\hat{g}_m \in HFF^*_g$ corresponding to g_m under the isomorphism of the statement).

Suppose $M > i + j$. Then let ft^r, $f \in H^*(\mathcal{N}_g)$, be the non-zero monomial in g_M with minimum r. So f has degree $M + 2r$. Pick $f' \in H^{6g-6-(M+2r)}(\mathcal{N}_g)$ with $\langle f, f'\rangle = -1$ in $H^*(\mathcal{N}_g)$. Let $z, z' \in \mathbb{A}(\Sigma)$ be the elements corresponding to $f, f' \in H^*(\mathcal{N}_g)$ under the splitting $\imath$. Then by (5.6)

$$\langle t^r \phi^w(A, ze^{t\Delta}), \phi^w(A, z'e^{t\Delta})\rangle = t^r + O(t^{r+1}),$$

so

$$\langle \hat{g}_M, \phi^w(A, z'e^{t\Delta})\rangle = t^r + O(t^{r+1}).$$

For $m < M$, it must be $\langle \hat{g}_m, \phi^w(A, z'e^{t\Delta})\rangle = O(t^{r+1})$ by (5.6) again, so finally

$$\langle \hat{e}_s\hat{e}_{s'}, \phi^w(A, z'e^{t\Delta})\rangle = t^r + O(t^{r+1}).$$

On the other hand, as $\deg(f_s) + \deg(f_{s'}) + \deg(f') < 6g - 6 - 2r$, it is

$$\langle \hat{e}_s\hat{e}_{s'}, \phi^w(A, z'e^{t\Delta})\rangle = D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'} z' e^{t\mathbb{CP}^1}) = O(t^{r+1}),$$

which is a contradiction. It must be $M \leq i + j$.

For $m = i + j$, put $g_m = G_{i+j} + tG_{i+j+2} + \cdots$, where

$$G_{i+j+2r} \in H^{i+j+2r}(\mathcal{N}_g).$$

Pick any $f_{s''}$ of degree $6g - 6 - m$. Clearly

$$D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'} z_{s''} e^{t\mathbb{CP}^1}) = -\langle f_s f_{s'}, f_{s''}\rangle + O(t).$$

Also

$$\begin{aligned} D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'} z_{s''} e^{t\mathbb{CP}^1}) =& \langle \hat{e}_s\hat{e}_{s'}, \hat{e}_{s''}\rangle \\ =& \langle \hat{g}_m, \hat{e}_{s''}\rangle + O(t) = -\langle g_m, f_{s''}\rangle + O(t). \end{aligned}$$

So $\langle G_{i+j}, f_{s''}\rangle = \langle f_s f_{s'}, f_{s''}\rangle$, for arbitrary $f_{s''}$, and hence $G_{i+j} = f_s f_{s'}$.

To check that $G_{i+j+2r}=0$ for $r>0$, pick any $f_{s''}$ of degree $6g-6-(m+2r)$. By (5.6) it is $D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'} z_{s''} e^{t\mathbb{CP}^1})=O(t^{r+1})$ and

$$\begin{aligned} D^{(w,\Sigma)}_{\Sigma\times\mathbb{CP}^1}(z_s z_{s'} z_{s''} e^{t\mathbb{CP}^1}) =&\langle \hat{e}_s\hat{e}_{s'},\hat{e}_{s''}\rangle \\ =&\langle \hat{g}_m,\hat{e}_{s''}\rangle + O(t^{r+1}) \\ =&-\langle G_{i+j+2r}, f_{s''}\rangle t^r + O(t^{r+1}), \end{aligned}$$

So $\langle G_{i+j+2r}, f_{s''}\rangle=0$, i.e., $G_{i+j+2r}=0$. q.e.d.

The structure of HFF^*_g is given by the following result.

Theorem 5.3. *Fix $g\geq 1$. Let $\Sigma=\Sigma_g$ be a Riemann surface of genus g. The Fukaya-Floer cohomology $HFF^*_g=HFF^*(\Sigma\times\mathbb{S}^1,\mathbb{S}^1)$ has a presentation*

$$HFF^*_g=\bigoplus_{k=0}^{g}\Lambda_0^k H^3\otimes\mathbb{C}[[t]][\hat{\alpha},\hat{\beta},\hat{\gamma}]/\mathcal{J}_{g-k}.$$

where $\mathcal{J}_r=(\mathcal{R}^1_r,\mathcal{R}^2_r,\mathcal{R}^3_r)$ and $\mathcal{R}^i_r$ are defined recursively by setting $\mathcal{R}^1_0=1$, $\mathcal{R}^2_0=0$, $\mathcal{R}^3_0=0$ and putting, for all $0\leq r\leq g-1$,

$$(5.7)\qquad \begin{cases} \mathcal{R}^1_{r+1}=(\hat{\alpha}+f_{11}(t))\mathcal{R}^1_r+r^2(1+f_{12}(t))\mathcal{R}^2_r+f_{13}(t)\mathcal{R}^3_r, \\ \mathcal{R}^2_{r+1}=(\hat{\beta}+(-1)^{r+1}8+f_{21}(t))\mathcal{R}^1_r \\ \qquad\qquad +f_{22}(t)\mathcal{R}^2_r+(\frac{2r}{r+1}+f_{23}(t))\mathcal{R}^3_r, \\ \mathcal{R}^3_{r+1}=\hat{\gamma}\mathcal{R}^1_r, \end{cases}$$

for some (unknown) functions $f^{r;g}_{ij}(t)$ lying in $t\mathbb{C}[[t]][\hat{\alpha},\hat{\beta},\hat{\gamma}]$, dependent on r and g. Moreover f_{ij} are such that $f_{11}\mathcal{R}^1_r+f_{12}\mathcal{R}^2_r+f_{13}\mathcal{R}^3_r$ and $f_{21}\mathcal{R}^1_r+f_{22}\mathcal{R}^2_r+f_{23}\mathcal{R}^3_r$ are both $\mathbb{C}[[t]]$-linear combinations of the monomials $\hat{\alpha}^a\hat{\beta}^b\hat{\gamma}^c$, $a+b+c<r+1$.

Proof. From (4.3) a basis of $H^*(\mathcal{N}_g)[[t]]$ as $\mathbb{C}[[t]]$-module is given by

$$\{x^{(k)}_i a^n b^m c^r / k=0,1,\ldots,g-1,\ n+m+r<g-k,\ i\in B_k\}.$$

Recalling $x^{(k)}_0=c_1c_2\cdots c_k\in\Lambda^k_0H^3$, a complete set of relations satisfied in $H^*(\mathcal{N}_g)$ are $x^{(k)}_0 q^i_{g-k}$, $i=1,2,3$, $0\leq k\leq g$, and the $\mathrm{Sp}\,(2g,\mathbb{Z})$-transforms of these. Now identifying H^3 with the $2g$-dimensional subspace of HFF^3_g generated by $\hat{\psi}_1,\ldots,\hat{\psi}_{2g}$, Proposition 5.2 implies that the set

$$\{x^{(k)}_i\hat{\alpha}^a\hat{\beta}^b\hat{\gamma}^c/k=0,1,\ldots,g-1,\ a+b+c<g-k,\ i\in B_k\},$$

where $x_i^{(k)} \in \Lambda_0^k H^3 \subset HFF_g^*$, is a basis for HFF_g^* as $\mathbb{C}[[t]]$-module, where Fukaya-Floer multiplication is understood. Also from Proposition 4.1, we can write

$$H^*(\mathcal{N}_g)[[t]] = \bigoplus_{k=0}^{g} \Lambda_0^k H^3 \otimes \frac{\mathbb{C}[[t]][a,b,c]}{(q_{g-k}^1, q_{g-k}^2, q_{g-k}^3)}.$$

The products in both $H^*(\mathcal{N}_g)[[t]]$ and HFF_g^* are $\mathrm{Sp}\,(2g,\mathbb{Z})$-equivariant, and the isomorphism in the statement of Proposition 5.2 is also $\mathrm{Sp}\,(2g,\mathbb{Z})$-equivariant. Then we can use the arguments in the proof of [24, Proposition 16] to write

$$HFF_g^* = \bigoplus_{k=0}^{g} \Lambda_0^k H^3 \otimes \frac{\mathbb{C}[[t]][\hat{\alpha},\hat{\beta},\hat{\gamma}]}{(\mathcal{R}_{g-k}^1, \mathcal{R}_{g-k}^2, \mathcal{R}_{g-k}^3)},$$

where if we put $x_0^{(k)} = \hat{\psi}_1\hat{\psi}_2\cdots\hat{\psi}_k \in \Lambda_0^k H^3$, then $x_0^{(k)}\mathcal{R}_{g-k}^i$, $i=1,2,3$, $0 \le k \le g$, and their $\mathrm{Sp}\,(2g,\mathbb{Z})$-transforms, are a complete set of relations for HFF_g^*. More explicitly, we decompose $HFF_g^* = \bigoplus_{k=0}^{g} \mathcal{V}_k$, where $\mathcal{V}_k = \Lambda_0^k H^3 \otimes \mathcal{F}_{g-k}$ is the image of

$$\Lambda_0^k H^3 \otimes \mathbb{C}[[t]][\hat{\alpha},\hat{\beta},\hat{\gamma}] \to HFF_g^*,$$

so in particular, the invariant part is $\mathcal{V}_0 = (HFF_g^*)_I$ and $\mathcal{V}_g = 0$. Then

$$HFF_g^* = \bigoplus_{k=0}^{g} \Lambda_0^k H^3 \otimes \mathcal{F}_{g-k},$$

where

$$\mathcal{F}_{g-k} = \frac{\mathbb{C}[[t]][\hat{\alpha},\hat{\beta},\hat{\gamma}]}{\mathcal{J}_{g-k}}, \tag{5.8}$$

for $0 \le k \le g$, where the generators of the ideal $\mathcal{J}_{g-k} \subset \mathbb{C}[[t]][\hat{\alpha},\hat{\beta},\hat{\gamma}]$ are obtained by writing q_{g-k}^1, q_{g-k}^2, q_{g-k}^3 in terms of the Fukaya-Floer product (see [28] for an analogous argument in the study of quantum cohomology)

$$q_{g-k}^i = \sum c_{abcd}^i \hat{\alpha}^a \hat{\beta}^b \hat{\gamma}^c t^d,$$

where the sum runs for $a+b+c < g-k$, $d \ge 0$, $2a+4b+6c-2d = \deg q_{g-k}^i - 4r$, $r > 0$ and $c_{abcd}^i \in \mathbb{C}$. So $\mathcal{J}_{g-k} = (\mathcal{R}_{g-k}^1, \mathcal{R}_{g-k}^2, \mathcal{R}_{g-k}^3)$ with

$$\mathcal{R}_{g-k}^i = q_{g-k}^i - \sum c_{abcd}^i \hat{\alpha}^a \hat{\beta}^b \hat{\gamma}^c t^d.$$

The elements $\mathcal{R}^i_{g-k}$ are uniquely defined by the following two conditions: $x_0^{(k)}\mathcal{R}^i_{g-k} = 0 \in HFF^*_g$ and $\mathcal{R}^1_{g-k}$ (respectively $\mathcal{R}^2_{g-k}$, $\mathcal{R}^3_{g-k}$) equals $\hat{\alpha}^{g-k}$ (respectively $\hat{\alpha}^{g-k-1}\hat{\beta}$, $\hat{\alpha}^{g-k-1}\hat{\gamma}$) plus terms of the form $\hat{\alpha}^a\hat{\beta}^b\hat{\gamma}^c t^d$ with $a+b+c<g-k$. Note that they might depend, in principle, not only on $g-k$ but also on the genus g (which was fixed throughout this section). In particular $\mathcal{R}^1_0 = 1$, $\mathcal{R}^2_0 = 0$ and $\mathcal{R}^3_0 = 0$.

