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A THEOREM OF CARTAN AND GUILLEMIN

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Introduction

In a paper [3] on the formal algebraic foundations of the theory of infinite pseudogroups, V. Guillemin proved that certain kinds of topological Lie algebras, which are naturally associated with such pseudogroups, possess a sort of Jordan-Hölder decomposition: if L is such a Lie algebra, there exists a sequence of closed L-ideals $L = L_0 \supset L_1 \supset \cdots \supset L_k = \{0\}$ such that for $i = 1, \cdots, k$ either (a) L_{i-1}/L_i is abelian or (b) there are no closed L-ideals lying properly between L_{i-1} and L_i; moreover, the non-abelian quotients L_{i-1}/L_i are unique except for the order in which they appear in the composition series. Guillemin was working in the category $\boldsymbol{L}_K$ of linearly compact topological Lie algebras over a field K of characteristic 0. An $L \in \boldsymbol{L}_K$ possessing an open subalgebra containing no L-ideals has a Jordan-Hölder decomposition.

Guillemin's result in the category $\boldsymbol{L}_K$ was conjectured by E. Cartan in [2] for his category of infinite infinitesimal groups. Moreover, Cartan had proved [2, Théorème XII] that the non-abelian quotients in the series (the simple intransitive groups) has a simple structure: they consist of "arbitrary" functions of a finite set of variables with values in a simple transitive group. Guillemin was able to prove a similar result for the category $\boldsymbol{L}_K$ [3, Theorem 7.1]; the proof is long and the methods are cohomological.

The main purpose of this paper is to give another proof of Guillemin's Theorem 7.1 using the method of produced representations which we developed in a previous paper [1]. The proof proper of the theorem (Theorem 2.4) is short and canonical. However, we are forced to impose certain restrictions on the field K. Our main tools are: (1) a modification (Theorem 1.2) of our Lie algebra analogue [1, Theorem 4(b)] of a theorem of G. W. Mackey to deal with linearly compact Lie algebra modules; and (2) a way of extending bilinear products between Lie algebra modules to the modules produced by them. In order to give an interpretation of Guillemin's theorem in terms of primitive actions, we also prove a theorem on "Induction in Stages" (Theorem 1.3). These technical preliminaries take up § 1. § 2 is devoted to the proof of Theorem 2.4 and to the construction of a representation which leads in § 3 to the interpretation in terms of primitive actions mentioned above. The main problem in § 3 is to

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show that every continuous derivation of a simple (or more generally, primitive) $L \in L_K$ differs from an inner derivation by a derivation of degree ≥ 0. This proof is messy and makes use of the classification of primitive Lie algebras. We also show that our primitive interpretation is essentially unique.

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1. Linear compactness, production, and products

Let K be a field, give K the discrete topology, and let V be a topological vector space $/K$. We shall denote the topological dual of V by V^* and give it the weak-* topology. The space continuous linear maps from one topological vector space V into another W will be denoted by $\mathrm{Hom}_K^c(V, W)$. Following [3], we shall say that V is *linearly compact* (l.c.) if V is separated and has a neighborhood base at 0 consisting of subspaces of V, and every family of closed affine subsets of V having the finite intersection property has non-void intersection. Proposition 1.2(d) of [3] asserts that the cartesian product of l.c. spaces is l.c. According to [3, Propositions 1.3 and 1.4], if V is l.c., then V^* is discrete and the canonical map of V into V^{**} is a topological isomorphism. Thus we can, and will, regard an l.c. V as the topological dual of the discrete space V^*.

Let V and W be l.c. Every $s \in \mathrm{Hom}_K^c(V, W)$ has a transpose ${}^ts \in \mathrm{Hom}_K(W^*, V^*)$ and ${}^{tt}s = s$ under the identification of V^{**} with V and W^{**} with W.

Let L be a Lie algebra over a field K, and V an l.c. vector space over K. V will be called a *linearly compact* L-module (l.c. L-module) if it is an L-module such that for every $x \in L$, the map $\gamma(x) \in \mathrm{Hom}_K(V, V)$ defined by $\gamma(x)v = xv$ for all $v \in V$ is continuous. We turn V^* into an L-module by setting $xv^* = -{}^t\gamma(x)v^*$. Then V is the L-module contragredient to V^* in the sense of [1, Proposition 1].

Let H be a Lie subalgebra of L, and V an l.c. H-module. As in [1], we may form the L-module $X = \mathrm{Hom}_{U(H)}(U(L), V)$, where U is the universal enveloping algebra functor. According to [1, Proposition 1], X is algebraically isomorphic to the L-module contragredient to $U(L) \otimes_{U(H)} V^*$. The corollary to that proposition asserts that X in the finite-open topology is homeomorphic under that isomorphism to $(U(L) \otimes_{U(H)} V^*)^*$, which is l.c. since $U(L) \otimes_{U(H)} V^*$ is discrete.

Lemma 1.1. *X is an l.c. L-module.*

Proof. We must show that for each $x \in L$, $u \mapsto xu$ is continuous from X to X. Let $a \in U(L)$, and $\{u_\alpha\}$ be a net in X with limit u. Then $\lim_\alpha (xu_\alpha)(a) = \lim_\alpha u_\alpha(ax) = u(ax) = (xu)(a)$ so that $\lim_\alpha xu_\alpha = xu$.

In what follows if V is any L-module and H is a subalgebra of V, we let V_H denote the H-module V becomes by restriction of the set of operators.

We may now reformulate Theorem 4 of [1] in our l.c. setting.

Definition. Let V be an l.c. L-module. V will be called *topologically irreducible* if V contains no nontrivial closed L-invariant subspaces. V will be called *topologically absolutely irreducible*, if it is topologically irreducible and the commuting ring of $\gamma(L)$ in $\mathrm{Hom}_K^c(V, V) = \{\text{multiplications by } K\}$.

