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A CUT POINT THEOREM FOR CAT(0) GROUPS

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Abstract

Let G be a group acting geometrically on a CAT(0) space X. We show that if $c \in \partial X$ is a cut point, then there is an infinite torsion subgroup of G which fixes c. In particular if G is virtually torsion free, if X is a Euclidean cube complex, or if X is 2-dimensional, then ∂X has no cut point.
We also show that if G is a group acting geometrically on a CAT(0) space X, then G has an element of infinite order.

Introduction

The purpose of this note is two-fold. First we will provide a proof of the cut point theorem for CAT(0) groups. (The notion of the boundary of a CAT(0) space will be defined in Section 2.)

Main Theorem. *Let G be a one-ended group acting properly discontinuously and cocompactly by isometries on a CAT(0) space X. If $c \in \partial X$ is a cut point, then there is an infinite torsion subgroup of G which fixes c.*

Secondly we will give a much shorter, largely self contained proof of the cut point theorem for word hyperbolic groups. The pieces of the original proof appear in [6], [7], [8], [9], [17], and [21].

Summary of Proof for both word hyperbolic and CAT(0)

We start with G, a group. In the case where G is word hyperbolic, let X be the Rips complex for G. In the CAT(0) setting, X will be a CAT(0) space admitting a geometric G-action. In either case G acts by

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homeomorphisms on the metric continuum ∂X. Also G is one-ended if and only if X is one-ended, if and only if ∂X is connected. Suppose ∂X is connected and has a cut point.

1. Construct from ∂X an $\mathbb{R}$-tree T. The action of G on ∂X gives rise to G-action on T.
2. We show that the action of G on T is nontrivial, stable, and nonnesting.
3. Construct from T, an $\mathbb{R}$-tree S on which G acts by isometries.
4. In the case where S is simplicial, we apply Bass-Serre theory to obtain a contradiction.
5. In the case where S is not simplicial, we apply the Rips' machine to this action to obtain a contradiction.

Step (3) is contained in [17]. If the reader is only interested in the word hyperbolic case he should skip Section 2 entirely and follow the *Word Hyperbolic Case Notes* through Sections 3 and 4.

In the word hyperbolic case, showing that ∂X has no cut point implies that ∂X is locally connected. This is not true in the CAT(0) setting. The first obvious non-Euclidean example $G = F_2 \times \mathbb{Z}$ is the counter example. The group G acts on a tree cross $\mathbb{R}$ which has the suspension of a Cantor set as its boundary Z. Notice that Z has no cut points, but is locally connected only at the two suspension points.

1. Continua and trees

The tree construction of this section also appears in [23] in the more general setting of Hausdorff continua, however the key steps in the proof of the cut point theorem involve analyzing the action of G on the constructed tree T and analyzing the structure of T itself. In the interest of sanity we have included a version of the tree construction for metric continua.

Definition. A continuum is a compact connected Hausdorff space. In a continuum Z, $c \in Z$ is a *cut point* if $Z = A \cup B$ where A and B are non-singleton continua and $A \cap B = \{c\}$. If in addition, $D \subset A - \{c\}$ and $E \subset B - \{c\}$, we say that c separates D from E.

Notation. For the remainder of this section, Z will be a metric continuum, G will be a group (possibly trivial) of homeomorphisms of Z, and $C \subset Z$ will be a G-equivariant ($GC = C$) set of cut points of Z.

Definition. For $a, b \in Z$ and $c \in C$, we define $c \in (a, b)$ if there exist non-singleton continua $A \ni a$ and $B \ni b$ with $A \cup B = Z$ and $A \cap B = \{c\}$. As one might imagine, (a, b) will be called an interval, and this relation an interval relation. We define the closed and half-open intervals in the obvious way. i,e., $[a, b] = \{a, b\} \cup (a, b)$, and $[a, b) = \{a\} \cup (a, b)$ for $a \neq b$ ($[a, a) = \emptyset$).

Notice that if $c \in (a, b)$ then for any subcontinuum $Y \subset Z$ from a to b ($a, b \in Y$), $c \in Y$.

Definition. For $a, b \in Z - C$ we say a is equivalent to b, $a \sim b$, if $(a, b) = \emptyset$. For $c \in C$, c is equivalent only to itself. This is clearly an equivalence relation, so let P be the set of equivalence classes of Z. We will abuse notation and say that $C \subset P$ since each element of C is its own equivalence class.

Observe that for $a, b, d \in Z$, if $a \sim b$ then $(a, d) = (b, d)$. We can therefore translate the interval relation on Z to P, and we also enlarge it as follows:

Definition. For $x, y, z \in P$, we say $y \in (x, z)$ if either

1. $y \in C$ where $y \in (a, b)$ for some $a, b \in Z$ with $a \in x$ and $b \in z$, or

2. $y \notin C$ and if $a, b, d \in Z$ with $a \in x$, $b \in y$, and $d \in z$, then $[a, b) \cap (b, d] = \emptyset$.

Since C was chosen to be G-invariant, the action of G on Z gives an action of G on P which preserves the interval structure (we have not given P a topology so it doesn't make sense to ask if the action is by homeomorphisms).

The continuum X

The pretree P and the continuum X

The interval structure on P

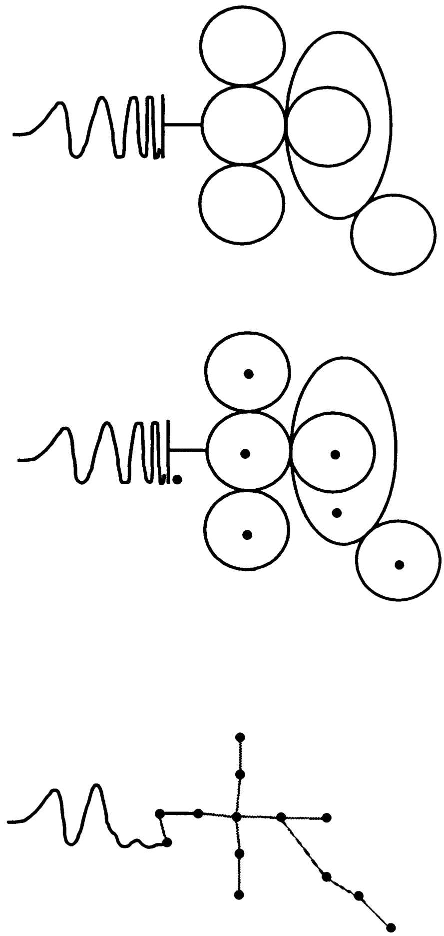

Figure 1

Example. Consider the 1-dimensional continuum Z in Figure 1, a locally finite tree of circles together with the closure of a topologist's sine curve. In this case C will be the set of all cut points of Z. An equivalence class of $Z - C$ will be a circle minus a finite number of points (those in C) or the limit set of the topologist's sine curve minus the point where it attaches to the circles. To represent P, take all the points of C together with a point in the interior of each circle to represent the equivalence class of that circle minus the points of C, and one point for the limit set of the sine curve minus the point of C in that limit set. The interval relation is obtained by "connecting the dots", that is, draw a dotted line segment from the point representing the equivalence class of each circle to the points of tangency (points of C) on that circle, and draw a dotted line segment from the point representing the equivalence class of the limit set of the sine curve to the cut point in the limit set of the sine curve. (See Figure 1.) The interval relation is that which P inherits as a subset of tree.

The following two Lemmas show that the interval relation on P satisfies the second and third axioms of a pretree. The first axiom (namely that $[x,y]=[y,x]$) is satisfied by the definition in our case. The fact that the interval relation on Z satisfies these properties is due to Bowditch [6].

Lemma 1. *For any $x,y,z\in P$, $[x,z]\subset[x,y]\cup[y,z]$.*

Proof. Let $u\in(x,z)$. It suffices to show $u\in[x,y]\cup[y,z]$.

First consider the case where $u\in C$. By definition, there exist A,B non-singleton subcontinua of Z with $A\cap B=\{u\}$, $A\cup B=Z$, $x\subset A$ and $z\subset B$. If $y=u$ then we are done, otherwise, y is contained in A or in B. With no loss of generality $y\subset A$. It follows that $u\in(y,z)$.

Next consider the case where $u\notin C$. Suppose that $u\notin[x,y]$. Choose representatives $a,b,d,e\in Z$ of u,x,y,z respectively. Since $u\notin[x,y]$, it follows that there exists $c\in[b,a)\cap(a,d]$. Since the case where $x=y$ is trivial, we may assume that $c\in C$. It can be shown that there are nonsingleton subcontinua $A,B\subset Z$ such that $a\in A$, $A\cap B=\{c\}$, $A\cup B=Z$, and $b,d\in B$. Since $u\in(x,z)$, it follows that $[b,a)\cup(a,e]=\emptyset$, and so $c\notin(a,e]$ implying that $e\in A-\{c\}$.

Suppose in addition that $u\notin[y,z]$. Arguing exactly as before, we have $c'\in C$ and nonsingleton subcontinua $A',B'\subset Z$ with $A'\cap B'=\{c'\}$, $A'\cup B'=Z$, $a,b\in A'-\{c'\}$, and $e,d\in B'$. Since $a,b\in A'$ it follows that $c\in A'$ (otherwise c couldn't separate a from b). However B' is a continuum from e to d missing c which is impossible since c separates e from d. Thus if $u\notin[x,y]$ then $u\in[y,z]$. q.e.d.

Lemma 2. *For any $x,y\in P$, if $z\in(x,y)$ then $x\notin(y,z)$.*

Proof. Given $z\in(x,y)$ it follows that x,y,z are distinct. Take representatives $a,b,d\in Z$ of x,y,z respectively.

If $d\in C$, then we have non-singleton subcontinua $A,B\subset Z$ with $A\cap B=\{d\}$, $A\cup B=Z$ and $a\in A$ and $b\in B$. Since $a\notin B$, it follows that if $a\in C$, then $a\notin(b,d)$ implying that $x\notin(y,z)$. Thus we are left with the case where $x\notin C$, but since $z\in[y,x)\cap(x,z]$, by definition we see that $x\notin(y,z)$.