Analogously to [24, Lemma 17], we prove the following chain of inclusions, for $0 \le r \le g-1$,

$$\hat{\gamma}\mathcal{J}_r \subset \mathcal{J}_{r+1} \subset \mathcal{J}_r.$$

It remains to prove the recurrence stated in (5.7), which is similar to [23, Theorem 10]. The inclusion $\hat{\gamma}\mathcal{J}_r \subset \mathcal{J}_{r+1}$ says that $\hat{\gamma}\mathcal{R}^1_r$ must be in $\mathcal{J}_{r+1}$, so it must coincide with $\mathcal{R}^3_{r+1}$. Now the inclusion $\mathcal{J}_{r+1} \subset \mathcal{J}_r$ implies the recurrence as written in (5.7) with $f_{ij} \in \mathbb{C}[[t]][\hat{\alpha},\hat{\beta},\hat{\gamma}]$. Lastly, the $\mathrm{Sp}\,(2g,\mathbb{Z})$-equivariant epimorphism $HFF^*_g \twoheadrightarrow HF^*_g$ yields that $\mathcal{R}^i_g$ reduces to R^i_g when we set $t=0$. Thus the functions f_{ij} are multiples of t.

The last sentence of the statement follows from the fact that $\mathcal{R}^1_{r+1}$ is written as a series with leading term $\hat{\alpha}^{r+1}$ plus terms of the form $\hat{\alpha}^a\hat{\beta}^b\hat{\gamma}^c t^d$, $a+b+c<r+1$, and that $\mathcal{R}^2_{r+1}$ is written as a series with leading term $\hat{\alpha}^r\hat{\beta}$ plus terms of the form $\hat{\alpha}^a\hat{\beta}^b\hat{\gamma}^c t^d$, $a+b+c<r+1$.

q.e.d.

We give the following two results, whose proofs are left to the reader, for completeness.

Corollary 5.4. *Fix $g \ge 1$. Let $\Sigma = \Sigma_g$ be a Riemann surface of genus g. Let $n \in \mathbb{Z}$. The Fukaya-Floer cohomology $HFF^*(\Sigma \times \mathbb{S}^1, n\mathbb{S}^1)$ has a presentation*

$$HFF^*(\Sigma \times \mathbb{S}^1, n\mathbb{S}^1) = \bigoplus_{k=0}^{g} \Lambda_0^k H^3 \otimes \mathbb{C}[[t]][\hat{\alpha},\hat{\beta},\hat{\gamma}]/\mathcal{J}_{g-k},$$

where $\mathcal{J}_r = (\mathcal{R}^1_r, \mathcal{R}^2_r, \mathcal{R}^3_r)$, and $\mathcal{R}^i_r$ are defined recursively by setting $\mathcal{R}^1_0 = 1$, $\mathcal{R}^2_0 = 0$, $\mathcal{R}^3_0 = 0$ and putting, for all $0 \le r \le g-1$,

$$\begin{cases} \mathcal{R}^1_{r+1} = (\hat{\alpha} + f_{11}(nt))\mathcal{R}^1_r + r^2(1+f_{12}(nt))\mathcal{R}^2_r + f_{13}(nt)\mathcal{R}^3_r, \\ \mathcal{R}^2_{r+1} = (\hat{\beta} + (-1)^{r+1}8 + f_{21}(nt))\mathcal{R}^1_r \\ \qquad\qquad + f_{22}(nt)\mathcal{R}^2_r + (\frac{2r}{r+1} + f_{23}(nt))\mathcal{R}^3_r, \\ \mathcal{R}^3_{r+1} = \hat{\gamma}\mathcal{R}^1_r. \end{cases}$$

In particular, for $n = 0$ we recuperate Proposition 4.2.

Proposition 5.5. *Let $n \in \mathbb{Z}$ and consider $HFF^*(\Sigma \times \mathbb{S}^1, n\mathbb{S}^1)$. Then for any $\alpha \in H_0(\Sigma)$ or $\alpha \in H_1(\Sigma)$, the action of $\mu(\alpha \times \mathbb{S}^1)$ in $HFF^*(\Sigma \times \mathbb{S}^1, n\mathbb{S}^1)$ is zero.*

6. Reduced Fukaya-Floer homology

In this section we give a detailed description of the subspace of HFF^*_g which controls the gluing theory of 4-manifolds with $b^+ > 1$ which are of strong simple type.

Suppose that X_1 is a 4-manifold with boundary $\partial X_1 = Y$, $w \in H^2(X; \mathbb{Z})$ satisfies $w|_Y = w_2 = \text{P.D.}[\mathbb{S}^1]$ and $X = X_1 \cup_Y A$ is a closed 4-manifold with $b^+ > 1$ and of strong simple type. Then

$$\phi^w(X_1, z_1 e^{tD_1}) \in \ker(\hat{\beta}^2 - 64) \cap \bigcap_{1 \le i \le 2g} \ker \hat{\psi}_i,$$

for any $z_1 \in \mathbb{A}(X_1)$ and any $D_1 \subset X_1$ with $\partial D_1 = \mathbb{S}^1$. Indeed

$$\langle (\hat{\beta}^2 - 64)\phi^w(X_1, z_1 e^{tD_1}), \hat{e}_s \rangle = D_X^{(w,\Sigma)}(16 z_1 z_s (x^2 - 4) e^{tD}) = 0,$$

for any $\hat{e}_s$ (defined by (5.1)), $s \in \mathcal{S}$, where $D = D_1 + \Delta \in H_2(X)$. Then $(\hat{\beta}^2 - 64)\phi^w(X_1, z_1 e^{tD_1}) = 0$. Analogously $\hat{\psi}_i \phi^w(X_1, z_1 e^{tD_1}) = 0$, for $1 \le i \le 2g$. So it is natural to give

Definition 6.1. We define the *reduced Fukaya-Floer homology of* $\Sigma \times \mathbb{S}^1$ to be

$$\overline{HFF}^*_g = HFF^*_g / (\hat{\beta}^2 - 64, \hat{\psi}_1, \ldots, \hat{\psi}_{2g}) \cong \ker(\hat{\beta}^2 - 64) \cap \bigcap_{1 \le i \le 2g} \ker \hat{\psi}_i,$$

where the last isomorphism is Poincaré duality. Note that $\overline{HFF}^*_g = (HFF^*_g)_I / (\hat{\beta}^2 - 64, \hat{\gamma})$.

The relevant structure theorem for $\overline{HFF}^*_g$ is given by

Theorem 6.2. *$\overline{HFF}^*_g$ is a free $\mathbb{C}[[t]]$-module of rank $2g - 1$. Moreover $\overline{HFF}^*_g = \bigoplus_{i=-(g-1)}^{g-1} R_{g,i}$, where $R_{g,i}$ are free $\mathbb{C}[[t]]$-modules of rank 1 such that for i odd, $\hat{\alpha} = 4i + 2t$ and $\hat{\beta} = -8$ in $R_{g,i}$. For i even, $\hat{\alpha} = 4i\sqrt{-1} - 2t$ and $\hat{\beta} = 8$ in $R_{g,i}$.*

For the proof we need to use the following result.

Proposition 6.3. *For each $0 \le k \le g-1$, there exists a non-zero vector $v \in (HFF_g^*)_I$ such that*

$$\begin{aligned} \hat{\alpha} v &= \begin{cases} (\pm 4(g-k-1)+2t)\, v, & g-k \text{ even}, \\ (\pm 4(g-k-1)\sqrt{-1}-2t)\, v, & g-k \text{ odd}, \end{cases} \\ \hat{\beta} v &= (-1)^{g-k-1} 8\, v, \\ \hat{\gamma} v &= 0. \end{aligned}$$

Proof. This is an extension of [23, Proposition 12]. We shall construct the vector corresponding to the plus sign, the other one being analogous. We have the following cases:

- $0 = k < g-1$. We shall look for $v \in (HFF_g^*)_I$ constructed as the relative invariants of a particular 4-manifold (see section 3). For finding such a vector v we use the same manifold as in the proof of [23, Proposition 12]. This is a 4-manifold $X = C_g$ with an embedded Riemann surface Σ of genus g and self-intersection zero, and $w \in H^2(X;\mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod 2$. Such X is of simple type, with $b_1 = 0$, $b^+ > 1$. Suppose for simplicity that $g-k = g$ is even (the other case is analogous). Then the Donaldson invariants of X are

$$D_X^{(w,\Sigma)}(e^{\alpha}) = -2^{3g-5} e^{Q(\alpha)/2} e^{K\cdot\alpha} + (-1)^g 2^{3g-5} e^{Q(\alpha)/2} e^{-K\cdot\alpha}, \tag{6.1}$$

for a single basic class $K \in H^2(X;\mathbb{Z})$ with $K \cdot \Sigma = 2g-2$. Let X_1 be X with a small open tubular neighbourhood of Σ removed, so that $X = X_1 \cup_Y A$. Consider $D \subset X$ intersecting transversely Σ in just one positive point. Let $D_1 = X_1 \cap D \subset X_1$, so that $\partial D_1 = \mathbb{S}^1$ and $D = D_1 + \Delta$. Then set

$$v = \phi^w(X_1, (\Sigma + 2g - 2 - t) e^{tD_1}) \in HFF^*(\Sigma \times \mathbb{S}^1, \mathbb{S}^1)_I.$$

Let us prove that this v does the job. For any $z_s = \Sigma^n x^m \gamma_{i_1} \cdots \gamma_{i_r}$, we compute from (6.1) that

$$\begin{aligned} \langle v, \hat{e}_s \rangle =& \langle \phi^w(X_1, (\Sigma + 2g-2-t)e^{tD_1}), \phi^w(A, z_s e^{t\Delta}) \rangle \\ =& D_X^{(w,\Sigma)}((\Sigma + 2g-2-t) z_s e^{tD}) \end{aligned}$$

is equal to

$$\begin{cases} 0, & r > 0, \\ -2^{3g-4}(2g-2)(2g-2+t)^n 2^m\, e^{Q(tD)/2+tK\cdot D}, & r = 0. \end{cases}$$

Then

$$\begin{aligned}\langle \hat{\alpha} v, \hat{e}_s\rangle =&\langle \phi^w(X_1, 2\,\Sigma(\Sigma+2g-2-t)e^{tD_1}), \phi^w(A, z_s e^{t\Delta})\rangle \\ =&D_X^{(w,\Sigma)}((\Sigma+2g-2-t)2\,\Sigma\, z_s e^{tD}) \\ =&(4g-4+2t)\langle v, \hat{e}_s\rangle,\end{aligned}$$

for all $s \in \mathcal{S}$. Thus $\hat{\alpha} v = (4g-4+2t)v$. Analogously, $\hat{\gamma} v = 0$ and $\hat{\beta} v = -8v$.