We observe that V is topologically absolutely irreducible if and only if V^* is algebraically absolutely irreducible. Let I be an ideal of L, and V a topologically absolutely irreducible l.c. I-module. Let H consist of all $x \in L$ for which there exists $s \in \mathrm{Hom}_K^c(V, V)$ such that $\gamma[x, z] = [s, \gamma(z)]$ for all $z \in I$.

Theorem 1.2. *Suppose K is of characteristic* 0. *Let W be a topologically (absolutely) irreducible l.c. H-module such that W_I is topologically module isomorphic to the cartesian product of copies of V. Then the L-module $\mathrm{Hom}_{U(H)}(U(L), W)$ is topologically (absolutely) irreducible.*

Proof. The proof is that of Theorem 4(b) in [1]. Write $W \simeq \prod_\alpha V_\alpha$, where the V_α are copies of V. Then $W^* \simeq \sum_\alpha V_\alpha^*$. Thus W^* is an H-module and W_I^* is a direct sum of copies of the absolutely irreducible H-module V^*. It follows from the properties of transpose and the definition of H that $H = \{x \in L : \exists s \in \mathrm{Hom}_K(V^*, V^*) : [x, z]v^* = szv^* - zsv^* \text{ for all } v^* \in V^* \text{ and } z \in I\}$. Moreover, W^* is an (absolutely) irreducible H-module. Applying Theorem 3 of [1], we see that $U(L) \otimes_{U(H)} W^*$ is (absolutely) irreducible, whence $\mathrm{Hom}_{U(H)}(U(L), W)$ is topologically (absolutely) irreducible.

We next prove a theorem on "Induction in Stages". Let L be a Lie algebra over K, M and N be subalgebras of L with $M \subseteq N$, and V be an M-module. If $w \in \mathrm{Hom}_{U(M)}(U(L), V)$, set $\hat{w}(a)b = w(ba)$ for $a \in U(L)$ and $b \in U(N)$.

Theorem 1.3. (1) *For each $a \in U(L)$, $\hat{w}(a) \in \mathrm{Hom}_{U(M)}(U(N), V)$.*

(2) $\hat{w} \in \mathrm{Hom}_{U(N)}(U(L), \mathrm{Hom}_{U(M)}(U(N), V))$.

(3) $\wedge$ *is an L-module isomorphism.*

(4) *If V is an l.c. M-module, $\wedge$ is a homeomorphism.*

Proof. (1) Let $c \in U(M)$. Then $\hat{w}(a)(cb) = w(cba) = cw(ba) = c\hat{w}(a)b$ for $b \in U(N)$ so that $\hat{w}(a) \in \mathrm{Hom}_{U(M)}(U(N), V)$.

(2) Let $c \in U(N)$. Then $\hat{w}(ca)b = w(bca) = \hat{w}(a)bc = [c\hat{w}(a)]b$ for $b \in U(N)$ so that $\hat{w} \in \mathrm{Hom}_{U(N)}(U(L), \mathrm{Hom}_{U(M)}(U(N), V))$.

(3) Suppose $\hat{w} = 0$. Then $w(a) = \hat{w}(a)1 = 0$ for $a \in U(L)$ so that $w = 0$. Thus $\wedge$ is injective. Let $c \in U(L)$. Then $(c\hat{w})(a)b = \hat{w}(ac)b = w(bac) = (cw)(ba) = (cw)^\wedge(a)b$ for $a \in U(L)$ and $b \in U(N)$ so that $c\hat{w} = (cw)^\wedge$. Thus $\wedge$ is an L-module homomorphism. Let $u \in \mathrm{Hom}_{U(N)}(U(L), \mathrm{Hom}_{U(M)}(U(N), V))$. Set $w(a) = u(a)1$, and let $c \in U(M)$. Then $w(ca) = u(ca)1 = [cu(a)]1 = u(a)c = c[u(a)]1 = cw(a)$ for $a \in U(L)$ so that $w \in \mathrm{Hom}_{U(M)}(U(L), V)$. Finally, $\hat{w}(a)b = w(ba) = u(ba)1 = [bu(a)]1 = u(a)b$ for $a \in U(L)$ and $b \in U(N)$ so that $\hat{w} = u$. Thus $\wedge$ is surjective.

(4) is obvious.

Let L be a Lie algebra with subalgebra H, and V_1, V_2 be H-modules. Then $V_1 \otimes V_2$ is an H-module. In [1], we introduced an operation $\otimes$

which maps $\mathrm{Hom}_{U(H)}(U(L), V_1) \times \mathrm{Hom}_{U(H)}(U(L), V_2)$ bilinearly into $\mathrm{Hom}_{U(H)}(U(L), V_1 \otimes V_2)$. If $w_i \in \mathrm{Hom}_{U(H)}(U(L), V_i)$, then $w_1 \otimes w_2$ is defined as follows: Let Δ denote the canonical diagonal homomorphism of $U(L)$ into $U(L) \otimes U(L)$ determined by $\Delta x = x \otimes 1 + 1 \otimes x$ for all $x \in L$. Then w_1 and w_2 determine a linear map $w_1 \boxtimes w_2$ of $U(L) \otimes U(L)$ into $V_1 \otimes V_2$, and we set $w_1 \otimes w_2 = (w_1 \boxtimes w_2) \circ \Delta$. Suppose now that $*$ is a bilinear map of $V_1 \times V_2$ into an H-module V_3 such that $hv_1 * v_2 + v_1 * hv_2 = h(v_1 * v_2)$ for $v_i \in V_i$ and $h \in H$, and let α be the corresponding linear map of $V_1 \otimes V_2$ into V_3. Then α is an H-homomorphism. Let $W_i = \mathrm{Hom}_{U(H)}(U(L), V_i)$, $i = 1, 2, 3$, and define a bilinear map, also called $*$, of $W_1 \times W_2 \to W_3$ by $w_1 * w_2 = \alpha \circ (w_1 \otimes w_2)$. It is obvious from [1, Proposition 3] that $xw_1 * w_2 + w_1 * x w_2 = x(w_1 * w_2)$ for $w_i \in W_i$ and $x \in L$.