If $a \in C$ then we can argue as in the previous paragraph, and so we are left with the case where $a, d \notin C$. Since $a \not\sim d$, there is $c \in C$ and non-singleton subcontinua $A, B \subset Z$ with $A \cup B = \{c\}$, $A \cup B = Z$, $a \in A$ and $d \in B$. Since $z \in (x, y)$, it follows that $c \notin (x, z) \cap (z, y)$ and so $c \notin (z, y)$ imply that $b \in B$. Thus $c \in (b, a) \cap (a, d)$ and by definition $x \notin (y, z)$. q.e.d.

The following two results are pretree results done first in [6]. We provide proofs for the sake of completeness.

Corollary. *If $x, y, z \in P$ with $y \in [x, z]$, then $[x, y] \subset [x, z]$.*

Proof. We may assume that $x \neq y \neq z$. Suppose $w \in [x, y] - [x, z]$. It follows $w \neq x, y, z$. By Lemma 1, $y \in (x, w) \cup (w, z)$ and $w \in (x, z) \cup (y, z)$ the latter of which implies that $w \in (y, z)$. However this in turn implies by Lemma 2 that $y \notin (w, z)$ and so by the first containment, $y \in (x, w)$. This however contradicts $w \in (x, y)$ by Lemma 2 and the proof is complete. q.e.d.

Lemma 3. *Let $[x, y]$ be an interval of P. The interval structure induces two linear orderings on $[x, y]$, one being the opposite of the other, with the property that if $<$ is one of the orderings, then for any $z, w \in [x, y]$ with $z < w$, $(z, w) = \{u \in [x, y] : z < u < w\}$. In other words the interval structure defined by one of the orderings is the same as our original interval structure.*

Proof. For $z, w \in [x, y]$, define $z < w$ if $z \in [x, w)$. We will show that this is a linear order and that the interval structure defined by it is the same as our original interval structure. The ordering we get by replacing x with y in the definition, is the opposite of this ordering because they define the same interval structure.

Using Lemma 2 and the fact that $[x, x) = \emptyset$, we see that $z \not< z$ for any $z \in [x, y]$.

We must next show that for any distinct $z, w \in [x, y]$ either $z < w$ or $w < z$. Suppose then that $w \notin [x, z)$. By Lemma 1, $w \in (z, y]$. By Lemma 2, $z \notin (w, y]$, and so by Lemma 1, $z \in [x, w)$. Hence $z < w$ as required.

Finally we must show transitivity. Let $z, u, w \in [x, y]$ with $z < u$ and $u < w$. In the language of intervals, $z \in [x, u)$ and $u \in [x, w)$. By the Corollary to Lemma 2, $z \in [x, w)$ and so $z < w$.

Now that we have shown that $<$ is a linear ordering on $[x, y]$ we will show that for $w, z \in [x, y]$ with $z < w$, $u \in (z, w)$ if and only if

$z < u < w$.

Let $u \in (z,w)$. It follows from the Corollary to Lemma 2 that $u \in [x,w)$, and so $u < w$. By Lemma 2, $z \notin (u,w)$, however $z < w$, so by Lemma 1, $z \in [x,u)$. Thus $z < u < w$.

Let $u \in [x,y]$ with $z < u$ and $u < w$ (in other words $z \in [x,u)$ and $u \in [x,w)$). By Lemma 1, $u \in [x,z) \cup (z,w)$. By Lemma 2, $u \notin [x,z)$, and so $u \in (z,w)$ as required. q.e.d.

Definition. We say distinct points $x, y \in P$ are *adjacent* if $(x,y) = \emptyset$.

Lemma 4. *If $x, y \in P$ are adjacent, then exactly one of them is in C, and the other is a nonsingleton equivalence class whose closure contains the first.*

Proof. If neither x nor y is in C, then an element c of C must separate them (otherwise $x = y$), and so $c \in (x,y)$ and x and y are not adjacent.

Suppose that both x and y are in C. For each $c \in C - \{x,y\}$, x and y will be contained in a single (quasi)component, D_c, of $X - \{c\}$. Let $Y = \bigcap_{c \in C - \{x,y\}} \overline{D_c}$. Thus Y is a subcontinuum from x to y and no point of C is a cut point of Y. Let $C' = C \cap Y$. It suffices to show that C' is countable, for Y is uncountable, and so $Y - C'$ will be an equivalence class in (x,y) contradicting the adjacency of x and y.

For each $c \in C'$, choose non-singleton subcontinua $A_c, B_c \subset Z$ with $A_c \cup B_c = Z$, $A_c \cap B_c = \{c\}$ and $x \in A_c$. Notice that for any other $d \in C'$, $B_d \subset A_c$, and so $\{(B_c - \{c\}) : c \in C'\}$ is a collection of disjoint nonempty open sets of Z, and therefore countable, since metric spaces are Lindeloef. Thus the set C' is countable and the proof is complete.

Hence one of them, say $x \in C$ and the other, $y \in P - C$. Repeating the above construction, we obtain a continua Y so that $Y - C \subset y$. Since the points $C' = C \cap Y$ are countable, every neighborhood of x contains points of $Y - C$. q.e.d.

Definition. We say $A \subset P$ is colinear if A is contained in some interval.

Lemma 5. *Any set A of three points of P, two of which are adjacent, is colinear. Also a union of two adjacent pairs is colinear.*

The proof is left to the reader.

Theorem 6. *A nested union of intervals of P is an interval of P.*

Proof. We may assume that all the intervals are closed. We first reduce to the case where each of the intervals shares an endpoint y. If they don't, then pick y, an element of the interior of one of them, and restrict to the intervals which contain y. Thus we can write each such interval $[x,z]=[x,y]\cup[y,z]$, and we deal with the two resulting nested unions separately.

Hence it suffices to show that if we have $\{[y,x_\alpha]:\alpha\in I\}$ where I is linearly ordered and $[y,x_\alpha]\subset[y,x_\beta]$ for $\alpha\leq\beta$, then $\cup[y,x_\alpha]$ is an interval of P. By taking a subnet, we may assume that all the x_α are distinct. If I has a last point the result is trivial, and otherwise we may assume that each $x_\alpha\in C$ (if not we replace x_α with an element of C in $(x_\alpha,x_\beta]$ where $\beta>\alpha$). Since Z is compact, the net (or sequence if you prefer) $\{x_\alpha\}$ has a convergent subnet. Since the union is nested, it will not change if we switch to a subnet, and so we may assume that $x_\alpha\to x\in Z$. We will abuse notation and use x both for the element of Z and the corresponding equivalence class of P. Since x_α separates y from $\{x_\beta|\beta>\alpha\}$, it follows that for $\gamma<\alpha$, x_γ separates y from the closure $\overline{\{x_\beta|\beta>\alpha\}}\ni x$. Thus $x_\gamma\in(y,x)$ for all $\gamma\in I$. Put the linear order on $[y,x]$ with $y<x$.

Case I. There is no last point of $[y,x)$.

We will show that $\cup[y,x_\alpha]=[y,x)$. Notice in Z, x_α separates y from x_β for all $\beta>\alpha$. Thus since $x_\alpha\to x$, it follows that $x_\alpha\in(y,x)$ for all α, and so by the Corollary to Lemma 2 $\cup[y,x_\alpha]\subset[y,x)$.

Now let $z\in[y,x)$. Since there is no last point of $[y,x)$, there is $c\in C$ such that $c\in(z,x)$. Since c cannot separate $\{x_\alpha\}$ from x, it follows that $c\in[y,x_\alpha)$ for $\alpha\gg 0$. This implies $z\in[y,x_\alpha)$.

Case II. The last point of $[y,x)$ is z.

Clearly $z\notin C$ (otherwise z would separate $\{x_\alpha\}$ from x). Similarly there can be no last point of $[y,z)$, for it would be in C. Arguing as in Case I, $\cup[y,x_\alpha]=[y,z)$. q.e.d.

Corollary. *Any interval of P has the supremuum property with respect to either of the linear orderings derived from the interval structure.*

Proof. Let $[x,y]$ be the interval in question with the linear order $x \le y$, and let $A \subset [x,y]$. Since A is linearly ordered, $\{[x,a] : a \in A\}$ is a set of nested intervals, and so by Theorem 6 its union will be an interval with one endpoint x, and the other endpoint $\sup A$. q.e.d.

Definition. For each pair $x, y \in P$ with x adjacent to y, let $\mathbb{R}_{x,y}$ be a copy of the real line. We will "sew in" $\mathbb{R}_{x,y}$ between x and y so that one of x, y (say x) is identified with $-\infty$ and the other, y, with ∞. The one identified with $-\infty$ will the first of the pair written, and so in $\mathbb{R}_{z,w}$, z is identified with $-\infty$ and w with ∞. Define

$$\overline{T} = P \cup \bigcup_{x,y \text{ adjacent}} \mathbb{R}_{x,y}$$

We now extend the interval relation to $\overline{T}$ in the obvious way. Namely:

1. For $x, y, z \in P$, then $z \in (x,y)$ if this was so in P.
2. For $x, y \in P$, then $\mathbb{R}_{w,z} \subset (x,y)$ if $w, z \in [x,y]$.
3. For $z \in \mathbb{R}_{x,y}$, then $(x,z) = (-\infty, z) \subset \mathbb{R}_{x,y}$ and $(z,y) = (z,\infty)$.
4. For $z \in \mathbb{R}_{x,y}$ and $w \in P$, by Lemma 5 $\{x,y,w\}$ is colinear, and so we may assume $x \in (y,w)$ in which case $(z,w) = (z,x) \cup [x,w)$.
5. For $z \in \mathbb{R}_{x,y}$ and $v \in \mathbb{R}_{u,w}$, by Lemma 5 $\{x,y,u,w\}$ is colinear and so we may assume $y, u \in (x,w)$ in which case $(z,v) = (z,y) \cup [y,u] \cup (u,v)$.

It is easily shown that $\overline{T}$ **satisfies Lemmas 1, 2, 3, Theorem 6, and also the corresponding Corollaries**. Notice also that there are no adjacent points in $\overline{T}$. Also we can extend the action of G on P to an action of G on $\overline{T}$ which preserves the interval relation on $\overline{T}$.

Definition. For $s \in \overline{T}$ and $E \subset \overline{T}$ finite, we define

$$U(s,E) = \{t \in \overline{T} | \, [s,t] \cap E = \emptyset\}.$$

Remark. Notice that if $t \in U(s,E)$, then by Lemma 1, $U(t,E) = U(s,E)$. Observe that by definition, $U(s,E) \cap U(s,F) = U(s, E \cup F)$.

