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RIEMANNIAN STRUCTURES OF PRESCRIBED GAUSSIAN CURVATURE FOR COMPACT 2-MANIFOLDS

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Let (M, g) denote a smooth (say C^3) compact two-dimensional manifold, equipped with some Riemannian metric g. Then, as is well-known, M admits a metric g_c of constant Gaussian curvature c; in fact the metrics g and g_c can be chosen to be conformally equivalent. Here, we determine sufficient conditions for a given non-simply connected manifold M to admit a Riemannian structure $\bar{g}$ (conformally equivalent to g) with arbitrarily prescribed (Hölder continuous) Gaussian curvature $K(x)$. If the Euler-Poincaré characteristic $\chi(M)$ of M is negative, the sufficient condition we obtain is that $K(x) < 0$ over M. Note that this condition is independent of g, and this result is obtained by solving an isoperimetric variational problem for $\bar{g}$. If $K(x)$ is of variable sign for $\chi(M) < 0$, or if $\chi(M) > 0$, then the desired critical point may not be an absolute minimum and our methods do not succeed. If $\chi(M) = 0$, our methods apply when $K(x)$ satisfies an integral condition with respect to the given metric g (see § 3); this result is perhaps not unreasonable since, for $\chi(\mathrm{M}) \leq 0$, distinct Riemannian structures on M need not be conformally equivalent.

1. Preliminaries

By passing (if necessary) to the orientable two-sheeted covering space of M, we may suppose M is orientable and admits a Riemannian structure with metric tensor g, Gaussian curvature $k(x)$, and volume element dV. If $K(x)$ is a given (Hölder continuous) function defined on M, we shall attempt to determine a metric tensor $\bar{g}$, conformal with g, whose Gaussian curvature $\bar{k}(x) = K(x)$ at each point of M, i.e., we shall seek a smooth function σ defined on M such that $\bar{g} = e^{2\sigma}g$ and $\bar{k}(x) = K(x)$. To find the equation which will determine σ in terms of the given data $K(x)$, $k(x)$ and g, we recall that in terms of isothermal parameters (u, v) on M an element of arc length can be written $ds^2 = \gamma\{du^2 + dv^2\}$, and the Gaussian curvature can be written

$$(1) \qquad k = -\tfrac{1}{2}\gamma^{-1}\{(\log \gamma)_{uu} + (\log \gamma)_{vv}\} .$$

Setting $\gamma' = \gamma \exp 2\sigma$, in place of γ in (1), we obtain the desired equation

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$$\Delta\sigma - k(x) + K(x)e^{2\sigma} = 0\,, \tag{2}$$

where Δ is the Laplace-Beltrami operator relative to g on M. Clearly a smooth solution σ of (2) defined on M corresponds to a metric $\bar{g} = e^{2\sigma}g$ with Gaussian curvature $K(x)$.

A few words concerning the solvability of (2) are in order at this stage. From the point of view of quasilinear elliptic partial differential equations, (2) is somewhat extraordinary in that its solutions do not admit any obvious pointwise a priori bounds (when either $k(x)$ or $K(x)$ is of variable sign). Consequently the methods of fixed point theory do not readily apply to the problem of the existence of a solution for (2). On the other hand, if the Euler-Poincaré characteristic $\chi(M)$ of M is not zero, it is easy and natural to formulate an isoperimetric variational problem whose solution (if such exists) yields a solution of (2). Indeed consider the totality S of functions u (belonging to some admissible class C) whose associated "integra curvatura" (with respect to $\bar{g}$) $\int_M K(x)e^{2u}\,dV = 2\pi\chi(M)$. Then the following result holds:

Lemma 1. *If $\chi(M) \neq 0$, any smooth (C^2) critical point of the functional $F(u) = \int_M (\frac{1}{2}|\nabla u|^2 + k(x)u)\,dV$ subject to the constraint S is a solution of (2).*

Proof. A smooth critical point u of the isoperimetric problem satisfies the Euler equation

$$\Delta u - k(x) + \beta K(x)e^{2u} = 0\,, \tag{3}$$

where β is some constant. To determine β, we integrate (3) over M to find $\int_M k(x)\,dV = \beta\int_M K(x)e^{2u}\,dV$. Thus since $\chi(M) \neq 0$, $\beta = 1$ and so any solution of (3) satisfies (2).

If $\chi(M) = 0$, an analogous but somewhat more involved isoperimetric problem can be used to solve (2); see § 3 below.