Let $V_1, \cdots, V_n$ be H-modules and set $W_i = \mathrm{Hom}_{U(H)}(U(L), V_i)$ for $i = 1, \cdots, n$. Let $\mathscr{P}$ be a set of products of the above type between the V_i: if $* \in \mathscr{P}$, then $*\colon V_{r(*)} \times V_{s(*)} \to V_{t(*)}$. Extending $* \in \mathscr{P}$ to the W_i we have $*\colon W_{r(*)} \times W_{s(*)} \to W_{t(*)}$.

Proposition 1.4. *Let $P_j\colon V_1 \times \cdots \times V_n \to V_{i_0}$ be a multilinear map which is a (non-associative) monomial using members af $\mathscr{P}$ as products, $j = 1, \cdots, m$, and $\tilde{P}_j\colon W_1 \times \cdots \times W_n \to W_{i_0}$ be the map obtained from P_j by replacing each $* \in \mathscr{P}$ by its extension to the W_i. Suppose $\sum_j c_j P_j = 0, c_j \in K$. Then $\sum_j c_j \tilde{P}_j = 0$.*

Proof. For each $* \in \mathscr{P}$, let $\alpha(*)\colon V_{r(*)} \otimes V_{s(*)} \to V_{t(*)}$ be the associated linear map. For each j, there is a permutation σ_j of $\{1, \cdots, n\}$ and linear maps $\tau_{j1}, \cdots, \tau_{jn-1}$ such that τ_{jk} is the tensor product (in some order) of $\alpha(*_{jk})$, where $*_{jk} \in \mathscr{P}$, and $k - 1$ identity maps, and such that $P_j(v_1, \cdots, v_n) = \tau_{j1} \circ \cdots \circ \tau_{jn-1}(v_{\sigma_j(1)} \otimes \cdots \otimes v_{\sigma_j(n)})$. For each j and k, let $\delta_{jk}\colon \otimes^k U(L) \to \otimes^{k+1} U(L)$ be the tensor product of $k - 1$ identity maps and the diagonal homomorphism Δ, where Δ stands in the same position in δ_{jk} as $\alpha(*_{jk})$ in τ_{jk}. Now $w_{\sigma_j(1)} \boxtimes \cdots \boxtimes w_{\sigma_j(n)}\colon \otimes^n U(L) \to V_{\sigma_j(1)} \otimes \cdots \otimes V_{\sigma_j(n)}$. One checks by working from the middle out to both ends that $\tilde{P}_j(w_1, \cdots, w_n) = \tau_{j1} \circ \cdots \circ \tau_{jn-1} \circ (w_{\sigma_j(1)} \boxtimes \cdots \boxtimes w_{\sigma_j(n)}) \circ \delta_{jn-1} \circ \cdots \circ \delta_{j1}$. But $\delta_{jn-1} \circ \cdots \circ \delta_{j1} = \Delta^{n-1}$, the unique homomorphism of $U(L) \to \otimes^n U(L)$ which, when restricted to L, is the diagonal map of $L \to \oplus^n L$. (This is just the coassociativity of the coproduct Δ.) Letting $\hat{\sigma}_j$ be the usual permutation linear map of $V_1 \otimes \cdots \otimes V_n \to V_{\sigma_j(1)} \otimes \cdots \otimes V_{\sigma_j(n)}$ and letting $\gamma_j = \tau_{j1} \circ \cdots \circ \tau_{jn-1} \circ \hat{\sigma}_j$, we have $P_j(v_1, \cdots, v_n) = \gamma_j(v_1 \otimes \cdots \otimes v_n)$. Therefore $\sum_j c_j \gamma_j = 0$. But then

$$\tilde{P}(w_1, \cdots, w_n) = \sum_j c_j \gamma_j \circ (w_1 \boxtimes \cdots \boxtimes w_n) \circ \Delta^{n-1} = 0.$$

Corollary. *If A is an associative (resp. Lie) algebra upon which H acts via derivations, then $B = \mathrm{Hom}_{U(H)}(U(L), A)$ is an associative (resp. Lie) algebra upon which L acts via derivations. If A is abelian, so is B. If, in addition, V*

is an H-module and an A-module such that $x(av) = (xa)v + a(xv)$ for $a \in A$, $x \in H$, and $v \in V$, then $W = \mathrm{Hom}_{U(H)}(U(L), V)$ is a B-module such that $y(bw) = (yb)w + b(yw)$ for $b \in B, y \in L$, and $w \in W$.

Lemma 1.5. *Let $V_i, i = 1, 2, 3$, be l.c. H-modules and suppose that $*: V_1 \times V_2 \to V_3$ is continuous. Then $*: W_1 \times W_2 \to W_3$ is also continuous.*

Proof. Let $\{w_{i\alpha}\}$ be a net in W_i such that $\lim_\alpha w_{i\alpha} = w_i, i = 1, 2$. Let $a \in U(L)$, and write $\Delta a = \sum_j b_j \otimes c_j, b_j$ and $c_j \in U(L)$. Then $(w_{1\alpha} * w_{2\alpha})(a) = \sum_j w_{1\alpha}(b_j) * w_{2\alpha}(c_j)$, whose limit is clearly $(w_1 * w_2)(a)$.