- $0 < k < g-1$. The same argument as above for genus $g-k$ produces a 4-manifold C_{g-k} with an embedded Riemann surface Σ_{g-k} of genus $g-k$ and self-intersection zero with a single basic class $K \in H^2(X;\mathbb{Z})$ with $K \cdot \Sigma_{g-k} = 2(g-k)-2$. Let now $X = C_{g-k} \# k\, \mathbb{S}^1 \times \mathbb{S}^3$ (performing the connected sum well apart from D). Consider the torus $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{S}^1 \times \mathbb{S}^3$ and the internal connected sum $\Sigma = \Sigma_g = \Sigma_{g-k} \# k\, \mathbb{S}^1 \times \mathbb{S}^1 \subset X = C_{g-k} \# k\, \mathbb{S}^1 \times \mathbb{S}^3$. When choosing the basis of $H_1(\Sigma;\mathbb{Z})$, we arrange $\gamma_1, \ldots, \gamma_k$ such that $\gamma_i = \mathbb{S}^1 \times \mathrm{pt}$ in the i-th copy $\mathbb{S}^1 \times \mathbb{S}^3$. Suppose for instance that $g-k$ is even. Then by Lemma 6.4 below, for any $\alpha \in H_2(C_{g-k}) = H_2(X)$, we have

$$\begin{aligned} D_X^{(w,\Sigma)}(\gamma_1 \cdots \gamma_k e^{\alpha}) =& c^k\, D_{C_{g-k}}^{(w,\Sigma)}(e^{\alpha}) \\ =& c^k \Big(-2^{3(g-k)-5} e^{Q(\alpha)/2} e^{K\cdot\alpha} \\ & \qquad + (-1)^{g-k} 2^{3(g-k)-5} e^{Q(\alpha)/2} e^{-K\cdot\alpha} \Big), \end{aligned}$$

with $w \in H^2(C_{g-k};\mathbb{Z})$ as in the first case. Write again $X = X_1 \cup_Y A$ and consider $D \subset X$ intersecting transversely Σ in one point with $D \cdot \Sigma = 1$, so that $D = D_1 + \Delta$ with $\partial D_1 = \mathbb{S}^1$. Then the element

$$v = \phi^w(X_1, (\Sigma + 2(g-k) - 2 - t)\gamma_1 \cdots \gamma_k e^{tD_1}) \in (HFF_g^*)_I$$

satisfies the required properties. Note that v is invariant since it has only non-zero pairing with elements in $(HFF_g^*)_I \subset HFF_g^*$.

- $k = 0$ and $g = 1$. Let S be the $K3$ surface, and let us fix an elliptic fibration for S, whose generic fibre is an embedded torus $\Sigma = \mathbb{T}^2$. The Donaldson invariants are, for $w \in H^2(S;\mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod 2$, $D_S^{(w,\Sigma)}(e^{tD}) = -e^{-Q(tD)/2}$. Fix $D \subset S$ which cuts Σ transversely in one point such that $\Sigma \cdot D = 1$. Then $D_S^{(w,\Sigma)}(\Sigma\, e^{tD}) = te^{-Q(tD)/2}$. Let S_1 be the complement of a small open tubular neighbourhood of Σ in S and $D_1 = S_1 \cap D \subset S_1$, so that $\partial D_1 = \mathbb{S}^1$. Then $v = \phi^w(S_1, e^{tD_1})$ generates HFF_1^* and $\phi^w(S_1, \Sigma\, e^{tD_1}) = -t\phi^w(S_1, e^{tD_1})$, so that $\hat{\alpha} v = -2tv$. Analogously $\hat{\beta} v = 8v$ and $\hat{\gamma} v = 0$.

- $0 < k = g - 1$. We use the same trick as in the second case, considering the $K3$ surface connected sum with k copies of $\mathbb{S}^1 \times \mathbb{S}^3$. q.e.d.

Lemma 6.4. *Let X be a 4-manifold with $b^+ > 1$, and $z \in \mathbb{A}(X)$. Consider $\tilde{X} = X \# \mathbb{S}^1 \times \mathbb{S}^3$ and $\gamma = \mathbb{S}^1 \times pt \subset \mathbb{S}^1 \times \mathbb{S}^3$ to be the natural generator of the fundamental group of $\mathbb{S}^1 \times \mathbb{S}^3$. We can view γ as an element of $\mathbb{A}(\tilde{X})$. For any $w \in H^2(X;\mathbb{Z}) = H^2(\tilde{X};\mathbb{Z})$, then we have $D^w_{\tilde{X}}(\gamma\, z) = c\, D^w_X(z)$, where c is a universal constant.*

Proof. Consider the moduli space $\mathcal{M}^{w,\kappa}_{\tilde{X}}$ of ASD connections over $\tilde{X}$ of dimension $d = 3 + \deg(z)$, κ denoting the charge [16]. Then there is a choice of generic cycles V_z, V_γ in $\mathcal{M}^{w,\kappa}_{\tilde{X}}$ such that

$$M = \mathcal{M}^{w,\kappa}_{\tilde{X}} \cap V_z$$

is smooth 3-dimensional and compact, and $D^w_{\tilde{X}}(\gamma\, z) = \#(M \cap V_\gamma)$. For metrics giving a long neck to the connected sum $\tilde{X} = X \# \mathbb{S}^1 \times \mathbb{S}^3$, the usual dimension counting arguments give that the only possible distribution of charges of limiting connections are κ on the X side and 0 on the $\mathbb{S}^1 \times \mathbb{S}^3$. Now recall that the moduli space of flat $SO(3)$-connections on $\mathbb{S}^1 \times \mathbb{S}^3$ is $\mathrm{Hom}(\pi_1(\mathbb{S}^1 \times \mathbb{S}^3), SO(3)) = SO(3)$, hence

$$M = (\mathcal{M}^{w,\kappa}_X) \cap V_z \times SO(3),$$

where $\mathcal{M}^{w,\kappa}_X \cap V_z$ consists of $D^w_X(z)$ points (counted with signs). The description of the cycle V_γ given in [16] implies that there is a universal constant $c = \# SO(3) \cap V_\gamma$ yielding the statement of the lemma. q.e.d.

Proof of Theorem 6.2. Since $\hat{\alpha}^a \hat{\beta}^b \hat{\gamma}^c$, $a + b + c < g$, form a basis for $(HFF_g^*)_I$, we have that $\hat{\alpha}^a \hat{\beta}^b$, $a + b < g$, $b = 0, 1$, generate $\overline{HFF}_g^*$

as $\mathbb{C}[[t]]$-module. Therefore the rank of $\overline{HFF_g^*}$ is less than or equal to $2g-1$. Now using Poincaré duality in HFF_g^*, we have that the dual of $\overline{HFF_g^*}$ is

$$\ker(\hat{\beta}^2 - 64) \cap \ker \hat{\psi}_1 \cap \cdots \cap \ker \hat{\psi}_{2g} \subset HFF_g^*. \tag{6.2}$$

By Proposition 6.3 there are at least $2g-1$ independent vectors in (6.2), so the rank of $\overline{HFF_g^*}$ is exactly $2g-1$ and it must be a free $\mathbb{C}[[t]]$-module. The $2g-1$ eigenvalues of $(\hat{\alpha}, \hat{\beta})$ given by Proposition 6.3 provide the decomposition in the statement. q.e.d.

7. Effective Fukaya-Floer homology

Parallel to our work in section 6, we now move on to find a description of the subspace of HFF_g^* which keeps track of the gluing theory of general 4-manifolds with $b^+ > 1$.

Definition 7.1. The *effective Fukaya-Floer homology of* $\Sigma \times \mathbb{S}^1$ is defined as the sub-$\mathbb{C}[[t]]$-module $\widetilde{HFF_g^*} \subset HFF_g^*$ generated by all $\phi^w(X_1, z_1 e^{tD_1})$, for all 4-manifolds X_1 with boundary $\partial X_1 = Y$ such that $X = X_1 \cup_Y A$ has $b^+ > 1$, $z_1 \in \mathbb{A}(X_1)$, $D_1 \subset X_1$ with $\partial D_1 = \mathbb{S}^1$ and $w \in H^2(X_1; \mathbb{Z})$ with $w|_Y = w_2 = \text{P.D.}[\mathbb{S}^1]$.

The action of $\text{Sp}\,(2g, \mathbb{Z})$ on HFF_g^* restricts to an action on $\widetilde{HFF_g^*}$. Also $\hat{\alpha}$, $\hat{\beta}$, $\hat{\psi}_1, \ldots, \hat{\psi}_{2g}$ (and hence $\hat{\gamma}$) act on $\widetilde{HFF_g^*}$ by multiplication. The main theorem of this section is

Theorem 7.2. *The eigenvalues of* $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ *acting by multiplication on* $\widetilde{HFF_g^*}$ *are* $(-2t, 8, 0)$, $(\pm 4 + 2t, -8, 0)$, $(\pm 8\sqrt{-1} - 2t, 8, 0)$, $\ldots$, $(\pm 4(g-1)\sqrt{-1}^g + (-1)^g 2t, (-1)^{g-1} 8, 0)$.

For a proof of this result we need some preliminary information. The following proposition is proved analogously to Proposition 4.3.

Proposition 7.3. *Let* $\bar{\mathcal{F}}_r = \mathcal{F}_r / \hat{\gamma} \mathcal{F}_r$, $0 \le r \le g$. *Then* $\bar{\mathcal{F}}_r$ *is a free* $\mathbb{C}[[t]]$*-module with basis* $\hat{\alpha}^a \hat{\beta}^b$, $a + b < r$. *We have* $\bar{\mathcal{F}}_r = \mathbb{C}[[t]][\hat{\alpha}, \hat{\beta}]/\bar{\mathcal{J}}_r$, *where* $\bar{\mathcal{J}}_r = (\bar{\mathcal{R}}_r^1, \bar{\mathcal{R}}_r^2)$, *and* $\bar{\mathcal{R}}_r^i$ *are determined by* $\bar{\mathcal{R}}_0^1 = 1$, $\bar{\mathcal{R}}_0^2 = 0$ *and, for* $0 \le r \le g-1$,

$$\begin{cases} \bar{\mathcal{R}}_{r+1}^1 = (\hat{\alpha} + \bar{f}_{11}) \bar{\mathcal{R}}_r^1 + r^2 (1 + \bar{f}_{12}) \bar{\mathcal{R}}_r^2, \\ \bar{\mathcal{R}}_{r+1}^2 = (\hat{\beta} + (-1)^{r+1} 8 + \bar{f}_{21}) \bar{\mathcal{R}}_r^1 + \bar{f}_{22} \bar{\mathcal{R}}_r^2, \end{cases}$$

for some functions $\bar{f}_{ij}(t) \in t\mathbb{C}[[t]][\hat{\alpha},\hat{\beta}]$. *Moreover* $\bar{f}_{ij}$ *are such that* $\bar{f}_{11}\bar{\mathcal{R}}_r^1 + \bar{f}_{12}\bar{\mathcal{R}}_r^2$ *and* $\bar{f}_{21}\bar{\mathcal{R}}_r^1 + \bar{f}_{22}\bar{\mathcal{R}}_r^2$ *are both* $\mathbb{C}[[t]]$*-linear combinations of the monomials* $\hat{\alpha}^a\hat{\beta}^b$, $a+b<r+1$.