In order to demonstrate the existence of critical points for the isoperimetric variational problem described above, it is convenient to restrict the admissible class C to an appropriate Hilbert space. To this end, we denote by $W_{1,2}(M, g)$ the set of functions $u(x)$ defined on M such that (relative to the Riemannian structure g) u and $\nabla u = \operatorname{grad} u$ are square integrable over M. Then $W_{1,2}(M, g)$ is a Hilbert space relative to the inner product

$$(u, v)_{1,2} = \int_M uv\,dV + \int_M \nabla u\cdot\nabla v\,dV\,. \tag{4}$$

We denote by $\overline{W}_{1,2}(M, g)$ the closed subspace of $W_{1,2}(M, g)$ consisting of functions $u \in W_{1,2}(M, g)$ of mean value zero. A well-known inequality of Friedrichs states that for $u \in W_{1,2}(M, g)$,

$$\int_M u^2 \, dV \le c \left\{ \int_M |\nabla u|^2 \, dV + \left| \int_M u \, dV \right|^2 \right\}, \tag{5}$$

where c is a constant independent of u. Thus the inner product in $\overline{W}_{1,2}(M, g)$ may be defined by

$$[u, v]_{1,2} = \int_M \nabla u \cdot \nabla v \, dV . \tag{6}$$

The inequalities of Sobolev imply that if $u \in W_{1,2}(M, g)$, then $u \in L_p(M, g)$ for all $p < \infty$. A sharper result [2] is that for $u \in W_{1,2}(M, g)$, the integral $\int_M e^{su} \, dV$ for any positive number s is bounded and, in fact, as a functional of u, is continuous with respect to weak convergence in $W_{1,2}(M, g)$. Furthermore, if $u \in \overline{W}_{1,2}(M, g)$, then there are finite positive constants c_1 and c_2 independent of u such that if $\|u\|_{W_{1,2}(M,g)} = 1$, then $\int_M \exp c_1 u^2 \, dV \le c_2$.

2. Manifolds with negative Euler-Poincaré characteristic

Here we prove the following:

Theorem 1. *Suppose the Euler-Poincaré characteristic $\chi(M)$ is negative. Then a sufficient condition for the existence of a Riemannian structure $(M, \bar{g})$ on M with given (Hölder continuous) Gaussian curvature $K(x)$ is that $K(x) < 0$ on M. Furthermore, if g denotes any given Riemannian metric defined on M, then $\bar{g}$ and g can be chosen to be conformally equivalent.*

Proof. To demonstrate the sufficiency condition stated in the theorem, we employ Lemma 1, i.e., we show that the functional

$$F(u) = \int_M [\tfrac{1}{2} |\nabla u|^2 + k(x) u] \, dV$$

subject to the constraint

$$S = \left\{ u \,|\, u \in W_{1,2}(M, g), \int_M K(x) e^{2u} \, dV = 2\pi\chi(M) \right\}$$

has *a* smooth critical point; more precisely we show that $a = \inf_S F(u)$ is a critical value for the isoperimetric problem. To this end, for $u \in S$, set $u = u_0 + u_m$ where $u_0 \in \overline{W}_{1,2}(M, g)$ and u_m is the mean value of u over M. Then

$e^{2u_m}\int_M K(x)e^{2u_0}\,dV = 2\pi\chi(M)$. Thus solving for u_m in terms of u_0 and substituting into (7) we find

$$\inf_S F(u) = \inf_{\overline{W}_{1,2}(M,g)} \left\{ \int_M [\tfrac{1}{2}|\nabla u_0|^2 + k(x)u_0]dV + \pi\chi(M)\left[\log 2\pi\chi(M) - \log\left|\int_M K(x)e^{2u_0}\,dV\right|\right]\right\} . \tag{7}$$

To show that $\inf_S F(u)$ is finite, we must estimate $-\chi(M)\log\left|\int_M K(x)e^{2u_0}\,dV\right|$ from below. For convenience we denote the norm $\|u\|_{W_{1,2}(M,g)}$ by $\|u\|$.