Suppose that $V_i, i = 1, 2, 3$, are M-modules, and M, N are subalgebras of L with $M \subseteq N$. As in Theorem 1.3 form $W_i = \mathrm{Hom}_{U(M)}(U(L), V_i)$ and $\hat{W}_i = \mathrm{Hom}_{U(N)}(U(L), \mathrm{Hom}_{U(M)}(U(N), V_i)), i = 1, 2, 3$. Let $*$ be a bilinear map of $V_1 \times V_2 \to V_3$ such that $xv_1 * v_2 + v_1 * xv_2 = x(v_1 * v_2)$ for $v_i \in V$ and $x \in M$. Extend $*$ as above to a map of $W_1 \times W_2 \to W_3$ and (in two stages) to a map of $\hat{W}_1 \times \hat{W}_2 \to \hat{W}_3$.

Proposition 1.6. $(w_1 * w_2)^\wedge = \hat{w}_1 * \hat{w}_2$.

Proof. Let $x_1, \cdots, x_r \in L$ and $y_1, \cdots, y_s \in N$. Using multi-index notation, we set $x^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r} \in U(L)$(resp. $y^{\alpha'} = y_1^{\alpha'_1} \cdots y_s^{\alpha'_s} \in U(N)$) for any r-tuple α (resp. s-tuple α') of nonnegative integers. Let $\varepsilon = (1, \cdots, 1)$(resp. $\varepsilon' = (1, \cdots, 1)$). Now $\Delta(y^{\varepsilon'} x^\varepsilon) = (\Delta y^{\varepsilon'})(\Delta x^\varepsilon)$. Hence

$$\begin{aligned}
(w_1 * w_2)^\wedge(x^\varepsilon)y^{\varepsilon'} &= (w_1 * w_2)(y^{\varepsilon'}x^\varepsilon) \\
&= \sum_{0 \le \alpha' \le \varepsilon'} \sum_{0 \le \alpha \le \varepsilon} w_1(y^{\alpha'}x^\alpha) * w_2(y^{\varepsilon'-\alpha'}x^{\varepsilon-\alpha}) \\
&= \sum_{0 \le \alpha' \le \varepsilon'} \sum_{0 \le \alpha \le \varepsilon} \hat{w}_1(x^\alpha)y^{\alpha'} * \hat{w}_2(x^{\varepsilon-\alpha})y^{\varepsilon'-\alpha'} \\
&= \Big(\sum_{0 \le \alpha \le \varepsilon} \hat{w}_1(x^\alpha) * \hat{w}_2(x^{\varepsilon-\alpha})\Big)y^{\varepsilon'} \\
&= (\hat{w}_1 * \hat{w}_2)(x^\varepsilon)y^{\varepsilon'}\ .
\end{aligned}$$

Since the products $x_1 \cdots x_r, x_i \in L, 0 \le r < \infty$ (resp. $y_1 \cdots y_s, y_i \in N, 0 \le s < \infty$) span $U(L)$(resp. $U(N)$), the proposition is proved.

2. A theorem of Cartan and Guillemin

Let L be an l.c. topological Lie algebra over a field K of characteristic 0, that is, a Lie algebra over K which is also an l.c. space such that the Lie bracket $[\cdot, \cdot]$ is continuous from $L \times L$ to L. Suppose L has a nonabelian minimal closed ideal I. According to [3, Proposition 6.1] I has a maximal proper ideal J. J is closed, and $\bar{I} = I/J$ is a nonabelian simple l.c. Lie algebra. Let $N = \mathrm{Norm}_L J$, the normalizer of J in L. N is open by [3, Proposition 6.2], and I is an l.c. L-module under the adjoint action of L on I. We denote this action by ad and extend it to an action of $U(L)$. Let φ be the canonical map of I onto $\bar{I}$. J is an N-submodule of I_N, so that $\bar{I}$ is an N-module. We will denote the action of N on $\bar{I}$ by $\overline{\mathrm{ad}}$. φ is an N-homomorphism. Thus, by [1], we have an L-homomorphism θ of I into $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$ given by the formula $\theta(x)a = \varphi((\mathrm{ad}\, a)x)$.

Lemma 2.1. *θ is a continuous injection.*

Proof. $\mathrm{ad}\, y$ is continuous on I for each $y \in L$ so that $\mathrm{ad}\, a$ is continuous for $a \in U(L)$. By the continuity of φ, $x \mapsto \theta(x)a$ is continuous for each $a \in U(L)$. But this is just the definition of the continuity of θ. Now let I_0 be the kernel of θ and let $x \in I_0$. If $y \in L$, then $\theta([y, x])a = \theta(x)(ay) = 0$ for all $a \in U(L)$ so that $\theta([y, x]) = 0$. Moreover, $\varphi(x) = \theta(x)1 = 0$ so that $x \in J$. Therefore I_0 is a closed ideal of L contained in J, which is properly contained in I. By the minimality of $I, I_0 = \{0\}$.

Let K' be the commuting ring in $\mathrm{Hom}_K^c(\bar{I}, \bar{I})$ of $\overline{\mathrm{ad}}(I)$. By [3, Proposition 4.4] K' is a finite algebraic extension of K.

