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MINIMAL SUBMANIFOLDS OF LOW COHOMOGENEITY

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Introduction

Let M be a Riemannian manifold and $I(M)$ its full isometry group. It was shown in [MST] that $I(M)$ is naturally a Lie group which acts differentiably on M. A Lie subgroup G of $I(M)$ is called an *isometry group of M*, and the codimension of the maximal dimensional orbits is defined to be the *cohomogeneity of G*. The cohomogeneity of $I(M)$ is called the *cohomogeneity of M*.

Many important Riemannian manifolds such as symmetric spaces, Stiefel manifolds and flag manifolds are of cohomogeneity zero, i.e., homogeneous. The local properties of any such space G/H are completely determined by those at one point and can actually be computed in terms of the infinitesimal structure of the pair $H \subseteq G$. However, many aspects of the global geometry of these spaces (in particular, many questions concerning compact totally geodesic or, more generally, compact minimal submanifolds) are poorly understood. The point of view here is that of using the inherent symmetries of these spaces as a tool in dealing with global questions.

From the point of view of transformation groups the full isometry group G on a Riemannian homogeneous space is too simple, orbit-wise, to be of interest. However, there are many actions induced from the transitive action of G (such as the natural actions on the orthonormal k-frame bundles or the restricted actions of the many subgroups of G) which shed light on the geometric structure of the space. In particular the action of each subgroup gives an interesting decomposition of the space into geodesically parallel orbits. To make proper use of the full isometry group one should study these decompositions, particularly for actions of low cohomogeneity.

The existence and global behavior of compact minimal submanifolds of a homogeneous Riemannian manifold is, in full generality, a difficult area of study. The nonlinearity of the problem makes even the construction of explicit examples reasonably difficult and, at the same time, makes such examples indispensable guidelines for research. Thus, it is natural to try to reduce the complexities of the situation by means of some isometry group G' of low cohomogeneity and, in particular, to look for the existence and general behavior of G'-invariant, minimal submanifolds. That is the purpose here.

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Let M be a Riemannian manifold and G a compact, connected group of isometries of M. Our first observation is: *a G-invariant submanifold N of M is minimal if and only if the volume of N is stationary with respect to compactly supported variations of N through G-invariant submanifolds*. The space of orbits M/G of G is naturally a differentiable stratified set with several natural, smooth metric structures. Our second observation, roughly stated, is that *a G-invariant submanifold N of M is minimal if and only if N/G is minimal in M/G with the appropriate metric*. From these two observations we proceed by elementary techniques to explicitly construct vast numbers of compact minimal submanifolds in nearly all homogeneous spaces. In many cases, for example Stiefel manifolds and group spaces, the set of (non-isometric) compact, minimal *hypersurfaces* is at least countably infinite. Furthermore, we give a detailed proof of the fact that *every compact, homogeneous space can be minimally immersed into some Euclidean sphere* [$H1$].

These considerations lead to the discovery of certain distinguished, minimal hypersurfaces, called "equators" in rank-one symmetric spaces, Stiefel manifolds and Grassmann manifolds and to conjectures concerning them, some of which have since been affirmatively settled [$L3$].

The next task undertaken is to classify the compact, minimal hypersurfaces in the sphere which have cohomogeneity-one, i.e., have self-congruence groups of cohomogeneity-one. To do this we first classify the compact, linear groups of cohomogeneity 2 and 3 or R^n together with their orbit spaces and the appropriate metrics. This classification is of interest in itself and should, moreover, be useful in studying questions in global geometry not considered here.

We then reduce the problem to the study of the geodesic structure of certain singular 2-manifolds. This study is carried out in detail to varying degrees.

In the case of the 3-sphere these methods give a very satisfactory result. The only group producing codimension-2 orbits on S^3 is $U(1)$. However, there are infinitely many inequivalent actions of $U(1)$ on S^3. For every action except one the orbit space with the appropriate metric is an ovaloid of revolution. The geodesics on the ovaloid, which correspond explicitly to invariant minimal submanifolds of S^3, can be expressed explicitly in terms of elliptic functions. For each action, including the exceptional one, the family of closed geodesics is countably infinite. The union of these families constitutes all compact minimal surfaces in S^3 of cohomogeneity-one and contains all those with nullity ≤ 5.

In general the procedures established here are useful for studying the behavior of high dimensional minimal varieties. For example this point of view gives geometric insight into certain counterexamples to the Bernstein conjecture (as shown below) and has led to a new proof of the non-interior-regularity for the Plateau Problem in dimension 8 [$L4$].

CHAPTER I

GENERALITIES

1. The fundamental theorem

Let M be a Riemannian manifold and G a compact, connected group of isometries of M. A submanifold $f\colon N \to M$ is called *G-invariant* if there exists a smooth action of G on N such that $gf = fg$ for all $g \in G$. The submanifold is said to be minimal if its mean curvature vector field vanishes identically (cf. [S]). The minimality of f is equivalent to the requirement that the induced volume of N be stationary with respect to compactly supported variations of f.

By an *equivariant variation* of a G-invariant submanifold $f\colon N \to M$ we mean a differentiable variation $f_t\colon N \to M$, $-\varepsilon < t < \varepsilon$, $f_0 = f$, through submanifolds such that $gf_t = f_t g$ for all $g \in G$ and all t. Here the action of G on N is independent of t. Hence, throughout the variation *the orbit structure of the submanifolds remains unchanged.*

Our fundamental observation is the following.

Theorem 1. *Let $f\colon N \to M$ be a G-invariant submanifold of M. Then $f\colon N \to M$ is minimal if and only if the volume of N is stationary with respect to all compactly supported, equivariant variations.*

Proof. Let $\mathscr{H}$ be the mean curvature vector field on N. Since $\mathscr{H}$ depends only on the immersion f, and f is G-invariant, we have that $g_*\mathscr{H} = \mathscr{H}$ for all $g \in G$.

Without loss of generality we may assume that the volume of N is finite. Let φ be a continuous, G-invariant, compactly supported function on N. We define a variation f_t, $-\varepsilon < t < \varepsilon$, of f by

$$f_t(x) = \exp_{\tilde{x}}\,(t\varphi(\tilde{x})\mathscr{H}(\tilde{x}))\ , \tag{1.1}$$

where $\tilde{x} = f(x)$, and exp is the usual exponential mapping on $T_{\tilde{x}}(M)$. We choose $\varepsilon > 0$ small enough that each f_t is an immersion. Observe that for each $g \in G$

$$\begin{aligned} g \circ f_t(x) &= g \circ \exp_{\tilde{x}}\,[t\varphi(\tilde{x})\mathscr{H}_{\tilde{x}}] = \exp_{g\tilde{x}}\,[g_*(t\varphi(\tilde{x})\mathscr{H}_{\tilde{x}})] \\ &= \exp_{g\tilde{x}}\,[t\varphi(\tilde{x})g_*\mathscr{H}_{\tilde{x}}] = \exp_{g\tilde{x}}\,[t\varphi(g\tilde{x})\mathscr{H}_{g\tilde{x}}] = f_t \circ g(x)\ , \end{aligned}$$

since $g\tilde{x} = gf(x) = fg(x)$. Hence each f_t is equivariant.

Let ω_t be the volume element of the metric induced by f_t, and set $V(t) = \int_N \omega_t$. Since the variation vector field is simply $\varphi\mathscr{H}$, the first variational formula has the form

$$\left.\frac{dV}{dt}\right|_{t=0} = -\int_N \varphi\,|\mathscr{H}|^2\,\omega_0 \ .$$

Clearly $\mathscr{H} \equiv 0$ if and only if $\left.\frac{dV}{dt}\right|_{t=0} = 0$ for all such variations. This proves the theorem.

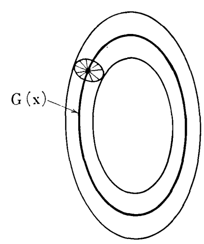
2. Differentiable transformation groups

To make proper use of this theorem we shall need some notions from the theory of differentiable transformation groups. For each $x \in M$ let G_x be the isotropy (stability) subgroup of x, and $G(x) \approx G/G_x$ be the orbit of x under G. The group G_x acts naturally on the normal vectors to $G(x)$ at x. This action φ_x, called the *slice representation* at x, associates a vector bundle $\alpha(\varphi_x)$ with the canonical G_x-bundle $G_x \to G \to G/G_x$. One of the important facts is the following [MY].

Theorem (*Slice*). *Let $\nu(G(x))$ denote the normal bundle of $G(x)$ in M. Then*

$$\nu(G(x)) = \alpha(\varphi_x) \ ,$$

and the exponential map maps a small disk bundle of $\nu(G(x))$ equivariantly and diffeomorphically onto an invariant tubular neighborhood of $G(x)$.



Two orbits, $G(x)$ and $G(y)$, are said to be of the *same type* if G_x and G_y are conjugate in G. The conjugacy classes of the subgroups $\{G_x\colon x \in M\}$ are called the *orbit types* of the G-space M. We partially order the orbit types as follows:

$$(H) \gtrsim (K) \Longleftrightarrow \exists g \in G \ni K \supseteq gHg^{-1} \ ,$$

where (H) denotes the conjugacy class of H. From the theorem above we have the following

Corollary (*Lower semi-continuity of orbit types*). *In a small, invariant neighborhood U of an orbit $G(x)$ there is only a finite number of orbit types, and for each $y \in U$*

$$(G_y) \gtrsim (G_x) \ .$$

Another important fact we shall need is the following [MSY].

Theorem (*Principal orbit type*). *Let M be a connected manifold with a differentiable G-action. Then there exists a unique orbit type (H) such that $(H) \gtrsim (K)$ for all orbit types (K) of the action. Moreover, the union of all orbits of type (H), namely $M^* = \{x \in M\colon G_x \in (H)\}$, is an open, dense, submanifold of M.*

Following [MY] we call (H) the *principal orbit type* of the G-space M. If $(H') \neq (H)$ but $\dim H' = \dim H$, then (H') will be called an *exceptional orbit type*. All orther orbit types will be called *singular*,

3. The singular set

From the discussion in § 2 we see that the *singular set* $M \sim M^*$ is a closed, nowhere dense set with a natural differentiable stratification. Let (H') be a non-principal orbit type, and let $M_{(H')} = \{x \in M\colon G_x \in (H')\}$. Then by the Slice Theorem each component of $M_{(H')}$ is a submanifold of M which by Theorem 1 must be minimal. More generally, we have

Corollary 1.1. *Let $\{(H_\alpha)\}_{\alpha \in A}$ be any collection of non-principal orbit types. Then wherever the set $M' = \{x \in M\colon G_x \in (H_\alpha)$ for some $\alpha \in A\}$ is a submanifold, it is a minimal submanifold.*

Hence, the singular set is a minimal variety stratified by minimal varieties.

In particular, we note that any orbit having no nearby orbit of the same type is automatically a closed minimal submanifold.

This stratification of the singular set by minimal varieties is clearly illustrated by the following examples. We denote by φ the action of G on M and define a G-space formally to be the triple (M, G, φ).

Example 1.2. Let $M = G$ with the bi-invariant metric, and set $\varphi_g(g') = gg'g^{-1}$. The conjugacy classes of G are the orbits of this action and have as a fundamental domain the Cartan polyhedron P. This principal orbits correspond to P^0, and the singular orbits to the linear complex ∂P. By Corollary 1.1 we have that *the inverse image of any subcomplex of ∂P under φ is a minimal variety in G.*

In particular, if h is the Cartan subalgebra of the Lie algebra of G, and $\{\alpha_k\}_{k=1}^m$ is any collection of roots, then the union of the orbits meeting the set

$$\{\exp(t)\colon t \in h \text{ and } \alpha_1(t) = \cdots = \alpha_m(t) = 0\}$$

is a closed minimal submanifold of G.

Example 1.3. Let $(M, G, \varphi) = (\boldsymbol{R}^{k_1} \times \cdots \times \boldsymbol{R}^{k_n}, SO(k_1) \times \cdots \times SO(k_n), \rho_{k_1} \oplus \cdots \oplus \rho_{k_n})$ where ρ_k is the standard representation of $SO(k)$ on $\boldsymbol{R}^k$. Let π_j denote the orthogonal projection of $\boldsymbol{R}^{k_1} \times \cdots \times \boldsymbol{R}^{k_n}$ onto the j^{th} factor. Then the singular set of φ is $\bigcup_j \pi_j^{-1}(0)$. The various orbit types and their corresponding strata in the singular set are evident.