Definition. A big arc is the homeomorphic image of a connected nonsingleton linearly ordered space. A separable big arc is an arc. A big tree is a uniquely big-arcwise connected locally connected Hausdorff topological space. If the big arcs are all arcs, we say the space is a real tree. A metrizable real tree is called an $\mathbb{R}$-tree.

Theorem 7. *The collection $\{U(s,E)\}$ is a basis for a topology on $\overline{T}$ such that G acts by homeomorphism on $\overline{T}$, and so that $\overline{T}$ is a regular big tree. If every interval of P contains only countable many adjacent pairs, then $\overline{T}$ is a real tree.*

Proof. First we show that the $U(s,E)$ form a basis. Clearly they cover, for example $U(s,\emptyset) = T$. Let $r \in U(s,E) \cap U(t,F)$. By the remark, $U(s,E) \cap U(t,F) = U(r,E) \cap U(r,F) = U(r, E \cup F)$. Thus the $U(s,E)$ form a basis for a topology on $\overline{T}$. Furthermore since the topology was defined in terms of the interval structure, which G preserves, it follows that G acts by homeomorphisms on $\overline{T}$. The local connectivity will follow from the fact that the $U(s,E)$ are big-arcwise connected.

To see that the $\overline{T}$ is Hausdorff, notice that for any $s,t \in \overline{T}$, $(s,t) \neq \emptyset$. Thus let $F \subset (s,t)$ be nonempty and finite. We will show that $U(s,F) \cap U(t,F) = \emptyset$. Suppose that $r \in U(s,F) \cap U(t,F)$. Then $F \cap [r,s] = \emptyset = F \cap [r,t]$. However by Lemma 1, $[s,t] \subset [r,s] \cup [r,t]$. This would mean $F \cap [s,t] = \emptyset$ contradicting the choice of F. Thus $\overline{T}$ is Hausdorff.

For regularity consider $U(s, \{x_1 \dots x_n\})$ where $s \neq x_i$. By the Remark, such sets form a basis at the point s. Since no points of $\overline{T}$ are adjacent, we may choose $y_i \in (s, x_i)$. By the Corollary to Lemma 2, $U(s,\{y_1, \dots y_n\}) \cup \{y_1, \dots y_n\} \subset U(s, \{x_1, \dots x_n\})$. The closure

$$\overline{U(s,\{y_1,\dots y_n\})} = U(s,\{y_1,\dots y_n\}) \cup \{y_1,\dots y_n\}$$

and so the topology is regular.

Consider a closed interval $[x,y]$ $(x \neq y)$ of $\overline{T}$. Use Lemma 3 to put a linear order on $[x,y]$. The subspace topology on $[x,y]$ will be exactly the order topology on $[x,y]$. By the Corollary to Theorem 6, $[x,y]$ has the supremuum property, and since no two points of $[x,y]$ are adjacent, it follows by standard results in linear topology that $[x,y]$ is connected and therefore a big arc. For any $z \in (x,y)$, as in the proof of Hausdorff, $U(x,\{z\}) \cap U(y,\{z\}) = \emptyset$. Also $\{U(t,\{z\}) | t \neq z\}$ is a collection of nonempty disjoint open sets whose union is $\overline{T} - \{z\}$. Thus any connected set of $\overline{T}$ which contains both x and y will also contain $[x,y]$, and so any big arc γ from x to y will contain $[x,y]$. However a big arc is an irreducible (maybe not metric) continuum between its endpoints, and therefore γ contains no sub-big arc from $[x,y]$. Thus $\overline{T}$ is uniquely big arcwise connected, and $\overline{T}$ is a big tree.

Now suppose that every interval of P contains only countably many adjacent pairs. We must show that each closed interval $[s,t]$ of $\overline{T}$ is an arc. By standard results in linear topology, we need only find a countable dense set in $[s,t]$ (since $[s,t]$ is connected). We can easily reduce to the case where $s,t \in P$. Since Z is a compact metric space, it follows that there exists a countable dense subset $\hat{D} \subset Z$. Let $D \subset P \subset \overline{T}$ be the set of equivalence classes of elements of $\hat{D}$. For any point $x \in \overline{T}$, since $\overline{T}$ is uniquely big arcwise connected, there is a unique point $\pi(x) \in [s,t]$ such that $[x, \pi(x)] \cap [s,t] = \{\pi(x)\}$. This defines a continuous function $\pi : \overline{T} \to [s,t]$.

Define

$$Q = \pi(D) \cup \bigcup_{\substack{x,y \in [s,t], \\ x,y \text{ adjacent in } P}} \mathbb{Q}_{x,y},$$

where $\mathbb{Q}_{x,y}$ is the set of rationals in $\mathbb{R}_{x,y}$. Since there are only countably many adjacent pairs in $[s,t] \cap P$, Q will be countable. We now will show that Q is dense in $[s,t]$. Let $u,v \in [s,t]$ with $u \in [s,v)$. If $(u,v) \not\subset P$, then by definition of $\overline{T}$, $(u,v) \cap \mathbb{R}_{x,y} \neq \emptyset$ for some adjacent pair x,y of $[s,t] \cap P$, and so $(u,v) \cap \mathbb{Q}_{x,y} \neq \emptyset$. Thus we may assume $(u,v) \subset P$. Since there are no adjacent points in $\overline{T}$, $(u,v) \neq \emptyset$ and in fact (u,v) is infinite since $[u,v]$ is connected. Every two nonsingleton equivalence class of Z will be separated by at least one element of C, and so it follows that there are distinct $c,c' \in (u,v) \cap C$. Let $p \in (c,c')$. With no loss of generality $c \in (u,p) \subset (s,p)$ and $c' \in (p,v) \subset (p,t)$. This implies that there are non-singleton subcontinua $A, B, A', B' \subset Z$ so that $A \cup B = Z = A' \cup B'$, $A \cap B = \{c\}$, $A' \cap B' = \{c'\}$, $s,u \subset A$, $t,v \subset A'$, and $p \subset B \cap B'$. The set $U = (B - \{c\}) \cap (B' - \{c'\})$ is open and non-empty ($p \in U$), and so there exists $\hat{d} \in U \cap (\hat{D})$. Let $d \in D$ be the corresponding equivalence class.

It will suffice to show that $\pi(d) \in [c,c']$. Since $\hat{d} \in B - \{c\}$, using definition we have $c \in (s,d)$, and similarly $c' \in (d,t)$. Thus by the Corollary to Lemma 2, $[s,c] \subset [s,d]$ and $[c',t] \subset [d,t]$. From definition it follows that $\pi(d) \notin [s,c)$ and $\pi(d) \notin (c',t]$, so that $\pi(d) \in [c,c'] \subset (u,v)$ as required. Therefore $[s,t]$ is separable, and hence an arc. q.e.d.

Corollary. *If C is countable, then $\overline{T}$ is a real tree.*

Proof. Let I be an interval of P. By Lemma 4 each adjacent pair of I will have exactly one point in C. For any $c \in C \cap I$, there are at most two points of I adjacent to c. Thus there is a function from the adjacent pairs of I to C which is at most 2 to 1, and set of adjacent pairs of I is countable. q.e.d.

Conjecture. *Every interval of P has only countable many adjacent pairs and so $\overline{T}$ is a real tree.*

Definition. The group G is called a convergence group if for each sequence of distinct elements of G, there exists a subsequence (g_i) and points $N, P \in Z$ such that for any neighborhood U of P and any compact $K \not\ni N$, $g_i(K) \subset U$ for all $i \gg 0$.

We will not discuss the properties of convergence groups here, but will refer one to [25].

Definition. A point of a big tree is terminal if it is not contained in the interior of any big arc.

Notation. Let $T = \overline{T} - \{\text{terminal points of } \overline{T}\}$. Since the set of terminal points of $\overline{T}$ is invariant under homeomorphism, T is invariant under the action of G.

Lemma 7. *Let H be a convergence group acting on the metric continuum Z with $\Lambda H = \{a, b\}$. If a and b are separated by an element of C, then the corresponding equivalence classes $[a], [b] \subset \overline{T}$ are terminal in $\overline{T}$, and no point of T is fixed by H. If on the other hand a and b are not separated by any element of C, then $|([a], [b]) \cap P| \leq 1$.*

Proof. First consider the case when a and b are separated by some $c \in C$. Thus we have $a \in A - \{c\}$, $b \in B - \{c\}$ where A and B are nonsingleton subcontinua with $A \cup B = Z$ and $A \cap B = \{c\}$. By [25, Lemmas 2Q and 2I] we may choose a $h \in H$ non-torsion so that $h^n(B) \to b$ as $n \to \infty$ and $h^n(A) \to a$ as $n \to -\infty$. Replacing h with a power, we may assume that $h(B) \subsetneq B$ and $h^{-1}(A) \subsetneq A$.

We now show that $[b]$ is terminal in P and therefore in $\overline{T}$; the argument for $[a]$ will be identical. Suppose not, then using Lemmas 1 and 2 there is $d \in Z$ such that $[b] \in ([a], [d])$. Since h fixes a and b, it follows that $h^n(c) \in ([a], [b])$ and so by Lemma 3 applied to $([a], [d])$, $h^n(c) \notin ([b], [d])$ for all n. This implies that $d \in h^n(B)$ for all n, but $\cap h^n(B) = \{b\}$. Thus $b = d$ contradicting $[b] \in ([a], [d])$. This completes the proof that b is terminal in $\overline{T}$.

To see that only $[a]$ and $[b]$ are fixed by the action of H on $\overline{T}$, notice that for any $x \in P - \{[a], [b]\}$ either $x \subset A$ or $x \subset B$. Without loss of generality $x \subset B$. Since the nested intersection $\cap h^n(B) = \{b\}$, it follows that $h^n(x) \to [b]$ as $n \to \infty$. Thus h does not fix x, and no element of T is fixed by H.

In the case where no element of C separates a from b, it follows that $([a],[b]) \cap P \subset (P - C)$. Furthermore no two points of $([a],[b]) \cap P$ are separated by an element of C, and so $([a],[b]) \cap P$ is at most a single point of $P - C$. q.e.d.

