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SHORTENING COMPLETE PLANE CURVES

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Abstract

It is shown that the curve shortening problem for a complete, properly embedded curve has a solution for all time provided the initial curve divides the plane into two regions of infinite area.

Introduction

In the curve shortening problem one studies the evolution of a plane curve γ_0 under the equation

$$\frac{\partial \gamma}{\partial t} = k\nu \tag{0.1}$$

where ν is a choice of unit normal and k is the curvature of $\gamma(\cdot,t)$ with respect to ν. When γ_0 is closed and convex, $\gamma(\cdot,t)$ shrinks under (0.1). In fact, it was proved by Gage and Hamilton [12] that a convex closed curve shrinks to a round point in finite time. Subsequently Grayson [13] showed that a closed embedded curve evolves into a convex curve before it shrinks to a point. Thus the curve shortening problem for closed embedded curves is completely solved. In this paper we study (0.1) for complete, noncompact embedded curves. Very few results in this direction are known. In [9] and [10] Ecker and Huisken studied the mean curvature flow for hypersurfaces which are either local or global graphs. Among other things they proved that the mean curvature flow for entire graphs exists for all time. Problem (0.1) is the mean curvature flow for curves. Their result asserts that for a complete initial curve which is a graph over a straight line, (0.1) has a solution for all $t \geq 0$.

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In general, hypersurfaces in $\mathbb{R}^{n+1} (n \geq 2)$ which are not graphs could develop singularities under the mean curvature flow. The notion of a viscosity solution has been used successfully to study the problem, see, Chen-Giga-Gota [6] and Evans-Spruck [11]. However, classical solutions are possible when $n = 1$. In this paper we shall establish a long time existence result which embraces a large class of complete curves. To state it let's recall that according to a fundamental theorem of Jordan a complete, properly embedded curve γ_0 divides the plane into two regions Ω_1 and Ω_2.

Theorem. *Let $\gamma_0 : \mathbb{R} \to \mathbb{R}^2$ be a locally Lipschitz continuous, complete, properly embedded curve dividing the plane into two regions Ω_1 and Ω_2. Suppose that the areas of Ω_1 and Ω_2 are infinite. Then there exists a solution $\gamma(\cdot, t)$ of (0.1) for all $t > 0$ such that*

(a) for each $t, \gamma(\cdot, t)$ is a smooth, complete, properly embedded curve which divides the plane into two regions of infinite area; and

(b) $\gamma(\cdot, t)$ converges to γ_0 uniformly on every compact subset of $\mathbb{R}$ as $t \downarrow 0$.

After the completion of this work, we learn that Polden [19], established the long time existence of (0.1) when γ_0 is, say, C^1-asymptotic to two distinct halflines. A short proof of this result can be found in Huisken [16] where an interesting distance comparison principle for (0.1) is discovered.

One does not expect (0.1) admits a solution for all time when one of the regions divided by the initial curve has finite area. In fact, we shall show that for complete curves with finite total absolute curvature the life span of the solution is precisely equal to the minimum of the areas of Ω_1 and Ω_2 divided by π. In a previous report [8] we established the theorem under a more stringent condition, namely, each $\Omega_i, i = 1, 2$, contains a semi-infinite strip. Now, we have found that this more general result follows by a slight modification of the original argument. We briefly describe the proof of the theorem, which is separated into two parts. First we establish the long time existence of a unique solution of (0.1) when γ_0 has finite total absolute curvature, a result interesting in itself. To prove this one needs to derive apriori estimates of the flow. We observe that in this case the total absolute curvature is nonincreasing in time. Outside a bounded region, each of the two ends of the curve can be represented as a graph over some semi-infinite axis. As a consequence one may use the localized estimates in [10] to bound the curvature. To control the curvature in the bounded region we employ a method of Hamilton [15], (see also Zhu [20]), which relies on an isoperimetric quantity and a blow up argument. We point out that the isoperimetric quantity is not defined for noncompact curves. We shall introduce a geometric quantity which is no longer isoperimetric. However, just like the isoperimetric quantity it is bounded positively from below, and this is sufficient for our purpose. One may use the ideas in Angenent [5] and Oaks [18] to bound the curvature as well. Anyway, we shall follow Hamilton's approach and we have tried to make it a self-contained presentation. After proving the long time existence for curves with finite total absolute curvature, we turn to the second half of the proof of the theorem, namely, we use the result in the first half to construct approximating solutions and foliations transversal to a given γ_0 satisfying the hypothesis of the theorem. The construction yields "gradient estimates" of the flow from which curvature estimates follow. The idea of using foliations to study (0.1) can be found in Grayson [14]. It was further developed in [5] and [18] to obtain gradient estimates which are local in time. We adopt the idea and apply it to a global setting. After obtaining the curvature estimates, long time existence can be deduced by an approximation argument.

This paper contains five sections. In Sections 1 to 3 we carry out the first part of the proof of the theorem. The second half of the proof occupies Section 4. Finally, Section 5 contains a remark on the uniqueness of solution to (0.1).

In concluding the introduction we state a useful lemma for (0.1) in case of graphs. In fact, when $\gamma(\cdot,t)$ is given by the graph of functions $y(x,t)$ over some interval in x, the equation is, up to tangential diffeomorphisms, equivalent to the following quasilinear parabolic equation:

$$y_t = \frac{y_{xx}}{1+y_x^2}. \tag{0.2}$$

We consider (0.2) in $Q_\delta = [R_1+\delta, R_2-\delta]\times[0,T), \delta \geq 0$ small and set $\gamma = \sup_{[R_1,R_2]} |y_x(x,0)|$. Then we have

Lemma 0.1. *Let y be a solution of (0.2) in Q_0. Then the following hold:*

(a) For each ε, $0<\varepsilon\leq\varepsilon_0=\gamma\delta$,

$$-\varepsilon - \frac{\pi\gamma}{4\varepsilon}t \;\leq\; y(x,t)-y(x,0) \;\leq\; \varepsilon + \frac{\pi\gamma}{4\varepsilon}t \;,$$

for all (x,t) in Q_δ.

(b) There exists a constant C_1 depending on δ such that

$$\sup_{Q_\delta} |y_x| \le C_1 \max\{ \sup_{[R_1,R_2]} |y_x(\cdot,0)|, \sup_{Q_0} |y|\}.$$

(c) There exists a constant C_2 depending on $\delta, \mu, \sup_{Q_0} |y|$, and $\sup_{Q_{\delta/2}} |y_x|$ such that

$$\sup_{[R_1+\delta,R_2-\delta]\times[\mu,T]} |y_{xx}| \le C_2 ;$$

there exists a constant C_3 depending on $\delta, \sup_{Q_0} |y|, \sup_{Q_{\delta/2}} |y_x|$ and $\sup_{[R_1,R_2]} |y_{xx}(\cdot,0)|$ such that

$$\sup_{Q_\delta} |y_{xx}| \le C_3 .$$

Proof. (a) For any x_0, α and C, the "grim reaper"

$$\bar{y}(x,t) = C^{-1} \log \sec C(x-x_0) + \alpha + Ct$$

is a solution of (0.2) in $(x_0 - \pi/2C, x_0 + \pi/2C) \times \mathbb{R}$ which blows up at the endpoints. When $0 < \varepsilon \le \varepsilon_0$, and $x_0 \in [R_1+\delta, R_2-\delta]$, the grim reaper with $C = \pi\gamma/4\varepsilon$ and $\alpha = y(x_0,0)+\varepsilon$ lies above $y(x,t)$ at $t=0$. By the maximum principle it follows that

$$y(x_0,t) \le y(x_0,0) + \varepsilon + \pi\gamma t/4\varepsilon$$

for $t \in [0,T)$. Similarly, one can prove the other inequality in (a).

The proof of (b) is standard. By looking at the maximum of the auxillary function $\varphi(x) y_x^2 + \lambda y^2$, where φ vanishes at R_1, R_2 and is equal to 1 in $[R_1+\delta, R_2-\delta]$, one can show that for large λ depending only on δ the maximum cannot attain in $(R_1,R_2)\times(0,T]$. Hence

$$\sup_{Q_\delta} |y_x| \le \max\Big\{ \sup_{[R_1,R_2]} \big(|y_x(\cdot,0)| + |y(\cdot,0)|\big), \sup_{\substack{t\in[0,T) \\ i=1,2}} |y(R_i,t)| \Big\}$$

and (b) follows.

Finally, (c) is a standard result from the theory of quasilinear parabolic equations, see, for instance, Chapter 6 in [17]. q.e.d.

1. Preliminary

Let γ_0 be a complete, noncompact properly embedded curve in the plane. Consider the curve shortening problem

$$(1.1) \qquad \begin{cases} \dfrac{\partial \gamma}{\partial t} &= k\nu \ , \\ \gamma(\cdot,0) &= \gamma_0 \ , \end{cases}$$

where ν is a choice of unit normal of $\gamma(\cdot,t)$, and k is the curvature of $\gamma(\cdot,t)$ with respect to ν. By a solution of (1.1) in $[0,T), T>0$, we mean a continuous map γ from $\mathbb{R}\times[0,T)$ to $\mathbb{R}^2$ which is smooth in $\mathbb{R}\times(0,T)$ and satisfies (a) γ solves the equation in (1.1) in $\mathbb{R}\times(0,T)$, (b) for each $t\in[0,T), \gamma(\cdot,t)$ is a complete, noncompact, properly embedded curve and (c) $\gamma(\cdot,t)$ converges uniformly to γ_0 on every compact subset of $\mathbb{R}$ as t tends to zero. First we state a local existence result.

Lemma 1.1. *Let γ_0 be a complete, noncompact, properly embedded curve in $C^{k+4,\alpha}$ (when parametrized by arc-length) where k is a nonnegative integer and $\alpha \in$ (0,1). There exists $T>0$ such that (1.1) admits a solution γ such that $\gamma(\cdot,t)-\gamma_0(\cdot)$ belongs to*

$$\tilde{C}^{k+2,\alpha}(\{\cdot\}\times[0,T]).$$

Moreover, if γ' is another solution of (1.1) with $\gamma'(\cdot,t)-\gamma_0(\cdot)$ in

$$\tilde{C}^{2}(\{\cdot\}\times[0,T]),$$

then γ' and γ coincide.

Here and in the following we use $C^{k,\alpha}(\mathbb{R})$ and $\tilde{C}^{k,\alpha}(\mathbb{R}\times[0,T])$ to denote the Banach spaces which consist of functions with finite $C^{k,\alpha}$-norm and "parabolic" $\tilde{C}^{k,\alpha}$-norm respectively, and we call a complete and noncompact curve γ_0 belongs to $C^{k+1,\alpha}$ if its derivative with respect to the arclength has finite $C^{k,\alpha}$-norm.