Example 1.4. Let $(M, G, \varphi) = (\boldsymbol{R}^5, SO(3), \Lambda^2\rho_3 - \theta)$ where θ denotes a 1-dimensional trivial representation. (This representation arises as the isotropy representation of the symmetric space $SU(3)/SO(3)$ (cf. Chapter II, § 3) and also as the action $O(A) = O^tAO$ of $SO(3)$ on the space of 3×3 traceless, symmetric matrices where $\langle A, B\rangle = \text{trace}\,(A^tB)$.) This representation gives an action of $SO(3)$ on $S^4 \subset \boldsymbol{R}^5$ having codimension 1 principal orbits $\approx SO(3)/Z_2 \times Z_2$, [$H2$]. The space of orbits is naturally a closed interval where the endpoints correspond to the two singulur orbits. By the above remarks these two orbits are automatically minimal. In fact, each orbit represents a minimal imbedding of the projective plane with constant curvature into S^4, the so-called Veronese surface.

4. The reduction theorem

We now discuss how to use groups of invariance to reduce questions concerning minimal submanifolds to a simpler form.

Let M and G be as above, and denote by M/G the orbit space and by $\pi: M \to M/G$ the canonical projection. The subset M^*/G of M/G is a manifold which carries a natural differentiable structure under which

$$G/H \longrightarrow M^* \xrightarrow{\pi} M^*/G$$

is a differentiable fibre bundle. (H denotes the principal isotropy subgroup.)

More generally, if we stratify M by the components of the sets $M_{(H')} = \{x \in M\colon G_x \in (H')\}$, then M/G carries a corresponding differentiable stratification such that the stratified morphism π, when restricted to components of $M_{(H')}$, is a differentiable bundle.

The set M^*/G carries a natural Riemannian structure as follows. Let $\mathscr{D}$ denote the distribution of normal planes of the orbits of G in M^*. Fix $x \in M^*/G$ and choose $x' \in \pi^{-1}(x)$. Then to each tangent vector X at x there corresponds a unique tangent vector X' in $\mathscr{D}_{x'}$ such that $\pi_*X' = X$. We define the metric g at x in M^*/G by

$$g(X, Y) = g'(X', Y') ,$$

where g' is the metric on M. g is a well defined Riemannian metric on M^*/G which satisfies the condition that the length of a curve γ in M^*/G equals the length of an orthogonal trajectory through the orbits of $\pi^{-1}(\gamma)$. Hence the distance between points in M^*/G is simply the distance between the corresponding orbits in M. (Note: With this metric, $\pi | M^*$ becomes a Riemannian submersion as defined by B. O'Neill [0].)

The same process will, in fact, define a "smooth" metric structure over all of M/G under which each of the strata is a Riemannian manifold.

We introduce on M/G a *volume function* $V\colon M/G \to \boldsymbol{R}$ as follows.

$$V(x) = \begin{cases} \operatorname{Vol}(\pi^{-1}(x))\ , & \text{if } \pi^{-1}(x) \text{ is a principal orbit}\ , \\ m\cdot \operatorname{Vol}(\pi^{-1}(x))\ , & \text{if } \pi^{-1}(x) \text{ is an exceptional orbit}\ , \\ 0\ , & \text{otherwise}\ , \end{cases}$$

where $m = \#(H'/H)$ for an appropriate $H' \in (G_{\tilde{x}})$, $\tilde{x} \in \pi^{-1}(x)$. Straightforward arguments using the Slice Theorem show that *the function V is continuous on M/G and differentiable on M^*/G.*

In many cases the volume function is a natural and easily computable function as the following examples illustrate.

Example 1.5. Let $SO(n)$ act on S^n by rotation about a fixed axis. This action commutes with the antipodal map and thereby produces an action on projective space $\boldsymbol{RP}^n$. The space $\boldsymbol{RP}^n/SO(n)$ can be (isometrically) considered as the interval $[0, \pi/2]$, the principal orbits $(0, \pi/2)$ are all standard spheres S^{n-1}, the endpoint 0 corresponds to the unique fixed point, and the endpoint $\pi/2$ corresponds to an exceptional orbit $\approx \boldsymbol{RP}^{n-1}$. At the latter orbit the slice representation is effectively $\boldsymbol{Z}_2$, and the $(n-1)$-dimensional volume drops by a half. However,

$$V(\theta) = c_{n-1} \sin^{n-1}\theta$$

for $\theta \in [0, \pi/2]$ where c_{n-1} is the volume of the unit $(n-1)$-sphere.

Example 1.6. Let $M = G$, and φ be as in Example 1.2. Let T be a maximal torus, and h the Cartan subalgebra, and let Δ_+ denote the set of positive roots. Define

$$Q(t) = \prod_{\beta \in \Delta_+} \sin(\pi\langle\beta, t\rangle)$$

for $t \in h$. The orbit space $G/\varphi \simeq T/(\text{Weyl group}) \simeq$ Cartan polyhedron $\simeq$ the closure of any component of $\{t \in h\colon Q(t) \neq 0\}$, and the volume function is

$$V(t) = cQ^2(t)$$

for some constant c.

Example 1.6′. Let G be semisimple, and denote by $\mathfrak{g}$ the Lie algebra of G with inner product $=$-(Killing form). By linearizing φ at the identity we produce an orthogonal action on $\mathfrak{g}$ (denoted Ad) whose orbit space is given naturally by the Weyl chamber of $\mathfrak{g}$ with the flat metric. The volume function is given by

$$V(t) = \prod_{\alpha \in \Delta_+} \alpha^2(t)\ .$$

Example 1.7. Let $(\boldsymbol{R}^n, G, \varphi)$ be an orthogonal representation of the compact group G, and V the volume function, and set $\tilde{V} = V \circ \pi$ where $\pi\colon \boldsymbol{R}^n \to \boldsymbol{R}^n/G$

is the standard projection. Then $\tilde{V}^2$ is a homogeneous, G-invariant polynomial in n-variables, and the set where $\tilde{V} = 0$ is precisely the union of all singular orbits. Hence, by Corollary 1.1 we have that

$$S^{n-1} \cap \{x \in \boldsymbol{R}^n : \tilde{V}(x) = 0\}$$

is an algebraic minimal variety in S^{n-1}.

Let $f: N \to M$ be a G-invariant submanifold of M, and for simplicity assume that $f(N) \cap M^* \neq 0$. (There is no loss of generality in this assumption since M may always be replaced by certain natural substrata for which the assumption holds and to which all subsequent arguments apply.) Let ν denote the common dimension of the principal orbits, namely $\dim G - \dim H$.

Definition. By the cohomogeneity of $f: N \to M$ in the G-space M we mean the integer

$$\dim N - \nu .$$

Thus a submanifold of cohomogeneity zero is a principal orbit, and more generally a submanifold of cohomogeneity k projects to a map $\tilde{f}: N/G \to M/G$ such that $(\tilde{f}\,|\,N^*/G): N^*/G \to M^*/G$ is a k-dimensional submanifold.

For each integer $k \geq 1$ we define a metric

$$g_k = V^{2/k} g \tag{1.2}$$

over M/G where g is the metric on M/G constructed above. Each g_k is a Riemannian metric over M^*/G which goes continuously to zero at the singular boundary.

Observe that *the volume of a G-invariant submanifold $f: N \to M$ of cohomogeneity k in M is equal to the volume of $\tilde{f}: N/G \to M/G$ in the manifold* $(M/G, g_k)$. Together with Theorem 1 this shows the following.

Theorem 2. *Let $f: N \to M$ be a G-invariant submanifold of cohomogeneity k, and let M/G be given the metric g_k defined in* (1.2). *Then $f: N \to M$ is minimal if and only if $\tilde{f}: N^*/G \to M^*/G$ is minimal.*

Remark. Theorem 2 also holds for cohomogeneity zero in which case g_k is replaced by the function V, and "minimal in M^*/G" is replaced by "critical point of V". In this simplest case we already have a non-trivial application. Let M and G be as above.

Corollary 1.8. *Suppose M is compact, and let* (H) *be any orbit type. Then the homogeneous space G/H can be minimally immersed into M.*

Proof. Any component M' of $M_{(H)} = \{x \in M : G_x \in (H)\}$ is a differentiable manifold, and in the above manner M'/G is also. By the Slice Theorem each point of $\overline{M'/G} \sim M'/G$ corresponds to an orbit type $(H') \precsim (H)$. If $\dim H' = \dim H$, then the orbits of type (H) pass in the limit to a finite covering of this orbit. (Up to conjugacy we have $H' \subset H$ and the covering is simply

$H'/H \to G/H \to G/H'$.) Let $\nu = \dim(G/H)$, and define *as above* the ν-dimensional volume function V on $\overline{M'/G}$. V is continuous as thus has a maximum at some point p. If $p \in M'/G$ we are done. If not, p must correspond to one of the exceptional orbit types, and again it is clear that $\pi^{-1}(p)$ is minimal. This completes the proof.

By a theorem of G. Mostow every compact, homogeneous space can be realized as the orbit of an orthogonal action on $\boldsymbol{R}^n$[*MO*]. Hence, we have the following theorem quoted in [*H*1] without proof.

Theorem 3. *Every compact homogeneous space can be minimally immersed into* S^n.

Note. The metric induced by this immersion will be G-invariant, but it may not be the "nicest" metric. For instance, the manifold $SO(3)/\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ in Example 1.4 does not have constant curvature.

While many special cases are treated in subsequent chapters, there is one case of current interest which deserves mentioning here.

Example 1.7. Let $G = SO(n) \times SO(m)$ acting on $\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}$ as in Example 1.3. Here the orbit space is naturally presented as $B = \{(x, y, z) \in \boldsymbol{R}^3:$ $x \geq 0$ and $y \geq 0\}$ where $\pi: \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R} \to B$ is defined by $\pi(X, Y, Z) =$ $(|X|, |Y|, Z)$. The singular set corresponds to $\{(x, y, z) \in B: xy = 0\}$. The volume function up to a constant is $V(x, y, z) = x^{n-1}y^{m-1}$, and the metric g is just $dx^2 + dy^2 + dz^2$. Observe that all non-parametric minimal hypersurfaces $Z = F(X, Y)$ in $\boldsymbol{R}^{n+m+1}$ which are G-invariant (cohomogeneity $= 2$) must, by Theorem 2, be precisely the inverse images under π of the surfaces $z = f(x, y)$ in B which are extremals of the area integral for the metric

$$g_2 = x^{n-1}y^{m-1}(dx^2 + dy^2 + dz^2) .$$

This integral is $A(f) = \displaystyle\int\!\!\int x^{n-1}y^{m-1}\sqrt{1 + f_x^2 + f_y^2}\,dxdy$ for which the Euler-Lagrange equations are

$$\begin{aligned} &(1 + f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1 + f_x^2)f_{yy} \\ &\qquad + \left[(n-1)\frac{f_x}{x} + (m-1)\frac{f_y}{y}\right](1 + f_x^2 + f_y^2) = 0 . \end{aligned} \tag{1.3}$$

It has been recently shown in a celebrated paper of Bombieri, De Giorgi, and Giusti [*BDG*] that when $n = m = 4$, global solutions to (1.3) can be found over the quadrant thus contradicting the long standing Bernstein conjecture in dimensions greater than eight.

Remark 1.8. The statement of Theorem 2 clearly remains true when the metric g_k is replaced by a positive scalar multiple of itself.

Remark 1.9. The above cohomogeneity is defined *with respect to the action of* G. However, if we define $I(M, f(N))$ to be the group of isometries of M, which leave $f(N)$ invariant, then this group depends only on the immersion f. When N is compact, $I(M, f(N))$ is compact and we can define the *absolute cohomogeneity* of N in M to be the cohomogeneity with respect to the action of $I(M, f(N))$.

5. Applications to homogeneous spaces

Theorem 2 is interesting and particularly simple when all the orbits of G are isometric. In this case $M/G \overset{\text{def}}{=} \tilde{M}$ is a non-singular manifold and $\pi\colon M \to \tilde{M}$ a fibre bundle. Moreover, the volume function is constant, and thus the metrics $g, g_1, g_2, g_3, \cdots$ are all equivalent (cf. Remark 1.8). Hence, if we provide $\tilde{M}$ with the natural metric g we have that *N is a minimal submanifold of $\tilde{M}$ if and only if $\pi^{-1}(N)$ is a minimal submanifold of M.* (In this section all mention of the immersion $f\colon N \to M$ will be suppressed.)

