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POLAR ACTIONS ON RANK-ONE SYMMETRIC SPACES

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Abstract

We give a complete classification up to orbit equivalence of polar actions of compact connected Lie groups on compact rank-one symmetric spaces. For polar actions on the complex projective spaces, we prove that they are orbit equivalent to the actions induced by isotropy representations of Hermitian symmetric spaces, while in the case of polar actions on the quaternionic projective spaces, we prove that they are orbit equivalent to the actions induced by products of k quaternion-Kähler symmetric spaces, where at least $k-1$ have rank one. For the Cayley projective plane $\mathbb{P}_2(\mathbb{O})$, we prove that the cohomogeneity of any polar action is either one or two, and we come up with a complete list of all compact connected subgroups of F_4 (up to conjugacy) acting polarly on $\mathbb{P}_2(\mathbb{O})$. The classification of polar actions on spheres and real projective spaces follows immediately from Dadok's paper [10].

0. Introduction

An isometric action of a compact Lie group G on a Riemannian manifold M is called *polar* if there exists a properly embedded, connected submanifold which meets every G-orbit orthogonally; any such submanifold is called a *section* and if the induced metric on a section is flat, then the action is called *hyperpolar*.

One should think of a section as a set of canonical forms for the polar action; see [24]. An example that demonstrates this viewpoint is the conjugation of symmetric matrices by elements of the orthogonal group. This action is polar with the diagonal matrices as a section.

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Another such example of a polar action is given by the action of a compact and connected Lie group G on itself by conjugation, when we endow G with a biinvariant metric. Here the section is a maximal torus and the action is hyperpolar. This example is a particular case of a wider class of hyperpolar actions; namely, if (G, H) is a symmetric pair, then the action of H on G/H is hyperpolar of cohomogeneity given by the rank k of the symmetric pair (G, H) and any k-flat is a section for this action (see [9], [6]).

Irreducible linear polar representations were classified by Dadok ([10]). It follows from his classification that any linear polar representation is orbit equivalent to the isotropy representation of a symmetric space. It is straightforward to see that polar actions on spheres are precisely the restrictions of linear polar actions, and similarly that polar actions on real projective spaces are orbit equivalent to those induced from polar actions on spheres.

Recently Kollross ([20]) classified hyperpolar actions on irreducible compact symmetric spaces up to orbit equivalence. Since in particular a cohomogeneity one action on such spaces is polar with a closed geodesic as a section, the work of Kollross generalizes [18], [28], [11] and [21].

Hyperpolar actions are particularly interesting since they are variationally complete in the sense of [5]; see [9]. It is proved in [6] that the orbits of variationally complete actions are *taut* in the sense that the energy functionals on the path spaces $P(M, \{p\} \times Gq) = \{\sigma \in H^1([0,1], M);\ \sigma(0) = p, \sigma(1) \in Gq\}$ are perfect Morse-Bott functions for all p and q in M. Polar actions are in general not variationally complete as one sees for example by considering any polar action of cohomogeneity at least two on a sphere (see [29, p.198]). Still Ewert could prove that regular orbits of polar actions and, more generally, submanifolds with parallel focal structure in simply connected compact symmetric spaces are taut; see [14]. We mention here the interesting, still unsolved problem, whether there exist polar actions that are not hyperpolar on irreducible compact symmetric spaces of rank greater than one.

The aim of this paper is to complete the classification of polar actions on compact rank-one symmetric spaces, the remaining cases being the complex and quaternionic projective spaces as well as the Cayley projective plane.

Our main result is the following

Theorem. *Let M be a compact rank-one symmetric space which is* *neither a sphere nor a real projective space. Given a polar action of a compact connected Lie group G on M, then a section is isometric to a real projective space $\mathbb{P}_k(\mathbb{R})$ for $k \geq 1$. Furthermore:*

1. *If M is a complex projective space, then the actions induced by isotropy representations of Hermitian symmetric spaces on M are polar. Conversely, a polar action on M is orbit equivalent to an action induced by the isotropy representation of a Hermitian symmetric space.*

2. *If M is a quaternionic projective space, then the actions induced on M by isotropy representations of products of k quaternion-Kähler symmetric spaces, where at least $k-1$ have rank one, are polar. Conversely, a polar action on M is orbit equivalent to an action induced by the isotropy representation of a product of k quaternion-Kähler symmetric spaces, where at least $k-1$ have rank one.*

3. *If M is the Cayley plane $\mathbb{P}_2(\mathbb{O}) = \mathrm{F}_4/\mathrm{Spin}(9)$, then any polar action has cohomogeneity k equal to one or two; moreover the closed connected subgroups of the full isometry group F_4 whose actions on M are polar are given (up to conjugation) in the following table.*

$k=1$	$\mathrm{Sp}(1)\cdot\mathrm{Sp}(3)$	$T^1\cdot\mathrm{Sp}(3)$	$\mathrm{Sp}(3)$	$\mathrm{Spin}(9)$
$k=2$	$\mathrm{Spin}(8)$	$\mathrm{T}^1\cdot\mathrm{Spin}(7)$	$\mathrm{SU}(2)\cdot\mathrm{SU}(4)$	$\mathrm{SU}(3)\cdot\mathrm{SU}(3)$

In Section 1, we give all the preliminaries, and we prove some basic facts which are repeatedly used in the paper.

In Section 2, we show how to construct examples of polar actions on complex and quaternionic projective spaces, using isotropy representations of Hermitian and quaternion-Kähler symmetric spaces. As for the Cayley plane, we prove that the cohomogeneity two actions on $\mathbb{P}_2(\mathbb{O})$ in the above theorem are polar with a projective plane as a section. We do not have to prove this for the cohomogeneity one actions since they are always polar.

In the remaining Sections 3, 4 and 5, we give the proof of our classification in the cases of complex projective spaces, quaternionic projective spaces and the Cayley plane respectively.

1. Preliminaries

1A. Polar actions

We will consider an isometric action of a compact Lie group G on a complete Riemannian manifold M. It is well known (see e.g. [7, pp.180-1]), that the orbits of the action of a compact Lie group on a differentiable manifold belong to one of the following three types: *regular* (or principal), *exceptional* and *singular*. Given a point $p \in M$, we say that the orbit Gp is regular if and only if the isotropy subgroup G_p acts trivially on the normal space $N_p(Gp)$ to the orbit; this action is called the *slice representation* of G_p at p. A point $p \in M$ is called regular if its orbit Gp is regular; the set of regular points is a dense subset of M and the regular orbits have maximal dimension. A point $q \in M$ and its orbit Gq are called exceptional if Gq has maximal dimension, but the slice representation at q is not trivial; a point $y \in M$ and its orbit Gy are called singular if they are neither regular nor exceptional.

An isometric action of a compact Lie group G on a complete Riemannian manifold M is said to be *polar* if there is a properly embedded connected submanifold Σ in M that meets every orbit of G and every intersection of an orbit of G with Σ is perpendicular. Such a submanifold Σ is called a *section*. Points on regular and exceptional orbits of polar actions lie in one and only one section. Furthermore, if the action is polar, then an orbit through p is regular if and only if the isotropy group G_p fixes the section through p pointwisely. It is easy to prove that a section must be totally geodesic; see [24], [25]. For the proof of the following lemma; see also [24], [25].

Lemma 1A.1. *Let M be a complete Riemannian manifold together with a polar action of a compact Lie group G. Let p be some point in M and $N_p(Gp)$ the normal space at p of the orbit of G through p. Then the action of the isotropy group G_p on $N_p(Gp)$ is polar with $T_p\Sigma$ as a section if Σ is a section of the G-action containing p. In particular, the isotropy subgroup G_p acts transitively on the set of all sections through p.*

A compact Lie group G acts isometrically and polarly on a Riemannian manifold if and only if the identity component G^o does (see [19, p.167]); so, from now on, we will suppose for simplicity, that the group G is connected.

The following Lemmas 1A.2 to 1A.4 will deal with the set of non-regular points, establishing important facts which will be repeatedly used in the sequel.

Lemma 1A.2. *Let M be a compact Riemannian manifold together with a polar action of a compact, connected Lie group G. If no covering of M is diffeomorphic to a product manifold, then the action has singular orbits.*

Proof. Let Σ be a section of the polar action. Notice that Σ must be compact since it is properly embedded in a compact manifold. Assume that the G-action has no singular orbits. Let p be a point in Σ such that the orbit Gp through p is regular. We now define a map $\phi : Gp \times \Sigma \to M$ by setting $\phi(g \cdot p, a) = g \cdot a$. Notice that $g \cdot p = h \cdot p$ implies that $g \cdot a = h \cdot a$, since p is regular. Hence the map is well defined. Since there are no singular orbits, it is easy to see that the map ϕ has maximal rank everywhere. This implies that ϕ is a covering map since $Gp \times \Sigma$ is compact. This is a contradiction because no covering of M is by assumption diffeomorphic to a product. This finishes the proof. q.e.d.

Lemma 1A.3. *Let M be a simply connected Riemannian symmetric space, which is acted on polarly by a compact connected Lie group G. Then the G-action has no exceptional orbits.*

Proof. Suppose $Q = Gq$ is an exceptional orbit and let Σ be a section through q. We can find a sufficiently small open ball B_r of radius $r < f(Q)$ around q such that for all $x \in B$, we have $G_x \subseteq G_q$ (see e.g. [7]), where $f(Q)$ denotes the focal radius of Q. Moreover, by the density of regular orbits, we may find a regular point $p \in B_r \cap \Sigma$ (see [24], [25]).

We write p as $p = \exp_q(v)$ for the unique $v \in N_q(Q)$ with $|v| < r$ and consider the geodesic $\gamma(t) = \exp_q(tv)$, $t \in [0,1]$. Notice that γ is perpendicular to Gp. We next prove that Gp does not have focal points along γ. To see this let $\gamma_s(t)$ be a variation of γ through geodesics normal to Gp, where $(s,t) \in I = (-\epsilon, \epsilon) \times [0,1]$ for some $\epsilon > 0$ and $\gamma_0(t) = \gamma(t)$ for all $t \in [0,1]$. Assume that the corresponding Jacobi field J does not vanish identically and has a zero in $t_0 \in (0,1)$. It is not too difficult to see that $r < f(Q)$ implies that $\gamma(t_0)$ and $\gamma(1)$ are not conjugate along γ (or one can simply assume that r was chosen so small that $\exp_q|_{\{v \in T_qM; ||v|| < r\}}$ is a diffeomorphism, since then, by the homogeneity of M, also $\exp_p|_{\{v \in T_pM; ||v|| < r\}}$ is a diffeomorphism). Thus $J(1) \neq 0$. The curve $s \to \gamma_s(t_0)$ lies in a tubular neighborhood of Q with radius smaller than the focal radius $f(Q)$ of Q. Therefore there is a unique variation $\eta_s(t)$, $(s,t) \in I$ of γ through geodesics normal to Q satisfying $\eta_s(t_0) = \gamma_s(t_0)$ for all s. Hence the Jacobi field I corresponding to the variation $\eta_s(t)$ has a zero in t_0. If we can show that $I(0) \neq 0$, then the Jacobi field I does not vanish identically, and $\gamma(t_0)$ is a focal point of Q contradicting the assumption that r is smaller than the focal radius $f(Q)$ of Q. Notice that for all $z \in Gp$, $\exp_z(N_z(Gp))$ is a section which meets Gq transversally. Thus there are small neighborhoods U of p in Gp and V of q in $Q = Gq$ and a diffeomorphism $\phi : U \to V$ defined by setting $\phi(z)$ for $z \in U$ equal to the point in the intersection $\exp_z(N_z(Gp)) \cap Gq$ closest to q. The geodesics γ_s and η_s both lie in $\exp_{\gamma_s(1)}(N_{\gamma_s(1)}(Gp))$ since $\gamma_s(t_0)$ lies in an orbit of maximal dimension. Therefore $\phi(\gamma_s(1)) = \eta_s(0)$ for sufficiently small s, and it follows that $I(0) = d\phi_p(J(1)) \neq 0$. We have hence proved that Gp does not have focal points along γ.

Since γ is perpendicular to Gp, γ is a critical point of the energy functional on $P(M, \{q\} \times Gp) = \{\sigma \in H^1([0,1], M);\ \sigma(0) = q, \sigma(1) \in Gp\}$. The index of the critical point γ is 0 by the Morse index theorem since Gp does not have any focal points along γ.

A regular orbit of a polar action has a parallel focal structure. Hence it follows from [14, Lemma 2.10], that there exists exactly one geodesic in Σ connecting q to Gp with index 0 as a critical point of the energy functional on the space $P(M, \{q\} \times Gp)$. On the other hand, since q is exceptional, G_p is a proper subgroup of G_q; hence we can find $g \in G_q \setminus G_p$, and $g \circ \gamma$ gives another geodesic in Σ of index 0, a contradiction. q.e.d.

We now recall the definition of the generalized Weyl group (see [24], [25]), which will be a fundamental tool throughout the following. If G is a connected compact Lie group acting polarly on a Riemannian manifold M, we fix a section Σ and consider the subgroups

$$N_G(\Sigma) = \{g \in G;\ g(\Sigma) = \Sigma\}$$

and

$$Z_G(\Sigma) = \{g \in G;\ g|_\Sigma = \mathrm{id}_\Sigma\}$$

of G. The quotient group $W_\Sigma := N_G(\Sigma)/Z_G(\Sigma)$, which is finite (see [24], [25]), is called the *generalized Weyl group*. It is not difficult to see that, if we start with another section Σ', we obtain a Weyl group $W_{\Sigma'}$ which is isomorphic to W_Σ (the isomorphism is actually induced by an inner automorphism of G), so that we will often simply refer to the Weyl group by W, dropping the subscript Σ, when it does not cause any confusion.

If Σ is a section and p is a point in Σ, then the intersection $Gp \cap \Sigma$ coincides with the W-orbit Wp (see [24], [25]). In [9], it is proved, among other things, that, if the manifold M is simply connected and if the action is hyperpolar, that is if the sections are flat in the induced metric, then the Weyl group is not trivial and acts on Σ as a finite reflection group.

If $p \in M$ is any point and Σ is any section through p, then we know from Lemma 1A.1, that $\tilde{\Sigma} = T_p\Sigma$ is a section for the slice representation of the isotropy subgroup G_p. Moreover, we can consider the *localized Weyl group* $W_{p,\Sigma} := (N_G(\Sigma) \cap G_p)/Z_G(\Sigma)$, which is in a natural way a subgroup of W. Since

$$N_G(\Sigma) \cap G_p = \{g \in G_p;\ dg_p(\tilde{\Sigma}) = \tilde{\Sigma}\},$$

we see immediately that $W_{p,\Sigma}$ coincides with the Weyl group for the G_p-action on the normal space $N_p(Gp)$.

The following Lemma was proved by [9] under the assumption that the sections are flat and M is a simply connected Riemannian manifold. A more general statement was proved by Ewert in [14, Section 2.3, pp. 28-30]. Instead of polar actions he considers submanifolds with parallel focal structure. The regular orbits of polar actions are examples of such submanifolds.

Lemma 1A.4. *Let M be a simply connected symmetric space on which a compact, connected Lie group G acts polarly. Let Σ be a section of the polar action and let $p \in \Sigma$ be such that the orbit through p is singular. Then there is a totally geodesic hypersurface H in Σ passing through p and consisting of singular points; moreover there exists a non-trivial element $g \in W_\Sigma$ which fixes H pointwisely.*

The set of singular points in Σ is a union of finitely many totally geodesic hypersurfaces $\{H_i\}_{i\in I}$ in Σ; if Σ is simply connected, then the Weyl group W_Σ is a Coxeter group, generated by reflections in the hypersurfaces $\{H_i\}_{i\in I}$.

Proof. We consider the slice representation of G_p on $N_p(Gp)$ which is polar with $\tilde{\Sigma} := T_p\Sigma$ as a section by Lemma 1A.1. We consider the localized Weyl group $W_{p,\Sigma}$. Since $\tilde{\Sigma}$ is flat, it follows from [9, Theorem III], that $W_{p,\Sigma}$ is nontrivial and generated by reflections in hyperplanes of $N_p(Gp)$. Let $g \in N_{G_p}(\tilde{\Sigma})$ be such that the action on $\tilde{\Sigma}$ is a reflection in the hyperplane $\tilde{H}$ in $\tilde{\Sigma}$. It is clear that g leaves Σ invariant, but not fixed. Let $\epsilon > 0$ be smaller than the injectivity radius of M at p. Then the fixed point set of g in $\Sigma \cap B_r(p)$ is equal to $\exp_p(\tilde{H}) \cap B_r(p)$. The connected component containing p of the fixed point set of g acting on Σ is therefore a hypersurface containing p. It is clear that all points in this hypersurface are non-regular, hence singular by Lemma 1A.3. So, the set of singular points in Σ is a union of totally geodesic hypersurfaces $\{H_i\}_{i\in I}$; since every localized Weyl group is a subgroup of the finite group W_Σ, we see that the index set I is finite. The last claim of the Lemma follows from [14], Theorem 2.23 and subsequent remark. q.e.d.

Remark. In the statement of Lemma 1A.4, it is in general not true that the hypersurface H coincides with the fixed point set of the element $g \in W_\Sigma$; indeed if Σ is isometric to a real projective plane, any non-trivial isometry of Σ which fixes a hypersurface, has disconnected fixed point set.

The following theorem of Dadok [10] will be fundamental in the proofs of our main theorems.