The proof of Lemma 1.1 is standard. We look for solution in the form

$$(1.2) \qquad \gamma(\cdot,t)=\gamma_0+u(\cdot,t)\nu_0,$$

where u satisfies

$$(1.3) \qquad \frac{\partial u}{\partial t}=k\nu\cdot\nu_0, \quad u(\cdot,0)\equiv 0.$$

It can be shown that (1.3) is equivalent to (1.1) up to tangential diffeomorphisms. For a similar situation see [10]. The equation in (1.3) is a quasilinear parabolic equation. By a straightforward computation, we have

$$u_t = a(x,u,u_x)u_{xx} + b(x,u,u_x) \ , \tag{1.3'}$$

where

$$a(x,u,u_x) = \frac{(1-k_0u)^2}{[(1-k_0u)^2+u_x^2]^2} \ , \quad \text{and}$$

$$b(x,u,u_x) = \frac{(1-k_0u)(k_0 - 2k_0^2u + k_0^3u^2 + 2k_0u_x^2 + k_{0x}uu_x)}{[(1-k_0u)^2+u_x^2]^2}.$$

Here k_0 is the curvature of γ_0. Notice that we have used the assumption that γ_0 is parametrized by arc-length.

Setting

$$\mathcal{F}[u] = a(x,u,u_x)u_{xx} + b(x,u,u_x) - u_t \ ,$$

$\mathcal{F}$ maps $\tilde{C}^{k+2,\alpha}(\mathbb{R}\times[0,T])$ to $\tilde{C}^{k,\alpha}(\mathbb{R}\times[0,T])$ for any $T>0$. Let $u_0 = b(x,0,0)t = k_0(x)t$ be the first approximation of the solution. The Fréchet derivative of $\mathcal{F}$ at u_0 is given by

$$D\mathcal{F}[u_0]v = A(x,t)v_{xx} + B(x,t)v_x + C(x,t)v - v_t,$$

where A, B and C are bounded in $\tilde{C}^{k,\alpha}(\mathbb{R}\times[0,T])$ provided γ_0 belongs to $C^{k+4,\alpha}(\mathbb{R})$. Moreover, if we restrict T to satisfy, say,

$$T < \frac{1}{2}\inf_x \frac{1}{1+k_0^2(x)} \ ,$$

it is also uniformly parabolic. According to linear parabolic theory (see, e.g. [17]),

$$D\mathcal{F}[u_0]v = f$$

establishes an isomorphism between $\tilde{C}^{k+2,\alpha}(\mathbb{R}\times[0,T])$ and $\tilde{C}^{k,\alpha}(\mathbb{R}\times[0,T])$. By the implicit function theorem (1.3)' has a unique solution in $\tilde{C}^{k+2,\alpha}(\mathbb{R}\times[0,T])$ provided $\mathcal{F}[u_0] = \mathcal{F}[k_0t]$ is sufficiently small in $\tilde{C}^{k,\alpha}(\mathbb{R}\times[0,T])$. But, this can be achieved by restricting to a smaller T, if necessary. So we have found a solution of (1.3)' in small time. Going back to the curve shortening problem, it is now routine to verify that requirements (a), (b) and (c) in the definition of a solution of (1.1) are all satisfied, and hence we have proved the solvability of (1.1) in small time. Finally we note that the uniqueness assertion in this lemma is a consequence of the maximum principle.

It is well-known that once exists, the solution γ is smooth for $t > 0$. By Lemma 1.1 we can extend the solution until at a time $t_{\max} \leq \infty$ where further extension is impossible. Clearly when $t_{\max}$ is finite, the $C^{4,\alpha}$-norm of $\gamma(\cdot, t)$ must blow-up. We claim that the curvature of $\gamma(\cdot, t)$ must blow up also. For, if $k(\cdot, t)$ is uniformly bounded in $[0, t_{\max})$, fix t_1 sufficiently close to $t_{\max}$ so that (1.2), with γ_0 replaced by $\gamma(\cdot, t_1)$, holds for $t \in [t_1, t_{\max})$, and we may also assume $\frac{\partial}{\partial x}k(\cdot, t_1)$ is bounded from the interior estimates of quasilinear parabolic theory. However, since the coefficients of (1.3)' are uniformly bounded and it is uniformly parabolic, by quasilinear parabolic theory we can bound all higher derivatives of u in $\mathbb{R} \times [t_1, t_{\max})$ in a uniform manner. Hence, in particular, the $C^{4,\alpha}$-norm of $\gamma(\cdot, t)$ cannot blow up, contradicting the assumption. So we have

Lemma 1.2. *Let γ be the maximal solution of (1.1) in $[0, t_{\max})$ with γ_0 in $C^{4,\alpha}$. If $t_{\max}$ is finite, then $k(\cdot, t)$ must become unbounded as t tends to $t_{\max}$.*

For closed curves this statement was first proved in [12].

In the remaining part of this section and the next two sections we shall be concerned with curves with flat ends. To be precise let's call a complete, regular C^1-curve γ *asymptotically flat* if $\nabla\gamma/|\nabla\gamma|$ converges uniformly to T_2 (resp. T_1) as $p \to \infty$ (resp. - ∞). T_1 and T_2 are called *asymptotic directions* and the angle Θ, $0 \leq \Theta \leq \pi$, between T_1 and T_2 the *asymptotic angle* of γ. For our study it is without loss of generality to assume $T_2 = (1, 0)$ and set

$$\Gamma(\Theta) = \left\{ \gamma : \begin{array}{l} \gamma \text{ is a complete, noncompact embedded, asym-} \\ \text{ptotically flat curve with } T_2 = (1,0) \text{ and} \\ \text{asymptotic angle } \Theta \end{array} \right\}.$$

Lemma 1.3. *Let γ be a solution of (1.1) where γ_0 belongs to $\Gamma(\Theta)$ for some Θ. Then for each $t, \gamma(\cdot, t)$ also belongs to $\Gamma(\Theta)$.*

Proof. Write one end of γ_0 as the graph of $y(x, 0)$ over $[R, \infty)$ for some $R > 0$. Then (0.2) is satisfied and the lemma follows from Lemma 0.1(a) and (b). q.e.d.

When the asymptotic angle is less than π it is clear that the distance between the ends become unbounded eventually. However, when Θ is equal to π, it may happen that the distance is finite. For any γ in $\Gamma(\pi)$ we may represent its ends over some $[R,\infty)$ as graphs of two functions $y^{\pm}, y^{+}(x) > y^{-}(x)$, and set

$$d(\gamma) = \varliminf_{x\to\infty} \{y^{+}(x) - y^{-}(x) \ : \ x \in [R,\infty)\}.$$

Notice by virtue of Lemma 1.3 we may write $d(t) = d(\gamma(\cdot,t))$ when $\gamma(\cdot,t)$ is a solution of (1.1).

Lemma 1.4. *Let $\gamma(\cdot,t)$ be a solution of (1.1). When $\Theta = \pi$ and $d(0) > 0$, $d(t) \geq d(0)$.*

Proof. Fix $T \in (0, t_{\max})$. For any $\delta > 0$, we shall show that $d(T) \geq (1-\delta)d(0)$. To see this we first observe that our assumption on γ_0 implies that, for any small $\varepsilon > 0$, there exists a large R such that

$$y^{+}(x,0) - y^{-}(x,0) \ \geq \ (1-\frac{\delta}{4})d(0), \quad x \geq R$$

and

$$|y_x^{\pm}| \ \leq \ \varepsilon \ \text{ in } \ [R,\infty)\times[0,T].$$

The function $z = y^{+} - y^{-}$ satisfies

$$z_t = \frac{z_{xx}}{1+y_x^{-2}} + \frac{y_{xx}^{+}(y_x^{+}+y_x^{-})}{(1+y_x^{+2})(1+y_x^{-2})} z_x.$$

By Lemma 0.1(a), we have

$$\begin{aligned} z(x,t) &= y^{+}(x,t) - y^{-}(x,t) \\ &\geq (1-\frac{\delta}{2})d(0) \end{aligned}$$

for (x,t) in $[R,\infty)\times[0,t_1]$ for some small t_1. Using (c) of the same lemma $y_{xx}^{\pm}$ are uniformly bounded in $[R,\infty)\times[t_1,T]$. Thus, the above equation can be written in the form

$$z_t = (1+a(x,t))z_{xx} + b(x,t)z_x \ , \quad (x,t)\in[R,\infty)\times[t_1,T] \ ,$$

where

$$|a(x,t)| \ , \ |b(x,t)| = O(\varepsilon)$$

and

$$z(x, t_1) \geq \left(1 - \frac{\delta}{2}\right) d(0). \tag{1.4}$$

Let φ be a cut-off function which is equal to one for $x \geq R + 1$ and vanishes for $x \leq R$. The function $w = \varphi z$ satisfies

$$w_t = w_{xx} + f(x, t),$$

where

$$f(x, t) = a\varphi z_{xx} + b\varphi z_x - \varphi_{xx} z - 2\varphi_x z_x.$$

Notice that φ_{xx} vanishes for $x \geq R + 1$. Henceforth,

$$\left|f(x, t)\right| = O(\varepsilon) \tag{1.5}$$

for $t \geq t_1$ and $x \geq R + 1$. Now we can write

$$w(x, t) = \int_{-\infty}^{\infty} \Gamma(x-y; t)\varphi(y)z(y, t_1)dy + \int_{t_1}^{t}\int_{-\infty}^{\infty} \Gamma(x-y; t-s)f(y, s)dyds$$

where $\Gamma(x, t) = (4\pi t)^{-1/2}\exp(-|x|^2/4t)$ is the heat kernel. It follows from (1.4) and (1.5) that for sufficiently large R',

$$w \geq (1 - \delta)d(0)$$

in $[t_1, T] \times [R', \infty)$. In particular,

$$d(T) \geq (1 - \delta)d(0).$$

q.e.d.

2. A geometric quantity

Let γ be a complete, noncompact, properly embedded C^1-curve in the plane and let Ω be one of the two domains bounded by γ. Consider the class

$$\mathcal{L}(\Omega) = \{\ell : \ell \text{ is a line segment in } \bar{\Omega} \text{ with endpoints on } \gamma\}.$$

Letting L be the length of ℓ and A the area (of the compact set) bounded by ℓ and γ, we define

$$G(\ell, \Omega) = L^2\left(\frac{1}{A} + 1\right)$$

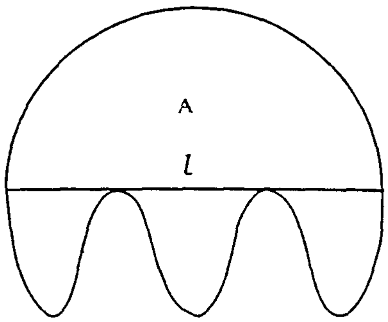
and set, when $\mathcal{L}(\Omega)$ is non-empty,

$$g(\Omega) = \inf\{G(\ell,\Omega) : \ell \in \mathcal{L}(\Omega)\}.$$

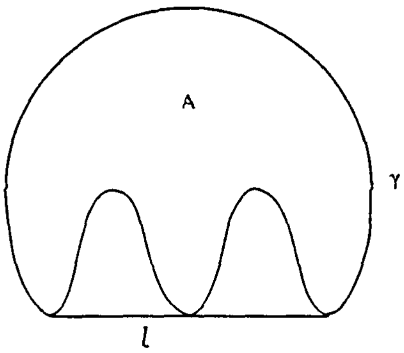
Notice that $\mathcal{L}(\Omega)$ does not contain a line segmemt with interior lying inside Ω if and only if $\mathbb{R}^2\backslash\Omega$ is convex.