2. CAT(0)

Recall that a CAT(0) space is a proper geodesic metric space with the property that every geodesic triangle is at least as thin as the corresponding Euclidean triangle (see [10]). For the remainder of this section, X will be a CAT(0) space and G will be a group which acts *geometrically* on X, that is G acts properly discontinuously, cocompactly by isometries on X. We review some properties of the action of G on X proven in [10]. It is assumed that the reader has a copy of [10] at hand. (Geodesic) arcs, rays, and lines are isometrically embedded connected subsets of $\mathbb{R}$. Unless otherwise stated, all rays will be parameterized by $[0, \infty)$.

Every element of G is either hyperbolic, or elliptic. The elliptic elements fix a point of X, and are therefore torsion. The hyperbolic elements act by translation on at least one line of X called an axis of the hyperbolic element, and all axes of a given hyperbolic element will be parallel.

The visual boundary, ∂X, is the set of equivalence classes of rays, where rays are equivalent if they fellow travel. Given a ray R and a point $x \in X$ there is a ray S emanating from x with $R \sim S$. Fixing a base point $\mathbf{0} \in X$, we define a topology on $\overline{X} = X \cup \partial X$ by taking the basic open sets of $x \in X$ to be the open metric balls about x. For $y \in \partial X$ and R a ray emanating from $\mathbf{0}$ representing y, we construct basic open sets $U(R, n, \epsilon)$ where R is a ray from $\mathbf{0}$ and $n, \epsilon > 0$. We say $z \in U(R, n, \epsilon)$ if the geodesic S parameterized by arc length and by $[0, d(\mathbf{0}, z)]$ from $\mathbf{0}$ to z has the property that $d(R(n), S(n)) < \epsilon$. These sets form a basis for a topology under which $\overline{X}$ and ∂X are compact metrizable.

Definition. For $g \in G$ we define $\tau(g) = \min_{x \in X} d(x, g(x))$. This minimum is realized and the set $Min(g) = \{x \in X \,|\, d(x, g(x)) = \tau(g)\}$ is nonempty.

We now need an easy result about geometric group actions, which nonetheless appeared in the authors thesis in a slightly different form.

The Zipper Lemma [22]. Let H act geometrically on the proper geodesic metric space Y. For every $\epsilon > 0$ there is N_ϵ with the property that for any $g, h \in G$ and any $x, y \in Y$, if $|\{g^n(x)\} \cap \mathrm{Nbh}(\{h^n(y)\}, \epsilon)| > N_\epsilon$ then $g^n = h^m$ for some $n, m \neq 0$.

Corollary. *If $g \in G$ is a hyperbolic element with axis L, then any element of G which fixes one endpoint of L fixes both endpoints of L.*

Proof. If h is an element which fixes one endpoint of L, then for some ϵ, there are subrays of L and $h(L)$ which fellow travel(ϵ). The lines L and $h(L)$ are axes of g and hgh^{-1} respectively, and we may apply the Zipper Lemma. q.e.d.

Definition. If $g \in G$ hyperbolic, then let L be an axis for g. We define $g^\infty \in \partial X$ to be the endpoint of L in the direction of g-translation, and $g^{-\infty}$ to be the other endpoint of L. Since all axes of g are parallel, this is independent of the choice of L.

Theorem 8. *If for some hyperbolic element $g \in G$, $h(g^\infty) = g^\infty$ for some $h \in G$, then for some $n \neq 0$, g^n and h commute. Thus $Stab(g^\infty) = \bigcup_{n>0} Z_{g^n}$.*

Proof. By the Corollary to the Zipper Lemma, there is an axis L of g which is parallel to an axis $h(L)$ of hgh^{-1}. Let $x \in L$ and $y \in h(L)$. We see that for some $\epsilon > 0$, $\{hg^nh^{-1}(y)\} \subset h(L) \subset \mathrm{Nbh}(\{g^n(x)\}, \epsilon)$. Thus by the Zipper Lemma $hg^mh^{-1} = g^n$ for some $n, m \neq 0$. Since the translation length $\tau(g) \neq 0$, $n = \pm m$ and since h fixes both endpoints of L, it follows that $n = m$. q.e.d.

The following is due to Bridson and Haeflinger [10].

Theorem 9. *If $g \in G$ is hyperbolic, then $Min(g) = Y \times \mathbb{R}$ where $\{y\} \times \mathbb{R}$ is an axis of g for each $y \in Y$.*

Definition. For $A \subset X$, define ΛA to be the set of limit points of A in ∂X. For $S \subset G$ we define $\Lambda S = \Lambda(Sx)$ for some $x \in X$. Observe that the definition of ΛS is independent of the choice of the point x.

The following can be viewed as a corollary to Theorem 9, but we provide an independent proof.

Lemma 10. *If $g \in G$ is hyperbolic then $Fix(g)$, the fixed point set of g acting on ∂X, is either equal to $\{g^{\pm\infty}\}$ or $Fix(g)$ is path connected. Moreover for each $x \in Fix(g)$, and any axis L of g, there is a half-flat P bounded by L with x a limit point of P.*

Proof. Let L be an axis for g, and suppose that $\mathrm{Fix}(g) \neq \{g^{\pm\infty}\}$. Let $p \in L$, and R be a ray representing a point of $\mathrm{Fix}(g) - \{g^{\pm\infty}\}$ which emanates from p. It follows that $g(R) \sim R$.

Let P be the convex hull of $\cup g^n(R)$. We first show that P is a Euclidean half-plane. We then show that every point of ΛP is in $\mathrm{Fix}(g)$. Since the points of ∂X represented by rays in a Euclidean half-plane form an arc, it follows every point of point of $\mathrm{Fix}(g)$ lies on an arc of $\mathrm{Fix}(g)$ joining $g^{-\infty}$ to g^{∞}.

The function $f(t) = d(g(R(t)), R(t))$ is a convex function from $[0,\infty) \to [0,\infty)$. Furthermore, since $(R(0), g(R(0)) \in L \subset Min(g)$, it follows that $f(0)$ is the minimum for f. Since $R \sim g(R)$, f is bounded and it follows that f is constant. For each $t \in [0,\infty)$ consider the piecewise geodesic

$$L_t = \bigcup_{n\in\mathbb{N}} [g^n(R(t)), g^{n+1}(R(t))].$$

If

$$\angle_{g^n(R(t))}(g^{n-1}(R(t)), g^{n+1}(R(t))) < \pi,$$

then there would be a point $x \in [g^{n-1}(R(t)), g^n(R(t))]$ with the property that $d(x, g(x)) < d(g^{n-1}(R(t)), g^n(R(t))) = \tau(g)$ which is absurd. Thus by [10] L_t is a geodesic line, and since g acts on it by translation, it follows that L_t is an axis. By the Flat Strip Theorem of [10], the convex hull of $L \cup L_t$ is a flat strip (isometric to $\mathbb{R} \times [0,b] \subset \mathbb{E}^2$). Letting $t \to \infty$ we deduce that P is isometric to a half-plane of $\mathbb{E}^2$. Notice since P is a union of axes of g, it follows that $g(P) = P$. For any ray $\rho \subset P$, $g(\rho) \sim \rho$, and thus $\partial P \subset \mathrm{Fix}(g)$. This completes the proof.

q.e.d.

Corollary. *For any hyperbolic element $g \in G$, $Fix(g) = \Lambda Min(g)$ is a suspension.*

Proof. By Theorem 9, any point of $\Lambda Min(g)$ lies in the limit set of a half-flat which is the union of axis of g. Thus $\Lambda Min(g) \subset \mathrm{Fix}(g)$. The other inclusion follows from Lemma 10. Since $Min(g)$ is a product, $\mathbb{R} \times Y$, its limit set is a suspension (perhaps of the empty set). q.e.d.

We now give proofs of two very elementary results. The first is apparently new, and the second has not be explicitly stated before but likely follows from work of Geoghegan [14].

Theorem 11. *If G acts geometrically on the CAT(0) space X, then G has an element of infinite order.*

Proof. Choose a geodesic ray $R \subset X$. Using cocompactness we find a sequence $(g_i) \subset G$ and an increasing sequence $r_i \in R$ having the following properties:

- $r_i \to R(\infty)$.
- $g_i(r_i) \to x \in X$.
- The rays $g_i(R)$ converge uniformly on compact subsets to a line L preserving the orientation.

For any $M > 0$ we can find $j > i$ so that $d(g_i(r_i), g_j(r_j)) < \frac{1}{M}$ and $g_j(R) \cap B(g_i(r_i), M) \subset \mathrm{Nbh}\left(g_i(R), \frac{1}{M}\right)$.

Let $h = g_i^{-1}g_j$. Now consider the angle $\theta = \angle_{h(r_i)}(r_i, h^2(r_i))$. Construct the picture of R, $h(R)$ and $h^2(R)$, and notice that $\theta \to \pi$ as $M \to \infty$. If h were torsion, the polygon with vertices $h^n(r_i)$ would be a regular polygon with interior angles all equal to θ. By [10, II 2.12(1)] θ would be no larger that the interior angle of the corresponding Euclidean polygon. Then the order of h is at least $\frac{2\pi}{\pi-\theta}$. There are only a finite number of conjugacy classes of torsion elements [10], so for M sufficiently large h is hyperbolic. q.e.d.

The following proof also works in the δ-hyperbolic setting if we choose $\epsilon > 2\delta$, and so gives an easier proof of a result in [24].

Theorem 12. *If X is a CAT(0) space which admits a cocompact action by isometries, then ∂X is finite dimensional.*

Proof. We first construct a sequence of covers of ∂X with mesh going to zero. Let $\mathbf{0} \in X$. Choose any $\epsilon > 0$. Let S_N be the set of points on the sphere of radius N about $\mathbf{0}$, through which rays from $\mathbf{0}$ pass. That is

$$S_N = \{R(N) : R \text{ is a geodesic ray emanating from } \mathbf{0}\}.$$

Since X is proper and S_N is closed and bounded, it follows that S_N is compact. Let E_N be a maximal ϵ-thin subset of S_N. In other words E_N is maximal with the property that if $x, y \in E_N$, then $d(x, y) \geq \epsilon$. The compactness of S_N forces E_N to be finite. Let $\mathcal{G}_N = \{U(R, N, \epsilon) : R$ is a ray from $\mathbf{0}$ with $R(N) \in E_N\}$. Thus the elements of $\mathcal{G}_N$ are centered on rays going through the points of E_N.