Theorem 1A.5 (Dadok). *Let G be a closed, connected subgroup of the special orthogonal group $SO(V)$ of a Euclidean space V, and assume that its action on V is polar. Then the following hold:*

(a) *If V splits as the sum of G-invariant subspaces $V_1 \oplus V_2$, then the G-action on each V_i is polar for $i = 1, 2$ and the cohomogeneity of the G-action on V is the sum of the cohomogeneities of the G-actions on V_i, $i = 1, 2$.*

(b) *If $\hat{G}$ denotes the maximal subgroup of $SO(V)$ having the same orbits as G, then the action of $\hat{G}$ on V is conjugate to the isotropy representation of a symmetric space.*

Remark. In Dadok's paper it is only claimed that there is a compact group $\hat{G}$ with a symmetric space representation that has the same orbits as G, but not that G is a subgroup in $\hat{G}$. The more precise statement follows from Theorems 9 and 10 in [10] and soon became a folklore. In the paper [12], this fact is explicitly worked out if the cohomogeneity of G is at least three. For cohomogeneity two actions the maximality of $\hat{G}$ is claimed in [18]; see Remark on p.16. One should notice that the isotropy representation of the symmetric space $\mathrm{G}_2/\mathrm{SO}(4)$ is missing in Theorem 5 on p.16 in [18]. The reader will notice that it is of fundamental importance in our proofs in sections three and four that G is a subgroup of $\hat{G}$.

We conclude by giving the definition of *orbit equivalence*, which will be used in our main statements.

Definition. Let G_1 and G_2 be two Lie groups acting isometrically on Riemannian manifolds M_1 and M_2 respectively. We say that these two actions are *orbit equivalent* if there exists an isometry of M_1 onto M_2 which maps G_1-orbits onto G_2-orbits.

Notice that the actions of G and $\hat{G}$ on V in Theorem 1A.5 are orbit equivalent.

1B. Compact rank-one symmetric spaces

A compact symmetric space has rank one if and only if all its geodesics are closed. We will always assume that the Riemannian metrics on these spaces are normalized such that their closed geodesics are of length 2π. These spaces are round spheres S^n, $n \geq 1$; the real projective spaces $\mathbb{P}_n(\mathbb{R})$, $n \geq 2$; the complex projective spaces $\mathbb{P}_n(\mathbb{C})$, $n \geq 2$; the quaternionic projective spaces $\mathbb{P}_n(\mathbb{H})$, $n \geq 2$; and the Cayley (or octonion) plane $\mathbb{P}_2(\mathbb{O})$. The normalization of the Riemannian metric implies that the maximum of the sectional curvature is equal to one except in the case of $\mathbb{P}_n(\mathbb{R})$ where it implies that the sectional curvature is equal to $1/4$.

Our main goal in this section is to prove the following Proposition.

Proposition 1B.1. *Let G be a compact Lie group acting polarly on a compact rank-one symmetric space M. Then a section Σ of the G-action has constant sectional curvature. In particular, if $M = \mathbb{P}_n(\mathbb{C})$, $n \geq 2$, then Σ is isometric to $\mathbb{P}_k(\mathbb{R})$ with $2 \leq k \leq n$ or $\Sigma = S^k$ with $1 \leq k \leq 2$. If $M = \mathbb{P}_n(\mathbb{H})$, $n \geq 2$, then Σ is isometric to $\mathbb{P}_k(\mathbb{R})$ with $2 \leq k \leq n$ or S^k with $1 \leq k \leq 4$. If $M = \mathbb{P}_2(\mathbb{O})$, then Σ is isometric to $\mathbb{P}_2(\mathbb{R})$ or S^k with $1 \leq k \leq 8$.*

Remark. We will improve Proposition 1B.1 in sections three to five and show that the section Σ cannot be a sphere S^k with $k \geq 2$.

Before we can start to prove Proposition 1B.1 we need to review the classification of totally geodesic submanifolds in the compact rank-one symmetric spaces; see [30].

Proposition 1B.2. *Let M be a connected compact symmetric space of rank-one, and let N be a connected totally geodesic submanifold of M. Then N is also a rank-one symmetric space and the following list shows to which standard space it can be isometric. All possibilities in the list occur.*

1. *$M = S^n$ and $N = S^k$ for $1 \le k \le n$; or*

2. *$M = \mathbb{P}_n(\mathbb{R})$, and either $N = S^1$ or $N = \mathbb{P}_k(\mathbb{R})$ for $2 \le k \le n$; or*

3. *$M = \mathbb{P}_n(\mathbb{C})$ and either $N = S^k$ for $1 \le k \le 2$ or $N = \mathbb{P}_k(\mathbb{R}), \mathbb{P}_k(\mathbb{C})$ for $2 \le k \le n$; or*

4. *$M = \mathbb{P}_n(\mathbb{H})$ and either $N = S^k$ for $1 \le k \le 4$ or $N = \mathbb{P}_k(\mathbb{R})$, $\mathbb{P}_k(\mathbb{C}), \mathbb{P}_k(\mathbb{H})$ for $2 \le k \le n$; or*

5. *$M = \mathbb{P}_2(\mathbb{O})$ and either $N = S^k$ for $1 \le k \le 8$ or $N = \mathbb{P}_2(\mathbb{R})$, $\mathbb{P}_2(\mathbb{C})$, $\mathbb{P}_2(\mathbb{H})$, $\mathbb{P}_2(\mathbb{O})$.*

If two connected totally geodesic submanifolds of M are homeomorphic, then there is an element in $I_o(M)$ that maps one into the other, where $I_o(M)$ denotes the identity component of the isometry group of M.

We are now in a position to prove Proposition 1B.1.

Proof of Proposition 1B.1. The compact rank-one symmetric spaces do not have product manifolds as coverings. It therefore follows from Lemmas 1A.2 that the action of G has a singular orbit Gp. A section Σ will meet Gp so that we can assume that $p \in \Sigma$. By Lemma 1A.4, there is an element $g \in G$ leaving Σ invariant and whose fixed point set in Σ has a component H that is a hypersurface in Σ. A section is totally geodesic. Hence Σ is a rank-one symmetric space. The components of fixed point sets of isometries are also totally geodesic. The only rank-one symmetric spaces that have totally geodesic hypersurfaces are the spheres and the real projective planes; see Proposition 1B.2. The rest of the proof now follows by going through the list in Proposition 1B.2. q.e.d.

The maximal totally geodesic spheres in $\mathbb{P}_n(\mathbb{C})$, $\mathbb{P}_n(\mathbb{H})$ and $\mathbb{P}_2(\mathbb{O})$ can be interpreted as the projective lines of the projective space structure of these spaces; see [8], Chapter IV, §3. There is a discussion of the projective plane structure of $\mathbb{P}_2(\mathbb{O})$ in [15] and chapter one in [26]; see also the remark at the end of this subsection. This implies that any two different points in these spaces lie in a unique maximal totally geodesic sphere. Hence two maximal geodesic spheres can at most meet in one point; a fact we will use in the proof of the next lemma and repeatedly later in the paper. We will frequently refer to maximal geodesic spheres in these spaces as *projective lines.*

Lemma 1B.3. *Let G be a compact Lie group acting on $M = \mathbb{P}_n(\mathbb{C})$, $\mathbb{P}_n(\mathbb{H})$ or $\mathbb{P}_2(\mathbb{O})$. Assume that the action is polar with sections isometric to a sphere S^k, $k \geq 2$. Let $p \in M$. Then $\exp_p(N_p(Gp))$ is a totally geodesic sphere which is the union over all sections of G passing through p.*

Proof. We can of course assume that p is not regular. We know from Lemma 1A.1, that G_p acts transitively on the sections of G containing p. Since $N_p(Gp)$ is a union of the sections of G_p, it follows that $\exp_p(N_p(Gp))$ is a union over all sections of G passing through p.

It is left to prove that $\exp_p(N_p(Gp))$ is a totally geodesic sphere. Let X and Y be two points in $N_p(Gp)$. We will prove that there is a sequence $\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_k$ of sections of G_p in $N_p(Gp)$ such that $X \in \tilde{\Sigma}_1$, $Y \in \tilde{\Sigma}_k$ and $\tilde{\Sigma}_1 \cap \tilde{\Sigma}_{i+1}$ has positive dimension for $i = 1, \ldots, k-1$. This can be used to finish the proof of the lemma as follows. The totally geodesic sphere $\exp(\tilde{\Sigma}_i)$ is contained in a unique maximal totally geodesic sphere S_i. The intersection $\exp(\tilde{\Sigma}_i) \cap \exp(\tilde{\Sigma}_{i+1})$ is a sphere of positive dimension lying in S_i and S_{i+1}. Two different maximal totally geodesic spheres can only meet in one point. It follows that $S_1 = \cdots = S_k$. Hence $\exp(N_p(Gp))$ lies in one totally geodesic sphere where it is clearly totally geodesic.

It is left to prove the existence of the sequence $\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_k$. This is an application of the K-cycles that Bott and Samelson ([6]) constructed for the orbits of variationally complete actions. We will use that G_p has the same orbits as an isotropy representation of a symmetric space; see Theorem 1A.5. It is proved in [6, Theorem II], that such representations are variationally complete. Let us fix a section $\tilde{\Sigma}_1$ containing X and let $\bar{X}$ be a regular point in $\tilde{\Sigma}_1$. There is a $g \in G_p$ such that $Y \in g(\tilde{\Sigma}_1)$. Let $\bar{Y}$ be a point in $G_p(\bar{X}) \cap g(\tilde{\Sigma}_1)$. Both $\bar{X}$ and $\bar{Y}$ lie in unique sections since they are regular. It is therefore enough to prove the claim for $\bar{X}$ and $\bar{Y}$ instead of X and Y. The fundamental cycle of the orbit $G_p(\bar{X})$ can be represented by a K-cycle. The geometric interpretation of the K-cycles for orbits of isotropy representations of symmetric spaces implies that there is a polygonal arc γ in $N_p(Gp)$ joining $\bar{X}$ and $\bar{Y}$ that does not pass through the origin and whose line segments lie in sections of G_p; see §4 and §5 of Chapter I in [6]. The edges of γ, if there are any, therefore lie in the intersections of sections. These intersections have positive dimensions since γ does not pass through the origin. This implies the existence of the sequence of sections $\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_k$. q.e.d.

Some comments about the Cayley projective plane are here in order. The full isometry group of $\mathbb{P}_2(\mathbb{O})$ is the exceptional compact simple Lie group F_4 (see e.g. [31, p.264]) and the isotropy group of a point in $\mathbb{P}_2(\mathbb{O})$ is conjugate to Spin(9) in F_4. We can therefore identify the Cayley projective plane $\mathbb{P}_2(\mathbb{O})$ with the coset space $F_4/\mathrm{Spin}(9)$. By Proposition 1B.2, any maximal, proper, totally geodesic submanifolds of $\mathbb{P}_2(\mathbb{O})$ is either isometric to the sphere S^8 or to the quaternionic projective plane $\mathbb{P}_2(\mathbb{H})$; and in both cases, they are orbits of maximal subgroups of maximal rank in F_4. More precisely, a totally geodesic sphere S^8 is the orbit of a group isomorphic to Spin(9), which fixes a point p in $\mathbb{P}_2(\mathbb{O})$, and the sphere S^8 coincides with the cut locus of this point p. As already mentioned above, a totally geodesic sphere will be called a projective line in $\mathbb{P}_2(\mathbb{O})$, since such projective lines satisfy the axioms for projective planes (see [8], [15] or [26]). A totally geodesic submanifold in $\mathbb{P}_2(\mathbb{O})$ that is isometric to the quaternionic projective plane $\mathbb{P}_2(\mathbb{H})$ is the orbit of a maximal subgroup of maximal rank G, locally isomorphic to $\mathrm{Sp}(1) \times \mathrm{Sp}(3)$ (see [30]), which acts on $\mathbb{P}_2(\mathbb{O})$ with cohomogeneity one (see [21], [20]).

1C. Factorizations of compact Lie groups

We will briefly recall the theory of factorizations of compact Lie groups, since we will use it in the proofs of our main theorems. We refer here to [22, §5 and §14], for a more detailed exposition.

Given a compact Lie group G and two connected Lie subgroups G', G'' of G, we shall say that the triple (G, G', G'') of groups is a *factorization* if $G = G'G''$. When G' (or G'') is a closed subgroup, this is equivalent to saying that G'' acts transitively on G/G' (or G' acts transitively on G/G''). A factorization (G, G', G'') will be called *irreducible*, if for any factorization (G, H', H''), with H', H'' connected normal subgroups of G', G'' respectively, we have $H' = G'$ and $H'' = G''$.

Moreover, given two compact Lie groups G, H and two factorizations (G, G', G'') and (H, H', H''), we say that they are *locally isomorphic* if there exists a Lie algebra isomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\phi(\mathfrak{g}') = \mathfrak{h}'$ and $\phi(\mathfrak{g}'') = \mathfrak{h}''$; we will say that they are *equivalent* if (G, G', G'') is locally isomorphic to (H, H', H'') or to (H, H'', H').

In a similar way we define a factorization for Lie algebras: given a Lie algebra $\mathfrak{g}$ and two Lie subalgebras $\mathfrak{g}', \mathfrak{g}''$ of $\mathfrak{g}$, we shall say that the triple $(\mathfrak{g}, \mathfrak{g}', \mathfrak{g}'')$ is a factorization if $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}''$.

We summarize the results we will need in the sequel in the following theorem due to Onishchik ([22, pp. 85-90, 226-228]).

Theorem 1C.1. *Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Then the following hold:*

1. *a triple* (G, G', G'') *is a factorization if and only if the triple* $(\mathfrak{g}, \mathfrak{g}', \mathfrak{g}'')$ *is a factorization, where* $\mathfrak{g}', \mathfrak{g}''$ *are the Lie algebras of* G', G'' *respectively;*

2. *if* G *is simple, then any non-trivial irreducible factorization* (G, G', G'') *is equivalent to one of the factorizations in the table below, where the connected component of the normalizer* $N_G(G'')^o$ *is also indicated (for* G' *we have* $N_G(G')^o = G'$*);*

3. *if* G *is simple, any non-trivial factorization is equivalent to a factorization* (G, G', G_1'')*, where* $G'' \subseteq G_1'' \subseteq N_G(G'')$ *and* (G, G', G'') *is one of the factorizations in the following table.*

G	G'	G''	$N_G(G'')^o$
$\mathrm{SU}(2n),\ n \geq 2$	$\mathrm{Sp}(n)$	$\mathrm{SU}(2n-1)$	$\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(2n-1))$
$\mathrm{SO}(7)$	G_2	$\mathrm{SO}(6)$	$\mathrm{SO}(6)$
		$\mathrm{SO}(5)$	$\mathrm{SO}(5) \times \mathrm{SO}(2)$
$\mathrm{SO}(2n)\ n \geq 4$	$\mathrm{SO}(2n-1)$	$\mathrm{SU}(n)$	$\mathrm{U}(n)$
$\mathrm{SO}(4n)\ n \geq 2$	$\mathrm{SO}(4n-1)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$
$\mathrm{SO}(16)$	$\mathrm{SO}(15)$	$\mathrm{Spin}(9)$	$\mathrm{Spin}(9)$
$\mathrm{SO}(8)$	$\mathrm{SO}(7)$	$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$

Remark. In the above table, we used the notation $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) = \frac{\mathrm{Sp}(n) \times \mathrm{Sp}(1)}{\mathbb{Z}_2}$. This notation will be repeatedly used in the sequel: If $K_1, \ldots, K_r$ are connected compact groups, then $K_1 \cdot \cdots \cdot K_r$ will denote $\frac{K_1 \times \cdots \times K_r}{\Gamma}$, where Γ is a finite central subgroup of $K_1 \times \cdots \times K_r$.

2. The examples of polar actions on $\mathbb{P}_n(\mathbb{C})$, $\mathbb{P}_n(\mathbb{H})$ and $\mathbb{P}_2(\mathbb{O})$

We will in this section give examples of polar actions on the spaces $\mathbb{P}_n(\mathbb{C})$, $\mathbb{P}_n(\mathbb{H})$ and $\mathbb{P}_2(\mathbb{O})$. It will then be proved in sections three to five that there are no further examples.

2A. The examples of polar actions on $\mathbb{P}_n(\mathbb{C})$ and on $\mathbb{P}_n(\mathbb{H})$

In this subsection, we will show how to construct examples of polar actions on the complex projective space $\mathbb{P}_n(\mathbb{C})$ and on the quaternionic projective space $\mathbb{P}_n(\mathbb{H})$; in sections three, and four, we will show that these examples exhaust the class of polar actions up to orbit equivalence.

We fix a Hermitian symmetric pair (U, K), where U is a compact semisimple Lie group and K is a compact connected Lie subgroup of U, which coincides with the centralizer in U of its non-discrete center. We split (U, K) as a product of irreducible Hermitian symmetric pairs $(U, K) = \Pi_{i=1}^k (U_i, K_i)$, where each U_i is a simple compact Lie group and $K_i \subset U_i$ is a compact subgroup with one dimensional center for $i = 1, \dots, k$.

We denote by Gothic letters the corresponding Lie algebras $\mathfrak{u}, \mathfrak{u}_i$ and $\mathfrak{k}, \mathfrak{k}_i$, and we consider the Cartan decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{u}_i = \mathfrak{k}_i + \mathfrak{p}_i$ for $i = 1, \dots, k$; we recall that the $\mathrm{Ad}(K_i)|_{\mathfrak{p}_i}$-action of K_i on $\mathfrak{p}_i$, which can be identified with the isotropy representation of K_i, is polar of cohomogeneity equal to the rank of the corresponding symmetric pair (see [24], [25]).