Lemma 7.4. *We have*

$$\bar{\mathcal{J}}_r/\bar{\mathcal{J}}_{r+1} = \ker(\bar{\mathcal{F}}_{r+1} \twoheadrightarrow \bar{\mathcal{F}}_r) = \bigoplus_{\substack{-r\le i\le r \\ i\equiv r \pmod 2}} R_{r+1,i},$$

where $R_{r+1,i}$ *is a free* $\mathbb{C}[[t]]$*-module of rank* 1. *For* r *even,* $\hat{\alpha} = 4i\sqrt{-1} + O(t)$ *and* $\hat{\beta} = 8 + O(t)$ *in* $R_{r+1,i}$. *For* r *odd,* $\hat{\alpha} = 4i + O(t)$, $\hat{\beta} = -8 + O(t)$ *in* $R_{r+1,i}$.

Proof. The natural map $HFF_g^* \twoheadrightarrow HF_g^*$ given by equating $t=0$ together with Lemma 4.4 yield the following commutative diagram with exact rows

$$\begin{array}{ccccc} \bar{\mathcal{J}}_r/\bar{\mathcal{J}}_{r+1} & \hookrightarrow & \bar{\mathcal{F}}_{r+1} & \twoheadrightarrow & \bar{\mathcal{F}}_r \\ \downarrow & & \downarrow & & \downarrow \\ \bar{J}_r/\bar{J}_{r+1} & \hookrightarrow & \bar{F}_{r+1} & \twoheadrightarrow & \bar{F}_r \end{array}$$

where

$$\mathrm{rk}_{\mathbb{C}[[t]]}(\bar{\mathcal{J}}_r/\bar{\mathcal{J}}_{r+1}) = \dim(\bar{J}_r/\bar{J}_{r+1}) = \binom{r+2}{2} - \binom{r+1}{2} = r+1.$$

Suppose for instance that r is odd. Then Lemma 4.4 implies that

$$P(\alpha) = \prod_{\substack{-r\le i\le r \\ i\equiv r \pmod 2}} (\alpha - 4i)$$

is the characteristic polynomial of the action of α on $\bar{J}_r/\bar{J}_{r+1}$. Therefore (and since all the roots are simple) the characteristic polynomial of the action of $\hat{\alpha}$ on $\bar{\mathcal{J}}_r/\bar{\mathcal{J}}_{r+1}$ is

$$P_t(\alpha) = \prod_{\substack{-r\le i\le r \\ i\equiv r \pmod 2}} (\alpha - 4i - f_i(t)),$$

for some $f_i(t) \in t\mathbb{C}[[t]]$. This yields that $\bar{\mathcal{J}}_r/\bar{\mathcal{J}}_{r+1} = \bigoplus_{\substack{-r\le i\le r \\ i\equiv r \pmod 2}} R_{r+1,i}$, where $R_{r+1,i}$ is a free $\mathbb{C}[[t]]$-module of rank 1 with $\hat{\alpha} = 4i + f_i(t)$. The eigenvalue of $\hat{\beta}$ on $R_{r+1,i}$ must be of the form $-8 + O(t)$. The case for even r is analogous. q.e.d.

Proof of Theorem 7.2. As $\hat{\gamma}\mathcal{J}_{r-1} \subset \mathcal{J}_r$, one has $\hat{\gamma}^g \in \mathcal{J}_g$, i.e., $\widetilde{\hat{\gamma}^g = 0}$ in HFF_g^*, so the only eigenvalue of $\hat{\gamma}$ on HFF_g^*, and hence on $\widetilde{HFF}_g^*$, is zero. To compute the eigenvalues of $\hat{\beta}$ on HFF_g^* we may restrict to $HFF_g^*/(\hat{\gamma})$, i.e., to every $\bar{\mathcal{F}}_{g-k}$. Using Lemma 7.4 recursively we find that all the eigenvalues of $\hat{\beta}$ on $\bar{\mathcal{F}}_r$ are of the form $\pm 8 + O(t)$. Thus all the eigenvalues of $\hat{\beta}$ on HFF_g^* are of the form $\pm 8 + O(t)$.

To get the eigenvalues of $\hat{\beta}$ on $\widetilde{HFF}_g^*$, let us argue by contradiction. Suppose that there is an eigenvalue different from ± 8. By definition of $\widetilde{HFF}_g^*$, there exists a vector $v = \phi^w(X_1, z_1 e^{tD_1}) \in \widetilde{HFF}_g^*$ such that $X = X_1 \cup_Y A$ is a 4-manifold with $b^+ > 1$, $z_1 \in \mathbb{A}(X_1)$, $D_1 \subset X_1$ with $\partial D_1 = \mathbb{S}^1$, $w \in H^2(X;\mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod 2$, satisfying $(\hat{\beta}^2 - 64)^N v \neq 0$, for arbitrarily large N. Then there is a polynomial $P(\hat{\beta}, t) = \prod(\hat{\beta} + (-1)^{\varepsilon_i} 8 - f_i(t))$ with $f_i(t) \in t\mathbb{C}[[t]]$, $f_i(t) \neq 0$, $\varepsilon_i = 0, 1$, such that

$$P(\hat{\beta}, t)(\hat{\beta}^2 - 64)^N v = 0,$$

for some $N \geq 0$. Substituting v by $(\hat{\beta}^2 - 64)^N v$, for suitable N, we can suppose that $N = 0$. Therefore $D_X^{(w,\Sigma)}(z_1 e^{tD+s\Sigma}) \neq 0$ and $D_X^{(w,\Sigma)}(P(-4x,t) z_1 e^{tD+s\Sigma}) = 0$, with $D = D_1 + \Delta \in H_2(X)$. Clearly we may also suppose that z_1 is homogeneous. As

$$D_X^w(z_1 e^{tD+s\Sigma}) = \frac{1}{2}\Big(D_X^{(w,\Sigma)}(z_1 e^{tD+s\Sigma}) \\ + \sqrt{-1}^{d_0 - \deg z_1/2} D_X^{(w,\Sigma)}(z_1 e^{\sqrt{-1}(tD+s\Sigma)})\Big),$$

for $d_0 = d_0(X,w) = -w^2 - \frac{3}{2}(1 - b_1 + b^+)$, we have $D_X^w(z_1 e^{tD+s\Sigma}) \neq 0$ and $D_X^w(Q(x,t) z_1 e^{tD+s\Sigma}) = 0$, with $Q(x,t) = P(-4x,t)P(4x, \sqrt{-1}t)$.

Moreover we can suppose that none of the homology classes appearing in $z_1 \in \mathbb{A}(X)$ has non-zero intersection with D (as it already happens with Σ), i.e., $z_1 \in \mathbb{A}(<\Sigma, D>^\perp)$ (write $z_1 = \sum_{m \geq 0} D^m z_1^m$, with $z_1^m \in \mathbb{A}(<\Sigma, D>^\perp)$ and substitute z_1 for one of the z_1^m). Substituting D by a linear combination $aD + b\Sigma$, $a \neq 0$, we can suppose that $D^2 = 0$, $D \in H_2(X;\mathbb{Z}) \subset H_2(X)$ and D is primitive, with $D \cdot \Sigma \neq 0$. Then $D_X^w(Q(x, at) z_1 e^{tD+s\Sigma}) = 0$. Also changing w by $w + \Sigma$ if necessary we can assume that $w \cdot D \equiv 1 \pmod 2$.

At this stage, we represent D by an embedded surface and invert the roles of D and Σ. This corresponds to changing the metric: we go from metrics giving a long neck when pulling Σ apart to metrics giving a long neck when pulling D apart. The Donaldson invariants of X do not change since $b^+ > 1$. Arguing as above, $D_X^w(Q'(x,s)z_1e^{tD+s\Sigma}) = 0$ for some polynomial $Q'(x,s)$. This time we do not bother on whether Q' is independent of s or not; we can take it to be just the characteristic polynomial of $\hat{\beta}$ acting on $HFF^*(D \times \mathbb{S}^1, \mathbb{S}^1)$. Now take the resultant of $Q(x,at)$ and $Q'(x,s)$, which is a series $R(s,t) \neq 0$. Then $D_X^w(R(s,t)z_1e^{tD+s\Sigma}) = 0$ implies $D_X^w(z_1e^{tD+s\Sigma}) = 0$, which is a contradiction. This proves that the only eigenvalues of $\hat{\beta}$ are ± 8.

Finally, to compute the eigenvalues of $\hat{\alpha}$ we can restrict to $\widetilde{HFF}_g^*/(\hat{\psi}_1, \ldots, \hat{\psi}_{2g}, \hat{\beta}^2 - 64)$. This is a subset of

$$\overline{HFF}_g^* = (HFF_g^*)_I/(\hat{\gamma}, \hat{\beta}^2 - 64),$$

which is computed in Theorem 6.2. Moreover all the eigenvalues in $\overline{HFF}_g^*$ are indeed eigenvalues of $\widetilde{HFF}_g^*$ as all the vectors constructed in Proposition 6.3 come from 4-manifolds with $b^+ > 1$. This completes the proof. q.e.d.

Remark 7.5. The author believes that the eigenvalues of $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ acting by multiplication on $\widetilde{HFF}_g^*$ given in Theorem 7.2 are indeed all the eigenvalues of $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ on HFF_g^*.

8. Fukaya-Floer homology $HFF_*(\Sigma \times \mathbb{S}^1, \delta)$

Now we deal with the Fukaya-Floer (co)homology of the 3-manifold $Y = \Sigma \times \mathbb{S}^1$ with the $SO(3)$-bundle with $w_2 = \text{P.D.}[\mathbb{S}^1] \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$ and loop $\delta \subset \Sigma \subset \Sigma \times \mathbb{S}^1$ representing a primitive homology class, and its $\mathbb{A}(\Sigma)$-module structure. Poincaré duality identifies the Fukaya-Floer homology $HFF_*(Y, \delta)$ with the Fukaya-Floer cohomology $HFF^*(Y, -\delta)$. The μ map gives an action of $\mathbb{A}(\Sigma)$ on $HFF_*(Y, \delta)$. Later we shall see that this gives in fact a structure of module over $HF_*(Y)$.