Lemma 2.1. *Let $g(\Omega)$ be finite and let ℓ be a minimizing line segment. Then the interior of ℓ is contained in Ω and, moreover, ℓ is transversal to γ at the endpoints.*

Proof. Let ℓ be minimizing. If it touches γ in its interior, it either touches γ from inside (Figure a) or outside (Figure b).



(a)



(b)

In the first case it is not hard to find a sub-segment ℓ' which together with γ bounds a larger area. This is impossible. In the second case we may decompose ℓ into two admissible, disjoint line segments ℓ_1 and ℓ_2. In obvious notation, we have

$$\begin{aligned} A &= A_1 + A_2, \\ L &= L_1 + L_2, \\ L_1^2\left(\frac{1}{A_1}+1\right) &\geq L^2\left(\frac{1}{A}+1\right), \text{ and} \\ L_2^2\left(\frac{1}{A_2}+1\right) &\geq L^2\left(\frac{1}{A}+1\right). \end{aligned}$$

Rewriting the last two inequalities as

$$\left(\frac{A+1}{A}\right)\left(\frac{A_1}{1+A_1}\right) \leq \frac{L_1^2}{L^2},$$

$$\left(\frac{A+1}{A}\right)\left(\frac{A_2}{1+A_2}\right) \leq \frac{L_2^2}{L^2}$$

and adding up, we have

$$\begin{aligned}\frac{A+1}{A}\left(\frac{A_1}{1+A_1}+\frac{A_2}{1+A_2}\right) &\leq 1-\frac{2L_1L_2}{L^2} \\ &< 1.\end{aligned}$$

However, by minimizing the left-hand side of this inequality for A_1 over [0, A], we see that the minimum is equal to 1. Again this is impossible. Hence ℓ cannot touch γ in its interior.

Now it is easy to see that ℓ must be transversal to γ. For, if not, we can always perturb ℓ slightly within $\mathcal{L}(\Omega)$ to decrease L^2 rapidly while keeping $1/A$ little changed. q.e.d.

Next we want to write down some necessary conditions for a minimizing line segment. For simplicity, let's place the segment in the vertical line $x = x_0$. By Lemma 2.1 near its top and bottom γ can be represented as the graphs of y^+ and y^- respectively, where $y^+(x) > y^-(x)$ for x near x_0. For $a, b \in \mathbb{R}$, let $l(\mu)$ be the line segment connecting $(x_0 + a\mu, y^-(x_0 + a\mu))$ and $(x_0 + b\mu, y^+(x_0 + b\mu))$ where μ is small. The square of the length of $l(\mu)$ is given by

$$L^2(\mu) = (b-a)^2\mu^2 + \big(y^+(x_0+b\mu) - y^-(x_0+a\mu)\big)^2$$

and the area bounded by γ and $l(\mu)$ is

$$\begin{aligned}A(\mu) &= A(0) + \int_{x_0}^{x_0+b\mu} y^+(x)dx - \int_{x_0}^{x_0+a\mu} y^-(x)dx \\ &\quad - \frac{1}{2}\big(y^+(x_0+b\mu) + y^-(x_0+a\mu)\big)(b-a)\mu.\end{aligned}$$

We have

$$
\begin{aligned}
\frac{dL}{d\mu} &= by_x^+(x_0) - ay_x^-(x_0), \\
\frac{d^2L}{d\mu^2} &= \frac{1}{L}\big[(b-a)^2 + L(b^2y_{xx}^+(x_0) - a^2y_{xx}^-(x_0))\big], \\
\frac{dA}{d\mu} &= \frac{b+a}{2}(y^+(x_0) - y^-(x_0)), \quad \text{and} \\
\frac{d^2A}{d\mu^2} &= ab(y_x^+(x_0) - y_x^-(x_0))
\end{aligned}
$$

at $\mu = 0$. The function $\log L^2(\mu)\big(\frac{1}{A(\mu)} + 1\big)$ attains minimum at $\mu = 0$. Therefore,

$$
\begin{aligned}
0 &= \frac{d}{d\mu}\log L^2(\mu)\Big(\frac{1}{A(\mu)} + 1\Big) \\
&= \frac{2}{L}\frac{dL}{d\mu} + \Big(\frac{1}{A+1} - \frac{1}{A}\Big)\frac{dA}{d\mu} \\
&= \frac{2}{L}\big(by_x^+(x_0) - ay_x^-(x_0)\big) \\
&\quad - \frac{b+a}{2}\frac{1}{A(A+1)}\big(y^+(x_0) - y^-(x_0)\big)
\end{aligned} \tag{2.1}
$$

and

$$
\begin{aligned}
0 &\le \frac{d^2}{d\mu^2}\log L^2(\mu)\Big(\frac{1}{A(\mu)} + 1\Big) \\
&= \frac{2}{L}\frac{d^2L}{d\mu^2} - \frac{2}{L^2}\Big(\frac{dL}{d\mu}\Big)^2 + \Big(\frac{1}{A+1} - \frac{1}{A}\Big)\frac{d^2A}{d\mu^2} \\
&\quad + \Big[\frac{1}{A^2} - \frac{1}{(A+1)^2}\Big]\Big(\frac{dA}{d\mu}\Big)^2 \\
&= \frac{2}{L^2}\big[(b-a)^2 + L(b^2y_{xx}^+(x_0) - a^2y_{xx}^-(x_0))\big] \\
&\quad - \frac{2}{L^2}\big(by_x^+(x_0) - ay_x^-(x_0)\big)^2 \\
&\quad - \frac{ab}{A(A+1)}\big(y_x^+(x_0) - y_x^-(x_0)\big) \\
&\quad + \Big[\frac{1}{A^2} - \frac{1}{(A+1)^2}\Big]\Big(\frac{b+a}{2}\Big)^2\big(y^+(x_0) - y^-(x_0)\big)^2
\end{aligned} \tag{2.2}
$$

at $\mu = 0$. Taking $a = -b = 1$ in (2.1), we deduce

$$
y_x^+(x_0) = -y_x^-(x_0). \tag{2.3}
$$

Substituting (2.3) back to (2.1) yields

$$y_x^+(x_0) = \frac{L^2}{4}\frac{1}{A(A+1)}. \tag{2.4}$$

Taking $a = b = 1$ in (2.2), by (2.3) and (2.4) we have

$$\begin{aligned} 0 &\le \frac{2}{L}\big(y_{xx}^+(x_0) - y_{xx}^-(x_0)\big) - \frac{2}{L^2}\big(y_x^+(x_0) - y_x^-(x_0)\big)^2 \\ &- \frac{1}{A(A+1)}\big(y_x^+(x_0) - y_x^-(x_0)\big) \\ &+ \Big[\frac{1}{A^2} - \frac{1}{(A+1)^2}\Big]\big(y^+(x_0) - y^-(x_0)\big)^2 \\ &\le \frac{2}{L}\big(y_{xx}^+(x_0) - y_{xx}^-(x_0)\big) - \frac{L^2}{A^2(A+1)^2} + L^2\Big[\frac{1}{A^2} - \frac{1}{(A+1)^2}\Big] \\ &= \frac{2}{L}\big(y_{xx}^+(x_0) - y_{xx}^-(x_0)\big) + \frac{2L^2}{A}\,\frac{1}{(A+1)^2}\ , \end{aligned}$$

that is,

$$-\frac{1}{L}\big(y_{xx}^+(x_0) - y_{xx}^-(x_0)\big) \le \frac{L^2}{A}\,\frac{1}{(A+1)^2}\ . \tag{2.5}$$

Lemma 2.2. *Let γ be a solution of (1.1) in $[0,T)$. Suppose for a choice of $\Omega(t), \mathcal{L}(\Omega(t))$ is non-empty and $g(\Omega(t))$ attains minimum for each t in $(0,t_1]$, $t_1 < T$. Then either $g(\Omega(t_1)) \ge \pi$ or $g(\Omega(t))$ is increasing in $(0,t_1]$.*

Notice that by requiring ν points inward or outward, there are two exactly choices of $\Omega(t)$.

Proof. Let ℓ be a minimizing line segment in $\mathcal{L}(\Omega(t_1))$. As before we may assume that it is contained in the vertical line $x = x_0$. Near the top and bottom of ℓ , $\gamma(\cdot,t)$ is represented as $y^+(x,t)$ and $y^-(x,t)$ respectively. The line segment connecting $(x_0, y^+(x_0,t))$ and $(x_0, y^-(x_0,t))$ belongs to $\mathcal{L}(\Omega(t))$ for t near t_1. Denote its length by $L(t)$ and the area bounded by $\gamma(\cdot,t)$ and this segment by $A(t)$. By (0.2), the formula

$$\frac{dA}{dt} = -\pi - \tan^{-1} y_x^+ + \tan^{-1} y_x^-$$

and (2.3), we have

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=t_1} \log L^2(t)\Big(\frac{1}{A(t)}+1\Big)\\
&= \frac{2}{L}\big(y_t^+(x_0,t_1)-y_t^-(x_0,t_1)\big)+\Big(\frac{1}{A+1}-\frac{1}{A}\Big)\frac{dA}{dt}\Big|_{t=t_1}\\
&= \frac{2}{L}\big(y_{xx}^+(x_0,t_1)-y_{xx}^-(x_0,t)\big)(1+y_x^{+2})^{-1}\\
&\quad+\frac{1}{A(A+1)}\big(\pi-2\tan^{-1}y_x^+(x_0,t_1)\big).
\end{aligned}$$

Use of (2.5) and (2.4) leads to

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=t_1} \log L^2(t)\Big(\frac{1}{A(t)}+1\Big)\\
&\geq -\frac{2L^2}{A}\frac{1}{(A+1)^2}\frac{1}{1+y_x^{+2}}+\frac{\pi}{A(A+1)}\\
&\quad-\frac{2}{A(A+1)}y_x^+(x_0,t_1)\\
&\geq -\frac{2L^2}{A}\Big(\frac{1}{A+1}\Big)^2+\frac{\pi}{A(A+1)}-\frac{L^2}{2A^2}\frac{1}{(A+1)^2}\\
&= \frac{1}{A(A+1)}\Big[\pi-\frac{L^2}{A+1}\Big(2+\frac{1}{2A}\Big)\Big]\\
&\geq \frac{1}{A(A+1)}\big(\pi-g(\Omega(t_1))\big).
\end{aligned}$$

Now, if $g(\Omega(t_1)) < \pi$, this inequality implies that $L^2(t)\big(A^{-1}(t)+1\big)$ increases in $[t_1-\delta, t_1]$ for some $\delta > 0$. It follows that $g(\Omega(t))$ is strictly increasing in this interval. The same argument can now be applied to $t_1-\delta$ until we conclude that $g(\Omega(t))$ is increasing in $(0, t_1]$. q.e.d.