It is well known that each $\mathfrak{p}_i$ admits up to a sign a unique $\mathrm{Ad}(K_i)$-invariant complex structure J_i, which can be expressed as $\mathrm{ad}(z_i)|_{\mathfrak{p}_i}$ for some $z_i \in \mathfrak{k}_i$ for $i = 1, \dots k$. If $\mathfrak{a}_i$ denotes a maximal abelian subalgebra of $\mathfrak{p}_i$, then $\mathfrak{a}_i$ provides a linear section for the K_i-action on $\mathfrak{p}_i$, and $\mathfrak{a}_i$ is totally real with respect to the complex structure J_i: indeed, if $X, Y \in \mathfrak{a}_i$ and if $<,>$ denotes the restriction of the Cartan-Killing form of $\mathfrak{u}_i$ on $\mathfrak{p}_i$, then

$$< J_i X, Y > = < \mathrm{ad}(z_i) X, Y > = < z_i, [X, Y] > = 0,$$

since $\mathfrak{a}_i$ is abelian.

We can now prove the following:

Proposition 2A.1. *Let $(U, K) = \Pi_{i=1}^k (U_i, K_i)$ be a Hermitian symmetric pair. If we identify $\mathfrak{p}$ with $\mathbb{C}^n$ by means of the complex structure $J = J_1 + \cdots + J_k$, where $n = \dim_{\mathbb{C}} U/K$, the $\mathrm{Ad}(K)$ -action on $\mathfrak{p}$ descends to a K-action on $\mathbb{P}_{n-1}(\mathbb{C})$, which is polar of cohomogeneity $r-1$, where r denotes the rank of the symmetric pair (U, K).*

Moreover, a section for the K-action on $\mathbb{P}_{n-1}(\mathbb{C})$ is homeomorphic to a real projective space.

Proof. We consider $\mathfrak{a} = \mathfrak{a}_1 + \cdots + \mathfrak{a}_k$ and define $\Sigma = \mathbb{P}(\mathfrak{a})$. We claim that Σ is a section for the K-action on $\mathbb{P}_{n-1}(\mathbb{C})$. Indeed, it is obvious that Σ meets every K-orbit and we have to prove that the intersections are orthogonal. We consider the canonical projection $\pi : S^{2n-1} \to \mathbb{P}_{n-1}(\mathbb{C})$, where S^{2n-1} denotes the unit sphere in $\mathfrak{p}$, and we consider $\hat{\mathfrak{a}} = \mathfrak{a} \cap S^{2n-1}$, so that $\Sigma = \pi(\hat{\mathfrak{a}})$. We fix a point $x \in \hat{\mathfrak{a}}$ and note that

$$T_x S^{2n-1} = \mathbb{R} \cdot Jx + \mathcal{H}_x,$$

where $\mathcal{H}_x$ denotes the orthogonal complement of $\mathbb{R} \cdot Jx$ in $T_x S^{2n-1}$. Note $T_x \hat{\mathfrak{a}} \subset \mathcal{H}_x$. Now, since $T_x(Kx)$ contains $\mathbb{R} \cdot Jx$, we have that

$$T_x(Kx) = \mathbb{R} \cdot Jx + (T_x(Kx)) \cap \mathcal{H}_x,$$

and $d\pi_x$ maps $(T_x(Kx)) \cap \mathcal{H}_x$ isometrically onto $T_{\pi(x)} K\pi(x)$. So, $T_{\pi(x)}(\pi(\hat{\mathfrak{a}})) = d\pi_x(T_x \hat{\mathfrak{a}})$ is orthogonal to $T_{\pi(x)} K\pi(x)$. q.e.d.

Remark. We note that the K-action on $\mathbb{P}_{n-1}(\mathbb{C})$ is not effective, since a one-dimensional factor of the center of K acts trivially on $\mathbb{P}_{n-1}(\mathbb{C})$. Here one should notice that it is not true that the product of the centers of the K_i act trivially on the complex projective space if (U, K) is reducible. Indeed, only a diagonal subgroup T^1 of that product acts trivially.

In order to describe the examples of polar actions on the quaternionic projective space $\mathbb{P}_n(\mathbb{H})$, we fix a quaternion-Kähler symmetric pair (U, K). This means by [4, p.408], that U is a compact simple Lie group and the subgroup K is a compact subgroup that can be written as $K = H \cdot \mathrm{Sp}(1)$, where H and $\mathrm{Sp}(1)$ are normal subgroups of K. Moreover the restriction of the isotropy representation to $\mathrm{Sp}(1)$ is equivalent to the representation of the normal subgroup $\mathrm{Sp}(1)$ of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ on $\mathbb{H}^n \cong \mathbb{R}^{4n}$, and the isotropy representation of H commutes with that of $\mathrm{Sp}(1)$ (here $\mathbb{H}^n$ is considered to be a right $\mathbb{H}$-module and $\mathrm{Sp}(1)$ acts by right multiplication by a unit quaternion). It is known that for each compact simple Lie group U (except $\mathrm{SU}(2)$) there exists exactly one quaternion-Kähler symmetric pair (U, K). We consider the Cartan decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ and we may identify $\mathfrak{p} \cong \mathbb{H}^n \cong \mathbb{R}^{4n}$ by considering the natural quaternionic structure Q induced by the $\mathrm{ad}(\mathfrak{sp}(1))$-action on $\mathfrak{p}$. As in the Hermitian case, we can prove that a linear section for the $\mathrm{Ad}(K)$-action on $\mathfrak{p}$ is totally real with respect to the complex structures $\{I, J, K\} \in Q$.

Proposition 2A.2.

(a) *Let (U, K) be a quaternion-Kähler symmetric pair, with Cartan decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$. If we identify $\mathfrak{p}$ with $\mathbb{H}^n$, where $4n = \dim U/K$, then the* $\mathrm{Ad}(K)$*-action of K on $\mathfrak{p}$ descends to a K-action on $\mathbb{P}_{n-1}(\mathbb{H})$, which is polar of cohomogeneity $r-1$, where r*

denotes the rank of the symmetric pair (U,K)*. Moreover a section for the* K*-action on* $\mathbb{P}_{n-1}(\mathbb{H})$ *is homeomorphic to a real projective space.*

(b) *If* (U_i, K_i) *are quaternion-Kähler symmetric pairs with Cartan decompositions* $\mathfrak{u}_i = \mathfrak{k}_i + \mathfrak{p}_i$ *for* $i = 1, \dots, m$*, the group* $H = H_1 \times \cdots \times H_m \times \mathrm{Sp}(1)$ *acts on the sum* $\mathfrak{p} = \mathfrak{p}_1 + \cdots + \mathfrak{p}_m$*, where* $\mathfrak{p}$ *is endowed with the natural quaternionic structure induced by the ones in each* $\mathfrak{p}_i$*, and the* $\mathrm{Sp}(1)$*-factor acts by right multiplication on* $\mathfrak{p}$*. The* H*-action on* $\mathfrak{p}$ *is polar if and only if all the symmetric pairs* (U_i, K_i) *have rank one, except possibly one of arbitrary rank. In this case the action of* H *on* $\mathfrak{p} \cong \mathbb{H}^n$ *descends to a polar action on* $\mathbb{P}_{n-1}(\mathbb{H})$ *of cohomogeneity equal to* $r-1$*, where* r *denotes the rank of the symmetric pair* $(U,K) = \Pi_{i=1}^m (U_i, K_i)$*.*

Remark. In contrast with the complex case, the action of $K_1 \times \cdots \times K_k$ does not descend to an action on the corresponding quaternionic projective space.

Proof. The claim in (a) can be proved in exactly the same way as Proposition 2A.1, so we leave it to reader.

Now, suppose we have quaternion-Kähler symmetric pairs (U_i, K_i), $i = 1, \dots, m$, where all (U_i, K_i) have rank one for $i = 1, \dots, m-1$, and (U_m, K_m) has arbitrary rank. In this case there exist integers $\{n_i\}_{i=1,\dots,m-1}$ such that $\mathfrak{u}_i = \mathfrak{sp}(n_i)$ and $\mathfrak{k}_i = \mathfrak{sp}(n_i - 1) + \mathfrak{sp}(1)$ for $i = 1, \dots, m-1$. Since the groups $H_i \cong \mathrm{Sp}(n_i - 1)$ and $K_i = \mathrm{Sp}(n_i - 1)\cdot\mathrm{Sp}(1)$ have the same orbits in $\mathfrak{p}_i$ for $i = 1, \dots, m-1$, it is clear that the action of $H = H_1 \times \cdots \times H_m \times \mathrm{Sp}(1)$ on $\mathfrak{p} = \mathfrak{p}_1 + \cdots + \mathfrak{p}_m$ is polar of cohomogeneity equal to the sum of the ranks. In this case the action of H descends to a polar action on the corresponding quaternionic projective space and a section is homeomorphic to a real projective space.

Vice versa, we suppose that the group H acts polarly on $\mathfrak{p}$. Then, by Dadok's Theorem 1A.5, we know that for any $i \neq j \in \{1, \dots, m\}$, the action of $H_{ij} = H_i \times H_j \times \mathrm{Sp}(1)$ acts polarly on $\mathfrak{p}_i + \mathfrak{p}_j$. We first prove the following useful Lemma.

Lemma 2A.3. *Let* G *be a connected compact Lie group acting polarly on a linear space* V*. If* V *splits as the sum of two invariant subspaces* $V = V_1 \oplus V_2$ *and if* L_1, L_2 *denote two regular isotropy subgroups of the* G*-action on* V_1*,* V_2*, respectively, then* $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$*, where Gothic letters denote the corresponding Lie algebras.*

Proof. We know that the G-action on each V_i is polar, and we can fix two sections Σ_1, Σ_2 in V_1, V_2 respectively, $Z_G(\Sigma_i) = L_i$. Since $\Sigma_1 + \Sigma_2$ is a section for the G-action on V, we can find a G-regular point $v = (v_1, v_2) \in V$ such that $G_{v_i} = L_i$. Then $G_v = L_1 \cap L_2$. Moreover, since the cohomogeneity of the G-action on V is the sum of the two cohomogeneities on each V_i, $i = 1, 2$, we get that

$$\dim V - \dim G/G_v = \dim V_1 - \dim G/L_1 + \dim V_2 - \dim G/L_2$$

and from this it follows that $\dim \mathfrak{g} = \dim \mathfrak{l}_1 + \dim \mathfrak{l}_2 - \dim(\mathfrak{l}_1 \cap \mathfrak{l}_2)$, which implies our claim. q.e.d.

We now denote by L_α a regular isotropy subgroup of $H_\alpha \times \mathrm{Sp}(1)$ acting on $\mathfrak{p}_\alpha$ for $\alpha = i, j$, and denote the corresponding Lie algebras by Gothic letters. Then by Lemma 2A.3, $\mathfrak{h}_1 + \mathfrak{h}_2 + \mathfrak{sp}(1) = \mathfrak{l}_1 + \mathfrak{l}_2$. If we denote by π the projection of $\mathfrak{h}_1 + \mathfrak{h}_2 + \mathfrak{sp}(1)$ onto $\mathfrak{sp}(1)$, then $\pi(\mathfrak{l}_1) + \pi(\mathfrak{l}_2) = \mathfrak{sp}(1)$. Since the non-trivial subalgebras of $\mathfrak{sp}(1)$ are one dimensional, it follows that one of the two subalgebras, say $\pi(\mathfrak{l}_i)$, coincides with $\mathfrak{sp}(1)$. If we now check through the list of isotropy subgroups of quaternion-Kähler symmetric spaces (see [4, p.406]) and their isotropy representations, we see that they are tensor products if the rank is greater than one. Hence the condition $\pi(\mathfrak{l}_i) = \mathfrak{sp}(1)$ is satisfied only if the rank of the corresponding symmetric space is one. Alternatively, our condition $\pi(\mathfrak{l}_i) = \mathfrak{sp}(1)$ is equivalent to saying that $K_i = H_i \times \mathrm{Sp}(1)$ and H_i have the same orbits in $\mathfrak{p}_i$, and it can be checked in [13] that this happens only for rank-one symmetric spaces. q.e.d.

2B. The examples of polar actions on $\mathbb{P}_2(\mathbb{O})$

We identify $\mathbb{P}_2(\mathbb{O})$ with the coset space $F_4/\mathrm{Spin}(9)$ and denote the point in $\mathbb{P}_2(\mathbb{O})$ corresponding to the colateral class $\mathrm{Spin}(9)$ by p. We also decompose the Lie algebra $\mathfrak{f}_4$ as $\mathfrak{f}_4 = \mathfrak{so}(9) + \mathfrak{p}$, where $\mathfrak{p}$ denotes the orthogonal complement of $\mathfrak{so}(9)$ with respect to the Cartan-Killing form; it is well known that the $\mathrm{Ad}(\mathrm{Spin}(9))$-action on $\mathfrak{p}$ is equivalent to the isotropy representation of $\mathrm{Spin}(9)$ at p.

We shall give examples of polar actions on $\mathbb{P}_2(\mathbb{O})$ in Corollary 2B.2 and Proposition 2B.4; the geometric interpretation of these actions are given in remarks after the proofs. In Section 5, we will actually prove that these examples exhaust the class of polar actions on $\mathbb{P}_2(\mathbb{O})$ with cohomogeneity greater than or equal to two. Notice that we do not have to deal with cohomogeneity one actions on $\mathbb{P}_2(\mathbb{O})$ since these were classified in [21]; see also [20].

We start with

Proposition 2B.1. *Let $G \subset Spin(9)$ be a compact Lie group acting on $\mathbb{P}_2(\mathbb{O})$ and suppose the linear isotropy representation of G on $\mathfrak{p}$ is polar with cohomogeneity two. Then the G-action on $\mathbb{P}_2(\mathbb{O})$ is polar with sections homeomorphic to the real projective plane.*

Before we start with the proof of the proposition, we review some well known facts on Jacobi fields on $\mathbb{P}_2(\mathbb{O})$ that will be used in the rest of this subsection. Let $x \in \mathfrak{p}$ be a unit vector. We consider the linear span of x, and decompose $\mathfrak{p}$ orthogonally into

$$\mathfrak{p} = \mathbb{R} \cdot x + \mathfrak{p}_1(x) + \mathfrak{p}_2(x),$$

where the subspaces $\mathfrak{p}_i(x)$ are the restricted root spaces with respect to the maximal abelian subalgebra $\mathbb{R} \cdot x$. The space $\mathfrak{p}_1(x)$ is seven dimensional and $\mathfrak{p}_2(x)$ is eight dimensional.

Let γ_x be the geodesic with $\dot{\gamma}_x(0) = x$. For a unit vector $y \in \mathfrak{p}_1(x) + \mathfrak{p}_2(x)$, we let J_y denote the Jacobi field along γ_x with $J_y(0) = 0$ and $J_y'(0) = y$, and let I_y denote the Jacobi field with $I_y(0) = y$ and $I_y'(0) = 0$. Furthermore, let E_y denote the parallel vector field along γ_x with $E_y(0) = y$. If $y \in \mathfrak{p}_1(x)$, then

$$J_y(t) = \sin t \; E_y(t), \qquad I_y(t) = \cos t \; E_y(t),$$

and if $y \in \mathfrak{p}_2(x)$, then

$$J_y(t) = 2\sin(\frac{t}{2}) \; E_y(t), \qquad I_y(t) = \cos(\frac{t}{2}) \; E_y(t);$$

see e.g. [8, Chapter IV, §2]. This implies that the two-plane spanned by x and y has sectional curvature 1 if $y \in \mathfrak{p}_1(x)$ and $1/4$ if y is in $\mathfrak{p}_2(x)$.

A two-plane $\tilde{\Sigma}$ spanned by x and an element $y \in \mathfrak{p}_2(x)$ is a Lie triple system and $\Sigma = \exp_p(\tilde{\Sigma})$ is a totally geodesic submanifold with curvature $1/4$ which is homeomorphic to the real projective plane.

The Jacobi fields relate to the exponential map through the well known formula

$$d(\exp_p)_{tx}(ty) = J_y(t).$$

Let us denote the unit sphere in $\mathfrak{p}$ by S^{15} and denote the cut locus of p by S^8. Define $h : S^{15} \to S^8$ by putting $h(x) = \exp_p(\pi x)$ for a point $x \in S^{15}$. It follows that $dh_x(y) = J_y(\pi)$. Hence $dh_x(y) = 0$ if $y \in \mathfrak{p}_1(x)$ and $|dh_x(y)| = 2|y|$ if $y \in \mathfrak{p}_2(x)$. The map h is therefore a submersion and even a Riemannian submersion up to the scaling factor $1/2$ on S^{15}.

Proof of Proposition 2B.1. Notice that the action of G leaves S^8 invariant. It cannot be transitive on S^8 since the only subgroup of Spin(9) that is transitive on S^8 is Spin(9) itself which acts with cohomogeneity one on $\mathbb{P}_2(\mathbb{O})$. If $\tilde{\Sigma}$ is a section of G in $\mathfrak{p}$, then it follows easily that $h(S)$ meets all orbits of G in S^8, where S is the unit circle in $\tilde{\Sigma}$. Thus the action of G on S^8 has cohomogeneity one. If we select a G-regular point $z \in S$, we can find a vector $v \in \mathfrak{p}_2(z)$ such that $dh_z(v)$ is orthogonal to the orbit $Gh(z)$. Thus

$$(T_z Gz)^{\perp} \cap T_z S^{15} = \mathbb{R} \cdot v.$$

Since for a cohomogeneity-one action a geodesic which meets one regular orbit orthogonally is automatically a normal geodesic, we get that $\tilde{\Sigma} = Span(z, v)$ is a linear section for G in $\mathfrak{p}$. We will parameterize the geodesic $\tilde{\Sigma} \cap S^{15}$ as $c(t)$, where $t \in [0, 2\pi]$ is an arc length parameter. We note that $c'(t) \in \mathfrak{p}_2(c(t))$ for all t.