8.1. The vector space $HFF_*(\Sigma \times \mathbb{S}^1, \delta)$. We can suppose that the basis $\{\gamma_1, \ldots, \gamma_{2g}\}$ of $H_1(\Sigma; \mathbb{Z})$ is chosen so that $[\delta] = \gamma_1$ (recall that $\gamma_i\gamma_{i+g} = \text{pt}$ for $1 \leq i \leq g$). The action of $\text{Sp}\,(2g, \mathbb{Z})$ on $\{\gamma_i\}$ restricts to an action of the subgroup $\text{Sp}\,(2g-2, \mathbb{Z})$ on $\gamma_2, \ldots, \gamma_g, \gamma_{g+2}, \ldots, \gamma_{2g}$. Any element of $\text{Sp}\,(2g-2, \mathbb{Z})$ can be realized by a diffeomorphism of $\Sigma \times \mathbb{S}^1$ fixing δ, hence it induces an automorphism of $HFF_*(Y, \delta)$. This gives an action of $\text{Sp}\,(2g-2, \mathbb{Z})$ on $HFF_*(Y, \delta)$.

We recall that for computing $HFF_*(Y, \delta)$ there is a spectral sequence whose E_3 term is $HF_*(Y) \otimes \hat{H}_*(\mathbb{CP}^\infty)$, with differential d_3 given by

$$\mu(\delta) : HF_i(Y) \otimes H_{2j}(\mathbb{CP}^\infty) \to HF_{i-3}(Y) \otimes H_{2j+2}(\mathbb{CP}^\infty),$$

and converging to $HFF_*(Y, \delta)$. The $\mathrm{Sp}\,(2g-2, \mathbb{Z})$ action on this E_3 term gives the action on $HFF_*(Y, \delta)$. Now we can use the description of $HF_g^* = HF^*(Y)$ gathered in Proposition 4.2, and the fact that $\mu(\delta)$ is multiplication by $\psi_1 = \phi^w(A, \gamma_1)$ to get a description of the E_4 term of the spectral sequence.

Proposition 8.1. *Consider $\psi_1 : HF_g^* \to HF_g^*$. Then*

$$\ker \psi_1 / \mathrm{im}\, \psi_1 = \bigoplus_{k=0}^{g-1} \Lambda_0^k H^3_{red} \otimes K_{g-k},$$

where $H^3_{red} = < \psi_2, \ldots, \psi_g, \psi_{g+2}, \ldots, \psi_{2g} >$ and $K_r = J_{r-1}/(J_r + \gamma J_{r-2})$.

Proof. The space H^3 has basis $\psi_1, \psi_2, \ldots, \psi_{2g}$, so we can write $H^3 = < \psi_1, \psi_{g+1} > \oplus H^3_{\mathrm{red}}$, where H^3_{red} is generated by $\psi_2, \ldots, \psi_g$, $\psi_{g+2}, \ldots, \psi_{2g}$ and it is the standard representation of $\mathrm{Sp}\,(2g-2, \mathbb{Z})$ ('red' stands for reduced and follows the notation of [20]). More intrinsically, we can identify $H^3_{\mathrm{red}} \cong < \psi_1 >^\perp / < \psi_1 >$. It is easy to check that $\Lambda_0^k H^3$ decomposes as

$$\Lambda_0^k H^3 = \gamma' \Lambda_0^{k-2} H^3_{\mathrm{red}} \oplus \Big(< \psi_1, \psi_{g+1} > \otimes \Lambda_0^{k-1} H^3_{\mathrm{red}} \Big) \oplus \Lambda_0^k H^3_{\mathrm{red}}$$

as $\mathrm{Sp}\,(2g-2, \mathbb{Z})$-representations, where $\gamma' = -g\psi_1 \wedge \psi_{g+1} + \gamma$. The reader can check this directly, noting that $\gamma' \in \Lambda_0^2 H^3$, or otherwise see [11, formula (25.36)].

As a shorthand, write $F_r = \mathbb{C}[\alpha, \beta, \gamma]/J_r = (HF_r^*)_I$. Then Proposition 4.2 says that

$$\begin{aligned} HF_g^* &= \bigoplus_{k=0}^{g-1} (\Lambda_0^k H^3 \otimes F_{g-k}) \\ &= \bigoplus_{k=0}^{g-1} \Lambda_0^k H^3_{\mathrm{red}} \otimes (F_{g-k} \oplus \gamma' F_{g-k-2}) \\ &\quad \oplus \bigoplus_{k=0}^{g-1} \Lambda_0^k H^3_{\mathrm{red}} \otimes (< \psi_1, \psi_{g+1} > \otimes F_{g-k-1}), \end{aligned} \tag{8.1}$$

as $\mathrm{Sp}\,(2g-2,\mathbb{Z})$-representations. Now multiplication by ψ_1 is $\mathrm{Sp}\,(2g-2,\mathbb{Z})$-equivariant and intertwines the two summands in (8.1), i.e.,

$$\begin{array}{ccc} F_{g-k}\oplus\gamma' F_{g-k-2} & \stackrel{\psi_1}{\to} & <\psi_1,\psi_{g+1}>\otimes F_{g-k-1} \\ x\oplus\gamma' y & \mapsto & \psi_1\otimes(x+\gamma y) \end{array} \tag{8.2}$$

and

$$\begin{array}{ccc} <\psi_1,\psi_{g+1}>\otimes F_{g-k-1} & \stackrel{\psi_1}{\to} & F_{g-k}\oplus\gamma' F_{g-k-2} \\ \psi_1\otimes z & \mapsto & 0 \\ \psi_{g+1}\otimes z & \mapsto & \frac{1}{g}\gamma z\oplus(-\frac{1}{g})\gamma' z. \end{array} \tag{8.3}$$

In (8.2),

$$\ker\psi_1=\{x\oplus\gamma' y\in F_{g-k}\oplus\gamma' F_{g-k-2}/x+\gamma y=0\in F_{g-k-1}\},$$

$\mathrm{im}\,\psi_1=\psi_1\otimes F_{g-k-1}$. In (8.3), $\ker\psi_1=\psi_1\otimes F_{g-k-1}$,

$$\mathrm{im}\,\psi_1=\{\gamma y\oplus(-\gamma' y)\in F_{g-k}\oplus\gamma' F_{g-k-2}\},$$

so

$$\ker\psi_1/\mathrm{im}\,\psi_1=\bigoplus_{k=0}^{g-1}\Lambda_0^k H^3_{\mathrm{red}}\otimes K_{g-k},$$

where

$$\begin{aligned} K_r &= \frac{\{x\oplus\gamma' y\in F_r\oplus\gamma' F_{r-2}/x+\gamma y=0\in F_{r-1}\}}{\{\gamma y\oplus(-\gamma' y)\in F_r\oplus\gamma' F_{r-2}\}} \\ &\cong \frac{\{x\in F_r/x=0\in F_{r-1}\}}{\{\gamma y/y=0\in F_{r-2}\}}=\frac{J_{r-1}/J_r}{\gamma(J_{r-2}/J_r)}=\frac{J_{r-1}}{J_r+\gamma J_{r-2}}. \end{aligned}$$

q.e.d.

Lemma 8.2. *As a $\mathbb{C}[\alpha,\beta,\gamma]$-module, $K_r=\bigoplus_{\substack{-(r-1)\le i\le r-1\\ i\equiv r-1 \pmod 2}} R_i$, where R_i is 1-dimensional, α acts as $4i\sqrt{-1}$ if i is even and as $4i$ if i is odd, β as $(-1)^i 8$ and γ as zero on R_i.*

Proof. K_r is generated, as $\mathbb{C}[\alpha,\beta,\gamma]$-module, by three elements R^1_{r-1}, R^2_{r-1} and R^3_{r-1}, which satisfy six relations $R^1_r=0$, $R^2_r=0$, $R^3_r=0$, $\gamma R^1_{r-2}=0$, $\gamma R^2_{r-2}=0$ and $\gamma R^3_{r-2}=0$. Therefore

$$\left\{\begin{array}{l} 0=\alpha R^1_{r-1}+(r-1)^2R^2_{r-1}, \\ 0=(\beta+(-1)^r 8)R^1_{r-1}+\frac{2(r-1)}{r}R^3_{r-1}, \\ 0=\gamma R^1_{r-1}. \end{array}\right.$$

Also $R^3_{r-1} = \gamma R^1_{r-2} = 0$. The first line allows to write R^2_{r-1} in terms of R^1_{r-1}, so K_r is generated by an element $k_r = R^1_{r-1}$, which satisfies $\gamma k_r = 0$ and $(\beta + (-1)^r 8)k_r = 0$. Therefore K_r is a module over $\mathbb{C}[\alpha,\beta,\gamma]/((\gamma, \beta + (-1)^r 8) + J_r)$. This is a quotient of $(HF^*_r)_I$ which has been computed in [23, Proposition 20] to be

$$S_r = \begin{cases} \mathbb{C}[\alpha]/((\alpha - 16(r-1)^2)(\alpha - 16(r-3)^2)\cdots(\alpha - 16\cdot 1^2)) & r \text{ even}, \\ \mathbb{C}[\alpha]/((\alpha + 16(r-1)^2)(\alpha + 16(r-3)^2)\cdots(\alpha + 16\cdot 2^2)\alpha) & r \text{ odd}. \end{cases}$$

So K_r is a quotient of S_r, being a cyclic module over this ring. In particular $\dim K_r \leq r$. On the other hand, if we consider the action of γ in F_r, [23, Corollary 18] says that $\ker \gamma = J_{r-1}/J_r$. Moreover $\ker \gamma^2 = J_{r-2}/J_r$, which is proved in the same fashion. So we can write $K_r = \frac{\ker \gamma}{\gamma \ker \gamma^2}$. Now $\dim \ker \gamma = \binom{r+1}{2}$, $\dim \ker \gamma^2 = \binom{r+1}{2} + \binom{r}{2}$. As the action of (multiplication by) γ vanishes on $\ker \gamma \subset \ker \gamma^2$, we have that $\dim(\gamma \ker \gamma^2) \leq \binom{r}{2}$. So $\dim(\ker \gamma/(\gamma \ker \gamma^2)) \geq \binom{r+1}{2} - \binom{r}{2} = r$, and thus K_r must equal S_r. q.e.d.