3. Curves with finite total absolute curvature

In this section we consider complete curves with finite total absolute curvature and establish a long time existence result for these curves. To begin with we first show that the total absolute curvature does not increase along the flow.

Lemma 3.1. *Let γ be a solution of (1.1) in $[0,T)$ where $\gamma(\cdot,t)$ is uniformly bounded in C^2-norm. Suppose that the total absolute curvature of γ_0 is finite. Then the total absolute curvature of $\gamma(\cdot,t)$ is nonincreasing in t.*

Proof. It suffices to show that the total absolute curvature of $\gamma(\cdot,t)$ is finite for $t>0$ and

$$\int_{\gamma(\cdot,t)} |k(\cdot,t)|ds \;\le\; \int_{\gamma_0} |k_0|ds. \tag{3.1}$$

To prove this we first observe that a complete, noncompact embedded curve with finite total absolute curvature is asymptotically flat, and consequently it belongs to some $\Gamma(\Theta)$. By Lemma 1.3 $\gamma(\cdot,t)$ belongs to the same $\Gamma(\Theta)$. Consider $\Theta\in[0,\pi)$ first. We arrange the coordinates so that the two ends of $\gamma(\cdot,t)$ eventually make angles with the positive x-axis equal to $\Theta/2$ and $-\Theta/2$ respectively.

Let

$$\eta(x,y) \;=\; \eta_N(x,y)=\begin{cases}1 & |x| \le N,\\ 0 & |x| \ge N+1,\end{cases}$$

be a smooth non-negative function. We compute

$$\begin{aligned}
&\frac{d}{dt}\int_{\gamma(\cdot,t)} \eta(\gamma(\cdot,t))(\varepsilon^2+k^2(\cdot,t))^{\frac12}ds\\
&= \int_\gamma \nabla\eta\cdot\gamma_t(\varepsilon^2+k^2)^{\frac12}ds+\int_\gamma \eta k k_t(\varepsilon^2+k^2)^{-\frac12}ds\\
&\quad -\int_\gamma \eta k^2(\varepsilon^2+k^2)^{\frac12}ds\\
&= \int_\gamma \nabla\eta\cdot\nu k(\varepsilon^2+k^2)^{\frac12}ds+\int_\gamma \eta k(\varepsilon^2+k^2)^{-\frac12}(k_{ss}+k^3)ds\\
&\quad -\int_\gamma \eta k^2(\varepsilon^2+k^2)^{\frac12}ds\\
&= \int_\gamma \nabla\eta\cdot\nu k(\varepsilon^2+k^2)^{\frac12}ds-\int_\gamma \nabla\eta\cdot T k(\varepsilon^2+k^2)^{-\frac12}k_s ds\\
&\quad -\int_\gamma \eta\varepsilon^2(\varepsilon^2+k^2)^{-\frac32}k_s^2ds-\int_\gamma \eta\varepsilon^2k^2(\varepsilon^2+k^2)^{-\frac12}ds\\
&= \int_\gamma \nabla\eta\cdot\nu k(\varepsilon^2+k^2)^{\frac12}ds+\int_\gamma \left[(\nabla^2\eta\cdot T)\cdot T+\nabla\eta\cdot\nu k\right](\varepsilon^2+k^2)^{\frac12}ds\\
&\quad -\int_\gamma \eta\varepsilon^2(\varepsilon^2+k^2)^{-\frac32}k_s^2ds-\int_\gamma \eta\varepsilon^2k^2(\varepsilon^2+k^2)^{-\frac12}ds\\
&\le \int_\gamma \left[2\nabla\eta\cdot\nu k+(\nabla^2\eta\cdot T)\cdot T\right](\varepsilon^2+k^2)^{\frac12}ds.
\end{aligned}$$

Using the fact that k is uniformly bounded and η vanishes outside the set $S_N = \{N \leq |x| \leq N+1\}$, we have

$$\frac{d}{dt}\int_\gamma \eta(\gamma)(\varepsilon^2+k^2)^{\frac{1}{2}}ds \leq C_1 \int\limits_{\gamma(\cdot,t)\cap S_N} (\varepsilon^2+k^2)^{\frac{1}{2}}ds.$$

As the length of the portion of $\gamma(\cdot,t)$ inside S_N is uniformly bounded, we may let ε go to zero to get

$$\begin{aligned}&\int\limits_{\gamma(\cdot,t)} \eta_N(\gamma(\cdot,t))|k(\cdot,t)|ds \\ &\qquad \leq \int\limits_{\gamma_0} \eta_N(\gamma_0)|k_0|ds + C_1\int_0^t\int_{\gamma\cap S_N}|k|dsdt.\end{aligned} \tag{3.2}$$

Now letting $N \to \infty$ we conclude that the total absolute curvature of $\gamma(\cdot,t)$ is uniformly bounded in $[0,T)$. By applying the dominated convergence theorem to the right-hand side of (3.2) as $N \to \infty$ we deduce (3.1) from (3.2).

When $\Theta = \pi$, (3.1) can be proved in a similar way if we place $\gamma(\cdot,t)$ in such a way that the two ends are eventually parallel to the x-axis. q.e.d.

According to Lemma 1.1, for a complete $C^{4,\alpha}$ initial curve, (1.1) admits a unique maximal solution in $[0,t_{\max})$. Now we give a sufficient condition for long time existence.

Proposition 3.2. *Let $\gamma(\cdot,t)$ be the maximal solution of (1.1) in $[0,t_{\max})$ where γ_0 is in $C^{4,\alpha}$ and satisfies the following conditions*

(a) γ_0 has finite total absolute curvature,

(b) γ_0 divides the plane into two domains with infinite areas, and

(c) $d(\gamma_0)$ is positive when $\Theta = \pi$.

Then $t_{\max} = \infty$. Furthermore, the curvature of $\gamma(\cdot,t)$ is uniformly bounded for all time.

We shall treat the case $d(\gamma_0) = 0$ later.

Proof. In view of Lemmas 1.1 and 1.2 it is sufficient to show that $k(\cdot,t)$ is uniformly bounded in $[0,t_{\max})$. We shall argue by contradictiion. Suppose this is not true. Then we can find $\{t_i\}, t_j \uparrow t_{\max}$, and $\{p_j\} \subset \mathbb{R}$ such that

$$|k|(p,t) \le |k|(p_j,t_j)+1, \quad \text{for all } 0 \le t \le t_j \quad \text{and} \quad p \in \mathbb{R},$$

where

$$|k|(p_j,t_j) \to \infty \quad \text{as} \quad j \to \infty.$$

Notice that $\{p_j\}$ exists since $k(\cdot,t)$ is bounded. Now we adopt a blow-up argument from Altschuler [1]. Define a sequence of complete curves $\{\gamma_j\}$ by

$$\gamma_j(p,t) = \Phi_j\big(\gamma(p_j+\varepsilon_j p, t_j+\varepsilon_j^2 t)\big) : \mathbb{R} \times [-t_j/\varepsilon_j^2, (t_{\max}-t_j)/\varepsilon_j^2] \longrightarrow \mathbb{R}^2$$

where

$$\begin{aligned} &\varepsilon_j = 1/|k|(p_j,t_j) \quad , \quad \text{and} \\ &\Phi_j(X) = (X-\gamma(p_j,t_j))/\varepsilon_j. \end{aligned}$$

Each γ_j satisfies

$$\frac{\partial \gamma_j}{\partial t} = k_j \nu,$$

where the curvature of γ_j, k_j, satisfies

$$|k_j(p,t)| \le 1+\varepsilon_j, \quad (p,t) \in \mathbb{R} \times [-t_j/\varepsilon_j^2, (t_{\max}-t_j)/\varepsilon_j^2],$$

and

$$|k_j(0,0)| = 1$$

for all j. By a standard argument we may choose a subsequence (still denoted by γ_j) which converges smoothly to some γ_∞ on every compact subset of $\mathbb{R}^2 \times (-\infty,0]$, and γ_∞ satisfies

$$\frac{\partial \gamma_\infty}{\partial t} = k_\infty \nu \quad \text{in} \quad \mathbb{R}^2 \times (-\infty,0],$$

with $|k_\infty| \le 1$ and $|k_\infty(0,0)| = 1$.

It is clear that $\gamma_\infty(\cdot,t)$ remains to be complete, noncompact and embedded. We claim that it is strictly convex. To prove this we first show that all inflection points of $\gamma_\infty(\cdot,t)$, if they ever exist, must be degenerate. To see this we follow the proof of Lemma 3.1 to get

$$\begin{aligned}
&\frac{d}{dt}\int_{\gamma_j}\eta(\gamma_j(\cdot,t))(\varepsilon^2+k_j(\cdot,t))^{\frac{1}{2}}ds\\
(3.3)\qquad &=\int_{\gamma_j}\left[\nabla\eta\cdot\nu k_j+(\nabla^2\eta\cdot T)\cdot T+\nabla\eta\cdot\nu k_j\right](\varepsilon^2+k_j^2)^{\frac{1}{2}}ds\\
&\quad-\int_{\gamma_j}\eta\varepsilon^2(\varepsilon^2+k_j^2)^{-\frac{3}{2}}k_{js}^2ds-\int_{\gamma_j}\eta\varepsilon^2k_j^2(\varepsilon^2+k_j^2)^{-\frac{1}{2}}ds.
\end{aligned}$$

Suppose at some $\bar{t}\le 0,\gamma_\infty(\cdot,\bar{t})$ has a non-degenerate inflection point at $\bar{p}$. Since γ_j tends to γ_∞ on every compact subset of $\mathbb{R}^2\times[\bar{t}-1,\bar{t}]$, for sufficiently large j_0 and sufficiently small $\delta>0$ the following holds : For each $t\in[\bar{t}-\delta,\bar{t}]$ and $j\ge j_0$, there exists $p_j(t)$ near $\bar{p}$ such that

$$k_j(p_j(t),t)=0,$$

and

$$\Big|\frac{\partial k_j}{\partial s}(p_j(t),t)\Big|\ge\frac{1}{2}\Big|\frac{\partial k_\infty}{\partial s}(\bar{p},\bar{t})\Big|>0.$$

Now, the second term on the right-hand side of (3.3) can be estimated as follows :

$$\begin{aligned}
&-\int_{\gamma_j}\eta\varepsilon^2(\varepsilon^2+k_j^2)^{-\frac{3}{2}}k_{js}^2ds\\
&\qquad\le\ -\int_{p_j(t)-\varepsilon}^{p_j(t)+\varepsilon}c_1\varepsilon^2\left[\varepsilon^2+c_2\big(s-p_j(t)\big)^2\right]^{-\frac{3}{2}}ds\\
&\qquad\le\ -c_1'\int_{-c_2'\varepsilon}^{c_2'\varepsilon}\varepsilon^2(\varepsilon^2+s^2)^{-\frac{3}{2}}ds\\
&\qquad=\ \frac{-2c_1'c_2'}{(1+c_2'^2)^{\frac{1}{2}}},
\end{aligned}$$

where the positive constants c_1,c_1',c_2,c_2' depend only on $\big|\partial k_\infty/\partial s(\bar{p},\bar{t})\big|$. On the other hand, the first term on the right-hand side of (3.3) is less than

$$C\left(\varepsilon+\int_{\gamma_j(\cdot,t)\cap\{N\le|x|\le N+1\}}|k_j|ds\right)$$