We now put $\Sigma = \exp_p(\tilde{\Sigma})$ and we prove that Σ is a section for the G-action on $\mathbb{P}_2(\mathbb{O})$. Notice that Σ is totally geodesic and homeomorphic to the real projective plane since $v \in \mathfrak{p}_2(z)$. It is clear that Σ meets all orbits of G since $g \cdot \exp_p(x) = \exp_p(g_* \cdot x)$. We will prove that all orbits in the complement B of the cut locus S^8 of p meet Σ perpendicularly. It then follows that this is also true for orbits in S^8 by continuity, and we have proved that the G-action on $\mathbb{P}_2(\mathbb{O})$ is polar. Let $\gamma_x(s) = \exp_p(sx)$ where $x = c(t)$ for some t and $|s| < \pi$. We can assume that x is a regular point of the G-action. Then $T_{sx}G(sx)$ is the orthogonal complement of $c'(t)$ in $\mathfrak{p}_1(x) + \mathfrak{p}_2(x)$. We choose an orthonormal basis $(X_1, \dots, X_{14})$ of $T_{sx}G(sx)$ such that $X_1, \dots X_7 \in \mathfrak{p}_1(x)$ and $X_8, \dots, X_{14} \in \mathfrak{p}_2(x)$. Set $Y_i = d(\exp_p)_{sx}(X_i)$ for $i = 1, \dots, 14$. It is clear that $(Y_1, \dots, Y_{14})$ is a basis of $T_{\gamma_x(s)}G\gamma_x(s)$. It follows from the Gauss Lemma that $Y_1, \dots Y_{14}$ are perpendicular to the geodesic γ_x. It is therefore left to prove that $Y_1, \dots Y_{14}$ are perpendicular to $d(\exp_p)_{sx}(c'(t))$. We consider the Jacobi fields J_{Y_i} along γ_x defined before the proof of this proposition. We have that $Y_i = J_{X_i}(s) = \alpha_i E_i(s)$ where E_i is a parallel vector field along γ_x with $E_i(0) = X_i$ and α_i is a real number. We also have that $d(\exp_p)_{sx}(c'(t)) = J_{c'(t)}(s) = E_0(s)$ where E_0 is the parallel vector field along γ_x with $E_0(0) = c'(t)$. It is now clear that $T_{\gamma_x(s)}G\gamma_x(s)$ is perpendicular to $T_{\gamma_x(s)}\Sigma$. This finishes the proof. q.e.d.

As a corollary of the previous proposition, we obtain the following examples of polar actions on $\mathbb{P}_2(\mathbb{O})$.

Corollary 2B.2. *The following maximal connected subgroups of* *maximal rank of* Spin(9) *act on* $\mathbb{P}_2(\mathbb{O})$ *polarly with cohomogeneity two:*

$$H_1 = \mathrm{Spin}(8),\ H_2 = \mathrm{SO}(2)\cdot\mathrm{Spin}(7),\ H_3 = \mathrm{SU}(2)\cdot\mathrm{SU}(4).$$

Proof. We will denote the Lie algebras of the subgroups H_i by $\mathfrak{h}_i$ for $i = 1, 2, 3$. We know that the linear isotropy representation of Spin(9) is equivalent to the spin representation Δ on $\mathfrak{p} \cong \mathbb{R}^{16}$. By Proposition 2B.1, it is enough to check that the restriction of Δ to each subalgebra $\mathfrak{h}_i$ for $i = 1, 2, 3$ is polar of cohomogeneity two. Indeed, we have:

1. $\Delta|_{\mathfrak{h}_1}$ splits as the sum of two inequivalent 8-dimensional representations of $\mathfrak{so}(8)$ (see e.g. [27, pp.108-116]) and is polar of cohomogeneity two (see e.g. [9], [21]);

2. $\Delta|_{\mathfrak{h}_2}$ is irreducible and $\mathfrak{p} \cong \mathbb{R}^2 \otimes \mathbb{R}^8$, where the $\mathbb{R}$-factor of $\mathfrak{h}_2$ acts on $\mathbb{R}^2$ in a standard way, while the $\mathfrak{so}(7)$-factor acts on $\mathbb{R}^8$ via the spin representation. Indeed $\mathfrak{so}(7) \subset \mathfrak{so}(8)$ and by (1), we know that $\mathfrak{p} = \mathbb{R}^8 + \mathbb{R}^8$ as a $\mathfrak{so}(8)$-module; when we restrict Δ to $\mathfrak{so}(7)$, each submodule $\mathbb{R}^8$ can either remain irreducible or split as $\mathbb{R} + \mathbb{R}^7$. If one of the two factors should split as $\mathbb{R} + \mathbb{R}^7$, then the Spin(7)-factor H_2^s of H_2 would have a non-zero fixed vector v; hence Spin(7) would be a subgroup of $K = \mathrm{Spin}(9)_v$. It is known (see e.g. [3]) that K is a subgroup isomorphic to Spin(7) and with discrete centralizer in Spin(9); so from $H_2^s \subseteq K$ we would have $H_2^s = K$ and K would have a nontrivial centralizer, a contradiction. So, each submodule $\mathbb{R}^8$ remains $\mathfrak{so}(7)$-irreducible and we have $\mathfrak{p} \cong \mathbb{R}^2 \otimes \mathbb{R}^8$. Since the spin representation of Spin(7) is transitive on the unit sphere S^7, the representation of $\mathbb{R}+\mathfrak{so}(7)$ has the same orbits as $\mathbb{R}+\mathfrak{so}(8)$ acting on $\mathbb{R}^2 \otimes \mathbb{R}^8$, and this last representation is polar of cohomogeneity two, since it is the isotropy representation of the rank-two symmetric space $\mathrm{SO}(10)/\mathrm{SO}(2)\times\mathrm{SO}(8)$;

3. $\Delta|_{\mathfrak{h}_3}$ is also irreducible and $\mathfrak{p} \cong [\mathbb{C}^2 \otimes \mathbb{C}^4]_{\mathbb{R}}$, as one can easily check using the same arguments as in (2). This representation is polar, since it has the same orbits as the isotropy representation of the rank-two symmetric space $\mathrm{SU}(6)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(4))$. q.e.d.

Remark. The action of the subgroup $H_1 = \mathrm{Spin}(8)$ on $\mathbb{P}_2(\mathbb{O})$ was already investigated in [21], although it was not noticed there that the action is polar. The action of Spin(8) has the following geometric interpretation: if S^8 denotes the cut locus of the point p and we fix a point $q \in S^8$, then the subgroup Spin(8) of F_4 is the stabilizer of the flag (q, S^8).

The subgroup $H_2 = \mathrm{SO}(2) \cdot \mathrm{Spin}(7)$ can be seen as the subgroup of F_4 which leaves the flag (γ, S^8) invariant, where γ denotes the trace of a geodesic in the projective line S^8, which is the cut locus of p.

The subgroup $H_3 = \mathrm{SU}(2) \cdot \mathrm{SU}(4)$ can be characterized as the stabilizer in F_4 of the flag (S^2, S^8), where S^2 denotes a totally geodesic two-sphere in S^8.

Among the maximal subgroups of maximal rank in Spin(9), all act polarly on $\mathbb{P}_2(\mathbb{O})$ with the exception of $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(2)$, as we will see in Section 5B.

Our next example is provided by the action on $\mathbb{P}_2(\mathbb{O})$ of a maximal subgroup of maximal rank G of F_4, locally isomorphic to $\tilde{G} = \mathrm{SU}(3) \times \mathrm{SU}(3)$. In order to describe it, we need to review some basic facts about the construction and classification of maximal subalgebras of maximal rank of $\mathfrak{f}_4$ (see [16, §8.3]).

We fix a maximal torus $T \subset \mathrm{Spin}(9)$ and we consider its Lie algebra $\mathfrak{t}$, so that $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{f}_4^{\mathbb{C}}$; the corresponding root system of $\mathfrak{f}_4$ will be denoted by Δ and we fix a system of simple roots $\{\alpha_1, \dots, \alpha_4\}$, corresponding to the Dynkin diagram

where the black roots $\{\alpha_1, \alpha_2\}$ are long, while the white ones $\{\alpha_3, \alpha_4\}$ are short. If $\alpha = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$ denotes the maximal root, we can form the extended Schäfli-Dynkin diagram

From the Borel-Siebenthal theory (see [16, §8.3]), we have that the subsystem $\Delta_1 = \{\alpha_2, \alpha_3, \alpha_4, \alpha\}$ is the system of simple roots of the subalgebra $\mathfrak{so}(9)$, while the subsystem $\Delta_2 = \{\alpha_1, \alpha_2, \alpha_4, \alpha\}$ is the system of simple roots of a subalgebra $\mathfrak{g}$ isomorphic to $\mathfrak{su}(3) + \mathfrak{su}(3)$. The connected Lie subgroup G of F_4 with Lie algebra $\mathfrak{g}$ is locally isomorphic to $\tilde{G} = \mathrm{SU}(3) \times \mathrm{SU}(3)$, and we have that $G = \tilde{G}/\mathbb{Z}_3$. In fact, the homogeneous space F_4/G is isotropy irreducible and the isotropy representation of G is $[S^2(\mathbb{C}^3) \otimes \mathbb{C}^3]_{\mathbb{R}}$ (see [31], table (8.10.13), p. 282a). Since the isotropy representation of G is faithful and the representation of $\tilde{G}$ on $[S^2(\mathbb{C}^3) \otimes \mathbb{C}^3]_{\mathbb{R}}$ has a kernel which is a finite central subgroup isomorphic to $\mathbb{Z}_3$, we have that $G = \tilde{G}/\mathbb{Z}_3$.

We first list the properties of the action of G on $\mathbb{P}_2(\mathbb{O})$ that we will need.

Lemma 2B.3.

(a) *The orbit Gp is totally geodesic and isometric to $\mathbb{P}_2\mathbb{C}$. The orbit Gp is also an orbit of one of the factors of $\tilde{G}$; the other one acts trivially on Gp. We will assume that the first factor of $\tilde{G}$ acts nontrivially, the second one trivially.*

(b) *We have $\tilde{G}_p = \mathrm{U}(2) \times \mathrm{SU}(3)$; the slice representation of G_p on $N_p(Gp)$ is polar with cohomogeneity two. A linear section for the G_p action on $N_p(Gp)$ meets a principal orbit in eight points.*

(c) *The action of G_p on the cut locus of p, that we will denote by S^8, has cohomogeneity one. A section of the action of G_p on S^8 meets a principal orbit in four points.*

(d) *Let H be the subgroup of $\tilde{G}_p$ leaving a given section in $N_p(Gp)$ fixed. Then H is a 2-torus, $H = H_1 \times H_2$, where $H_1, H_2 \cong T^1$ and the isotropy representation of H_1 on $T_p(Gp)$ does not have a trivial factor.*

Proof. (a) We first recall that the symmetry σ_p at p belongs to the center of Spin(9). Moreover the intersection $G \cap \mathrm{Spin}(9)$ has maximal rank, since both groups share the same maximal torus; it then follows that $\sigma_p \in G$ and the orbit Gp is a totally geodesic submanifold. Moreover we see that $\Delta_1 \cap \Delta_2$ is the system of simple roots for the semisimple part of the intersection $G \cap \mathrm{Spin}(9)$; from this we see that $\mathfrak{g} \cap \mathfrak{so}(9) = \mathbb{R} + \mathfrak{su}(2) + \mathfrak{su}(3)$. Therefore one SU(3)-ideal of G acts trivially on the orbit Gp, which is homeomorphic to a complex projective space $\mathbb{P}_2(\mathbb{C})$.

(b) We denote by $\mathfrak{k}$ the Lie subalgebra $\mathfrak{g} \cap \mathfrak{so}(9)$, which we know to be isomorphic to $\mathbb{R} + \mathfrak{su}(2) + \mathfrak{su}(3)$. We now decompose the Lie algebra $\mathfrak{f}_4 = \mathfrak{g} + \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{f}_4$; we know that $\mathfrak{m} = [S^2\mathbb{C}^3 \otimes \mathbb{C}^3]_{\mathbb{R}}$ as an $\mathrm{ad}(\mathfrak{g})$-module. We restrict the $\mathrm{ad}(\mathfrak{g})|_{\mathfrak{m}}$-representation to $\mathfrak{k}$ and we get that $\mathfrak{m} = [\mathbb{C}^3 + S^2(\mathbb{C}^2) \otimes \mathbb{C}^3 + \mathbb{C}^2 \otimes \mathbb{C}^3]_{\mathbb{R}}$. Therefore we can decompose the Lie algebra $\mathfrak{f}_4$ into $\mathrm{ad}(\mathfrak{k})$-submodules as

$$\mathfrak{f}_4 = \mathfrak{k} + [\mathbb{C}^2]_{\mathbb{R}} + [\mathbb{C}^3 + S^2(\mathbb{C}^2) \otimes \mathbb{C}^3 + \mathbb{C}^2 \otimes \mathbb{C}^3]_{\mathbb{R}},$$

and we note that all this submodules are mutually nonequivalent as $\mathrm{ad}(\mathfrak{k})$-modules. (This also follows from the fact that $\mathfrak{k}$ coincides with the centralizer in $\mathfrak{f}_4$ of a nonzero element in $\mathfrak{t}$ since the isotropy representation of a generalized flag manifold decomposes into mutually nonequivalent irreducible submodules.) On the other hand, we may write $\mathfrak{f}_4 = \mathfrak{k}+\mathfrak{m}_1+\mathfrak{p}$, where $\mathfrak{k}+\mathfrak{m}_1 = \mathfrak{so}(9)$. By comparing this two decompositions of $\mathfrak{f}_4$ and using the fact that all the submodules are nonequivalent, we get that $\mathfrak{p}$ splits as $\mathfrak{p} = [\mathbb{C}^2 + \mathbb{C}^2 \otimes \mathbb{C}^3]_{\mathbb{R}}$ as an $\mathrm{ad}(\mathfrak{k})$-module. Therefore the slice representation of $\mathfrak{k}$ is given by $\mathbb{C}^2 \otimes \mathbb{C}^3$, which is polar of cohomogeneity two, since it is the isotropy representation of the symmetric space $\mathrm{SU}(5)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$. The second assertion follows from the fact that a principal orbit of the Weyl group of this symmetric space consists of eight points.

(c) This follows from the fact that the subalgebra $\mathfrak{k}$ of $\mathfrak{so}(9)$ is embedded as $\mathfrak{k} = \mathfrak{su}(2)+(\mathbb{R}+\mathfrak{su}(3)) \subset \mathfrak{su}(2)+\mathfrak{so}(6) \subset \mathfrak{so}(9)$. Thus the standard action of $\mathrm{Spin}(9)$ on $\mathbb{R}^9$, when restricted to $\tilde{G}_p$, splits as $\mathbb{R}^9 = \mathbb{R}^3 + \mathbb{R}^6$ and the action of $\tilde{G}_p$ on $\mathbb{R}^9$ has cohomogeneity two, hence one on S^8. The principal orbits are product embeddings $S^2(r_1) \times S^5(r_2) \subset S^8$, $r_1^2 + r_2^2 = 1$, which clearly meet a section in four points.

(d) By dimension reasons, we have that the regular isotropy subgroup H of $\tilde{G}_p$ for the slice representation is two-dimensional and compact, hence its identity component is a 2-torus. Moreover, the regular orbits are simply connected, so that H is connected. It is moreover easy to check that the Lie algebra $\mathfrak{h}$ of H has a nontrivial projection on the $\mathfrak{su}(2)$-factor of $\mathfrak{k}$. Therefore, if we write H as $H = H_1 \times H_2$, where each H_i is a one-dimensional torus with Lie algebra $\mathfrak{h}_i$ for $i = 1, 2$, then at least one of the $\mathfrak{h}_i$, say $\mathfrak{h}_1$, has nontrivial projection on the $\mathfrak{su}(2)$-factor of $\mathfrak{k}$. Since the isotropy representation of $\tilde{G}_p$ on $T_p\mathbb{P}_2(\mathbb{C})$, restricted to the SU(2)-factor, is the standard representation of SU(2) on $\mathbb{C}^2$, it follows that the isotropy representation of H_1 on $T_p\mathbb{P}_2(\mathbb{C})$ has no fixed vector. q.e.d.

Proposition 2B.4. *The action of the group $\tilde{G}$ on $\mathbb{P}_2(\mathbb{O})$ is polar of cohomogeneity two with sections homeomorphic to the real projective plane.*

Proof. First notice that G_p acts transitively on the unit sphere in the tangent space $T_p(Gp)$. We consider the action of G_p on $T_p\mathbb{P}_2(\mathbb{O}) = T_p(Gp) + N_p(Gp)$. The action is polar with cohomogeneity three. We let S^{15} denote the unit sphere in $\mathbb{P}_2(\mathbb{O})$. Let $h : S^{15} \to S^8$ be as above. Again we have that $h(G_p(p)) = G_p(h(p))$. Let $q \in S^8$ be given. The preimage of q under h is a seven-dimensional great sphere S in S^{15} that must meet $N_p(Gp)$ which is twelve-dimensional. There is therefore a point $r \in S^{15} \cap N_p(G(p))$ such that the orbit $G_p(r)$, which lies completely in $N_p(Gp)$, maps onto the orbit $G_p(r)$ in S^8. Now let $\tilde{\Sigma}$ be a two-dimensional section of G_p in $N_p(Gp)$ and c its intersection with S^{15}. Notice that c is a great circle in S^{15}. A closed segment d of c from a singular orbit to a singular orbit has length $\pi/4$, since a principal orbit meets a section eight times. Such a segment meets every orbit exactly once. It follows that $h(d)$ meets every orbit of G_p in S^8 once. Such a curve must have length $\pi/2$ at least, since a section of G_p on S^8 meets the principal orbits four times and the orbits are equidistant. The length of $h(d)$ can be at most $\pi/2$ since $|dh_x(y)| \leq 2|y|$ for all $y \in T_x S^{15}$; see above. A curve of length $\pi/2$ meeting all orbits must be a great circle arc. It follows that a semicircle of c is a horizontal lift (up to a factor $1/2$ in the parameterization of c) of a section of G_p in S^8. This implies that the image Σ of $\tilde{\Sigma}$ under $\exp_p$ is a real projective plane, since it is spanned by $c(t)$ and $c'(t) \in \mathfrak{p}_2(c(t))$; see above. The calculations in the proof Proposition 2B.1 above imply that $\Sigma = \exp_p(\tilde{\Sigma})$ meets the orbits of G_p within a ball B of radius $r \leq \pi$ around p perpendicularly whenever they meet. (Notice that the principal orbits of G_p in B are thirteen dimensional and Σ is two dimensional. Hence not all orbits can meet Σ. In fact, only those orbits that are images of orbits in $N_p(Gp)$ will meet Σ.)