One particularly important class of examples of this sort arises in the following way. Let H be a compact, connected subgroup of a connected Lie group G, and suppose G has a left invariant metric g' which at the identity is Ad_H-invariant. Then H acts isometrically on G by multiplication from the right, and all the orbits of this action are isometric. Furthermore, *the canonical metric on the homogeneous space G/H associated with g'* (cf. [KN, p. 200]) *is exactly the orbit space metric g defined in* § 4. Thus if $\pi\colon G \to G/H$ is the canonical projection, we have

Proposition 1.10. *N is a minimal submanifold of G/H if and only if $\pi^{-1}(N)$ is a minimal submanifold of G.*

We now consider a somewhat more general situation. Let $H_1 \subset H_2$ be compact, connected subgroups of a connected Lie group G, $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{h}_1, \mathfrak{h}_2$ the subalgebras corresponding to H_1 and H_2. Suppose $\langle \cdot, \cdot \rangle$ is any positive definite Ad_{H_2}-invariant symmetric bilinear form on $\mathfrak{g}$. Then the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{h}_1^\perp$ and $\mathfrak{h}_2^\perp$ gives rise canonically to G-invariant Riemannian metrics on the homogeneous spaces G/H_1 and G/H_2 respectively [KN, p. 200]. Given such metrics (together with the above connectedness assumptions) we call the fibration

$$H_2/H_1 \to G/H_1 \to G/H_2$$

a *regular fibration of Riemannian homogeneous spaces.*

Theorem 4. *Let $H_2/H_1 \to G/H_1 \xrightarrow{\pi} G/H_2$ be a regular fibration of Riemannian homogeneous spaces. Then N is a minimal submanifold of G/H_2 if and only if $\pi^{-1}(N)$ is a minimal submanifold of G/H_1.*

Proof. Let g' be the left invariant metric on G associated with the form $\langle \cdot, \cdot \rangle$. With this metric both of the fibrations $\pi_1\colon G \to G/H_1$ and $\pi_2\colon G \to G/H_2$ satisfy the hypotheses of Proposition 1.10. Applying this proposition to the commutative diagram

$$\begin{tikzcd} & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G/H_1 \arrow[rr, "\pi"] & & G/H_2 \end{tikzcd}$$

proves the theorem.

This theorem is quite useful because most natural fibrations of Riemannian homogeneous spaces are regular. If, say, G is compact and $\langle \cdot, \cdot \rangle$ is Ad_G-invariant (so the resulting metric on G is bi-invariant), then all associated fibrations are regular.

Beginning with known examples in spheres, Theorem 4 produces a vast collection of closed minimal submanifolds in compact homogeneous spaces. Closed minimal varieties in spheres have been constructed in [$H2$] and [$L2$] and many more will be found below. If we observe that the spaces $SO(n)/SO(n-1)$, $SU(n)/SU(n-1)$, $Sp(n)/Sp(n-1)$, $G_2/SU(3)$, $\mathrm{Spin}(7)/G_2$, Spin (9)/Spin (7) (non-standard Spin (7)-subgroup) all naturally represent Euclidean spheres, we see how many examples can be produced. For example, let $\xi_{m,k}$ be a compact minimal surface of genus mk imbedded in S^3[$L2$]. By letting S^3 lie geodesically in S^{4n-1} we can consider $\xi_{m,k}$ minimally imbedded there. For each closed subgroup G of $SO(4n-1)$ (resp. $SU(2n-1), Sp(n-1)$) the inverse image of $\xi_{m,k}$ under the canonical projection is a closed, non-1-connected minimal submanifold imbedded in $SO(n)/G$ (resp. $SU(n)/G, Sp(n)/G$. In particular, we can produce a number of interesting minimal varieties in Stiefel manifolds.

Similarly, let G/K be a Hermitian symmetric space, and suppose $K/H \to G/H \to G/K$ is a regular Riemannian fibration. Then for any Kähler submanifold $N \subset G/K$ (automatically minimal [S]) $\pi^{-1}(N)$ is minimal in G/H.

Example 1.11. Let

$$\begin{aligned} &S^1 \longrightarrow S^{2n+1} \xrightarrow{\pi} CP^n \,, \\ &S^3 \longrightarrow S^{4n+3} \xrightarrow{\pi} QP^n \end{aligned} \tag{1.4}$$

be the standard Hopf fibrations of complex and quaternionic projective spaces. With the usual symmetric space metrics these fibrations, which can be rewritten

$$\begin{aligned} U(1) &\to U(n+1)/U(n) \to U(n+1)/U(n) \times U(1) \,, \\ \mathrm{Sp}\,(1) &\to \mathrm{Sp}\,(n+1)/\mathrm{Sp}\,(n) \to \mathrm{Sp}\,(n+1)/\mathrm{Sp}\,(n) \times \mathrm{Sp}\,(1) \,, \end{aligned}$$

are regular. Hence, if N is any complex submanifold of CP^n, then $\pi^{-1}(N)$ is minimal in S^{2n+1}. This gives a number of algebraic minimal varieties in spheres. We also note that $QP^1 = S^4$ with the standard metric. Hence, letting M denote the minimal hypesurface in S^4 of type $SO(3)/Z_2 \times Z_2$ discussed in [$H2$], we have that $\pi^{-1}(M)$ is a minimal hypersurface of S^7. Similarly, for each compact minimal surface $\xi_{m,k} \subset S^3 \subset S^4$ discussed in [$L2$], $\pi^{-1}(\xi_{m,k})$ is a closed, 5-dimensional minimal variety in S^7.

Thinking in the reverse direction is equally interesting. One can ask: Are there distinguished hypersurfaces in $\boldsymbol{C}P^n$ and $\boldsymbol{Q}P^n$ which generalize the equators of spheres? For $n > 1$ these spaces have no totally geodesic hypersurfaces. However, the above picture provides the following natural candidates in the class of minimal surfaces. In S^{n+1} the hypersurfaces

$$M_{p,q} = S^p(\sqrt{p/n}) \times S^q(\sqrt{q/n}) \tag{1.5}$$

for $p + q = n$ and $p = 0, \cdots, [n/2]$ have been variously characterized by their common geometric properties [CDK], [$L1$], [S]. Note that for each pair of integers $p, q \geq 0$ such that $p + q = n$ there are surfaces

$$M_{2p-1,2q-1} \subset S^{2n-1} ,$$
$$M_{4p-1,4q-1} \subset S^{4n-1} ,$$

which are invariant under the respective actions of $U(1)$ and Sp (1) in (1.4). These surfaces project to minimal hypersurfaces $M^*_{p,q}$ and $M^{**}_{p,q}$ of $\boldsymbol{C}P^{n-1}$ and $\boldsymbol{Q}P^{n-1}$ respectively, which, when $n = 2$, are in fact the equatorial hyperspheres. It has been shown in [$L3$] that these "generalized equators" admit strong intrinsic characterizations which distinguish them in the class of minimal hypersurfaces.

Let H and K be compact, connected subgroups of a compact connected Lie group G. Fix a bi-invariant metric on G and assume that

$$H \longrightarrow G \xrightarrow{\pi_1} G/H ,$$
$$H \longrightarrow G \xrightarrow{\pi_2} K\backslash G$$

are regular fibrations of Riemannian homogeneous spaces, where $K\backslash G$ is the space of right cosets. Then K acts isometrically on G/H by mutiplication from the left, and H acts isometrically on $K\backslash G$ by multiplication from the right. Of course $K \times H$ acts isometrically on G by multiplying by K from the left and H from the right.

Following the above reasoning produces a useful observation.

Corollary 1.12. *Let N be a K-invariant minimal submanifold in G/H. Then $\pi_1^{-1}(N)$ is a $K \times H$ invariant minimal submanifold in G and $\pi_2(\pi_1^{-1}(N))$ is an H-invariant minimal submanifold in $K\backslash G$.*

Hence we can take invariant minimal submanifolds in one homogeneous space, lift them to the group and then push them into completely different homogeneous spaces. If the original submanifold had cohomogeneity k, then the other two will have the same cohomogeneity k. Thus each family of cohomogeneity one minimal hypersurfaces in S^n constructed below produces similar families of hypersurfaces in homogeneous spaces which are not fibred over S^n.

We note that taking this viewpoint produces good candidates for generalized equators in Grassmann manifolds. Let $M_{p-1,q-1}$ be the generalized Clifford surface defined by (1.5). The manifold is an extremal of the standard action of $SO(p) \times SO(q)$ on the sphere. Let $G_{p,q}(\boldsymbol{R})$ be the real Grassmannian $SO(p) \times SO(q) \backslash SO(p+q)$. As above we have fibrations

$$\begin{tikzcd} & SO(p+q) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S^{p+q-1} & & G_{p,q}(\boldsymbol{R}) \end{tikzcd}$$

and by Corollary 1.12 the manifold $\tilde{M}_{p,q} = \pi_2(\pi_1^{-1}(M_{p-1,q-1}))$ is a homogeneous minimal hypersurface of $G_{p,q}(\boldsymbol{R})$. This procedure similarly produces distinguished minimal hypersurfaces in the complex and quaternionic Grassmann manifolds. These surfaces should admit geometric characterizations similar to those in [L3].

6. Nullity and Jacobi fields

Let $f\colon N \to M$ be a compact minimal submanifold. The second variation of f produces a symmetric bilinear form II defined on the space $\mathcal{N}$ of normal vector fields on N. This bilinear form generalizes the Hessian form associated with geodesics (cf. [M]) and relates similarly to a generalized Morse theory for minimal submanifolds (see [S]). The (finite dimensional) null space $\mathcal{N}_0$ of II consists of the zeros of a certain elliptic partial differential operator on $\mathcal{N}$, the so-called *Jacobi fields* [S]. The dimension of $\mathcal{N}_0$ is called the *nullity* of the immersion.

The space $\mathcal{N}_0$ corresponds to the variations of f which preserve area to second order. If $f_t\colon N \to M$ is a variation of f through minimal immersions, the normal component of $(f_0)_*(\partial/\partial t)$ is a Jacobi vector field. Using Cartan-Kähler theory D. Leung has recently proved that locally *every Jacobi field on N is obtained in this way*. Suppose $f\colon N \to M$ is a G-invariant minimal submanifold of cohomogeneity k, and let $f\colon N^*/G \to M^*/G$ be the associated minimal immersion (with metric g_k on M^*/G). There is a natural one-to-one correspondence between G-invariant normal vector fields J' on N and normal vector fields J on N^*/G (with proper extendability to the singular set.) This correspondence relates J' to J where $\pi_* J'_x = J_{\pi(x)}$ for each $x \in N^*$.

Let J' and J be such a pair of normal vector fields.

Proposition 1.13. *J' is a Jacobi field on N if and only if J is a Jacobi field on N^*/G.*

Proof. Assume J is a Jacobi field. We first observe that J' is a Jacobi field if $II(J', V') = 0$ for all *invariant* normal fields V', since if V is any normal vector field, then $II(J', V) = \int_G II(g_*J', g_*V)dg = \int_G II(J', g_*V)dg = II\left(J', \int_G g_*Vdg\right) = 0$. However, for any invariant field V' (with corresponding field $\bar{V}$ on N^*/G) we can construct, by appropriate use of the exponential mapping, an equivariant variation $f_{s,t}\colon N \to M, 0 \leq s, t \leq 1$, of $f = f_{0,0}$ such that $(f_{0,0})_*(\partial/\partial s) = J'$ and $(f_{0,0})_*(\partial/\partial t) = V'$. This projects to a variation $\bar{f}_{s,t}$ of $\bar{f}$ having the same area function $\mathscr{A}(s,t)$. Hence, $II(J', V') = \left.\frac{\partial^2 \mathscr{A}}{\partial s \partial t}\right|_{t=s=0} = II(J, V) = 0$, and J' is a Jacobi field. The converse is trivial.

Jacobi fields come from variations through minimal varieties. Hence a special class of Jacobi fields is produced by the 1-parameter groups of isometries of M. To be precise these are the normal components of the restrictions to N of the Killing vector fields on M and are designated as *Killing-Jacobi fields*. The dimension of the subspace of such fields is called the *Killing nullity*. (For the special hypersurfaces $M_{p,q}, M^*_{p,q}$ and $M^{**}_{p,q}$ discussed in § 5 the nullity = the Killing Nullity = $(p+1)(q+1), 2pq$, and $4pq$ respectively.)

For a compact minimal submanifold N of M it is not difficult to see that *the Killing nullity of* $N = \dim (I(M)/I(M, N))$(cf, Remark 1.9). In [S] J. Simons characterized the geodesic subsphere of spheres as minimal submanifolds of least nullity. In order to classify closed minimal submanifolds of small, but not necessarily least, nullity it is clear that one must first classify those of *small cohomogeneity*. This program is carried out successfully for the 3-sphere in Chapter IV.

CHAPTER II

ISOMETRY GROUPS OF LOW COHOMOGENEITY

By Theorem 2 the problem of classifying closed G-invariant cohomogeneity-one minimal submanifolds in a manifold M is reduced to finding all "closed" geodesics on the singular manifold $(M/G, g_1)$ where by "closed" we mean compact and without boundary relative to the singular set $(M \sim M^*)/G$. As seen below whenever a geodesic meets a "nice" part of the singular set, the geodesic meets the part transversely and lifts to a non-singular submanifold upstairs.

From now on the purpose of this paper will be to classify, by use of the above observation, all minimal hypersurfaces in S^n of absolute cohomogeneity one. To do this we first classify all linear actions of compact groups on $\boldsymbol{R}^n$, which have low co-dimensional principal orbits, and then compute the metrics, volume functions, etc. for the orbit spaces and analyse their geodesic structures.