Thus by integrating (3.3) from $\bar{t}-\delta$ to $\bar{t}$, we have

$$\begin{aligned}\int_{\gamma_j(\cdot,\bar{t})} \eta(\varepsilon^2+k_j^2)^{\frac{1}{2}}ds \leq \int_{\gamma_j(\cdot,\bar{t}-\delta)} &\eta(\varepsilon^2+k_j^2)^{\frac{1}{2}}ds \\ &+ C\left(\varepsilon\delta + \int_{\bar{t}-\delta}^{\bar{t}}\int_{\gamma_j(\cdot,t)\cap\{N\leq|x|\leq N+1\}} |k_j|ds\right) \\ &- \frac{2c_1'c_2'\delta}{(1+c_2'^2)^{\frac{1}{2}}}\,.\end{aligned}$$

Letting $N \to \infty$ it follows from Lemma 3.1 and the dominated convergence theorem that

$$\int_{\gamma_j(\cdot,\bar{t})} (\varepsilon^2+k_j^2)^{\frac{1}{2}}ds \leq \int_{\gamma_j(\cdot,\bar{t}-\delta)} (\varepsilon^2+k_j^2)^{\frac{1}{2}}ds - \frac{2c_1'c_2'\delta}{(1+c_2'^2)^{\frac{1}{2}}},$$

and, after letting $\varepsilon \to 0$,

$$\int_{\gamma_j(\cdot,\bar{t})} |k_j|ds \leq \int_{\gamma_j(\cdot,\bar{t}-\delta)} |k_j|ds - \frac{2c_1'c_2'\delta}{(1+c_2'^2)^{\frac{1}{2}}}.$$

In other words,

$$\int_{\gamma(\cdot,t_j+\varepsilon_j^2\bar{t})} |k|ds - \int_{\gamma(\cdot,t_j+\varepsilon_j^2(\bar{t}-\delta))} |k|ds \leq -\frac{2c_1'c_2'\delta}{(1+c_2'^2)^{\frac{1}{2}}}.$$

However, the left-hand side of this inequality tends to zero as $j \to \infty$, – contradiction. Hence γ_∞ cannot have any non-degenerate inflection points.

Now we can show that γ_∞ is strictly convex. For, if not, then for some t we can find p_1, p_2 such that $k(p_1,t)k(p_2,t) < 0$, and there is a point q between p_1 and p_2 satisfying $k(q,t) = 0$. However, by applying Sturm comparison theorem (cf: [2]) to the equation satisfied by the curvature k we infer that for $t' > t, t' - t$ small, γ_∞ has a non-degenerate inflection point near q. This is impossible. So γ_∞ cannot have any inflection points at any time. In fact, by the strong maximum principle we know that k_∞ is always positive.

It is easy to see that the total curvature of γ_∞ is always equal to π. For, suppose γ_∞ belongs to some $\Gamma(\Theta)$ for some $\Theta \neq \pi$. We can represent γ_∞ as graphs over the entire x-axis. Using the interior curvature estimate from [10] we have

$$|k_\infty(p,t)| \leq \frac{C}{t-t'} \quad , \quad p \in (-\infty, +\infty) \quad ,$$

where C depends on the gradient of γ_∞, which is uniformly bounded for all t. Letting $t' \to -\infty$ we get $k_\infty(0,0) = 0$, contradicting $k_\infty(0,0) = 1$. Therefore, $\gamma_\infty(\cdot,t)$ is the graph of $y = y(x,t)$ over $(a,b) \times (-\infty, 0]$, $-\infty \leq a < b \leq \infty$, and we have

$$\lim_{x \to a} y_x(x,t) = -\infty \quad , \quad \lim_{x \to b} y_x(x,t) = \infty.$$

As a result, for given $\varepsilon > 0$, we can choose a horizontal line segment l_ε with endpoints on $\gamma_\infty(\cdot,0)$ such that

$$\frac{L_\varepsilon^2}{A_\varepsilon} \leq \frac{\varepsilon}{2} \ ,$$

where L_ε is the length of l_ε, and A_ε is the area bounded by l_ε and $\gamma_\infty(\cdot,0)$. Since $\gamma_j(\cdot,0) = \Phi_j(\gamma(\cdot,t))$ converges to $\gamma_\infty(\cdot,0)$ on compact sets, there is a choice of $\Omega(t_j)$ such that

$$\begin{aligned} g(\Omega(t_j)) &\leq (\varepsilon_j L_\varepsilon)^2 \left(\frac{1}{\varepsilon_j^2 A_\varepsilon} + 1\right) \\ &\leq \frac{\varepsilon}{2} + \varepsilon_j^2 L_\varepsilon^2 \\ &< \varepsilon \end{aligned}$$

for large j.

We claim that we can trace back in time from t_j to 0 during which $g(\Omega(t))$ is well-defined and attained. For, by the definition of g, the asymptotic flattness of $\gamma(\cdot,t)$ and Lemma 1.4 we know that once we fix $\varepsilon < \min\{\pi, d_0^2\}$, $g(\Omega(t))$ attains minimum as long as $\mathcal{L}(\Omega(t))$ is non-empty. Clearly, this holds for those $t < t_j$ with $t_j - t$ small. We claim that in fact $\mathcal{L}(\Omega(t))$ is non-empty for t in $(0, t_j]$. For, if at some t^*, $\mathcal{L}(\Omega(t))$ is empty, then $\mathbb{R}^2 \backslash \Omega(t^*)$ is strictly convex. Since (1.1) preserves convexity, it implies that $k(\cdot, t_j)$ is negative, contradicting $k_\infty(0,0) = 1$. Therefore now we can apply Lemma 2.2 to conclude that $g(\Omega(t))$ is increasing in $(0, t_j]$. Nevertheless, if we set $\tilde{t} = \frac{1}{2} t_{\max}$ when $t_{\max}$ is finite and $\tilde{t} = 1$ when $t_{\max} = \infty$, and take $\varepsilon < \min\{\pi, d_0^2, g(\Omega(\tilde{t}))\}$, then

$$\begin{aligned} \varepsilon &> g(\Omega(t_j)) \\ &> g(\Omega(\tilde{t})) \quad , \end{aligned}$$

contradiction holds. So $k(\cdot, t)$ must be uniformly bounded in $[0, t_{\max})$, and the proof of Proposition 3.2 is completed. q.e.d.

Next we treat the case $d(\gamma_0) = 0$.

Proposition 3.3. *Let $\gamma(\cdot, t)$ be a solution of (1.1) where $\gamma_0 \in C^{4,\alpha}$ satisfies:*

(a) γ_0 has finite total absolute curvature,

(b) γ_0 divides the plane into two domains with infinite areas, and

(c) γ_0 belongs to $\Gamma(\pi)$ and $d(\gamma_0) = 0$.

Then $t_{\max} = \infty$. When (b) is replaced by

(b)' one domain bounded by γ_0 has finite area A,

then under (a), (b)' and (c), $t_{\max} = A/\pi$.

In both cases, the curvature of $\gamma(\cdot, t)$ becomes unbounded as $t \uparrow t_{\max}$.

When $d(\gamma_0) = 0$, the geometric quantity g defined in Section 2 may tend to zero. Hence the proof of Proposition 3.2 cannot apply to the present situation. We need to modify g.

Let us consider the case where (a), (b) and (c) hold first. Fix $T > 0$. For sufficiently large R the two ends of γ_0 are given by the graphs of y^+ and $y^-, y^+ > y^-$, in $[R, \infty)$, and

$$\frac{dy^{\pm}}{dx}(x, 0) \to 0 \quad \text{as} \quad x \to \infty. \tag{3.4}$$

We may also assume

$$A_R(t) \geq \delta > 0 \ , \quad \text{all} \quad t \in [0, T) \ , \tag{3.5}$$

where $A_R(t)$ is the area bounded by γ_0 and $x = R$, and δ is some positive number. We can do this because

$$\frac{dA_R(t)}{dt} \ \geq \ -2\pi \ ,$$

for any sufficiently large R, and so

$$A_R(t) \geq A_R(0) - 2\pi T \geq \delta > 0$$

for large R.

Using the facts that $\gamma(\cdot,0)$ intersects $x = R', R' \geq R$, at exactly two points and the number of intersections does not increase in time (see Angenent [2]), it follows from (3.5) that during $[0,T)$ $\gamma(\cdot,t)$ can be represented as the graphs of $y^{\pm}, y^{+}(x,t) > y^{-}(x,t)$ in $[R,\infty)$. Since both y^{+} and y^{-} satisfy (0.2), by Lemma 0.1 y^{+} and y^{-} extend smoothly to a solution of (0.2) in $[R+1,R+2]\times[T/2,T]$. By the strong maximum principle,

$$(3.6) \quad y^{+} - y^{-} \geq \delta_1 > 0 \quad , \quad \text{all} \quad (x,t) \in [R+1, R+2] \times [T/2, T]$$

for some δ_1. We set, for $t \in [0,T)$,

$$\gamma_R(\cdot,t) = \gamma(\cdot,t) \cap \{(x,y) \quad : \quad x < R+2\}$$

and define

$$\tilde{g}(\Omega(t)) = \inf\left\{ L^2(\frac{1}{A}+1) \ : \ \begin{array}{l} l \text{ is a line segment lying inside the} \\ \text{domain bounded by } \gamma_R(\cdot,t) \text{ and } x =. \\ R+2 \text{ with endpoints on } \gamma_R(\cdot,t) \end{array} \right\}$$

where the domain $\Omega(t)$ is between y^{+} and y^{-}. Using (3.4) and (3.6) we know that for sufficiently large R, $\tilde{g}(\Omega(t))$ attains minimum provided $\tilde{g}(\Omega(t)) < \delta_2$ for some δ_2 depending on δ_1 and T. Following the reasoning in the proof of Lemma 2.2 one can show that $\tilde{g}(\Omega(t))$ is increasing in $(0,t_1]$ if $\tilde{g}(\Omega(t_1)) < \pi$.

Now we can conclude the proof of this proposition by following the proof of Proposition 3.2. In case the horizontal line segment, when scaled back, belongs to $\Omega(t_j)$, we argue using $\tilde{g}$ instead of g. By further restricting ε to $\varepsilon < \delta_2$, the same contradiction can be drawn. Hence γ exists up to T. Since T is arbitrary, we conclude that $t_{\max} = \infty$.