Now we will consider the whole group G and show that its orbits meet Σ perpendicularly. Let q be an element in Σ that is not in S^8. Then there is an $x \in \tilde{\Sigma}$ such that $q = \exp_p(x)$. Let y be an element in $\mathfrak{p}_2(x)$ such that x and y span the section $\tilde{\Sigma}$. We consider the subgroup $K \cong \mathrm{G}_2$ of Spin(9) leaving $\tilde{\Sigma}$ invariant. The representation of K on $T_p\mathbb{P}_2(\mathbb{O})$ splits into invariant submodules as follows:

$$T_p\mathbb{P}_2(\mathbb{O}) = \mathbb{R} \cdot x + \mathbb{R} \cdot y + \mathfrak{p}_1(x) + \mathfrak{q}_2(x,y),$$

where $\mathfrak{q}_2(x,y)$ is the orthogonal complement of y in $\mathfrak{p}_2(x)$. Now consider the group $H = G_p \cap K$. As we saw in Lemma 2B.3,(d), if we restrict the H-action to $T_p(Gp)$ and divide out the kernel, then we get an action of S^1 on $T_p(Gp)$ without trivial factors, i.e., we have a decomposition of $T_p(Gp)$ into two irreducible two-dimensional modules

$$T_p(Gp) = V_1 + V_2.$$

The module V_i is for $i = 1,2$ either contained in $\mathfrak{p}_1(x)$ or $\mathfrak{q}_2(x,y)$. We can therefore find an orthonormal basis $(X_1, \ldots, X_4)$ of $T_p(Gp)$ such that X_i is either in $\mathfrak{p}_1(x)$ or $\mathfrak{p}_2(x)$ for $i = 1,2,3,4$. We write $\hat{\mathfrak{p}}$ for $T_p(Gp)$.

Then $\mathfrak{su}(3) = (\mathbb{R} + \mathfrak{su}(2)) + \hat{\mathfrak{p}} \subset \mathfrak{f}_4$ is the Cartan decomposition of the ideal of $\mathfrak{su}(3) + \mathfrak{su}(3)$, that is nontrivial on $\hat{\mathfrak{p}} = T_pGp$. We consider the variations $V_i(s,t) = \exp_G(sX_i) \cdot \gamma_x(t)$ of the geodesic γ_x and the corresponding Jacobi fields I_i. We have to show that

$$\frac{d}{dt}\bigg|_{s=0} \exp_G(sX_i) \cdot x = I_i(1)$$

is perpendicular to

$$d(\exp_p)_x(y) = J_y(1).$$

Notice that $I_i(0) = X_i$ and $I_i'(0) = 0$. Hence $I_i = I_{X_i}$ in the notation above. Since X_i lies in $\mathfrak{p}_1(x)$ or $\mathfrak{p}_2(x)$ there is a parallel vector fields E_i along γ_x with the property that

$$I_i(t) = \cos(\alpha t) E_i(t).$$

As we have seen above there is a parallel vector field E along γ_x such that

$$J_y(t) = 2\sin(\frac{t}{2})E(t).$$

Now it is clear that $I_i(1)$ and $J_y(1)$ are perpendicular since $E_i(0) = X_i$ and $E(0) = y$ are perpendicular. This finishes the proof that the action of $\tilde{G}$ on $\mathbb{P}_2(\mathbb{O})$ is polar. q.e.d.

Remark. The maximal subgroup $G = \mathrm{SU}(3) \cdot \mathrm{SU}(3)$ of F_4 can be seen as the subgroup which leaves a totally geodesic complex projective plane invariant. Therefore the homogeneous space F_4/G (which is isotropy irreducible but not symmetric) can be seen as the space of all totally geodesic complex projective planes in $\mathbb{P}_2(\mathbb{O})$.

3. Polar actions on complex projective spaces

Theorem 3.1. *Let G be a compact and connected Lie group acting polarly on the complex projective space $\mathbb{P}_n(\mathbb{C})$. Then the action of G is orbit equivalent to the action induced by a Hermitian symmetric space.*

Proof. We can assume that the G-action is effective by factoring out the kernel of the action. The group G can then be considered to be a subgroup of $\mathrm{SU}(n+1)/Z(\mathrm{SU}(n+1))$ which is the identity component of the isometry group of $\mathbb{P}_n(\mathbb{C})$. There is therefore a finitely sheeted covering $\pi : G' \to G$ where G' is a connected subgroup of $\mathrm{SU}(n+1)$.

Let Σ be a section of the G-action. By Proposition 1B.1, the section Σ is either isometric to a sphere S^2 or to a real projective space $\mathbb{P}_k(\mathbb{R})$ where $1 \leq k \leq n$. Notice that our point of view here is to consider a one-dimensional section to be isometric to $\mathbb{P}_1(\mathbb{R})$, not to S^1 as in Proposition 1B.1.

We first show that Σ cannot be isometric to S^2. Assume that it is the case. Then Σ is a projective line in $\mathbb{P}_n(\mathbb{C})$, i.e., $\Sigma \cong \mathbb{P}_1(\mathbb{C})$. A real codimension-two submanifold M of $\mathbb{P}_n(\mathbb{C})$ with the property that there is a projective line meeting M perpendicularly in p for every p in M is a complex hypersurface. Any two complex hypersurfaces in $\mathbb{P}_n(\mathbb{C})$ intersect. It follows that any two regular orbits of G intersect and there can therefore only be one regular orbit, which is of course a contradiction.

We shall now discuss the case where Σ is isometric to a real projective space $\mathbb{P}_k(\mathbb{R})$. By Proposition 1B.2, there is an element in the identity component of the isometry group of $\mathbb{P}_n(\mathbb{C})$ that maps Σ to the standard embedding $\mathbb{P}_k(\mathbb{R}) \subset \mathbb{P}_n(\mathbb{R}) \subset \mathbb{P}_n(\mathbb{C})$. We can therefore assume that $\Sigma = \mathbb{P}_k(\mathbb{R})$. Let $\pi_{\mathbb{R}} : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}_n(\mathbb{R})$ be the canonical projection. Set $\tilde{\Sigma} = \pi_{\mathbb{R}}^{-1}(\Sigma) \cup \{0\}$. Notice that $\tilde{\Sigma}$ is a linear subspace of $\mathbb{R}^{n+1}$.

The finitely-sheeted covering $\pi : G' \to G$ clearly satisfies $\pi_{\mathbb{C}}(g \cdot v) = \pi(g) \cdot \pi_{\mathbb{C}}(v)$ for all $g \in G'$ and all $v \in \mathbb{C}^{n+1}$ where $\pi_{\mathbb{C}} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}_n(\mathbb{C})$ is the canonical projection. We consider the group $\tilde{G} = G' \times T^1$ acting on $\mathbb{C}^{n+1}$, where T^1 is simply the multiplication by unit complex numbers. We claim that the $\tilde{G}$-action on $\mathbb{C}^{n+1}$ is polar with $\tilde{\Sigma}$ as a section.

Indeed, it is clear that $\tilde{\Sigma}$ meets every $\tilde{G}$-orbit. We now choose a point $p \in \tilde{\Sigma}$. Since $\tilde{\Sigma}$ lies in a totally real subspace of $\mathbb{C}^{n+1}$, the T^1-orbit through p will be orthogonal to $\tilde{\Sigma}$; moreover we can decompose the tangent space $T_p(\tilde{G}p)$ orthogonally as

$$T_p(\tilde{G}p) = T_p(T^1 \cdot p) \oplus \mathcal{H}_p,$$

and we note that the horizontal subspace $\mathcal{H}_p$ maps isometrically onto the tangent space $T_{\pi_{\mathbb{C}}(p)}(G\pi_{\mathbb{C}}(p))$. Since $T_{\pi_{\mathbb{C}}(p)}(G\pi_{\mathbb{C}}(p))$ is orthogonal to $T_{\pi_{\mathbb{C}}(p)}\Sigma$, our claim follows.

By Dadok's theorem ([10]); see Theorem 1A.5, there is a symmetric pair (U, K) with Lie algebra decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$, so that $\mathfrak{p}$ can be identified with $\mathbb{C}^{n+1}$ in such a way that $\tilde{G}$ is a subgroup of K, the isotropy representation of K on $\mathfrak{p}$ has the same orbits as $\tilde{G}$ and the actions of $\tilde{G}$ on $\mathfrak{p}$ and $\mathbb{C}^{n+1}$ coincide up to the identification of $\mathfrak{p}$ and $\mathbb{C}^{n+1}$.

We claim that we can choose the symmetric pair (U, K) to be Hermitian symmetric. Notice that this implies the claim in Theorem 3.1.

We first assume that the $\tilde{G}$-module $\mathbb{C}^{n+1}$ is reducible as a real representation. We split $\mathbb{C}^{n+1} = V_1 \oplus \cdots \oplus V_k$, where the V_i's are $\tilde{G}$-irreducible subspaces; no factor is trivial, since the T^1-action has no fixed points. Moreover the V_i's are complex subspaces because of the T^1-action. Since each $\tilde{G}$-action on V_i is polar, the corresponding symmetric space will split as $(U, K)= \Pi_i^k(U_i, K_i)$; see [10], and the actions of $\tilde{G}$ and K_i on each V_i have the same orbits. So, we can confine ourselves to the irreducible case.

We note that the symmetric pair $(\mathrm{SO}(2p+1), \mathrm{SO}(2p))$ $(p \geq 2)$ is not Hermitian symmetric, but K contains the subgroup $\mathrm{U}(p)$ and $\mathrm{Ad}(K)_{\mathfrak{p}}$ and $\mathrm{Ad}(\mathrm{U}(p))_{\mathfrak{p}}$ have the same orbits; in this case we can substitute the symmetric pair $(\mathrm{SO}(2p+1), \mathrm{SO}(2p))$ with the Hermitian symmetric pair $(\mathrm{SU}(p+1), \mathrm{U}(p))$. Our claim that we can choose the symmetric pair (U, K) to be Hermitian symmetric follows from the following lemma.

Lemma 3.2. *Let* $(\mathfrak{u}, \mathfrak{k})$ *be an effective irreducible orthogonal symmetric pair with* $(\mathfrak{u}, \mathfrak{k}) \neq (\mathfrak{so}(2p+1), \mathfrak{so}(2p))$, *and assume the following conditions are satisfied:*

1. *there exists an element* $Z \in \mathfrak{k}$ *such that* $\mathrm{ad}(Z)|_{\mathfrak{p}}$ is a complex structure on $\mathfrak{p}$;
2. *there exists a compact Lie subgroup* $H \subset K$ *such that* $\mathrm{Ad}(H)$ and $\mathrm{Ad}(K)$ *have the same orbits in* $\mathfrak{p}$*;*
3. *the* $\mathrm{Ad}(H)$*-action on* $\mathfrak{p}$ *commutes with* $\mathrm{ad}(Z)|_{\mathfrak{p}}$.

Then $(\mathfrak{u}, \mathfrak{k})$ *is Hermitian.*

Remark. This Lemma generalizes Lemma 2.3 in [28].

Proof. First of all we show that rank$(\mathfrak{u})$=rank$(\mathfrak{k})$. Indeed, we may choose a maximal abelian subalgebra $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$, where $\mathfrak{t}_1$ is maximal abelian in $\mathfrak{k}$ and contains Z, while $\mathfrak{t}_2$ is maximal abelian in $\mathfrak{p}$. If $\mathfrak{k}$ does not have maximal rank in $\mathfrak{u}$, then $\mathfrak{t}_2 \neq \{0\}$; but this contradicts the fact that $Z \in \mathfrak{t}_1$ acts on $\mathfrak{p}$ as a complex structure, hence fixed point free.

We now consider the maximal rank subalgebra $\mathfrak{h}' = \mathfrak{z}(Z)$ given by the centralizer of Z; this subalgebra is contained in $\mathfrak{k}$, since $\mathrm{ad}(Z)|_{\mathfrak{p}}$ has trivial kernel. Moreover $\mathfrak{h} \subseteq \mathfrak{h}'$, where $\mathfrak{h}$ denotes the Lie algebra of H. Indeed, if $X \in \mathfrak{h}$ and $Y \in \mathfrak{p}$, then, using the Jacobi identity and property (3), we have

$$[[X, Z], Y] = -\mathrm{ad}(Z)\mathrm{ad}(X)Y + \mathrm{ad}(X)\mathrm{ad}(Z)Y = 0.$$

Hence $\mathrm{ad}([X, Z])|_{\mathfrak{p}} = 0$, and therefore $[X, Z] = 0$, since the $\mathrm{ad}(\mathfrak{k})$-representation on $\mathfrak{p}$ is faithful. We fix an $ad(\mathfrak{k})$-regular element $A \in \mathfrak{p}$ and note that property (2) implies

$$[\mathfrak{k}, A] = [\mathfrak{h}, A] = [\mathfrak{h}', A].$$

If we denote by $\mathfrak{c} = \mathfrak{z}_{\mathfrak{k}}(A)$ the centralizer of A in $\mathfrak{k}$, then we have that

$$(*) \qquad \mathfrak{k} = \mathfrak{h}' + \mathfrak{c}.$$

The triple of Lie algebras $(\mathfrak{k}, \mathfrak{h}', \mathfrak{c})$, with $\mathfrak{h}', \mathfrak{c}$ subalgebras of $\mathfrak{k}$, is therefore a factorization of $\mathfrak{k}$. The claim of the lemma is equivalent to $\mathfrak{k} = \mathfrak{h}'$.

We are now going to exclude all non-Hermitian irreducible symmetric pairs $(\mathfrak{u}, \mathfrak{k})$ with rank($\mathfrak{u}$)=rank($\mathfrak{k}$) by a case by case inspection. We now list all such irreducible pairs, indicating also the dimension of $\mathfrak{p}$ and the cohomogeneity r of the $\mathrm{ad}(\mathfrak{k})$-action on $\mathfrak{p}$, namely the rank of the symmetric pair. We can immediately rule out the case of adjoint orbits, simply because they all have positive Euler characteristic, whereas $\mathrm{ad}(Z)$ induces a fixed point free action on the orbits of the $\mathrm{Ad}(H)$-action on $\mathfrak{p}$.

$n.$	$\mathfrak{u}$	$\mathfrak{k}$	$\dim \mathfrak{p}$	r
1	$\mathfrak{so}(2p+2q)$	$\mathfrak{so}(2p) + \mathfrak{so}(2q),\ p, q > 1$	$4pq$	$min(2p, 2q)$
2	$\mathfrak{so}(2p+2q+1)$	$\mathfrak{so}(2p) + \mathfrak{so}(2q+1),\ p > 1, q \geq 1$	$2p(2q+1)$	$min(2p, 2q+1)$
3	$\mathfrak{sp}(p+q)$	$\mathfrak{sp}(p) + \mathfrak{sp}(q)$	$4pq$	$min(p, q)$
4	$\mathfrak{e}_6$	$\mathfrak{su}(6) + \mathfrak{su}(2)$	40	4
5	$\mathfrak{e}_7$	$\mathfrak{su}(8)$	70	7
6	$\mathfrak{e}_7$	$\mathfrak{so}(12) + \mathfrak{su}(2)$	64	4
7	$\mathfrak{e}_8$	$\mathfrak{so}(16)$	128	8
8	$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{su}(2)$	112	4
9	$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{su}(2)$	28	4
10	$\mathfrak{f}_4$	$\mathfrak{so}(9)$	16	1
11	$\mathfrak{g}_2$	$\mathfrak{su}(2) + \mathfrak{su}(2)$	8	2

This table and the corresponding isotropy representations that we will need in the proof can be extracted from the tables on pp. 311-4 in [4].

Cases (1)-(2): We will deal with these two cases in a unified way. First of all, we put $\mathfrak{k} = \mathfrak{k}_1 + \mathfrak{k}_2$, where $\mathfrak{k}_1 = \mathfrak{so}(2p)$ and $\mathfrak{k}_2 = \mathfrak{so}(2q)$ or

$\mathfrak{so}(2q+1)$; accordingly, we decompose $Z = Z_1 + Z_2 \in \mathfrak{k}_1 + \mathfrak{k}_2$. Since the representation $\mathrm{ad}(\mathfrak{k})|_{\mathfrak{p}}$ is given by the tensor product of the standard representations of $\mathfrak{k}_1$ and $\mathfrak{k}_2$ on $V = \mathbb{R}^{2p}$ and $W = \mathbb{R}^{2q}, \mathbb{R}^{2q+1}$ respectively, the condition that $\mathrm{ad}(Z)^2 = -I$ implies that either $Z_1 = 0$ or $Z_2 = 0$: indeed suppose that $Z_1 \not\equiv 0$ and select a vector $v \in V$ with $Z_1^2 v = \lambda v$ for some $\lambda \neq 0$; then for every $w \in W$ we have

$$\lambda v \otimes w + 2Z_1 v \otimes Z_2 w + v \otimes Z_2^2 w = -v \otimes w,$$

which implies that $Z_2 w = 0$, hence $Z_2 = 0$.