Lemma 2.2. *Suppose $K' = K$. Then θ is an l.c. L-module isomorphism. Moreover, I is a topologically absolutely irreducible L-module.*

Proof. Since θ is a continuous injection, $\theta(I)$ is a closed L-submodule of $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$. But we have assumed that $\bar{I}$ is a topologically absolutely irreducible I-module; *a fortiori*, it is a topologically absolutely irreducible N-module. Thus Theorem 1.2 will tell us that $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$ is a topologically irreducible L-module providing we can show that N is the algebra H of that theorem. Plainly $N \subseteq H$, since for each $x \in N$, we can let $s = \overline{\mathrm{ad}}\, x$. Conversely, let $x \in H$ and choose $s \in \mathrm{Hom}_K^c(\bar{I}, \bar{I})$ corresponding to it. If $z \in J$, then $\overline{\mathrm{ad}}[x, z] = s\, \overline{\mathrm{ad}}\, z - (\overline{\mathrm{ad}}\, z)s = 0$ because $\overline{\mathrm{ad}}\, z = 0$. Since $\bar{I}$ has no center, $\varphi([x, z]) = 0$; i.e., $[x, z] \in J$. But this just says that $x \in N$. Therefore $N = H$, as desired. Hence $\theta(I) = \mathrm{Hom}_{U(N)}(U(L), \bar{I})$ and the lemma is proved.

Now $\bar{I}$ is an l.c. Lie algebra upon which N acts via derivations. It follows from the corollary to Proposition 1.4 that $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$ is an l.c. Lie algebra upon which L acts via derivations. Let V be any N-module such that J annihilates V. Then V becomes an $\bar{I}$-module. Plainly the hypotheses of the corollary to Proposition 1.4 are satisfied. Therefore $W = \mathrm{Hom}_{U(N)}(U(L), V)$ is a $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$-module as well as an L-module.

Lemma 2.3. *Let $x \in I$ and $w \in W$. Then $xw = (\theta x)w$.*

Proof. Let $y_1, \cdots, y_r \in L$. Using the multi-index notation of Proposition 1.6, we have $(xw)y^\varepsilon = w(y^\varepsilon x)$. Applying the main antiautomorphism [1, p. 458] to [1, Lemma 7], we have

$$y^\varepsilon x = \sum_{0 \leq \alpha \leq \varepsilon} ((\mathrm{ad}\, y^\alpha)x)y^{\varepsilon - \alpha} .$$

Therefore

$$(xw)y^\varepsilon = \sum_{0 \leq \alpha \leq \varepsilon} ((\mathrm{ad}\, y^\alpha)x)w(y^{\varepsilon-\alpha}) = \sum_{0 \leq \alpha \leq \varepsilon} ((\theta x)y^\alpha)w(y^{\varepsilon-\alpha}) = ((\theta x)w)y^\varepsilon .$$

Since the products $y_1 \cdots y_r, y_i \in L, 0 \leq r < \infty$ span $U(L)$, our lemma is proved.

Corollary. *θ is a homomorphism of I into* $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$.

Proof. Let $V = \bar{I}$. If $x_1, x_2 \in I$, then $\theta([x_1, x_2]) = x_1(\theta x_2) = [\theta x_1, \theta x_2]$.

The foregoing proves the following special case of Guillemin's version [3, Theorem 7.1] of a theorem of Cartan [2, Théorème XII] concerning simple intransitive groups:

Theorem 2.4. *I is a Lie algebra and l.c. L-module isomorphic via θ to a closed subalgebra L-submodule of* $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$. *If $K' = K$, the isomorphism is surjective.*

To translate Theorem 2.4 into the terms of Guillemin's paper, use Proposition 7 of [1] to see that $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$ is isomorphic as a Lie algebra to $\mathrm{Hom}_K(S(L/N), \bar{I})$; it is easy to see that this is isomorphic $K[[(L/N)^*]] \hat{\otimes} \bar{I}$ in the notation of [3](cf. the corollary to Proposition 7 in [1]). Observing that the filtration subalgebras I_s of I as defined in [3, § 6.3] are nothing more than $I_s = \{x \in I\colon (\mathrm{ad}\, a)x \in J \text{ for all } a \in U_{s-1}(L)\}$, we see that θ preserves filtrations. Thus, if $K' = K$, then I is isomorphic to the completion of $\mathrm{gr}\, I$ where gr is the gradation functor. In this way we obtain in short order all of the results of § 7 of [3] in the case $K' = K$. This includes the important case where K is algebraically closed.

We close this section by constructing a representation of L which is associated with the structure of I as given by Theorem 2.4. Let L be any l.c. Lie algebra over K. As in [3], we will call a proper open subalgebra A of L *primitive* if it is maximal and contains no non-trivial ideals of L. L is called *primitive* if it contains a primitive subalgebra. Every simple L is primitive.

Let L, I, J, and N be as above, $\bar{A}$ be a primitive subalgebra of $\bar{I}$, and A be its complete inverse image in I. A lies properly between I and J. Let $M = \mathrm{Norm}_L A$, the normalizer of A in L. The proof of Proposition 6.2 of [3] shows that $M \subseteq N$. M is open by [3, Lemma 6.1]. We consider the l.c. L-module $\mathrm{Hom}_{U(M)}(U(L), K)$, where K is the 1-dimensional trivial M-module. According to Theorem 1.3, this is isomorphic via $\wedge$ as an l.c. L-module to $\mathrm{Hom}_{U(N)}(U(L), \mathrm{Hom}_{U(M)}(U(N), K))$. Now K is trivially an abelian l.c. algebra over K with jointly continuous product. Moreover, M acts on K via derivations, namely, by the trivial action. According to the corollary to Proposition 1.4 and Lemma 1.5, $\mathrm{Hom}_{U(M)}(U(L), K)$, $\mathrm{Hom}_{U(M)}(U(N), K)$, and $\mathrm{Hom}_{U(N)}(U(L), \mathrm{Hom}_{U(M)}(U(N), K))$ are also abelian l.c. algebras with jointly continuous product. According to Proposition 1.6, the above isomorphism is an algebra isomorphism.