1. Classification of compact linear groups with cohomogeneity two

In the important special case that M is the euclidean n-space $\boldsymbol{R}^n, ISO(\boldsymbol{R}^n)$ also preserves the linear structure of $\boldsymbol{R}^n$. The action of any compact subgroup G of $ISO(\boldsymbol{R}^n)$ necessarily has a fixed point and hence, up to conjugation, we may assume that $G \subseteq O(n) \subseteq ISO(\boldsymbol{R}^n)$. The following are some useful observations of [H3, pp. 87–90] for the classification of linear groups with low cohomogeneity.

Observation 2.1. Let ϕ_1, ϕ_2 be two representations of G with $(H_1), (H_2)$ as their principal orbit types respectively, and set

$$\begin{aligned} r_1 &= \dim \phi_1 - \dim (G/H_1) , \\ r_2 &= \dim \phi_2 - \dim (G/H_2) , \\ r &= \dim (\phi_1 + \phi_2) - \dim (G/H) , \end{aligned}$$

where (H) is the principal orbit type of $\phi_1 + \phi_2$. Then one has $r \geq r_1 + r_2$.

Observation 2.2. Let $G \subseteq G' \subseteq O(n)$ be compact linear groups. Then the cohomogeneity of G' is smaller or equal to that of G. From the viewpoint of geometry, one may assume that there does not exist G' such that $G \subsetneq G' \subseteq O(n)$ with the same cohomogeneity as that of G.

Observation 2.3. Suppose ϕ is an irreducible, real representation of a compact connected Lie group $G = G_1 \times G_2$. Then there exist real irreducible representations ϕ_1, ϕ_2 of G_1, G_2 such that

$$\phi | G = k_1 \cdot \phi_1 , \qquad \phi | G_2 = k_2 \cdot \phi_2$$

respectively and $\dim \phi \leq \dim \phi_1 \cdot \dim \phi_2 \leq 4 \dim \phi$.

Observation 2.4. Let ϕ be a *faithful* representation of G, and H_1 a principal isotropy subgroup of ϕ. We define a series of subgroups and representations as follows:

$$\phi | H_1 = (\mathrm{Ad}_G | H_1 - \mathrm{Ad}_{H_1}) + r_1 \theta = \phi_1 + r_1 \theta .$$

H_{i+1} is a principal isotropy subgroup of ϕ_i, which is a representation of H_i, and ϕ_{i+1} is given by

$$\phi_i | H_{i+1} = \phi_{i+1} + r_{i+1} \cdot \theta .$$

We observe that:

(a) H_i is a principal isotropy subgroup of $i \cdot \phi$.

(b) Since ϕ is faithful, ϕ_i are all faithful, and hence $\dim \phi_i \neq 0$ unless $H_i = \{e\}$.

(c) $\dim \phi_i = \dim \phi_{i+1} + r_{i+1}$, and if $\dim \phi_i \neq 0$, then $r_{i+1} > 0$. Hence $\sum r_i \leq \dim \phi$, and $H_i = \{e\}$, $\dim \phi_i = r_i = 0$ for $i > \dim \phi$.

(d) The codimension of the principal orbits of $k \cdot \phi$ is given by

$$kr_1 + (k-1)r_2 + \cdots + r_k .$$

Observation 2.5. If the representation (K, φ) can be realized as the isotropy representation of K on a symmetric space G/K, then the cohomogeneity of (K, φ) equals rank (G/K). For example, if $K = SO(n) \times SO(m)$ and $\varphi = \rho_n \otimes \rho_m$, then the cohomogeneity $=$ rank $(G_{n,m}(\boldsymbol{R})) = \min(m, n)$.

Theorem 5. *Let (G, ϕ) be a maximal compact connected linear group of cohomogeneity two. Then there are the following possibilities*:

(i) *If ϕ is reducible, then $G = SO(k)$, $\phi = \rho_k + \theta$ or $G = SO(k_1) \times SO(k_2)$ and $\phi = \rho_{k_1} + \rho_{k_2}$.*

(ii) *If G is non-simple and irreducible, then either*

	$G = SO(2) \times SO(k)$;	$\phi = \rho_2 \otimes_{\boldsymbol{R}} \rho_k$,
or	$G = S(U(2) \times U(k))$;	$\phi = [\mu_2 \otimes_{\boldsymbol{C}} \mu_k]_{\boldsymbol{R}}$,
or	$G = Sp(2) \times Sp(k)$;	$\phi = \nu_2 \otimes_{\boldsymbol{Q}} \nu_k{}^*$,
or	$G = U(1) \times Spin(10)$;	$\phi = [\mu_1 \otimes_{\boldsymbol{C}} \Delta_{10}^+]_{\boldsymbol{R}}$,
or	$G = U(5)$;	$\phi = [\Lambda^2 \mu_5]_{\boldsymbol{R}}$.

(iii) *If G is simple and irreducible, then the possibilities are given by the following table.*

TABLE 1. *Simple irreducible linear groups of cohomogeneity* 2

G	ϕ	$\dim \phi$	Principal orbit type
$SO(3)$	$(S^2\rho_3 - \theta)$	5	$SO(3)/Z_2 + Z_2$
$SU(3)$	Ad	8	$SU(2)/T^2$
$Sp(3)$	$(\Lambda^2\nu_3 - \theta)$	14	$Sp(3)/Sp(1)^3$
$Sp(2)$	Ad	10	$Sp(2)/T^2$
G_2	Ad	14	G_2/T^2
F_4	φ_4	26	$F_4/Spin(8)$

Remark. It is a rather amusing coincidence that the above linear representations are exactly those *isotropy representations of various symmetric spaces of rank* 2.

Proof of Theorem 5. Since the principal orbit types of all simple linear groups have been classified in [*HH*2], it is not difficult to reduce the proof of the above theorem to those tables of [*HH*2] by means of the observations (2.1)–(2.4); we leave this to the reader.

2. Linear groups of cohomogeneity 3

(A) *Reducible compact linear groups of cohomogeneity* 3

Thenrem 6. *Let* (G, ψ) *be a reducible maximal compact linear group of cohomogeneity* 3. *Then there are the following possilities*;

(i) $\psi = \psi' + \theta$, *and* (G, ψ') *is a compact linear group of cohomogeneity* 2 (cf. Theorem 5).

(ii) $G = SO(k) \times G', \psi = \rho_k + \psi'$, *and* (G', ψ') *is a compact linear group of cohomogeneity* 2 (cf. Theorem 5).

(iii) $G = SO(k), \psi = 2\rho_k$, *or* $G = SO(2) \times SU(k), \psi = \rho_2 \otimes_{\boldsymbol{R}} [\mu_k]_{\boldsymbol{R}}$.

(iv) G *is a circle group acting on* $\boldsymbol{R}^4$.

Proof. From Observation 1 we have that $\psi = \psi_1 + \psi_2$ where $r_1 = 1$ and $r_2 = 1$ or 2. We first discount the special cases (i) and (iv). Then by a theorem of [*MS*] there exists a normal subgroup $G_1 \subseteq G$ which acts transitively on the unit sphere S_1 of the representation space of ψ_1. If G_1 acts trivially on the representation space of ψ_2, then, by Observation 2, we may further assume $G_1 = SO(k)$ and $\psi_1 = \rho_k, \psi = \rho_k + \psi_2$, which is case (ii).

If G_1 acts non-trivially on the representation space of ψ, then it follows from the classification of linear groups of cohomogeneity 1 and case by case checking that the only possible cases are

$$\begin{aligned} &G = SO(k) \text{ or } G_2 \text{ or } Spin(7)\,;\ \psi = 2\cdot\rho_k \text{ or } 2\cdot\varphi_1 \\ &(\dim \varphi_1 = 7) \quad \text{or} \quad \psi = 2\cdot\Delta_7 (\dim \Delta_7 = 8)\,; \\ &G = SO(2) \times SU(k)\ , \qquad \psi = \rho_2 \otimes_{\boldsymbol{R}} [\mu_k]_{\boldsymbol{R}}\ . \end{aligned}$$

However, the cases $G = G_2$ or $Spin(7)$ are not maximal linear groups of cohomogeneity 3. Hence, if G_1 acts non-trivially on the representation space of ψ_2, then it is exactly the case (iii).

(B) *Irreducible, non-simple compact linear groups of cohomogeneity* 3

Theorem 7. *Let* (G, ψ) *be an irreducible non-simple maximal compact linear group of cohomogeneity* 3. *Then there are the following possibilities*;

(i) *If G is semi-simple, then either*

$$G = SO(3) \times SO(n)\ , \quad n \geq 3\ , \quad \psi = \rho_3 \oplus_{\boldsymbol{R}} \rho_n$$

or

$$G = Sp(3) \times Sp(n)\ , \quad n \geq 3\ , \quad \psi = \nu_3 \otimes_{\boldsymbol{Q}} \nu_n^*\ .$$

(ii) *If G is not semi-simple, then either*

$$G = S(U(3) \times U(n))\ , \quad n \geq 3\ , \quad \psi = [\mu_3 \otimes_{\boldsymbol{C}} \mu_n]_{\boldsymbol{R}}$$

or $$G = SO(2) \times Spin(9)\ , \quad \psi = \rho_2 \otimes_{\boldsymbol{R}} \Delta_9\ , \quad \dim \Delta_9 = 16$$

or $$G = U(1) \times E_6\ , \quad \psi = [\mu_1 \otimes_{\boldsymbol{C}} \varphi_1]_{\boldsymbol{R}}\ , \quad \dim_{\boldsymbol{C}} (\varphi_1) = 27$$

or $$G = U(3)\ , \quad \psi = [S^2 \mu_3]_{\boldsymbol{R}}$$

or $$G = U(6), U(7), \quad \psi = [\Lambda^2 \mu_6]_{\boldsymbol{R}}, [\Lambda^2 \mu_7]_{\boldsymbol{R}}\ .$$

Proof. If G is *semi-simple* and *non-simple*, then G consists of exactly two simple normal factors, say $G = G' \times G''$. For otherwise one can show that $\phi(G) \subseteq \rho_n \otimes \rho_m(SO(n) \times SO(m)) \subseteq SO(n\cdot m)$ where $\min(m, n) \geq 4$, and thus by Observation 2.5 the cohomogeneity of G must be ≥ 4. Hence either $\phi = \phi' \otimes_{\boldsymbol{R}} \phi''$ where ϕ' and ϕ'' are real representations of G' and G'' or $\phi = \phi' \otimes_{\boldsymbol{Q}} \phi''$ where ϕ' and ϕ'' are irreducible quaternionic representations of G' and G'' respectively. In the former case, it follows easily from Observation 2.2 that $G = SO(3) \times SO(n), n \geq 3, \phi = \rho_3 \otimes_{\boldsymbol{R}} \rho_n$. In the later case, it is not difficult to check through Table A of [*HH*2] to show that the only possibility is $G = Sp(3) \times Sp(n), n \geq 3, \phi = \nu_3 \otimes_{\boldsymbol{Q}} \nu_n^*$.

If G is not semi-simple, then it follows easily from the Schur lemma and the irreducibility of G that the connected center of G is a circle group S^1, i.e., $G \sim S^1 \times \tilde{G}$ and $\tilde{G}$ is semi-simple. It is easy to show that if $\tilde{G}$ is non-simple, then $G = S(U(3) \times U(n)), n \geq 3$ and $\phi = [\mu_3 \otimes_{\boldsymbol{C}} \mu_n]_{\boldsymbol{R}}$. The remaining case that $\tilde{G}$ is simple follows easily from Table A of [*HH*2].

(C) *Irreducible, simple, compact linear groups of cohomogeneity* 3

It follows directly from Table A of [*HH*2] that they are as follows:

(i) $G = SO(6), SO(7)$ or $Sp(3), \phi = Ad_G$,

(ii) $G = Sp(4), \phi = (\Lambda^2\nu_4 - \theta), \dim_{\boldsymbol{R}}(\phi) = 27$.

Remark. Again, it is interesting to notice that most maximal compact linear groups of cohomogeneity 3 are given by the *isotropy representations of various symmetric spaces of rank* 3. The following four cases are the *only exceptions*:

(1) $G = SO(n), \phi = 2\cdot\rho_n$.

(2) $G = SO(2) \times SU(k), \phi = \rho_2 \otimes_{\boldsymbol{R}} [\mu_k]_{\boldsymbol{R}}$.

(3) $G = SO(2) \times Spin(9), \phi = \rho_2 \otimes_{\boldsymbol{R}} \Delta_9$.

(4) G is a circle group acting on $\boldsymbol{R}^4$.

The above fact is rather useful in the later discussion of the "orbit structure" of linear groups of cohomogeneity 2 and 3.