When (b) is replaced by (b)', we first observe that solution cannot exist for all time. For, the following formula

$$\frac{dA(t)}{dt} = -\pi \ ,$$

where $A(t)$ is the area of the domain $\Omega(t)$ bounded by y^{+} and y^{-}, implies that $t_{\max} \leq A/\pi$. However, we note that the above proof works as long as $T < A/2\pi$. Hence, under (a), (b)' and (c) the solution exists in $[0,T_1]$ where $T_1 = A/2\pi - \varepsilon/2$, for any small $\varepsilon > 0$. Now, we can use $\gamma(\cdot,T_1)$ as the initial curve and apply the same reasoning to conclude that it extends up to $T_2 = A_1/2\pi - \varepsilon/2^2$ where $A_1 = A - \pi T_1$ is the area bounded by the curve at time T_1. Keep doing this and in general we have

$$\begin{aligned} T_{j+1} &= A_j/2\pi - \varepsilon/2^{j+1} , \\ A_{j+1} &= A_j - \pi T_{j+1} , \\ A_0 &= A. \end{aligned}$$

Therefore, the solution exists up to

$$\begin{aligned} \sum_{\nu=1}^{j+1} T_\nu &= \frac{1}{2\pi}\sum_{\nu=0}^{j} A_\nu - \varepsilon\sum_{\nu=0}^{j}\frac{1}{2^{\nu+1}} \\ &= \frac{A}{2\pi}\sum_{\nu=0}^{j}\frac{1}{2^\nu} + \frac{\varepsilon}{2}\sum_{\nu=0}^{j}\frac{\nu}{2^\nu} - \varepsilon\sum_{\nu=0}^{j}\frac{1}{2^{\nu+1}} \\ &\to \frac{A}{\pi} \end{aligned}$$

as $j \to \infty$. So we conclude that $t_{\max} = A/\pi$ in this case. The proof of Proposition 3.3 is completed.

4. Long time existence

In this section we prove the theorem stated in the introduction. Recall that we assume γ_0 is a locally Lipschitz continuous, complete and properly embedded curve dividing the plane into two regions Ω_1 and Ω_2, and each of them has infinite area. First we consider γ_0 to be smooth and approximate it by a sequence of smooth, properly embedded curves $\{\gamma_0^j\}, j = 1, 2, \cdots$, whose derivatives have finite $C^{4,\alpha}$-norms when parametrized in arclength such that

$$\text{(4.1)} \qquad \gamma_0^j = \gamma_0 \quad \text{on} \quad D_j(0) = \{(x,y) \ : \ x^2 + y^2 \le j^2\}$$

and,

(4.2) for each j, there exists a compact subset of the plane outside which γ_0^j consists of two disjoint semi-infinite lines.

By Proposition 3.2, the problem (0.1) has a unique solution of all time satisfying $\gamma(\cdot, t) = \gamma_0^j$. Denote it by γ^j. Let T and R be arbitrary positive fixed number. We want to obtain uniform estimates on $\gamma^{j'}s$ and their curvatures on $D_{R/2}(0)$ and t in $[0, T]$.

Since by proper embeddedness $\gamma_0(p)$ goes to infinity as $p \to \pm\infty$, it is clear that we can choose two points, q_1 and q_2 in the interiors of $\Omega_1 \backslash D$ and $\Omega_2 \backslash D$, $D = D_{\sqrt{R^2+2T}}(0)$, respectively and two smooth arcs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, whose interiors are disjoint, connecting q_1 and q_2 so that each $\tilde{\Gamma}_i, i = 1,2$, intersects tranversally with γ_0 at one and only one point. Furthermore, the region Ω bounded between $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ contains the disk D.

It is easy to see that each region $\Omega_i \backslash \Omega$, $i = 1,2$, has infinite area. We can place a smooth, closed, embedded curve $\tilde{\gamma}_1$ (resp. $\tilde{\gamma}_2$) inside $\Omega_1 \backslash \Omega$(resp. $\Omega_2 \backslash \Omega$), such that the area bounded by $\tilde{\gamma}_1$ (resp. $\tilde{\gamma}_2$) is not less than $2\pi(T+1)$. Since each set $\Omega_i \backslash (\Omega \cup \{\text{the region enclosed by } \tilde{\gamma}_i\}), i = 1,2$, remains as a region, we can find a smooth arc $\tilde{\Gamma}_3$ connecting q_1 and q_2 in the region $\mathbb{R}^2 \backslash (\Omega \cup \{\text{the region enclosed by } \tilde{\gamma}_1\} \cup \{\text{the region enclosed by } \tilde{\gamma}_2\})$ so that $\tilde{\Gamma}_3$ intersects transversally with γ_0 at exactly one point, and the closed, embedded curve consisting of the arcs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_3$ (resp. $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_3$) bounds the two regions enclosed by $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. By modifying these two curves slightly, we obtain two smooth, closed, embedded curves Γ_1 and Γ_2 satisfying:

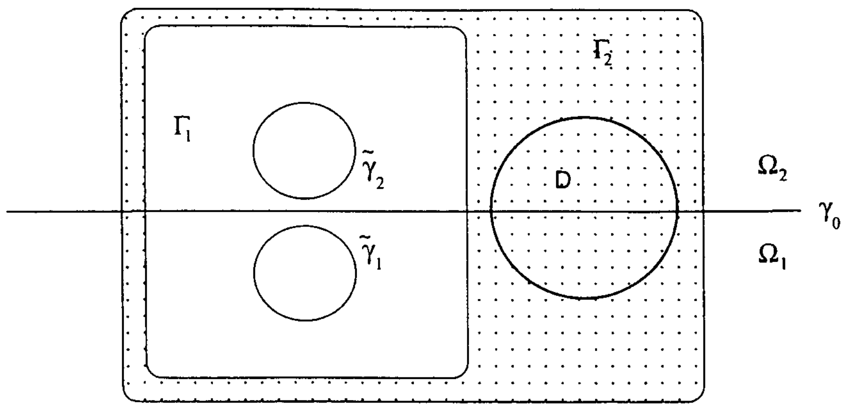
(1) each $\Gamma_i, i = 1,2$, intersects transversally with γ_0 at exactly two points;

(2) each $\Gamma_i, i = 1,2$, bounds the two regions enclosed by $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$;

(3) Γ_2 encloses Γ_1 and they are disjoint;

(4) the region $\tilde{\Omega}$ bounded between Γ_1 and Γ_2 contains the disk D.

A "topological" picture of the present situation is shown below.



$\tilde{\Omega}$ = dotted region bounded between Γ_1 and Γ_2.

Observe that $\tilde{\Omega}$ is diffeomorphic to the standard annulus $S^1 \times [1, 2]$. We can fill up $\tilde{\Omega}$ by a family of smooth, closed and embedded curves $\Gamma_\mu, \mu \in [1, 2]$, satisfying:

(a) each Γ_μ intersects γ_0 transversally at exactly two points;

(b) each $\Gamma_\mu, \mu \in [1, 2]$, is disjoint from the others;

(c) when $\mu = 1, 2$, they coincide with Γ_1 and Γ_2 defined before.

The family $\mathfrak{T} = \{\Gamma_\mu : \mu \in [1, 2]\}$ forms a foliation of $\tilde{\Omega}$. For each μ, let $\Gamma_\mu(\cdot, t)$ be the smooth curve which evolves according to the curve shortening problem (0.1) with Γ_μ as its initial datum. Since each Γ_μ bounds an area not less than $4\pi(T+1)$, all $\Gamma_\mu(\cdot, t)$ exist for $t \in [0, 2(T+1))$ according to Grayson's theorem. By the same reason, the evolving curve $\tilde{\gamma}_1(\cdot, t)$ (resp. $\tilde{\gamma}_2(\cdot, t)$) with $\tilde{\gamma}_1$ (resp. $\tilde{\gamma}_2$) as initial curve exists over $[0, T+1)$. ¿From the smooth dependence of the solution on parameters for (0.1) for closed curves (see, e.g., Angenent [3], [4]), the family

$$\mathfrak{T}(t) = \{\Gamma_\mu(\cdot, t) : \mu \in [1, 2]\} .$$

forms a foliation of the region bounded by $\Gamma_1(\cdot, t)$ and $\Gamma_2(\cdot, t)$.

We now turn to the approximating solutions $\{\gamma^j(\cdot, t)\}$. By the construction of the foliation $\mathfrak{T}$, there exists a sufficiently large J depending on R and T such that for all $j \geq J, \gamma_0^j$ is equal to γ_0 in the region enclosed by Γ_2, so then each leaf of $\mathfrak{T}$ intersects γ_0^j transversally at exactly two points. Since at $t = 0, \tilde{\gamma}_1$ (resp. $\tilde{\gamma}_2$) is disjoint from γ_0^j for all $j \geq J$, it follows from the strong maximum principle that for each t in $[0, T]$, the evolving curve $\tilde{\gamma}_1(\cdot, t)$ (resp. $\tilde{\gamma}_2(\cdot, t)$) is still disjoint from $\gamma^j(\cdot, t)$ for all $j \geq J$. Meanwhile, $\tilde{\gamma}_1(\cdot, t)$ and $\tilde{\gamma}_2(\cdot, t)$ continue to be enclosed by $\Gamma_\mu(\cdot, t)$ for each μ. Since at $t = 0$ each $\gamma_0^j, j \geq J$, separates the region enclosed by Γ_μ into two parts such that one part contains $\tilde{\gamma}_1$ and the other contains $\tilde{\gamma}_2$, we deduce that for each t in $[0, T]$ and $j \geq J, \gamma^j(\cdot, t)$ continues to separate the region enclosed by $\Gamma_\mu(\cdot, t)$ into two parts such that one part contains $\tilde{\gamma}_1(\cdot, t)$ and the other contains $\tilde{\gamma}_2(\cdot, t)$. This implies that for each μ in $[1, 2], t$ in $[0, T]$, and $j \geq J$, the curve $\gamma^j(\cdot, t)$ intersects $\Gamma_\mu(\cdot, t)$ at least at two points. Using the Sturm comparison theorem [2] and the fact that the number of intersections of γ_0^j and Γ_μ is exactly two, we know that $\gamma^j(\cdot, t)$ intersects the leaf $\Gamma_\mu(\cdot, t)$ transversally at exactly two points. By a compactness argument we conclude that there is a positive θ_0 depending on T, R, the minimum of angles of the intersections of γ_0 with $\Gamma_\mu, \mu \in [1, 2]$, which depends on the Lipschitz constant of γ in the compact region enclosed by Γ_2 and the foliation $\mathfrak{T}$, such that for all t in $[0, T]$ and $j \geq J$, the angles at the intersections of $\gamma^j(\cdot, t)$ and the leaves of $\mathfrak{T}(t)$ have a positive lower bound θ_0.