Now we are able to write down the E_4 term of the spectral sequence. Decompose $\ker \psi_1 = \operatorname{im} \psi_1 \oplus (\ker \psi_1/\operatorname{im} \psi_1)$, where $\operatorname{im} \psi_1 \subset \ker \psi_1$ is the null part for the intersection pairing on $\ker \psi_1$. Then

$$E_4 = (\operatorname{im} \psi_1 \oplus (\ker \psi_1/\operatorname{im} \psi_1)) \times (\ker \psi_1/\operatorname{im} \psi_1)t \times (\ker \psi_1/\operatorname{im} \psi_1)\frac{t^2}{2!} \times \cdots$$

So Lemma 8.2 gives

$$E_4 = \operatorname{im} \psi_1 \oplus \bigoplus_{i,k} \Lambda^k_0 H^3_{\text{red}} \otimes R_i \otimes \mathbb{C}[[t]],$$

where $0 \leq k \leq g-1$, $-(g-k-1) \leq i \leq g-k-1$ and $i \equiv g-k-1 \pmod 2$. We can write $E_4 = \operatorname{im} \psi_1 \oplus \tilde{E}_4$, where the intersection pairing vanishes on the first summand. In order to compute Donaldson invariants, this first summand is ineffective, so we will ignore its behaviour through the spectral sequence, and look henceforth to the spectral sequence given by $\tilde{E}_4$.

Proposition 8.3. *The spectral sequence $\tilde{E}_n$, $n \geq 4$, collapses at the fourth stage, i.e., $d_n = 0$, for all $n \geq 4$.*

Proof. There is a well-defined $\mathbb{A}(\Sigma)$-module structure in the spectral sequence, since it is defined at the chain level in section 3. Also any $f \in \operatorname{Sp}(2g-2, \mathbb{Z})$ induces $f : HFF_*(\Sigma \times \mathbb{S}^1, \delta) \to HFF_*(\Sigma \times \mathbb{S}^1, \delta)$ which can be defined at the chain level and therefore also appears through the spectral sequence. Therefore every differential d_n is $\mathrm{Sp}\,(2g-2,\mathbb{Z})$-equivariant, $\mathbb{C}[\alpha,\beta,\gamma]$-linear and $\mathbb{C}[[t]]$-linear. Now

$$\tilde{E}_4 = \bigoplus_{i,k} \Lambda_0^k H^3_{\mathrm{red}} \otimes R_i \otimes \mathbb{C}[[t]]$$

is a direct sum of inequivalent irreducible modules for the ring

$$\mathbb{C}\mathrm{Sp}\,(2g-2,\mathbb{Z}) \otimes \mathbb{C}[\alpha,\beta,\gamma] \otimes \mathbb{C}[[t]]$$

(where $\mathbb{C}\mathrm{Sp}\,(2g-2,\mathbb{Z})$ is the group algebra of $\mathrm{Sp}\,(2g-2,\mathbb{Z})$). So d_n has to send every summand to itself, and $d_n^2=0$ on it implies $d_n=0$. The proposition follows. q.e.d.

Henceforth we will only consider

$$\overline{HFF}_*(Y,\delta) = \bigoplus_{i,k} \Lambda_0^k H^3_{\mathrm{red}} \otimes R_i \otimes \mathbb{C}[[t]] \subset HFF_*(Y,\delta), \tag{8.4}$$

which coincides with $HFF_*(Y,\delta)/$null part.

8.2. The HF^*_g-module $HFF_*(\Sigma\times\mathbb{S}^1,\delta)$. In order to determine the $\mathbb{A}(\Sigma)$-module structure on $\overline{HFF}_*(Y,\delta)$, we consider the natural cobordism between $(Y,\delta)\sqcup(Y,\emptyset)$ and (Y,δ). It gives the map (3.3)

$$\cdot : HF_*(Y)\otimes HFF_*(Y,\delta) \to HFF_*(Y,\delta).$$

Now for any $\phi\in HFF_*(Y,\delta)$, it is

$$\alpha\cdot\phi = \phi^w(A,2\,\Sigma)\cdot\phi = 2\mu(\Sigma)(\mathbf{1})\cdot\phi = \mathbf{1}\cdot 2\mu(\Sigma)(\phi) = 2\mu(\Sigma)(\phi),$$

with $\mathbf{1}=\phi^w(A,1)$. Therefore the action of $2\mu(\Sigma)$ is multiplication by α. Analogously for $\mu(\mathrm{pt})$ and $\mu(\gamma_j)$. Therefore the $\mathbb{A}(\Sigma)$-module structure reduces to an $HF^*(Y)$-module structure on $HFF_*(Y,\delta)$, and hence on $\overline{HFF}_*(Y,\delta)$. In (8.4), $i\equiv g-k-1 \pmod 2$, so the action of $\mu(\gamma_j)$ vanishes. Therefore we have proved

Theorem 8.4. *Let $Y=\Sigma\times\mathbb{S}^1$ and $\delta\subset\Sigma\subset Y$ a loop representing a primitive homology class. Let $\overline{HFF}_*(Y,\delta)$ be $HFF_*(Y,\delta)$ modulo its null part under the intersection pairing. Then $\overline{HFF}_*(Y,\delta)$ is an $HF^*(Y)$-module and*

$$\overline{HFF}_*(Y,\delta) = \bigoplus_{i,k} \Lambda_0^k H^3_{red} \otimes R_i \otimes \mathbb{C}[[t]], \tag{8.5}$$

where $0 \leq k \leq g-1$, $-(g-k-1) \leq i \leq g-k-1$ *and* $i \equiv g-k-1 \pmod 2$. *The* R_i *are* 1*-dimensional.* $\alpha = 2\mu(\Sigma)$ *acts as* $4i\sqrt{-1}$ *if* i *is even and as* $4i$ *if* i *is odd,* $\beta = -4\mu(pt)$ *acts as* $(-1)^i 8$ *and the action of* $\psi_j = \mu(\gamma_j)$ *is zero.*

Theorem 8.4 gives us the action of $H_*(\Sigma)$ on $\overline{HFF}_*(Y,\delta)$, but to get a more intrinsic picture which does not need explicitly the isomorphism $Y \cong \Sigma \times \mathbb{S}^1$, we have to give the action of the full $H_*(Y)$ on the Fukaya-Floer cohomology. This is provided by the following Proposition:

Proposition 8.5. *Consider* $\overline{HFF}_*(Y,\delta)$ *as given in* (8.5). *Then on* $R_i \otimes \mathbb{C}[[t]]$, $-4\mu(pt)$ *acts as* $(-1)^i 8$, $\mu(a) = 0$ *for any* $a \in H_1(Y)$ *and, for* $a \in H_2(Y)$, $2\mu(a)$ *is* $4(a \cdot \mathbb{S}^1)i\sqrt{-1} - 2(a \cdot \delta)t$ *if* i *is even and* $4(a \cdot \mathbb{S}^1)i + 2(a \cdot \delta)t$ *if* i *is odd.*

Proof. As $Y = \Sigma \times \mathbb{S}^1$ is a (trivial) circle bundle over Σ, we may consider an automorphism of Y as a circle bundle. This is classified by an element $f \in H^1(\Sigma;\mathbb{Z})$, so we shall put $\varphi_f : Y \to Y$. The action in homology $\varphi_f : H_*(Y) \to H_*(Y)$ is $\varphi_f(\mathrm{pt}) = \mathrm{pt}$, $\varphi_f(\gamma_j) = \gamma_j + (f[\gamma_j])\mathbb{S}^1$, $\varphi_f(\Sigma) = \Sigma + \mathrm{P.D.}[f] \times \mathbb{S}^1$ and $\varphi_f(\alpha \times \mathbb{S}^1) = \alpha \times \mathbb{S}^1$, for any $\alpha \in H_*(\Sigma)$. In particular,

$$\delta_f = \varphi_f(\delta) = \delta + n\mathbb{S}^1, \qquad \text{where } n = f[\delta].$$

So $\varphi_f : HFF_*(Y,\delta) \xrightarrow{\cong} HFF_*(Y,\delta_f)$ and hence

$$\overline{HFF}_*(Y,\delta + n\mathbb{S}^1) = \bigoplus_{i,k} \Lambda_0^k H^3_{\mathrm{red}} \otimes R_i \otimes \mathbb{C}[[t]]. \tag{8.6}$$

Now there is a natural cobordism between $(Y,\delta_f) \sqcup (Y, n\mathbb{S}^1)$ and (Y,δ_f), which, in the same fashion as above, gives an $HFF^*(Y,n\mathbb{S}^1)$-module structure to $HFF_*(Y,\delta_f)$. This goes down to a module structure over the reduced Fukaya-Floer homology

$$\overline{HFF}^*(Y,n\mathbb{S}^1) = HFF^*(Y,n\mathbb{S}^1)/(\hat{\beta}^2 - 64, \hat{\psi}_1, \ldots, \hat{\psi}_{2g}).$$

Corollary 5.4 (and the description of the eigenvalues of $\overline{HFF}^*_g$ given in Theorem 6.2) yields that on the summand $R_i \otimes \mathbb{C}[[t]]$ of $\overline{HFF}_*(Y,\delta + n\mathbb{S}^1)$, $2\mu(\Sigma)$ must act as $4i\sqrt{-1} - 2nt$ if i is even and as $4i + 2nt$ if i is odd, $-4\mu(\mathrm{pt})$ as $(-1)^i 8$ and $\mu(\gamma_j)$ as zero. Finally we go back under the isomorphism $\varphi_f : Y \to Y$. So on the summand $R_i \otimes \mathbb{C}[[t]]$ of (8.5), the μ-actions are as follows

$$\begin{cases} 2\mu(\varphi_f^{-1}(\Sigma)) = 2\mu(\Sigma - \text{P.D.}[f] \times \mathbb{S}^1)) = \begin{cases} 4i\sqrt{-1} - 2nt & i \text{ even}, \\ 4i + 2nt & i \text{ odd}, \end{cases} \\ -4\mu(\varphi_f^{-1}(\text{pt})) = -4\mu(\text{pt}) = (-1)^i 8, \\ \mu(\varphi_f^{-1}(\gamma_j)) = \mu(\gamma_j - (f[\gamma_j])\mathbb{S}^1) = 0. \end{cases}$$

This implies that $\mu(\mathbb{S}^1)$ acts as zero and $\mu(\gamma_j \times \mathbb{S}^1)$ acts as $(-1)^i(\gamma_j \cdot \delta)t$. The proposition follows. q.e.d.

9. Applications of Fukaya-Floer homology

In this section we are going to give a number of remarkable applications from the knowledge of the structure of the Fukaya-Floer homology groups of $\Sigma \times \mathbb{S}^1$. The author expects to extend the techniques to be able to get the general shape of the Donaldson invariants of 4-manifolds not of simple type with $b^+ > 1$.

9.1. 4-manifolds are of finite type. In [17] it is conjectured that any 4-manifold with $b^+ > 1$ is of finite type. In [13], Frøyshov gives a proof of the finite type condition for any simply connected 4-manifold by studying the general properties of the map $\mu(\text{pt})$ on the Floer homology of 3-manifolds. In [30], Wieczorek also proves the finite type condition for simply connected 4-manifolds by studying configurations of embedded spheres of negative self-intersections. Here we give a proof of the finite type condition for arbitrary 4-manifolds with $b^+ > 1$ by using the effective Fukaya-Floer homology $\widetilde{HFF}_g^*$.