Clearly $\mathcal{G}_N$ is an open covering of ∂X, for if S is any ray from $\mathbf{0}$ then, by the maximality of E_N, $d(S(N), E_N) < \epsilon$. For fixed $m, \mu > 0$, if $N >> m$, then $U(S, N, \epsilon) \subset U(S, m, \mu)$ for any ray S. It follows that mesh $\mathcal{G}_N \to 0$.

The dimension of the nerve of $\mathcal{G}_N$, $dim\,\mathcal{G}_N$, is by definition one less than the maximal number of elements of $\mathcal{G}_N$ meeting in a single point. It suffices to show that $\{dim\,\mathcal{G}_N : N \in \mathbb{N}\}$ is bounded.

Suppose to the contrary that for each m there exists N_m with $dim\,\mathcal{G}_{N_m} \geq m$. Thus for each m there exist distinct points $x_0^m, x_1^m \ldots x_m^m \in E_{N_m}$ with $d(x_i^m, x_j^m) < 2\epsilon$. Since X admits a cocompact action, there is a compact $K \subset X$ such that any point of X can be moved into K by an isometry. For each m choose an isometry g_m so that $g_m(x_0^m) \in K$. Passing to a subsequence we may assume that $g_m(x_0^m) \to x_0$ where x_0 is some point of K. Passing to a smaller subsequence, we may assume that $g_m(x_1^m) \to x_1$. Continuing we get a sequence of distinct points (x_i) such that $\epsilon \leq d(x_i, x_j) \leq 2\epsilon$ when $i \neq j$. It follows that (x_i) is a bounded sequence with no convergent subsequence. This is a contradiction as X is a proper metric space. q.e.d.

The following is due to Ruane [20]. It was stated only for hyperbolic elements but the proof doesn't use that hypothesis.

Theorem 13. *For any $g \in G$, Z_g acts geometrically on $Min(g)$.*

The proof of the following is exactly the same as the proof that a normal subgroup of the fundamental group corresponds to a regular covering space.

Lemma 14. *If a group G acts geometrically on X and $N \lhd G$, then G/N acts geometrically on X/N.*

Definition. For $A \subset X$, a closed convex set, and $x \in X$ we define $\pi_A(x)$ to be the unique point of A closest to x.

Definition. A subgroup $H < G$ is called *convex* if there is a nonempty closed convex $A \subset X$ with $HA = A$, and so H acts cocompactly on the CAT(0) space A. Notice that every finite subgroup is convex.

The following is a special case of [10, II 2.5 (1)].

Lemma 15. *If $A \subset X$ is closed and convex, then so is $Nbh(A, \epsilon) = \{x \in X : d(x, A) \leq \epsilon\}$ for any $\epsilon > 0$.*

Theorem 16. *If $H, K < G$ are convex, then*

1. *$H \cap K$ is convex ,*

2. $\Lambda H \cap \Lambda K = \Lambda(H \cap K)$.

Proof. We first show (1). Let A and B be nonempty closed convex subsets of X on which H and K respectively act cocompactly. Fix some $\epsilon > 0$ with $D_\epsilon = \text{Nbh}(A, \epsilon) \cap \text{Nbh}(B, \epsilon) \neq \emptyset$. The intersection of closed convex sets is a closed and convex set, and so D_ϵ is closed and convex. Any element $g \in H \cap K$ leaves A and B invariant, and so leaves D_ϵ invariant.

We must now show that $H \cap K$ acts cocompactly on D_ϵ. Let C_A and C_B be compact sets whose translates by H and K respectively cover A and B respectively. Let $f : X \to X/(H \cap K)$ be the projection map. Suppose $f(D_\epsilon)$ is not compact. Then there is a sequence of points $d_i \in D_\epsilon$ with $f(d_i)$ having no convergent subsequence. Let $a_i = \pi_A(d_i)$ and $b_i = \pi_B(d_i)$.

Choose $a \in A$ and $b \in B$ with $d(a, b) \leq 2\epsilon$. For some $\mu > 0$, every point of A has an H-translate inside the ball $B(a, \mu)$, and similarly every point of B has a K-translate inside the ball $B(b, \mu)$. For each i choose $h_i \in H$ and $k_i \in K$ with $h_i(a_i) \in B(a, \mu)$ and $k_i(b_i) \in B(b, \mu)$. Thus letting $E = \text{Nbh}(B(b, \mu) \cup B(a, \mu), 2\epsilon)$ we see that $k_i^{-1}(E) \cap h_i^{-1}(E) \neq \emptyset$, so $h_i k_i^{-1}(E) \cap E \neq \emptyset$. Since E is compact and G acts properly discontinuously, $h_i k_i^{-1} = h_j k_j^{-1}$ for infinitely many i, j, and so taking a subsequence we may assume that $h_j^{-1} h_i = k_j^{-1} k_i$ for all i, j. In particular $h_j^{-1} h_1 \in H \cap K$ for all j. Notice that $d(d_j, h_j^{-1} h_1(h_1^{-1}(a))) < \mu + \epsilon$. It follows that in $X/(H \cap K)$, $d(f(d_j), f(h_1^{-1}(a))) \leq \mu + \epsilon$ for j. Thus the sequence $f(d_j)$ is bounded and must have a convergent subsequence. We have a contradiction which completes the proof of (1).

Now for (2). Clearly $\Lambda(H \cap K) \subset \Lambda H \cap \Lambda K$. Notice that $\Lambda H = \Lambda A$ and $\Lambda K = \Lambda B$. Let R and S be rays of A and B respectively with $R \sim S$. There is an δ with $R \subset \text{Nbh}(S, \delta)$ and $S \subset \text{Nbh}(R, \delta)$. For $\epsilon > \delta$, $R \subset D_\epsilon$ and so the point at infinity represented by R will be in $\Lambda D_\epsilon = \Lambda(H \cap K)$. Thus $\Lambda H \cap \Lambda K \subset \Lambda(H \cap K)$. q.e.d.

Theorem 17. *If $H < G$ is an infinite torsion subgroup which fixes a point of $b \in \partial X$, then there is a CAT(0) space Y and a group K acting geometrically on Y with $\dim \partial Y < \dim \partial X$ where K contains an infinite torsion subgroup.*

Proof. By Theorem 12, $dim \partial X < \infty$. Since there are only finitely many conjugacy classes of finite subgroups, it follows that there are elements $h_1, \ldots h_n \in H$ which generate an infinite subgroup and we may assume that $H = \langle h_1, \dots h_n \rangle$. Let

$$Z = \bigcap_{i=1}^{n} Z_{h_i},$$

where Z_{h_i} is the centralizer of h_i. By Theorem 13, Z_{h_i} acts geometrically on $Min(h_i)$. Thus $b \in \mathrm{Fix}(h_i) = \Lambda Z_{h_i}$ for each $1 \leq i \leq n$. By Theorem 16, $b \in \Lambda Z$ and Z is a convex subgroup. In particular Z is infinite. By Theorem 11, Z has a hyperbolic element g. By [20] Z_g acts geometrically on the $Min(g) = Y \times \mathbb{R}$. Since $< g > \lhd Z_g$ it follows from Lemma 14 that $Z_g / < g >$ acts geometrically on Y. The torsion subgroup H commutes with non-torsion element g, and so the quotient map embeds H into $Z_g / < g >$. The result follows. q.e.d.

Corollary. *If $H < G$ then $|\Lambda H| \neq 1$.*

Proof. If H contains a hyperbolic element, then ΛH has at least two points, namely the limit points of the hyperbolic element. Thus we may assume that H is torsion. If $\{b\} = \Lambda H$, then H fixes b, and we are in the setting of the theorem. Following the proof of the theorem we see that H stabilizes the convex subset Y and so $\Lambda H \subset \Lambda Y$. Since $dim \partial Y < dim \partial X$, by induction we may reduce to the case where $dim \partial X = 0$. However in that case G is virtually free and has no infinite torsion subgroups. q.e.d.

Definition. For $a, b \in \partial X$, we define $\angle(a, b) = \sup_{x \in X} \angle_x(a, b)$, where $\angle_x(a, b)$ is the angle between the two rays emanating from x, which represent a and b. This forms a metric on ∂X which gives a topology at least as fine as the visual topology on ∂X. We define Td to be the path metric corresponding to the angle metric on ∂X. The set ∂X with the metric Td will be called the Tits boundary TX which is a CAT(1) space, but need not be connected and can be a discrete.

One of the main differences between negative and positive curvatures is in the action on the boundary. In negative curvature, if a sequence $(g_n) \subset G$ with $g_n(x) \to a \in \partial X$ for some (any) $x \in X$, then all but one point of the boundary ∂X will also be sent to a by (g_n). This is equivalent to saying that G acts as a convergence group on $\overline{X}$ with limit set ∂X. This is not the case in even the simplest example of non-positive curvature. In the example of $\mathbb{Z}^2$ acting on the Euclidean plane, $\mathbb{E}^2$, the action of $\mathbb{Z}^2$ fixes the boundary $\partial \mathbb{E}^2 = S^1$. If we take two points in $d, e \in \partial\mathbb{E}^2 = S^1$ with $\angle(d,e) = \theta$, then we can choose a sequence of $g_n \in \mathbb{Z}^2$ so that $g_n(\mathbf{0}) \to a$, where a is the point of $\partial\mathbb{E}^2$ at greatest distance from $\{d, e\}$, that is $\angle(d,a) = \angle(e,a) = \pi - \frac{\theta}{2}$. In the next lemma we see that this is the worst that can happen in any CAT(0) space.