$I \subseteq N$ so that I acts on $\mathrm{Hom}_{U(M)}(U(N), K)$ via continuous derivations.

Lemma 2.5. *The ideal of I annihilating* $\mathrm{Hom}_{U(M)}(U(N), K)$ *is J.*

Proof. J is an ideal of N contained in M, and M annihilates K. Application of the main anti-automorphism [1, p. 458] to [1, Lemma 7] shows that $U(N)J = JU(N)$. Therefore J annihilates $\mathrm{Hom}_{U(M)}(U(N), K)$. Now $M \not\supseteq I$, or else A would be a proper ideal of I properly containing J, contradicting the maximality of J. The argument at the bottom of p. 459 of [1] shows that elements of N not in M certainly act non-trivially on $\mathrm{Hom}_{U(M)}(U(N), K)$. Hence I does not annihilate everything, and the annihilating ideal is J by the maximality of J in I.

Theorem 2.6. *For each $x \in I$ and $w \in \mathrm{Hom}_{U(M)}(U(L), K)$, $(xw)^\wedge = (\theta x)\hat{w}$.*

Proof. Note that $(xw)^\wedge = x\hat{w}$ by Theorem 1.3(3). Because of Lemma 2.5, we can then apply Lemma 2.3 to obtain our result.

3. Geometric interpretation

In this section we will suppose, for simplicity, that K is algebraically closed and has characteristic 0. In the notation of [3], Theorems 2.4 and 2.6 say the following: $I \simeq K[[(L/N)^*]] \hat{\otimes} \bar{I}$. L operates by derivations on $K[[(L/M)^*]]$, which is isomorphic to $K[[(L/N)^*]] \hat{\otimes} K[[(N/M)^*]]$. $\bar{I}$ acts on $K[[(N/M)^*]]$. The action of I on $K[[(L/M)^*]]$ is just the action one would expect of $K[[(L/N)^*]] \hat{\otimes} \bar{I}$ on $K[[(L/N)^*]] \hat{\otimes} K[[(N/M)^*]]$.

Suppose that M contains no L-ideals (which we can always arrange for by dividing out by the largest L-ideal in M-this leaves I, A, and J essentially unchanged) and suppose $K = \boldsymbol{C}$. Using the terminology and notation of [5], the above statements correspond to the following geometrical situation: L is the formal algebra at some $p \in X$, X a complex manifold, of a transitive holomorphic Lie algebra sheaf $\mathscr{L}$. M is the closure of the image in L of $\mathscr{L}^0$. I determines a normal subsheaf $\mathscr{I}$ of $\mathscr{L}$, J determines a normal subsheaf $\mathscr{J}$ of $\mathscr{I}$, and N determines a completely integrable differential system $\mathscr{D}$. $\mathscr{D}$ is invariant under $\mathscr{I}$. Let Y be a local leaf (integral submanifold) of $\mathscr{D}$. Then $\mathscr{J}|_Y = \{0\}$, so that we have a weak Lie algebra sheaf $\mathscr{I}|_Y/\mathscr{J}|_Y$ on each such Y, which we denote by $\bar{\mathscr{I}}_Y$. We can choose a Y through p, a local cross-section Z to $\mathscr{D}$ through p, and a diffeomorphism ψ of a neighborhood of y onto $Z \times Y$ in such a way that $\psi_*(\mathscr{I})$ consists of all holomorphic functions on Z with values $\bar{\mathscr{I}}_Y$.

We would like to show that $\bar{\mathscr{I}}_Y$ is transitive and primitive. In formal terms, this amounts to showing that the action of $\bar{I}$ on $K[[(N/M)^*]]$ is transitive and primitive. Looking at the action of I on $\mathrm{Hom}_{U(M)}(U(N), K)$, we see that the subalgebra of I giving rise to derivations of degree ≥ 0 is $I \cap M$. Clearly $I \cap M \supseteq A$. Since A is not an ideal of $I, I \cap M \neq I$. Therefore $I \cap M = A$ by the maximality of A in I. Thus the action of $\bar{I}$ will be primitive if it is transitive. For transitivity we must show that $I + M = N$. Let $\bar{N} = N/J$ and $\bar{A} = A/J$. Then $\bar{M} = M/J$ is the normalizer of $\bar{A}$ in $\bar{N}$, and $\bar{A}$ is a primitive subalgebra of $\bar{I}$. Looking at the derivations $\mathrm{ad}_{\bar{I}}\, x, x \in \bar{N}$, it suffices to show (with a change of notation) the following:

Theorem 3.1. *Let A be a primitive subalgebra of the l.c. Lie algebra L over the algebraically closed field K of characteristic* 0, *and d be a continuous derivation of L. Then there exists an $x \in L$ such that $(d - \mathrm{ad}\, x)A \subseteq A$.*