Proposition 9.1. *Let X be a 4-manifold with $b^+ > 1$ and $\Sigma \hookrightarrow X$ an embedded surface of self-intersection zero. Suppose there is $w \in H^2(X;\mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod 2$. Then there exists $n \geq 0$ such that $D_X^w((x^2-4)^n z) = 0$ for any $z \in \mathbb{A}(X)$.*

Proof. If the genus g of Σ is zero, then the Donaldson invariants vanish identically, so the statement is true with $n = 0$. Suppose then that $g \geq 1$. Thus we split $X = X_1 \cup_Y A$, where A is a small tubular neighbourhood of Σ. Let $D \in H_2(X)$ such that $D \cdot \Sigma = 1$. Represent D by a 2-cycle intersecting transversely Σ in one positive point and put $D = D_1 + \Delta$, with $D_1 \subset X_1$ and $\partial D_1 = \mathbb{S}^1$. Then for any $z \in \mathbb{A}(X_1)$ it is $\phi^w(X_1, ze^{tD_1}) \in \widetilde{HFF}_g^*$ by Definition 7.1. By Theorem 7.2 there is some $n > 0$ such that $(\hat{\beta}^2 - 64)^n = 0$ on $\widetilde{HFF}^*_g$. Using Lemma 5.1,

$$D_X^{(w,\Sigma)}(z(x^2-4)^n e^{tD}) = \langle \frac{1}{16^n}(\hat{\beta}^2 - 64)^n \phi^w(X_1, ze^{tD_1}), \phi^w(A, e^{t\Delta})\rangle = 0.$$

So $D_X^w(z(x^2-4)^n D^m) = 0$ for all $m \geq 0$. This is equivalent to the statement. q.e.d.

Now we are ready to give a proof of Theorem 1.2.

Theorem 9.2. *Let X be a 4-manifold with $b^+ > 1$. Then X is of w-finite type, for any $w \in H^2(X;\mathbb{Z})$.*

Proof. First note that if $\tilde{X} = X \# \overline{\mathbb{CP}}^2$ is the blow-up of X with exceptional divisor E, then X is of w-finite type if and only if $\tilde{X}$ is of w-finite type if and only if $\tilde{X}$ is of $(w+E)$-finite type. This is a consequence of the general blow-up formula [9]. It means that, after possibly blowing-up, we can suppose w is odd. Then there exists $x \in H_2(X;\mathbb{Z})$ with $w \cdot x \equiv 1 \pmod 2$. As $b^+ > 0$, there is $y \in H_2(X;\mathbb{Z})$ with $y \cdot y > 0$. Consider $x' = x + 2ny$ for n large. Then $x' \cdot x' > 0$ and $w \cdot x' \equiv 1 \pmod 2$. Represent x' by an embedded surface Σ' and blow-up X at $N = x' \cdot x'$ points in Σ' to get a 4-manifold $\tilde{X} = X \# N \overline{\mathbb{CP}}^2$ with an embedded surface $\Sigma \subset \tilde{X}$ such that $\Sigma \cdot \Sigma = 0$ and $w \in H^2(X;\mathbb{Z}) \subset H^2(\tilde{X};\mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod 2$. Then Proposition 9.1 implies that $\tilde{X}$ is of w-finite type and hence X is of w-finite type. q.e.d.

Proposition 9.3. *Let X be a 4-manifold with $b^+ > 1$ and containing an embedded surface Σ of genus g and self-intersection zero such that there is $w \in H^2(X;\mathbb{Z})$ with $w \cdot \Sigma \equiv 1 \pmod 2$. Then X is of w-finite type of order less than or equal to*

$$\sum_{i=1}^{g} \left(\left[\frac{2g-2i}{4} \right] + 1 \right),$$

where $[x]$ denotes the integer part of x. If furthermore X has $b_1 = 0$, then X is of w-finite type of order less than or equal to

$$\left[\frac{2g-2}{4} \right] + 1. \tag{9.1}$$

Proof. The result is obvious for $g = 0$. We can thus suppose $g \geq 1$. We only need to find the minimum $n \geq 0$ such that $(\hat{\beta}^2 - 64)^n = 0$ in $\widetilde{HFF}^*_g$ (see proof of Proposition 9.1). Consider the element

$$e_r = (\hat{\beta} + (-1)^r 8)(\hat{\beta} + (-1)^{r-1} 8) \overset{(r)}{\cdots} (\hat{\beta} - 8),$$

for $1 \leq r \leq g$. Using Lemma 7.4 we prove by induction that there are polynomials $P_r(\hat{\beta}, t) \in \mathbb{C}[[t]][\hat{\beta}]$ such that $e_r P_r = 0 \in \bar{\mathcal{F}}_r$ and $P_r(\pm 8, t) \neq 0$ (indeed P_r collects all the eigenvalues of $\hat{\beta}$ different from ± 8). Thus $e_r P_r$ is a multiple of $\hat{\gamma}$ in $\mathcal{F}_r$. Now the inclusion $\hat{\gamma}\mathcal{J}_r \subset \mathcal{J}_{r+1}$ yields that $e_r P_r \mathcal{J}_r \subset \mathcal{J}_{r+1}$ and, by recurrence, that $\prod_{r=1}^{g} e_r P_r \in \mathcal{J}_g$. We conclude that $\prod_{r=1}^{g} e_r P_r = 0$ in HFF_g^*. As P_r are isomorphisms over $\widetilde{HFF}_g^*$ by Theorem 7.2, we have that $\prod_{r=1}^{g} e_r = 0$ in $\widetilde{HFF}_g^*$. This means that we may take $n = \sum_{i=1}^{g}(\left[\frac{2g-2i}{4}\right] + 1)$ to get $(\hat{\beta}^2 - 64)^n = 0$ on $\widetilde{HFF}_g^*$.

In the case $b_1 = 0$, we use that $e_g P_g$ is a multiple of $\hat{\gamma}$ in $\mathcal{F}_g$. As P_g is an isomorphism over $\widetilde{HFF}_g^*$, e_g is a multiple of $\hat{\gamma}$ on $\widetilde{HFF}_g^*$. The result follows easily. q.e.d.

Remark 9.4. The bound in (9.1) is in agreement with the conjecture in [17]. Let us check some simple cases in which Proposition 9.3 was already known to hold. For $g = 0$, we get that X is of zeroth-order finite type, i.e., that the Donaldson invariants vanish identically. For $g = 1$, we get that X is of simple type [19] [18]. For $g = 2$ we get that X is of second order finite type [19, Theorem 5.16]. If $b_1 = 0$ and $g = 2$, X is again of simple type.

9.2. Connected sums along surfaces of 4-manifolds with $b_1 = 0$. We are going to apply the description of the Fukaya-Floer homology of section 8 to the problem of determining the Donaldson invariants of a connected sum along a Riemann surface of 4-manifolds with $b_1 = 0$ (but not necessarily of simple type). This has been extensively studied in [21].

Let $\bar{X}_1$ and $\bar{X}_2$ be 4-manifolds with $b_1 = 0$ and containing embedded Riemann surfaces $\Sigma = \Sigma_i \hookrightarrow \bar{X}_i$ of the same genus $g \geq 1$, self-intersection zero and representing odd homology classes. Put X_i for the complement of a small open tubular neighbourhood of Σ_i in $\bar{X}_i$ so that $\bar{X}_i = X_i \cup_Y A$, X_i is a 4-manifold with boundary $\partial X_i = Y = \Sigma \times \mathbb{S}^1$. Let $\phi : \partial X_1 \to \overline{\partial X_2}$ be an identification (i.e., a bundle isomorphism) and put $X = X(\phi) = X_1 \cup_\phi X_2 = \bar{X}_1 \#_\Sigma \bar{X}_2$ for the connected sum of $\bar{X}_1$ and $\bar{X}_2$ along Σ. As we are only dealing with one identification, we may well suppose that $\phi = \mathrm{id}$. Recall [21, remark 8] that homology orientations of both $\bar{X}_i$ induce a homology orientation of X. Also choose $w_i \in H^2(X_i; \mathbb{Z})$, $i = 1, 2$, and $w \in H^2(X; \mathbb{Z})$ such that $w_i \cdot \Sigma_i \equiv 1 \pmod 2$, $w \cdot \Sigma \equiv 1 \pmod 2$, in a compatible way (i.e., the restriction of w to $X_i \subset X$ coincides with the restriction of w_i to $X_i \subset \bar{X}_i$). Also as $b_1(X_1) = b_1(X_2) = 0$ it is $b_1(X) = 0$ and $b^+(X) > 1$. Moreover there is an exact sequence [19, subsection 2.3.1]

$$\begin{aligned} 0 \to H_2(Y) \to H_2(X) \to H_2(X_1, \partial X_1) \oplus H_2(X_2, \partial X_2) \\ \to H_1(Y) \to 0. \end{aligned} \tag{9.2}$$

Now we pass on to give a proof of Theorem 1.3. This result gives a strong restriction on the invariants of X and complements the results of [21]. It is also in accordance with the case $g = 2$ studied in [20].

Theorem 9.5. *The* 4*-manifold* $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ *is of simple type with* $b_1 = 0$ *and* $b^+ > 1$. *Let* $\mathbb{D}_X = e^{Q/2} \sum a_i e^{K_i}$ *be its Donaldson series. Then for all basic classes* K_i, *we have* $K_i \cdot \Sigma \equiv 2g - 2 \pmod 4$.

Proof. Fix $D_S \in H_2(X)$ with $D_S|_Y = [\mathbb{S}^1] \in H_1(Y)$. Now for any $\delta \in H_1(\Sigma; \mathbb{Z})$ which is primitive we consider any $D \in H_2(X)$ with $D|_Y = \delta$. Represent $D + nD_S$ as $D_1 + D_2$, with $D_i \subset X_i$ and $\partial D_1 = \delta + n\mathbb{S}^1$, where $n \in \mathbb{Z}$. The Fukaya-Floer homology $\overline{HFF}_*(Y, \delta + n\mathbb{S}^1)$ has been determined in (8.6) and in particular $\hat{\beta}^2 - 64 = 0$. So for any $z_1 \in \mathbb{A}(X_1)$ and $z_2 \in \mathbb{A}(X_2)$

$$\begin{aligned} & D_X^{(w,\Sigma)}(z_1 z_2 (x^2 - 4) e^{t(D + nD_S)}) \\ & \qquad = \langle \phi^w(X_1, (x^2 - 4) z_1 e^{tD_1}), \phi^w(X_2, z_2 e^{tD_2}) \rangle \\ & \qquad = \langle \frac{1}{16} (\hat{\beta}^2 - 64) \phi^w(X_1, z_1 e^{tD_1}), \phi^w(X_2, z_2 e^{tD_2}) \rangle \\ & \qquad = 0. \end{aligned}$$

By continuity this implies that $D_X^{(w,\Sigma)}(z(x^2 - 4)e^{tD}) = 0$ for any $D \in H_2(X)$. So X is of w-simple type, and hence of simple type.