3. Orbit structures and the induced metrics of the orbit spaces of linear groups of low cohomogeneity

Let M be a differentiable manifold with a given differentiable action of a compact Lie group G. The given G-action provides a decomposition of M into orbits which are homogeneous spaces of G of certain types. Such a decomposition is usually called *the orbit structure of the given G-action* and the decomposition space is called the *orbit space* M/G. As mentioned in Chapter I, the orbits of *a given type* forms a regular invariant submanifold and there is a natural "differentiable stratification" of M by such invariant submanifolds. Furthermore, the orbit space M/G is a differentiable stratified set endowed naturally with a smooth metric structure under which each stratum is a Riemannian manifold. Our purpose here is to compute those orbit spaces $\boldsymbol{R}^n/G$ as such "Riemannian" stratified sets where G is one of the linear groups of cohomogeneity 2 or 3 that we classified in § 1 and § 2.

Since most compact linear groups of cohomogeneity ≤ 3 are given by the isotropy representations of symmetric spaces of rank ≤ 3, it seems proper to recall here some known results on the orbit structure of isotropy representations of symmetric spaces (cf. [*HE*, Chapter VII]).

(A) Let G/K be a compact symmetric space, $\mathfrak{G}$ and $\mathfrak{K}$ be the Lie algebra of G and K respectively, and $\mathfrak{G} = \mathfrak{K} + \mathfrak{g}$ be the usual decomposition. Then the isotropy representation of G/K is conveniently given by $Ad_K(\mathfrak{g})$. Let A be a principal isotropy subgroup of the K-action on $\mathfrak{g}$, and $\mathfrak{h}_\mathfrak{g} = F(A;\mathfrak{g})$ be the fixed point set of A in $\mathfrak{g}$. Then $\mathfrak{h}_\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$ which is usually called the Cartan subalgebra of the above symmetric pair. Let $N(A)$ be the normalizor of A in K, and let $W(G,K) = N(A)/A$. Then $W(G,K)$ is called the Weyl group of the above symmetric pair, and $W(G,K)$ acts naturally on $\mathfrak{h}_\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{G}$ containing $\mathfrak{h}_\mathfrak{g}$, Δ be the root system of $\mathfrak{G}$, and $\Delta_\mathfrak{g}$ be the set of roots with *non-zero* restriction on $\mathfrak{h}_\mathfrak{g}$. Then $W(G,K)$ is generated by reflections $\{s_\alpha;\ \alpha \in \Delta_\mathfrak{g}\}$ where s_α is the reflection with respect to the perpendicular hyperplane of α in $\mathfrak{h}_\mathfrak{g}$. These hyperplanes divide the space $\mathfrak{h}_\mathfrak{g}$ into finitely many connected components, called the Weyl chambers. It is well known that $W(G,K)$ acts simply transitively on the set of Weyl chambers, and the closure of each Weyl chamber is a *fundamental domain* of the $W(G,K)$-action on $\mathfrak{h}_\mathfrak{g}$. The importance of the closed Weyl chamber for our purpose is that *it is also the fundamental domain* of the K-action on $\mathfrak{g}$, which we are interested in. Hence

Proposition 2.1. *Let $\phi = Ad_K\mathfrak{g}$ be the isotropy representation of the symmetric space G/K. Then the orbit space $\mathfrak{g}/\phi(K)$ of the linear K-action ϕ on $\mathfrak{g}$ is given by the Weyl chamber as a Riemannian stratified set. In particular, the natural induced metric is flat and its stratification is given by the walls and their intersections.*

Examples.

(i) $K = SO(n)$(resp. $SU(n), Sp(n)$), $\phi = (S^2\rho_n - \theta)$(resp. $Ad_{SU(n)}, (\Lambda^2\nu_n - \theta)$), $G/K = SU(n)/SO(n)$ (resp. $SU(n) \times SU(n)/SU(n), SU(2n)/Sp(n)$). Then the principal isotropy subgroups A form the conjugacy class of

$$Z_2^{n-1} \text{ (resp. } T^{n-1}, Sp(1)^{n-1}) ,$$

and the Cartan subalgebra $\mathfrak{h}_\mathfrak{g} = F(A,\mathfrak{g}) \approx R^{n-1}$ is an $(n-1)$-dimensional vector space. With suitable coordinates,

$$\mathfrak{h}_\mathfrak{g} = R^{n-1} = \{(\alpha_1\alpha_2, \cdots, \alpha_n) \mid \Sigma\alpha_i = 0\} .$$

The Weyl group $W(G,K)$ acts as permutations of α_i, and the hyperplanes are given by $\{\alpha_i = \alpha_j, i \neq j\}$. Hence the following domain

$$D_0 = \{\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n ; \Sigma\alpha_i = 0\}$$

is a closed Weyl chamber.

(ii) $K = SO(n) \times SO(m)$(resp. $S(U(n) \times U(m)), Sp(n) \times Sp(m)), n \leq m$, $\psi = \rho_n \otimes_{\boldsymbol{R}} \rho_m$ (resp. $[\mu_n \otimes_{\boldsymbol{C}} \mu_m]_{\boldsymbol{R}}, \nu_n \otimes_{\boldsymbol{Q}} \nu_m^*$) and $G/K = SO(n+m)/SO(n) \times SO(m)$ (resp. $SU(n+m)/S(U(n) \times U(m)), Sp(n+m)/Sp(n) \times Sp(m)$). Then the principal isotropy subgroups are of the form $A = \boldsymbol{Z}_2^{n-1} \times SO(m-n)$ (resp. $S(U(1)^n \times U(m-n)), Sp(1)^n \times Sp(m-n)$) and the Cartan subalgebra $= F(A, \mathfrak{g}) \approx \boldsymbol{R}^n$ is an n-dimensional vector space

$$\boldsymbol{R}^n = \{(\alpha_1, \alpha_2, \cdots, \alpha_n) ; \alpha_i \in \boldsymbol{R}\} .$$

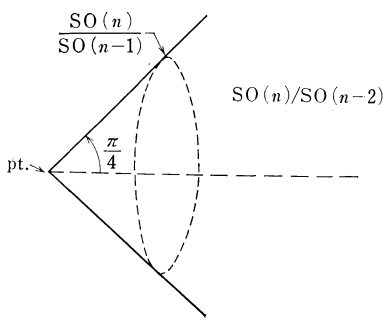
The Weyl group $W(G, K)$ acts as permutations of the α_i with arbitrary changing of signs. The hyperplanes are given by $\pm\alpha_i \pm \alpha_j = 0$ and $\alpha_i = 0$. Hence the following domain

$$D_0 = \{\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0\}$$

is a closed Weyl chamber.

(B) *The exceptional cases*

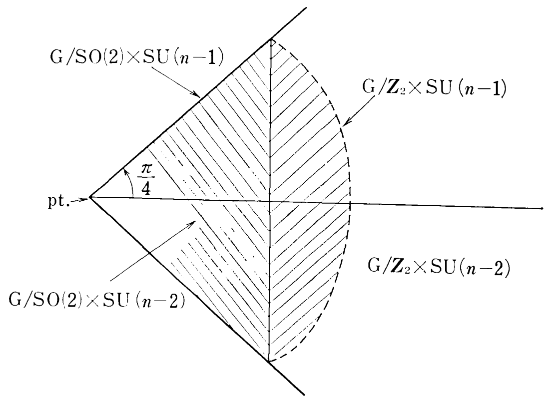
(1) $G = SO(n), \psi = 2\rho_n$: The principal orbit type is the second Stiefel manifold $SO(n)/SO(n-2)$, the singular orbit types are $SO(n)/SO(n-1)$ and point (the origin), and the orbit space $R^{2n}/SO(n)$ with the natural metric is a solid circular cone of angle $\pi/4$.



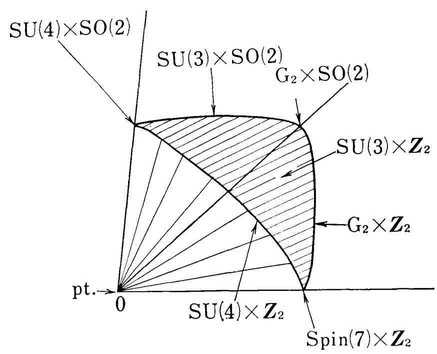
(2) $G = SO(2) \times SU(n), \psi = \rho_2 \otimes_{\boldsymbol{R}} [\mu_k]_{\boldsymbol{R}}$: The orbit space with the induced metric is a half of the above solid circular cone as in the following figure (i).

(3) $G = SO(2) \times Spin(9), \psi = \rho_2 \otimes_{\boldsymbol{R}} \Delta_9$: The orbit structure is as in the following figure (ii). Since we will not need the Riemannian structure of this orbit space, we leave its derivation to the reader.

(4) G is a circle group acting on $\boldsymbol{R}^4$: (cf. Chapter IV).



(i)



(ii)

4. Volume functions

In order to consider the equivariant variational problems of low cohomogeneity on Euclidean spaces and spheres we need to compute the volume functions of those linear groups of cohomogeneity ≤ 3 which we classified in § 1 and § 2. The following observations are useful in such computations:

(A) If $(\boldsymbol{R}^m, G)$ is a compact linear transformation group, then $\boldsymbol{R}^m \xrightarrow{\pi} \boldsymbol{R}^m/G \xrightarrow{V^2} \boldsymbol{R}^+$ is clearly an invariant polynomial function and hence can be expressed in terms of the basic invariant functions.

(B) By definition the volume function V must vanish on the singular set of the Riemannian G-space. In the linear case, V^2 is an element of the ideal corresponding to the singular set. We shall delete the detailed calculations and simply list the results in Tables 2 and in the appendix.

CHAPTER III

CLOSED COHOMOGENEITY-ONE HYPERSURFACES OF S^n

The problem of classifying and understanding the behavior of closed cohomogeneity-one minimal hypersurfaces of S^n has been reduced to studying "closed" geodesics on the orbit spaces, with metric g_1, given in Chapter II. (If $X \subset \boldsymbol{R}^3$ is an orbit space of a cohomogeneity 3 action, the space S^{n-1}/G is represented by $X \cap S^2$.) By "closed" we mean compact with possible boundary at regular points of the singular set.

The orbit spaces in Table 3 fall into two distinct catagories: 1) The orbit space is a disk with rotationally invariant metric. 2) The orbit space is a region of the Euclidean 2-sphere bounded by two or three great circular arcs, and the volume function is the restriction to the sphere of a homogeneous polynomial in $\boldsymbol{R}^3$.

Examples of type 1) will be treated uniformly and completely. For those of type 2) we shall discuss generic cases.

There is one type of action which does not fit into these catagories, namely, circle actions on S^3. These will be treated in Chapter IV.

1. The cases $\rho_n \otimes 2\theta$ and $\rho_n \otimes \rho_n$: Otsuki manifolds

In this section we shall investigate the structure of geodesics on a class of orbit spaces arising from actions of $SO(n)$ on spheres. There are two distinct cases.

Case 1. $SO(n)$ acts on S^{n+1} by $\rho_n \oplus 2\theta$.

Case 2. $SO(n)$ acts on S^{2n-1} by $\rho_n \oplus \rho_n$.

In both cases the orbit space with the natural metric g is a closed circular disk on the constant curvature 2-sphere. There are only two orbit types: the principal orbits ($\approx S^n$ in Case 1 and $\approx SO(n)/SO(n-2)$ in Case 2) which correspond to the interior of the disk, and the singular orbits ($\approx$points in Case 1 and $\approx S^{n-1}$ in Case 2) which correspond to the boundary circle.

We can choose parameters (φ, θ); $0 \leq \varphi < \pi/2$ and $0 \leq \theta < 2\pi$, for these orbit spaces so that the metric g (cf. Chapter I, § 4) has the form $\cos^2 \varphi d\theta^2 + d\varphi^2$ and so that the volume function has the form $\sin^{n-1} \varphi$ for Case 1 and $\sin^{n-2} \varphi$ for Case 2.

Hence minimal submanifolds invariant under the above actions are represented naturally by geodesics on the disk with metric

$$ds^2 = \sin^{2k} \varphi(\cos^2 \varphi d\theta^2 + d\varphi^2) . \tag{3.1}$$

Here θ is the rotational parameter, and φ the radial parameter with $\varphi = 0$ corresponding to the boundary.

Observe that the metric is invariant under rotations in θ. These rotations correspond to the normalizer of the action in the full isometry group (namely, $\rho_n \oplus \rho_2(SO(n) \times SO(2))$ in Case 1 and $\rho_n \otimes \rho_2(SO(n) \times SO(2))$ in Case 2). Geodesics congruent under these rotations correspond to congruent submanifolds in the sphere.