Consider the region $\tilde{\Omega}(t)$ bounded by $\Gamma_1(\cdot, t)$ and $\Gamma_2(\cdot, t)$. By the smooth dependence on parameters of (0.1) for closed curves again, there is a natural diffeomorphism

$$F \ : \ (u, \mu) \to (x, y) \ = \ \Gamma_\mu(u, t)$$

between the annulus $S^1 \times [1, 2]$ and $\tilde{\Omega}(t)$ such that $F(u, \mu, t)$ maps the circles onto the leaves $\Gamma_\mu(\cdot, t)$. Moreover, F is at least $\tilde{C}^{3,\alpha}$ in $S^1 \times [1, 2] \times [0, T]$. Since we have already shown that $\gamma_j(\cdot, t)$ meets each leaf of $\mathfrak{T}(t)$ transversally at exactly two points as $t \in [0, T]$ and $j \geq T$, the restriction of $\gamma^j(\cdot, t)$ to $\tilde{\Omega}(t)$ has exactly two connected components. And when we lift S^1 to its universal cover $\mathbb{R}$, under the diffeomorphism F each component corresponds to a graph of a $\tilde{C}^{3,\alpha}$-function

$$u \ = \ f^j(\mu, t) \ , \ \mu \in [1, 2] \text{ and } t \in [0, T] \ .$$

The fact that the angles between $\gamma^j(\cdot, t)$ and $\mathfrak{T}(t)$ have a uniform positive lower bound tells us that the gradients of $f^j, \partial f^j / \partial \mu(\mu, t)$, is uniformly bounded for all $j \geq J$ and $(\mu, t) \in [1, 2] \times [0, T]$. In order to obtain higher order estimates we need to write down the evolution equation for f^j. Each connected component of $\gamma^j(\cdot,t)$ relative to $\tilde{\Omega}(t)$ is given implicitly by

$$\{(x,y) \;=\; F(f^j(\mu,t)\,\mu,t) \;:\; \mu\in[1,2]\}\ .$$

The curvature is given by

$$\begin{aligned} k &= \det\Big|\frac{dF}{d\mu}\ ,\ \frac{d^2F}{d\mu^2}\Big|/\Big|\frac{dF}{d\mu}\Big|^3 \\ &= \Big|\frac{dF}{d\mu}\Big|^{-3}\big[(F^1_\mu+F^1_u f^j_\mu)(F^2_{\mu\mu}+2F^2_{\mu u}f^j_\mu+F^2_{uu}(f^j_\mu)^2+F^2_u f^j_{\mu\mu}) \\ &\quad -(F^2_\mu+F^2_u f^j_\mu)(F^1_{\mu\mu}+2F^1_{\mu u}f^j_\mu+F^1_{uu}(f^j_\mu)^2+F^1_u f^j_{\mu\mu})\big]\ , \end{aligned}$$

where

$$\Big|\frac{dF}{d\mu}\Big|^2 \;=\; (F^1_\mu+F^1_u f^j_\mu)^2+(F^2_\mu+F^2_u f^j_\mu)^2$$

is uniformly bounded from below by using the facts that f^j_μ is uniformly bounded and the Jacobian

$$\Big|\frac{\partial(F^1,F^2)}{\partial(u,\mu)}\Big| \;=\; |F^1_\mu F^2_u - F^1_u F^2_\mu|$$

is uniformly bounded from below on $(u,\mu,t)\in S^1\times[1,2]\times[0,T]$. So

$$\begin{aligned} k \;&=\; \frac{\partial(F^1,F^2)}{\partial(\mu,u)}\cdot\Big|\frac{dF}{d\mu}\Big|^{-3} f^j_{\mu\mu} \\ &\quad+\Big|\frac{dF}{d\mu}\Big|^{-3}\cdot\big[(F^1_\mu+F^1_u f^j_\mu)(F^2_{\mu\mu}+2F^2_{\mu u}f^j_\mu+F^2_{uu}(f^j_\mu)^2) \\ &\quad-(F^2_\mu+F^2_u f^j_\mu)(F^1_{\mu\mu}+2F^1_{\mu u}f^j_\mu+F^1_{uu}(f^j_\mu)^2)\big]\ . \end{aligned} \tag{4.3}$$

The unit normal ν is given by

$$\nu \;=\; \Big|\frac{dF}{d\mu}\Big|^{-1}(-(F^2_\mu+F^2_u f^j_\mu),\ F^1_\mu+F^1_u f^j_\mu)\ .$$

Equation (0.1) becomes

$$\begin{aligned} &-(F^1_t+F^1_u\frac{\partial f^j}{\partial t})(F^2_\mu+F^2_u f^j_\mu)+(F^2_t+F^2_u\frac{\partial f^j}{\partial t})(F^1_\mu+F^1_u f^j_\mu) \\ &\quad=\;\frac{\partial(F^1,F^2)}{\partial(\mu,u)}\cdot\Big|\frac{dF}{d\mu}\Big|^{-2} f^j_{\mu\mu} \\ &\quad\quad+\Big|\frac{dF}{d\mu}\Big|^{-2}\cdot\big[(F^1_\mu+F^1_u f^j_\mu)(F^2_{\mu\mu}+2F^2_{\mu u}f^j_\mu+F^2_{uu}(f^j_\mu)^2) \\ &\quad\quad-(F^2_\mu+F^2_u f^j_\mu)(F^1_{\mu\mu}+2F^1_{\mu u}f^j_\mu+F^1_{uu}(f^j_\mu)^2)\big]\ , \end{aligned}$$

that is,

$$
\begin{aligned}
\frac{\partial f^j}{\partial t} &= \left|\frac{dF}{d\mu}\right|^{-2} f^j_{\mu\mu} \\
&\quad + \left(\frac{\partial(F^1,F^2)}{\partial(\mu,u)}\right)^{-1} \cdot \left|\frac{dF}{d\mu}\right|^{-2} \big[(F^1_\mu + F^1_u f^j_\mu) \\
&\qquad\qquad \cdot (F^2_{\mu\mu} + 2F^2_{\mu u} f^j_\mu + F^2_{uu}(f^j_\mu)^2) \\
&\quad - (F^2_\mu + F^2_u f^j_\mu)(F^1_{\mu\mu} + 2F^1_{\mu u} f^j_\mu + F^1_{uu}(f^j_\mu)^2)\big] \\
&\quad + \left(\frac{\partial(F^1,F^2)}{\partial(\mu,u)}\right)^{-1} \big[F^1_t(F^2_\mu + F^2_u f^j_\mu) - F^2_t(F^1_\mu + F^1_u f^j_\mu)\big] \,.
\end{aligned} \tag{4.4}
$$

Taking derivative with respect to μ in the above equation, we get the evolution equation of f^j_μ ,

$$
\begin{aligned}
\frac{\partial f^j_\mu}{\partial t} &= \frac{\partial}{\partial \mu}\left(\left|\frac{dF}{d\mu}\right|^{-2} \frac{\partial f^j_\mu}{\partial \mu}\right) + \phi(\mu,t,f^j(\mu,t),f^j_\mu(\mu,t)) \frac{\partial f^j_\mu}{\partial \mu} \\
&\quad + \psi(\mu,t,f^j(\mu,t),f^j_\mu(\mu,t))
\end{aligned} \tag{4.5}
$$

for some $\tilde{C}^\alpha$ functions ϕ, ψ, where $\phi(\mu,t,f^j(\mu,t),f^j_\mu(\mu,t))$ and $\psi(\mu,t,f^j(\mu,t),f^j_\mu(\mu,t))$ are uniformly bounded for $(\mu,t) \in [1,2] \times [0,T]$. Using, e.g., Theorem 3.1 of Chapter 5 in [17] , we conclude that for any fixed small $\delta > 0$, the $\tilde{C}^{1,\alpha}$ norm of the function $f^j_\mu(\mu,t)$ on $[1+\delta, 2-\delta] \times [\delta, T]$ is uniformly bounded for all $j \geq J$.

Since at $t = 0$, $\mathfrak{T}$ fills up the disk $D = D_{\sqrt{R^2+2T}}(0)$, it follows from a comparison with the evolution of circles that $\mathfrak{T}(t)$ fills up $D_R(0)$ for all $t \in [0,T]$. It is clear that if δ is small enough, the region $\hat{\Omega}(t)$ between $\Gamma_{1+\delta}(\cdot,t)$ and $\Gamma_{2-\delta}(\cdot,t)$ covers the disk $D_{R/2}(0)$ for all $t \in [0,T]$. Therefore the curvature of $\gamma^j(\cdot,t)$, when restricted to the region $\hat{\Omega}(t)$ which contains $D_{R/2}(0)$, is uniformly bounded and uniformly Hölder continuous for all $j \geq J$.

Next we show that there are sufficiently large constants R_0 and J_0 such that all $\gamma^j(\cdot,t), j \geq J_0$, must intersect with the disk $D_{R_0/2}(0)$. Recall γ_0 divides the plane into two regions Ω_1, Ω_2 of infinite area. Draw two smooth, closed embedded curves $\hat{\gamma}_1$ and $\hat{\gamma}_2$ such that one lies in Ω_1 and the other in Ω_2, and each of them encloses an area not less than $2\pi(T+1)$. Choose R_0 so large that $D_{R_0/2}(0)$ covers the two regions enclosed by $\hat{\gamma}_1$ and $\hat{\gamma}_2$ and determine J_0 large enough such that as $j \geq J_0$, γ^j_0 coincide with γ_0 on $D_{R_0/2}(0)$. Then let $\hat{\gamma}_1$, $\hat{\gamma}_2$ and the boundary circle of $D_{R_0/2}(0)$ evolving according to the curve shortening equation (0.1).

Since each $\gamma_0^j (j \geq J_0)$ goes through $D_{R_0/2}(0)$ and seperates $\hat{\gamma}_1$ and $\hat{\gamma}_2$, it follows from the strong maximum principle that $\gamma^j(\cdot,t)$ continues to seperate $\hat{\gamma}_1(\cdot,t)$ and $\hat{\gamma}_2(\cdot,t)$, the evolutions of $\hat{\gamma}_1$ and $\hat{\gamma}_2$, for all $t \in [0,T]$. Therefore $\gamma^j(\cdot,t)(j \geq J_0)$ intersects with $D_{R_0/2}(0)$ for all $t \in [0,T]$.

Remember that the restriction of $\gamma^j(\cdot,t)$ to $\tilde{\Omega}(t)$ has exactly two components. Let $R \geq R_0$ and focus our attention to the component which meets $D_{R_0/2}(0)$. The further restriction of this component on $\hat{\Omega}(t)$, the region between $\Gamma_{1+\delta}(\cdot,t)$ and $\Gamma_{2-\delta}(\cdot,t)$, is clearly also a connected component. So by passing to a convergent subsequence we get a solution $\gamma_R(\cdot,t)$ of the curve shortening equation (0.1) in the domain $\{(p,t)|p \in \hat{\Omega}(t), t \in (0,T]\}$ which contains $D_{R/2}(0) \times (0,T]$. The solution $\gamma_R(\cdot,t)$ remains to be embedded. Since the approximating solutions $\gamma^j(\cdot,t)$ are embedded, the limit $\gamma_R(\cdot,t)$ has at most tangent self-intersections. But applying the maximum principle to the evolution equation of $\gamma_R(\cdot,t)$ excludes this possibility. Meanwhile, $\gamma_R(\cdot,t)$ is also a connected arc with one endpoint on $\Gamma_{1+\delta}(\cdot,t)$ and the other endpoint on $\Gamma_{2-\delta}(\cdot,t)$. This says that the two endpoints of $\gamma_R(\cdot,t)$ stay outside the disk $D_{R/2}(0)$.