Lemma 18. *Let X be a CAT(0) space with G acting by isometries on X, and suppose that $d, e \in \partial X$ with $\angle(d,e) = \theta$. If there is (g_n) a sequence of group elements with $g_n(x) \to a \in \partial X$ for some (any) $x \in X$, $g_n(d) \to \tilde{b}$, and $g_n(e) \to \hat{b}$, then $\angle(a,\hat{b}) + \angle(a,\tilde{b}) \leq 2\pi - \theta$.*

Proof. Suppose not. Then

$$\angle(\tilde{b},a) + \angle(\hat{b},a) > 2\pi - \theta + 2\epsilon$$

for some $\epsilon > 0$. Using a ray representing a, and [10, II 9.8 (2)] we can find a point $y \in X$ with

$$\angle_y(\tilde{b},a) + \angle_y(\hat{b},a) > 2\pi - \theta + 2\epsilon.$$

By definition, we can find $x \in X$ with $\angle_x(d,e) > \theta - \epsilon$. For any n, consider the two angles of $\angle_{g_n(x)}(y, g_n(d))$ and $\angle_{g_n(x)}(y, g_n(e))$. By the triangle inequality for angles [10, I 1.14],

$$\angle_{g_n(x)}(y, g_n(d)) + \angle_{g_n(x)}(y, g_n(e)) \geq \angle_{g_n(x)}(g_n(d), g_n(e)) > \theta - \epsilon.$$

Since $g_n(x) \to a$, the sequence of segments $[y, g_n(x)) \to [y, a)$, the ray from y to a. Similarly $[y, g_n(d)) \to [y, \tilde{b})$ and $[y, g_n(e)) \to [y, \hat{b})$. It follows that $\angle_y(g_n(d), g_n(x)) \to \angle_y(\tilde{b}, a)$ and $\angle_y(g_n(e), g_n(x)) \to \angle_y(\hat{b}, a)$ Thus for $n \gg 0$, $\angle_y(g_n(d), g_n(x)) + \angle_y(g_n(e), g_n(x)) > 2\pi - \theta + 2\epsilon$.

Consider the semi-ideal triangles Δ_1 with vertices y, $g_n(x)$, and $g_n(d)$ and Δ_2 with vertices y, $g_n(x)$, and $g_n(e)$. The sum of the angle sums of Δ_1 and Δ_2 will be at least $(2\pi - \theta + 2\epsilon) + (\theta - \epsilon) > 2\pi$ for n sufficiently large. Thus one of the semi-ideal triangles Δ_1, Δ_2 violates [10, II 9.3] and we have a contradiction. q.e.d.

Lemma 19. *If $R : [0, \infty)$ is a ray representing $a \in \partial X$, $b \in \partial X$ with $\angle(a,b) < \pi$, and the image of R in X/H is bounded for some $H < G$, then there is a sequence $(g_n) \subset H$ with:*

1. *$g_n(R) \to \hat{R}$, a line;*

2. $g_n(a) \to \hat{a}$*;*

3. $g_n(b) \to \hat{b}$*;*

4. for any $t \in \mathbb{R}$, $\angle_{\hat{R}(t)}(\hat{a}, \hat{b}) = \angle(\hat{a}, \hat{b}) = \angle(a, b)$*;*

5. There is a half-flat B *bounded by* $\hat{R}$ *with* $\hat{b} \in \Lambda\hat{B}$*.*

Proof. Since the image of R in X/H is bounded, we may find a sequence $(g_n) \subset H$ with $\{d(R(0), g_n(R(n))) : n \in \mathbb{N}\}$ bounded. Using compactness and passing to a subsequence n_j we obtain (1), (2), and (3).

The equality (4) has the same proof as [10, II 9.5 (3)] and we leave the conversion of notation to the anal retentive reader.

For $s, t \in \mathbb{R}$ and $i \in \mathbb{N}$, consider now the triangle Δ with vertices $\hat{R}(t)$, $\hat{R}(s)$ and $\hat{b}$. Using [10, I 1.14] we can show that the angle sum of Δ is at least π. Thus by [10, II 9.3] the angle sum is exactly π and the convex hull of $[R(s), \hat{b}) \cup [R(t), \hat{b})$ is isometric to the convex hull of a triangle in the Euclidean plane with those angles and one vertex at infinity. Taking the nested union of such things we obtain the half-flat $\hat{B}$ and this completes (5). q.e.d.

3. CAT(0) boundary cut points

Definition. An action on a real tree S is non-nesting if no arc of S is mapped to a proper subset of itself. A non-nesting action is stable if for each arc γ of S, there is a non-trivial subarc α of γ so that $\mathrm{Fix}(\alpha) = \mathrm{Fix}(\beta)$ for all β nonsingleton subarc of α. An action on an $\mathbb{R}$-tree is *trivial* if the action has a global fixed point.

Theorem 20. *Suppose that* $\partial X = A \cup B$ *where* A *and* B *are nonsingleton subcontinuum with* $|A \cap B| = 1$*. If* $g \in G$ *with* $g(A) \subsetneq A$*, then* $\cap g^n(A) = \{g^\infty\}$*,* $\cap g^n(B) = \{g^{-\infty}\}$ *and* $\langle g \rangle$ *acts as a convergence group on* ∂X*.*

Proof. Clearly the two compact sets $\cap g^n(A)$ and $\cap g^n(B)$ are nonempty, as they are nested intersections of compact sets.

We first show that given $a \in \cap g^n(A)$ and $b \notin \cap g^n(A)$, there is a line in X with endpoints a and b. By [10, II 9.21 (1)] it suffices to show that $Td(a, b) = \infty$. Let $\{c\} = A \cap B$. There is $N > 0$, such that for all $n \geq N$, $g^{n+1}(c)$ separates a from $g^n(c)$, and $g^n(c)$ separates b from $g^{n+1}(c)$. Let γ be an arc of TX from a to b (if there is no such arc, then $Td(a,b) = \infty$). For each $n \geq N$ there is a subarc γ_n of γ irreducible from $g^n(c)$ to $g^{n+1}(c)$. Let $\epsilon = \angle(c, g(c)) > 0$. Since G acts by isometries on the angle metric of ∂X, it follows that $\epsilon = \angle(g^n(c), g^{n+1}(c))$ for all $n \in \mathbb{Z}$. Thus the length $\ell(\gamma_n) \geq \epsilon$. However for $n \neq m$, the set $\gamma_n \cap \gamma_m$ has at most one point, and so $\ell(\gamma) \geq \sum_{n \geq N} \ell(\gamma_n) = \infty$. Since the length $\ell(\gamma) = \infty$ for all arcs of ∂X joining a to b, it follows by definition that $dT(a,b) = \infty$.

We now show that $g^\infty \in \cap g^n(A)$. By compactness

$$g^{n_i}(c) \to a \in \cap g^n(A)$$

for some increasing sequence of integers n_i. Notice that $g^{n_i}(x) \to g^\infty$ for any $x \in X$ and $g^{n_i}(g^{-\infty}) \to g^{-\infty}$. Since $\angle(c, g^{-\infty}) \neq 0$, by Lemma 18 $\angle(a, g^\infty) < \pi$. Thus by the previous paragraph, $g^\infty \in \cap g^n(A)$ and similarly $g^{-\infty} \in \cap g^n(B)$.

Next we see that $\cap g^n(A) = \{g^\infty\}$. The argument will be similar to the previous one. Let $a \in \cap g^n(A)$. By compactness $g^{-n_i}(a) \to \hat{a}$ for some increasing sequence of integers n_i. As $\cap g^n(A)$ is closed and g-invariant, it follows that $\hat{a} \in \cap g^n(A)$. For any $x \in X$, $g^{-n_i}(x) \to g^{-\infty}$ and $g^{-n_i}(g^\infty) \to g^\infty$. Since $\hat{a}, g^\infty \in \cap g^n(A)$, and $g^{-\infty} \notin \cap g^n(A)$, we have $\angle(g^{-\infty}, \hat{a}) = \pi = \angle(g^{-\infty}, g^\infty)$. By Lemma 18, $\angle(g^\infty, a) = 0$, in other words $g^\infty = a$.

To show that the action of $\langle g \rangle$ on ∂X is a convergence action, let U be a neighborhood of g^∞ in ∂X, and C be a compact set of $\partial X - \{g^{-\infty}\}$. For some $n > 0$, $C \cap g^{-n}(B) = \emptyset$. Thus $C \subset g^{-n}(A)$. For some $m > 0$, $g^m(A) \subset U$. Thus $g^{n+m}(C) \subset g^m(A) \subset U$, and the action of $\langle g \rangle$ on ∂X is a convergence action. q.e.d.

Word Hyperbolic Case. Theorem 20 is trival in the word hyperbolic case (a subgroup of a convergence group is a convergence group). The proof of the following Corollary is the same in the word hyperbolic case.

Corollary. *Let C be a countable G-invariant set of cut points of ∂X. Let T be the real tree (without terminal points) constructed in the previous section from ∂X and C. The action of G on T is non-nesting.*

Proof. Suppose not, then there exist $g \in G$ and an arc $[c,a] \subset T$ with $g([c,a]) \subsetneq [c,a]$. Replacing g with g^2 if need be, we may assume that g

preserves the orientation of $[c, a]$. By applying the Brower fixed point theorem, we may assume that $g(a) = a$. Thus $g(c) \in (c, a)$. From the construction of $\overline{T}$, we may assume $c \in C$. By definition of C, $\partial X = A \cup B$ where A and B are non-degenerate continuum and $A \cap B = \{c\}$. The points of ∂X corresponding to a will lie in one of these continuum, A. It follows from the construction of $\overline{T}$ that $g(A) \subsetneq A$. By Theorem 20, the action of $\langle g \rangle$ on ∂X is a convergence action. Therefore Lemma 7 shows that the action of $\langle g \rangle$ on T is non-nesting (T has no terminal points) contradicting the choice of g. q.e.d.

Word Hyperbolic Case. In the word hyperbolic case, the fixed point set in ∂X of an infinite order element (which can only be hyperbolic) is the two points $g^{\pm\infty}$. Thus the following result is trivial in that case.

Lemma 21. *For any hyperbolic g either $Fix(g)$ is not separated in ∂X by any point of C, or $Fix(g) = \{g^{\pm\infty}\}$ and g has no fixed points in it action on T.*

Proof. Let $Min(g) = \mathbb{R} \times Y$. First consider the case where Y is unbounded. By [20] the centralizer Z_g acts geometrically on the convex subspace $Min(g)$. Furthermore $\langle g \rangle < Z_g$. It follows from Lemma 14 that $Z_g/\langle g \rangle$ acts geometrically on $Min(g)/\langle g \rangle$. Using [10, II 6.8 (4)] We see that $Min(g)/\langle g \rangle = S^1 \times Y$. Since Z_g preserves the product structure, $Z_g/\langle g \rangle$ acts geometrically on the CAT(0) space Y. By Theorem 11, Y contains the axis of some hyperbolic element and so $|\Lambda Y| > 1$. The only suspension which has a cut point is the suspension of a single point, so in this case Fix(g) is not separated in ∂X by any point of C.