We note also that the metric (3.1) is invariant under the reflection

$$\theta_0 + \theta \to \theta_0 - \theta \tag{3.2}$$

for any θ_0. Hence the diameters $\theta =$ constant are all geodesics. In Case 1 they correspond to geodesic hyperspheres and in Case 2 to the manifolds $M_{n-1,n-1}$ given by (1.5).

By uniqueness for geodesics no other geodesic can be tangent to a diameter, and in particular no other geodesic can pass through the center of the disk. Hence all other geodesics can be expressed as $\varphi = \varphi(\theta)$ where θ is now allowed to vary over all real numbers.

If $\varphi = \varphi(\theta)$ corresponds to a closed geodesic, then there is some point, which we may assume is $\theta = 0$, at which φ assumes a maximum. Hence we may assume that φ satisfies the initial conditions

$$\varphi(0) = a \qquad \text{for } a \in (0, \pi/2)\ , \tag{3.3}$$

$$\varphi'(0) = 0\ , \tag{3.4}$$

$$\varphi''(0) \leq 0\ . \tag{3.5}$$

Using procedures discussed in [C] the general equation for φ is found to be

$$\theta = \pm \int_a^{\varphi(\theta)} \frac{d\varphi}{\cos\varphi \sqrt{\dfrac{\sin^{2k}\varphi \cos^2\varphi}{\sin^{2k} a \cos^2 a} - 1}}\ . \tag{3.6}$$

Thus

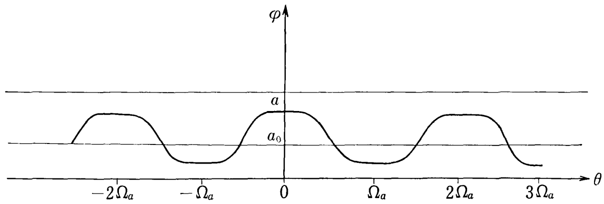
$$\frac{d\varphi}{d\theta} = \pm \cos\varphi \sqrt{\frac{\sin^{2k}\varphi \cos^2\varphi}{\sin^{2k} a \cos^2 a} - 1}\ , \tag{3.7}$$

and

$$\frac{d^2\varphi}{d\theta^2}(0) = \operatorname{ctn} a((k + 1)\cos^2 a - 1)\ . \tag{3.8}$$

From (3.5) and (3.8) we see that $a \geq \cos^{-1}(1/\sqrt{k+1}) = a_0$. When $a = a_0$ we have $\varphi(\theta) \equiv a_0$. This geodesic is the extremal orbit of the rotation group acting on the disk, and the inverse image of this geodesic is the extremal orbit of the normalizer actions mentioned above. In Case 1 this is the generalized Clifford surface $M_{1,n-1}$ and in Case 2 it is the surface $\approx SO(n) \times SO(2)/SO(n - 2) \times Z_2$ discussed in [$H2$].

Suppose $a > a_0$, and let $a' \in (0, a_0)$ be the number such that $\sin^{2k} a' \cos^2 a' = \sin^{2k} a \cos^2 a$. As θ increases from 0, $\varphi(\theta)$ will strictly decrease until $\theta = \Omega_a$ where $\varphi(\Omega_a) = a'$. (The existence of Ω_a is guaranteed by the convergence of the integral (3.6) for $\varphi = a'$.) Then, using the symmetries (3.2), we can continue $\varphi(\theta)$ as a geodesic by reflection at the points $\{n\Omega_a\}_{n\in Z}$.

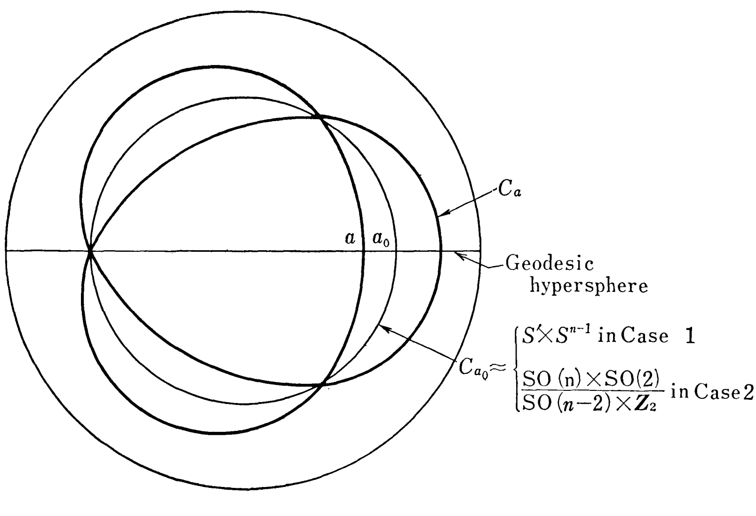


Note that the geodesic $\varphi = \varphi(\theta)$ is closed on the disk if and only if the period

$$\Omega_a = \int_{a'}^{a} \frac{1}{\sqrt{\dfrac{\sin^{2k}\varphi\cos^2\varphi}{\sin^{2k} a\cos^2 a} - 1}} \frac{d\varphi}{\cos\varphi}$$

is a rational multiple of π. Since Ω_a is a non-constant, continuous function of a, we have that there exist countably many such closed geodesics. Thus, letting $A_n = \{a \in (a_0, \pi/2) \colon (1/\pi)\Omega_a \text{ is rational}\} \cup \{a_0\}$ a letting C_a denote the inverse image in the sphere of the closed geodesic satisfying (3.3) and (3.4) (and given locally by (3.6) if $a \neq a_0$), we have

Theorem 8. *The closed, minimal hypersurfaces in S^{n+1} (resp. S^{2n-1}) of cohomogeneity-one with respect to the action $\rho_n \oplus 2\theta$ (resp. $\rho_n \oplus \rho_n$) of $SO(n)$ are the totally geodesic hypersphere and the countably infinite family $\{C_a\}_{a\in A_n}$.*



Note. The examples of Case 1 were first discussed in detail by T. Otsuki [*OT*].

Observe that the question of whether the submanifold C_a is imbedded reduces to the question of whether the period Ω_a ever assumes the value π/m for some integer $m \geq 1$.

When $k = 1$ (the case of $\rho_2 \oplus 2\theta$ acting on S^3 or $\rho_3 \oplus \rho_3$ acting on S^5) we have

$$\Omega_a = \int_a^{\pi/2-a} \frac{d\theta}{\cos\theta\sqrt{\dfrac{\sin^2\theta\cos^2\theta}{\sin^2 a\cos^2 a} - 1}}$$

for $a \in (0, \pi/2)$. (Here a corresponds to a minimum point.) In terms of the complete elliptic integrals K and Π (see [*BF*, pp. 9–10]) we find that

$$\Omega_a = \sin a[K(k) + \tan^2 a\Pi(\alpha^2, k)] ,$$

where

$$k^2 = \frac{\cos^2 a - \sin^2 a}{\cos^2 a} = \alpha^2\cos^2 a .$$

Calculations from tables indicate that in this case all $C_a, a \neq a_0$, have self-intersections.

2. The case $\varphi = \varphi' \oplus \theta$

Let $(\boldsymbol{R}^N, G, \varphi)$ be an (orthogonal) representation, where $\varphi = \varphi' \oplus \theta$ and φ' has cohomogeneity 2. Then the orbit space $\boldsymbol{R}^n/G$ can, after an appropriate transformation, be represented either by $H = \{(x, y, z) \in \boldsymbol{R}^3 \colon x \geq 0\}$ with metric $g_1 = x^n\{(x^2 + y^2)^\alpha(dx^2 + dy^2) + dz^2\}$ for some integer $n > 0$ and some $\alpha < 0$, or by $Q = \{(x, y, z) \in \boldsymbol{R}^3 \colon x \geq 0$ and $y \geq 0\}$ with metric $g_1 = x^p y^p\{(x^2 + y^2)^\beta \cdot(dx^2 + dy^2) + dz^2\}$ for integers $p, q > 0$ and $\beta = 0$ or $-1/2$.

The corresponding orbit space S^{N-1}/G can be represented as either $H \cap S^2$ or $Q \cap S^2$ with the induced metric. Choosing coordinates (φ, θ) for these spaces so that $x = \sin\theta\sin\varphi, y = \cos\theta\sin\varphi,\ z = \cos\varphi$, for $-\pi/2 \leq \varphi \leq \pi/2$ and $0 \leq \varphi \leq \pi$ (or the same with θ replaced by $\theta/2$), we then represent S^{N-1}/G by a square in the (θ, φ)-plane.

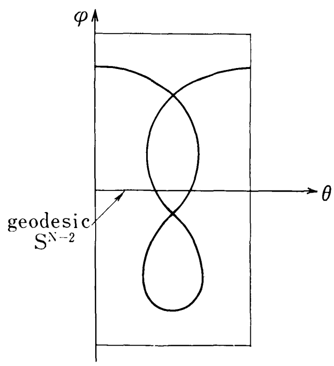
Straightforward computations show the following:

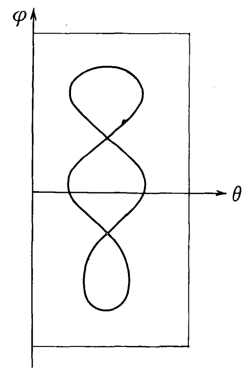
(a) Any geodesic, which meets the singular boundary $\theta = 0$ or $\theta = \pi$, meets it orthogonally. For any point on this boundary, there is exactly one geodesic which emanates from p.

(b) If $\varphi = \varphi(\theta)$ is a geodesic such that $\varphi'(\theta_0) = 0$ and $\varphi(\theta_0) > 0$ (resp. <0), then $\varphi''(\theta_0) < 0$ (resp. >0).

(c) If $\theta = \theta(\varphi)$ is a geodesic such that $\theta'(\varphi_0) = 0$ and $\theta(\varphi_0)$ is close to $\theta = 0$ (resp. $\theta = \pi$), then $\theta''(\varphi_0) < 0$ (resp. >0).

Thus geodesics either meet the boundary orthogonally or are constantly repelled by the boundary. Again one can show that there exist countably many closed geodesics, examples of which are shown in the figures:

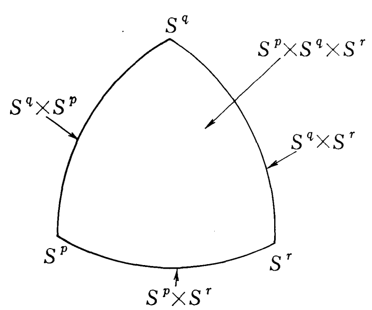




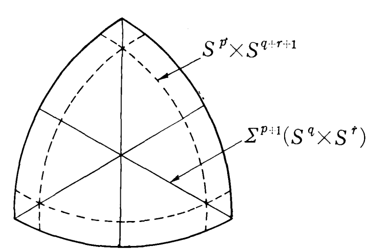
3. The other cases

The orbit spaces of the remaining representations (with the exception treated in Chapter IV) are all given isometrically by the cone in $\boldsymbol{R}^3$ over a geodesic triangle T on S^2. The volume function V is a homogeneous polynomial which vanishes on the planes determining this cone. T and $V \mid T$ represent the orbit space and volume function of the action on the sphere.

Typical of the general case is the following simple example. Let $G = SO(p+1) \times SO(q+1) \times SO(r+1)$ acting on $\boldsymbol{R}^{p+q+r+3}$ by $\rho_{p+1} \oplus \rho_{q+1} \oplus \rho_{r+1}$. The orbit space $X = \{(x, y, z) \in \boldsymbol{R}^3 \colon x \geq 0, y \geq 0, z \geq 0\}$ and $V = x^p y^q z^r$. The orbit types represented by points of $T = X \cap S^2$ are as shown in Figure (a).



(a)



(b)

Each simplex of ∂T corresponds to a geodesic sphere of the appropriate dimension. Each solid line in Figure (b) (representing a geodesic on S^2) corresponds to a minimal variety homeomorphic to a k-fold suspension of a product of spheres, and these lines meet at a point corresponding to the minimal surface $S^p(\sqrt{p/n}) \times S^q(\sqrt{q/n}) \times S^r(\sqrt{r/n})$ where $n = p + q + r$. The dotted lines (also geodesics on S^2) represent minimal surfaces of type $S^p(\sqrt{p/(n+1)}) \times S^{q+r+1}(\sqrt{(q+r+1)/(n+1)})$, etc.

General geodesics on T again have property (a) of the previous section. Moreover, any geodesic which does not meet the boundary orthogonally is eventually turned away from the boundary, and the closer it manages to come the more sharply it is turned.