Now we claim that $\gamma_R(\cdot,t)$ takes γ_0 as initial value. For, given any $\varepsilon > 0$, one can find $t_\varepsilon > 0$ such that $\gamma^j(\cdot,t)\big|_{\hat{\Omega}(t)}$ is contained in the ε-neighborhood of $\gamma_0\big|_{\hat{\Omega}(0)}$ for all $t \in [0,t_\varepsilon)$ and $j \geq J$. See Lemma 0.1 (a). Thus when $j \to \infty, \gamma_R(\cdot,t)$ is contained in the ε-neighborhood of $\gamma_0\big|_{\hat{\Omega}(0)}$, and so $\gamma_R(\cdot,0) = \gamma_0$ on $\hat{\Omega}(0)$.

Since for given T we can find large R and a subsequence of $\gamma^j(\cdot,t)$ which converges uniformly with bounded curvature in $[0,T]$, by taking a diagonal sequence we conclude the existence of a global solution $\gamma(\cdot,t)$ of (0.1) satisfying $\gamma(\cdot,0) = \gamma_0$. By the same reason as before, the solution $\gamma(\cdot,t)$ remains to be complete and embedded. Further we claim that $\gamma(\cdot,t)$ is still proper. First we observe that the two endpoints of $\gamma_R(\cdot,t)$ stay outside $D_{R/2}(0)$, so there must be some sequences $u_j^+ \to +\infty$ and $u_j^- \to -\infty$ such that both $\gamma(u_j^+,t)$ and $\gamma(u_j^-,t)$ go to infinity as $j \to +\infty$. Suppose for some $t_1 > 0$, $\gamma(\cdot,t_1)$ is not proper. That is, there exist a positive $R_1 < +\infty$ and a sequence $v_j \to \infty$ such that $\gamma(v_j,t_1) \in D_{R_1}(0)$ for all j. Let's look back the initial curve γ_0 which divides the plane into two infinite area regions Ω_1 and Ω_2. Similarly as before, we can draw two smooth, closed and embedded curves $\bar{\gamma}_1$ and $\bar{\gamma}_2$ in Ω_1 and Ω_2 respectively such that each of them bounds an are at least $2\pi(t_1+1)$. And we can draw a smooth, closed and embedded curve $\bar{\Gamma}$ satisfying :

(i) $\bar{\Gamma}$ encloses the two regions bounded by $\bar{\gamma}_1$ and $\bar{\gamma}_2$;

(ii) $\bar{\Gamma}$ encloses the disk $D_{\sqrt{R_1^2+2(t_1+1)}}(0)$;

(iii) $\bar{\Gamma}$ intersects with γ_0 at exactly two points.

Now let $\bar{\gamma}_1, \bar{\gamma}_2$ and $\bar{\Gamma}$ evolve according to (0,1) to get $\bar{\gamma}_1(\cdot,t), \bar{\gamma}_2(\cdot,t)$ and $\bar{\Gamma}(\cdot,t)$. From Grayson's theorem and the comparison principle, these evolving curves exist for $t \in [0, t_1+1)$ and $\bar{\Gamma}(\cdot,t)$ encloses the disk $D_{R_1}(0)$ for all $t \in [0, t_1+1)$. Exactly as before, since at $t=0$, γ_0 separates $\bar{\gamma}_1$ and $\bar{\gamma}_2$ in the region bounded by $\bar{\Gamma}, \gamma(\cdot,t)$ continues to separate $\bar{\gamma}_1(\cdot,t)$ and $\bar{\gamma}_2(\cdot,t)$ in the region bounded by $\bar{\Gamma}(\cdot,t)$. Then the Sturm Comparison theorem implies that $\gamma(\cdot,t)$ intersects with $\bar{\Gamma}(\cdot,t)$ at exactly two points for all $t \in [0, t_1+1)$. On the other hand, we have already known that $\gamma(u_j^+, t_1)$ and $\gamma(u_j^-, t_1)$ go to infinity for some $u_j^+ \to +\infty$ and $u_j^- \to -\infty$, and $\gamma(v_j, t_1) \in D_{R_1}(0)$ for all j. By continuity and embeddedness, these imply that $\gamma(\cdot,t_1)$ must intersect with $\bar{\Gamma}(\cdot,t_1)$ at infinitly many points. Contradiction holds. So the solution $\gamma(\cdot,t)$ is proper.

Finally, as seen from the construction of the above foliation, the second and higher order derivatives estimates on each compact subset of $\mathbb{R}^2 \times (0,T]$ depend only on the initial Lipschitz constant on a suitable larger subset of $\mathbb{R}^2$ and T, an approximation argument yields a global solution of (0.1) for locally Lipschitz continuous initial data. The proof of the theorem is completed.

5. A uniqueness property

In this section we shall establish a strong uniqueness property of (0.1) for a class of complete curves.

Proposition 5.1 *Let γ_0 be a locally Lipschitz continuous, properly embedded curve whose two ends are representable as graphs over some semi-infinite lines. Then (0.1) has at most one solution taking γ_0 as its initial value.*

More precisely, we mean that if γ_1 and γ_2 are solutions of (0.1) which tend to γ_0 uniformly on compact subsets of $\mathbb{R}$ as t goes to 0, then γ_1 and γ_2 must coincide. At first glance this uniqueness property is rather surprising. In fact, when restricted to graphs, (0.1) is equivalent to

$$u_t = \frac{u_{xx}}{1+u_x^2} \,. \tag{5.1}$$

We have shown in Chou-Kwong [7] that the Cauchy problem to (5.1) as well as to certain quasilinear parabolic equations has a unique solution without any assumption concerning the growth of the initial data.

Lemma 5.2 *Let u and v be two solutions of (5.1) in $[a,\infty)\times[0,T_0)$ with $u(x,0) = v(x,0)$. Then there exist constants C and $T \le T_0$ depending only on the initial data that*

$$\int_{a+1}^{\infty} (u(x,t)-v(x,t))^2 dx \le Ct\ ,\ \ t \le T\ .$$

Proof. Let $\zeta = \zeta_R$ be a smooth, non-negative function in $[a,\infty)$ such that $\zeta = 1$ in $(a,R), \zeta = 0$ in $(2R,\infty)$ and $|\zeta_x| \le 2/R$. By (5.1) we have

$$\frac{1}{2}\frac{d}{dt}\zeta^2(u-v)^2 \ = \zeta^2(u-v)(\text{arc tan } u_x - \text{arc tan } v_x)_x\ .$$

Integrating this equality yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{a+1}^{\infty}\zeta^2(u-v)^2dx \le &-\int_{a+1}^{\infty}\zeta^2(u_x-v_x)(\text{arc tan } u_x - \text{arc tan } v_x)\\ &-2\int_{a+1}^{\infty}\zeta\zeta_x(u-v)(\text{arc tan } u_x - \text{arc tan } v_x)\\ &+\zeta^2(u-v)(\text{arc tan } u_x - \text{arc tan } v_x)\Big|_{a+1}^{\infty}\\ &\le \frac{16\pi}{\sqrt{R}}\sqrt{\int_{a+1}^{\infty}\zeta^2(u-v)^2} + 4\pi\big|(u-v)(a+1,t)\big|\\ &\le \frac{16\pi}{\sqrt{R}}\sqrt{\int_{a+1}^{\infty}\zeta^2(u-v)^2} + Ct^{1/2}\end{aligned}$$

for t in $[0,T]$. It immediately implies that

$$\int_{a+1}^{\infty}(u-v)^2(x,t)dx \le C't\ ,\ \ t\in[0,T]\ ,$$

after letting R go to ∞. Here T is some positive constant less than T_0 and C' only depends on the initial data and T. q.e.d.

By our assumption both ends of γ_0 are graphs over two semi-infinite lines l_1 and l_2. One may agree that as p increases from $-\infty$ to some p_1, each vertical line from l_1 cuts $\{\gamma_0(p) : p \le p_1\}$ transversally at exactly one point. Extend this family of line segments for p after p_1 by line segments transversal to γ_0 at $\gamma_0(p)$. By Lipschitz continuity this is possible. When p increases beyond some p_2, by our assumption again, the line segments become vertical lines to l_2. In such a way we have constructed a foliation of a tubular neighborhood of γ_0. Using the foliation we can define a local diffeomorphism $G : \mathbb{R}^2 \to \mathbb{R}^2$ such that

(i) $G(x, 0)$ is a complete curve which coincides with l_1 for $x \le -x_0$ and with l_2 for $x \ge x_0$;

(ii) G is a constant linear transformation on $\{x \le -x_0\}$ and another constant linear transformation on $\{x \ge x_0\}$;

(iii) there exists a locally Lipschitz continuous function u_0 such that $G(\cdot, u_0(\cdot)) = \gamma_0$; and

(iv) for each $x \in [-x_0, x_0]$, the set $\{G(x, y) \mid y \in [u_0(x) - \delta, u_0(x) + \delta]\}$ is a line segment for some $\delta > 0$.

By continuity and the Sturm comparison theorem it is not hard to see that for sufficiently small t, say, in $(0, T)$, there exists a unique $u(x, t)$ such that $G(\cdot, u(\cdot, t)) = \gamma(\cdot, t)$. We note that u satisfies a quasilinear parabolic equation of the form

$$u_t = a(x, u, u_x)u_{xx} + b(x, u, u_x) \ , \tag{5.2}$$

similar to (4.4). Moreover, outside $(-x_0, x_0)$ the equation reduces to (5.1).

Suppose γ_1 and γ_2 are two solutions of (0.1) with $\gamma_1(\cdot, 0) = \gamma_2(\cdot, 0) = \gamma_0$. Their corresponding functions u_1 and u_2 satisfy (5.2). The function $w = u_2 - u_1$ satisfies a parabolic equation of the form

$$w_t \ = \ Aw_{xx} + Bw_x + Cw \tag{5.3}$$

when A, B and C are functions of (x, t) in $\mathbb{R} \times (0, T)$. Notice that C vanishes outside $[-x_0, x_0]$. Furthermore, by the quasilinear parabolic theory there exists a constant λ depending only on T and u_0 in $[-x_1, x_1]$ for some fixed $x_1 > x_0$ such that $|C(x, t)| \le \lambda$. Now, given any small $\varepsilon > 0$, and suppose that $(u_2 - u_1)(z, t_0) > \varepsilon$ for some z and $t_0 \in (0, T)$. Consider the set $\{(x, t) : (u_2 - u_1)(x, t) > e^{-\lambda T}\varepsilon\}$ and denote its connected component containing (z, t_0) by Ω. By Lemma 5.2 Ω is compact and either $u_2 - u_1 = e^{-\lambda T}\varepsilon$ or $u_2 = u_1$ on the parabolic boundary of Ω. It follows from the maximum principle that

$$e^{-\lambda t_0}(u_2 - u_1)(z, t_0) \le e^{-\lambda T}\varepsilon .$$

But then

$$\begin{aligned} \varepsilon &< (u_2 - u_1)(z, t_0) \\ &\le \varepsilon , \end{aligned}$$

contradiction holds. We conclude that $u_2 \le u_1$. By switching the roles of u_1 and u_2 we also have $u_1 \le u_2$. Hence γ_1 and γ_2 must be identical. The proof of the proposition is completed. We point out that by slightly modifying this proof Proposition 5.1 actually holds for properly immersed curves.

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