Since $\chi(M) < 0$ and M is compact, we use the fact that $\max_m K(x) \leq -\delta < 0$ for some constant δ. Then $\int_M K(x)e^{2u_0}\,dV \leq -\delta\int_M e^{2u_0}dV$. Since $e^x \geq 1 + x$, we have

$$-\chi(M)\log\left|\int K(x)e^{2u_0}\,dV\right| \geq |\chi(M)|\log\left\{\delta\int_M e^{2u_0}\,dV\right\} \geq |\chi(M)|(\log\delta + \log c_1)\,,$$

where c_1 is the volume of M. Thus

$$F(u) \geq \int_M (|\nabla u_0|^2 + k(x)u_0)dV - c_2\,, \tag{8}$$

where c_2 is a positive constant independent of $u_0 \in \overline{W}_{1,2}(M,g)$. Hence by virtue of (5) and the Schwarz inequality,

$$\int_M k(x)u_0\,dV \leq c_3\varepsilon\|u_0\|^2 + c_4\varepsilon^{-1}\,, \tag{9}$$

where c_3 and c_4 are positive constants independent of u_0 and ε, so that choosing $c_3\varepsilon = 1$, we find $a = \inf_S F(u) \geq -c_4c_3 - c_2$.

We are now in a position to complete our proof. First note that by hypothesis the set S mentioned in Lemma 1 is nonvacuous. Indeed, since $K(x)$ is continuous and strictly negative everywhere on M, we can easily find a C^∞ function w such that $\int_M K(x)e^{2w}\,dV = 2\pi\chi(M)$. Now let $u^{(n)}$ be a sequence of functions belonging to S such that $F(u^{(n)}) \to a$ and $F(u^{(n)}) \leq a + 1$. Then by (8) and (9) choosing $c_3\varepsilon = \frac{1}{2}$ we find $a + 1 \geq \frac{1}{2}\|u_0^{(n)}\|^2 - c_5$ where c_5 is some constant independent of n. Consequently $\|u^{(n)}\|_{W_{1,2}(M,g)}$ are uniformly bounded and so possess a weakly convergent subsequence with weak limit $\bar{u}$. Now the functional $\int_M (|\nabla u|^2 + k(x)u)dV$ is lower semi-continuous with respect to weak convergence in $W_{1,2}(M,g)$. Thus $F(\bar{u}) \leq a$. On the other hand, by the remarks made at the end of § 1, the functional $\int_M K(x)e^{2u}\,dV$ is continuous with respect to weak convergence in $W_{1,2}(M,g)$, so $\bar{u} \in S$, and then $F(\bar{u}) = a$, so that $\bar{u}$ is the desired critical point.

It remains to show that $\bar{u}$ is smooth enough to satisfy equation (2). Since the functionals $F(u)$ and $\int_M K(x)e^{2u}\,dV$ are differentiable in $W_{1,2}(M,g)$ and $\min_S F(u) = F(\bar{u})$,

$$\int_M (\nabla\bar{u}\cdot\nabla\sigma + k(x)\phi - \beta K(x)e^{2\bar{u}}\phi)dV = 0$$

for all $\phi \in \overline{W}_{1,2}(M,g)$ and some constant β, so that $\bar{u}$ can be regarded as a weak solution of the equation $\Delta u = f$ where $f \in L_p(M)$ for all $p < \infty$. Thus $\bar{u} \in W_{2,p}(M,g)$ for all $p < \infty$ by the L_p regularity theory for linear elliptic partial differential equations, and $\bar{u} \in C_{1,\alpha}(M,g)$ by the Sobolev imbedding theorem after a possible redefinition on a set of measure zero (on M). Hence $\bar{u} \in C_2(M,g)$ by the Schauder regularity theory provided $K(x)$ is Hölder continuous over M.

3. Manifolds with vanishing Euler-Poincaré characteristic

Theorem 2. *Suppose $\chi(M) = 0$, and let g denote a given Riemannian metric on M with volume element dV. Then a necessary condition for the existence of a Riemannian metric $\bar{g}$ on M with given (Hölder continuous) Gaussian curvature $\pm K(x)$ is that on M either $K(x)$ vanishes identically or $K(x)$ changes sign, and a sufficient condition that $\bar{g}$ can be chosen conformally equivalent to g is that in addition either $\int K(x)e^{2u_0}\,dV \neq 0$ where u_0 is any solution of the Poisson equation $\Delta u = k(x)$ on M, or $K(x) \equiv 0$ on M.*

Before proving this theorem we prove an analogue of Lemma 1, using the notation of Theorem 2.