Proof. This theorem is trivially true if L is a finite dimensional semi-simple

Lie algebra since every derivation is inner. So suppose L is not finite dimensional semi-simple. Let $\{L_i\}$ be the Weisfeiler filtration on L associated with A: L_{-1} is a minimal subspace of L properly containing A and invariant under $\operatorname{ad} A$; $L_j, j \leq -2$ is defined inductively by $L_j = L_{j+1} + [L_{j+1}, L_{j+1}]$; $L_0 = A$; $L_j, j \geq 1$ is defined inductively as $\{x \in L_{j-1} : \operatorname{ad} x(L_{-1}) \subseteq L_{j-1}\}$. Then (see [4] for details): (1)$\{L_j\}$ makes L into a filtered Lie algebra; (2) the $\{L_j\}$ form an open neighborhood base at 0; (3)$L_j = L$ for $j \leq -2$; (4) if $\operatorname{gr} L$ is the corresponding graded algebra, then $\operatorname{gr}_0 L$ acts irreducibly on $\operatorname{gr}_{-1} L$; (5) if $x \in \operatorname{gr}_j L$, $j \geq 0$, and $[\operatorname{gr}_{-1} L, x] = \{0\}$, then $x = 0$; and (6) if L is infinite dimensional, then L is isomorphic to the completion of $\operatorname{gr} L$ in the topology of its downward filtration, that is, $L \simeq \prod_j \operatorname{gr}_j L$. If L is finite dimensional but not semi-simple, let I be a nontrivial abelian ideal of L. Then one shows easily that $A + I = L$, $A \cap I = \{0\}$, I is irreducible under $\operatorname{ad} A$, and A acts faithfully on I via ad_I. It follows that $L_{-1} = L, L_0 = A, L_1 = \{0\}, \operatorname{gr}_{-1} L \simeq I$, and $\operatorname{gr}_0 L \simeq A$. Thus (6) holds in this case also. If $x \in L$, then x_j will denote the component of x in $\operatorname{gr}_j L$, where $\operatorname{gr}_j L$ is regarded as a subspace of L by (6).

We define maps d_n on $\operatorname{gr} L$: $d_n x = (dx)_{n+j}$ if $x \in \operatorname{gr}_j L$. It is trivial that d_n is a derivation of $\operatorname{gr} L$, homogeneous of degree n. Now properties (4) and (5) imply that $\operatorname{gr}_0 L$ is either semi-simple or else has a 1-dimensional center acting faithfully as multiplication by scalars on $\operatorname{gr}_{-1} L$ by a well-known theorem of E. Cartan and N. Jacobson.

Cace I. $\operatorname{gr}_0 L$ has a one-dimensional center. Choose $z \in$ the center of $\operatorname{gr}_0 L$ such that $[z, x] = x$ for $x \in \operatorname{gr}_{-1} L$. Then (see the proof of Proposition 7.2 in [4]) $\operatorname{ad} z$ acts as multiplication by $-j$ on $\operatorname{gr}_j L$. If $x \in \operatorname{gr}_j L$, then $-jd_n x = d_n[z, x] = [d_n z, x] + [z, d_n x] = [d_n z, x] - (j + n)d_n x$. Therefore $\operatorname{ad}(d_n z) = nd_n$ on $\operatorname{gr}_j L$. It follows that $d_n = \frac{1}{n}\operatorname{ad}(d_n z)$ on $\operatorname{gr} L$ for $n \neq 0$. Since d is continuous, $d = \sum_{n \neq 0} \frac{1}{n} \operatorname{ad}(d_n z)$ modulo derivations of degree 0 on L, and the theorem holds in this case.

Case II. $\operatorname{gr}_0 L$ is semi-simple. Then (a) L is finite dimensional, (b) L is the algebra of divergence free formal vector fields, or (c) L is the Hamiltonian algebra (see [4, §6] and [5, Chapter V]). We first claim that $d_n = 0$ if $n \leq -2$. This is evident in subcase (a).

Lemma 3.2. *Let L be a graded Lie algebra with $\operatorname{gr}_j L = \{0\}$ for $j \leq -2$ and also satisfying* (5), *and d be a derivation of degree n, $n \leq -1$. If $d = 0$ on $\operatorname{gr}_{-n-1} L$, then $d = 0$.*

Proof. Since $\operatorname{gr}_j L = \{0\}$ for $j \leq -2, d = 0$ on $\operatorname{gr}_j L$ for $j \leq -n - 1$. Suppose inductively that $d = 0$ on $\operatorname{gr}_j L$ for $j \leq m$ with $m \geq -n - 1$, and let $x \in \operatorname{gr}_{m+1} L$. Then for every $y \in \operatorname{gr}_{-1} L, 0 = d[y, x] = [y, dx]$, because $d = 0$ on $\operatorname{gr}_{-1} L$. Now $dx \in \operatorname{gr}_{n+m+1} L$ and $n + m + 1 \geq 0$. By property (5), $dx = 0$. q.e.d.

Now in subcases (b) and (c), $\mathrm{gr}_j L = \{0\}$ for $j \leq -2$, [4, § 6]. Let $n \leq -2$. We note that $d_n = 0$ on $\mathrm{gr}_0 L$ so that $d_n[x, y] = [x, d_n y]$ for all $x \in \mathrm{gr}_0 L$ and $y \in \mathrm{gr}_{-n-1} L$; i.e., d_n intertwines the representations of $\mathrm{gr}_0 L$ on $\mathrm{gr}_{-n-1} L$ and $\mathrm{gr}_{-1} L$.

In subcase (b) $\mathrm{gr}_j L \subseteq V \otimes S^{j+1}(V^*)$ where $V = \mathrm{gr}_{-1} L$, $\dim V \geq 2$, and the action of $\mathrm{gr}_0 L$ is the natural one induced by the action of $\mathrm{gr}_0 L = sl(V)$ on V (see [5, § 1.10]). Since every finite dimensional representation of $sl(V)$ is completely reducible, $d_n | \mathrm{gr}_{-n-1} L$ is the restriction of an $sl(V)$-homomorphism σ of $V \otimes S^{-n}(V^*)$ into V. σ may be regarded as an $sl(V)$-invariant of $V \otimes (V \otimes S^{-n}(V^*))^* \simeq V \otimes V^* \otimes S^{-n}(V)$, and this invariant may be regarded as an $sl(V)$-homomorphism τ of $\mathrm{Hom}_K(V, V)$ into $S^{-n}(V)$. Now $\mathrm{Hom}_K(V, V)$ is the direct sum of the one-dimensional trivial $sl(V)$-module and the adjoint $sl(V)$-module. These modules, as well as $S^{-n}(V)$, are well-known to be irreducible and non-isomorphic. Thus $\tau = 0$ so that $\sigma = 0$ and $d_n = 0$ on $\mathrm{gr}_{-n-1} L$.