Now consider the case where Y is bounded and so Fix(g) $= \{g^{\pm\infty}\}$. If Fix(g) is separated by a cut point $c \in C$, then since g doesn't fix c, we may apply Theorem 20 to show that $\langle g \rangle$ acts as a convergence group on ∂X. The result now follows from Lemma 7. q.e.d.

Word Hyperbolic Case. In the word hyperbolic case, if h and g are hyperbolic and Fix(h) $\cap$ Fix(g) $\neq \emptyset$, then Fix(h) $=$ Fix(g) $= \{g^{\pm\infty}\}$ [25, 2G].

Theorem 22. *If g, h are hyperbolic elements of G with $Fix(g) \cap Fix(h) \neq \emptyset$, then either $|Fix(g) \cup Fix(h)| = 2$ or the subspace $Fix(g) \cup Fix(h)$ has no cut point.*

Proof. The first case is where both $\mathrm{Fix}(g)$ and $\mathrm{Fix}(h)$ are two point sets. That is, $\mathrm{Fix}(g)$ is just the endpoints of an axis L of g, and similarly $\mathrm{Fix}(h)$ is the endpoints of an axis $\hat{L}$ of h. Since L and $\hat{L}$ share an endpoint, we may apply the Zipper Lemma to $x \in L$ and $y \in \hat{L}$ to see that $g^n = h^m$ for some $m, n \neq 0$. Thus $\mathrm{Fix}(g) = \mathrm{Fix}(h)$.

For the other case, by Lemma 21, we may assume that $\mathrm{Fix}(g)$ has no cut point. By the Corollary to Lemma 10, $\mathrm{Fix}(g) = \Lambda Z_g$ and $\mathrm{Fix}(h) = \Lambda Z_h$. Since $Min(g)$ and $Min(h)$ are convex, it follows from [20] that Z_g and Z_h are convex subgroups of G. Thus by Theorem 16, $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) = \Lambda(Z_g \cap Z_h)$. Since $Z_g \cap Z_h$ is an infinite convex subgroup, Theorem 11 implies that $|\mathrm{Fix}(g) \cap \mathrm{Fix}(h)| > 1$. Either $\mathrm{Fix}(h) \subset \mathrm{Fix}(g)$ in which case the result is clear, or $|\mathrm{Fix}(h)| > 2$ and so $\mathrm{Fix}(h)$ has no cut point. In this latter case, since $|\mathrm{Fix}(g) \cap \mathrm{Fix}(h)| > 1$, $\mathrm{Fix}(g) \cup \mathrm{Fix}(h)$ will have no cut point. q.e.d.

Word Hyperbolic Case. In the word hyperbolic case, the only way a point $c \in C$ can be fixed by a hyperbolic element is if $c = g^{\pm\infty}$. In fact if $p \in P$ and $\{g^{\pm\infty}\} \cap \overline{p} = \emptyset$, then using the convergence action of $\langle g \rangle$ on ∂X, we see that $g(p) \neq p$. Thus the following result is trivial in the word hyperbolic case.

Lemma 23. *If an interval I of T has points of C in its interior, then $Stab(I)$ is torsion.*

Proof. Suppose there is a hyperbolic $g \in \mathrm{Stab}(I)$, with axis L oriented so that $L(\infty) = g^\infty$. If $|I \cap C| > 2$, then there are $c_1, c_2, c_3 \in I \cap C$ with c_2 separating c_1 from c_3. However $c_1, c_2, c_3 \in \mathrm{Fix}(g)$, and by Lemma 21 $\mathrm{Fix}(g)$ is not separated by any cut points. Thus $|I \cap C| \leq 2$.

If $J \subset I$, then $\mathrm{Stab}(I) < \mathrm{Stab}(J)$. Using the construction of T, we can reduce to the case where $I = [a, b]$ and $[a, b] \cap P = \{a, c, b\}$ for some $c \in C$. Since g fixes a point of T, by Lemma 21, $\mathrm{Fix}(g)$ is not separated by any point of C. Thus there is a nonsingleton equivalence class $d \in P$ with $\mathrm{Fix}(g) \subset \overline{d}$.

We may assume that $d \neq b$, and will show that g doesn't stabilize b. Since $b, d \in P - C$, $(b, d) \cap C$ is nonempty and so there is $\hat{c} \in (b, d) \cap C$. By Lemma 1, $\hat{c} \in [b, c] \cup [c, d]$. However $(b, c) \cap P = \emptyset = (c, d) \cap P$. Thus $\hat{c} = c$ and c separates b from d. There are two cases based on how close c is to some point of b in the Tit's metric dT of TX.

CASE I. There is $\hat{b} \in b$ with $dT(c, \hat{b}) < \frac{\pi}{2}$. By Lemma 10, c is in the limit set of a half-flat containing an axis of g. It follows that $dT(c, \{g^{\pm\infty}\}) \leq \frac{\pi}{2}$. Thus we may assume $\theta = dT(\hat{b}, g^{-\infty}) = \angle(\hat{b}, g^{-\infty}) < \pi$. Since $\langle g \rangle$ acts cocompactly on L, we may apply Lemma 19 to get a sequence (g_n) of $\langle g \rangle$ with

- $g_n(c) \to c$ (since $c \in \mathrm{Fix}(g)$),
- $g_n(\hat{b}) \to b'$,
- $g_n(L) \to L$ (since L is an axis of g),
- L bounding a half-flat containing b' in its boundary.

The limit set of this half-flat is an arc γ in ∂X from g^{∞} to $g^{-\infty}$ which passes through b'. Since g stabilizes b, $b' \in \bar{b} - \{c\}$. This is impossible as the arc γ would have pass through c twice.

CASE II. For some $\hat{b} \in b$, $dT(c, \hat{b}) = \infty$. Since Td is a path metric and c separates d from b, it follows that $Td(\{g^{\pm\infty}\}, \hat{b}) = \infty$. We may assume that $c \neq g^{-\infty}$. By [10, II 9.21] there is a line R from $g^{-\infty}$ to $\hat{b}$. Let $\epsilon = d(R, L(0))$. Now consider the sequence of lines $g^n(R)$ for $n > 0$. Clearly $d(g^n(R), g^n(L(0))) = \epsilon$. Since $g^{\infty} = L(\infty)$ and $R(-\infty) = g^{-\infty}$, by convexity of the metric $d(g^n(R), L(0)) \leq \epsilon$. Thus some subsequence of $g^{n_i}(R)$ will converge to a line S. Since $g^{-\infty}$ is fixed by g, $S(-\infty) = g^{-\infty}$. However for any $m > n$, $d(g^m(R), g^n(L(0))) \leq \epsilon$. It follows that S is parallel to L, and so $g^{n_i}(\hat{b}) \to g^{\infty}$. The isometry g stabilizes $b \ni \hat{b}$, so g^{∞} is a limit point of the set b, which is absurd since c separates g^{∞} from b. q.e.d.

4. Cut points in geometric boundaries

Word Hyperbolic Case. The following result follows from [25, 2R] in the word hyperbolic case since the action on ∂X is a convergence action with limit set ∂X.

Theorem 24. *The action of G on the tree T has no global fixed point.*

Proof. Suppose that G fixes a point $p \in T$. By construction we may assume that $p \in P$, the set of equivalence classes of ∂X defined in Section 1.

If p were a singleton equivalence class of ∂X, then G would fix the point $p \in \partial X$. It follows from Theorem 11 that G has an element of infinite order. As all hyperbolic elements would fix p, using Theorem 22 and Theorem 20 it can be shown that there would be $\hat{p} \in P$, whose closure contained all fixed point sets of hyperbolic elements. Thus the non-singleton equivalence class $\hat{p}$ would also be fixed by G. We may then assume that p is a non-singleton equivalence class in P.

Choose distinct $d, e \in \overline{p}$ and let $\theta = \angle(d, e) > 0$. For any $a \in \partial X$, we can find a sequence $(g_n) \subset G$ and with $g_n(x) \to a$ for some (any) $x \in X$. Since G leaves p invariant, passing to a subsequence, we get $g_n(d) \to \tilde{b} \in \overline{p}$ and $g_n(e) \to \hat{b} \in \overline{p}$. Applying Lemma 18 we see that $dT(a, \overline{p}) \leq \pi - \frac{\theta}{2}$. Thus for every $a \in \partial X$, $dT(a, \overline{p}) < \pi$.

Let $c \in C$ with A and B nonsingleton subcontinuum such that $A \cup B = \partial X$ and $A \cap B = \{c\}$. By definition, the equivalence class p is not separated by any point of C, so with no loss of generality $p \subset A$. By [10, II 9.13], the Tit's boundary, TX, is a CAT(1) space (Recall that TX is the set ∂X with the usually finer metric Td). Since an arc in TX is an arc in ∂X, B is a convex subset of TX, and therefore a CAT(1) space. Since $\overline{p} \subset A$, the Tit's distance from any point of B to c is less than π. Thus we can find a maximal geodesic segment $[c, b]$ of the CAT(1) space B. By [10, II 1.4 (1)], for each $a \in B$, there is a unique segment $[c, a]$ which varies continuously with its endpoints. We can then geodesically contract B to c. Since $[c, b]$ was a maximal geodesic segment, it follows that for each $a \in B - \{b\}$, there is a unique geodesic segment $[c, a]$, and we can also contract $B - \{b\}$ geodesically to c.

Notice that $B \subset B(c, \pi)$ where the ball is in the Tit's metric Td. Since the Tit's metric is proper, B is a compact set of the Tit's boundary TX. By [10, II 9.7], the obvious map from $TX \to \partial X$ is continuous, however a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. It follows that the Tit's metric on B gives rise to the same topology as the subspace topology from ∂X. Thus B and $B - \{b\}$ are both contractable as subspaces of ∂X.