We first assume that $Z_2 = 0$, so that Z_1 acts as a complex structure on V. Then we have that $\mathfrak{h}' = \mathfrak{u}(p) + \mathfrak{k}_2$, where $\mathfrak{u}(p)$ denotes the Lie algebra of the unitary group $\mathrm{U}(p) \subset \mathrm{SO}(2p)$. We now consider an element t of $V \otimes W$ of type $t = v_1 \otimes w_1 + v_2 \otimes w_1 + v_2 \otimes w_2$ for some pairs of orthonormal vectors $v_1, v_2 \in V$ and $w_1, w_2 \in W$ and we note that the isotropy subalgebra $\mathfrak{k}_t$ projects into $\mathfrak{so}(2p-2) \subset \mathfrak{k}_1$; indeed we have that for $(A, B) \in \mathfrak{k}_t$,

$$A(v_1 + v_2) \otimes w_1 + Av_2 \otimes w_2 + (v_1 + v_2) \otimes Bw_1 + v_2 \otimes Bw_2 = 0,$$

which easily implies that $Av_1 = Av_2 = 0$. The subalgebra $\mathfrak{c}$ is conjugate to a subalgebra of $\mathfrak{k}_t$, so it projects to a proper subalgebra $\mathfrak{c}'$ in $\mathfrak{k}_1$. Hence we have a nontrivial factorization $\mathfrak{k}_1 = \mathfrak{u}(p) + \mathfrak{c}'$, which contradicts Theorem 1C.1. If $Z_1 = 0$, we can argue similarly.

Case (3): In this case the representation of K on $\mathfrak{p}$ will be also a tensor product, and we can see as in the cases (1)-(2) that the projection of $\mathfrak{c}$ into either of the two factors of $\mathfrak{k}$ is not onto. Since $\mathfrak{h}'$ has maximal rank in $\mathfrak{k}$, we may split it as $\mathfrak{h}' = \mathfrak{h}_1 + \mathfrak{h}_2$ with $\mathfrak{h}_1 \subset \mathfrak{sp}(p), \mathfrak{h}_2 \subset \mathfrak{sp}(q)$ (see e.g [23, p.183]). Since $\mathfrak{h}'$ is not semisimple, we can assume that $\mathfrak{h}_1 \neq \mathfrak{sp}(p)$. Then $\mathfrak{sp}(p) = \mathfrak{h}_1 + \mathfrak{c}_1$, where $\mathfrak{c}_1$ is the projection of $\mathfrak{c}$ into $\mathfrak{sp}(p)$. This is not possible by Theorem 1C.1.

Case (4): The dimension of $\mathfrak{su}(6) + \mathfrak{su}(2)$ is 38 and the dimension of a regular orbit in $\mathfrak{p}$ is 36. Hence $\mathfrak{c}$ is two-dimensional, i.e., $\mathfrak{c} \cong \mathbb{R}^2$. We look for a subalgebra $\mathfrak{h}'$ such that $\mathfrak{su}(6) + \mathfrak{su}(2) = \mathfrak{h}' + \mathfrak{c}$. We have $\dim \mathfrak{k}/\mathfrak{h}' \leq 2$. The group $\mathrm{SU}(6)$ admits no non-trivial action on a one- or two-dimensional manifold since it is simple. Hence $\mathfrak{su}(6) \subset \mathfrak{h}'$. Since $\mathfrak{h}'$ has maximal rank in $\mathfrak{k}$, it must have the form $\mathfrak{h}' = \mathfrak{su}(6) + \mathbb{R}$ for some $\mathbb{R} \subset \mathfrak{su}(2)$. But then the projection of $\mathfrak{c}$ on the $\mathfrak{su}(2)$-factor would be a two-dimensional subalgebra, which does not exist.

Case (5): The dimensions of both $\mathfrak{su}(8)$ and a regular orbit in $\mathfrak{p}$ is 63. Hence it follows immediately that $\mathfrak{h}' = \mathfrak{su}(8)$ which is a contradiction since $\mathfrak{h}'$ is not simple.

Case (6): In this case $\dim \mathfrak{c} = 9$. Since any compact Lie group acting almost effectively on a manifold of dimension less or equal to 9 has dimension at most 45, it follows that $\mathfrak{so}(12) \subset \mathfrak{h}'$, so that $\mathfrak{h}' = \mathfrak{so}(12) + \mathbb{R}$, for some $\mathbb{R} \subset \mathfrak{su}(2)$: indeed $\mathfrak{h}'$ has non-trivial center, which prevents it from coinciding with $\mathfrak{k}$. Now we claim that the subalgebra $\mathfrak{c}$ projects trivially on the $\mathfrak{su}(2)$-factor contradicting the factorization (*). Actually, the representation of K on $\mathfrak{p}$ is given by $\mathrm{Spin}(12) \otimes \mathrm{SU}(2)$ on $\mathbb{R}^{16} \otimes \mathbb{R}^4$. The isotropy subalgebra of $\mathfrak{k}$ at an element $v \otimes w \in \mathbb{R}^{16} \otimes \mathbb{R}^4$ has trivial projection on $\mathfrak{su}(2)$ and contains a conjugate copy of $\mathfrak{c}$, so that $\mathfrak{c}$ projects trivially on $\mathfrak{su}(2)$.

Case (7): The dimension of both $\mathfrak{so}(16)$ and a regular orbit in $\mathfrak{p}$ is 120. Hence it follows immediately that $\mathfrak{h}' = \mathfrak{so}(16)$ which is a contradiction since $\mathfrak{h}'$ is not simple.

Case (8): The representation of K on $\mathfrak{p}$ is given by $\Lambda^2 E_7 \otimes SU(2)$ on $\mathbb{R}^{28} \otimes \mathbb{R}^4$. The same argument as in case (6) shows that $\mathfrak{c}$ has trivial projection on the $\mathfrak{su}(2)$-factor. Therefore, $\mathfrak{h}'$, which has maximal rank in $\mathfrak{k}$, must split as $\mathfrak{h}' = \mathfrak{h}'' \oplus \mathfrak{su}(2)$, with $\mathfrak{h}'' \subset \mathfrak{e}_7$ of maximal rank. The dimension of $\mathfrak{h}'$ is at least equal to the dimension of the regular orbits in $\mathfrak{p}$ which is 108. Hence $\dim \mathfrak{h}'' \geq 105$ and $\mathrm{rank}(\mathfrak{h}'')= 7$. Moreover $\mathfrak{h}''$ can be written as $\mathfrak{h}'' = \mathfrak{h}''' \oplus \mathbb{R}$, where $\mathfrak{h}'''$ has rank 6 and dimension greater then or equal to 104; but there is no such compact algebra.

Case (9): The dimension of $\mathfrak{sp}(3) + \mathfrak{su}(2)$ and the regular orbits in $\mathfrak{p}$ is 24. Hence $\mathfrak{h}' = \mathfrak{sp}(3) + \mathfrak{su}(2)$ which is a contradiction since $\mathfrak{h}'$ is not semisimple.

Case (10) is not possible since the only subgroup of Spin(9) acting transitively on the sphere S^{15} coincides with Spin(9): indeed from the classification of compact Lie groups acting transitively on the sphere (see e.g. [4, p. 179]), we see that the groups acting transitively on S^{15} are either SO(16) or contain SU(8) or Sp(4) as subgroups and can therefore not be subgroups of Spin(9).

Case (11): The dimension of $\mathfrak{su}(2) + \mathfrak{su}(2)$ and the regular orbits in $\mathfrak{p}$ is 6. Hence $\mathfrak{h}' = \mathfrak{su}(3) + \mathfrak{su}(3)$ which is a contradiction since $\mathfrak{h}'$ is not semisimple. q.e.d.

This also finishes the proof of Theorem 3.1. q.e.d.

Remark. Let (G, K) be an irreducible symmetric pair and $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ the corresponding decomposition. In the paper [13] the question which subgroups K' of K have the same orbits in $\mathfrak{p}$ as K is answered. If we had used the classification in [13] in the proof of our last lemma, we would only have had to consider the symmetric pair $(\mathrm{SO}(11), \mathrm{SO}(8) \times \mathrm{SO}(3))$.

By their result one can easily check that the isotropy representation of this space restricted to the subgroup Spin(7) $\times$ SO(3) does not leave a complex structure invariant. We have preferred to give a proof that is independent of [13] since most cases are rather elementary and some quite trivial. The same remark also applies to the next section, but notice that there the classification in [13] would leave us with more cases.

1. 4. Polar actions on quaternionic projective spaces

Theorem 4.1. *Let G be a compact and connected Lie group acting polarly on a quaternionic projective space $\mathbb{P}_n(\mathbb{H})$, $n \geq 2$. Then the action of G is orbit equivalent to the action induced by a product of k quaternion-Kähler symmetric spaces, where at least $k-1$ have rank one.*

Remark. The one-dimensional quaternionic projective space is isometric to the 4-dimensional sphere S^4 of constant curvature, so that we can restrict ourselves to the case $n \geq 2$. On the other hand the statement of the theorem does not hold when $n = 1$; indeed the action of T^1 on $\mathbb{R}^5 = \mathbb{R}^3 + \mathbb{R}^2$ given by the sum of the trivial and the standard representations respectively, induces a cohomogeneity-three polar action on S^4, which is not orbit equivalent to any action induced by products of quaternion-Kähler symmetric spaces.

Proof. We shall denote by Σ a section for the G-action. By Lemma 1B.1, we know that Σ is isometric to a sphere S^k, $2 \leq k \leq 4$, lying in a quaternionic line $\mathbb{P}_1(\mathbb{H})$ or to a real projective space $\mathbb{P}_k(\mathbb{R})$ for some $1 \leq k \leq n$.

We will first show that Σ cannot be isometric to a sphere S^k, $2 \leq k \leq 4$, lying in a fixed quaternionic line $\mathbb{P}_1(\mathbb{H})$. We will examine the cases $k = 2, 3, 4$ separately.

Lemma 4.2. *A section Σ is not isometric to S^4.*

Proof. If $\Sigma \cong S^4$, then the sections are quaternionic lines. Hence any regular orbit is a quaternion-Kähler submanifold of $\mathbb{P}_n(\mathbb{H})$, since the normal space at any point is quaternionic. Using the well known fact that any quaternion-Kähler submanifold is totally geodesic (see e.g. [2]), we have that any regular orbit is a quaternionic hyperplane. Since any two quaternionic hyperplanes intersect non-trivially, we have a contradiction. q.e.d.

Lemma 4.3. *A section Σ is not isometric to S^3.*

Proof. Assume that $\Sigma \cong S^3$. By Lemmas 1A.2 and 1A.4, we know that the set of singular points in Σ is a non-empty union of finitely many totally geodesics 2-spheres in Σ. We select a singular point $q \in \Sigma$ which lies in one of these 2-spheres, which we denote by S. Then Lemma 1B.3 tells us that $\exp(N_q(Gq))$ is a totally geodesic sphere. Hence it is contained in a projective line. It must then coincide with the projective line since $\dim(\exp(N_q(Gq))) > 3$. As in the proof of Lemma 4.2, it follows that Gq is a quaternionic hyperplane. Let q' be a point in the 2-sphere S that is not in Gq. Then Gq' is also a quaternionic hyperplane. It follows that Gq and Gq' intersect, which is a contradiction. q.e.d.

Lemma 4.4. *A section Σ is not isometric to S^2.*

Proof. We consider a section Σ which is isometric to a 2-sphere S^2 and we recall that the set of singular points in Σ is a union of a finitely many great circles by Lemmas 1A.2 and 1A.4. We take a point q which lies in only one of these singular circles, say S; note that a suitable neighborhood U of q in S consists of singular points with the same type of orbit. Then by the arguments used in the proof of Lemma 4.3, we see that $\dim(N_q(Gq)) = 3$ or 4.

Assume first that the codimension of Gq is four. Then as above, Gq is a quaternionic hyperplane. Since the intersection $Gq \cap \Sigma$ is an orbit of the Weyl group, which is finite, we may select a point $q' \in U$ such that $Gq \neq Gq'$. It follows that Gq' is also a quaternionic hypersurface and must meet Gq, which is of course a contradiction.

Now assume that $\dim(N_q(Gq)) = 3$. We first show that there exist more than one singular great circle in Σ. Indeed, suppose there is only one such circle. Then there are only two different types of orbits. The singular orbits have codimension three, hence a vanishing Euler characteristic. A regular orbit fibers over a singular one with typical fiber S^1, so that it has vanishing Euler characteristic too. This contradicts the fact that a maximal torus T of G has a fixed point on $\mathbb{P}_n(\mathbb{H})$, so that there must be an orbit with positive Euler characteristic.

We can therefore choose a point q' which belongs to at least two singular great circles S_1, S_2 in Σ. The codimension of Gq' will be at least one dimension less than the orbit of a point that lies only in one of the singular great circles. Hence the codimension is at least four. The dimension of $\exp_{q'}(N_{q'}(Gq'))$ is at most four by Lemma 1B.3. Hence it is four and $\exp_{q'}(N_{q'}(Gq'))$ is a projective line. As in the proof of Lemma 4.2 we see that Gq' is a quaternionic hyperplane. A quaternionic hyperplane meets a quaternionic line that it does not contain in at most one point. Hence Gq' meets Σ only in q'. Let q'' be the point antipodic to q' in Σ. It is clear that q'' also lies in both S_1 and S_2. Therefore the above arguments imply that Gq'' is a quaternionic hyperplane. It follows that the two different hyperplanes Gq' and Gq'' must intersect, which is a contradiction. This finishes the proof. q.e.d.

We now deal with the case where a section Σ is isometric to the real projective space $\mathbb{P}_k(\mathbb{R})$ for some $1 \leq k \leq n$. By Proposition 1B.2, there is an element in the identity component of the isometry group of $\mathbb{P}_n(\mathbb{H})$ that maps Σ into the standard embedding $\mathbb{P}_k(\mathbb{R}) \subset \mathbb{P}_n(\mathbb{R}) \subset \mathbb{P}_n(\mathbb{H})$ induced by $\mathbb{R}^k \subset \mathbb{R}^{n+1} \subset \mathbb{H}^{n+1}$.

We will consider $\mathbb{H}^{n+1}$ as a right $\mathbb{H}$-module. We can assume that G acts effectively by factoring out the kernel of the action and identify it with a subgroup of $\mathrm{PSp}(n) = \mathrm{Sp}(n)/\{I, -I\}$. A covering G' of G will therefore act linearly on $\mathbb{H}^{n+1}$, and we may enlarge it to the group $\tilde{G} = G' \times \mathrm{Sp}(1)$, where the group of unit quaternions $\mathrm{Sp}(1)$ acts on $\mathbb{H}^{n+1}$ by right multiplication. If we lift Σ to $\tilde{\Sigma} \subset \mathbb{R}^{n+1}$, then the same arguments as in the proof of Theorem 3.1 together with the fact that $\tilde{\Sigma}$ is totally real, hence orthogonal to the $\mathrm{Sp}(1)$-orbit through any of its point, will show that $\tilde{\Sigma}$ is a section for the $\tilde{G}$-action.

By Dadok's result (Theorem 1A.5) there is a symmetric pair (U, K) with Lie algebra decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$, so that the isotropy representation of K on $\mathfrak{p}$ is orbit-equivalent to the $\tilde{G}$-action on $\mathbb{H}^{n+1}$, after the identification $\mathfrak{p} \cong \mathbb{H}^{n+1}$. Moreover we can assume that $\tilde{G}$ is a subgroup of K.

We claim that we can choose the symmetric pair (U, K) to be a product of k quaternion-Kähler symmetric pairs.

We first assume that the $\tilde{G}$-module $\mathbb{H}^{n+1}$ is reducible as a real representation. We split $\mathbb{H}^{n+1} = V_1 \oplus \cdots \oplus V_k$, where the V_i's are $\tilde{G}$-irreducible subspaces; no factor is trivial, since the $\mathrm{Sp}(1)$-action has no fixed points. Moreover the V_i's are quaternionic subspaces because of the $\mathrm{Sp}(1)$-action. Since each $\tilde{G}$-action on V_i is polar, the corresponding symmetric space will split as $(U, K) = \Pi_i^k(U_i, K_i)$, according to Dadok's Theorem (see 1A.5), and the actions of $\tilde{G}$ and K_i on each V_i have the same orbits.

We note that the symmetric pair $(\mathrm{SO}(4p+1), \mathrm{SO}(4p))$ is not quaternionic Kähler, but $K = \mathrm{SO}(4p)$ contains the subgroup $H = \mathrm{Sp}(p)\mathrm{Sp}(1)$ which has the same orbits in $\mathfrak{p}$ as K; in this case we can replace the symmetric pair $(\mathrm{SO}(4p+1), \mathrm{SO}(4p))$ with the quaternion-Kähler symmetric pair $(\mathrm{Sp}(p+1), \mathrm{Sp}(p)\mathrm{Sp}(1))$. Our claim will then follow from the next lemma, which is the quaternionic version of Lemma 3.2.

Lemma 4.5. *Let (U, K) be an effective irreducible orthogonal symmetric pair, with canonical Lie algebra decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ and with $(\mathfrak{u}, \mathfrak{k}) \neq (\mathfrak{so}(4p+1), \mathfrak{so}(4p))$. We assume the following conditions are satisfied:*

1. *there exists a compact subgroup $A \subset K$ with $A \cong \mathrm{Sp}(1)$, such that $\mathrm{Ad}(A)|_{\mathfrak{p}}$ induces a quaternionic structure on $\mathfrak{p}$;*
2. *there exists a compact Lie subgroup $H \subset K$ such that $\mathrm{Ad}(H)$ and $\mathrm{Ad}(K)$ have the same orbits in $\mathfrak{p}$;*
3. *the $\mathrm{Ad}(H)$-action on $\mathfrak{p}$ normalizes $\mathrm{Ad}(A)|_{\mathfrak{p}}$.*

Then (U, K) is quaternion-Kähler.