To analyse the general geodesic behavior in detail we shall consider the special case above where $p = q = r$. In this case each of the distinguished three geodesics which meet in the center is a *line of reflection symmetry for the orbit space T*. We choose parameters (φ, θ) for T by setting $x = \cos\theta\cos\varphi, y = \cos\theta\sin\varphi, z = \sin\varphi$ for $0 \leq \theta, \varphi \leq \pi/2$. The metric g_1 can then be written

$$ds^2 = \sin^p 2\theta \cos^{2p}\varphi \sin^p\varphi(\cos^2\varphi \, d\theta^2 + d\varphi^2) \, .$$

Any geodesic can be locally expressed as either $\varphi = \varphi(\theta)$ or $\theta = \theta(\varphi)$ where these functions satisfy the appropriate Euler-Lagrange equations for the arc-length integral. These equations show:

(i) If $\varphi = \varphi(\theta)$ and $\varphi'(\theta_0) = 0$, then

$$\varphi''(\theta_0) = -(3p + 1)\sin\varphi\cos\varphi + p\tan\varphi \, .$$

Hence $\varphi'' > 0 \Longleftrightarrow \sin^2\varphi < p/(3p + 1)$.

(ii) If $\theta = \theta(\varphi)$ and $\theta'(\varphi_0) = 0$, then $\theta''(\varphi_0) = 2p \operatorname{ctn}(2\theta)/\cos^2\varphi_0$. Hence $\theta'' > 0 \Longleftrightarrow \theta < \pi/4$.

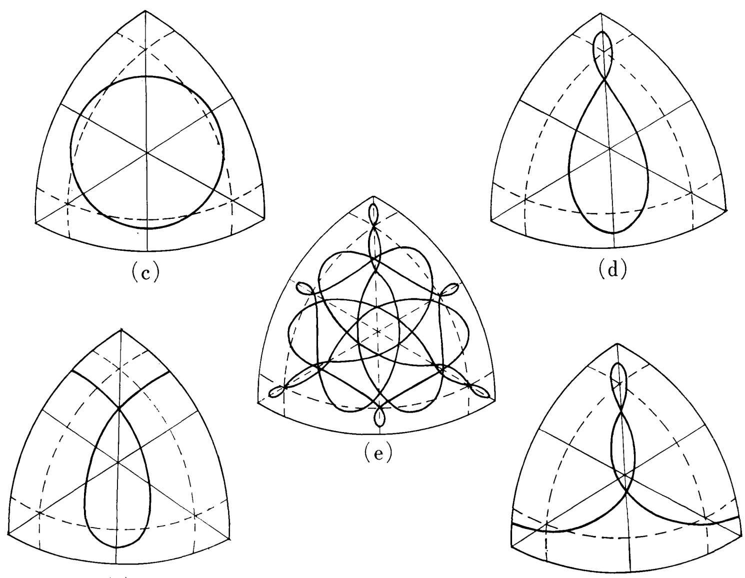
(iii) $\varphi = \varphi(\theta)$ and $\varphi(\theta)$ is close to zero, then

$$\varphi'' \approx \frac{p}{\sin\varphi}[1 + (\varphi')^2]^2 - 2p\operatorname{ctn}^2\theta[1 + (\varphi')^2]\varphi' \, .$$

Using the symmetries of the space together with these three facts allows us to draw strong conclusions about the global behavior of geodesics. We note first that the geodesic $\sin^2\varphi \equiv p/(3p + 1)$ corresponds to the dotted lines of Figure (b). Moreover, each of the six geodesics in Figure (b) divides T into two regions to which either above Observation (i) or (ii) applies (by the rotational symmetry of T). Thus in the small triangle bounded by dotted lines the geodesics behave very regularly. Near the boundary they will obey the equation in (iii).

To find closed geodesics we consider curves which are initially of the type $\varphi = \varphi(\theta)$ where $\varphi(\pi/4) = a, \varphi'(\pi/4) = 0$. To show that any such geodesic is closed it is sufficient to show that it eventually meets one of the lines of reflection symmetry (e.g., $\theta = \pi/4$) orthogonally. The geodesic is then closed by a finite number of successive reflections. Due to the smooth dependence of geodesics on their initial values and our our knowledge of the general behavior of geodesics on T it is straightforward to conclude that *there are countably many distinct closed geodesics on T. Moreover, there exists a closed geodesic on T without self-intersections.* General descriptions of these appear in Figures (c), (d) and (e). By considering the family of geodesics which meets one of the boundary arcs (orthogonally) we similarly conclude that there are countably many such closed geodesics (see Figures (f) and (g)).

For general orbit spaces of the type considered in this section the geodesic behavior will be a distorted version of the one studied here in detail.



(c) (d) (e) (f) (g)

CHAPTER IV

CLASSIFICATION FOR S^3

In the case of S^3 the above procedures give a complete and very pretty classification of low cohomogeneity minimal surfaces.

Let $S^3 = \{(z, w) \in \boldsymbol{C}^2 \colon |z|^2 + |w|^2 = 1\}$. For each pair of integers $m \geq k$ where $(m, k) = 1$ we define an effective unitary representation of the circle group on $\boldsymbol{C}^2$ by the mapping

$$(z, w) \rightarrow (e^{im\theta}z, e^{ik\theta}w)$$

for $\theta \in [0, 2\pi)$, and denote the corresponding subgroup of $SO(4)$ by $G_{m,k}$. Each $G_{m,k}$ represents a distinct action of S^1 on S^3.

In fact, up to conjugation the only closed, connected subgroups of $SO(4)$ are: $SO(3)$, the maximal torus T^2 and the groups $G_{m,k}$. Both $SO(3)$ and T^2 produce codimension 1 orbits on S^3 where the extremal orbits are the geodesic 2-sphere and the *Clifford torus*

$$T = \{(z, w) \in S^3 \colon |z|^2 = |w|^2 = 1/2\} . \tag{4.1}$$

These are the only minimal surfaces of cohomogeneity zero on S^3. There are, however, a remarkable number of minimal surfaces of cohomogeneity-one with respect to each $G_{m,k}$. To understand them it is sufficient to study the closed geodesics on the orbit spaces $X_{m,k} = S^3/G_{m,k}$ with the metric g_1.

The space $X_{1,0}$ is topologically a disk and corresponds to one of the cases treated in detail in Chapter III, § 1.

The spaces $X_{m,k}$ for $m, k \geq 1$ are all homeomorphic to S^2. All but two points of $X_{m,k}$ correspond to principal orbits having trivial isotropy subgroups. The exceptional points correspond to the exceptional orbits

$$c_1 = \{(z, 0) \in C^2 \colon |z| = 1\} ,$$
$$c_2 = \{(0, w) \in C^2 \colon |w| = 1\} ,$$

which have isotropy groups Z_m and Z_k respectively. For these cases the map $\pi \colon S^3 \to X_{m,k} \approx S^2$ is a Seifert fibration.

The normalizer $N(G_{m,k})$ of $G_{m,k}$ in $SO(4)$ is the torus acting on S^3 by

$$(z, w) \to (e^{iu}z, e^{iv}w)$$

for $(u, v) \in R^2$. The group $N(G_{m,k})/G_{m,k}$ acts isometrically on $X_{m,k}$ with metric g_1 (and geodesics on $X_{m,k}$ which are congruent under this group lift to congruent minimal surfaces in S^3), Hence $X_{m,k}$ can be thought of as an ovaloid of revolution with c_1 and c_2 corresponding to axis points.

Note also that in the special case $m = k = 1$ there are, in fact, no exceptional orbits. The map $\pi \colon S^3 \to X_{1,1}$ is the Hopf fibration, and as discussed in Chapter I, § 5; the metric g_1 has constant curvature. The closed geodesics on the orbit space are just great circles, and each of them lifts to a congruent image of the Clifford torus.

To understand the geodesic structure of $X_{m,k}$ for $mk > 1$ we will need to compute the metric. We begin by choosing natural parameters as follows. Let

$$I^2 = \{(\theta, \varphi) \in R^2 \colon 0 \leq \theta < 2\pi \quad \text{and} \quad 0 \leq \varphi \leq \pi\} ,$$

and define $\psi \colon I^2 \to S^3$ by

$$\psi(\theta, \varphi) = (\cos \varphi/2, e^{i\theta} \sin \varphi/2) .$$

Observe that ψ maps the lines $\varphi = 0$ and $\varphi = \pi$ respectively into the exceptional orbits c_1 and c_2. Each of the remaining points is mapped into exactly one of each of the principal orbits. Thus, following ψ by projection onto $X_{m,k}$ gives a coordinate chart for $X_{m,k}$ (in int (I^2)).

The metric g (cf. Chapter I, § 4) in these coordinates is computed as follows. Let X and Y be tangent vectors at $p \in I^2$, and let $\bar{g}$ denote the metric on S^3. Then

$$g(X, Y) = \bar{g}((\psi_* X)^N, (\psi_* Y)^N) ,$$

where $(\ )^N$ denotes the component of the vector normal to the orbit through $\psi(p)$. Following this prescription we get that up to a constant multiple

$$g = \left(\frac{m^2 \sin^2 \varphi}{m^2 \cos^2 \varphi/2 + k^2 \sin^2 \varphi/2}\right) d\theta^2 + d\varphi^2 .$$

The volume function $V(\theta, \varphi)$ is easily seen to be proportional to $(m^2 \cos^2 \varphi/2 + k^2 \sin^2 \varphi/2)^{1/2}$, and thus, up to a constant multiple, the orbit space metric $g_1 = V^2 g$ is given by

$$g_1 = m^2 \sin^2 \varphi d\theta^2 + (m^2 \cos^2 + k^2 \sin^2 \varphi/2) d\varphi^2 .$$

This is the metric of an ovaloid of revolution where φ parameterizes the longitudes. The curvature of the metric is

$$K = \frac{(1/2)(m^2 + k^2)}{(m^2 \cos^2 \varphi/2 + k^2 \sin^2 \varphi/2)^2} .$$

It is immediate that the longitudes $\theta =$ constant are closed geodesics on $X_{m,k}$ whose inverse images are congruent in S^3. This congruence class corresponds to the minimal surface

$$T_{m,k} = \{(z, w) \in S^3 \colon \operatorname{Re}\{z^k \bar{w}^m\} = 0\}$$

first discussed in [$L2$]. This represents an immersed surface of Euler characteristic zero which is non-orientable if and only if $2 | (mk)$. It is furthermore a ruled surface and is algebraic of degree $m + k$.

Let γ be a geodesic not of the form $\theta =$ constant. Then by the uniqueness of geodesics, γ is never tangent to the longitudes and, in particular, never passes through the (exceptional) axis points. Hence γ may be expressed as a function $\varphi = \varphi(\theta)$ where θ now varies over all real numbers.

If $\varphi = \varphi(\theta)$ represents a closed geodesic, then φ takes a minimum value at some point which we may assume to be $\theta = 0$. Assume $\varphi \neq$ constant, and let $\theta = \Omega$ be the first critical point of $\varphi(\theta)$ after $\theta = 0$. Then $\varphi(\theta)$ is increasing on $[0, \Omega]$ and satisfies $\varphi'(0) = \varphi'(\Omega) = 0$. Since the reflections

$$\theta_0 + \theta \to \theta_0 - \theta \qquad \text{for} \quad \theta_0 \in \boldsymbol{R}$$

all represent isometries of $X_{m,k}$, the curve $\varphi(\theta)$ can be continued as a geodesic by successively reflecting the arc $\varphi([0, \Omega])$ at the points $n\Omega$; $n \in \boldsymbol{Z}$. Hence $\varphi(\theta)$ is closed on $X_{m,k}$ if and only if Ω/π is rational.

The congruence classes of closed geodesics on $X_{m,k}$ therefore correspond to the geodesics $\varphi = \varphi(\theta)$ for which

$$\varphi(0) = a\,, \qquad \text{for some} \quad a \in (0, \pi)\,, \tag{4.2}$$

$$\varphi'(0) = 0\,, \tag{4.3}$$

$$\varphi''(0) \leq 0\,, \tag{4.4}$$

and for which $\Omega = \Omega_a$ is a rational multiple of π. Denote these geodesics by γ_a.