Lemma 1′. *Suppose $\chi(M) = 0$ and $K(x)$ is a given function defined on M such that relative to some Riemannian metric g defined on M, $\int_M K(x)e^{2u_0}\,dV \neq 0$. Then the (smooth) critical points of the functional $F(u)$ subject to the constraint*

$$S' = \left\{ u \mid u \in W_{1,2}(M,g),\ \int_M u\, dV = 0,\ \int_M K(x)e^{2u}\, dV = 0 \right\}$$

are (apart from a constant) solutions of the equation

$$\Delta u - k(x) \pm K(x)e^{2u} = 0\,, \tag{10}$$

where $k(x)$ is the Gaussian curvature of (M, g).

Proof. A smooth critical point u of the isoperimetric variational problem satisfies the Euler equation

$$\Delta u - k(x) + \beta_1 K(x)e^{2u} = \beta_2\,, \tag{11}$$

where β_1 and β_2 are constants. Since $\int_M K(x)e^{2u_0}\, dV \neq 0$, both β_1 and β_2 cannot be zero. To show $\beta_2 = 0$ we integrate (11) over M to find

$$\int_M k(x)dV + \beta_1 \int_M K(x)e^{2u}\, dV = \beta_2 \mu(M)\,.$$

Since $\int_M k(x)dV = 0$ and $u \in S'$, $\beta_2 = 0$. Since $\beta_1 \neq 0$, there is a constant c such that $\pm e^{2c} = \beta_1$. Hence $\bar{u} = u + c$ satisfies $\Delta \bar{u} - k(x) \pm K(x)e^{2\bar{u}} = 0$.

Proof of Theorem 2. Again the necessity of the condition stated in the theorem is a consequence of the Gauss-Bonnet formula. To demonstrate the sufficiency condition, we consider the following two cases:

Case I. Suppose relative to a given Riemannian structure on M, $\int_M K(x)e^{2u_0}\, dV \neq 0$. Then we can employ Lemma 1′ as in the proof of Theorem 1. In this case it is quite easy to show that $a' = \inf_{S'} F(u)$ is bounded below. Indeed, since $u \in S'$, $u \in \overline{W}_{1,2}(M, g)$ and therefore, by (5) and (9), $F(u) \geq (1 - c_3\varepsilon)\|u\|^2 - c_4\varepsilon^{-1}$.
So choosing $\varepsilon = c_3^{-1}$, we have $a' = \inf_{S'} F(u) \geq -c_3c_4$. Again, suppose for the moment that the set S' is nonvacuous and any minimizing sequence $u^{(n)} \in S'$ with $F(u^{(n)}) \leq a' + 1$ has a weakly convergent subsequence with weak limit $\bar{u}$ such that $\bar{u} \in S'$ and $F(\bar{u}) = a'$. Furthermore the proof given of Theorem 1 shows that, after a possible redefinition on a set of measure zero on $M, \bar{u}$ is C^2 and satisfies equation (10) on M as required.

To show S' is nonvacuous, suppose $\int_M K(x)dV = A$. If $A = 0$, then $u(x) \equiv 0 \in S'$. If $A \neq 0$, since $K(x)$ changes sign on M we can find a point $x_0 \in M$ and a small neighborhood $N(x_0)$ of x_0 such that $K(x_0) = -\delta \operatorname{sgn} A$ where $\delta > 0$.

Hence, if v is a C^∞ function vanishing outside $N(x_0)$ and such that $v(x_0) = 1$, then $f(t) = \int_M K(x)e^{2tv}dV$ is a smooth function of t with $f(0) = A$. On the other hand, if $N(x_0)$ is sufficiently small as $t \to \infty$, then $f(t) \sim A - \delta \operatorname{sgn} A\, e^{2t}$ and hence $f(t_0) = 0$ for some $t_0 \in (0, \infty)$. Finally, set $u = t_0 v$ and let u_m denote the mean value of u over M. Then $w = u - u_m$ has mean value zero and $\int_M K(x)e^{2w}dV = 0$. Hence w is an element of S'.

Case II. $K(x) \equiv 0$. In this case equation (2) reduces to $\Delta\sigma - k(x) = 0$. Since $\int_M k(x)dV = 0$, this equation is easily solved as in Case I by minimizing $F(u)$, defined above, among all functions $u \in \overline{W}_{1,2}(M, g)$.