Now X has $b_1 = 0$ and $b^+ > 1$, so we have $\mathbb{D}_X = e^{Q/2} \sum a_i e^{K_i}$. Also

$$\phi^w(X_1, e^{tD_1}) \in \overline{HFF}_*(Y, \delta + n\mathbb{S}^1)_I = \bigoplus_{\substack{-(g-1) \le i \le g-1 \\ i \equiv g-1 \pmod 2}} R_i \otimes \mathbb{C}[[t]].$$

Put

$$p(\Sigma) = \begin{cases} (\Sigma^2 - (2g-2)^2)(\Sigma^2 - (2g-6)^2) \cdots (\Sigma^2 - 2^2) & g \text{ even}, \\ (\Sigma^2 + (2g-2)^2)(\Sigma^2 - (2g-6)^2) \cdots (\Sigma^2 + 4^2)\, \Sigma & g \text{ odd}, \end{cases}$$

so that in $\overline{HFF}_*(Y, \delta + n\mathbb{S}^1)_I$, for odd g $p(\alpha/2 + nt) = 0$, and $p(\alpha/2 - nt) = 0$ for even g (see Proposition 8.5). Suppose for concreteness that

g is even (the other case is analogous). Then

$$\begin{aligned} p(\frac{\partial}{\partial s} - nt) & D_X^{(w,\Sigma)}(e^{t(D+nD_S)+s\Sigma}) \\ &= D_X^{(w,\Sigma)}\big(p(\Sigma - nt)e^{t(D+nD_S)+s\Sigma}\big) \\ &= \langle p(\alpha/2 - nt)\phi^w(X_1, e^{tD_1}), \phi^w(X_2, e^{tD_2+s\Sigma})\rangle = 0. \end{aligned}$$

On the other hand, as

$$Q(t(D+nD_S)+s\Sigma) = Q(t(D+nD_S)) + 2nts,$$

[20, Proposition 12] implies

$$\begin{aligned} & D_S^{(w,\Sigma)}(e^{t(D+nD_S)+s\Sigma}) \\ &= e^{Q(t(D+nD_S))/2+nts} \sum_{K_i\cdot\Sigma\equiv 2 \pmod 4} a_{i,w} e^{K_i\cdot(D+nD_S)t+(K_i\cdot\Sigma)s} \\ &+ e^{-Q(t(D+nD_S))/2-nts} \sum_{K_i\cdot\Sigma\equiv 0 \pmod 4} a_{i,w} e^{\sqrt{-1}K_i\cdot(D+nD_S)t+\sqrt{-1}(K_i\cdot\Sigma)s}, \end{aligned}$$

which is a sum (over $\mathbb{C}[[t]]$) of exponentials of the form $e^{nts+2rs}$, $-(g-1) \le r \le g-1$, $r \equiv 1 \pmod 2$, and $e^{-nts+2r\sqrt{-1}s}$, $-(g-1) \le r \le g-1$, $r \equiv 0 \pmod 2$. So for $D_X^{(w,\Sigma)}(e^{t(D+nD_S)+s\Sigma})$ to be a solution of the ordinary differential equation $p(\frac{\partial}{\partial s} - nt)$, the only exponentials appearing should be $e^{nts+2rs}$, with $-(g-1) \le r \le g-1$, $r \equiv 1 \equiv g-1 \pmod 2$. The result follows. q.e.d.

From [21, Corollary 13], the sum of the coefficients of all basic classes K_i of X with $K_i \cdot \Sigma = 2r$ is zero whenever $|r| < g-1$. It is natural to expect that actually these basic classes do not appear. Theorem 9.5 shows that this is in fact true for $r \not\equiv g-1 \pmod 2$.

9.3. Donaldson invariants of $\Sigma_g \times \Sigma_h$. Our final intention is to give the Donaldson invariants of the 4-manifold which is given as the product of two Riemann surfaces of genus $g \ge 1$ and $h \ge 1$. Let $S = \Sigma_g \times \Sigma_h$. Then $b^+ = 1 + 2gh > 2$, so the Donaldson invariants are well-defined. Recall that a 4-manifold X is of w-strong simple type if $D_X^w(\gamma z) = 0$ for any $\gamma \in H_1(X)$, $z \in \mathbb{A}(X)$, and also $D_X^w((x^2-4)z) = 0$ for any $z \in \mathbb{A}(X)$. The structure theorem of [16] is also valid in this case (see [25] for a proof using Fukaya-Floer homology groups).

Proposition 9.6 ([16], [25]). *Let X be a manifold of w-strong simple type for some w and $b^+ > 1$. Then X is of strong simple type and we* *have* $\mathbb{D}_X^w = e^{Q/2} \sum (-1)^{\frac{K_i \cdot w + w^2}{2}} a_i e^{K_i}$, *for finitely many* $K_i \in H^2(X;\mathbb{Z})$ *(called basic classes) and rational numbers* a_i *(the collection is empty when the invariants all vanish). These classes are lifts to integral cohomology of* $w_2(X)$. *Moreover, for any embedded surface* $S \hookrightarrow X$ *of genus* g, *with* $S^2 \geq 0$ *and representing a non-torsion homology class, one has* $2g-2 \geq S^2 + |K_i \cdot S|$.

Now suppose we are in the following situation: $\bar{X}_1$ and $\bar{X}_2$ are 4-manifolds containing embedded Riemann surfaces $\Sigma = \Sigma_i \hookrightarrow \bar{X}_i$ of the same genus $g \geq 1$, self-intersection zero and representing odd elements in homology. Consider $X = \bar{X}_1 \#_\Sigma \bar{X}_2$, the connected sum along Σ (for some identification). Suppose that $\bar{X}_i$ are of strong simple type and moreover that there is an injective map

$$\begin{array}{rcl} H_2(X) & \to & H_2(\bar{X}_1) \oplus H_2(\bar{X}_2), \\ D & \mapsto & (\bar{D}_1, \bar{D}_2) \end{array}$$

satisfying $D^2 = \bar{D}_1^2 + \bar{D}_2^2$ and $D|_{X_i} = \bar{D}_i|_{X_i}$, $i = 1, 2$. Then we have

Proposition 9.7. *In the above situation* X *is of strong simple type. Write* $\mathbb{D}_{X_1} = e^{Q/2} \sum a_j e^{K_j}$ *and* $\mathbb{D}_{X_2} = e^{Q/2} \sum b_k e^{L_k}$ *for the Donaldson series for* X_1 *and* X_2, *respectively. If* $g \geq 2$ *then*

$$\begin{aligned} \mathbb{D}_X(e^{tD}) = & e^{Q(tD)/2} \Big(\sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = 2g-2} 2^{7g-9} a_j b_k \, e^{(K_j \cdot \bar{D}_1 L_k \cdot \bar{D}_2 + 2\Sigma \cdot D)t} \\ & + \sum_{K_j \cdot \Sigma = L_k \cdot \Sigma = -(2g-2)} (-1)^{g-1} \, 2^{7g-9} a_j b_k \, e^{(K_j \cdot \bar{D}_1 + L_k \cdot \bar{D}_2 - 2\Sigma \cdot D)t} \Big). \end{aligned}$$

If $g = 1$ *then*

$$\mathbb{D}_X(e^{tD}) = e^{Q(tD)/2} \sum_{K_j, L_k} a_j b_k \, e^{(K_j \cdot \bar{D}_1 + L_k \cdot \bar{D}_2)t} (\sinh(\Sigma \cdot D)t)^2.$$

Proof. Let us see first that X is of strong simple type. Choose $w_i \in H^2(X_i;\mathbb{Z})$, $i = 1, 2$, and $w \in H^2(X;\mathbb{Z})$ such that $w_i \cdot \Sigma_i \equiv 1 \pmod 2$, $w \cdot \Sigma \equiv 1 \pmod 2$, in a compatible way. For any $D \in H_2(X)$ with $D \cdot \Sigma = 1$, put $D = D_1 + D_2$ with $\bar{D}_i = D_i + \Delta \subset X_i$. As $\bar{X}_1$ is of strong simple type, $D_{\bar{X}_1}^{(w,\Sigma)}((x^2-4)e^{t\bar{D}_1} z_s) = 0$, for any $s \in \mathcal{S}$, so $\phi^w(X_1, e^{tD_1})$ is killed by $\hat{\beta}^2 - 64$. Analogously $\phi^w(X_1, e^{tD_1})$ is killed by ψ_i, for $1 \leq i \leq 2g$. Therefore

$$D_X^{(w,\Sigma)}((x^2-4)e^{tD}) = \langle \phi^w(X_1, (x^2-4)e^{tD_1}), \phi^w(X_2, e^{tD_2}) \rangle = 0.$$

Analogously we see $D_X^{(w,\Sigma)}(\gamma_i e^{tD}) = 0$, $1 \leq i \leq 2g$. We leave to the reader the other $\gamma \in H_1(X)$ not in the image of $H_1(\Sigma) \to H_1(X)$.

For the second assertion, suppose now $g \geq 2$. Then $\phi^w(X_1, e^{tD_1})$ lives in the reduced Fukaya-Floer homology $\overline{HFF}_g^*$ of section 6, which is found in Theorem 6.2 to be isomorphic to $\mathbb{C}^{2g-1}[[t]]$. Actually it is the space $\mathbb{C}^{2g-1}[[t]] \subset V[[t]]$ of [21, page 794]. In [21] the intersection pairing restricted to $\mathbb{C}^{2g-1}[[t]]$ is computed and then

$$D_X^{(w,\Sigma)}(e^{tD}) = \langle \phi^w(X_1, e^{tD_1}), \phi^w(X_2, e^{tD_2}) \rangle$$

is found. So the arguments in [21] carry over to our situation and the result in [21, Theorem 9] is true for X. The statement follows.

The result for $g = 1$ is in [19, Theorem 4.13] and [18]. q.e.d.

We conclude with the proof of Theorem 1.4.

Theorem 9.8. *Let $S = E \times F$ be the product of two Riemann surfaces of genus $g, h \geq 1$, i.e., $E = \Sigma_g$ and $F = \Sigma_h$. Arrange so that $h \leq g$. Then S is of strong simple type and the Donaldson series are as follows:*

$$\begin{cases} \mathbb{D}_S = 4^g e^{Q/2} \sinh^{2g-2} F & \text{if } h = 1, \\ \mathbb{D}_S = 2^{7(g-1)(h-1)+3} \sinh K & \text{if } g, h > 1, \text{ both even}, \\ \mathbb{D}_S = 2^{7(g-1)(h-1)+3} \cosh K & \text{if } g, h > 1, \text{ at least one odd}, \end{cases}$$

where $K = K_S = (2g-2)F + (2h-2)E$ is the canonical class.

Proof. The result is a simple consequence of Proposition 9.7 noting that $S = \Sigma_1 \times \Sigma_1$ is of strong simple type (we leave the proof of this to the reader using the description of HFF_1^*) and also making use of the Donaldson series $\mathbb{D}_S = 4e^{Q/2}$ given in [29]. q.e.d.

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