Using Hamilton-Jacobi theory (cf. [C]) we find that $\varphi(\theta)$ is given by the equation

$$\theta = \pm \frac{1}{m} \int_a^{\varphi(\theta)} \sqrt{\frac{m^2 \cos^2 \varphi/2 + k^2 \sin^2 \varphi/2}{\dfrac{\sin^2 \varphi}{\sin^2 a} - 1}} \frac{d\varphi}{\sin \varphi}\,. \tag{4.5}$$

Hence

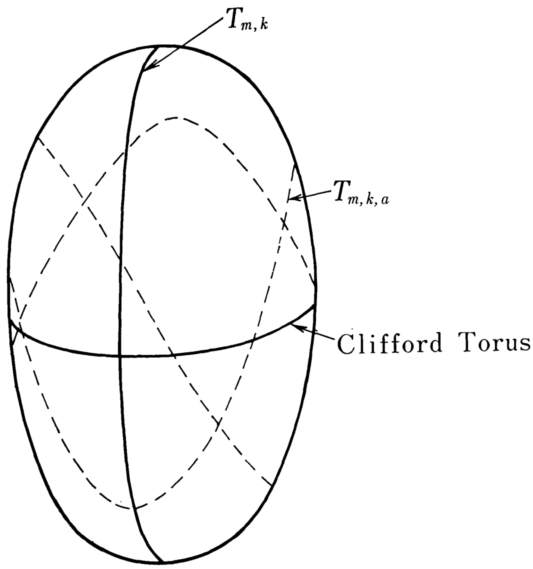
$$\frac{d\varphi}{d\theta} = \pm \frac{m \sin \varphi}{\sqrt{m^2 \cos^2 \varphi/2 + k^2 \sin^2 \varphi/2}} \left(\frac{\sin^2 \varphi}{\sin^2 a} - 1 \right)^{1/2}, \tag{4.6}$$

$$\frac{d^2\varphi}{d\theta^2}(0) = \frac{m^2}{2} \frac{\sin 2a}{m^2 \cos^2 a/2 + k^2 \sin^2 a/2}\,.$$

Note that $\varphi''(0) \geq 0$ if and only if $a \leq \pi/2$, and equality holds only when $a = \pi/2$. When $a = \pi/2$, we have $\varphi = \pi/2$. This geodesic represents the Clifford torus.

When $a < \pi/2$ we see from (4.6) that φ has critical values a and $\pi - a$. Hence $\varphi(\theta)$ oscillates between the curves $\varphi = a$ and $\varphi = \pi - a$ with period $2\Omega_a$ where

$$\Omega_a = \frac{1}{m} \int_a^{\pi - a} \sqrt{\frac{m^2 \cos^2 \varphi/2 + k^2 \sin^2 \varphi/2}{\dfrac{\sin^2 \varphi}{\sin^2 a} - 1}} \frac{d\varphi}{\sin \varphi}\,.$$



$X_{m,k}$

Since Ω_a is a non-constant continuous function of a, we have that there exist countably many non-congruent closed geodesics on $X_{m,k}$. In particular, let $A_{m,k} = \{a \in (0, \pi/2) \colon \Omega_a/\pi \text{ is rational}\}$, and denote by $T_{m,k,a}$ the minimal submanifold of S^3, which is the inverse image under π of the geodesic γ_a. Then, letting $\{C_a\}_{a \in A_2}$ denote the family discussed in Chapter III, § 1 and setting $T_{m,k} = T_{m,k,0}$ for convenience, we have

Theorem 9. *The closed minimal surfaces in S^3 of absolute cohomogeneity-one are exactly, up to congruences, the surfaces, of the family $\mathscr{S} = \{C_a\}_{a \in A_2} \cup \{T_{m,k,a} \colon a \in A_{m,k} \cup \{0\}$, and $m > k \geq 1$ are integers such that $(m, k) = 1\}$. Each of the subfamilies $\{T_{m,k,a}\}_{a \in A_{m,k}}$ is countably infinite.*

Referring to the discussion of Chapter I, § 6 we now have the following.

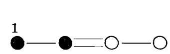
Theorem 10. *Let M be a closed minimal surfacce in S^3 of Killing nullity ν. Then*

$$\nu = 3 \Longleftrightarrow M = \text{a totally geodesic 2-sphere},$$
$$\nu = 4 \Longleftrightarrow M = T\ ,$$
$$\nu = 5 \Longleftrightarrow M \in \mathscr{S}\ .$$

Otherwise $\nu = 6$. In the first two cases ν is the full nullity.

Note. It is an open question whether the manifolds $T_{m,k,a}$, $a \neq 0$, are algebraic. We conjecture that in general they are not.

TABLE II. Linear actions with

K	ϕ	$\dim \phi$	Prin. isotropy subgp. type, H
$SO(n) \times SO(m)$	$\rho_n + \rho_m$	$n + m$, $(n \geq m \geq 1)$	$SO(n-1) \times SO(m-1)$
$SO(2) \times SO(m)$	$\rho_2 \otimes \rho_m$	$2m$, $(m \geq 3)$	$\boldsymbol{Z}_2 \times SO(m-2)$
$S(U(2) \times Um))$	$[\mu_2 \otimes_{\boldsymbol{C}} \mu_m]_{\boldsymbol{R}}$	$4m$, $(m \geq 2)$	$T^2 \times SU(m-2)$
$Sp(2) \times Sp(m)$	$\nu_2 \otimes_{\boldsymbol{Q}} \nu_m^*$	$8m$, $(m \geq 2)$	$Sp(1)^2 \times Sp(m-2)$
$SO(3)$	$S^2\rho_3 - \theta$	5	$\boldsymbol{Z}_2 + \boldsymbol{Z}_2$
$SU(3)$	Ad	8	T^2
$Sp(3)$	$\Lambda^2\nu_3 - \theta$	14	$Sp(1)^3$
$Sp(2)$	Ad	10	T^2
G_2	Ad	14	T^2
$U(1) \times Spin(10)$	$[\mu_1 \otimes_{\boldsymbol{C}} \Delta_{10}^+]_{\boldsymbol{R}}$	32	$T^1 \times SU(4)$
F_4		26	$Spin(8)$
$U(5)$	$[\Lambda^2\mu_5]_{\boldsymbol{R}}$	20	$SU(2) \times SU(2) \times T^1$

codimension 2 principal orbit type

Orbit Space X (As a linear cone in $F(H)$)	volume function f	Associated Sym. space G/K
$x \geq 0; y \geq 0$	$c \cdot x^{n-1} y^{m-1}$	$\dfrac{SO(n+1) \times SO(m+1)}{SO(n) \times SO(m)}$
$y \geq 0, (x-y) \geq 0$	$c \cdot (xy)^{n-2}(x^2 - y^2)$	$\dfrac{SO(m+2)}{SO(2) \times SO(m)}$
$y \geq 0, (x-y) \geq 0$	$c \cdot (xy)^{2n-3}(x^2 - y^2)^2$	$\dfrac{SU(m+2)}{S(U(2) \times U(m))}$
$y \geq 0, (x-y) \geq 0$	$c \cdot (xy)^{4n-5}(x^2 - y^2)^4$	$\dfrac{Sp(m+2)}{Sp(2) \times Sp(m)}$
$\{(\lambda_1, \lambda_2, \lambda_3); \sum \lambda_i = 0$ $\lambda_1 \geq \lambda_2 \geq \lambda_3\}$	$c \cdot (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)$	$\dfrac{SU(3)}{SO(3)}$
$\{(\lambda_1, \lambda_2, \lambda_3); \sum \lambda_i = 0$ $\lambda_1 \geq \lambda_2 \geq \lambda_3\}$	$c \cdot (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_3)^2$	$\dfrac{SU(3) \times SU(3)}{SU(3)}$
$\{(\lambda_1, \lambda_2, \lambda_3\}; \sum \lambda_i = 0$ $\lambda_1 \geq \lambda_2 \geq \lambda_3\}$	$c \cdot (\lambda_1 - \lambda_2)^4(\lambda_2 - \lambda_3)^4(\lambda_1 - \lambda_3)^4$	$\dfrac{SU(6)}{Sp(3)}$
$y \geq 0, (x-y) \geq 0$	$c \cdot (xy)^2(x^2 - y^2)^2$	$\dfrac{Sp(2) \times Sp(2)}{Sp(2)}$
$\{(\lambda_1, \lambda_2, \lambda_3); \sum \lambda_i = 0$ $\lambda_1 \geq \lambda_2 \geq 0\}$	$c \cdot (\lambda_1\lambda_2\lambda_3)^2(\lambda_1 - \lambda_2)^2$ $\cdot (\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2$	$\dfrac{G_2 \times G_2}{G_2}$
$\{(\lambda_1, \lambda_2, \lambda_3); \sum \lambda_i = 0$ $\lambda_1 \geq \lambda_2 \geq \lambda_3\}$	$c \cdot (\lambda_1 - \lambda_2)^{10}(\lambda_2 - \lambda_3)^{10}(\lambda_3 - \lambda_1)^{10}$	$\dfrac{E_6}{U(1) \times Spin(10)}$
$\{(\lambda_1, \lambda_2, \lambda_3); \sum \lambda_i = 0$ $\lambda_1 \geq \lambda_2 \geq \lambda_3\}$	$c \cdot (\lambda_1 - \lambda_2)^8(\lambda_2 - \lambda_3)^8(\lambda_3 - \lambda_1)^8$	$\dfrac{E_6}{F_4}$
$y \geq 0, (x-y) \geq 0$	$c \cdot (xy)^5(x^2 - y^2)^4$	$\dfrac{SO(10)}{U(5)}$

TABLE III. Irreducible linear actions with

K	ϕ	dim ϕ	Prin. isotropy subgp. type, H.
$SO(3) \times SO(n)$	$\rho_3 \otimes_{\boldsymbol{R}} \rho_n$	$3n$, $(n \geq 3)$	$\boldsymbol{Z}_2^2 \times SO(n-2)$
$S(U(3) \times U(n))$	$[\mu_3 \otimes_{\boldsymbol{C}} \mu_n]_{\boldsymbol{R}}$	$6n$, $(n \geq 3)$	$T^3 \times SU(n-3)$
$Sp(3) \times Sp(n)$	$\nu_3 \otimes_{\boldsymbol{Q}} \nu_n^*$	$12n$, $(n \geq 3)$	$Sp(1)^3 \times Sp(n-3)$
$SU(4)$	Ad	15	T^3
$Sp(3)$	Ad	21	T^3
$SO(7)$	Ad	21	T^3
$Sp(4)$	$\Lambda^2\nu_4 - \theta$	27	$Sp(1)^4$
$U(6)$	$[\Lambda^2\mu_6]_{\boldsymbol{R}}$	30	$(SU(2))^3$
$U(7)$	$[\Lambda^2\mu_7]_{\boldsymbol{R}}$	42	$(SU(2))^3 \times U(1)$
$U(1) \times E_6$	$[\mu_1 \otimes_{\boldsymbol{C}} \varphi_1]_{\boldsymbol{R}}$	54	$Spin(8)$
$U(3)$	$[S^2\mu_3]_{\boldsymbol{R}}$	12	$\boldsymbol{Z}_2^3$
$SO(2) \times Spin(9)$	$\rho_2 \otimes_{\boldsymbol{R}} \Delta_9$	32	$SU(3) \times \boldsymbol{Z}_2$

codimension 3 principal orbit type

Orbit Space X, (As a linear cone in $F(H)$)	volume function f	Associated Sym. space G/K
$x_1 \geq x_2 \geq x_3 \geq 0$	$c\cdot(x_1x_2x_3)^{(n-2)}(x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)$	$\frac{SO(n+3)}{SO(3) \times SO(n)}$
$x_1 \geq x_2 \geq x_3 \geq 0$	$c\cdot(x_1x_2x_3)^{2n-5}(\Pi(x_i^2 - x_j^2))^2$	$\frac{SU(n+3)}{S(U(3) \times U(n))}$
$x_1 \geq x_2 \geq x_3 \geq 0$	$c\cdot(x_1x_2x_3)^{4n-9}(\Pi(x_i^2 - x_j^2))^4$	$\frac{Sp(n+3)}{Sp(3) \times Sp(n)}$
$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$	$c\cdot\Pi(\lambda_i - \lambda_j)^2$	$\frac{SU(4) \times SU(4)}{SU(4)}$
$x_1 \geq x_2 \geq x_3 \geq 0$	$c\cdot(x_1x_2x_3)^2\Pi(\lambda_i^2 - \lambda_j^2)^2$	$\frac{Sp(3) \times Sp(3)}{Sp(3)}$
$x_1 \geq x_2 \geq x_3 \geq 0$	$c\cdot(x_1x_2x_3)^2\Pi(\lambda_i^2 - \lambda_j^2)^2$	$\frac{SO(7) \times SO(7)}{SO(7)}$
$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$	$c\cdot\prod_{i<j}(\lambda_i - \lambda_j)^4$	$\frac{SU(8)}{Sp(4)}$
$x_1 \geq x_2 \geq x_3 \geq 0$	$c\cdot x_1x_2x_3\{\Pi(x_i^2 - x_j^2)\}^4$	$\frac{SO(12)}{U(6)}$
$x_1 \geq x_2 \geq x_3 \geq 0$	$c\cdot(x_1x_2x_2)^5\{\Pi(x_i^2 - x_j^2)\}^4$	$\frac{SO(14)}{U(7)}$
$x_1 \geq x_2 \geq x_3 \geq 0$	$c\cdot(x_1x_2x_3)\{\Pi(x_i^2 - x_j^2)\}^8$	$\frac{E_7}{U(1) \times E_6}$
$x_1 \geq x_2 \geq x_3 \geq 0$	$c\cdot(x_1x_2x_3)\Pi(x_i^2 - x_j^2)$	$\frac{Sp(3)}{U(3)}$
		None

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