From the long exact sequence for a pair, we see that $H_i(B, B - \{b\}) = 0$ for all i and any coefficients. However $B - \{c\}$ is an open set containing b and so by excision, $H_i(\partial X, \partial X - \{b\}) = 0$ for all i. The action of G on X need not be free, but we can still use X to compute the rational cohomology groups $H^*(G, \mathbb{Q}G)$, and in fact the result [3, 1.10] holds with coefficients in $\mathbb{Q}$ even though the action of G on X is not free. Thus by [3, 1.10 (2)] $H^i(G, \mathbb{Q}G)$ is a finite dimensional vector space for all i. Now using [3, 1.10 (1)], the dual of $H^i(G, \mathbb{Q}G)$ is trivial for all i. It follows that $H^i(G, \mathbb{Q}G) = 0$ for all i. Hence $cd_{\mathbb{Q}}G = 0$, and so by [12, IV 3.5] G is finite, contradicting the one-endedness of both X and G. q.e.d.

Word Hyperbolic Case. The proof of the following is the same in the word hyperbolic case. Word hyperbolic groups have no infinite torsion subgroups (see [15]) and so there can be no cut points.

Main Theorem. *If G acts geometrically on the one-ended CAT(0) space X, and $c \in \partial X$ is a cut point, then there is an infinite torsion subgroup $H < G$ fixing c.*

Proof. Suppose that ∂X has a cut point c, and let $C = Gc$. Let T be the $\mathbb{R}$-tree constructed in Section 1 from C. By Theorem 24, G acts without global fixed points on T. By taking a minimal invariant subtree, we may assume that T is minimal (T contains no proper invariant subtree).

Apply [17] to the $\mathbb{R}$-tree T. We obtain an $\mathbb{R}$-tree S on which G acts by isometries, and an onto G-function $f : T \to S$ which is continuous on arcs of T. Furthermore the image of an arc of T will be an arc or a point of S. Reading [17] we see that there are two possibilities:

1. For any interval I of T, $Gc \cap I$ is discrete. In this case, f is a bijection, and the tree S is a simplicial tree. The vertices of S are the points of P (the set of equivalence classes defined in Section 1), and the edges of S are the adjacent pairs of P.

2. The set $Gc \subset I$ is not discrete for some interval I of T. In this case, the image $f(C)$ of $C = Gc$ is dense in every arc of S.

We first consider (1). Every adjacent pair of P consists of one vertex of C and one of $P - C$, and so S has a G-invariant bipartite structure. Thus G acts on T without edge inversion. The cut point c is a vertex of S, and we choose an edge e of S having c as one of its endpoints and $p \in P$ as the other. Since G has one end, all edge stabilizers must be infinite. By hypothesis, we may assume that there is a hyperbolic element $g \in \mathrm{Stab}(e)$. Since $c \in \mathrm{Fix}(g)$, it follows that there is a nonsingleton equivalence class $p' \in P$ adjacent to c with $\mathrm{Fix}(g) \in \overline{p}'$. By Lemma 23, $p = p'$. Choose another edge $\hat{e} \neq e$ containing c. Any hyperbolic element h which fixes c has $\mathrm{Fix}(h) \subset \overline{p}$ by Theorem 22. From Lemma 23 we know that $\mathrm{Stab}(\hat{e})$ is torsion, and therefore infinite torsion. This completes the proof for (1).

We are left with the case where S is an $\mathbb{R}$-tree every arc of which contains infinitely many images of C. Notice that if I is an arc of T, with $f(I)$ not a point, then the stabilizer $\mathrm{Stab}(f(I)) < \mathrm{Stab}(I)$. By Lemma 23 we see that all arc stabilizers of S are torsion. By hypothesis we may assume that all arc stabilizers of S are finite. CAT(0) groups have only finitely many conjugacy classes of finite subgroups [10]. Thus there is a bound on the size of a finite subgroup and the action of G on S is stable. This action is also fixed point free by Theorem 24.

We will now analyze the action of G on S using the following which follows from [16, 14.12.5; 14.12.2; 14.9.2; 14.12.6] and also from [4].

Rips Machine. Let G be a finitely presented group acting minimally, stably and nontrivially by isometries on an $\mathbb{R}$-tree S. If G doesn't split over an arc stabilizer of S, then one of the following is true:

1. There is a line $L \subset S$ acted on by a subgroup $H < G$, and $N \triangleleft H$, the kernal of the action of H on S, so that H/N is virtually $\mathbb{Z}^n$ for some $n > 1$.

2. There is a closed hyperbolic cone 2-orbifold F and a normal subgroup $N \triangleleft G$ with $\pi_1(F) \cong G/N$. Furthermore the action of G on S factors through $\pi_1(F)$.

3. There is a finite graph of groups decomposition, Γ, of G with $H < G$ a vertex group having the following properties.

   (a) There is F, a cone 2-orbifold with boundary, and a normal subgroup $N \triangleleft H$ with $H/N \cong \pi_1(F)$, so that the action of H on S factors through $\pi_1(F)$.

   (b) The edge groups of H in Γ correspond to the peripheral subgroups of F.

   (c) All edge groups act trivially.

First consider (1). Since the arc stabilizers of S are finite, H is virtually $\mathbb{Z}^n$ for some $n > 1$. From the flat torus theorem of [10] it follows that $\Lambda H \cong S^{n-1}$, which has no cut points. By Theorem 20 every hyperbolic element h which acts by translation on a line of T (or S) has the property that the points $h^{\pm\infty}$ are seperated by a cut point in C. None of the hyperbolic elements of H can act by translation on the line L, so the finite index $\mathbb{Z}^n$-subgroup of H fixes L, contradicting the fact that arc stabilizers of S are finite.

In the case of (2), since the arc stabilizers are finite, G act geometrically on $\mathbb{H}^2$ and so G is word hyperbolic and $\partial X = \partial G \cong S^1$.

Thus for both (1) and (2) ∂X has no cut points.

Now we need only consider (3). The normal subgroup $N \triangleleft H$ will be finite since the edge stabilizers of S are finite. Since H acts non-trivially on S, there will be an $h \in H$ which acts by translation on a line L of S. Let γ be a loop in F representing the image of h in $\pi_1(F)$. Since the edge groups of Γ act trivially on S, γ is not peripheral in F.

Consider the covering F_γ of F corresponding to the subgroup $\langle[\gamma]\rangle < \pi_1(F)$. Since F has no reflections and γ is not peripheral in F, it follows that F_γ has more than one end. Using the graph of groups decomposition Γ for G, one can show that $e(G, \langle h\rangle) > 1$. Thus the quotient X_h of X by the action of $\langle h\rangle$ has more than one end.

By Theorem 20, $\langle h\rangle$ acts as a convergence group on both ∂X and $\overline{X}$ with limit set $\{g^{\pm\infty}\}$. Hence $\langle h\rangle$ acts properly discontinuously on $\hat{X} = \overline{X} - \{h^{\pm\infty}\}$.

Geodesic rays of X_h lift to geodesic rays representing points of $\partial X - \{g^{\pm\infty}\}$. Using this we see that the quotient, $\hat{X}_h$, of $\hat{X}$ by the action of $\langle h\rangle$ is compact. Since X_h has more than one end, it follows that the quotient of $\partial X - \{h^{\pm\infty}\}$ by the action of $\langle h\rangle$ has more than one component. Thus the two point set $\{h^{\pm\infty}\}$ separates ∂X. This is absurd. By Theorem 20, $h^{\pm\infty}$ are singleton equivalence classes which are terminal in the tree $\overline{T}$, and so $\{g^{\pm\infty}\}$ separates nothing. q.e.d.

Corollary. *If any one of the following conditions implies that ∂X has no cut point, then:*

1. *G is virtually torsion free,*
2. *X is a cube complex,*
3. *X is 2-dimensional.*

Proof. For (1) notice that any torsion free subgroup H of finite index in G also acts geometrically on X.

G. Niblo, L. Reeves, and M. Roller showed that finitely generated infinite torsion groups do not act properly discontinuously on CAT(0) cube complexes [18],[19], and this proves (2).

In the case of (3), assume we do have a cut point c. There is an infinite torsion subgroup H which fixes c. Applying Theorem 17, we obtain a group K, containing an infinite torsion subgroup, with K acting geometrically on the one dimensional CAT(0) space Y. One dimension CAT(0) spaces are also known as $\mathbb{R}$-trees, and no group acting geometrically on a $\mathbb{R}$-tree contains an infinite torsion subgroup. q.e.d.

We now give an example, due to Bestvina, of a homology boundary of $BS(1,2)$ with a cut point.

Example. Consider the group $G = BS(1,2) = \langle a, b | \, aba^{-1} = b^2 \rangle$. We construct a 2-complex for G as follows: In the upper half plane model, $\{(x,y) \in \mathbb{R}^2 | y > 0\}$, for $\mathbb{H}^2$, consider the D region bounded by $x = 0, x = 1, y = 1, y = 2$. In the hyperbolic metric, the vertical "sides" of D are geodesics, but the horizontal sides are not. Furthermore, the bottom "side" is twice as big as the top "side". We now label the sides of D. The left and right we label with a pointing upwards. The top is labeled by b pointing from left to right, and the bottom is labeled by b^2 pointing from left to right. We can now tile $\mathbb{H}^2$ with copies of D so that the edges match up. The quotient, X_G of D by the edge identifications is a $K(\pi, 1)$ for G, and we take its universal cover X for our geometry. Each translate of D in X is contained in an isometrically embedded copy of $\mathbb{H}^2$. These "sheets" bifurcate as you go upwards in X. Any two of these sheets coincide except on a vertical horo-disk in each.

Thus the visual boundary is a non-hausdorff compactuum obtained from a circle by replacing one point with a cantor set. That is, a neighborhood of a point on the circle is the same as before replacement, and a neighborhood of a point p in the cantor set is a neighborhood in the cantor set (before replacement) together with a neighborhood of the replaced point of the circle (minus the replaced point of course). We now alter this boundary as follows: We collapse the circle minus a point to a single point c and blow up each point in the cantor set to an open arc which has c as both of its endpoints. The resulting geometric boundary Z is the Cantor Hawaiian earring. The usual construction for the Cantor Hawaiian earring is to start with the standard cantor set in the plane, and take the union of all circles B with center on the x-axis going through the origin and one other point on the Cantor set.

There is only one cut point in Z, namely c. The tree $\overline{T}$ we obtain is simplicial having one vertex of uncountable valence, and an uncountable number of valence 1 vertices. The action of G on $\overline{T}$ interchanges valence 1 vertices, but leaves c fixed.

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