In subcase (c), $\mathrm{gr}_j L \simeq S^{j+2}(V)$ where $V = \mathrm{gr}_{-1} L$, $\dim V = 2k \geq 2$, and $\mathrm{gr}_0 L = sp(V)$(see [5, § 1.12]). Since the action of $sp(V)$ on $S^{j+2}(V)$ is the natural one, d_n interwines the non-isomorphic irreducible $sp(V)$-module $S^{-n+1}(V)$ and V and hence is zero on $\mathrm{gr}_{-n-1} L$.

Applying Lemma 3.2 we see that $d_n = 0$ for $n \leq -2$ in subcases (b) and (c).

Our final step in Case II is to show that $d_{-1} = \mathrm{ad}\, z$ on $\mathrm{gr}_0 L$ for some $z \in \mathrm{gr}_{-1} L$. It will then follow by Lemma 3.2 that $(d - \mathrm{ad}\, z)_n = 0$ for $n \leq -1$ so that $(d - \mathrm{ad}\, z)A \subseteq A$ by the continuity of d as required by the theorem. So let $x, y \in \mathrm{gr}_0 L$. Then $[x, d_{-1} y] - [y, d_{-1} x] - d_1[x, y] = 0$; i.e., d_{-1} is a cocycle for the natural representation of $\mathrm{gr}_0 L$ on $\mathrm{gr}_{-1} L$. Since $\mathrm{gr}_0 L$ is semi-simple, d_{-1} is a coboundary, i.e., for some $z \in \mathrm{gr}_{-1} L$, $d_{-1} = \mathrm{ad}\, z$ on $\mathrm{gr}_0 L$, as desired. This completes the proof of Theorem 3.1.

Remark. It would be desirable to have a proof of Theorem 3.1 which did not use the classification of primitive Lie algebras. The machinery used here is all out of proportion to the result proved.

Returning now to the situation and notation of the beginning of this section, we know that $I + M = N$ so that $\bar{I}$ is represented transitively and primitively on $\mathrm{Hom}_{U(M)}(U(N,)K)$. We next show that the open subalgebras N and M are completely determined by the requirements (1) that $M \subseteq N$, (2) that the action of I on $\mathrm{Hom}_{U(M)}(U(L), K)$ leaves the "differential system" determined by N "leafwise" invariant, (3) that I acts on each "leaf" via the primitive action of $\bar{I}$ determined by $\bar{A}$, and (4) that the actions on the "leaves" are completely independent of each other. In our formal setting, $\mathrm{Hom}_{U(M)}(U(L), K) \simeq \mathrm{Hom}_{U(N)}(U(L), \mathrm{Hom}_{U(M)}(U(N), K))$. (2) clearly requires that $I \subseteq N$. As for (3), if I is to act on $\mathrm{Hom}_{U(M)}(U(N), K)$ transitively, then $M + I = N$; if this action is to come from the primitive action of $\bar{I}$ determined by $\bar{A}$, then $M \cap I = A$. This implies that $M \subseteq \mathrm{Norm}_L A \subseteq \mathrm{Norm}_L J$. Since $I \subseteq \mathrm{Norm}_L J$, we have that $N \subseteq \mathrm{Norm}_L J$. We will be finished if we can show $N = \mathrm{Norm}_L J$, for then the vector space isomorphisms $N/M \simeq I/A \simeq \mathrm{Norm}_L J/\mathrm{Norm}_L A$ will prove that $M = \mathrm{Norm}_L A$.

Now requirement (4) is taken to mean that the set of operators I on $\mathrm{Hom}_{U(N)}(U(L), \mathrm{Hom}_{U(M)}(U(N), K))$ is precisely the set of operators $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$. With our new N and M we can define the maps θ and $\wedge$ exactly as in § 2, for those constructions and the results of Lemmas 2.1, 2.3, and 2.5, and Theorem 2.6 depend only on $N \subseteq \mathrm{Norm}_L J, J \subseteq M \subseteq N$, and $M \not\supseteq I$. Thus I is a subalgebra of the set of operators $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$ on $\mathrm{Hom}_{U(N)}(U(L), \mathrm{Hom}_{U(M)}(U(N), K))$. But if $N \neq \mathrm{Norm}_L J$, then $\mathrm{Hom}_{U(\mathrm{Norm}_L J)}(U(L), \bar{I})$ is a proper closed subspace of $\mathrm{Hom}_{U(N)}(U(L), \bar{I})$(see [1, p. 459]). We know that θ maps I into $\mathrm{Hom}_{U(\mathrm{Norm}_L J)}(U(L), \bar{I})$. Therefore we get the full set of operators only if $N = \mathrm{Norm}_L J$. This proves the correctness of our characterization of M and N.

Finally, we mention that I is not in general the full ideal leaving the "differential system" determined by N "leafwise" invariant even if M contains no L-ideals. In fact, let L be any infinite dimensional l.c. nonsimple primitive Lie algebra; e.g., let L be the algebra of all formal vector fields of constant divergence [5, p. 112]. Let I be the intersection of all closed nonzero ideals of L. According to [4, Proposition 3.4], I is nonabelian and simple, and if A is the unique primitive subalgebra of I, then M is the unique primitive subalgebra of L. Thus M contains no L-ideals. Since $J = \{0\}, N = L$. Therefore the L-ideal leaving the "differential system" determined by N "leafwise" invariant is L itself.

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