4. Remarks on the simply connected case

If M is simply connected, we have been unable to solve the isoperimetric problem defined in Lemma 1, since, in this case it is not clear that the infimum of $F(u)$ is attained on the set S. Indeed, for simplicity suppose $k(x) = 1$; then by virtue of (7), there is an absolute constant C_0 such that

$$\inf_S F(u) \geq \inf_{\overline{W}_{1,2}(M,g)} \left\{\frac{1}{2}\int_M |\nabla u_0|^2 dV - C_0 - 2\pi \log \int_M K(x)e^{2u_0}dV\right\} .$$

Since $K(x)$ is bounded, it suffices to bound $\log \int e^{2u_0}dV$ in terms of $\int_M |\nabla u_0|^2 dV = \|u_0\|^2$. To this end, setting $u_0 = \|u_0\| v$ and using the remarks made at the end of § 1, we obtain $\int \exp 2u_0\, dV \leq \int \exp\, (\|u_0\|^2/c_1 + c_1 v^2)dV$, so that $\log \int \exp 2u_0 dV \leq \|u_0\|^2/c_1 + \log c_2$. Hence

$$\inf_S F(u) \geq \inf_{\overline{W}_{1,2}(M,g)} \tfrac{1}{2}\|u_0\|^2 - \frac{2\pi}{c_1}\|u_0\|^2 - c_3 ,$$

where c_3 is another absolute constant. In order for $F(u)$ to be bounded below we must have $c_1 \geq 4\pi$. However we have only been able to show that in this case $c_1 \geq 4\pi/e - \varepsilon$ for any $\varepsilon > 0$.

Lemma. *If $(M, g) = (S^2, g_1)$, then $c_1 \geq 4\pi/e - \varepsilon$ for any $\varepsilon > 0$.*

Proof. Let x, y be two points on S^2, and $G(x, y)$ denote the Green's function for the Laplacian on S^2. If $\int u dV = 0$, then

$$u(x) = \int_{S^2} G(x, y)\Delta u(y) dV = -\int_{S^2} \nabla_y G(x, y)\nabla u(y) dV .$$

Since $G(x,y) = -\frac{1}{2\pi}\log\left(2\sin\frac{r(x,y)}{2}\right)$, [1, p. 182], where $r(x,y)$ is the geodesic distance along S^2 from x to y, we have

$$|u(x)| \le \frac{1}{2\pi}\int_{S^2}\frac{|\nabla u(y)|\,dV}{r(x,y)} = \frac{1}{2\pi}\int_{S^2}(|\nabla u|^{2/N}r^{-1/N})(|\nabla u|^{1-2/N})(r^{-1+1/N})dV\,,$$

and therefore

$$|u(x)| \le \frac{1}{2\pi}\left(\int_{S^2}\frac{|\nabla u|^2}{r}dV\right)^{1/N}\left(\int_{S^2}|\nabla u|^2 dV\right)^{(N-2)/(2N)}\left(\int_{S^2}r^{-2+2/N}dV\right)^{1/2}$$

by Hölder's inequality. Hence

$$\begin{aligned}\int_{S^2}|u|^N dV &\le \frac{1}{(2\pi)^N}(\|\nabla u\|_{0,2})^{(N-2)/2}\int_{S^2}\left(\int_{S^2}\frac{|\nabla u|^2}{r}dV\right)\left(\int_{S^2}r^{-2+2/N}dV\right)^{N/2}\\ &\le \frac{c}{(2\pi)^N}\|\nabla u\|^{N/2}\pi^{N/2}N^{N/2}\,,\end{aligned}$$

where c is a constant independent of N and u. Thus, if $\|\nabla u\|_{0,2} = 1$ and $\int u = 0$, then

$$\int \exp c_1u^2 dV = \sum\frac{1}{N!}\int c_1{}^N|u|^{2N}dV \le c\sum\frac{N^N}{N!}\left(\frac{c_1}{4\pi}\right)^N.$$

This last series converges provided $c_1 < 4\pi/e$.

The Gauss-Bonnet theorem implies that a necessary condition for (M,g) to admit a metric $\bar{g}$ (conformal to g) with Gaussian curvature $K(x)$ is that $K(x) > 0$ for some $x_0 \in M$. Frank Warner and the author conjecture that this condition is also sufficient.

Added in proof. In a paper soon to appear in the Indiana Univ. Math. J., Jürgen Moser has proven that the constant $c_1 = 4\pi$, so that $F(u)$ is bounded below on S. Nevertheless the question of the attainment of inf $F(u)$ on S is still open.

Bibliography

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