Proof. We will denote by $\mathfrak{a}$ the Lie algebra of A; in $\mathfrak{a}$ we can find elements I, J, K such that $\{\mathrm{ad}(I), \mathrm{ad}(J), \mathrm{ad}(K)\}$ are complex structures on $\mathfrak{p}$ satisfying the multiplication rules of the standard generators of the quaternions.

First of all we show that $\mathrm{rank}(\mathfrak{u})=\mathrm{rank}(\mathfrak{k})$. Indeed, we may choose a maximal abelian subalgebra $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$, where $\mathfrak{t}_1$ is maximal abelian in $\mathfrak{k}$ and contains I, while $\mathfrak{t}_2$ is maximal abelian in $\mathfrak{p}$. If $\mathfrak{k}$ does not have maximal rank in $\mathfrak{u}$, then $\mathfrak{t}_2 \neq \{0\}$; but this contradicts the fact that $I \in \mathfrak{t}_1$ acts on $\mathfrak{p}$ as a complex structure, hence fixed point free.

We now consider the normalizer $\mathfrak{h}' = \mathfrak{n}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{u}$ that we split as $\mathfrak{h}' = \mathfrak{a} \oplus \mathfrak{z}(\mathfrak{a})$, where $\mathfrak{z}(\mathfrak{a})$ denotes the centralizer of $\mathfrak{a}$. The subalgebra $\mathfrak{h}'$ is contained in $\mathfrak{k}$, since for instance $\mathrm{ad}(I)|_{\mathfrak{p}}$ has trivial kernel. Moreover $\mathfrak{h} \subseteq \mathfrak{h}'$, where $\mathfrak{h}$ denotes the Lie algebra of H, as can be seen from property (3) and the fact that $\mathrm{Ad}(K)|_{\mathfrak{p}}$ is faithful.

We fix an $\mathrm{ad}(\mathfrak{k})$-regular element $v \in \mathfrak{p}$ and note that, by property (2),

$$[\mathfrak{k}, v] = [\mathfrak{h}, v] = [\mathfrak{h}', v].$$

Therefore, if we denote by $\mathfrak{c} = \mathfrak{z}_{\mathfrak{k}}(v)$ the centralizer of A in $\mathfrak{k}$, we have that

$$\mathfrak{k} = \mathfrak{h}' + \mathfrak{c}. \tag{$*$}$$

The claim of the lemma is equivalent to $\mathfrak{k} = \mathfrak{h}'$. We are now going to exclude all irreducible symmetric pairs $(\mathfrak{u}, \mathfrak{k})$ which are not quaternionic-Kähler and have $\mathrm{rank}(\mathfrak{u})=\mathrm{rank}(\mathfrak{k})$ by a case by case inspection. We now list all such irreducible pairs, indicating also the dimension of $\mathfrak{p}$ and the cohomogeneity r of the $\mathrm{ad}(\mathfrak{k})$-action on $\mathfrak{p}$, namely the rank of the symmetric pair. We can immediately rule out the case of adjoint orbits, simply because they all have positive Euler characteristic, while $\mathrm{Ad}(\mathrm{Sp}(1))$ acts freely on each $\mathrm{Ad}(K)$-orbit.

$n.$	$\mathfrak{u}$	$\mathfrak{k}$	$\dim\mathfrak{p}$	r
1	$\mathfrak{so}(2p+2q)$	$\mathfrak{so}(2p)+so(2q),\ 1\leq p,q\neq 2$	$4pq$	$min(2p,2q)$
2	$\mathfrak{so}(2p+2q+1)$	$\mathfrak{so}(2p)+\mathfrak{so}(2q+1),\ 1\leq p\neq 2, 2\leq q$	$2p(2q+1)$	$min(2p,2q+1)$
3	$\mathfrak{sp}(p+q)$	$\mathfrak{sp}(p)+\mathfrak{sp}(q), p,q>1$	$4pq$	$min(p,q)$
4	$\mathfrak{su}(p+q)$	$\mathbb{R}+\mathfrak{su}(p)+\mathfrak{su}(q), p,q\neq 2$	$2pq$	$min(p,q)$
5	$\mathfrak{so}(2n)$	$\mathbb{R}+\mathfrak{su}(n), n\geq 3$	$n(n-1)$	$[\frac{n}{2}]$
6	$\mathfrak{sp}(n)$	$\mathbb{R}+\mathfrak{su}(n)$	$n(n+1)$	n
7	$\mathfrak{e}_6$	$\mathbb{R}+\mathfrak{so}(10)$	32	2
8	$\mathfrak{e}_7$	$\mathfrak{su}(8)$	70	7
9	$\mathfrak{e}_7$	$\mathbb{R}+\mathfrak{e}_6$	54	3
10	$\mathfrak{e}_8$	$\mathfrak{so}(16)$	128	8
11	$\mathfrak{f}_4$	$\mathfrak{so}(9)$	16	1

We will use the same notation as above, and also refer to the arguments used in the proof of Lemma 3.2.

In cases (1)-(2), we put $\mathfrak{k}=\mathfrak{k}_1+\mathfrak{k}_2$, where $\mathfrak{k}_1=\mathfrak{so}(2p)$ and $\mathfrak{k}_2=\mathfrak{so}(2q)$ or $\mathfrak{so}(2q+1)$. The same argument as in the proof of Lemma 3.2 shows that any element $I\in\mathfrak{k}$ acting on $\mathfrak{p}$ as a complex structure must lie in either $\mathfrak{k}_1$ or $\mathfrak{k}_2$; since $\mathfrak{a}$ is not abelian, we must therefore have $\mathfrak{a}\subset\mathfrak{k}_1$ or $\mathfrak{a}\subset\mathfrak{k}_2$. Suppose $\mathfrak{a}\subset\mathfrak{k}_1$, so that $\mathfrak{h}'=\mathfrak{n}_{\mathfrak{k}_1}(\mathfrak{a})+\mathfrak{k}_2$; we should have then

$$\mathfrak{so}(2p)=\mathfrak{n}_{\mathfrak{so}(2p)}(\mathfrak{a})+\mathfrak{c}', \tag{4.1}$$

where $\mathfrak{c}'$ denotes the projection of $\mathfrak{c}$ into $\mathfrak{so}(2p)$. We now note that $p\geq 3$ and that the dimension of $\mathfrak{c}'$ is less than or equal to $\dim\mathfrak{so}(2p-2)$, so that (4.1) contradicts Theorem 1C.1.

In case (3), we put again $\mathfrak{k}_1=\mathfrak{sp}(p),\mathfrak{k}_2=\mathfrak{sp}(q)$, with $p,q\geq 2$. The same argument as in the previous case shows that $\mathfrak{a}$ must lie in either $\mathfrak{k}_1$ or $\mathfrak{k}_2$ (say $\mathfrak{k}_1$) and that we have a factorization

$$\mathfrak{sp}(p)=\mathfrak{n}_{\mathfrak{sp}(p)}(\mathfrak{a})+\mathfrak{c}',$$

where $\mathfrak{c}'$ denotes the projection of $\mathfrak{c}$ in $\mathfrak{sp}(p)$. Now, since $\mathfrak{sp}(p)$ is simple, the subalgebra $\mathfrak{n}_{\mathfrak{sp}(p)}(\mathfrak{a})$ is proper and the dimension of $\mathfrak{c}'$ is strictly less than $\dim\mathfrak{sp}(p)$. This contradicts the fact that there exists no nontrivial factorization of $\mathfrak{sp}(p)$; see Theorem 1C.1.

In case (4), we can argue in the same way as above, proving that $\mathfrak{a}$ must be contained in one of the simple factors of $\mathfrak{k}$, say in $\mathfrak{su}(p)$. Therefore again we should have

$$\mathfrak{su}(p) = \mathfrak{n}_{\mathfrak{su}(p)}(\mathfrak{a}) + \mathfrak{c}',$$

where the projection $\mathfrak{c}'$ can be shown to be a proper subalgebra. But Theorem 1C.1 shows that no such factorization exists if $p \geq 3$. Here we have used that $\mathfrak{n}_{\mathfrak{su}(p)}(\mathfrak{a})$ contains $\mathfrak{sp}(1)$ as a factor.

In case (5), we should have that $\mathfrak{a}$ is contained in the semisimple part of $\mathfrak{k}$, that is in $\mathfrak{su}(n)$. So, we should have the factorization $\mathfrak{su}(n) = \mathfrak{n}_{\mathfrak{su}(n)}(\mathfrak{a}) + \mathfrak{c}'$, where again $\mathfrak{c}'$ denotes the projection of $\mathfrak{c}$ in $\mathfrak{su}(n)$. Now $\mathfrak{c}'$ is a proper subalgebra, while Theorem 1C.1 shows that there is no factorization of $\mathfrak{su}(n)$ in which one of two factors contains $\mathfrak{sp}(1)$ as an ideal, as is the case for $\mathfrak{n}_{\mathfrak{su}(n)}(\mathfrak{a})$, unless $n = 2$.

In case (6), the subalgebra $\mathfrak{a}$ should be contained in the semisimple part of $\mathfrak{k}$, namely $\mathfrak{su}(n)$; but there is no element $I \in \mathfrak{su}(n)$ which acts on $\mathfrak{p}$ as a complex structure. Indeed, suppose there is such an element $I \in \mathfrak{su}(n)$. Since $\mathrm{Tr}(I) = 0$, there exist $v, w \in \mathbb{C}^n$ which are eigenvectors of I with different eigenvalues λ, μ respectively. Since the representation of $\mathfrak{k}$ on $\mathfrak{p}$ is given by $S^2(\mathbb{C}^n)$, we must have that $I^2(v\cdot v) = 4\lambda^2 v\cdot v = -v\cdot v$ hence $\lambda = \pm\frac{i}{2}$ and similarly $\mu = \pm\frac{i}{2}$. We can assume that $\lambda = \frac{i}{2}$ and $\mu = -\frac{i}{2}$. Then we have

$$I^2(v \cdot w) = (\lambda^2 + 2\lambda\mu + \mu^2) v \cdot w = 0,$$

which is a contradiction.

In case (7), the subalgebra $\mathfrak{a}$ should be contained in the semisimple part of $\mathfrak{k}$ which is $\mathfrak{so}(16)$; so $\mathfrak{h}' = \mathbb{R} + \mathfrak{n}_{\mathfrak{so}(16)}(\mathfrak{a})$. Since $\mathfrak{h}'' = \mathfrak{n}_{\mathfrak{so}(16)}(\mathfrak{a})$ does not coincide with $\mathfrak{so}(16)$ because $\mathfrak{so}(16)$ is simple, we should have a factorization $\mathfrak{so}(16) = \mathfrak{h}'' + \mathfrak{c}'$, where $\mathfrak{c}'$ denotes the projection of $\mathfrak{c}$ on $\mathfrak{so}(16)$. Since $\dim \mathfrak{c}' \leq \dim \mathfrak{c} = 16$, $\mathfrak{c}'$ would be a proper subalgebra of $\mathfrak{so}(16)$ and this contradicts Theorem 1C.1.

In the cases (8) and (10) the dimension of the regular orbits is equal to $\dim \mathfrak{k}$, so $\mathfrak{h}' = \mathfrak{k}$ which is a contradiction, since $\mathfrak{h}'$ is not simple.

Case (9) can be dealt by using similar arguments, since the simple Lie algebra $\mathfrak{e}_6$ does not admit any non-trivial factorization.

Case (11) is not possible by the same argument as in Lemma 3.2.

q.e.d.

So far we have proved that there are quaternion-Kähler symmetric pairs (U_i, K_i), $i = 1, \ldots, k$, so that we may identify $\mathbb{H}^{n+1}$ with $\mathfrak{p} =$

$\mathfrak{p}_1+\cdots+\mathfrak{p}_k$ and the group $\tilde{G}\cdot\mathrm{Sp}(1)$, which is a subgroup of $K=\Pi_{i=1}^k K_i$, has the same orbits in $\mathfrak{p}$ as K. Now, if we denote by $\phi : G' \to \mathrm{SO}(V_i)$ the homomorphism determined by the restriction of the G'-action on each V_i for $i=1,\ldots,k$, then we know that $\phi_i(G') \subseteq \mathrm{Sp}(n_i)$ where $4n_i = \dim_{\mathbb{R}} V_i$. Moreover, if we split K_i as $K_i = H_i \cdot \mathrm{Sp}(1)$ using the same notation as in §2, we have that $\phi_i(G')\cdot\mathrm{Sp}(1) \subseteq H_i \cdot \mathrm{Sp}(1)$, hence $\phi_i(G') \subseteq (H_i\cdot\mathrm{Sp}(1)) \cap \mathrm{Sp}(n_i) = H_i$. It then follows that $G' \subseteq \Pi_{i=1}^k H_i$ and $\tilde{G} \subseteq (\Pi_{i=1}^k H_i)\cdot\mathrm{Sp}(1)$, so that $\tilde{G}$ and $(\Pi_{i=1}^k H_i)\cdot\mathrm{Sp}(1)$ have the same orbits. Since the $\tilde{G}$-action is polar, by Proposition 2A.2, (2), the set $\{i \in \{1,\ldots,k\};\ \mathrm{rank}(U_i,K_i)=1\}$ has cardinality at least $k-1$. This concludes the proof of Theorem 4.1. q.e.d.

5. Polar actions on the Cayley Plane

In this section we will classify the polar actions on the Cayley projective plane $\mathbb{P}_2(\mathbb{O}) = \mathrm{F}_4/\mathrm{Spin}(9)$. More precisely, we will prove the following.

Theorem 5.1. *Let G be a connected, compact subgroup of* F_4 *acting polarly on $\mathbb{P}_2(\mathbb{O})$ with cohomogeneity $k \geq 1$. Then $k=1$ or $k=2$. Moreover all such subgroups are listed, up to conjugation, in the following table.*

$k=1$	$\mathrm{Sp}(1)\cdot\mathrm{Sp}(3)$	$T^1\cdot\mathrm{Sp}(3)$	$\mathrm{Sp}(3)$	$\mathrm{Spin}(9)$
$k=2$	$\mathrm{Spin}(8)$	$\mathrm{T}^1\cdot\mathrm{Spin}(7)$	$\mathrm{SU}(2)\cdot\mathrm{SU}(4)$	$\mathrm{SU}(3)\cdot\mathrm{SU}(3)$

Remark. The cohomogeneity-one actions on $\mathbb{P}_2(\mathbb{O})$ were classified by Iwata ([21]) and later by Kollross ([20]).

If Σ denotes a section for the G-action, we know from Proposition 1B.1 that Σ can be homeomorphic to a sphere S^k $(1 \leq k \leq 8)$ or to a real projective plane. Since the cohomogeneity-one case is already known, we will confine ourselves to the case when the cohomogeneity $k \geq 2$. The proof of Theorem 5.1 will follow from the results achieved in subsections 5A and 5B: in subsection 5A we will prove that a polar action can not have a section homeomorphic to a sphere and therefore $k=2$, while in subsection 5B we will deal with the case $k=2$, providing a complete classification of the subgroups of F_4 that act polarly on $\mathbb{P}_2(\mathbb{O})$.

Throughout the following we will keep the same notation as in Section 2B.

5A. The case when a section is isometric to a round sphere

We will start by establishing some useful lemmas which will allow us to show that a section Σ can not be isometric to a round sphere.

We assume in Lemmas 5A.1 to 5A.4 that the section is isometric to a round sphere S^k with $k \geq 2$.

Lemma 5A.1. *Given any point $p \in \mathbb{P}_2(\mathbb{O})$, the normal space to the orbit Gp is contained in the tangent space at p of the unique projective line containing a section through p.*

Proof. This follows immediately from Lemma 1B.3 q.e.d.

Lemma 5A.2. *The Weyl group W acting on Σ does not have a fixed point. Therefore every orbit meets Σ in at least two points.*

Proof. Assume there is a fixed point $p \in \Sigma$. Then $Gp \cap \Sigma = \{p\}$. Since the isotropy subgroup G_p acts transitively on the set of sections through p, we have that $Gp \cap \Sigma' = \{p\}$ for any section Σ' through p. We consider the submanifold $S = \exp_p(N_p(Gp))$, which is, by Lemma 1B.3, a totally geodesic sphere. Since S is a union over all sections through p, we have that $Gp \cap S = \{p\}$ and the orbit Gp and S meet transversally. It follows that the intersection number of Gp and S is equal to ± 1. Since $\mathbb{P}_2(\mathbb{O})$ only has non-trivial homology in dimensions 0, 8 and 16, we see that both Gp and S are eight dimensional. A class having intersection number ± 1 with a projective line S^8 is $\pm[S^8]$. Hence Gp represents the homology class $\pm[S^8]$.

The antipodal point $\bar{p}$ of p in Σ is also a fixed point of W; hence $G\bar{p}$ also represents $\pm[S^8]$. Therefore Gp and $G\bar{p}$ must intersect and since they are orbits, we have $Gp = G\bar{p}$. It follows that Gp meets Σ in more than one point and therefore p can not be a fixed point of W. q.e.d.

Lemma 5A.3. *There is a point p such that the orbit type of Gp is isolated, i.e., the slice representation at p has no nontrivial fixed points.*

Proof. The section Σ can be embedded as a hypersphere in some Euclidean space V, and the Weyl group W is then a finite reflection group acting on V. By the previous lemma, the origin is the only fixed point of W. It is then enough to consider a point $p \in \Sigma$ which lies in at least k mirrors that intersect in a line, where $k+1$ is the dimension of V. We refer to Chapters 4 and 5 of the book [17] for the facts on finite reflection groups needed here. q.e.d.

Lemma 5A.4. *We have that* $\dim \Sigma \leq 4$.

Proof. By Lemma 5A.3, we can fix a singular point p such that the slice representation N_p at p has no fixed nontrivial points in the normal space $N_p(Gp)$. By Lemma 5A.1, we know that $\dim N_p(Gp) \leq 8$. The slice representation N_p is polar by Lemma 1A.1 and orbit equivalent to the isotropy representation of a symmetric space X of dimension at most eight by Dadok's Theorem; see Theorem 1A.4. The symmetric space does not have a Euclidean factor since N_p has no fixed points. Hence the rank of X is at most four. The claim in the lemma now follows since $\dim \Sigma$ is equal to the rank of X. q.e.d.

So we are left with the case where the cohomogeneity of the G-action is at most four. The following lemma will be useful also later on. We do not assume in this lemma that the section Σ is a sphere.

Lemma 5A.5. *Let G be a compact connected Lie group acting on $\mathbb{P}_2(\mathbb{O})$ polarly with cohomogeneity k where $2 \leq k \leq 4$. Then either G has a fixed point or G is locally isomorphic to $SU(3) \times SU(3)$.*

Proof. We know that the regular orbits have dimension at least 12. The rank r of G is less than or equal to four, since $G \subset \mathrm{F}_4$; moreover $r \geq 2$, because $\dim G \geq 12$. We will now consider the possible ranks $r = 2, 3, 4$ separately. Throughout the following, we will use the classification of maximal subalgebras of maximal rank in the Lie algebra $\mathfrak{f}_4$ of F_4: these are (up to conjugation) $\mathfrak{so}(9)$, $\mathfrak{su}(3) + \mathfrak{su}(3)$ and $\mathfrak{sp}(1) + \mathfrak{sp}(3)$ (see [16, p. 414]).

(i) Case $r = 2$. Because of the dimension restriction, the only possibility is $G = \mathrm{G}_2$. We show that any subgroup G of F_4 that is isomorphic to G_2 has a fixed point in $\mathbb{P}_2(\mathbb{O})$. Indeed, it is known (see [1, chapter 16]) that F_4 acts irreducibly on $\mathbb{R}^{26}$ with induced action on the sphere S^{25} of cohomogeneity one and a singular orbit isometric to $\mathbb{P}_2(\mathbb{O})$. It is not difficult to see that the antipodal map interchanges the singular orbits. Hence they are both isometric to $\mathbb{P}_2(\mathbb{O})$. Now, the lowest dimensional irreducible representations of G_2 have dimensions $7, 14, 27$, so that G must fix some point p in S^{25}. One can clearly choose the point p in one of the singular orbits. Hence G is conjugate to a subgroup of $\mathrm{Spin}(9)$.

(ii) Case $r = 3$. If G is not simple, then a finite covering of G must be isomorphic to $\mathrm{T}^1 \times \mathrm{G}_2$ or $\mathrm{SU}(2) \times \mathrm{G}_2$ or to $\mathrm{SU}(2) \times \mathrm{Sp}(2)$. If $G \cong \mathrm{T}^1 \times \mathrm{G}_2$, then we know from the previous case, that there is an orbit Gp in $\mathbb{P}_2(\mathbb{O})$ of dimension at most one. By Lemma 5A.1, G must act with cohomogeneity 2 with a section isometric to $\mathbb{P}_2(\mathbb{R})$. But the slice representation at p has cohomogeneity greater than 2 as one can easily see using the fact that the irreducible representations of G_2 of dimension less or equal to 16 have dimensions 7 or 14.

If G is locally isomorphic to $\mathrm{SU}(2) \times \mathrm{G}_2$ and has no fixed point, then we would have an orbit Gp of dimension 2 or 3. If $\dim Gp = 3$, the slice representation would have cohomogeneity at least 3; if $\dim Gp = 2$, then the stabilizer G_p is locally isomorphic to $\mathrm{T}^1 \times \mathrm{G}_2$, and the slice representation has cohomogeneity 2 only if the T^1-factor acts trivially on the normal space $N_p(Gp)$ of dimension 14. But this would mean that the fixed point set of the T^1-action has dimension at least 14, hence T^1 would act trivially on $\mathbb{P}_2(\mathbb{O})$, since a maximal totally geodesic submanifold of $\mathbb{P}_2(\mathbb{O})$ is eight dimensional.

If G is locally isomorphic to $\mathrm{SU}(2) \times \mathrm{Sp}(2)$, we claim that the Lie algebra $\mathfrak{g}$ of G lies in some maximal subalgebra of maximal rank in $\mathfrak{f}_4$. Indeed, we recall that a Lie subalgebra is called an R-subalgebra if it is contained in a proper regular subalgebra, which by definition is normalized by some Cartan subalgebra, and is called a S-subalgebra otherwise. We know (see [23, p.207]) that a maximal S-subalgebra in $\mathfrak{f}_4$ is isomorphic to $\mathfrak{su}(2)$ or to $\mathfrak{su}(2)+\mathfrak{g}_2$, and therefore $\mathfrak{g}$ is a R-subalgebra. We claim that $\mathfrak{g}$ is contained in a maximal subalgebra of maximal rank of $\mathfrak{f}_4$. Indeed, $\mathfrak{g}$ is contained in some regular subalgebra $\mathfrak{h} \subset \mathfrak{f}_4$ of rank at least three; our claim is equivalent to saying that we can choose $\mathfrak{h}$ of maximal rank. If $\mathfrak{h}$ has rank three, then the regularity of $\mathfrak{h}$ implies that there exists $X \in \mathfrak{f}_4 \setminus \mathfrak{h}$ which centralizes $\mathfrak{h}$; hence $\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{z}_{\mathfrak{f}_4}(X)$ and $\mathfrak{z}_{\mathfrak{f}_4}(X)$ has maximal rank in $\mathfrak{f}_4$. Since $\mathfrak{su}(3)$ does not contain a subalgebra isomorphic to $\mathfrak{sp}(2)$, it follows that either $\mathfrak{g} \subset \mathfrak{so}(9)$ or $\mathfrak{g} \subset \mathfrak{sp}(1)+\mathfrak{sp}(3)$. If $\mathfrak{g} \subset \mathfrak{sp}(1)+\mathfrak{sp}(3)$, a finite covering of G is contained in $\mathrm{Sp}(1) \times \mathrm{Sp}(3)$, which has an orbit isometric to $\mathbb{P}_2(\mathbb{H})$ and G fixes a point in this orbit.

If G is simple, then $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(4)$ or $\mathfrak{sp}(3)$ or $\mathfrak{so}(7)$. The same arguments as in the previous paragraph using the classification of S-subalgebras of $\mathfrak{f}_4$ shows that $\mathfrak{g}$ must be contained in a maximal subalgebra of maximal rank, and an easy inspection of the possible immersions shows that $\mathfrak{g}$ must be conjugate to a subalgebra of $\mathfrak{so}(9)$.

(iii) Case $r = 4$. In this case $\mathfrak{g}$ can be conjugated into one of the standard maximal subalgebras of maximal rank. If G has no fixed point, then $\mathfrak{g}$ can be conjugated into $\mathfrak{su}(3) + \mathfrak{su}(3)$ or $\mathfrak{sp}(1) + \mathfrak{sp}(3)$. If $\mathfrak{g}$ is a proper subalgebra of $\mathfrak{su}(3) + \mathfrak{su}(3)$, then it is conjugate to $\mathfrak{su}(3) + \mathbb{R} + \mathfrak{su}(2) = (\mathfrak{su}(3) + \mathfrak{su}(3)) \cap \mathfrak{so}(9)$, so G would have a fixed point. If $\mathfrak{g}$ can be conjugated into $\mathfrak{sp}(1) + \mathfrak{sp}(3)$, then $\mathfrak{g}$ is a proper subalgebra, since $\mathrm{Sp}(1) \times \mathrm{Sp}(3)$ acts on $\mathbb{P}_2(\mathbb{O})$ with cohomogeneity one. So $\mathfrak{g}$ lies in $\mathfrak{sp}(1) + \mathfrak{sp}(1) + \mathfrak{sp}(2)$ or in $\mathfrak{sp}(1) + \mathbb{R} + \mathfrak{su}(3)$: in the first case G has a fixed point in the orbit of $\mathrm{Sp}(1) \times \mathrm{Sp}(3)$ isometric to a quaternionic projective plane; in the second case $\mathfrak{g} = \mathfrak{sp}(1) + \mathbb{R} + \mathfrak{su}(3)$ by dimension reasons and G acts by cohomogeneity four. Now G is a subgroup of the maximal subgroup $\mathrm{Sp}(1) \cdot \mathrm{Sp}(3)$, which has an eight dimensional orbit $\mathbb{P}_2(\mathbb{H})$, and G has orbits inside $\mathbb{P}_2(\mathbb{H})$ of dimension less than eight, contradicting Lemma 5A.1.

So $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(3) + \mathfrak{su}(3)$. q.e.d.

The next lemma will allow us to conclude the discussion of polar actions with round spheres as sections.

Lemma 5A.6. *If the G-action is polar and has a fixed point, then the section is not a sphere of dimension greater than or equal to* 2.

Proof. Let p be a fixed point for the G-action. We assume that the section is a sphere of dimension greater than or equal to 2. We consider the cut locus of the point p, which is a G-invariant projective line; we claim that this projective line L is a G-orbit. Indeed, if Σ is a section, then $L = G(L \cap \Sigma)$; but the intersection $L \cap \Sigma$ consists of only one point, so that L is a G-orbit. Now, the only subgroup of $\mathrm{Spin}(9)$ which acts transitively on S^8 is the full $\mathrm{Spin}(9)$, which acts on $\mathbb{P}_2(\mathbb{O})$ by cohomogeneity one. q.e.d.

Summing up, we have proved the following.

Proposition 5A.7. *Let G be a connected compact Lie group acting isometrically and polarly on $\mathbb{P}_2(\mathbb{O})$. Then a section is not isometric to a round sphere. In particular, the cohomogeneity of the G-action is one or two.*

Proof. Indeed, by Lemmas 5A.5 and 5A.6, we know that a section can not be isometric to a round sphere, at least if $\mathfrak{g}$ is not isomorphic to $\mathfrak{su}(3) + \mathfrak{su}(3)$. On the other hand, if $\mathfrak{g}$ is conjugate to the maximal subalgebra $\mathfrak{su}(3) + \mathfrak{su}(3)$, then we know from Proposition 2B.4 that a section of the G-action is the real projective plane. q.e.d.

5B. Polar actions of cohomogeneity 2

By the previous subsection, we know that any polar action on $\mathbb{P}_2(\mathbb{O})$ must have cohomogeneity 1 or 2 and the sections homeomorphic to a circle or the real projective plane $\mathbb{P}_2(\mathbb{R})$.

Moreover, by Lemma 5A.5, we know that either G has a fixed point or it is locally isomorphic to $\mathrm{SU}(3) \times \mathrm{SU}(3)$. We will now discuss the case where G has a fixed point p, finishing the proof of Theorem 5.1.

Throughout the following we will suppose that G is a subgroup of Spin(9); the slice representation of G at p is polar with cohomogeneity two. Furthermore, the cut locus of p, which is a projective line S^8, is acted on by G with cohomogeneity one since a section intersects S^8 in a great circle.

Let r denote the rank of G. We will distinguish between the two cases $r \leq 3$ and $r = 4$. We will exclude the possibility $r \leq 3$, while for $r = 4$ we will find all the cases enumerated in Theorem 5.1.

(i) Case $r \leq 3$. Simply by dimension counting, we have that G is locally isomorphic to G_2, $\mathrm{T}^1 \times \mathrm{G}_2$, $\mathrm{SU}(2) \times \mathrm{G}_2$, SU(4), Spin(7), Sp(3). But, $\mathrm{SU}(2) \times \mathrm{G}_2$ and Sp(3) admit no non-trivial homomorphisms into Spin(9), since they have no almost faithful representations in dimension less than or equal to 9.

If G is locally isomorphic to G_2 or to $\mathrm{T}^1 \times \mathrm{G}_2$, then the slice representation at p would have cohomogeneity at least 3, as one can see by looking at the lowest dimensional representations of G_2.

If G is locally isomorphic to SU(4), we look at the lowest dimensional representations of SU(4) and we see that the action of G on S^8 must have a fixed point; this means that SU(4) is conjugate to a subgroup of Spin(8) and it should act with the same orbits as Spin(8), since Spin(8) acts on $\mathbb{P}_2(\mathbb{O})$ by cohomogeneity two. Since a regular Spin(8)-orbit is $\mathrm{Spin}(8)/\mathrm{G}_2$ (see [21]), we would have a factorization $\mathrm{Spin}(8) = \mathrm{SU}(4)\mathrm{G}_2$ and this contradicts Theorem 1C.1 (see also [22, p. 228]).

If G is locally isomorphic to Spin(7), the same arguments as in the previous case show that it has a fixed point in S^8 and therefore it is conjugate to a subgroup of Spin(8); then the slice representation, restricted to G, splits as $\mathbb{R}^8 \oplus \mathbb{R}^8$ and it has cohomogeneity three (a regular isotropy subgroup is isomorphic to SU(3)).

(ii) Case $r = 4$. From the classification of maximal subalgebras of maximal rank in $\mathfrak{so}(9)$ (see [16, p.412]), we see that $\mathfrak{g}$ can be conjugated into one of the following maximal subalgebras:

$$\mathfrak{so}(8), \mathbb{R} + \mathfrak{so}(7), \mathfrak{su}(2) + \mathfrak{su}(4), \mathfrak{su}(2) + \mathfrak{su}(2) + \mathfrak{sp}(2).$$

We will now consider each case separately, proving that $\mathfrak{g}$ must coincide with one of the first three subalgebras. This will then conclude the proof of Theorem 5.1.

If $\mathfrak{g}$ can be conjugated into $\mathfrak{so}(8)$, we can suppose $G \subseteq \mathrm{Spin}(8)$. Then G and Spin(8) have the same orbits; since a regular isotropy subgroup of Spin(8) is G_2 and since there is no non-trivial factorization of Spin(8) with one factor G_2 (see Theorem 1C.1), we have that $G = \mathrm{Spin}(8)$.

If $\mathfrak{g}$ is conjugate to a subalgebra of $\mathbb{R} + \mathfrak{so}(7)$, we have by [13] that $\mathfrak{g} = \mathbb{R} + \mathfrak{so}(7)$. Alternatively, this can be seen in the following way. Suppose that G is a subgroup of $\mathrm{SO}(2) \cdot \mathrm{Spin}(7)$ with the same orbits as $\mathrm{SO}(2) \cdot \mathrm{Spin}(7)$ and also acting with cohomogeneity two. We may write $\mathfrak{g} = \mathbb{R} + \mathfrak{g}_1$, where $\mathfrak{g}_1 \subseteq \mathfrak{so}(7)$. We also recall that the Spin(7)-action on $\mathfrak{p} \cong \mathbb{R}^{16}$ splits as $\mathbb{R}^8 + \mathbb{R}^8$ and is transitive on the unit sphere in each factor. So, $\mathfrak{g}_1$ corresponds to a connected Lie subgroup of Spin(7), acting transitively on the sphere S^7. The only such rank-three subgroups are $\mathrm{T}^1 \cdot \mathrm{Sp}(2)$ and $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$ (see [4, p.179]). But $G = \mathrm{T}^1 \cdot \mathrm{T}^1 \cdot \mathrm{Sp}(2)$ has dimension 12 and can be ruled out. If $G = \mathrm{T}^1 \cdot \mathrm{SU}(4)$, then the action has cohomogeneity three and can also be ruled out.

If $\mathfrak{g}$ is conjugate to a proper subalgebra of $\mathfrak{su}(2) + \mathfrak{su}(4)$, a simple inspection of the dimensions of the maximal subalgebras of maximal rank in $\mathfrak{su}(4)$ and $\mathfrak{sp}(2)$ (see [16]) shows that $\dim \mathfrak{g} \leq 12$, a contradiction.

If $\mathfrak{g}$ is conjugate to a subalgebra of $\mathfrak{h} = \mathfrak{su}(2) + \mathfrak{su}(2) + \mathfrak{sp}(2)$, then a simple inspection of the dimensions of maximal subalgebras of maximal rank in $\mathfrak{sp}(2)$ shows that $\mathfrak{g}$ must coincide with $\mathfrak{h}$. We now show that the corresponding subgroup H of Spin(9) does not act polarly with cohomogeneity two. Indeed, we consider the restriction of the spin representation Δ of $\mathfrak{so}(9)$ to $\mathfrak{h}$ and we suppose that such a representation is polar with cohomogeneity two. Now, if $\Delta|_{\mathfrak{h}}$ is irreducible, then the action of H on $\mathfrak{p} \cong \mathbb{R}^{16}$ would have the same orbits as the isotropy representation of an irreducible symmetric space of rank two and dimension 16: the only such isotropy representation are given by $\mathrm{SO}(2) \times \mathrm{SO}(8)$ acting on $\mathbb{R}^2 \otimes \mathbb{R}^8$ and $\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(4))$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^4$ and none of these groups contains H. So $\Delta|_{\mathfrak{h}}$ should be reducible; since H is supposed to act with cohomogeneity two, then $\mathfrak{p}$ should split as $\mathbb{R}^8 + \mathbb{R}^8$, where the action of $\mathfrak{h}$ on each $\mathbb{R}^8$ factor has a kernel given by one factor $\mathfrak{su}(2)$ and $\mathfrak{su}(2) + \mathfrak{sp}(2)$ acts on $\mathbb{R}^8$ in the standard way. But this representation is easily seen not to be polar, applying the criterium given in Lemma 2A.3. q.e.d.

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